Elliptic Special Weingarten Surfaces of Minimal Type with Finite Total Second Fundamental Form

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To Yulieth, my wife... my whole family

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Abstract

The aim of this work is to study Elliptic Special Weingarten Surfaces of Minimal Type in the Euclidean space $\mathbb{R}^3$, ESWMT-surfaces in short, that is, those whose mean curvature, H, and Gaussian curvature, K, satisfy a relation of the form $H = f(H^2 - K)$, where $f : [0, \infty) \to \mathbb{R}$ is a continuous function of class C^1 in $(0, +\infty)$ satisfying

$$4t(f'(t))^2 < 1 \text{ for all } t \in (0, \infty) \quad \text{and} \quad f(0) = 0.$$

Our main result establishes sufficient conditions for an embedded ESWMT-surface to be rotationally symmetric. This result, together with the characterization of rotationally symmetric examples due to Sa Earp and Toubiana, yields a Schoen type theorem for ESWMT-surfaces. Specifically, we prove

Main Theorem. *Let M be a complete connected ESWMT-surface embedded in $\mathbb{R}^3$ with finite total second fundamental form and two ends. Suppose that the function f associated to M is non-negative and Lipschitz. Then M must be rotationally symmetric. In fact, there exists $\tau > 0$ such that M is the surface of revolution M_τ determined by Sa Earp and Toubiana in [ST95].*

In order to obtain this result, we analyze the asymptotic growth of the height function on the ends of an ESWMT-surface using a conformal parameter and then, using the Kenmotsu representation formula, we are able to describe these ends as graphs of functions of logarithmic order. Finally, we use the Aleksandrov reflection method through tilted planes to obtain the rotational symmetry.

As a corollary of the main theorem, we can recover the original Schoen theorem, namely,

Corollary. *Let M be a complete connected embedded minimal surface in $\mathbb{R}^3$ with finite total curvature and two ends. Then M is a catenoid.*

Resumo

O objetivo deste trabalho é estudar as Superfícies Elípticas Especiais de Weingarten do Tipo Mínimo no espaço euclideano $\mathbb{R}^3$, também chamadas ESWMT-superfícies em referência à nomenclatura inglesa, isto é, aquelas que sua curvatura meia, H, e a curvatura Gaussiana, K, satisfazem uma relação da forma $H = f(H^2 - K)$, onde $f : [0, \infty) \to \mathbb{R}$ é uma função contínua de classe C^1 em $(0, +\infty)$ satisfazendo

$$4t(f'(t))^2 < 1 \text{ para todo } t \in (0, \infty) \quad \text{e} \quad f(0) = 0.$$

Nosso principal resultado estabelece condições suficientes para que uma ESWMT-superfície mergulhada seja rotacionalmente simétrica. Este resultado, junto com a caracterização dos exemplos rotacionalmente simétricos devidos a Sa Earp e Toubiana, resulta em um teorema do tipo Schoen para ESWMT-superfícies. Especificamente provamos que,

Teorema Principal. *Seja M uma ESWMT-superfície conexa, completa e mergulhada em $\mathbb{R}^3$ com segunda forma fundamental total finita e dois fins. Suponha a função f associada à M é não negativa e lipschitziana. Então M deve ser rotacionalmente simétrica. De fato, existe $\tau > 0$ tais que M é a superfície de revolução M_τ determinada por Sa Earp e Toubiana em [ST95].*

Para obter este resultado, nós analisamos o crescimento assintótico da função altura nos fins duma ESWMT-superfície usando um parâmetro conforme e então, usando a formula de representação de Kenmotsu, fomos capazes de descrever os fins como gráficos de funções de ordem logarítmica. Finalmente usamos o método de reflexão de Aleksandrov através de planos inclinados para obter a simetria rotacional.

Como corolário do teorema principal, podemos recuperar o teorema original de Schoen, a saber,

Corolário. *Seja M uma superfície mínima conexa completa e mergulhada em $\mathbb{R}^3$ com curvatura total finita e dois fins. Então M é um catenoide.*

Contents

Introduction

Two of the most remarkable theorems in the study of differential geometry in the large of surfaces immersed in the Euclidean space $\mathbb{R}^3$ are the Liebmann theorem, asserting that an immersed sphere with constant Gaussian curvature is a round sphere; and the Hopf theorem, which states that an immersed sphere with constant mean curvature is a round sphere. Constant Gaussian curvature surfaces and constant mean curvature surfaces can be seen as particular cases of a bigger family of surfaces, the so-called Weingarten surfaces.

Generally speaking, a Weingarten surface (W-surface) is a surface immersed in a three-dimensional manifold such that the principal curvatures, k_1 and k_2, of the immersion satisfy certain relation, i.e., there is a smooth function W of two variables such that $W(k_1, k_2) = 0$, see [Che45; HW54]. This family of surfaces was introduced by Weingarten in the context of finding all surfaces isometric to a given surface of revolution, see [Wei61; Wei63]. Note that a Weingarten relation can be rewritten (generically) as a relation between the mean curvature, H, and the Gaussian curvature, K, of the surface, which we denote in the same way, $W(H, K) = 0$. Closed Weingarten surfaces in $\mathbb{R}^3$ were widely studied by Hartman and Wintner [HW54], Chern [Che45; Che55b], Hopf [Hop83] and Bryant [Bry11] obtaining, among others, a generalization of Hopf's theorem and Liebmann's theorem.

When the function W is linear, that is, when we are considering a Weingarten relation of the form $aH + bK = c$, where a, b, c are real constants, W-surfaces are called Linear Weingarten surfaces, abbreviated by LW-surfaces. These surfaces have been studied by several authors. For instance, Rosenberg and Sa Earp in [RS94] showed that if $b = 1$ and $a, c > 0$, the annular ends of a properly embedded LW-surface converge to Delaunay ends. Gálvez et al. in [Gá+03] obtained a harmonic representation of LW-surfaces of elliptic type, and optimal height and curvature estimates, moreover, they characterized the spherical caps as the only LW-surfaces of elliptic type achieving these bounds. On the other hand, López gave a classification of rotational LW-surfaces of hyperbolic type, see [Ló08b]. Immersed LW-surfaces in other ambient spaces have been also analyzed, to mention, Aledo and Espinar studied LW-surfaces in the de Sitter space reaching a classification of complete LW-surfaces with non-negative Gaussian curvature, see [AE07]; and Barros et al. in [Bar+12] presented a complete description of all rotational LW-surfaces in the Euclidean sphere $\mathbb{S}^3$.

It is a special case when the Weingarten relation $W(H, K) = 0$ can be written as $H = f(H^2 - K)$, where $f : [0, \infty) \to \mathbb{R}$ is a continuous function of class C^1 in $(0, +\infty)$. In this case, the W-surfaces are called Special Weingarten surfaces or simply SW-surfaces and we will refer to f as the associated function of the SW-surface. An interesting class of SW-surfaces is obtained when the function f satisfies $4t(f'(t))^2 < 1$ for all $t \in (0, \infty)$. This class of surfaces is called Special Weingarten surfaces of elliptic type and denoted by ESW-surfaces or f-surfaces, see [ST93; ST99b; ST01; Ale+10]. Rosenberg and Sa Earp in [RS94] proved that there exists a large family of LW-surfaces contained in the family of ESW-surfaces, namely when $b = 1$, $a \geq 0$ and $c > 0$. One of the most useful results about the family of ESW-surfaces in $\mathbb{R}^3$ is that it satisfies the maximum principle, a result achieved by Rosenberg and Sa Earp in [RS94], and Braga and Sa Earp in [BS97]. Aledo et al. in [Ale+10] classified ESW-surfaces for which the Gaussian curvature does not change sign, obtaining that, if an ESW-surface is complete with $K \geq 0$ then it must be either a totally umbilical sphere, a plane or a right circular cylinder, and, if an ESW-surface is properly embedded with $K \leq 0$, then it is a right circular cylinder or $f(0) = 0$. In [Gá+16], Gálvez et al. improved the above result in the case $K \leq 0$ removing the properly embedded hypothesis and assuming completeness.

The family of ESW-surfaces can be divided in two classes, namely $f(0) \neq 0$ and $f(0) = 0$. In the first case, ESW-surfaces are called of constant mean curvature type, and in the second case, are called Elliptic Special Weingarten surfaces of minimal type, ESWMT-surfaces in short. For the first family of surfaces, Aledo et al. in [Ale+10] extended the results obtained in the theory of constant mean curvature surfaces by Korevaar et al. in [Kor+89], specifically, they showed that if an ESW-surface of constant mean curvature type is properly embedded, in $\mathbb{R}^3$ or $\mathbb{H}^3$, has finite topology and k ends, then, $k \geq 2$, is rotationally symmetric if $k = 2$, and is contained in a slab if $k = 3$. On the other hand, about ESWMT-surfaces, Sa Earp and Toubiana in [ST95] proved existence and uniqueness of rotational symmetric ESWMT-surfaces. They also proved, under some additional condition on the function f, that there exists an one-parameter family of ESWMT-surfaces that behaves as the family of minimal catenoids, allowing them to prove a half-space theorem in this context. ESWMT-surfaces in other ambient spaces have been studied by Morabito and Rodríguez in [MR13], where they established necessary and sufficient conditions for existence and uniqueness of complete rotational ESWMT-surfaces in $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$ and showed the existence of non-complete examples of such surfaces. Also, de Almeida et al. proved in [de +09] the existence of rotational ESWMT-surfaces in $\mathbb{S}^3$.

Trivially, any plane is an ESWMT-surface. This basic fact together with the maximum principle make us think in the theory of ESWMT-surfaces in the same way as the theory of minimal surfaces. Hence, developing the understanding of ESWMT-surfaces we are contributing to improve the knowledge of a broader class of surfaces. With this premise in mind, it is natural to ask: which of the classical theorems of minimal surfaces in $\mathbb{R}^3$ can be extended to ESWMT-surfaces?

The theory of minimal surfaces is huge, nevertheless we will focus for a moment on complete minimal surfaces with finite total curvature in $\mathbb{R}^3$. Finite total curvature means that the Gaussian curvature, K, is integrable on M, i.e.,

$$\int_M K\,dA < \infty.$$

A classical result, due to Chern and Osserman in [CO67], states that each of these surfaces is of finite topological type. Specifically, it is conformally equivalent to a compact Riemann surface with a finite number of points removed. These points are the representation of the ends of the minimal surface. In particular, each end is conformally a punctured disk. Moreover, the Gauss map of the immersion extends to a meromorphic function defined over the whole disk, which means that the Gauss map can be defined globally over the compact Riemann surface associated to the minimal surface. Jorge and Meeks III, in [JM83], proved that a surface of finite topological type, such that the Gauss map is well defined at infinity, is properly embedded and, viewed from infinity, looks like a finite collection of planes passing through the origin. A nice way to interpret this result is to think that the ends of such surfaces behave like the ends of either a catenoid or a plane at infinity. Using this result they showed that an embedded minimal surface with finite total curvature must be a flat plane or it has at least two ends; in particular, if the surface is not a plane, it cannot be simply connected. Furthermore, they proved that the total curvature of a complete minimal surface M with finite total curvature, Euler characteristic $\chi(M)$ and k ends must satisfies

$$\int_M K\,dA = 2\pi\,(\chi(M) - k)$$

if, and only if, the ends are embedded. As a consequence of this formula, they obtained a characterization of the catenoid as the unique embedded minimal surface with finite total curvature, two ends and genus zero.

The characterization obtained for the catenoid by Jorge and Meeks III was improved by Schoen in [Sch83] assuming the embedding hypothesis just for the ends and without any assumption on the topology of the surface. More precisely, he proved that a complete minimal immersion of a hypersurface in $\mathbb{R}^n$, which is regular at infinity and has two ends, must be a pair of hyperplanes or a catenoid. The concept of regular at infinity defined by Schoen means, roughly speaking, that each end of the surface is well behaved in the sense that the "growth at infinity" from some fixed plane is controlled and almost radial with respect to some axis. In other words, that the ends are quite similar to the ends of either a catenoid or a plane. The precise definition will be presented in Chapter 3. In the Euclidean space $\mathbb{R}^3$, Schoen proved that the property of being regular at infinity for an immersed minimal surface M is equivalent to finite total curvature and embedded ends. There is a version of Schoen's theorem in $\mathbb{H}^2 \times \mathbb{R}$ due to Hauswirth et al., see [Hau+15]. In this paper, they proved that a complete minimal surface immersed in $\mathbb{H}^2 \times \mathbb{R}$ with finite total curvature and two ends, each one asymptotic to a vertical geodesic plane, must be a horizontal catenoid.

Taking into account the existence and uniqueness of revolution ESWMT-surfaces, we pursuit to obtain a Schoen type theorem, which establishes these examples as the only ones with finite total curvature.

Schoen's proof can be described in three parts.

- As we have mentioned, complete minimal surfaces with finite total curvature are of finite topological type and the Gauss map extends conformally to infinity, see [CO67; Oss69; JM83].
- There is a well-known link between minimal surfaces and complex analysis. This relation is given by the Weierstrass representation, which says that any minimal surface can be represented by complex holomorphic functions, called Weierstrass data. By Osserman's results, the Weierstrass data can be extended to the ends as meromorphic functions. Schoen used these extensions and the Weierstrass representation to derive an asymptotic expansion for embedded ends, concluding that each end must be geometrically asymptotic either to a plane or to a catenoid. By applying the flux formula he proved that if the surface is connected and has two ends, then the ends should be parallel, and by the half-space theorem, both ends have (simultaneously) logarithmic growth, in opposite directions.
- Finally, Schoen used the Aleksandrov reflection method to prove that the surface has a horizontal plane of symmetry and a vertical axis of symmetry. This means that the minimal surface is rotationally symmetric, hence a catenoid or a pair of planes.

As we can see, Schoen's proof relies strongly on the fact that the surface is minimal and on the connection of the theory of minimal surfaces with complex analysis. This reasoning cannot be translated directly to ESWMT-surfaces because these surfaces have no natural connection with complex analysis. This is one of the obstacles to overcome. We are going to introduce some techniques that helped us to use a similar scheme as Schoen, but avoiding the minimal hypothesis.

The first point to deal with is the finite total curvature hypothesis. Thanks to the work of Huber it is well known that surfaces with finite total curvature has a natural compactification, see [Hub57]. In this paper, he proved that a complete surface with finite total curvature is isometric to a compact Riemannian surface with finitely many singular points, each admitting a neighborhood isometric to the punctured disk equipped with a conformal flat metric, see [HT92]. Unfortunately, the hypothesis of finite total curvature does not imply that the Gauss map can be extended to the singularity, that is, the Gauss map can have an undesirable behavior at infinity, see [Whi87; Con+15] for examples of such kind of ends. To settle this situation, we consider a stronger assumption namely finite total second fundamental form, which means that the norm of the second fundamental form, $|II|$, of an immersed surface M in $\mathbb{R}^3$, is L^2 on the surface, that is,

$$\int_M |II|^2 \, dA < \infty.$$

White proved in [Whi87] that if a surface has finite total second fundamental form, then it has finite total curvature and also the Gauss map extends continuously to infinity. Furthermore, it can be proved that if the end is embedded, then the end is a graph over a plane orthogonal to the Gauss map at infinity. White asked whether finite total second fundamental form implies properly immersed, this was answered positively by Müller and Šverák in [MŠ95]. Hence, for ESWMT-surfaces, we will adopt finite total second fundamental form as a natural hypothesis.

Once we have adopted the finite total second fundamental form hypothesis, the second point to deal with is the growth of an embedded end. In 1915, Bernstein proved that a C^2 function $u(x,y)$ defined in the whole x,y-plane such that $u_{xx}u_{yy} - u_{xy}^2 \leq 0$ and $u_{xx}u_{yy} - u_{xy}^2 \not\equiv 0$ cannot be bounded, see [Ber15]. The original proof had a gap of topological nature. Hopf, in 1950, bridged this gap and improved the result of Bernstein, see [Hop50a; Hop50b]. The theorem of Hopf establishes that a function with the conditions imposed by Bernstein cannot be $o(r)$ where r is the distance from an arbitrary point to a chosen fixed point. The proof presented by Hopf was adapted by Chan to prove that there is no isometric immersion in $\mathbb{R}^3$ from a complete oriented, one ended, non-positively curved surface with an isolated set of parabolic points and finite total second fundamental form, such that the end is embedded, see [Cha00]. We adapt Chan's proof to demonstrate that if the graph of a real valued function, defined outside a compact subset of $\mathbb{R}^2$, is an ESWMT-surface with finite total second fundamental form, then the function grows infinitely just in one direction, that is, it has either a finite lower or an upper bound. This fact may seem natural by the hypothesis on the curvatures, however, we can find in the literature examples of surfaces with finite total curvature that have highly complicated ends, as for instance the examples exhibited by Connor et al. in [Con+15].

The study of the Gauss map of an immersed surface in $\mathbb{R}^3$ by isothermal coordinates can derive in representation formulas analogous to the Weierstrass representation, see [Ken79; GM00]. This was the idea developed in [Ken79] by Kenmotsu to obtain a representation formula for immersed surfaces of prescribed mean curvature by means of the Gauss map. Nevertheless, since the Gauss map just extends continuously to singularities, the functions involved in the Kenmotsu representation may not have an extension to infinity as meromorphic functions.

Using the fact that ends of ESWMT-surfaces with finite total second fundamental form cannot grow infinitely in opposite directions, we are able to use some results of Taliaferro about the growth of a positive superharmonic function near an isolated singularity, see [Tal99; Tal01a], for obtaining a Bôcher type theorem that gives us a full description of the height function defined on a conformal parameter. In order to use the Kenmotsu representation, it is necessary to have some additional information about the Gauss map. It is known that the Gauss map, g, satisfies a Beltrami equation $g_{\bar{z}} = \mu g_z$, see [Ken79]. This differential equation was used by Gauss to prove the local existence of isothermal coordinates on a surface with analytic Riemannian metric, in particular, Gauss proved the existence and uniqueness of a solution in the case that μ is real analytic. Ahlfors noticed a close relation between the Beltrami equation and quasiconformal maps, a generalization of conformal maps. The notion of a quasiconformal map, was introduced by Grötzsch in 1928, see [Grö28], and is based on the fact that there is no conformal map from a square to a rectangle, not square, which maps vertices on vertices. Grötzsch asks for the most nearly conformal map of this kind. This calls for a measure of approximate conformality and, in supplying such a measure, Grötzsch took the first step toward the creation of the theory of quasiconformal maps. Roughly speaking, a quasiconformal map is an homeomorphism between complex domains with positive Jacobian that takes infinitesimal circles into infinitesimal ellipses with eccentricity uniformly bounded. It was pointed by Sa Earp and Toubiana that strengthening the elliptic condition on the Weingarten relation, the Gauss map is a quasiconformal map. We prove this assertion and use it together with the Mori theorem, see [Mor56], to control the behavior of the Gauss map at infinity. This allows us to use the Kenmotsu representation to describe the growth of the ends in Cartesian coordinates, obtaining a kind of regularity at infinity for the ends in the sense of Schoen.

