A Note on Matrix Valued Bispectral Triples

Brian David Vasquez Campos [1],

Jorge Passamani Zubelli [2]

Instituto Nacional de Matemtica Pura e Aplicada-IMPA

Estrada Dona Castorina 110/ Zip code: 22460-320

bridava@impa.br

1 Abstract

We consider the triples (L, ψ, B) satisfying system of equations

$$L\psi(x,z) = \psi(x,z)F(z) \quad (\psi B)(x,z) = \theta(x)\psi(x,z) \tag{1}$$

with $L = L(x, \partial_x)$, $B = B(z, \partial_z)$ linear matrix differential operators, i.e $L = \sum_{i=0}^{l} a_i(x)\partial_x^i$, $B = \sum_{j=0}^{m} b_j(z)\partial_z^j$. The functions $a_i, b_j, , F, \theta$ and the common eigenfunction ψ are in principle arbitrary matrix valued functions. A triple (L, ψ, B) satisfying (1) is called a bispectral triple. We fix the normalized [1] operator L and eigenfunctions $\psi(\cdot, z)$. We are interested in the bispectral pairs of $L = L(x.\partial_x)$, i.e operators $B = B(z, \partial_z)$ such that such that $(\psi B)(x,z) = \theta(x)\psi(x,z)$ for some function $\theta = \theta(x)$.

2 Introduction

Classical orthogonal polynomials as well many important special functions satisfy remarkable relations both in the physical as well as the spectral variables [DG04]. More precisely, they are eigenfunctions of an operator in the physical variable (say x) with eigenvalues depending on the spectral variable (say z) as well as the other way around, eigenfunctions of an operator in z with x–dependent eigenvalues. Such bispectral property was explored in the scalar case in the work of J. J. Duistermaat and F. A. Grünbaum [DG86]. It turned out to have deep connections with many problems in Mathematical Physics. Indeed, it could be arranged in suitable manifolds which were naturally parameterized by the flows of the Korteweg de-Vries (KdV) hierarchy or its master-symmetries [ZM91]. It led to generalizations associated to the Kadomtsev-Petviashvili (KP) hierarchy [Zub92, Wil93].

3 Goals

The main goals of this article are: Firstly, to establish a method to verify whether an algebra of matrix polynomials is bispectral or not. Secondly, to obtain an isomorphism between the algebra of (matrix) eigenvalues and spectral-parameter operators. Thirdly, we give a presentation of each (bispectral) algebra using generators and its Gröbner basis. Thus describing the ideal of relations. In this case one eigenvalue is scalar and the other is matrix. For the scalar eigenvalue fixed the algebra of corresponding matrix eigenvalues is characterized, moreover the isomorphism between the matrix eigenvalues and the corresponding operator is given explicitly.

Our results give positive answers to the three conjectures in [Grü14].

4 Results

We first note that θ has to be a polynomial with matrix valued coefficients. The proof follows closely an argument in the original paper of [DG86]. Clearly the set

$$\mathbb{A} = \{\theta | \exists \mathcal{B} = \mathcal{B}(z.\partial_z), (\psi\mathcal{B})(x,z) = \theta(x)\psi(x,z)\} \tag{2}$$

is a noncommutative $\mathbb{C}$-algebra and the previous observation implies that $\mathbb{A} \subset M_n(\mathbb{C})[x]$.

Theorem 1. *Let (L, ψ, B) a bispectral triple, with F scalar and L normalized. Consider the left-bispectral algebra (2) and suppose that there exists an algebra $\Gamma \subset M_n(\mathbb{C})[x]$ with the following properties:*
(1)There exists a finite dimensional vector space E such that $\Gamma = E \bigoplus x^{k_0} M_n(\mathbb{C})[x]$ for some $k_0 \in \mathbb{Z}^+$.
(2) The algebra generated by E is Γ ($Alg(E) = \Gamma$).
(3) $\mathbb{A} \bigcap \bigoplus_{k=0}^{k_0-1} M_n(\mathbb{C})[x]_k = E$. Then $\Gamma = \mathbb{A}$.

