New examples of static spacetimes with a unique standard decomposition

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1. Standard static spacetimes

A connected, oriented Lorentzian manifold admitting a global timelike unitary vector field is time-oriented, so it is a spacetime. If in addition such a vector field is also Killing the spacetime is said to be *stationary*. Among all the stationary spacetimes, those for which the Killing vector field is irrotational are known in the literature as *static spacetimes*. From the physical point of view, those spaces are static relative to the observer field $Z = \frac{1}{\|K\|}K$ determined by K.

It is not difficult to see that for each point p in a static spacetime M, there is a local isometry of a neighbourhood at p in M into a warped product $(I\,_\alpha\!\times \Sigma, g)$, g being

$$g = -\alpha(\pi_\Sigma)^2\pi_I^*(dt^2) + \pi_\Sigma^*(h),$$

where (Σ, h) is an n-dimensional Riemannian manifold, and π_I and π_Σ denote the projections of $I\times\Sigma$ onto I and Σ, respectively. Moreover, the Killing vector field K is transformed into $\partial_t := \partial/\partial t$.

In the case where the above relation is a global isometry, the spacetime (M, g) is called a *standard static spacetime*, and we will identify M with its warped product representation. In order to abbreviate, we will simply write

$$M = \mathbb{R}\,_\alpha\!\times\Sigma \quad \text{and} \quad g = -\alpha^2 dt^2 + h. \tag{1}$$

Moreover, we will say that an $(n+1)$-dimensional spacetime admits a *standard static splitting* (K, Σ) if it admits an splitting as in (1) with $\partial_t = K$.

In any standard static spacetime there is a remarkable family of totally geodesic spacelike hypersurfaces, namely its spacelike *slices* or *leaves* $\Sigma_{t_0} = \{t_0\}\times\Sigma$, $t_0\in I$. Moreover, if Σ is a complete Riemannian manifold, it is well known that M is then globally hyperbolic and such slices are actually Cauchy hypersurfaces.

On the one hand, from a physical standpoint the existence of a global splitting allows the observers in Z to agree in a kind of average time t, or compromise time. On the other hand, standard static spacetimes, and more specifically splitting type theorems appear naturally in the context of singularity theorems. Therefore, a splitting of a manifold as a standard static spacetime provides essential information for both, the geometry and the physical properties of the model. Notice that in the case where there is a unique standard static splitting, such a splitting singles a distinguised timelike direction out. Furthermore, in this case Z is the only complete and integrable observer field, whose observers perceive its common physical space as static.

A spacetime is said to be *spatially closed* if it admits a compact spacelike hypersurface. Equivalently, a standard static spacetime is spatially closed if and only if its base Σ is compact. Otherwise, the spacetime is said to be *spatially open*.

2. Uniqueness results in the spatially closed case

2.1. Sánchez and Senovilla approach

In [3] Sánchez and Senovilla proved the uniqueness of the splitting for a spatially closed standard static spacetime by means of an analysis of the properties of totally geodesic embedded manifolds on such models. Concretely, let N be a totally geodesic embedded submanifold of a spatially closed standard static spacetime $M = \mathbb{R}\,_\alpha\!\times\Sigma$, and let τ be the restriction of the time function on M to N, that is, $\tau = \pi_\mathbb{R}|_N$. Then, they showed that

$$\int_N \alpha^2\|d\tau\|^2 = 0,$$

deducing that $\tau = t_0$ should be constant. Consequently, N is contained in the slice $\Sigma_{t_0} = \{t_0\}\times\Sigma$.

Their result strongly relies on the use of the divergence theorem.

2.2. Aledo, Romero and Rubio approach

The approach followed on [2] is radically different. Given (K,Σ) and $(\tilde K, \tilde\Sigma)$ two different standard static splittings with compact bases, Aledo-Romero-Rubio's approach makes a comparison between the volumes of Σ and $\tilde\Sigma$. It is worth mentioning that this comparison only requires that both Σ and $\tilde\Sigma$ have finite volume. In fact, following the same ideas as in [2] we can deduce the following lemma:

Lemma 1 *Let $(M.g)$ be a standard static spacetime admitting a standard static splitting (K,Σ) with Σ a complete, orientable and simply connected n-Riemannian manifold with finite volume. If $\tilde\Sigma$ is a complete spacelike hypersurface in M, then $\tilde\Sigma$ must have finite volume.*

And, consequently, we can get the following uniqueness result,

Theorem 2 *Let $(M.g)$ be a spacetime admitting a standard static splitting (K,Σ) with Σ a complete and simply connected n-Riemannian manifold with finite volume. Then, every irrotational Killing vector field $\tilde K$ on M must be proportional to K. Equivalently, if (M,g) is a standard static spacetime such that its base has finite volume, then the splitting is unique.*

3. Standard static spacetimes admitting several splittings

In the general case there is an easy way to construct examples of non necessarily flat spacetimes admitting several standard static spacetimes:

Example 1 *Given any complete Riemannian manifold (N, g_N), we can consider the spacetime (M,g) determined by*

$$M = \mathbb{L}^2\times N, \quad g = \alpha^2 g_L + g_N,$$

where g_L is the usual metric in the Lorentz-Minkowski plane $\mathbb{L}^2$ and α a positive smooth function on N. It immediately follows that any standard static splitting of $\mathbb{L}^2$ naturally defines a standard static splitting on M.