Finally, in order to establish that an embedded ESWMT-surface with finite total second fundamental form and two ends is rotationally symmetric, we have to introduce the Aleksandrov reflection method for non-compact surfaces. In 1956 Aleksandrov published one of the most renowned characterization of the sphere, known as the Aleksandrov theorem. This theorem establishes that spheres are the the only closed connected embedded surfaces in $\mathbb{R}^3$ with constant (non-zero) mean curvature. However, more significant than the theorem could be the procedure introduced by Aleksandrov proving it, called in the literature as the Aleksandrov reflection method, based on the maximum principle. This method was adapted by Korevaar et al. in [Kor+89], using tilted planes, to deal with non-compact surfaces, specifically they proved that a complete connected properly embedded constant mean curvature surface in $\mathbb{R}^3$ with two ends is rotationally symmetric; namely, a Delaunay surface. In fact, we will use the extension presented by Aledo et al., in [Ale+10], for Special Weingarten surfaces of elliptic type.

Summarizing, we have several background to set our main theorem, a Schoen type theorem for embedded ESWMT-surfaces in the Euclidean space $\mathbb{R}^3$ with finite total second fundamental form and two ends.

We are going to describe the results obtained in this thesis.

In Chapter 2, we obtain a representation formula for a SW-surface immersed in $\mathbb{R}^3$, using a special Codazzi pair, that depends of the Gauss map, g, of the immersion and the function associated to the surface, Theorem 2.16. Furthermore, we show that the Gauss map satisfies a second order differential equation which is a complete integrability condition for this representation formula:

Theorem 2.16. *Let M be an oriented SW-surface, then the Gauss map g satisfies*

$$\frac{2}{c-aH}\left(g_{z\bar{z}} - 2g_z g_{\bar{z}}\frac{\bar{g}}{1+|g|^2}\right) = -\left(\frac{a+1}{c-aH}\right)_{\bar{z}} g_z + \left(\frac{a-1}{c-aH}\right)_z g_{\bar{z}},$$

where the functions a and c are given by (2.13) *and* (2.14) *respectively.*

The first result in Chapter 3 is called the growth lemma, Lemma 3.10. It establishes that an embedded end of an ESWMT-surface in $\mathbb{R}^3$ with finite total second fundamental form cannot grow infinitely in opposite directions. Formally speaking, we prove that if the Gauss map at infinity is vertical and the end is the graph of a function ϕ defined outside some compact set in a horizontal plane, then after a vertical translation of the end, either $\phi \leq 0$ or $\phi \geq 0$.

Lemma 3.10 (Growth lemma)**.** *Let E be an end of an immersed ESWMT-surface. Suppose there exists $R > 0$ such that E is obtained as the graph of a C^2 class function ϕ defined in $\mathbb{R}^2 \setminus B(R)$. Suppose also that $|\nabla\phi| \to 0$ as $r \to \infty$, where r is the distance from an arbitrary point to the origin. Then at least one of the quantities $\inf\limits_{|(x,y)|\geq R} \phi(x,y)$ or $\sup\limits_{|(x,y)|\geq R} \phi(x,y)$ is finite.*

As a consequence of the growth lemma we have an obstruction for non-planar one ended ESWMT-surfaces with finite total second fundamental form in $\mathbb{R}^3$. This is an evidence about the premise that the theory of ESWMT-surfaces is very similar to the theory of minimal surfaces, even more if we note that there is not such obstruction for harmonic surfaces with finite total curvature, see [Con+15].

Theorem 3.11. *Let M be a complete connected ESWMT-surface in $\mathbb{R}^3$ with finite total second fundamental and one embedded end. Suppose that the function f associated to M is non-negative, Lipschitz at 0 and satisfies $\liminf\limits_{t\to 0^+} 4t\big(f'(t)\big)^2 < 1$. Then M is a plane.*

The main result of Chapter 3, Theorem 3.21, is a full representation of the height function in the growth lemma and can be interpreted as a kind of regularity at infinity for ends of ESWMT-surfaces with finite total second fundamental form.

Theorem 3.21. *Let M be a complete ESWMT-surface in $\mathbb{R}^3$ with finite total second fundamental form and embedded ends. Suppose that the function f associated to M is non-negative and Lipschitz. Then each end E_i is the graph of a function ϕ_i over the exterior of a bounded region in some plane Π_i. Moreover, if x_1, x_2 are the coordinates* *in Π_i, then the function ϕ_i have the following asymptotic behaviour for $r = \sqrt{x_1^2 + x_2^2}$ large*

$$\phi_i(x_1, x_2) = \beta \log r + a_0 + \frac{a_1 x_1}{r^{2/\epsilon}} + \frac{a_2 x_2}{r^{2/\epsilon}} + O\left(r^{-2/\epsilon}\right) + o\left(\log r\right),$$

where $\epsilon, \beta, a_0, a_1, a_2$ are real constants depending on i and $0 < \epsilon \leq 1$.

The main result of this work can be found in Chapter 4. It establishes that the rotational examples of ESWMT-surfaces, characterized by Sa Earp and Toubiana, are the only ones embedded with finite total second fundamental form and two ends. This result can be understood as a Schoen type theorem for ESWMT-surfaces, it reads as follows:

Theorem 4.4 (Main Theorem)**.** *Let M be a complete connected ESWMT-surface embedded in $\mathbb{R}^3$ with finite total second fundamental form and two ends. Suppose that the function f associated to M is non-negative and Lipschitz. Then M must be rotationally symmetric. In fact, there exists $\tau > 0$ such that M is the surface of revolution M_τ determined by Sa Earp and Toubiana in [ST95].*

In the subsection 3.4.3 we will show that the Lipschitz condition for the function f in the Main Theorem can be weakened to the condition $\limsup_{t \to 0^+} 4t\left(f'(t)\right)^2 < 1$.

As a direct consequence of the last theorem, we can recover the original Schoen theorem.

Corollary 4.6. *Let M be a complete connected embedded minimal surface in $\mathbb{R}^3$ with finite total curvature and two ends. Then M is a catenoid.*

Thesis Structure

This document is organized in four chapters that develop, step by step, all the tools required to prove Theorem 4.4.

Chapter 1

The first chapter is devoted to the introductory material that we will use in the remaining chapters. It starts with a brief presentation of the first and second fundamental forms of an immersion. Also, here we present an alternative for the Weierstrass representation used in the Schoen theorem, namely the Kenmotsu representation. We explore other types of finite total curvature surfaces based on the work of White in [Whi87]. We recall the Aleksandrov reflection method for ESWMT-surfaces. Finally, we introduce a Codazzi pair, which is a pair of bilinear forms, one of them positively defined, satisfying the Codazzi equation. This is a concept that generalizes the first and second fundamental forms.

Chapter 2

In order to show the similarities between minimal surfaces and ESWMT-surfaces, we prove that an ESWMT-surface satisfies some basic properties that come from minimal surface theory, as the convex hull property, for instance. We summarize the results of Sa Earp and Toubiana in [ST95] that characterize the rotational examples of ESWMT-surfaces. Besides, following the work of Kenmotsu in [Ken79] and using Codazzi pairs we give a representation formula for SW-surfaces, which is based on the Weingarten relation.

Chapter 3

Here, we use the finite total second fundamental form hypothesis and the Kenmotsu representation to prove that the height function, in a conformal parameter, associated to an end of an ESWMT-surface is of logarithmic order.

Chapter 4

The final chapter uses the Aleksandrov reflection method to show that an embedded ESWMT-surface with finite total second fundamental form and two ends must be rotational. This fact joined with the results of Sa Earp and Toubiana give us a Schoen type theorem for ESWMT-surfaces.

1 Preliminaries

> *Mathematicians do not study objects,*
> *but relations between objects.*
>
> — **Henri Poincaré**

This chapter contains basic concepts that we will use along this thesis. At a first glance, the material might seem quite independent, however, throughout the document these contents will relate among themselves to shape the main result of this work. Readers familiarized with the topics presented here can skip the entire chapter.

This chapter is based on [Ken79; Mil80; Whi87; Kor+89; PS07; Esp08; Ale+10].

Along this document we will consider M as a connected oriented surface endowed with a Riemannian metric, which is isometrically immersed in the Euclidean space $\mathbb{R}^3$. Since every immersion is locally embedded, for any point at M there exists a neighborhood in M such that its image by the immersion is a regular surface contained in $\mathbb{R}^3$. This fact allows us, in order to simplify the notation, to locally identify M with its image in the Euclidean space and every tangent vector of M with its image by the differential of the immersion. That is, if $\mathbf{f} : M \to \mathbb{R}^3$ is an isometric immersion, then for any $p \in M$ there exists a neighborhood $U \subset M$ of p such that $\mathbf{f}(U) \subset \mathbb{R}^3$ is a regular surface and we will identify p with $\mathbf{f}(p)$, U with $\mathbf{f}(U)$ and $v \in T_pM$ with $d\mathbf{f}_p(v) \in T_{\mathbf{f}(p)}\mathbb{R}^3 \simeq \mathbb{R}^3$. Using these identifications and the scalar product of $\mathbb{R}^3$ we have the following decomposition at any point $p \in M$

$$\mathbb{R}^3 \simeq T_p\mathbb{R}^3 = T_pM \oplus N_pM,$$

where N_pM is the orthogonal complement of T_pM in $\mathbb{R}^3$. In this way we can now consider the normal bundle NM of M, whose fiber at p is N_pM. We will denote by $\mathfrak{X}(M)$ the set of tangent vector fields on M, that is, $X \in \mathfrak{X}(M)$ if $X : M \to TM$ is a smooth map; in the same way $\mathfrak{X}(M)^\perp$ will be the set of normal vector fields on M. Since M is oriented, there exists a unit normal vector field $N \in \mathfrak{X}(M)^\perp$. Given $X, Y \in \mathfrak{X}(M)$ and $\bar{X}, \bar{Y}$ local extensions to $\mathbb{R}^3$ of these fields, we have that the Levi-Civita connection of $\mathbb{R}^3$ and M, denoted by $\overline{\nabla}$ and ∇ respectively, are related by the equation

$$\nabla_X Y = \left(\overline{\nabla}_{\bar{X}}\bar{Y}\right)^T.$$

1.1 Fundamental forms of an immersion

We will consider on M two bilinear forms, denoted by I and II, induced naturally. We are talking about the first and second fundamental forms of M that are defined by the Euclidean metric induced on the surface, and the bilinear form associated to the Weingarten endomorphism, respectively. We will formalize these ideas and fix some notations.

We will denote by $\mathfrak{D}(M)$ the set of C^∞ maps on M valued in $\mathbb{R}$. Let $N \in \mathfrak{X}(M)^\perp$ be a unit normal vector field. We define $I, II : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{D}(M)$ for $X, Y \in \mathfrak{X}(M)$ as

- $I(X,Y) = \langle X, Y\rangle$, where $\langle , \rangle$ is the scalar product of $\mathbb{R}^3$.
- $II(X,Y) = \langle -\overline{\nabla}_X N, Y\rangle$.

The bilinear forms I and II are called first and second fundamental forms of M, respectively.

On the one hand, note that the first fundamental form is definite, that is, the quadratic form associated to I is definite, actually positive definite. On the other hand, it is easy to show that $\overline{\nabla}_X N$, where $X \in \mathfrak{X}(M)$, is a tangent field on M and that the endomorphism S of $\mathfrak{X}(M)$ defined by $SX = -\overline{\nabla}_X N$ is self-adjoint; this means that the second fundamental form is symmetric. The map S is known as the Weingarten endomorphism or shape operator. The second fundamental form and the Weingarten endomorphism are mutually determined, that is, one of them determines in a unique way the other one by the equation $II(X,Y) = I(SX,Y)$ for all $X, Y \in \mathfrak{X}(M)$.

The first and second fundamental forms are related by the Codazzi equation,

$$\nabla_X SY - \nabla_Y SX = S[X,Y], \tag{1.1}$$

where $X, Y \in \mathfrak{X}(M)$ and $[X,Y]$ is the bracket of the tangent fields X and Y. The Codazzi equation is a direct consequence of the flatness of the Euclidean space, that is, the fact that the curvature tensor, R, of $\mathbb{R}^3$ vanishes identically, where R is defined as $R(X,Y,Z) = \overline{\nabla}_X\overline{\nabla}_Y Z - \overline{\nabla}_Y\overline{\nabla}_X Z - \overline{\nabla}_{[X,Y]}Z$, where X, Y, Z are vector fields in $\mathbb{R}^3$.

1.1.1 Curvatures of an immersion

Let $p \in M$ and $v \in T_pM$. Since the value $\overline{\nabla}_X N(p)$ depends only on the value $X(p)$, we can consider the self-adjoint linear transformation $S_p : T_pM \to T_pM$ defined by $S_p(v) = -\overline{\nabla}_X N(p)$, where $X \in \mathfrak{X}(M)$ is any tangent vector field such that $X(p) = v$. This observation allows us to write

$$S_p(v) = -\overline{\nabla}_v N = -dN_p(v).$$

By the symmetry of S_p, there exists an orthonormal basis $\{v_1, v_2\}$ of T_pM and $k_1(p), k_2(p) \in \mathbb{R}$ such that

$$S_p(v_1) = k_1(p)v_1 \quad \text{and} \quad S_p(v_2) = k_2(p)v_2.$$

The principal curvatures of M at p are defined as the eigenvalues $k_1(p), k_2(p)$ of S_p and can be obtained as the maximum and the minimum of the second fundamental form at p restricted to the unit circle of T_pM, see [do 76].

A point $p \in M$ is called umbilical if $k_1(p) = k_2(p)$. A surface such that all points are umbilical is named totally umbilical, or simply, umbilical. An interested fact about totally umbilical surfaces in $\mathbb{R}^3$ is that they are, even locally, spheres and planes, see [do 76; Daj90].

The Gaussian curvature, K, and the mean curvature, H, of M at p are defined as the determinant and the normalized trace of S_p, respectively. That is, in terms of the principal curvatures

$$K(p) = k_1(p)k_2(p) \quad \text{and} \quad H(p) = \frac{1}{2}(k_1(p) + k_2(p)).$$

Remark 1.1. *One of the most remarkable theorems of Gauss says that for an immersed surface in $\mathbb{R}^3$ the Gaussian curvature, K, is invariant under local isometries. Actually this quantity depends only on the metric structure of the surface, that is, the first fundamental form. In other words, the "extrinsic" Gaussian curvature is an intrinsic invariant of the surface.*

It is known that the determinant and the trace of a linear map are invariants of the map, i.e., they do not dependent on the choice of the basis. This fact gives us an alternative way to compute the Gaussian and mean curvature. Considering the matrices associated to the first and second fundamental forms in any basis of T_pM, and denoting this matrices in the same way, $I(p)$ and $II(p)$ respectively, it is easy to show that the Gaussian and mean curvature are determined by

$$K(p) = \frac{\det(II(p))}{\det(I(p))} \quad \text{and} \quad H(p) = \frac{1}{2}\big(\operatorname{tr}(I(p)^{-1}II(p))\big). \tag{1.2}$$

A straightforward computation shows that we can recover the principal curvatures from the mean and Gaussian curvatures,

$$k_1(p) = H(p) + \sqrt{H(p)^2 - K(p)} \quad \text{and} \quad k_2(p) = H(p) - \sqrt{H(p)^2 - K(p)}. \tag{1.3}$$

Note that

$$q(p) = H(p)^2 - K(p) = \frac{1}{4}\big(k_1(p) - k_2(p)\big)^2, \tag{1.4}$$

which means that $q(p) = 0$ if, and only if, $p \in M$ is an umbilical point. The value $q(p)$ is called the skew curvature of M at p.

The norm of the second fundamental form at $p \in M$ is defined using the principal curvatures as

$$|II(p)| = \sqrt{k_1(p)^2 + k_2(p)^2}. \tag{1.5}$$

From the above definition we have that

$$|II|^2 = 4H^2 - 2K. \tag{1.6}$$

1.1.2 Fundamental forms in coordinates

Let $(U, \mathbf{x} = (x, y))$ be a local chart of M. The fundamental forms (I, II) can be written using these coordinates as

$$I = Edx^2 + 2Fdxdy + Gdy^2 \quad \text{and} \quad II = edx^2 + 2fdxdy + gdy^2,$$

where E, F, G, e, f, g are smooth functions defined on U. Using these functions and (1.2) we can compute the Gaussian and mean curvature of the immersion as

$$K = \frac{eg - f^2}{EG - F^2} \quad \text{and} \quad H = \frac{Eg - 2Ff + Ge}{2(EG - F^2)}.$$

Now consider the complex parameters $(z, \bar{z})$ on U, where $z = x + iy$; then defining $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$ and $\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$, M can be viewed as a Riemann surface, and $\mathbb{C}TM$ as the complexification of the tangent bundle of M, generated by the set $\{\partial_z, \partial_{\bar{z}}\}$. We rewrite the first and second fundamental forms as

$$I = Pdz^2 + 2\lambda|dz|^2 + \bar{P}d\bar{z}^2 \quad \text{and} \quad II = Qdz^2 + 2\rho|dz|^2 + \bar{Q}d\bar{z}^2,$$

where P, λ, Q, ρ are complex functions. In these coordinates, the curvatures can be determined as

$$K = \frac{|Q|^2 - \rho^2}{|P|^2 - \lambda^2} \quad \text{and} \quad H = \frac{P\bar{Q} - 2\lambda\rho + \bar{P}Q}{2(|P|^2 - \lambda^2)}.$$

The equations that relate the corresponding functions in the $(x, y)-$coordinates and the $(z, \bar{z})-$coordinates are:

$$\begin{aligned} P &= \frac{1}{4}(E - G - 2iF), & Q &= \frac{1}{4}(e - g - 2if), \\ \lambda &= \frac{1}{2}(E + G), & \rho &= \frac{1}{2}(e + g). \end{aligned}$$

Note that λ and ρ are real functions.

