A result about Global bifurcation on Time scales

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INTRODUCTION

We consider the following semilinear parametrized operator equation

$$Lx+\lambda h(x)+k(x)=0, \qquad \lambda \geq 0. \tag{1}$$

The problem related to equation (1) here is the existence of an atypical bifurcation point. We make a topological approach, based on the notion of topological degree for compact perturbations on nonlinear Fredholm maps in Banach spaces. Our objective is to give a result about global bifurcation for the nonlinear equation (1) on time scales.

TIME SCALES

- A time scale $\mathbb{T}$ is a closed, nonempty set of the real numbers.
- $f: T \to X$ is rd-continuous if it is continuous at right-dense points, and left-sided limits exist at left-dense points.
- A subset $\mathscr{F} \subset \mathscr{C}_{rd}([a,b]_{\mathbb{T}}, E)$ is equiregulated, if given $\varepsilon > 0$, $t_0 \in [a,b]_{\mathbb{T}}$, there exists $\delta > 0$ such that

(i) for any $t_0-\delta < s < t_0$, $\|x(s)-x(t_0^-)\| < \varepsilon$;

(ii) for any $t_0 < \tau < t_0+\delta$, $\|x(\tau)-x(t_0^+)\| < \varepsilon$.

- The set $\mathscr{F}$ is rd-equicontinuous, if $\mathscr{F}$ is equiregulated, and for every $\varepsilon > 0$ and for all right-dense $t' \in [a,b]_{\mathbb{T}}$, there exists $\delta > 0$ such that

$$\|f(t)-f(t')\| < \varepsilon, \qquad f \in \mathscr{F}, \quad t \in [t', t'+\delta) \cap [a,b]_{\mathbb{T}}.$$

Proposition 1 (Arzelà-Ascoli). *Let E be a Banach space. The set $H \subset \mathscr{C}_{rd}([a,b]_{\mathbb{T}}, X)$ is relatively compact if and only if, H is rd-equicontinuous and for each $t \in [a,b]_{\mathbb{T}}$, $H(t)$ is relatively compact set in E.*

ORIENTATION AND DEGREE

- $L: E \to F$ is Fredholm if $\ker L$ and $\operatorname{coker} L$ have finite dimension, and the index is defined by $n = \dim\ker L - \dim\operatorname{coker} L$.
- $g: \Omega \to F$ is a Fredholm map if it is C^1 and $g'(x)$ is a Φ_0-operator.
- $f: \Omega \to F$ is quasi-Fredholm if there exists a Φ_0-map g, called smoothing map of f, such that $g-f$ is locally compact.
- $H: \Omega \times [0,1] \to F$ of the form $H(x,\lambda) = G(x,\lambda) - K(x,\lambda)$, where G is a Φ_0-homotopy and K is locally compact, is called a quasi-Fredholm homotopy and G a smoothing map of H.

Definition 2. *An orientation of a Φ_0-map $g: \Omega \subset E \to F$ is an orientation of $g': x \mapsto g'(x)$, denoted by $\alpha(x)$, such that for any x there exists $A \in \alpha(x)$ which is a positive corrector of $h(y)$ for any y in a neighborhood of x.*

Definition 3 (Topological degree). *Let g be a positively oriented smoothing map of f, V an open nbh of $f^{-1}(y) \cap U$ such that $\overline{V} \subseteq U$, g is proper on $\overline{V}$ and $(f-g)|_{\overline{V}}$ has relatively compact image, and $\xi: \overline{V} \to F$ having bounded finite dimensional image and such that*

$$\|g(x)-f(x)-\xi(x)\| < dist_F(y, f(\partial V)), \quad x \in \partial V,$$

where $f(\partial V)$ is closed. Then, define

$$\deg(f,U,y) = \deg(g-\xi, V, y).$$

ATYPICAL BIFURCATION

Definition 4. *An atypical bifurcation point is a point $p \in \Omega \cap \ker L$ such that any neighborhood of $(p,0)$ contains solutions (x,λ) with $\lambda \neq 0$.*

Theorem 5. *Let $H: \Omega \times [0,\infty) \to F$ be an oriented map, defined by*

$$H(x,\lambda) = Lx + \lambda\,(h(x)+k(x)),$$

where $G(x,\lambda) = Lx+\lambda h(x)$ is a Φ_0-homotopy, and $K(x,\lambda) = -\lambda k(x)$ is locally compact. Let

$$v: \Omega \cap \ker L \to F_1, \qquad v(p) = \pi h(p) + \pi k(p).$$

Let $W \subset \Omega \times [0,+\infty)$ be open, and denote by $W_\lambda = \{x \in \Omega : (x,\lambda) \in W\}$. If the Brouwer degree $\deg_B(v, W_0 \cap \ker L, 0)$ is well defined and nonzero, then there exists in W a connected set of nontrivial solutions of the equation (1) whose closure in W is not compact and intersects $\ker L \times \{0\}$.

RESULTS ON TIME SCALES

We consider the following particular case of (1), defined on time scales

$$\begin{cases} x^{\Delta}(t) + \lambda\phi(t, x(t), x^{\Delta}(t)) + \lambda\psi(t, x(t)) = 0 \\ x(0) = x(T). \end{cases} \tag{2}$$

To obtain a global bifurcation theorem on time scales, we assume:

(A1) $\mathbb{T}$ is an unbounded T-periodic time scale containing 0;

(A2) $\phi(t,x(t),x^{\Delta}(t))$ and $\psi(t,x(t))$ are rd–continuous and T-periodic with respect to t, and continuous with respect to x and x^{Δ};

(A3) ϕ is differentiable with respect to the second and third variable, and

$$(t,u,v) \mapsto \partial_2\phi(t,u,v), \qquad (t,u,v) \mapsto \partial_3\phi(t,u,v)$$

are rd-continuous w.r.t. t, and continuous w.r.t. the variables u and v.

Proposition 6. *The map $h(x)(t) = \phi(t,x(t),x^{\Delta}(t))$ is C^1, and*

$$h'(x)q(t) = \partial_2\phi(t,x(t),x^{\Delta}(t))q(t) + \partial_3\phi(t,x(t),x^{\Delta}(t))q^{\Delta}(t).$$

Proposition 7. *Let $G(x,\lambda)(t) = x^{\Delta}(t)+\lambda\phi(t,x(t),x^{\Delta}(t))$. A necessary and sufficient condition for $G'_\lambda(x)$ to be a Fredholm operator of index zero is*

$$\det(I+\lambda M_x(t)) \neq 0, \qquad t \in [0,T]_{\mathbb{T}}.$$

Proposition 8. *The map $k(x)(t) = \psi(t,x(t))$, is compact.*

Theorem 9. *Consider $w: \mathbb{R}^n \to \mathbb{R}^n$ given by*

$$w(p) = \frac{1}{T}\int_0^T \phi(t,p,0) + \psi(t,p)\,\Delta t.$$

Let $W \subset C^1 \times [0,+\infty)$ open, and $W_0 = \{p \in \mathbb{R}^n : (p,0) \in W\}$. Assume that $\deg_B(w, W_0, 0)$ is defined and different from zero. Then, W contains a connected set of nontrivial solutions of problem ..., whose closure in W is not compact and intersects $\ker L \times \{0\} \simeq \mathbb{R}^n$.

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