Spectral Representation of the Covariance Function

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Abstract

We are interested in stationary random processes and their applications to modelling and predicting data collected sequentially in time. The spectral representation of the covariance function (Theorem of Herglotz) is a tool for studying these processes. Informally, the covariance function of every stationary (wide sense) random sequence with zero mean can be represented by one distribution function F, called the spectral distribution function, (up to normalization), whose support is concentrated on $[-\pi, \pi)$.

Introduction

Stochastic processes have been widely used to model some natural and social phenomena, specially the stationary random sequences. In the analysis of time-series (the realizations of a process), many authors are interested in estimators (asymptotic behavior) for the data. At this point, an important tool is the spectral representation of the covariance function of stationary processes. That's the main result of this poster, the Theorem of Herglotz and some classical examples.

Definition 1: A sequence of complex random variables $\xi = (\xi_n)_{n\in\mathbb{Z}}$ with $E|\xi_n|^2 < \infty, n \in \mathbb{Z}$, is stationary in wide sense if, for all $n \in \mathbb{Z}$,

$$E\xi_n = E\xi_0$$

and

$$\operatorname{cov}(\xi_{k+n}, \xi_k) = \operatorname{cov}(\xi_n, \xi_0) := R(n), k \in \mathbb{Z},$$

where $R(n)$ is the covariance function. We shall always suppose that $E\xi_0 = 0$.

Example 1: (White noise) Let $\varepsilon = (\varepsilon_n)$ be an orthonormal random sequence, $E\varepsilon_n = 0, E\varepsilon_i\overline{\varepsilon_j} = \delta_{ij}$. Such a sequence is evidently stationary, and

$$R(n) = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases}$$

Observe that $R(n)$ can be represented in the form $R(n) = \int_{\pi}^{-\pi} e^{i\lambda n} dF(\lambda)$ where

$$F(\lambda) = \int_{-\pi}^{\lambda} f(v)dv; f(\lambda) = \frac{1}{2\pi}, -\pi \leq \lambda < \pi.$$

Example 2: (ARMA) Let $\varepsilon = (\varepsilon_n)$ be the white noise introduced before. The random sequence $\xi = (\xi_n)$ is an autoregressive model of order q if

$$\xi_n + b_1\xi_{n-1} + \ldots + b_q\xi_{n-q} = \varepsilon_n.$$

Suppose we have this below, then $\xi = (\xi_n)$ is a moving average of order p:

$$\xi_n = a_0\varepsilon_n + a_1\varepsilon_{n-1} + \ldots + a_p\varepsilon_{n-p}.$$

Now, we obtain an autoregression and moving average of order (p,q):

$$\xi_n + b_1\xi_{n-1} + \ldots + b_q\xi_{n-q} = a_0\varepsilon_n + a_1\varepsilon_{n-1} + \ldots + a_p\varepsilon_{n-p},$$

It's possible to show that it has the stationary solution $\xi = (\xi_n)$ for which the covariance function is

$$R(n) = \int_{-\pi}^{\pi} e^{i\lambda n} dF(\lambda); F(\lambda) = \int_{-\pi}^{\lambda} f(v)dv; f(\lambda) = \frac{1}{2\pi}\left|\frac{P(e^{-i\lambda})}{Q(e^{-i\lambda})}\right|^2.$$

Results

Theorem (Herglotz) 1: Let $R(n)$ be the covariance function of a stationary random sequence with zero mean. Then, there is, on $([-\pi,\pi), \mathcal{B}([-\pi,\pi)))$, a finite measure $F = F(B), B \in \mathcal{B}([-\pi,\pi))$, such that for every $n \in \mathbb{Z}$,

$$R(n) = \int_{-\pi}^{\pi} e^{i\lambda n} F(d\lambda).$$

Proof 1: For $N \geq 1$ and $\lambda \in [-\pi, \pi]$, define

$$f_N(\lambda) = \frac{1}{2\pi N}\sum_{k=1}^{N}\sum_{l=1}^{N} R(k-l)e^{-i\lambda k}e^{i\lambda l}.$$

Using the nonnegative definiteness of $R(n)$ and doing basic calculations, we can see that f_N is a nonnegative real function and write for $k - l = m$

$$f_N(\lambda) = \frac{1}{2\pi}\sum_{|m|<N}^{N}\left(1 - \frac{|m|}{N}\right)R(m)e^{-i\lambda m}.$$

Let $F_N(B) = \int_B f_N(\lambda)d\lambda$, $B \in \mathcal{B}([-\pi,\pi))$. Then

$$\int_{-\pi}^{\pi} e^{i\lambda n}F_N(d\lambda) = \int_{-\pi}^{\pi} e^{i\lambda n} f_N(\lambda)d\lambda = \begin{cases}\left(1 - \frac{|n|}{N}\right)R(n) & \text{if } |n| < N, \\ 0 & \text{if } |n| \geq N. \end{cases}$$

Note that the measures F_N, $N \geq 1$, are supported on the interval $[-\pi, \pi]$ and $F_N([-\pi,\pi]) = R(0) < \infty$ for all $N \geq 1$. Consequently, by Prokhorov's Theorem, there is a sequence $\{N_k\} \subset \{N\}$ and a measure F such that $F_{N_k} \xrightarrow{w} F$. It then follows that

$$\int_{-\pi}^{\pi} e^{i\lambda n}F(d\lambda) = \lim_{N_k\to\infty}\int_{-\pi}^{\pi} e^{i\lambda n}F_{N_k}(d\lambda) = R(n).$$

To complete the proof, we just have to transfer the "mass" $F(\{\pi\})$, which is concentrated in π, to $-\pi$. The resulting new measure (again denoted by F) will be supported on $[-\pi, \pi)$.

Remark 1: If $\sum|R(n)| < \infty$, it's possible to show that the spectral function $F(\lambda)$ has density $f(\lambda)$ given by

$$f(\lambda) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} e^{-i\lambda n}R(n).$$

Conclusion

The analysis of stationary processes by means of their spectral representation provides complete information on the "spectrum" of the sequence ξ. In addition, the spectral analysis together with the study of robust estimators for the covariance function are big steps towards constructing a better mathematical model for the data.

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