Isothermal (conformal) parameters

We say that (x, y) are isothermal parameters for the first fundamental form I, if $E = G > 0$ and $F = 0$, thus $I = E(dx^2 + dy^2)$. We also say that the complex parameter $z = x + iy$ is a conformal parameter for I if $P = 0$, so $I = 2\lambda|dz|^2$. Thereby, from the equations presented above, the parameter $z = x + iy$ is a conformal parameter for I if, and only if, (x, y) are isothermal parameters for I.

It is well known that, when the surface is regular enough, at any point of the surface there exists a local isothermal chart, see for instance [Che55a]. Also, if the surface is oriented, a positive oriented atlas of isothermal charts gives to the surface a structure of a Riemann surface.

Lemma 1.2. *Let I, II be the first and second fundamental forms, respectively, and z be a conformal parameter for I. Then*

$$I = 2\lambda|dz|^2 \quad \textit{and} \quad II = Qdz^2 + 2\lambda H|dz|^2 + \bar{Q}d\bar{z}^2, \tag{1.7}$$

and the following equations hold,

$$K = H^2 - \frac{|Q|^2}{\lambda^2} \quad \textit{and} \quad |Q|^2 = q\lambda^2. \tag{1.8}$$

The equation on the right in (1.8) says that $p \in M$ is umbilical if, and only if, $Q(p) = 0$. In particular, if Q vanishes identically on M, then M is totally umbilical in $\mathbb{R}^3$.

The Hopf theorem

Let us observe that, by (1.7), the Weingarten endomorphism is determined by the equalities,

$$S\partial_z = H\partial_z + \frac{Q}{\lambda}\partial_{\bar{z}} \quad \text{and} \quad S\partial_{\bar{z}} = \frac{\bar{Q}}{\lambda}\partial_z + H\partial_{\bar{z}}. \tag{1.9}$$

From (1.9), it is easy to conclude that the Codazzi equation (1.1) is equivalent to

$$Q_{\bar{z}} = \lambda H_z, \tag{1.10}$$

which implies that the mean curvature, H, is constant if, and only if, the quadratic differential form Qdz^2, which does not depend on the chosen parameter z, is holomorphic. Hence, if H is a constant and M is a topological sphere, the differential Qdz^2 must vanish identically on M. We have proven the celebrated Hopf theorem, which asserts

Theorem 1.3 (Hopf Theorem). *Any topological sphere immersed in the Euclidean space $\mathbb{R}^3$ with non-zero constant mean curvature must be a round sphere.*

The quadratic differential form Qdz^2 is called the Hopf differential, see [Hop83; AR04; Esp08; Ale+10].

1.2 The Gauss map and the representation of surfaces in the Euclidean space

Given a unit normal vector field $N \in \mathfrak{X}(M)^{\perp}$, called Gauss map, we can consider N as a function into the unit sphere of $\mathbb{R}^3$, and think this sphere as the standard Riemann sphere $\mathbb{C} \cup \{\infty\}$. From this point of view, the Gauss map can be rewritten as the composition of the usual stereographic projection with N, we will denote this function by $g : M \to \mathbb{C} \cup \{\infty\}$, which will be also called the Gauss map of M.

The Gauss map of an immersion in $\mathbb{R}^3$ has been of great interest in Differential Geometry. Several authors have shown that this map encodes a large amount of information about the surface, and, in some cases it can determine the existence and uniqueness of an immersion with a given conformal structure, see for example [Oss69; AE75; Ken79; HO83; HO85]. There are two well-known representations of a surface based on the Gauss map: the Weierstrass-Enneper representation and the Kenmotsu representation. However, there are other representations, as for example the one obtained by Gálvez and Martínez in [GM00] for a surface with positive Gaussian curvature using the conformal structure given by the second fundamental form. In the following we will introduce briefly the Weierstrass-Enneper and Kenmotsu representations.

1.2.1 The Weierstrass-Enneper representation

An oriented surface M immersed in $\mathbb{R}^3$ is called minimal if the mean curvature vanishes everywhere. Minimal surfaces are one of the most studied topic in Differential Geometry; they are intimately related with harmonic and complex functions. The Weierstrass-Enneper representation establishes this relation and it says that any minimal surface may be represented by an holomorphic function and a meromorphic function, called Weierstrass data.

Let $(U, \mathbf{x} = (x, y))$ be a local chart of M, where (x, y) are isothermal parameters for the first fundamental form, and consider the complex parameter $z = x + iy$, which is conformal. By Lemma 1.2 we know that $I = 2\lambda|dz|^2$. It is well known that $\Delta\mathbf{x} = 2\lambda H N$, where Δ is the Laplace-Beltrami operator (see [do 76]). This allows us to conclude that M is minimal in $\mathbb{R}^3$ if, and only if, the coordinate functions are harmonic.

In order to establish the relation between minimal surfaces and analytic functions of a complex variable we define

$$(\phi_1, \phi_2, \phi_3) = \Phi = \partial_z \mathbf{x} \quad \text{and} \quad ((\Phi, \Phi)) = \phi_1^2 + \phi_2^2 + \phi_3^2.$$

A straightforward computation shows that z is a conformal parameter if, and only if, $((\Phi,\Phi)) = 0$, and $\Delta\mathbf{x} = 0$ if, and only if, $\partial_{\bar{z}}\Phi = 0$, see [Oss69]. Note that the condition $\partial_{\bar{z}}\Phi = 0$ says that all the components of Φ are holomorphic, and in this case we will say that Φ is holomorphic.

We can recover the immersion $\mathbf{x}$, on simply connected domains, from Φ by the Cauchy theorem as

$$\mathbf{x}(z) = \Re \int_{z_0}^{z} \Phi(w)\, dw,$$

then, in order to get locally minimal surfaces, we focus on holomorphic solutions of the equation $((\Phi,\Phi)) = 0$. The next lemma characterizes those solutions.

Lemma 1.4 (Osserman, [Oss69]). *Let $U \subset \mathbb{C}$ be a domain and $g : U \to \mathbb{C}$ a meromorphic function. Let $f : U \to \mathbb{C}$ be a holomorphic function having the property that each pole of g of order m is a zero of order at least $2m$. Then,*

$$\Phi = \left(\frac{1}{2}f(1-g^2), \frac{i}{2}f(1+g^2), fg\right) \tag{1.11}$$

is holomorphic and satisfies $((\Phi,\Phi)) = 0$. Conversely, every holomorphic Φ satisfying $((\Phi,\Phi)) = 0$ may be represented in the form (1.11)*, except for $\Phi = (\phi, -i\phi, 0)$.*

The lemma above implies the next theorem which gives us a local representation of minimal surfaces in $\mathbb{R}^3$.

Theorem 1.5 (Weierstrass-Enneper representation). *Every minimal surface immersed in $\mathbb{R}^3$ can be locally represented in the form*

$$\mathbf{x}(z) = \Re\left\{\int_0^z \Phi(\omega)\, d\omega\right\} + c \tag{1.12}$$

where c is a constant in $\mathbb{R}^3$, Φ is defined by (1.11)*, the functions (f,g) having the properties stated in Lemma 1.4, the domain U being either the unit disk or the entire complex plane, and the integral being taken along an arbitrary path from the origin to the point z. The surface $\mathbf{x}(U)$ will be regular if, and only if, f vanishes only at the poles of g, and the order of each zero is exactly twice the order of this point as pole of g.*

Remark 1.6. *It is easy to check that if an immersed minimal surface is represented by* (1.12)*, the Gauss map is exactly the meromorphic function g.*

As an example of this representation we can obtain the catenoid, see Figure 1.1, for the functions $f, g : \mathbb{C} \setminus \{0\} \to \mathbb{C}$, where $f(z) = e^{-z}$ and $g(z) = e^z$. Then by (1.11), we get

$$\Phi(\omega) = \left(\frac{1}{2e^{\omega}}(1-e^{2\omega}), \frac{i}{2e^{\omega}}(1+e^{2\omega}), 1\right).$$

The immersion is achieved as

$$\begin{aligned}
\mathbf{x}(z) &= \Re \int_0^z \left(\frac{e^{-\omega} - e^{\omega}}{2}, \frac{i(e^{-\omega} + e^{\omega})}{2}, 1 \right) \\
&= \Re \int_0^z (-\sinh(\omega), i\cosh(\omega), 1) \\
&= \Re\,(-\cosh(z), i\sinh(z), z) \\
\mathbf{x}(x,y) &= (-\cosh(x)\cos(y), -\cosh(x)\sin(y), y)\,.
\end{aligned}$$

It is well known that the catenoid is the only non-planar minimal surface that is rotationally symmetric. Besides, it is properly embedded in $\mathbb{R}^3$, and conformally the punctured plane, therefore parabolic.

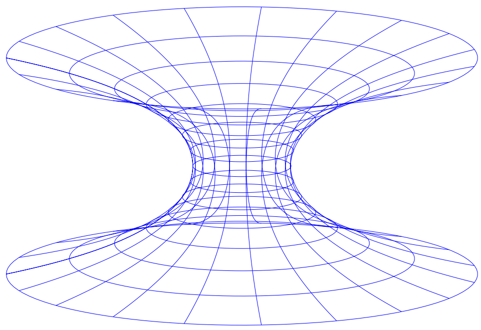

Figure 1.1: Catenoid.

1.2.2 The Kenmotsu representation

A natural question is: can we generalize the Weierstrass-Enneper representation for any other kind of surface? The answer is yes, we can have a representation quite similar to the Weierstrass-Enneper one for any immersed surface in $\mathbb{R}^3$, however, the complex functions involved in the representation may not be holomorphic. One of this kind of representations was presented by Kenmotsu in [Ken79].

Let M be an oriented surface immersed in $\mathbb{R}^3$ and $z = x + iy$ be a conformal parameter for the first fundamental form, I. We consider the smooth unit normal vector field N defined by $N = \frac{1}{E}(\partial_x \wedge \partial_y)$, where $\wedge$ is the exterior product in $\mathbb{R}^3$. The vector field $N_z = \partial_z(N)$ is tangent to the surface. The following lemma establishes, using Lemma 1.2, the representation of this tangent vector field in the basis $\{\partial_z, \partial_{\bar{z}}\}$.

Lemma 1.7. $-N_z = H\partial_z + \frac{Q}{\lambda}\partial_{\bar{z}}$.

Proof. Since $-N_z$ is a tangent vector field, there exist complex functions c_1 and c_2 such that $-N_z = c_1\partial_z + c_2\partial_{\bar{z}}$. Then $I(-N_z, \partial_z) = c_2\lambda$ and $I(-N_z, \partial_{\bar{z}}) = c_1\lambda$. Since $I(-N_z, \partial_z) = II(\partial_z, \partial_z) = Q$ and $I(-N_z, \partial_{\bar{z}}) = II(\partial_z, \partial_{\bar{z}}) = \lambda H$, we obtain the result. □

The coordinates (x, y) are isothermal, then from the definition of N we have that $N \wedge \partial_x = \partial_y$ and $N \wedge \partial_y = -\partial_x$. Thus,

$$N \wedge \partial_z = N \wedge \left(\frac{1}{2}(\partial_x - i\partial_y)\right) = \frac{1}{2}(\partial_y + i\partial_x) = i\partial_z \quad \text{and} \quad N \wedge \partial_{\bar{z}} = -i\partial_{\bar{z}}. \quad (1.13)$$

This computations will allow us to generate the tangent bundle using the information obtained from the unit normal vector field N and its derivative N_z, at any point outside of umbilical points and minimal points. Specifically, let us verify that the vectors N_z and $iN \wedge N_z$ are linearly independent whenever HQ does not vanish. From Lemma 1.7 and (1.13) we get

$$iN \wedge N_z = iN \wedge \left(-H\partial_z - \frac{Q}{\lambda}\partial_{\bar{z}}\right) = H\partial_z - \frac{Q}{\lambda}\partial_{\bar{z}}. \quad (1.14)$$

Then, it is very easy to see that the vectors make a tangent basis under the condition $HQ \neq 0$.

Now we can recover the natural basis induced by the immersion at any point with non-zero mean curvature as we show in the next Lemma.

Lemma 1.8. $\partial_z = \frac{1}{2H}(iN \wedge N_z - N_z)$.

Proof. It follows immediately from (1.14) and Lemma 1.7. □

As we mentioned above, we are considering the stereographic projection of the unit normal $N = (N_1, N_2, N_3)$ from the north pole, that is,

$$g(z) = \frac{N_1(z)}{1 - N_3(z)} + i\frac{N_2(z)}{1 - N_3(z)}.$$

We will call this function, again, the Gauss map of the surface M. Hence,

$$N = \frac{1}{1 + g\bar{g}}(g + \bar{g}, -i(g - \bar{g}), -1 + g\bar{g}),$$

and $N_z = g_z\bar{\xi} + \bar{g}_z\xi$, where ξ is the vector field defined as

$$\xi = \frac{1}{(1 + g\bar{g})^2}(1 - g^2, i(1 + g^2), 2g).$$

A straightforward computation shows that $iN \wedge \xi = -\xi$, and then by conjugation $iN \wedge \bar{\xi} = \bar{\xi}$. This helps us to obtain

$$iN \wedge N_z = iN \wedge \left(g_z\bar{\xi} + \bar{g}_z\xi\right) = g_z\bar{\xi} - \bar{g}_z\xi,$$

which means that we can obtain ∂_z from g, g_z and $\bar{g}_z$. Specifically,

Theorem 1.9. $\partial_z = -\frac{1}{H}\bar{g}_z\xi$.

Proof. Directly from Lemmas 1.8 and the above equations. □

When the surface is minimal, it is well known that the Gauss map is a holomorphic function. In general, it might be not true that the Gauss map is holomorphic, however, this map satisfies a Beltrami equation.

Theorem 1.10 (Beltrami equation). *The Gauss map g of an immersed surface M in $\mathbb{R}^3$ must satisfy a Beltrami equation $Hg_z = \frac{Q}{\lambda} g_{\bar{z}}$.*

Proof. Lemma 1.7 says that $-N_z = H\partial_z + \frac{Q}{\lambda}\partial_{\bar{z}}$, and then by conjugation we have that $-N_{\bar{z}} = \frac{\bar{Q}}{\lambda}\partial_z + H\partial_{\bar{z}}$. Note that $\{N_z, N_{\bar{z}}\}$ is also a tangent basis, then we recover ∂_z in this basis as

$$\partial_z = -\frac{1}{K}\left(HN_z - \frac{Q}{\lambda}N_{\bar{z}}\right). \tag{1.15}$$

From $N_z = g_z\bar{\xi} + \bar{g}_z\xi$, we obtain by conjugation $N_{\bar{z}} = \bar{g}_{\bar{z}}\xi + g_{\bar{z}}\bar{\xi}$. Then, using this expressions and Theorem 1.9 in (1.15) we get,

$$\frac{K}{H}\bar{g}_z\xi = -K\partial_z = \left(H\bar{g}_z - \frac{Q}{\lambda}\bar{g}_{\bar{z}}\right)\xi + \left(Hg_z - \frac{Q}{\lambda}g_{\bar{z}}\right)\bar{\xi}. \tag{1.16}$$

Since the vector fields ξ and $\bar{\xi}$ are linearly independent, we conclude that the coefficient of $\bar{\xi}$ in (1.16) vanishes, that is, $Hg_z - \frac{Q}{\lambda}g_{\bar{z}} = 0$, as desired. □

Note that when the immersion is minimal, the Beltrami equation implies that $g_{\bar{z}} = 0$, which means that the Gauss map is a holomorphic function; this fact was pointed by Osserman in [Oss69]. Hence, the Beltrami equation can be seen as a generalization of the Cauchy-Riemann equations for g in the minimal case.

The Gauss map also satisfies a second order differential equation, which is a complete integrability condition for the equation in Theorem 1.9. This differential equation is presented in the next theorem.

Theorem 1.11. *The Gauss map g of an immersed surface M in $\mathbb{R}^3$ must satisfy*

$$H\left(g_{z\bar{z}} - \frac{2\bar{g}}{1+g\bar{g}}g_z g_{\bar{z}}\right) = H_z g_{\bar{z}}. \tag{1.17}$$

The proof of the last theorem can be found in [Ken79], and it is mainly a consequence of the Codazzi equation (1.1) and the Theorems 1.9 and 1.10.

Now we have all elements to recover the immersion from the Gauss map and the non-zero mean curvature. This means that we can have a representation formula for immersed surfaces prescribing the mean curvature and the Gauss map. Such representation, presented bellow, can be seen as a generalization of the Weierstrass-Enneper formula for minimal surfaces. This result was proved by Kenmotsu in [Ken79].

Theorem 1.12. *Let M be a simply connected 2-dimensional smooth manifold and $H : M \to \mathbb{R}$ be a non-zero C^1–function. Let $g : M \to \mathbb{C} \cup \{\infty\}$ be a smooth map satisfying* (1.17) *for H. Then there exists a branched immersion $\mathbf{x} : M \to \mathbb{R}^3$ with Gauss map g and mean curvature H. Moreover, the immersion is unique up to similarity transformations of $\mathbb{R}^3$ and it can be recovered using the equation*

$$\mathbf{x} = \Re\left\{\int -\frac{1}{H}\bar{g}_z \xi \, dz\right\} + c, \tag{1.18}$$

where c is a constant in $\mathbb{R}^3$ and the integrals are taken along a path from a fixed point to a variable point. Moreover, if $g_{\bar{z}} \neq 0$, then $\mathbf{x}(M)$ is a regular surface.

Proof. This follows from Theorem 1.9 and Theorem 1.11. □

1.3 Finite total curvature

This section is based on the work of Huber and White, especially on [Hub57; Hub66] and [Whi87]. The aim is to know the relation between the topology and the conformal structure of a non-compact complete surface and the finiteness of some geometric quantities. The first result in this direction is due to Cohn-Vossen in [CV33], where he proved that for a finitely connected and complete Riemannian surface, the integral of the Gaussian curvature is finite and at most 2π times the Euler characteristic of the surface. This integrability of the Gaussian curvature is known in the literature as finite total curvature, see for example [CV33; Fin65; HT92; LT91; Oss69; Shi85].

Definition 1.13. *An immersed surface M in $\mathbb{R}^3$ is said to have finite total curvature if*

$$\left|\int_M K \, dA\right| < +\infty. \tag{1.19}$$

Huber in [Hub57] proved that finite total curvature implies finite connectivity, which means that the hypothesis finitely connected in the Cohn-Vossen theorem can be replaced by connected. His theorem reads as

Theorem 1.14 (Huber, [Hub57])**.** *If M has finite total curvature, then M is of finite topological type, moreover M is conformally equivalent to $\bar{M} \setminus \{p_1, \cdots, p_k\}$, where $\bar{M}$ is a compact Riemann surface.*

Here we will work with the integral of the squared norm of the second fundamental form, recall (1.5) and (1.6).

