Generalized Stirling numbers and an analog of exponent in characteristic $p > 0$

Alexandre Grichkov & Ilya Zaharevich

University of São Paulo, IME

shuragri@gmail.com

impa Instituto de Matemática Pura e Aplicada

Overview

Introduction

Results

Consider the following counting problem of selection. Suppose that we have the set of balls such that every ball a is labeled by a pair of numbers (a_1, a_2). For every finite set N of such balls and $a \in N$ we define the numbers

$$t_N(a) = |\{b \in N | b_1 = a_1\}|$$
$$\alpha_i(N) = \frac{|\{a \in N | t_N(a) = i\}|}{i}$$

and a vector $\alpha(N) = (\alpha_1(N), \alpha_2(N), \ldots) \in P = \{(\alpha_1, \alpha_2, \ldots) | \alpha_i \in \mathrm{Z}_+, m, \forall n > m : \alpha_n = 0\}$. We shall call the vector $\alpha(N)$ *the type of the set* N. Now consider n numerated boxes with our balls such that the $i-$th box contains only i balls: $(1, i), (2, i), \ldots, (i, i)$.

Definition 1. *Let $\alpha \in P$ and denote by K^n_α the number of ways to choose a set of balls of the type α from the given n boxes such that from one box we can take no more than one ball.*

We note that if $\alpha = (n, 0, ..., 0)$, then $K^m_\alpha = S(n, m)$ is the classical Stirling numbers of the second type.

In the present work we begin the study of the function K^n_α as the function of n and α. We shall find the recurrence relation for this function and prove some properties of it as the function of n and α.

1 Preliminaries

For a vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in P$ we define the following numbers:

$$|\alpha| = \sum_{i=1}^{m} \alpha_i, \qquad ||\alpha|| = \sum_{i=1}^{m} i\alpha_i, \qquad \langle\alpha\rangle = \prod_{i=1}^{m} \alpha_i!((i+1)!)^{\alpha_i}$$

and operators ∂_i, $(i = 1, 2, \ldots)$ such that

$$\partial_i\alpha = \begin{cases} (\alpha_1 - 1, \alpha_2, \ldots), & \text{if } i = 1 \text{ and } \alpha_1 > 0; \\ (\alpha_1, \ldots, \alpha_{i-1} + 1, \alpha_i - 1, \alpha_{i+1}, ..) & \text{if } i > 1 \text{ and } \alpha_i > 0. \end{cases} \tag{1}$$

For every x and a positive integer k set:

$$x^{[k]} = x(x-1)\cdots(x-k+1).$$

Let p be a positive integer number, denote by $\mathrm{Q}_{(p)}$ the ring of the p-integers numbers, recall that a number $a = n/m, n, m \in \mathrm{Z}$ is called p-integer if $(n, p) = 1$

Theorem 1. *Function K^n_α satisfies the following recurrence relation:*

$$K^{n+1}_\alpha = K^n_\alpha + \sum_{i=2}^{m} (\alpha_{i-1} + 1) K^n_{\partial_i\alpha} + (n - ||\alpha|| + 2) K^n_{\partial_1\alpha} \tag{2}$$

where $\alpha \in \mathrm{Z}^{\mathrm{m}}_+$.

Moreover there exists a unitary polynomial $f_\alpha(x) \in \mathrm{Q}[x]$ of degree $|\alpha| - 1$ such that

$$K^n_\alpha = \frac{(n+1)^{[||\alpha||+1}f_\alpha(n)}{\langle\alpha\rangle} \tag{3}$$

and $f_\alpha(x) \in \mathrm{Q}_{(p)}[x]$, if p is a prime number, $p > ||\alpha|| + 1$.

The crucial role in the proof of this theorem plays the following result.

Theorem 2. *Let $R = \mathrm{Z_p}[[x]]$ be a ring of power series of x over the field of p elements for prime p. Then for every $f \in R$ we have*

$$\underbrace{(\ldots(f'f)'f)'\ldots f)'f}_{p-1} = -(f^{p-1})^{(p-2)}. \tag{4}$$

As corollary of this theorem we have the following proposition.

