# A Family of Algebraic Lattices in dimension $2^n$ via Quaternion Algebra


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### Introduction

It has been shown that algebraic lattices, i.e., lattices constructed via the canonical embedding of an algebraic number field, provide an efficient tool for designing lattice codes [2].

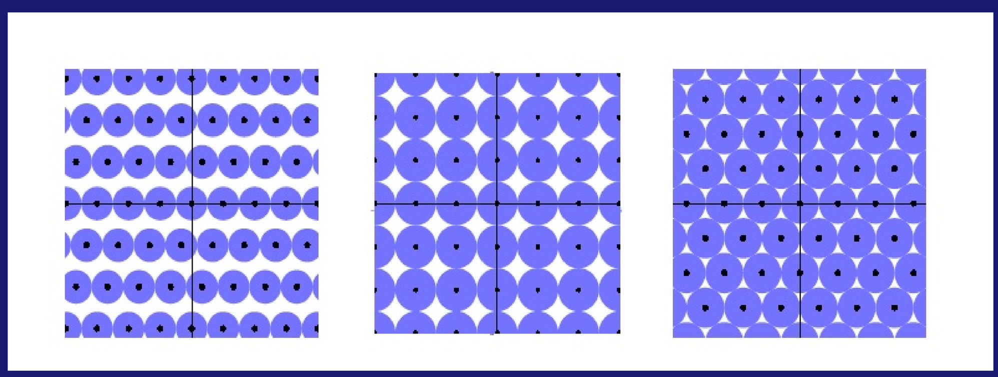

**Figure 1:** Packing density of different lattices

In this work we propose an algebraic construction of lattices in dimension $4n$ via maximal orders of quaternion algebras. Based in [3], we show that we can define algebraic lattices from quaternion algebra in the same way that we define algebraic lattices from number fields. By using the quaternion algebra $\mathcal{A} = (-1,-1)_{\mathbb{K}}$, where $\mathbb{K} = \mathbb{Q}(\zeta_{2^r} + \zeta_{2^r}^{-1})$ we characterize the associated maximal order and construct a family of algebraic lattices in dimension $2^n$, $n > 0$.

### Algebraic Lattices via Quaternion Algebra

A *quaternion algebra* $\mathcal{A} = (a,b)_{\mathbb{K}}$ over a field $\mathbb{K}$ is a central simple algebra of dimension 4 with basis $\{1, i, j, k\}$ satisfying

$$i^2 = a, \; j^2 = b \text{ and } k = ij = -ji,$$

where $a, b \in \mathbb{K}/\{0\}$.

Let $\mathbb{K}$ be a totally real number field with degree $n$, $\alpha$ a totally positive element of $\mathbb{K}$ and let $\mathcal{A}$ be a definite quaternion algebra over $\mathbb{K}$. If $x = x_1 + x_2 i + x_3 j + x_4 k \in \mathcal{A}$ then we define a twisted embedding $\sigma_{\alpha\mathcal{A}}$ from $\mathcal{A}$ to $\mathbb{R}^{4n}$ by

$$\sigma_{\alpha\mathcal{A}}(x) = \Big(\sqrt{2\sigma_1(\alpha)}\sigma_1(x_1), \cdots, \sqrt{2\sigma_n(\alpha)}\sigma_n(x_1), \cdots, \sqrt{2\sigma_1(\alpha)}\sigma_1(x_4), \cdots, \sqrt{2\sigma_n(\alpha)}\sigma_n(x_4)\Big), \quad (1)$$

where $\sigma_1, \cdots, \sigma_n$ are the $n$-embedding of $\mathbb{K}$ in $\mathbb{R}$.

Let $\mathbb{K}$ be a totally real number field with degree $n$, $\alpha$ a totally positive element of $\mathbb{K}$ and let $\mathcal{A}$ be a definite quaternion algebra over $\mathbb{K}$. If $\mathcal{I} \subseteq \mathcal{M}$ is a right ideal of a maximal quaternion order $\mathcal{M}$ of $\mathcal{A}$ with $\mathbb{Z}$-basis $\{w_1, \cdots, w_{4n}\}$, then $\sigma_{\alpha\mathcal{A}}(\mathcal{I})$ is a lattice with basis $\{\sigma_{\alpha\mathcal{A}}(w_1), \cdots, \sigma_{\alpha\mathcal{A}}(w_{4n})\}$ and volume

$$\mathcal{V}ol(\sigma_{\alpha\mathcal{A}}(\mathcal{I})) = \left(N_{\mathbb{K}/\mathbb{Q}}(\alpha) d_{\mathbb{K}}\right)^2 \left(N_{\mathbb{K}/\mathbb{Q}}\left(\det(Trd(v_s \bar{v}_{s'}))_{s,s'=1}^4\right)\right)^{1/2} \quad (2)$$

where $N_{\mathbb{K}/\mathbb{Q}}(\alpha)$ is the norm of $\alpha$, $d_{\mathbb{K}}$ is the discriminant of $\mathbb{K}$ and $\{v_1, v_2, v_3, v_4\}$ is a $\mathcal{O}_{\mathbb{K}}$-basis of $\mathcal{I}$.