Definition 1.15. *An immersed surface M in $\mathbb{R}^3$ is said to have finite total second fundamental form if*

$$\int_M |II|^2 \, dA < +\infty. \tag{1.20}$$

This name is taken from [CM99], however, this concept is also usually named as finite total curvature, see for example [Whi87; Che94; MŠ95; CM11; AL13]. We prefer to avoid confusions and use those names separately.

Note that a minimal surface in $\mathbb{R}^3$ has finite total second fundamental form if, and only if, has finite total curvature, since

$$|II|^2 = k_1^2 + k_2^2 = (k_1 + k_2)^2 - 2k_1k_2 = 4H^2 - 2K = -2K.$$

This equivalence is also true for quasiconformal harmonic immersions in $\mathbb{R}^3$, see [AL13].

In general, finite total second fundamental form is stronger than finite total curvature; a fact proved by White in [Whi87].

Theorem 1.16 (White, [Whi87])**.** *If M has finite total second fundamental form, then M has finite total curvature, indeed $\int_M K\, dA = 2\pi m$ for some integer m.*

In 1964, Osserman in [Oss64] showed that, for minimal surfaces in $\mathbb{R}^3$, the total curvature is related to the distribution of the normals. Specifically, he proved that if a complete minimal surface has finite total curvature, then the surface is conformally equivalent to a compact Riemann surface minus a finite number of points, and the Gauss map extends to a meromorphic function at those points. It is possible to obtain an Osserman's type theorem removing the hypothesis of being minimal. The topological conclusion can be deduced without the minimal hypothesis, Theorem 1.14; the second implication was obtained by White in [Whi87] assuming finite total second fundamental form and non-positive Gaussian curvature, in this case the extension is, a priori, only continuous.

Theorem 1.17 (White, [Whi87])**.** *Suppose M has finite total second fundamental form, in particular, $M \approx \bar{M} \setminus \{p_1, \cdots, p_k\}$ as in Theorem 1.14. Let U_i be a (punctured) neighborhood of p_i in M. If K does not change sign in U_i, then the Gauss map extends continuously to p_i, $i = 1, \ldots, k$, and M is properly immersed near p_i.*

1.4 Maximum principles and geometric applications

The maximum principle is a classical result on the theory of second order elliptic partial differential equations, it leads us to know information about solutions of differential equations and inequalities without any explicit knowledge of the solutions themselves. Initially, it was proposed separately by Gauss and Earnshaw for harmonic and subharmonic functions (1839) and, from then, several mathematicians have generalized those results as, for example, Paraf (1892), Moutard (1894), Picard (1905), Picone (1927) and Hopf (1952).

Here we are interested in the application of the maximum principle in geometry; it becomes a powerful tool since surfaces are locally graphs of functions defined in a domain of the tangent plane and elliptic partial differential equations arise naturally in the study of the curvatures of the surface. This was the idea of Aleksandrov [Ale56] for proving that, *a closed connected embedded constant (non-zero) mean curvature surface in the Euclidean three space must be a round sphere*; theorem known nowadays as the Aleksandrov theorem.

1.4.1 The Hopf Maximum principle

The maximum principle we will present here is due to Hopf, see [Hop27; Hop52]. For a very detailed proof of it, we refer the reader to [PS07]. It is based on the observation that if a function of class C^2 on a domain attains a maximum value at certain point, then at this point the gradient of the function must vanish and the Hessian matrix must be non-positive definite.

Let $\Omega \subset \mathbb{R}^n$ be a domain and set L a second order partial differential operator of the form

$$L = \sum_{i,j} a_{ij}(x)\partial^2_{x_i x_j} + \sum_i b_i(x)\partial_{x_i}, \tag{1.21}$$

for given real continuous functions a_{ij} and b_i defined on Ω satisfying $a_{ij} = a_{ji}$.

The partial differential operator L is called uniformly elliptic if there exists a positive constant δ such that

$$\sum_{i,j} a_{ij}(x)y_i y_j \geq \delta|y|^2, \tag{1.22}$$

for any $x \in \Omega$ and $y = (y_1, \dots, y_n) \in \mathbb{R}^n$. We say that L is locally uniformly elliptic if (1.22) holds on any compact subset of Ω, where the constant δ is now depending of the compact subset.

Note that the ellipticity of L means that for each point $x \in \Omega$, the symmetric matrix $(a_{ij}(x))$ is positive definite, with smallest eigenvalue greater than or equal to δ.

Theorem 1.18 (Maximum principle). *Let u be a $C^2(\Omega)$ function which satisfies the differential inequality $Lu \geq 0$ in the domain Ω, where L is a locally uniformly elliptic operator. If u takes a maximum value C in Ω, then $u \equiv C$ in Ω.*

The Interior maximum principle does not apply when the function u attains its maximum value at the boundary of the domain Ω. The following theorem treats this case.

Theorem 1.19 (Hopf Lemma). *Let u be a $C^2(\Omega)$ function continuous in $\partial\Omega$ which satisfies the differential inequality $Lu \geq 0$ in the domain Ω, where L is a locally uniformly elliptic operator. Suppose that Ω satisfies the interior sphere condition and let $x_0 \in \partial\Omega$ such that*

(i) u is of class C^1 in x_0.

(ii) $u(x_0) \geq u(x)$ *for all* $x \in \Omega$.

(iii) $\partial_\nu u(x_0) \geq 0$*, where* ν *is the inward normal direction of* $\partial\Omega$.

Then u *is constant.*

1.4.2 Comparison principle for surfaces

In order to apply Theorems 1.18 and 1.19 in a geometric context, we will give the definitions of interior tangency point and boundary tangency point that will help us to situate the local description of surfaces.

Definition 1.20. *Let* M_1 *and* M_2 *be immersed surfaces in* $\mathbb{R}^3$ *and* $p \in M_1 \cap M_2$*. The point* p *is called a tangent point if* $T_pM_1 = T_pM_2$*, and either* p *is an interior point of both surfaces or* $p \in \partial M_1 \cap \partial M_2$ *and the interior conormals vectors of* ∂M_1 *and* ∂M_2 *coincide at* p*.*

Given a surface M immersed in $\mathbb{R}^3$ and $p \in M$, we can describe locally the surface M in a neighborhood of p as a graph of a smooth function defined on its tangent plane at p, thinking of this linear space as an affine plane of $\mathbb{R}^3$ passing through p. That is, there exists a neighborhood $V \subset M$ of p, a domain $\Omega \subset T_pM$ and a function $u : \Omega \to \mathbb{R}$ of class C^2 on Ω such that

$$M \cap V = \mathrm{Gra}\,(u) = \{q + u(q)N(p) : q \in \Omega\},$$

where N is a unit normal vector field defined on M. Note that $u(p) = 0$. When the surface M has no empty boundary and $p \in \partial M$, the function u is defined on $\bar{\Omega}$, $p \in \partial\Omega$ and $u(\partial\Omega) = \partial M \cap V$.

Note that when the surfaces M_1 and M_2 have a common tangency point p (at the interior or at the boundary), then locally those surfaces can be obtained as graphs (over the same plane) of functions u_1 and u_2 respectively, defined on the same domain (Ω or $\bar{\Omega}$ according if either p is an interior or a boundary point respectively).

Let us denote by $M_1 \geq_p M_2$ if M_1 and M_2 have the point p as a common tangency point and $u_1 \geq u_2$ in a neighborhood of p in Ω (or $\bar{\Omega}$), which means geometrically that locally the surface M_1 is above of the surface M_2 around p.

Now we state a geometric maximum principle using the functions that locally describe the surfaces.

Definition 1.21. *Let* M_1 *and* M_2 *be surfaces immersed in* $\mathbb{R}^3$*. We say that the surfaces satisfy the maximum principle if, for every common tangent point* $p \in M_1 \cap M_2$*, the function* $w = u_2 - u_1$ *satisfies a differential inequality* $Lw \geq 0$ *for some locally uniformly elliptic operator* L *in the form* (1.21) *and, in addition, the conditions in the hypothesis of Theorem 1.19 when the tangent point is at the common boundary of the surfaces. Here* u_i *is the function such that* M_i *is locally, around* p*, represented as the graph of* u_i*, for* $i = 1, 2$*.*

The intention of the last definition is to apply Theorem 1.18 and Theorem 1.19 thinking just in the geometric fact that two surfaces are touching tangentially each other. Hence, using the Definition 1.21 and the theorems mentioned we obtain the following geometric comparison principle.

Theorem 1.22 (Geometric comparison principle)**.** *Let M_1 and M_2 be surfaces immersed in $\mathbb{R}^3$ satisfying the maximum principle. If $M_1 \geq_p M_2$ then $M_1 = M_2$.*

It is common to work with families of surfaces satisfying the maximum principle, see for example [Ale+10; Esp08].

Definition 1.23. *Let $\mathfrak{F}$ be a family of oriented immersed surfaces in $\mathbb{R}^3$. We say that $\mathfrak{F}$ satisfies the maximum principle if the following properties are fulfilled:*

(i) *$\mathfrak{F}$ is invariant under isometries of $\mathbb{R}^3$.*

(ii) *If $M \in \mathfrak{F}$ and $\widetilde{M}$ is another surface contained in M, then $\widetilde{M} \in \mathfrak{F}$.*

(iii) *Any two surfaces in $\mathfrak{F}$ satisfy the maximum principle, in the sense of Definition 1.21.*

As examples of families of surfaces satisfying the maximum principle we would like to mention the family of minimal surfaces and the family of surfaces with non-zero constant mean curvature. In both cases, they achieve the conditions of Definition 1.23 mainly because the equation $H = c$, where c is a constant ($c = 0$ for the case of minimal surfaces), gives a quasilinear elliptic equation when the surface is written as a graph. Although these families come from a similar equation they are quite different; for example there are no compact minimal surfaces while the sphere of radius $\frac{1}{c}$ is the unique compact embedded surface of constant mean curvature c (the Aleksandrov Theorem).

1.4.3 The Aleksandrov reflection method

As we have mentioned, Aleksandrov characterized spheres as the only closed connected surface embedded in $\mathbb{R}^3$ with constant (non-zero) mean curvature. Although this is a major theorem, the procedure introduced by Aleksandrov have had a remarkable impact in the development of Differential Geometry. This technique is called the Aleksandrov reflection method and it is based in the geometric comparison principle, Theorem 1.21.

Aleksandrov's idea, roughly speaking, was to compare the surface with successive reflections of parts of the surface looking for a possible first tangency point, and finding at this time a plane of symmetry for the original surface. With the aim of applying this method later we are going to present the Aleksandrov reflection method following [Kor+89; Ale+10; Esp08], which is a generalization to non-compact surfaces.

Let M be a connected properly embedded surface in $\mathbb{R}^3$, then the surface divides $\mathbb{R}^3$ into two connected components. Let denote by N a unit normal vector field globally defined on M. Set $\mathcal{C}(N)$ the component of $\mathbb{R}^3 \setminus M$ pointed by N. Note that $\partial\mathcal{C}(N) = M$.

Consider a fixed plane $\mathcal{P} \subset \mathbb{R}^3$ with normal unit vector n. For $t \in \mathbb{R}$ we will denote by $\mathcal{P}_t$ the parallel plane to $\mathcal{P}$ at signed distance t, that is, $\mathcal{P}_t = \mathcal{P} + tn$. Hence, the family $\{\mathcal{P}_t\}_{t\in\mathbb{R}}$ is a foliation of $\mathbb{R}^3$ by parallel planes to $\mathcal{P}$. Set $\mathcal{P}_{t^-}$ and $\mathcal{P}_{t^+}$ the closed half spaces, lower and upper respectively, from the plane $\mathcal{P}_t$, i.e.,

$$\mathcal{P}_{t^-} = \bigcup_{s\leq t} \mathcal{P}_s \quad \text{and} \quad \mathcal{P}_{t^+} = \bigcup_{s\geq t} \mathcal{P}_s.$$

For any set $G \subset \mathbb{R}^3$, let G_{t^+} be the portion of G above the plane $\mathcal{P}_t$ and let $G^*_{t^+}$ be the reflection of this portion through the plane $\mathcal{P}_t$, then we can write

$$G_{t^+} = G \cap \mathcal{P}_{t^+} \quad \text{and} \quad G^*_{t^+} = \{p + (t - r)n : p \in \mathcal{P},\ p + (t + r)n \in G\}. \tag{1.23}$$

We define analogously the sets G_{t^-} and $G^*_{t^-}$.

Given an open set $W \subset \mathcal{C}(N)$ consider the portion S of the surface M obtained by $S = \partial W \cap M$. If the set S_{t^+} is not empty and $S^*_{t^+} \subset \bar{W}$ we write $S^*_{t^+} \geq S_{t^-}$, which means that the reflection of the set S_{t^+} thought the plane $\mathcal{P}_t$ is above the set S_{t^-}, considering the height from the plane $\mathcal{P}$.

When S is not empty, we will define a function Λ_1, called Aleksandrov function associated to S.

We start with the definition of the domain of the function Λ_1. Given a point $p \in \mathcal{P}$, set L_p the perpendicular line to $\mathcal{P}$ passing through p, therefore we can parametrize this line as $L_p = \{p+tn : t \in \mathbb{R}\}$. Suppose that $L_p\cap\bar{W} \neq \emptyset$ but L_p is disjoint from $\bar{W}$ for all sufficiently large t, then there exists a value $t_1(p)$ such that $p+tn \notin \bar{W}$ for any $t > t_1(p)$ and $\mathbf{P}_1(p) = p+t_1(p)n \in \bar{W}$. We say that $\mathbf{P}_1(p)$ is the first point of contact of the set $L_p\cap\bar{W}$ as t decreases from $+\infty$. Now we are going to suppose additionally that $\mathbf{P}_1(p) \in S$, then the intersection of L_p and S can be either tangential or else transversal. In the first case we will denote $\mathbf{P}_2(p) = \mathbf{P}_1(p)$ and $t_2(p) = t_1(p)$. When the intersection is transversal, we are going to look for the next point where L_p tries to leave $\bar{W}$, if exists, that is, we look for a point $\mathbf{P}_2(p) = p + t_2(p)n$ such that $p+tn \in \bar{W}$ for any $t_1(p) \geq t \geq t_2(p)$ and $\mathbf{P}_2(p) \in S$. We say that $\mathbf{P}_2(p)$ is the second point of contact of the set $L_p \cap \bar{W}$ as t decreases from $+\infty$.

Let us denote by $\mathcal{D}$ the domain of the Aleksandrov function associated to S and define it as the set of points $p \in \mathcal{P}$ such that either there exists just a first contact point or there exist the first and second contact points of the set $L_p \cap \bar{W}$ as t decreases from $+\infty$, and those points belongs to S. Geometrically, $\mathcal{D}$ contains all points p in the plane $\mathcal{P}$ such that the lines L_p, coming from infinity, enter $\bar{W}$ through S either tangentially or transversally and leaving $\bar{W}$ also through S.

We define $\Lambda_1 : \mathcal{D} \to \{-\infty\} \cup \mathbb{R}$ the Aleksandrov function associated to S by

$$\Lambda_1(p) = \begin{cases} \dfrac{t_1(p) + t_2(p)}{2}, & \text{if there exist } \mathbf{P}_1(p) \text{ and } \mathbf{P}_2(p), \\ -\infty, & \text{if there exist just } \mathbf{P}_1(p). \end{cases} \tag{1.24}$$

Note that $\Lambda_1(p)$ is the value such that the reflection of the point $\mathbf{P}_1(p)$ through the plane $\mathcal{P}_{\Lambda_1(p)}$ is $\mathbf{P}_2(p)$.

Definition 1.24. *A point $p \in \mathcal{D}$ is called a local interior maximum for Λ_1 if there exists a neighborhood U of p in $\mathcal{P}$ such that for any $q \in U$ either $q \in \mathcal{D}$, $\mathbf{P}_1(q), \mathbf{P}_2(q) \notin \partial S$ and $\Lambda_1(q) \le \Lambda_1(p)$, or else $L_q \cap \bar{W} = \emptyset$. Any other local maximum of Λ_1 will be called a local boundary maximum.*

Definition 1.25. *A first local point of reflection for S with respect to the plane $\mathcal{P}$ with normal n is defined to be a point $\mathbf{P}_2(p)$ such that $p \in \mathcal{D}$ and is a local maximum of Λ_1, that is, there exists a neighborhood U of p in $\mathcal{P}$ such that $\Lambda_1(q) \le \Lambda_1(p)$ for any $q \in U \cap \mathcal{D}$.*

The above definitions are justified through the next lemma which shows that the Aleksandrov reflection method can be applied to non-compact surfaces.

Lemma 1.26 (Aleksandrov reflection method)**.** *Let $\mathfrak{F}$ be a family that satisfies the maximum principle and $M \in \mathfrak{F}$ be a connected surface. Let $\mathcal{P}$ be a plane in $\mathbb{R}^3$ with normal n. If, relative to the subsets $S \subset M$ and $W \subset \mathcal{C}(N)$, the Aleksandrov function Λ_1 has a local interior maximum value t_0 at $p \in \mathcal{D}$, then the plane $\mathcal{P}_{t_0}$ is a plane of symmetry for M.*

Proof. We are going to compare the surface S with the reflection $S^*_{t_0^+}$ of $S_{t_0^+}$ through the plane $\mathcal{P}_{t_0}$, keeping in mind the fact that S and $S^*_{t_0}$ are surfaces in $\mathfrak{F}$. Since $\Lambda_1(p) = t_0$ we get that $\mathbf{P}_1(p)$ reflects to $\mathbf{P}_2(p)$. The local maximality hypothesis implies that there exists a neighborhood $U \subset \mathcal{P}$ of p such that any $q \in U$ verifies that either $q \in \mathcal{D}$ and $\Lambda_1(q) \le \Lambda_1(p)$, or else $L_q \cap \bar{W} = \emptyset$. Note that in the second case we have that $L_q \cap S = \emptyset$, thus we can assume $q \in \mathcal{D}$. Using the definition of the Aleksandrov function (1.24) we get

$$t_2(q) \le t_0 - (t_1(q) - t_0). \tag{1.25}$$

Looking at (1.23), we infer that inequality (1.25) means that the reflection of $\mathbf{P}_1(q)$ through the plane Π_{t_0} lies above the point $\mathbf{P}_2(q)$ and then $\mathbf{P}_1(q)^* \in \bar{W}$.