Proposition 1. *In the ring $\mathrm{Z}[x_1, \ldots, x_{p-1}]$ the following equality holds:*

$(x_1 + \cdots + x_{p-1})(x_1 + \cdots + x_{p-1} - 1)\cdots(x_1 + \cdots + x_{p-1} - p + 2) \equiv$

$$\sum_{\sigma \in S_{p-1}} x_{\sigma 1}(x_{\sigma 1} + x_{\sigma 2} - 1)\cdots(x_{\sigma 1} + \cdots + x_{\sigma(p-1)} - p + 2) \pmod p. \tag{5}$$

Definition 2. *Let $\mathrm{Z}_p[[x]]$ be a ring of series over Z_p. Then $e(x) \in 1 + x\mathrm{Z}_p[[x]]$ is called a Z_p-exponent if the following equation holds*

$$e'(x) = \Phi(x^{p-1})e(x), \; \Phi(x) \in \mathrm{Z}_p[[x]]. \tag{6}$$

Denote by $\mathcal{E}_p$ the set of Z_p-exponents.

For example, the famous Artin-Hasse function

$$AH_p(x) = exp(\sum_{i=0}^{\infty} \frac{x^{p^i}}{p^i})$$

is Z_p-exponent.

Definition 3. *Let $\mathrm{Z}_p[[x]]$ be a ring of series over Z_p. Then $e(x) \in 1 + x\mathrm{Z}_p[[x]]$ is called a Z_p-exponent if the following equation holds*

$$e'(x) = \Phi(x^{p-1})e(x), \; \Phi(x) \in \mathrm{Z}_p[[x]]. \tag{7}$$

Denote by $\mathcal{E}_p$ the set of Z_p-exponents.

For example, the famous Artin-Hasse function

$$AH_p(x) = exp(\sum_{i=0}^{\infty} \frac{x^{p^i}}{p^i})$$

is Z_p-exponent.

It is interesting to find some algebraic solution of equation (7). We try to construct a solution of the equation (7) in the form

$$\begin{aligned} &e(x) = \textstyle\sum_{i=0}^{p-1} x^i/i! + \sum_{i=p+1} e_i x^i, \\ &e_{j(p-1)+1} = 0, j = 0, \ldots. \end{aligned} \tag{8}$$

In this case a function $\Phi(x^{(p-1)})$ may be expressed in terms of $e(x)$. Indeed, we can write uniquely $e(x) = E_0 + E_1 + ... + E_{p-2}$, where $E_i \in x^i\mathrm{Z}_p[[x^{p-1}]]$. Then by (7) we have

$$\begin{aligned} &e'(x) = E'_0 + ... + E'_{p-2} = \Phi(x^{p-1})e(x) = \\ &\Phi(x^{p-1})E_{p-2} + \Phi(x^{p-1})E_0 + ... + \Phi(x^{p-1})E_{p-3}. \end{aligned} \tag{9}$$

By construction of $e(x)$ (see (8)) we have $E_1 = x$, hence by (9) we get

$$x = E_0E'_2 = E_0(E_0E'_3)' = ... = (...(E'_0E_0)'E_0)'E_0)'...E_0)'E_0, \tag{10}$$

where E_0 appears $(p-1)$ times. If the equation (10) has a solution in $\mathrm{Z}_p[[x^{p-1}]]$ then we can construct a solution of (7) in the form (8) with $\Phi(x^{p-1}) = E_0^{-1}(x^{p-1})$.

We note that the first step to solve (10) over $\mathrm{Z}_p[[x^{p-1}]]$ is to find a solution of the equation reduced module p. It is the Theorem 2 above.

Acknowledgement

The first author would like to thank FAPESP for the financial support of this work (process n. 2018/223690-6).