New constructions of finite geometries with four points on a line

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Overview

This work is about finite geometries with finite points on every line and related loops and quasigroups. Steiner triple systems as special block designs are a major part of combinatorics, and there are many interesting connections developed between these combinatorial structures and their algebraic aspects. In this work we consider Steiner triple systems from algebraic point of view, i.e., we study the corresponding Steiner loops. In the case of 4 points on a line we construct new series of such geometries as central extensions of corresponding non-commutative Steiner quasigroups. We conjecture that those geometries are universal in some sense.

Introduction

A set L with a binary operation $L \times L \to L : (x, y) \mapsto x \cdot y$ is called a *loop*, if for given a, b, the equations $a \cdot y = b$ and $x \cdot a = b$ are uniquely solvable, and there is an element $e \in L$ such that $e \cdot x = x \cdot e = x$ for all $x \in L$. A loop is called *diassociative* if every two elements generate a group. A loop L is called *Steiner loop* if $x \cdot (x \cdot y) = y$ holds for all $x, y \in L$ and $x^2 = e$ for all $x \in L$, where e is the identity of L.

Results

Let's remind that $3-$geometries are actually the same objects as Steiner Triple Systems, so $3-$geometries are connected with Steiner loops and Steiner quasigroups. We note that the set of all $Q-$loops is not a variety, but only quasivariety, since (ii) is not identity, but quasi-identity. There exists a relation between $Q-$loops and $Qq-$quasigroups as between Steiner loops and quasigroups.

Proposition 1. *If P is a $Q-$loop, then*
i) $S = P \setminus \{e\}$ is a Steiner system of the type $(2,4)$,(it means that every line contains four points) where a line that passed by x, y is $l(x, y) = \{x, y, xy, yx\}$.
ii) S is a $Qq-$quasigroup with multiplication: $x \circ x = x$, $x \circ y = x \cdot y$, if $x \neq y$.

Central extension of Qq-quasigroups

The simplest example of $Qq-$quasigroup is a F_4-vector space V, where $\mathrm{F}_4 = \{0, 1, \tau, \tau^2 = 1 + \tau\}$ is the field of 4 elements. In this case $v \cdot w = w + \tau(v + w) = \tau v + \tau^2 w$.

Lemma 1. *The set V with multiplication defined above is a $Qq-$quasigroup.*

Definition 1. *Let Q be a $Qq-$quasigroup. It is called $group-like$ if $Q = (V, .)$, where V is a F_4-space with multiplication as defined above.*

If $Q = (V, .)$ is group-like $Qq-$quasigroup then in the corresponding $4-$geometry the line $l(x, y)$, $x, y \in V$, is the line of the affine geometry on $V = \mathrm{F}_4^n : l(x, y) = \{x, y, \tau x + \tau^2 y, \tau y + \tau^2 x\}$.

Let V, W be group-like $Qq-$quasigroups and $\psi : V \times V \to W$ a map (not necessary linear!) $U = V \oplus W$ is a direct sum of the corresponding F_4-spaces. We can define on U a multiplication

$$(v_1, w_1) \cdot (v_2, w_2) = (v_1 \cdot v_2, w_1 \cdot w_2 + \psi(v_1, v_2)).$$

Definition 2. *A map $\psi : V \times V \to W$ is a cocycle if the set $U = V \oplus W$ with multiplication as above is $Qq-$quasigroup.*

If $w_0, ..., w_{N-1}$ is a basis of the F_4-vector space W then any element $w \in W$ has the unique form $w = \sum_{i=0}^{N-1} \alpha_i w_i,\ \alpha_i \in \mathrm{F}_4$.

We can write any element of F_4 in the form τ^s, $s = 0, 1, 2, 3$, where, by definition $\tau^0 = 0$. Hence every $w \in W$ has the unique form $w = \sum_{i=0}^{N-1} \tau^{n_i} w_i$, $n_i \in \{0, 1, 2, 3\}$. Then we have the bijection

$\phi : W \to W_N = \{(n_{N-1} n_{N-2} ... n_1 n_0) | n_i \in \{0, 1, 2, 3\}\}$, $\phi(w) = n_{N-1} ... n_1 n_0$. Below we will use this identification W and W_N. Let $B = B_N$ be a set of lines in W which does not contain 0. Note that a line l contains 0 iff $l = \{0, v, \tau v, \tau^2 v\}$, $v \in W \setminus \{0\}$. For any $l \in B$ consider a $1-$dimensional F_4-space U_l with a basis a_l. We identify U_l with $\mathrm{F}_4 = \mathrm{F}_4 a_l$. Let $T = \{(i, j) | i \neq j \in \{0, 1, 2, 3\}\}-$ the set of all ordered pairs from $\{0, 1, 2, 3\}$. We fix a symmetric map $\psi_l = \psi : T \to U_l$, $\psi(i, j) = \psi(j, i)$, and a partial map $\lambda : W \times W \to T$, $\psi(1, 0) = 1$, $\psi(2, 0) = \tau^2$, $\psi(3, 0) = \tau$, $\psi(2, 1) = \tau$, $\psi(3, 1) = \tau^2$, $\psi(3, 2) = 1$.

The map $\lambda : W \times W \to T$, is defined only if $v \neq w, \tau w, \tau^2 w \in W$. By definition $\lambda(v, w) = (i, j)$, if $\phi(v) = n_{N-1} ... n_1 n_0$, $\phi(w) = m_{N-1} ... m_1 m_0$, $n_{N-1} = m_{N-1}, ..., n_t = m_t$, $n_{t-1} = i \neq j = m_{t-1}$, $t \leq N - 1$.

Finally, we define a map $\pi : W_N \times W_N \to U_N = \sum_{l \in B_N} \oplus U_l$, such that

$\pi(n, m) = 0$, iff $n, m \in \{0, k, k^\tau, k^{\tau^2}\}$ for some k.

$\pi(n, m) = \psi_l(\lambda(n, m))$, if $n, m \in l = \{n, m, n{\cdot}m, m{\cdot}n\} \in B_N$. We can consider U_N as a group like $Qq-$quasigroup. We define a central extension of W_N with U_N by $S_N = W_N \oplus U_N$,

$$(v, x) \cdot (w, y) = (v \cdot w, x \cdot y + \pi(v, w)).$$

Theorem 1. *The set S_N with multiplication above is a non-commutative Steiner quasigroup.*

Lemma 2. *The map π is cocycle iff*
(1) $\tau^2 \pi(v, w) = \pi(w, vw)$,
(2) $\tau \pi(v, w) = \pi(vw, v)$,
(3) $\pi(v, w) + \pi(v, wv) + \pi(w, wv) = 0$.

We aim to soon give an answer to the following:

Conjecture 1. *Let ψ be a cocycle $\psi : V \times V \to Z$, where V, Z are $Qq-$quasigroups of group-like type. Then $\psi(v, w) = \psi(w, v)$.*

Conjecture 2. *Let S be a $Qq-$quasigroup and $S \not\simeq S_1 \times S_2$ for any non-trivial $Qq-$quasigroups S_1 and S_2. Moreover, S is an central extension of two group-like $Qq-$quasigroups V, W. Then $\dim W \leq \frac{(4^n - 1)(4^{n-1} - 1)}{3}$, where $\dim V = n$.*

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