Summarizing, a neighborhood of $S^*_{t_0^+}$ containing $\mathbf{P}_2(p)$ is contained in $\bar{W}$. We have shown that $S^*_{t_0} \ge_{\mathbf{P}_2(p)} S$ or equivalently $M^*_{t_0} \ge_{\mathbf{P}_2(p)} M$; in particular $\mathbf{P}_2(p)$ is an interior tangent point when $\mathbf{P}_1(p) \ne \mathbf{P}_2(p)$ and a boundary tangent point when $\mathbf{P}_1(p) = \mathbf{P}_2(p)$. By Theorem 1.22, we conclude that $M = M^*_{t_0}$, thus $\mathcal{P}_{t_0}$ is a plane of symmetry of M. $\square$

The Aleksandrov reflection method is very useful any time we get a local interior maximum for the Aleksandrov function. Although the function Λ_1 is not continuous in general, it is upper semicontinuous and it will attain its supremum in any compact domain. This will be an important fact to be used later in Chapter 4.

The next lemma shows that the Aleksandrov function Λ_1 is upper semicontinuous with respect to planes as well as points.

Lemma 1.27. *Let S be a closed surface and a parameter $\varepsilon \to 0$. Suppose that there exists a sequence of points $p^\varepsilon \to p$ and a sequence of planes $\mathcal{P}^\varepsilon \to \mathcal{P}$ such that $p^\varepsilon \in \mathcal{P}^\varepsilon$ and $p \in \mathcal{P}$. Let Λ_1^ε and Λ_1 be the corresponding Aleksandrov functions. If $\Lambda_1^\varepsilon(p^\varepsilon)$ and $\Lambda_1(p)$ exist, then*

$$\limsup_{\varepsilon\to 0} \Lambda_1^\varepsilon(p^\varepsilon) \leq \Lambda_1(p). \tag{1.26}$$

Proof. Assuming $\Lambda_1^\varepsilon(p^\varepsilon)$ exists for $\varepsilon \to 0$, then there is a sequence $(\mathbf{P}_1(p^\varepsilon), \mathbf{P}_2(p^\varepsilon))$ of pair of points of S. Since S is closed, a subsequence of this sequence converges to a pair of points of S, possibly identical, (Q_1, Q_2). Note that clearly $Q_1, Q_2 \in L_p$ since $L_{p^\varepsilon} \to L_p$. We call t_1 and t_2 the height of these points above $\mathcal{P}$.

The values $t_1(p^\varepsilon)$ and $t_2(p^\varepsilon)$ converge to the height of Q_1 and Q_2 above $\mathcal{P}$, respectively. By definition of first and second contact points we conclude that the points $\mathbf{P}_1(p)$ and $\mathbf{P}_2(p)$ must be at least as hight as the points Q_1 and Q_2 respectively, that is, $t_1(p) \geq t_1$ and $t_2(p) \geq t_2$. Hence $\Lambda_1(p) \geq \frac{t_1+t_2}{2}$ and (1.26) is verified. □

The Aleksandrov theorem

Aleksandrov communicated his theorem and sketched the proof in a lecture given in Zurich in July 1955. The proof was published in March 1956, see [Ale56], depending on two parts, the first one presented here by Theorem 1.18, Theorem 1.19 and Lemma 1.26, and a second one based on the following well-known characterization of spheres, whose proof can be found in [Hop83].

Lemma 1.28. *If a compact surface M is embedded in $\mathbb{R}^3$ and has a plane of symmetry in every direction, then M is a totally umbilical sphere.*

Now we can establish a proof for the Aleksandrov Theorem.

Corollary 1.29 (Aleksandrov Theorem)**.** *Let $\mathfrak{F}$ be a family that satisfies the maximum principle and $M \in \mathfrak{F}$ be a connected compact surface embedded in $\mathbb{R}^3$. Then M is a totally umbilical sphere.*

Proof. If M is compact and embedded, then for any plane $\mathcal{P}$ the domain of the Aleksandrov function Λ_1 is a compact set. Therefore, Λ_1 reaches its maximum value at some point $p \in \mathcal{D}$. By the Aleksandrov reflection method (Lemma 1.26) a plane parallel to $\mathcal{P}$ is a plane of symmetry of M. Since $\mathcal{P}$ is an arbitrary plane of $\mathbb{R}^3$ we conclude by Lemma 1.28 that M is a totally umbilical sphere. □

1.5 Codazzi pairs

In this section we will generalize the concepts introduced in Section 1.1. Here we are going to consider M as a smooth oriented surface, not necessarily immersed in the Euclidean space or any other manifold, with a pair of real symmetric bilinear forms prescribed. The outline is to imitate some situations involving the first and second fundamental forms of an immersion, especially the proof presented of Hopf's theorem.

We denote by $\mathcal{Q}(M)$ the set of symmetric bilinear forms on M and by $\mathcal{R}(M)$ the subset of those forms that are positive definite. A pair $(A, B) \in \mathcal{R}(M) \times \mathcal{Q}(M)$ is called a fundamental pair.

Once a bilinear form $A \in \mathcal{R}(M)$ is fixed, there exists a bijective correspondence between $\mathcal{Q}(M)$ and the set of self-adjoint endomorphisms of $\mathfrak{X}(M)$ denoted by $\mathcal{S}(M, A)$; this correspondence is given by the equation $B(X, Y) = A(SX, Y)$, $X, Y \in \mathfrak{X}(M)$, where $B \in \mathcal{Q}(M)$ and $S \in \mathcal{S}(M, A)$. We will say that S is the endomorphism associated to the fundamental pair (A, B).

Definition 1.30. *Let (A, B) be a fundamental pair on the surface M and S the endomorphism associated to this pair. We define the Codazzi tensor associated to the fundamental pair (A, B) as the map $T_S : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ given by*

$$T_S(X, Y) = \nabla_X SY - \nabla_Y SX - S[X, Y],$$

where ∇ is the Levi-Civita connection associated to (M, A).

In the case that the surface M is immersed in $\mathbb{R}^3$, the pair (I, II) formed by the first and second fundamental forms is, in this context, a fundamental pair on M, that is, $(I, II) \in \mathcal{R}(M) \times \mathcal{Q}(M)$. Moreover, considering the Weingarten endomorphism S we have that the Codazzi tensor, T_S, associated to the pair (I, II) vanishes identically by the Codazzi equation (1.1). This fact motivates the next definition.

Definition 1.31. *Let (A, B) be a fundamental pair on the surface M and S the endomorphism associated to this pair. We say that (A, B) is a Codazzi pair if, and only if, the Codazzi tensor T_S associated to this pair vanishes identically.*

For a fundamental pair (A, B) we can define real functions that generalize the mean and Gaussian curvature, the principal curvatures and the skew curvature of an immersed surface in the Euclidean space. To do this, we replicate the formulas given by (1.2)–(1.4).

Definition 1.32. *Let (A, B) be a fundamental pair on the surface M. We define the mean and extrinsic curvatures of the pair (A, B) as*

$$H(A, B) = \frac{1}{2}\,\mathrm{tr}(A^{-1}B) \quad \textit{and} \quad K_e(A, B) = \frac{\det(B)}{\det(A)}.$$

Also, we define the principal curvatures and the skew curvature of the pair as

- $k_1(A,B) = H(A,B) - \sqrt{H(A,B)^2 - K_e(A,B)}$,
- $k_2(A,B) = H(A,B) + \sqrt{H(A,B)^2 - K_e(A,B)}$,
- $q(A,B) = H(A,B)^2 - K_e(A,B) = \frac{1}{4}\Big(k_1(A,B) - k_2(A,B)\Big)^2$.

The explicit dependence of the fundamental pair (A,B) in the above notations will be used only when strictly necessary and it helps to avoid misunderstandings, that is, in general we will just write H, K_e, k_1, k_2 and q for the curvatures defined above.

We say that the fundamental pair (A,B) is umbilical at the point $p \in M$, if $q(p) = 0$, and that is minimal at p, if $H(p) = 0$.

Using a local isothermal chart on M, we have that all equalities in Subsection 1.1.2 hold for a fundamental pair. Furthermore, the Codazzi tensor vanishes identically if, and only if, (1.10) holds. Thus, we have the next theorem, which can be considered as an abstract version of the Hopf theorem. It was established by Milnor in [Mil80].

Theorem 1.33. *Let (A,B) be a fundamental pair. Then any of the two following statements imply the third one.*

- *(A,B) is a Codazzi pair.*
- *H is constant.*
- *The $(2,0)-$part of B is holomorphic to the conformal structure given by A.*

2 Elliptic Special Weingarten Surfaces of Minimal Type

> *Calculating does not equal mathematics. It's a subsection of it. In years gone by it was the limiting factor, but computers now allow you to make the whole of mathematics more intellectual.*
>
> — **Conrad Wolfram**

In this chapter we will present some results on the theory of Elliptic Special Weingarten Surfaces of Minimal Type that will be used in the next chapters, following mainly [ST95; ST93; BS97; RS94; Ale+10] and references therein.

An immersed surface M in the Euclidean space $\mathbb{R}^3$ is called a Special Weingarten surface (SW-surface) if its mean curvature, H, and Gaussian curvature, K, satisfy a relation of the form $H = f(H^2 - K)$, where $f : [0, \infty) \to \mathbb{R}$ is a continuous function of class C^1 in $(0, +\infty)$; in this case the function f is called the associated function of the surface M. When the function f satisfies that

$$4t(f'(t))^2 < 1 \text{ for all } t \in (0, \infty), \tag{2.1}$$

a SW-surface is called a Special Weingarted surface of elliptic type, or an Elliptic Special Weingarten surface (ESW-surface). We will refer to the condition (2.1) as the elliptic condition for the function f, and in the case that f satisfies it we will say that f is an elliptic function. An ESW-surface is also denoted in the literature as f-surface, see [ST93; ST99b; ST01], here we will use also this notation.

Elliptic means that, the family of f-surfaces satisfies the maximum principle in the sense of the Definition 1.23, see [RS94; BS97]; hence, for ESW-surfaces we are able to apply the geometric comparison principle (Theorem 1.22) and the Aleksandrov reflection method (Lemma 1.26). This fact gives us an analogy between ESW-surfaces and constant mean curvature surfaces. Actually, ESW-surfaces are divided in two classes depending on whether $f(0) \neq 0$ or $f(0) = 0$. Note that in the first case, any sphere of radius $\frac{1}{|f(0)|}$ is an ESW-surface, and in the second case any plane is an ESW-surface. We are interested in the second family of solutions, the family containing all planes of $\mathbb{R}^3$. Later, the reader will note that this family behaves as the family of minimal surfaces.

Definition 2.1. *Let M be an immersed surface in $\mathbb{R}^3$. We say that M is an Elliptic Special Weingarten surface of minimal type, ESWMT-surface in short, if M is an f-surface such that $f(0) = 0$.*

Note that, by (1.4), we can rewrite the Weingarten relation using the skew curvature as $H = f(q)$.

2.1 General facts

In this section we will recall some general properties of ESWMT-surfaces. These facts begin to show the similarities between ESWMT and minimal surfaces.

Let f be an elliptic function such that $f(0) = 0$, we denote by $\mathcal{S}(f)$ the family of all ESWMT-surfaces satisfying $H = f(q)$. As we mentioned above, any plane in $\mathbb{R}^3$ is an element of $\mathcal{S}(f)$, since $f(0) = 0$.

The first property to mention is the maximum principle. Rosenberg and Sa Earp in [RS94] proved that any pair of ESW-surfaces, satisfying the same Weingarten relation, satisfies the maximum principle in the sense of Definition 1.21. Another proof of this fact can be found in [BS97]. As a consequence we have the next proposition.

Proposition 2.2. *The family $\mathcal{S}(f)$ satisfies the maximum principle.*

One of the immediately consequences of the last proposition is the non-existence of elliptic points in ESWMT-surfaces, as we show in the next proposition.

Proposition 2.3. *Let M be an ESWMT-surface in $\mathbb{R}^3$. Then the Gaussian curvature of M is non-positive.*

Proof. By contradiction, suppose that there exists a point $p \in M$ such that $K(p) > 0$. Then there exists a neighborhood V of p in M such that all points of V are in the same side of the tangent plane T_pM. Since $p \in M \cap T_pM$ we conclude that $M \geq_p T_pM$. Due to the fact that both surfaces T_pM and M are in $\mathcal{S}(f)$, we can apply the geometric comparison principle locally, around p, to conclude that both surfaces agree in some neighborhood of p. This allows us to infer that $K(p) = 0$, which is a contradiction. $\square$

The next proposition, taken from [ST93], sets another property of non-flat ESWMT-surfaces: the set of parabolic points cannot have accumulation points.

Proposition 2.4. *Let M be an ESWMT-surface in $\mathbb{R}^3$. Then either the zeros of the Gaussian curvature are isolated, or M is contained in a plane.*

Recall that the convex hull of a set in the Euclidean space can be defined as the smallest closed convex set containing the original set. It is well known that any compact minimal surface with boundary immersed in $\mathbb{R}^3$ is contained in the the convex hull of its boundary. Considering that this fact relies on the maximum principle, we have the same property for ESWMT-surfaces as we display below.

Proposition 2.5. *Let M be a compact ESWMT-surface in $\mathbb{R}^3$ with boundary. Then M is contained inside the convex hull of its boundary.*

Proof. This is a direct consequence from the fact that $\mathcal{S}(f)$ has the maximum principle property and that any plane is an element of this family. Note that for any plane we can find a parallel plane far away from M. Then, moving the plane along its normal line approaching M, the first contact point of the plane with the surface must be at the boundary, since the maximum principle does not allow to have a first contact point at the interior of M. $\square$

One of the consequences of the last proposition is that there is no compact surfaces without boundary in the family $\mathcal{S}(f)$.

2.2 Rotational examples

As examples of ESWMT-surfaces we have all minimal surfaces, this is for the specific case when $f \equiv 0$. In general, Sa Earp and Toubiana proved in [ST95] the existence and uniqueness of rotational ESWMT-surfaces assuming that the function f satisfies

$$\liminf_{t \to 0^+} 4t\big(f'(t)\big)^2 < 1. \tag{2.2}$$

The first theorem about the existence of rotationally symmetric examples says:

Theorem 2.6 (Existence of rotational ESWMT-surfaces)**.** *Let f be an elliptic function satisfying $f(0) = 0$ and* (2.2)*, and let $\tau > 0$ be a real number satisfying*

$$\frac{1}{\tau} < \lim_{t \to \infty} \Big(t - f(t^2)\Big). \tag{2.3}$$

Then there exists a unique complete ESWMT-surface, M_τ, satisfying $H = f(q)$ such that M_τ is of revolution. Moreover, the generatrix curve is the graph of a convex function of class C^3, strictly positive, with τ as a global minimum, and symmetric.

Theorem 2.6 determines pretty well the geometry of rotational examples of ESWMT-surfaces. We have two cases, if the generatrix is or is not bounded. In the first case, the ESWMT-surface M_τ is between two parallel planes and, in the second case, all coordinates functions of M_τ are proper functions. See Figure 2.1.

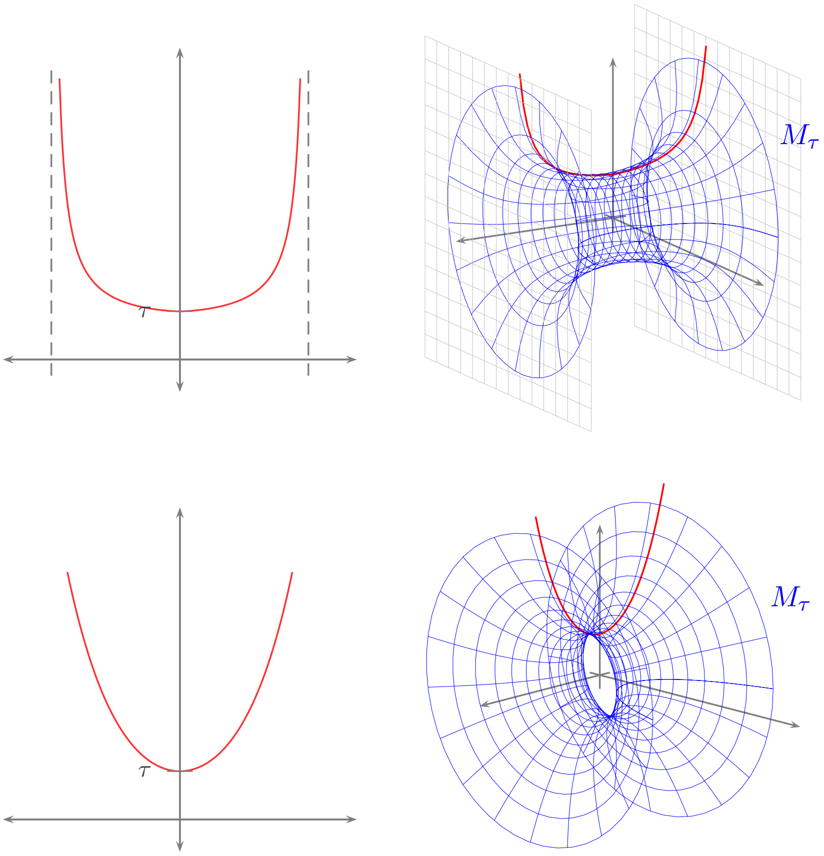

Figure 2.1: Generatrix (left) and complete ESWMT-surface M_τ (right).

Theorem 2.7 (Sa Earp and Toubiana, [ST95])**.** *Let f be an elliptic function satisfying $f(0) = 0$ and* (2.2)*, and let $\tau > 0$ be a real number satisfying* (2.3)*. If f is Lipschitz at 0, then the ESWMT-surface M_τ, obtained by Theorem 2.6, is not asymptotic to any plane in $\mathbb{R}^3$. Moreover, there exists a catenoid C such that outside of a compact set, M_τ is in the component of $\mathbb{R}^3 - C$ containing the revolution axis of C.*

Theorem 2.8 (Sa Earp and Toubiana, [ST95])**.** *Let f be a non-negative elliptic function satisfying $f(0) = 0$ and* (2.2)*. If f is Lipschitz at 0 and satisfies*

$$\lim_{t\to\infty}\left(t - f(t^2)\right) = +\infty, \tag{2.4}$$

then, as τ tends to 0, the generatrix of M_τ tends to a ray orthogonal to the rotation axis.

The main consequence of the last theorem is that the family of ESWMT-surfaces $\{M_\tau : \tau > 0\}$ behaves quite similar to the family of minimal catenoids. This means, in addition with the geometric comparison principle, that we have a half-space theorem for ESWMT-surfaces (with the appropriate requirements for f). The proof of this is the same as for minimal surfaces, see [HM90].

