We consider the triple $(L, \psi, B)$ satisfying system of equations
\[ L\psi(x, z) = \psi(x, z)F(z), \quad (\psi B)(x, z) = \theta(x)\psi(x, z) \quad (1) \]
with $L = L(x, \partial_z)$, $B = B(z, \partial_z)$ linear matrix differential operators, i.e $L = \sum_{i=0}^{\infty} a_i(x)\partial_z^i$, $B = \sum_{i=0}^{\infty} b_i(z)\partial_z^i$. The functions $a_i, b_i, F, \theta$ and the common eigenfunction $\psi$ are in principle arbitrary matrix valued functions. A triple $(L, \psi, B)$ satisfying (1) is called a bispectral triple. We fix the normalized 1 operator $L$ and eigenfunctions $\psi_x(z)$. We are interested in the bispectral pairs of $L = L_x(z, \partial_z)$, i.e operators $B = B(z, \partial_z)$ such that such that $(\psi B)(x, z) = \theta(x)\psi(x, z)$ for some function $\theta(x)$.

2 Introduction

Classical orthogonal polynomials as well many important special functions satisfy remarkable relations both in the physical as well as the spectral variables [DG04]. More precisely, they are eigenfunctions of an operator in the physical variable (say $x$) with $x$-dependent eigenvalues. In this case one eigenvalue is scalar and the other is spectral) algebra using generators and its Gröbner basis. Thus describing the ideal of relations. In this case one eigenvalue is scalar and the other is spectral) algebra using generators and its Gröbner basis. Thus describing the ideal of relations. In this case one eigenvalue is scalar and the other is spectral) algebra using generators and its Gröbner basis. Thus describing the ideal of relations. In this case one eigenvalue is scalar and the other is spectral) algebra using generators and its Gröbner basis. Thus describing the ideal of relations.

3 Goals

The main goals of this article are: Firstly, to establish a method to verify if an algebra of matrix polynomials is bispectral or not. Secondly, to obtain an isomorphism between the algebra of (matrix) eigenvalues and spectral-parameter operators. Thirdly, we give a presentation of each (bispectral) algebra using generators and its Gröbner basis. Thus describing the ideal of relations. In this case one eigenvalue is scalar and the other is spectral. For the scalar eigenvalue fixed the algebra of corresponding matrix eigenvalues is characterized, moreover the isomorphism between the matrix eigenvalues and the corresponding operator is given explicitly.

Our results give positive answers to the three conjectures in [Grü14]. We consider the three conjectures.

\[ \psi(x, z) = e^{\frac{xz}{(x^2+1)}} \left( \frac{x^2-2x+1+xz+2z+2z^2}{x^2+z^2} \right) \left( x^2-2xz+1 \right) \]

it is easy to check that $\psi B = \theta \psi$ for
\[ B = \partial_z^2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \partial_x^2 \left( -\frac{2}{x+1} \right) + \partial_x \left( \frac{1}{2(x-1)} \right) + \left( -\frac{z^{-1}}{6z^{-3}} \right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]
and
\[ \theta(x) = \begin{pmatrix} x & 0 \\ x^2(x-2) & x \end{pmatrix} \]

The following proposition characterize the left bispectral algebra $\Lambda$ of all polynomial $F$ such that there exist $L = L(x, \partial_x)$ with $L\psi = \psi F$.

Proposition 1. Let $\Gamma$ be the sub-algebra of $M_2(C) \{z\}$ of the form
\[ \left( \begin{array}{cc} a & 0 \\ b - a & b \end{array} \right) + \left( \begin{array}{cc} c & -c \\ -c & c \end{array} \right) z + \left( \begin{array}{cc} a - b & c \\ c & a - b \end{array} \right) z^2 + z^3 p(z) \]
where $p \in M_2(C)[z]$ and all the variables $a, b, c, d, e$ are arbitrary.

Then $\Gamma = \Lambda$.

References


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