Theorem 2. *We assume the equation $L\psi(x,z) = \psi(x,z)F(z)$ and the following properties:*
(1) $\{F^n\}_{n\in\mathbb{N}}$ is a linearly independent set.
(2) there exist $(x_0, z_0) \in \mathbb{C}_\infty \times \mathbb{C}_\infty$ and some scalar function $e = e(x,z)$ such that $e(\cdot, z_0)\psi(\cdot, z_0)$ and $e(x_0, \cdot)\psi(x_{0,\cdot})$ are not zero divisors.
Then we can define the map $\Phi : \mathbb{A} \longrightarrow Bp(z, L)$, $\psi\Phi(\theta) = \theta\psi$, which is an isomorphism of algebras. (Here $Bp(z, L)$ is the set of bispectral pairs of L).

5 Conclusions

Positive answers to the three conjectures given in [Grü14]. We consider the three conjecture. Let

$$\psi(x,z) = \frac{e^{xz}}{(x-2)xz}\begin{pmatrix} \frac{x^3z^2-2x^2z^2-2x^2z+3xz+2x-2}{xz} & \frac{1}{x} \\ \frac{xz-2}{z} & x^2z-2xz-x+1 \end{pmatrix}$$

it is easy to check that $\psi B = \theta\psi$ for

$$B = \partial_z^3 . \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \partial_z^2 . \begin{pmatrix} 0 & 0 \\ -\frac{2z+1}{z} & 0 \end{pmatrix} + \partial_z . \begin{pmatrix} 1 & 0 \\ \frac{2(z-1)}{z^2} & 1 \end{pmatrix} + \begin{pmatrix} -z^{-1} & 0 \\ 6z^{-3} & z^{-1} \end{pmatrix}$$

and

$$\theta(x) = \begin{pmatrix} x & 0 \\ x^2(x-2) & x \end{pmatrix}$$

The following proposition characterize the left bispectral algebra $\mathbb{A}$ of all polynomial F such that there exist $L = L(x, \partial_x)$ with $L\psi = \psi F$.

Proposition 1. *Let Γ be the sub-algebra of $M_2(\mathbb{C})[z]$ of the form*

$$\begin{pmatrix} a & 0 \\ b-a & b \end{pmatrix} + \begin{pmatrix} c & c \\ a-b-c & -c \end{pmatrix} z + \begin{pmatrix} a-b-c & c+a-b \\ d & e \end{pmatrix} \frac{z^2}{2} + z^3 p(z)$$

where $p \in M_2(\mathbb{C})[z]$ and all the variables a, b, c, d, e are arbitrary. Then $\Gamma = \mathbb{A}$

[]

References

[BL08] Carina Boyallian and Jose I Liberati. Matrix-valued bispectral operators and quasideterminants. *Journal of Physics A: Mathematical and Theoretical*, 41(36):365209, aug 2008.

[DG86] J.J. Duistermaat and F.A. Grünbaum. Differential equations in the spectral parameter. *Communications in Mathematical Physics*, 103(2):177–240, 1986.

[DG04] A. J. Durán and F.A Grünbaum. Orthogonal matrix polynomials satisfying second-order differential equations. *Int. Math. Res. Not.*, (10), 2004.

[Gei17] Emil; Yakimov Milen Geiger, Joel; Horozov. Noncommutative bispectral darboux transformations. *Trans. Amer. Math. Soc.*, 369(2):5889–5919, 2017.

[Grü14] F. A. Grünbaum. Some noncommutative matrix algebras arising in the bispectral problem. *SIGMA*, 10:078–088, 2014.

[Wil93] George Wilson. Bispectral commutative ordinary differential operators. *Journal fr die Reine und Angewandte Mathematik*, 442, 01 1993.

[ZM91] J.P. Zubelli and F. Magri. Differential equations in the spectral parameter, Darboux transformations and a hierarchy of master symmetries for KdV. *Communications in Mathematical Physics*, 141(2):329–351, 1991.

[Zub92] Jorge P. Zubelli. On the polynomial τ-functions for the KP hierarchy and the bispectral property. *Lett. Math. Phys.*, 24(1):41–48, 1992.

Acknowledgements

The author acknowledge support from CAPES-Brazil and IMPA.

[1] If $L = L(x, \partial_x)$, $L = \sum_{i=0}^{l} a_i(x)\partial_x^i$ with a_l constant and scalar, $a_{l-1} = 0$, then L is called normalized.