Corollary 2.9 (Half-space theorem for ESWMT-surfaces)**.** *Let f be a non-negative elliptic function Lipschitz at 0 satisfying $f(0) = 0$,* (2.2) *and* (2.4)*. Let M be a* *complete connected properly immersed ESWMT-surface verifying $H = f(q)$. If M is contained in a half space of $\mathbb{R}^3$, then M is flat.*

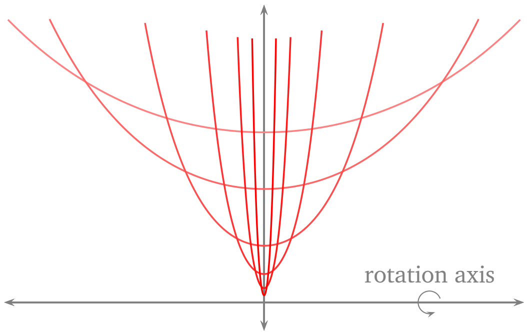

Figure 2.2: Generatrixes of rotational ESWMT-surfaces when the function f is non-negative, Lipschitz at 0 and satisfies (2.4).

The uniqueness of these examples is obtained in the next theorem.

Theorem 2.10 (Sa Earp and Toubiana, [ST95]). *Let f be an elliptic function satisfying $f(0) = 0$ and* (2.2). *If M is a complete ESWMT-surface satisfying $H = f(q)$ and rotationally symmetric, then there exists $\tau > 0$ such that M is the surface of revolution M_τ determined in Theorem 2.6.*

2.3 Special Codazzi pairs

We start this section recalling that a pair (A, B) of symmetric bilinear forms defined on M is called a fundamental pair if A is positive definite, and it is called a Codazzi pair if the Codazzi tensor, Definition 1.30, associated to this pair vanishes identically.

Consider a SW-surface M satisfying a relation of the form $H = f(q)$ and the fundamental pair (I, II) given by its first and second fundamental forms. Around any non umbilical point or any interior umbilical point, there exist local doubly orthogonal parameters (u, v) given by lines of curvature such that we are able to rewrite the Codazzi pair as:

$$I = Edu^2 + Gdv^2 \quad \text{and} \quad II = k_1 Edu^2 + k_2 Gdv^2,$$

where k_1 and k_2 are the principal curvatures of the surface M. Note that without loss of generality we can assume that $k_1 \geq k_2$.

In these coordinates, the Codazzi equation satisfied by the fundamental pair is equivalent to the following equations

$$(k_1)_v E = -tE_v, \quad \text{and} \quad (k_2)_u G = tG_u, \tag{2.5}$$

where $t = \frac{1}{2}(k_1 - k_2)$. Note that $t^2 = q = H^2 - K$.

Now we will use the relation $H = f(H^2 - K)$ in order to get a new Codazzi pair on the SW-surface M. Set

$$\phi(r) = \int_0^r 2f'(s^2)ds, \tag{2.6}$$

and define the following symmetric bilinear forms

$$I_f = e^{\phi(t)}Edu^2 + e^{-\phi(t)}Gdv^2 \quad \text{and} \quad II_f = te^{\phi(t)}Edu^2 - te^{-\phi(t)}Gdv^2. \tag{2.7}$$

The bilinear form I_f is positive definite since the functions E, G, $e^{\phi(t)}$ and $e^{-\phi(t)}$ are positive on M, and clearly the bilinear form II_f is symmetric. Therefore (I_f, II_f) is a fundamental pair.

The metric I_f was defined by Bryant, see [Bry11; Ale+10]. Furthermore, he used the complex structure of I_f to find a holomorphic quadratic form defined globally on M, whose zeros are the umbilic points of the immersion. This quadratic form allowed Bryant to prove a Hopf type theorem for SW-surfaces.

We are going to use the notation H_f for the mean curvature of the fundamental pair (I_f, II_f), that is, $H_f = H(I_f, II_f)$. We will simplify in the same way the notation for the extrinsic and skew curvature of this new fundamental pair. A quick computation shows that these curvatures are

$$H_f = 0, \tag{2.8}$$

$$K_f = -q, \tag{2.9}$$

$$q_f = q \tag{2.10}$$

Lemma 2.11. *If M is a SW-surface satisfying $H = f(H^2 - K)$, then (I_f, II_f) is a Codazzi pair on M.*

Proof. The Codazzi tensor associated to the pair (I_f, II_f) vanishes identically if, and only if, the following equalities hold

$$(te^{\phi(t)}E)_v = (e^{\phi(t)}E)_v H_f \tag{2.11}$$

$$(-te^{-\phi(t)}G)_u = (e^{-\phi(t)}G)_u H_f. \tag{2.12}$$

Since $H_f = 0$, we will show that $(te^{\phi(t)}E)_v = (-te^{-\phi(t)}G)_u = 0$. In the next computation we will use (2.5) and the equalities $\phi'(t) = 2f'(t^2)$ and $H_v = 2f'(t^2)tt_v$, the last one obtained by differentiate the Weingarten relation.

$$\begin{aligned}
(te^{\phi(t)}E)_v &= e^{\phi(t)}\Big(t_vE + t\phi'(t)t_vE + tE_v\Big) \\
&= e^{\phi(t)}E\Big(t_v + 2f'(t^2)tt_v - (k_1)_v\Big) \\
&= e^{\phi(t)}E\Big(t + H - k_1\Big)_v \\
&= 0.
\end{aligned}$$

To prove the other equality, we use an analogous computation. ☐

Remark 2.12. *The pair (I_f, II_f) can be defined globally in M; note that in an umbilical point, $I_f = I$ and $II_f \equiv 0$. In fact, we can define this pair without using doubly orthogonal coordinates.*

The last lemma allows us to apply Theorem 1.33 to the fundamental pair (I_f, II_f), then we conclude that the $(2,0)-$part of II_f is holomorphic for the conformal structure given by I_f. From Lemma 1.2, we already know that the zeros of this holomorphic form are the umbilic points of the fundamental pair (I_f, II_f), this is, the points where $q_f = 0$, but (2.10) shows that they are the umbilic points of the immersion.

Consider an isothermal parameter μ to I_f around a non umbilical point, then by Lemma 1.2 we can write

$$I_f = 2\lambda|d\mu|^2 \quad \text{and} \quad II_f = Qd\mu^2 + \bar{Q}d\bar{\mu}^2,$$

since $H_f = 0$. The fact that $Qd\mu^2$ is holomorphic implies that there exists a conformal parameter z for I_f such that $cQd\mu^2 = dz^2$ for a non-zero constant c, see [Mil80]. If we consider $c = 2$, we get that $2|Q||d\mu|^2 = |dz|^2$ is a flat metric.

By Lemma 1.2, $|Q|^2 = q\lambda^2$ and then

$$\lambda = \frac{|Q|}{\sqrt{q}} = \frac{|Q|}{t},$$

which means that $tI_f = 2|Q||d\mu|^2$.

We summarize this in the next lemma.

Lemma 2.13. *Let M be a SW-surface satisfying a relation of the form $H = f(q)$. The metric $\sqrt{q}I_f$ is flat on $M \setminus M_u$, where M_u is the set of umbilic points of M. Moreover, there exists a local conformal parameter z such that*

$$\sqrt{q}I_f = |dz|^2 \quad \textit{and} \quad II_f = \frac{1}{2}\left(dz^2 + d\bar{z}^2\right).$$

2.4 The Gauss map of Special Weingarten surfaces

Let M be an oriented SW-surface satisfying a relation of the form $H = f(q)$. In the last section we have defined the Codazzi pair (I_f, II_f), away from umbilical points, using a primitive of the function $2f'(s^2)$. Actually, the metric I_f can be well defined at umbilic points whenever the function $\frac{1}{t}\sinh\phi(t)$ can be extended to $t = 0$. To justify this assertion, we will define the fundamental pair in an equivalent way. Consider the functions

$$a = \cosh\phi(t) - \frac{H}{t}\sinh\phi(t), \quad b = \tfrac{1}{t}\sinh\phi(t) \tag{2.13}$$

$$c = t\sinh\phi(t) - H\cosh\phi(t), \quad d = \cosh\phi(t). \tag{2.14}$$

A straight computation exhibits that

$$I_f = aI + bII \quad \text{and} \quad II_f = cI + dII. \tag{2.15}$$

Note that $ad - bc = 1$, so we can recover the first and second fundamental forms as

$$I = dI_f - bII_f \quad \text{and} \quad II = -cI_f + aII_f. \tag{2.16}$$

Using the conformal parameter obtained in Lemma 2.13, we rewrite (2.16) as

$$\begin{aligned} I &= \frac{1}{2}\left(-b\,dz^2 + 2\frac{d}{t}|dz|^2 - b\,d\bar{z}^2\right), \\ II &= \frac{1}{2}\left(adz^2 - 2\frac{c}{t}|dz|^2 + ad\bar{z}^2\right). \end{aligned}$$

With the notation of the Subsection 1.1.2, we can write

$$P = -\frac{b}{2}, \quad \lambda = \frac{d}{2t}, \quad Q = \frac{a}{2} \quad \text{and} \quad \rho = \frac{c}{2t}. \tag{2.17}$$

By Lemma 1.7 we know that $-N_z = H\partial_z + t\partial_{\bar{z}}$ and that this vector field vanishes only at umbilical points. Since $I(N_z, N_z) = -\frac{1}{2}(c - aH)$, the quantity $c - aH$ vanishes only when $q = 0$.

We are going to compute the norm of the vector field $\partial_z \wedge \partial_{\bar{z}}$.

$$\cos\left(\measuredangle(\partial_z, \partial_{\bar{z}})\right) = \frac{I(\partial_z, \partial_{\bar{z}})}{|\partial_z||\partial_{\bar{z}}|} = \frac{\lambda}{|P|} = \frac{d}{tb},$$

then

$$\sin\left(\measuredangle(\partial_z, \partial_{\bar{z}})\right) = \sqrt{1 - \cos^2\left(\measuredangle(\partial_z, \partial_{\bar{z}})\right)} = \frac{\sqrt{t^2b^2 - d^2}}{tb} = \frac{i}{\sinh\phi(t)},$$

finally,

$$|\partial_z \wedge \partial_{\bar{z}}| = |\partial_z||\partial_{\bar{z}}|\sin\left(\measuredangle(\partial_z, \partial_{\bar{z}})\right) = |P|\frac{i}{\sinh\phi(t)} = \frac{bi}{2\sinh\phi(t)} = \frac{i}{2t}.$$

Since $N = \frac{1}{|\partial_z \wedge \partial_{\bar{z}}|}\partial_z \wedge \partial_{\bar{z}}$, we conclude that

$$I(N, \partial_z \wedge \partial_{\bar{z}}) = |\partial_z \wedge \partial_{\bar{z}}| = \frac{i}{2t}. \tag{2.18}$$

There exist complex functions α and β such that $-iN \wedge \partial_z = \alpha\partial_z + \beta\partial_{\bar{z}}$. Hence using (2.18) we obtain

$$0 = I(iN \wedge \partial_z, \partial_z) = I(\alpha\partial_z + \beta\partial_{\bar{z}}, \partial_z) = P\alpha + \lambda\beta,$$

$$\frac{1}{2t} = -iI(N, \partial_z \wedge \partial_{\bar{z}}) = I(-iN \wedge \partial_z, \partial_{\bar{z}}) = I(\alpha\partial_z + \beta\partial_{\bar{z}}, \partial_{\bar{z}}) = \lambda\alpha + \bar{P}\beta.$$

Solving the linear system we have that

$$\alpha = \frac{-\lambda\frac{1}{2t}}{|P|^2-\lambda^2} = \frac{d}{d^2-t^2b^2} = d \quad \text{and} \quad \beta = \frac{P\frac{1}{2t}}{|P|^2-\lambda^2} = \frac{tb}{d^2-t^2b^2} = tb.$$

Summarizing,

$$iN \wedge \partial_z = -\cosh\phi(t)\partial_z - \sinh\phi(t)\partial_{\bar z}, \tag{2.19}$$

$$iN \wedge \partial_{\bar z} = \sinh\phi(t)\partial_z + \cosh\phi(t)\partial_{\bar z}. \tag{2.20}$$

The equality (2.20) is obtained from (2.19) by conjugation.

Lemma 2.14. $iN \wedge N_z = c\partial_z + ta\partial_{\bar z}$.

Proof. By Lemma 1.7 and equalities (2.19) and (2.20) we have

$$\begin{aligned} iN \wedge N_z &= i(N \wedge (H\partial_z + t\partial_{\bar z})) \\ &= H(iN \wedge \partial_z) + t(iN \wedge \partial_{\bar z}) \\ &= -H(\cosh\phi(t)\partial_z + \sinh\phi(t)\partial_{\bar z}) + t(\sinh\phi(t)\partial_z + \cosh\phi(t)\partial_{\bar z}) \\ &= (t\sinh\phi(t) - H\cosh\phi(t))\partial_z + (t\cosh\phi(t) - H\sinh\phi(t))\partial_{\bar z} \\ &= c\partial_z + ta\partial_{\bar z}. \end{aligned}$$

□

Now we have all the elements to recover the natural basis induced by the immersion in terms of N_z and $iN \wedge N_z$.

Lemma 2.15. $\partial_z = \dfrac{1}{c-aH}(aN_z + iN \wedge N_z)$.

Proof. Immediate from Lemma 1.7 and Lemma 2.14. □

Recalling the expressions for N, N_z and $iN \wedge N_z$ in terms of the Gauss map of the surface M obtained in the Subsection 1.2.2, we conclude, after a directly computation, that

Theorem 2.16. $\partial_z = \dfrac{a-1}{c-aH}\bar g_z \xi + \dfrac{a+1}{c-aH} g_z \bar\xi$.

Following the ideas presented in [GM00], imposing that $\partial_{\bar z}(\partial_z) = \partial_z(\partial_{\bar z})$ we obtain a complete integrability condition from Theorem 2.16. We synthesize this in the next theorem.

Theorem 2.17. *Let M be an oriented SW-surface satisfying a relation of the form $H = f(q)$, then the Gauss map g satisfies*

$$\frac{2}{c-aH}\left(g_{z\bar z} - 2g_z g_{\bar z}\frac{\bar g}{1+g\bar g}\right) = -\left(\frac{a+1}{c-aH}\right)_{\bar z} g_z + \left(\frac{a-1}{c-aH}\right)_z g_{\bar z}, \tag{2.21}$$

where the functions a and c are given by (2.13) *and* (2.14) *respectively.*

Proof. In order to simplify the following equations, we will denote $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$, $A = \frac{a-1}{c-aH}$ and $B = \frac{a+1}{c-aH}$.

From Theorem 2.16 we get that

$$\begin{aligned}(\mathbf{x}_1)_{z\bar{z}} &= A_{\bar{z}}\frac{1-g^2}{(1+g\bar{g})^2}\bar{g}_z - 2A\frac{g+\bar{g}}{(1+g\bar{g})^3}\bar{g}_z g_{\bar{z}} - 2A\frac{g(1-g^2)}{(1+g\bar{g})^3}\bar{g}_z\bar{g}_{\bar{z}} + A\frac{1-g^2}{(1+g\bar{g})^2}\bar{g}_{z\bar{z}} \\ &\quad + B_{\bar{z}}\frac{1-\bar{g}^2}{(1+g\bar{g})^2}g_z - 2B\frac{g+\bar{g}}{(1+g\bar{g})^3}g_z\bar{g}_{\bar{z}} - 2B\frac{\bar{g}(1-\bar{g}^2)}{(1+g\bar{g})^3}g_z g_{\bar{z}} + B\frac{1-\bar{g}^2}{(1+g\bar{g})^2}g_{z\bar{z}},\end{aligned}$$

$$\begin{aligned}(\mathbf{x}_1)_{\bar{z}z} &= A_z\frac{1-\bar{g}^2}{(1+g\bar{g})^2}g_{\bar{z}} - 2A\frac{g+\bar{g}}{(1+g\bar{g})^3}\bar{g}_z g_{\bar{z}} - 2A\frac{\bar{g}(1-\bar{g}^2)}{(1+g\bar{g})^3}g_z g_{\bar{z}} + A\frac{1-\bar{g}^2}{(1+g\bar{g})^2}g_{z\bar{z}} \\ &\quad + B_z\frac{1-g^2}{(1+g\bar{g})^2}\bar{g}_{\bar{z}} - 2B\frac{g+\bar{g}}{(1+g\bar{g})^3}g_z\bar{g}_{\bar{z}} - 2B\frac{g(1-g^2)}{(1+g\bar{g})^3}\bar{g}_z\bar{g}_{\bar{z}} + B\frac{1-g^2}{(1+g\bar{g})^2}\bar{g}_{z\bar{z}},\end{aligned}$$

$$\begin{aligned}(\mathbf{x}_2)_{z\bar{z}} &= iA_{\bar{z}}\frac{1+g^2}{(1+g\bar{g})^2}\bar{g}_z + 2iA\frac{g-\bar{g}}{(1+g\bar{g})^3}\bar{g}_z g_{\bar{z}} - 2iA\frac{g+g^3}{(1+g\bar{g})^3}\bar{g}_z\bar{g}_{\bar{z}} + iA\frac{1+g^2}{(1+g\bar{g})^2}\bar{g}_{z\bar{z}} \\ &\quad - iB_{\bar{z}}\frac{1+\bar{g}^2}{(1+g\bar{g})^2}g_z + 2iB\frac{g-\bar{g}}{(1+g\bar{g})^3}g_z\bar{g}_{\bar{z}} + 2iB\frac{\bar{g}+\bar{g}^3}{(1+g\bar{g})^3}g_z g_{\bar{z}} - iB\frac{1+\bar{g}^2}{(1+g\bar{g})^2}g_{z\bar{z}},\end{aligned}$$

$$\begin{aligned}(\mathbf{x}_2)_{\bar{z}z} &= -iA_z\frac{1+\bar{g}^2}{(1+g\bar{g})^2}g_{\bar{z}} + 2iA\frac{g-\bar{g}}{(1+g\bar{g})^3}\bar{g}_z g_{\bar{z}} + 2iA\frac{\bar{g}+\bar{g}^3}{(1+g\bar{g})^3}g_z g_{\bar{z}} - iA\frac{1+\bar{g}^2}{(1+g\bar{g})^2}g_{z\bar{z}} \\ &\quad + iB_z\frac{1+g^2}{(1+g\bar{g})^2}\bar{g}_{\bar{z}} + 2iB\frac{g-\bar{g}}{(1+g\bar{g})^3}g_z\bar{g}_{\bar{z}} - 2iB\frac{g+g^3}{(1+g\bar{g})^3}\bar{g}_z\bar{g}_{\bar{z}} + iB\frac{1+g^2}{(1+g\bar{g})^2}\bar{g}_{z\bar{z}},\end{aligned}$$

$$\begin{aligned}(\mathbf{x}_3)_{z\bar{z}} &= 2A_{\bar{z}}\frac{g}{(1+g\bar{g})^2}\bar{g}_z + 2A\frac{1-g\bar{g}}{(1+g\bar{g})^3}\bar{g}_z g_{\bar{z}} - 4A\frac{g^2}{(1+g\bar{g})^3}\bar{g}_z\bar{g}_{\bar{z}} + 2A\frac{g}{(1+g\bar{g})^2}\bar{g}_{z\bar{z}} \\ &\quad + 2B_{\bar{z}}\frac{\bar{g}}{(1+g\bar{g})^2}g_z + 2B\frac{1-g\bar{g}}{(1+g\bar{g})^3}g_z\bar{g}_{\bar{z}} - 4B\frac{\bar{g}^2}{(1+g\bar{g})^3}g_z g_{\bar{z}} + 2B\frac{\bar{g}}{(1+g\bar{g})^2}g_{z\bar{z}},\end{aligned}$$

$$\begin{aligned}(\mathbf{x}_3)_{\bar{z}z} &= 2A_z\frac{\bar{g}}{(1+g\bar{g})^2}g_{\bar{z}} + 2A\frac{1-g\bar{g}}{(1+g\bar{g})^3}\bar{g}_z g_{\bar{z}} - 4A\frac{\bar{g}^2}{(1+g\bar{g})^3}g_z g_{\bar{z}} + 2A\frac{\bar{g}}{(1+g\bar{g})^2}g_{z\bar{z}} \\ &\quad + 2B_z\frac{g}{(1+g\bar{g})^2}\bar{g}_{\bar{z}} + 2B\frac{1-g\bar{g}}{(1+g\bar{g})^3}g_z\bar{g}_{\bar{z}} - 4B\frac{g^2}{(1+g\bar{g})^3}\bar{g}_z\bar{g}_{\bar{z}} + 2B\frac{g}{(1+g\bar{g})^2}\bar{g}_{z\bar{z}}.\end{aligned}$$