Gutiérrez and Olea gave in [4] a rigidity result, proving that under some reasonable assumptions any spacetime admitting two different standard static splittings is isometric to the above ones. Specifically,

Theorem 3 [4, Proposition 4.7] *Let (M,g) be a standard static spacetime as in (1). If M admits a different standard static splitting $(\tilde K,\tilde\Sigma)$ such that ∂_t and $\tilde K$ are linearly independent at some point, then M decomposes as*

$$M = Q\times\mathbb{R}^2, \quad g = g_Q + \beta^2 ds^2 - \alpha^2 dt^2,$$

where β is a smooth and positive function on Q and $\tilde K = \partial_s$; or Σ decomposes as

$$\Sigma = Q\times\mathbb{R}, \quad g_Q + \beta^2 ds^2,$$

and $\alpha = c\,\beta$ for a certain positive $c\in\mathcal{C}^\infty(\mathbb{R})$. Equivalently, M is the warped product $Q\times_\alpha(\mathbb{R}\times_c\mathbb{R}_1)$.

Furthermore, under a suitable curvature hypothesis, they were able to classify static spacetimes admitting more than one static decomposition, and they presented a new uniqueness result under a certain condition on the lightlike sectional curvature.

4. A global approach

We present a new approach for the uniqueness problem for standard static splittings.

Let us begin by considering (M,g) a standard static spacetime as in (1), $\tilde\Sigma$ and immersed spacelike hypersurface on M, and $\tau:\tilde\Sigma\to\mathbb{R}$ defined as $\tau = \pi_\mathbb{R}|_{\tilde\Sigma}$. From a straightforward computation, using Gauss and Weingarten formulae together with the fact that ∂_t is Killing, we can compute the Laplacian of τ in $\tilde\Sigma$, $\tilde\Delta\tau$. And, in the particular case where $\tilde\Sigma$ is maximal, it can be expressed as the following linear elliptic partial differential equation:

$$(\mathcal{L}(\tau) :=)\,\alpha\tilde\Delta\tau + 2\tilde h(\tilde\nabla\alpha, \tilde\nabla\tau) = 0,$$

$\tilde\nabla$ and $\tilde h$ being the gradient operator and the metric induced on $\tilde\Sigma$, respectively. As a consequence of the strong maximum principle applied to the previous operator we get the following result,

Theorem 4 *Let (M,g) be a spatially closed standard static spacetime as in (1). Then, there exists no other maximal Cauchy hypersurface but the slices Σ_{t_0}.*

Therefore, as an inmmediate consequence we get

Corollary 5 *A spacetime (M,g) admitting a compact Cauchy hypersurface admits at most one standard static splitting.*

Thus, we obtain an alternative proof for the uniqueness of static splittings in the spatially closed case.

Previous result only makes special use of the fact that $\tilde\Sigma$ is maximal. However, the existence of another standard static splitting $(\tilde K,\tilde\Sigma)$ will require not just $\tilde\Sigma$ to be a maximal hypersurface, but a totally geodesic one. This stronger condition derives on stronger restrictions for the existence of several splittings. In fact, let (M,g) be a spacetime admitting a standard static splitting (K,Σ) and let $\tilde K$ be another Killing vector field. Given any point $p=(t,x)\in M$, let $\tilde K_\Sigma|_p$ be the horizontal projection of $\tilde K|_p$. Then, if $\tilde K_\Sigma|_p = 0$ for some point $p=(t,x)\in M$ both K and $\tilde K$ are proportional at p, so they have the same orthogonal subspace. This assertion, jointly with the fact that both Σ and $\tilde\Sigma$ are totally geodesic hypersurfaces derives on the following uniqueness result,

Theorem 6 *Let (M,g) be a complete spacetime admitting two standard static splittings (K,Σ) and $(\tilde K,\tilde\Sigma)$. If $\tilde K_{t_0}$ has some zero on Σ_{t_0} for some $t_0\in\mathbb{R}$, then both splittings coincide.*

By a classical result, the orthogonal projection of a Killing vector field $\tilde K$ on Σ is also Killing. Let us denote by $\mathrm{Iso}(\Sigma)$ the set of isometries on M, and by $\mathrm{Iso}_0(\Sigma)$ its connected component containing the identity. The flow of a Killing vector field in Σ naturally defines a one-parameter family of isometries living on $\mathrm{Iso}_0(\Sigma)$, and fixed points of this family are related with zeroes of the Killing field. Following this idea, in case that $\mathrm{Iso}_0(\Sigma)$ is a compact group, we can obtain the following uniqueness result:

Theorem 7 *Let (M,g) be a complete and simply connected spacetime admitting a standard static splitting (K,Σ). Let us assume that $\mathrm{Iso}_0(M)$ is a compact group, and that either Σ has non-positive sectional curvature, or Σ has positive sectional curvature. Then, the previous standard static splitting is unique.*

Let us observe that there is a big variety of Riemannian manifolds Σ satisfying the above assumptions, even in the simplest case where Σ is an embedded surface in $\mathbb{R}^3$. In fact, on the one hand any ruled surface in $\mathbb{R}^3$ has non-positive Gaussian curvature. Therefore, we can consider any simply connected ruled surface in $\mathbb{R}^3$ with compact isometry group, as it is the hyperbolic paraboloid. On the other hand, any simply connected rotational surface in $\mathbb{R}^3$ generated by a strictly convex curve has positive Gaussian curvature and compact isometry group. In this family we can consider, for instance, one of the two connected components of the two-sheeted hyperboloid.

Finally, we can present the following uniqueness result in the 3-dimensional case, [1].

Theorem 8 *Let (M,g) be a 3-dimensional spacetime admitting a standard static splitting (K,Σ), Σ being a complete, non-compact and simply connected Riemannian surface. If the horizonal projection onto some slice Σ_{t_0} of any other Killing timelike vector field $\tilde K$ in M has at least a closed orbit, then the splitting is unique.*

References

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