Let $\mathcal{A} = (a,b)_{\mathbb{K}}$ be a definite quaternion algebra over a totally real number field $\mathbb{K}$ of degree $n$. If $x \in \mathcal{A}$ then

$$|\sigma_{\alpha\mathcal{A}}(x)|^2 = Tr_{\mathbb{K}/\mathbb{Q}}\left(Trd(\alpha x \bar{x})\right),$$

where $\sigma_{\alpha\mathcal{A}}$ is the embedding defined in (1).

Therefore, by Theorem and Proposition , the center density of the algebraic lattice $\Lambda = \sigma_{\alpha\mathcal{A}}(\mathcal{I})$ is given by

$$\delta(\sigma_{\alpha\mathcal{A}}(\mathcal{I})) = \frac{\left(\sqrt{t_\alpha}\right)^n}{2^n (N_{\mathbb{K}/\mathbb{Q}}(\alpha) d_{\mathbb{K}})^2 \left(N_{\mathbb{K}/\mathbb{Q}}\left(\det(Trd(v_s \bar{v}_{s'}))_{s,s'=1}^4\right)\right)^{1/2}} \quad (3)$$

where $t_\alpha = \min\{Tr_{\mathbb{K}/\mathbb{Q}}\left(Trd(\alpha x \bar{x})\right), \; x \in \mathcal{I}, \; x \neq 0\}$.

Now, we will consider a family of lattices in dimension $2^r$, $r \geq 1$, by using a maximal quaternion order in the quaternion algebra $\mathcal{A}_r = (-1,-1)_{\mathbb{K}_r}$, where $\mathbb{K}_r = \mathbb{Q}\left(\zeta_{2^r} + \zeta_{2^r}^{-1}\right)$.

Let $\mathcal{A}_r = (-1,-1)_{\mathbb{K}_r}$ be the quaternion algebra over $\mathbb{K}_r = \mathbb{Q}\left(\zeta_{2^r} + \zeta_{2^r}^{-1}\right) = \mathbb{Q}(\eta_r)$. Then, $\mathcal{M}_r \supseteq (-1,-1)_{\mathbb{O}_{\mathbb{K}_r}}$ characterized by the basis

$$\mathcal{B}_r = \left\{1, \frac{p_{r-1}(\eta_r)}{2}(1+i), \frac{p_{r-1}(\eta_r)}{2}(1+j), \frac{1+i+j+k}{2}\right\}, \quad (4)$$

is a maximal quaternion order in $\mathcal{A}_r$, where $p_{r-1}(x)$ is the minimal polynomial of $\eta_{r-1}$ and $\mathbb{O}_{\mathbb{K}_r} = \mathbb{Z}[\eta_r]$ is the ring of integers of $\mathbb{K}_r$.

Let $\mathcal{A}_4 = (-1,-1)_{\mathbb{K}}$ be a quaternion algebra, where $\mathbb{K}_4 = \mathbb{Q}(\eta_4)$, $\eta_4 = \left(\zeta_{2^4} + \zeta_{2^4}^{-1}\right) = \sqrt{2+\sqrt{2}}$. A maximal order associated with this algebra is characterized by the basis,

$$\mathcal{B} = \left\{1, \frac{1}{\sqrt{2}}(1+i), \frac{1}{\sqrt{2}}(1+j), \frac{1+i+j+k}{2}\right\}.$$

Choosing $\alpha = (2-\eta_4)^3$ and $\mathcal{I} = \mathcal{M}$, by Theorem , we obtain a 16-dimensional lattice $\Lambda = \sigma_{\alpha\mathcal{A}_4}(\mathcal{I})$ with volume

$$\mathcal{V}ol(\Lambda) = \left(2^3 \cdot 2^{11}\right)^2 \sqrt{1} = 2^{28}.$$

We have that

$$t_\alpha = \min\{Tr_{\mathbb{K}/\mathbb{Q}}\left(Trd(\alpha x \bar{x})\right), \; x \in \mathcal{I}, \; x \neq 0\} = 32.$$

Then, by (3) the center density of $\Lambda$ is

$$\delta(\Lambda) = \frac{\left(\sqrt{32}\right)^{16}}{2^{16} \cdot 2^{28}} = \frac{2^{40}}{2^{44}} = \frac{1}{16}.$$

Therefore, $\Lambda = \sigma_{\alpha\mathcal{A}_4}(\mathcal{I})$ has the same center density of $\Lambda_{16}$ lattice.

In Table 1, we compare the center density of the family of lattices obtained with $\mathbb{Z}^n$ and BW-$n$ lattices.

**Table 1:** Center Density

| $n$ | $\mathbb{Z}^n$ | $\Lambda_n$ | BW-$n$ |
|---|---|---|---|
| 4 | 0.06265 | 0.125 | 0.125 |
| 8 | 0.00390625 | 0.0625 | 0.0625 |
| 16 | $2^{-16}$ | 0.0625 | 0.0625 |
| 32 | $2^{-32}$ | $2^{-16}$ | 1 |
| 64 | $2^{-64}$ | $2^{-32}$ | $2^{16}$ |
| 128 | $2^{-128}$ | $2^{-64}$ | $2^{64}$ |

### References


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