From the equations above, we infer that $(\mathbf{x}_i)_{\bar{z}z} = (\mathbf{x}_i)_{z\bar{z}}$ $(i = 1, 2, 3)$ if, and only if,

$$\begin{aligned}0 &= (1-g^2)\Big(A_{\bar{z}}\bar{g}_z - B_z\bar{g}_{\bar{z}}\Big) - (1-\bar{g}^2)\Big(A_z g_{\bar{z}} - B_{\bar{z}}g_z\Big) \\ &\quad + \frac{4}{(c-aH)(1+g\bar{g})}\Big((g-g^3)\bar{g}_z\bar{g}_{\bar{z}} - (\bar{g}-\bar{g}^3)g_z g_{\bar{z}}\Big) \\ &\quad + \frac{2}{c-aH}\Big((1-\bar{g}^2)g_{z\bar{z}} - (1-g^2)\bar{g}_{z\bar{z}}\Big),\end{aligned} \tag{2.22}$$

$$
\begin{aligned}
0 \quad = \quad & (1+g^2)\Big(A_{\bar{z}}\bar{g}_z - B_z\bar{g}_{\bar{z}}\Big) + (1+\bar{g}^2)\Big(A_z g_{\bar{z}} - B_{\bar{z}} g_z\Big) \\
& + \frac{4}{(c-aH)(1+g\bar{g})}\Big((g+g^3)\bar{g}_z\bar{g}_{\bar{z}} + (\bar{g}+\bar{g}^3)g_z g_{\bar{z}}\Big) \\
& - \frac{2}{c-aH}\Big((1+\bar{g}^2)g_{z\bar{z}} + (1+g^2)\bar{g}_{z\bar{z}}\Big),
\end{aligned}
\tag{2.23}
$$

$$
\begin{aligned}
0 \quad = \quad & g\Big(A_{\bar{z}}\bar{g}_z - B_z\bar{g}_{\bar{z}}\Big) - \bar{g}\Big(A_z g_{\bar{z}} - B_{\bar{z}} g_z\Big) \\
& + \frac{4}{(c-aH)(1+g\bar{g})}\Big(g^2\bar{g}_z\bar{g}_{\bar{z}} - \bar{g}^2 g_z g_{\bar{z}}\Big) \\
& + \frac{2}{c-aH}\Big(-g\bar{g}_{z\bar{z}} + \bar{g}g_{z\bar{z}}\Big).
\end{aligned}
\tag{2.24}
$$

If we take (2.22) minus (2.23) plus $2g$ times (2.24), then we obtain that

$$
0 = -(1+g\bar{g})\Big(A_z g_{\bar{z}} - B_{\bar{z}} g_z\Big) - \frac{4}{(c-aH)(1+g\bar{g})}\bar{g}(1+g\bar{g})g_z g_{\bar{z}} + \frac{2}{c-aH}(1+g\bar{g})g_{z\bar{z}}
$$

Rewriting the above equation we have (2.21).

□

3 Asymptotic behaviour of ends of Elliptic Special Weingarten Surfaces of Minimal Type

> *Mathematics is not a careful march down a well-cleared highway, but a journey into a strange wilderness, where the explorers often get lost. Rigour should be a signal to the historian that the maps have been made, and the real explorers have gone elsewhere.*
>
> — **William Anglin**

This chapter is devoted to study the ends of an ESWMT-surface with finite total second fundamental form. Roughly speaking, we want to prove that those ends are regular at infinity, which means that the growth of each end is at most logarithmic. This is the situation for minimal surfaces, in fact, being regular at infinity is equivalent to have finite total curvature; this was proved by Schoen in [Sch83].

We begin with the concept of regular at infinity and the equivalence with finite total curvature for minimal surfaces.

3.1 Regular at infinity for minimal surfaces

Although Schoen developed this concept for minimal hypersurfaces, see [Sch83], the definition can be used for any hypersurface.

Definition 3.1. *A complete immersion* $\mathbf{f} : M^n \to \mathbb{R}^{n+1}$ *is said to be regular at infinity if there is a compact subset* $K \subset M$ *such that* $M \setminus K$ *consists of* r *components* $M_1, \cdots, M_r$ *such that each* M_i *is the graph of a function* u_i *with bounded slope over the exterior of a bounded region in some hyperplane* Π_i*. Moreover, if* $x_1, \cdots, x_n$ *are the coordinates in* Π_i*, we require the* u_i*'s have the following asymptotic behaviour for* $|x|$ *large and* $n = 2$*:*

$$u_i(x) = a \log|x| + b + \frac{c_1 x_1}{|x|^2} + \frac{c_2 x_2}{|x|^2} + O(|x|^{-2}); \tag{3.1}$$

while for $n \geq 3$ *we require*

$$u_i(x) = a + b|x|^{2-n} + \sum_{i=1}^{n} c_j x_j |x|^{-n} + O(|x|^{-n}), \tag{3.2}$$

for constants a, b, c_j depending on i.

The components M_i in the last definition are the ends of the hypersurface M. Observe that this definition requires embedded ends, but does not prohibit the possibility that they intersect.

As we mentioned at the beginning of this chapter, this definition, for minimal surfaces, is related to finite total curvature. More precisely,

Theorem 3.2 (Schoen, [Sch83])**.** *A complete minimal immersion $\mathbf{f} : M^2 \to \mathbb{R}^3$ is regular at infinity if, and only if, M has finite total curvature and each end of M is embedded.*

The proof presented by Schoen depends strongly on the fact that M is a minimal surface, especially it depends on the Weierstrass representation. This is the first limitation to deal in order to generalize the classification obtained by Schoen to ESWMT-surfaces.

For higher dimensions, $n \geq 3$, it is proven that regularity at infinity is a consequence of minimality and bounded slope. This theorem and its proof have a different perspective in contrast to the case of surfaces.

Theorem 3.3 (Schoen, [Sch83])**.** *Assume $n \geq 3$, and $\mathbf{f} : M^n \to \mathbb{R}^{n+1}$ is a minimal immersion with the property that $M \setminus K$, for some compact K, is the union of $M_1, \cdots, M_r$ where each M_i is a graph of bounded slope over the exterior of a bounded region over a hyperplane Π_i. Then M is regular at infinity.*

The main idea of the proof is to use the bounded slope to show that the tangent planes have some "limit plane" at infinity and to obtain a nice growth of the height of each end. Using elliptic theory, the desired expansion of the functions modeling each end is achieved.

With this characterization we get,

Theorem 3.4 (Schoen, [Sch83])**.** *The only connected complete minimal immersions $\mathbf{f} : M^n \to \mathbb{R}^{n+1}$, which are regular at infinity and have two ends, are the catenoids.*

3.2 ESWMT-surfaces with finite total second fundamental form

First we are going to prove that the ends of an ESWMT-surface with finite total second fundamental form are, outside of some compact set, graphs of functions defined over planes orthogonal to the Gauss map at infinity. We will use results presented in Section 1.3.

Theorem 3.5. *Let M be a complete connected oriented ESWMT-surface immersed in $\mathbb{R}^3$. If M has finite total second fundamental form, then M is of finite topology, properly immersed, the Gauss map extends continuously at infinity and any end is a multigraph over some plane. Moreover, if an end is embedded, this end is a graph over some plane.*

Proof. By Theorem 1.16, we get that M has finite total curvature, then using Theorem 1.14, we conclude that M is of finite topology. Additionally, Proposition 2.3 says that $K \leq 0$, hence Theorem 1.17 gives that M is properly immersed and the Gauss map extends continuously at infinity. Let E be an end of M and suppose that the Gauss map goes (at this end) to the unitary vector $\nu \in \mathbb{R}^3$ at infinity, then considering the plane $\Pi_\nu = \{x \in \mathbb{R}^3 : \langle x, \nu \rangle = 0\}$ we infer that the tangent planes are uniformly near to Π_ν going far out on E. This last information guaranties that the projection of E over the plane Π_ν is a covering map outside a big enough cylinder, i.e., there exists $R > 0$ such that the projection $p_\nu : E \setminus C_\nu(R) \to \Pi_\nu \setminus B_{\Pi_\nu}(R)$ defined by $p_\nu(x) = x - \langle x, \nu \rangle \nu$ is a covering map, where $C_\nu(R) = \{x \in \mathbb{R}^3 : ||p_\nu(x)|| \leq R\}$ and $B_{\Pi_\nu}(R) = \{x \in \Pi_\nu : ||x|| \leq R\}$. Since M is properly immersed, E cannot be infinitely sheeted, thereby E (outside the compact $E \cap C_\nu(R)$) is a multigraph. Besides, if E is embedded, then $E \setminus C_\nu(R)$ is a graph over $\Pi_\nu \setminus B_{\Pi_\nu}(R)$. □

From now on, we follow the notation of the last proof and we work on an embedded end E of M, which is actually a graph over a domain contained in the plane Π_ν and such that the tangent planes are uniformly near to Π_ν. We denote by $N : E \to \mathbb{S}^2$ the Gauss map and $\theta : E \to [-1, 1] \subset \mathbb{R}$ the angle function defined by $\theta(x) = \langle N(x), \nu \rangle$ for each $x \in E$. From Theorem 3.5, there exists $0 < \epsilon < 1$ such that $1 - \epsilon \leq \theta(x) \leq 1$ for all $x \in E$. Let $\iota : E \to \mathbb{R}^3$ be the inclusion map, then it is well known that $\Delta^E \iota = 2\mathbf{H}$, where $\mathbf{H}$ is the mean curvature vector of E, that is, $\mathbf{H} = HN$, H the mean curvature and Δ^E the Laplacian operator on E. Multiplying by ν this identity we get $\Delta^E \langle \iota, \nu \rangle = 2H\theta$. Since E is a graph, we can write $E = \{x + \phi(x)\nu : x \in \Pi_\nu \setminus B_{\Pi_\nu}(R)\}$ for some smooth function $\phi : \Pi_\nu \setminus B_{\Pi_\nu}(R) \to \mathbb{R}$, and then abusing notation we write $\langle \iota, \nu \rangle = \phi$ and $\Delta^E \phi = 2H\theta$.

Theorem 1.14 says that the end E is conformally equivalent to the punctured disk $\mathbb{D}^\circ = \{z \in \mathbb{C} : 0 < |z| \leq 1\}$, that is, there exists a diffeomorphism $\alpha : \mathbb{D}^\circ \to E$ such that the metrics g_0 and $g_1 = \alpha^* g$ are pointwise conformal. Here we are denoting as g_0 and g, the flat metric in $\mathbb{D}^\circ$ and the Euclidean metric restricted to E, respectively. It follows that there exists a smooth positive function $\lambda : \mathbb{D}^\circ \to \mathbb{R}$ such that we can write $g_1 = \lambda g_0$, and, after a straightforward computation, $\Delta^{g_1} = \lambda^{-1} \Delta^{g_0}$. Define the function $u : \mathbb{D}^\circ \to \mathbb{R}$ by $u(z) = \langle \alpha(z), \nu \rangle$, due to the fact that α is an isometry from $(\mathbb{D}^\circ, g_1)$ onto (E, g) we have that $\Delta^{g_1} u = (\Delta^E \langle \iota, \nu \rangle) \circ \alpha$; joining this with the relation between the Laplicians and the expression obtained for $\Delta^E \phi$ we achieve

$$\Delta^{g_0} u = 2\lambda H\theta. \tag{3.3}$$

The right hand side of (3.3) must be read as a function on $\mathbb{D}^\circ$, it means that there is an implicit composition with the diffeomorphism α.

We think on the function u as the height of the end E, over the plane Π_ν, given by the conformal immersion $\alpha : \mathbb{D}^\circ \to E$, see Figure 3.1. In the following section we will study the growth near the origin of this height, and in Section 3.4 we will prove that $u(z)$ is asymptotically radial as $|z| \to 0$.

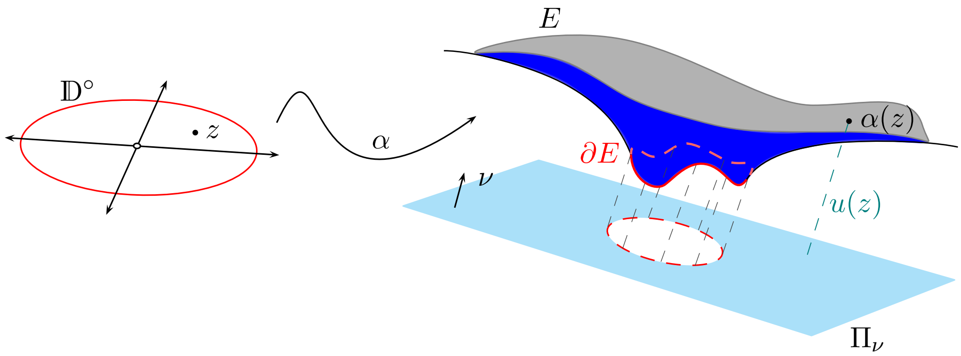

Figure 3.1: Parametrizing the end E with a conformal parameter.

3.3 Growth of an ESWMT-end

The first claim about the height function u of an end E is that, after a vertical translation of the end E, either $u \geq 0$ or $u \leq 0$. This means that the end E cannot grow infinitely in both directions ν and $-\nu$ simultaneously. This fact is intuitively true, but it is no trivial.

We would like to emphasize that there are harmonic immersions of compact Riemann surfaces with finitely many punctures that are complete, properly embedded, negatively curved, with finite total curvature and highly complicated ends, see [Con+15; AL13]. Some of those examples are presented in Figure 3.2. Although the family of harmonic surfaces has similarities with the family of minimal surfaces, for example harmonic surfaces satisfy the maximum principle and have non-positive Gaussian curvature; some differences appear, for instance there are embedded examples of any genus with just one end.

We will devote this section to prove the claim we made above, Corollary 3.12. The proof presented here is based principally on the work of Chan, see [Cha00; CT01], and Hopf, see [Hop50b].

With the purpose of fixing some notations and hypothesis, let E be an embedded end of an ESWMT-surface M immersed in $\mathbb{R}^3$ with finite total second fundamental form. We already know that the Gauss map goes, up to a rigid motion, to the unitary vector $\nu = (0,0,1)$ at infinity and that, outside some cylinder $C_\nu(R)$, the end E is the graph of a function $\phi : \mathbb{R}^2 \setminus B(R) \to \mathbb{R}$.

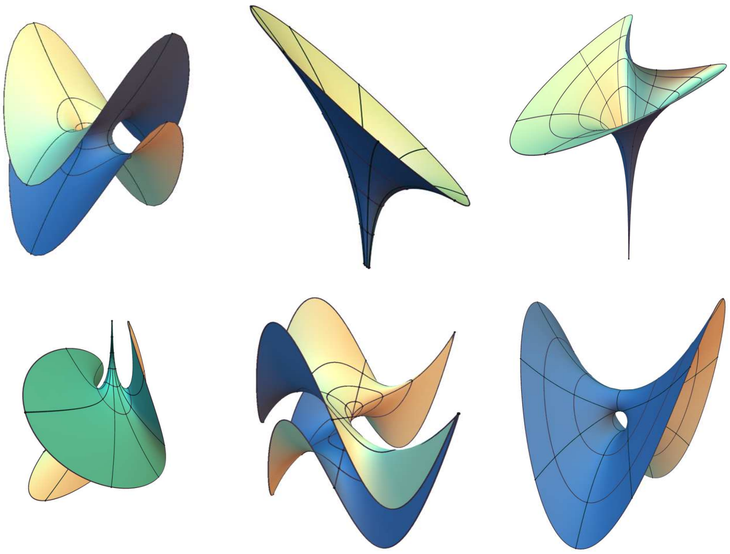

Figure 3.2: Examples of properly embedded harmonic surfaces with finite total curvature[1].

Consider a circle γ contained in the domain of ϕ. Let P_γ denote the linear function that best approximates the function ϕ on the circle γ in the supremum norm. If we define

$$m(\gamma) = \inf_{a,b,c \in \mathbb{R}} \left\{ \sup_{(x,y)\in\gamma} |\phi(x,y) - (ax + by + c)| \right\}, \tag{3.4}$$

then $m(\gamma) = \sup\limits_{(x,y)\in\gamma} |\phi(x,y) - P_\gamma(x,y)|$. Note that $m(\gamma) = 0$ if, and only if, $\phi(\gamma)$ is a torsion free curve.

The next lemma, under the hypothesis that ϕ is not linear, guaranties the existence of a circle such that its best approximating plane is not horizontal.

Lemma 3.6 (Chan and Treibergs, [CT01]). *If ϕ is not a linear function, there exists a circle γ_0 surrounding $\partial B(R)$ such that $m(\gamma_0) \neq 0$ and $\nabla P_{\gamma_0} \neq 0$.*

The Tchebychev equioscillation theorem, see [Ach92, p. 57], says that if a polynomial of degree n is the best approximation of a function defined on a compact interval, in the supremum norm, then there exist $n+2$ ordered points on the interval where this distance is attained and the polynomial oscillates around the function. There is a special version of the Tchebychev equioscillation theorem for periodic functions approximated by trigonometric functions, see [Ach92, pp. 64–65]. This version immediately guaranties the following lemma.

Lemma 3.7. *For any circle γ contained in the domain of ϕ, there exist at least four points, ordered as, β_1^+, β_1^-, β_2^+, β_2^- along γ, such that for $i = 1, 2$ we have $\phi(\beta_i^+) = P_\gamma(\beta_i^+) + m(\gamma)$ and $\phi(\beta_i^-) = P_\gamma(\beta_i^-) - m(\gamma)$.*

This lemma means that on the circle γ the function $\phi - P_\gamma$ alternates between $m(\gamma)$ and $-m(\gamma)$ at the points $\beta_i^\pm$.

[1] From `http://www.indiana.edu/~minimal/archive/Harmonic/index.html`, by Weber.

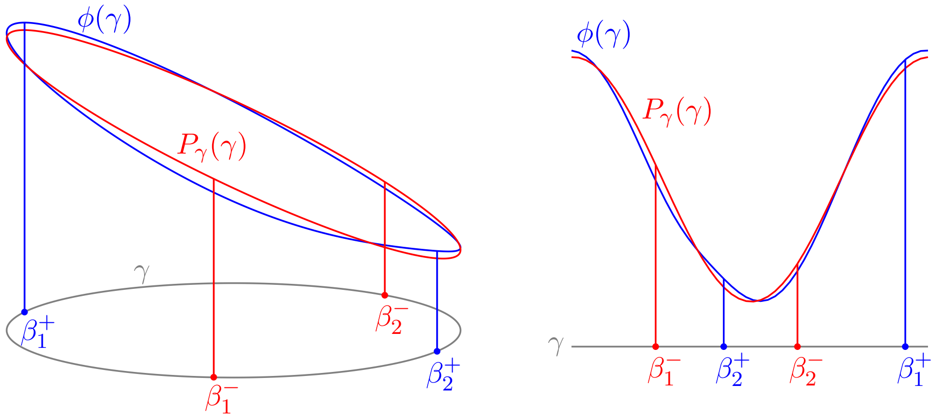

Figure 3.3: The Tchebychev theorem.

The next lemma is due to Bernstein.

Lemma 3.8 (Bernstein, [Ber15]). *Let ϕ be of class C^2 in a connected open set Ω and let $\phi_{xx}\phi_{yy} - \phi_{xy}^2 \leq 0$ in Ω. Assume $\phi > 0$ in Ω and $\phi = 0$ on $\partial\Omega$. Suppose that Ω can be placed in a sector of angle less than π. Let $M(r)$ be the maximum of ϕ on the part of the circle of radius r, centered at some fixed point, contained in $\overline{\Omega}$. Then there exist a positive constant c such that $M(r) > cr$ for all r sufficiently large.*

The original proof of Bernstein contained a gap. Hopf in [Hop50a] proved a topological lemma, which is the key to fulfill the gap detected in Bernstein's work. This lemma is also important in the proof of our result and it reads as follows.

Lemma 3.9 (Hopf, [Hop50a]). *If $\Omega \subset \mathbb{R}^n$ is a bounded open set and $\Omega' \subset \Omega$ is a connected open subset whose boundary has at least one point in common with $\partial\Omega$, then the set C of all of this common boundary points contains a set accessible from within Ω. If C is the union of two disjoint closed sets then each of these sets contains a set accessible from within Ω.*

We are ready to state the main result of this section.

Lemma 3.10 (Growth lemma). *Let E be an end of an immersed ESWMT-surface. Suppose there exists $R > 0$ such that E is obtained as the graph of a C^2 class function ϕ defined in $\mathbb{R}^2 \setminus B(R)$. Suppose also that $|\nabla\phi| \to 0$ as $r \to \infty$, where r is the distance from an arbitrary point to the origin. Then at least one of the quantities $\inf\limits_{|(x,y)|\geq R} \phi(x,y)$ or $\sup\limits_{|(x,y)|\geq R} \phi(x,y)$ is finite.*

Proof. The Gaussian curvature K of E is then given by

$$K = \frac{\phi_{xx}\phi_{yy} - \phi_{xy}^2}{(1 + \phi_x^2 + \phi_y^2)^2},$$

therefore by Proposition 2.3 the function ϕ satisfies $\phi_{xx}\phi_{yy} - \phi_{xy}^2 \leq 0$. If K vanishes identically, then E is contained in a plane and, hence, ϕ is linear. From the hypothesis on the gradient of ϕ, we conclude that $\phi \equiv 0$, and the proposition is proven. Consequently we will consider that K does not vanish identically.

Now, assume that

$$\inf_{|(x,y)|\geq R} \phi(x,y) = -\infty \text{ and } \sup_{|(x,y)|\geq R} \phi(x,y) = \infty, \tag{3.5}$$

with the purpose of obtaining a contradiction.

Claim A. *The function ϕ has sublinear growth.*

Proof. For $r \geq R$, let $\gamma(r)$ be the circle centered at the origin and radius r, $m(r)$ and $M(r)$ be the minimum and the maximum value of ϕ on $\gamma(r)$ respectively. Taking r big enough, (3.5) implies that

$$m(r) < \min\{0, m(R)\} \text{ and } M(r) > \max\{0, M(R)\}.$$

By Lemma 3.7, there exist ordered points p_1, q_1, p_2, q_2 in $\gamma(r)$ such that $\phi(p_1) = m(r)$, $\phi(p_2) = M(r)$ and $\phi(q_1) = \phi(q_2) = 0$. By the mean value theorem, there exist points w_1, w_2 in $\gamma(r)$ such that

$$\begin{aligned} -m(r) &= |\phi(p_1) - \phi(q_1)| \leq \pi r |\nabla\phi(w_1)| \\ M(r) &= |\phi(p_2) - \phi(q_2)| \leq \pi r |\nabla\phi(w_2)|. \end{aligned}$$

Thus, for any p in $\gamma(r)$, we have

$$-r\epsilon(r) \leq -\pi r|\nabla\phi(w_1)| \leq m(r) \leq \phi(p) \leq M(r) \leq \pi r|\nabla\phi(w_2)| \leq r\epsilon(r),$$

where $\epsilon(r) = \dfrac{1}{\pi}\max_{\gamma(r)}|\nabla\phi|$.

To obtain a non-increasing function $\epsilon(r)$ and the strict inequality it is sufficient to change $\epsilon(r)$ by $\sup\{\epsilon(\bar{r}) : \bar{r} \geq r\} + \frac{1}{r^2}$; this proves the claim. ☐

From now on, we redefine R, if necessary, in order to argue that the function ϕ has sublinear growth in $\mathbb{R}^2 \setminus B(R)$.

According to Lemma 3.6, there exists a circle γ_0 of radius $r_0 > R$ looping around $\partial B(R)$ such that $m(\gamma_0) \neq 0$ and $|\nabla P_{\gamma_0}| \neq 0$. After a translation and a rotation in the domain of ϕ, we can assume that the center of γ_0 is the origin and $P_{\gamma_0} = q_0 y$ with $q_0 > 0$. We would like to emphasize that all points of the set $\phi(\gamma_0)$ are interior points of the end E. Let B denote the set obtained after the translation and rotation of $B(R)$.

We define the function $\psi(x,y) = \phi(x,y) - P_{\gamma_0}(x,y)$ for any $(x,y) \in \mathbb{R}^2 \setminus B$. Note that $\psi_{xx}\psi_{yy} - \psi_{xy}^2 \leq 0$. From Lemma 3.7 there exist four points, ordered as, β_1^+, β_1^-, β_2^+, β_2^- along γ_0 such that

$$\psi(\beta_1^+) = \psi(\beta_2^+) = m(\gamma_0) > 0 \ \text{ and } \ \psi(\beta_1^-) = \psi(\beta_2^-) = -m(\gamma_0) < 0.$$

Let $\bar{r} > r_0$ big enough such that $\epsilon(r) < q_0$ for all $r \geq \bar{r}$. We define the subset G of $\mathbb{R}^2$ as

$$G = B(\bar{r}) \cup \left\{ (x,y) \in \mathbb{R}^2 \setminus B(\bar{r}) : |y| < \frac{r\epsilon(r)}{q_0} \right\}.$$

Let us analyze the set G in each quadrant of $\mathbb{R}^2$. On the first quadrant, using polar coordinates, the equation $|y| = \frac{r\epsilon(r)}{q_0}$ becomes $\sin\theta = \frac{\epsilon(r)}{q_0}$. Therefore outside $B(\bar{r})$, for each $r \geq \bar{r}$, there is a unique θ satisfying the last equation, moreover, since $\frac{\epsilon(r)}{q_0}$ is non-increasing, the curve determined by the initial equation is a graph over the $x-$line (outside $B(\bar{r})$), see Figure 3.4.

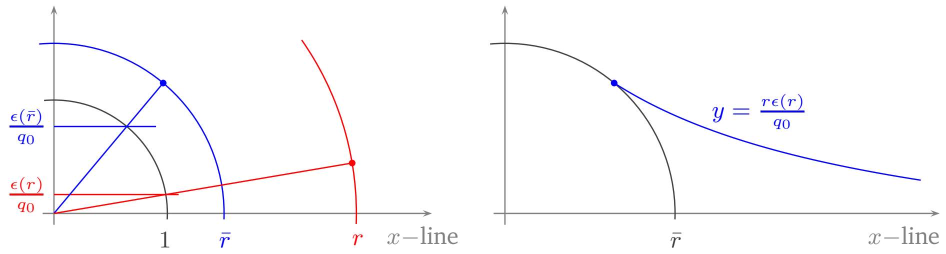

Figure 3.4: The set ∂G in the first quadrant.

We must observe that ∂G is symmetric with respect the x-axis and y-axis. Therefore, ∂G is completely determined by the points in the curve solution to $|y| = \frac{r\epsilon(r)}{q_0}$ contained in the first quadrant, see Figure 3.5.

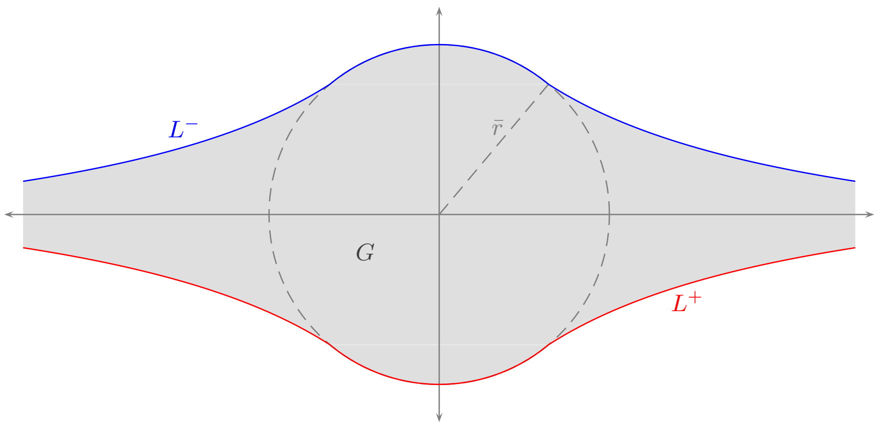

Figure 3.5: The set G.

We have obtained that the set G is bounded by two curves L^+ and L^-. The curve L^- is the union of the curves solution of $|y| = \frac{r\epsilon(r)}{q_0}$ in the first and the second quadrants and the arc of $\partial B(\bar{r})$ (in these quadrants) joining these curves. This arc can be described as the set of points $(x, y) \in \partial B(\bar{r})$ such that $y \geq \frac{\bar{r}\epsilon(\bar{r})}{q_0}$, see the left side of Figure 3.4 . The curve L^+ is the reflection of L^- with respect to the $x-$axis. A sketch of the set G is presented in the Figure 3.5.

On the one hand, any $(x, y) \in L^-$ satisfies either $y = \frac{r\epsilon(r)}{q_0}$ or $y \geq \frac{\bar{r}\epsilon(\bar{r})}{q_0}$. Therefore on L^- either $\psi(x, y) = \phi(x, y) - q_0 y = \phi(x, y) - r\epsilon(r) < 0$ or $|(x, y)| = \bar{r}$ and $\psi(x, y) = \phi(x, y) - q_0 y \leq \phi(x, y) - \bar{r}\epsilon(\bar{r}) < 0$. Thus, the function ψ is negative on L^-. Analogously we can prove that ψ is positive on L^+. On the other hand, any point (x, y) above the curve L^- satisfies $y > \frac{r\epsilon(r)}{q_0}$, then we can conclude that $\psi(x, y) = \phi(x, y) - q_0 y < r\epsilon(r) - r\epsilon(r) = 0$. In the same way, any point (x, y) below the curve L^+ satisfies $\psi(x, y) > 0$.

This shows that the set $\{(x, y) \in \mathbb{R}^2 \setminus B : \psi(x, y) = 0\}$ must be strictly contained in G. Clearly this set is not empty since ψ alternates between $-m(\gamma_0)$ and $m(\gamma_0)$ on $\partial B(r_0)$. Let us denote the sets

$$
\begin{aligned}
\Omega^+ &= \{(x, y) \in \mathbb{R}^2 \setminus B(r_0) : \psi(x, y) > m(\gamma_0)\}, \\
\Omega^- &= \{(x, y) \in \mathbb{R}^2 \setminus B(r_0) : \psi(x, y) < -m(\gamma_0)\}.
\end{aligned}
$$

Claim B. *Neither Ω^+ nor Ω^- is empty.*

Proof. Suppose that the set Ω^+ is empty; then $\psi(x, y) \leq m(\gamma_0)$ for all $(x, y) \in \mathbb{R}^2 \setminus B(r_0)$. Recalling that $\psi(x, y) = \phi(x, y) - q_0 y$ we infer that $\phi(x, y) \leq q_0 y + m(\gamma_0)$, this means that the end E is placed below the plane $q_0 y + m(\gamma_0)$ which is above plane parallel to P_{γ_0} at distance $m(\gamma_0)$, see Figure 3.6. The intersection between the end and the plane has no interior points of the end, by the geometric comparison principle. Consequently, the intersection occurs only on $\partial B(\bar{r})$ and this intersection is transversal, again by the geometric comparison principle. Moreover, as $r \to \infty$, the distance between the end E and the plane $q_0 y + m(\gamma_0)$ goes to infinity since the function ϕ has sublinear growth and the Gauss map goes to ν at infinity. Now we can decrease the inclination of the plane keeping the end E below until we obtain either an interior first contact point between the end and some plane, or a horizontal plane such that E is below of this plane, see Figure 3.7. The first case cannot happen, by the geometric comparison principle, and the second case contradicts the assumption (3.5). Analogously, the statement $\Omega^- = \varnothing$ contradicts the supposition of that E grows also infinitely in the direction $-\nu$. $\square$

Claim C. *Each component of Ω^+ and Ω^- is unbounded.*

Proof. Suppose that a component of Ω^+ is bounded; geometrically this means that the part of the end E, corresponding to that component, above the plane $q_0 y + m(\gamma_0)$ is compact. Hence for a positive constant

c the plane $q_0 y + m(\gamma_0) + c$ and the end E have an interior contact point, again the geometric comparison principle leads to a contradiction. Analogously for the components of Ω^-. $\square$

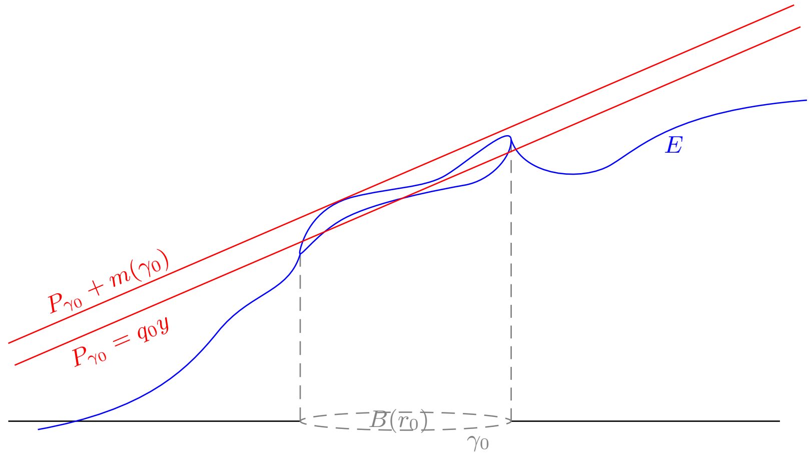

Figure 3.6: Assuming that Ω^+ is empty.

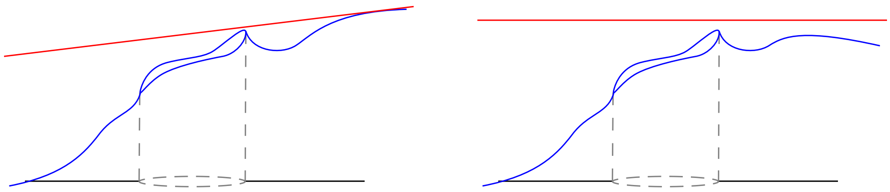

Figure 3.7: Decreasing the inclination of the plane.

Claim D. *Each set Ω^+ and Ω^- has at least two connected unbounded components.*

Proof. Each component of the sets Ω^+ and Ω^- is unbounded by Claim C. Note that $\beta_1^+, \beta_2^+ \in \overline{\Omega^+}$ and $\beta_1^-, \beta_2^- \in \overline{\Omega^-}$. Let us denote by $\Omega_i^\pm$ the connected component of $\Omega^\pm$ such that $\beta_i^\pm \in \partial\Omega_i^\pm$. Assume $\Omega_1^+ = \Omega_2^+$, then we can connect the points β_1^+ and β_2^+ by a curve contained in $\Omega_1^+ = \Omega_2^+$, moreover, we can connect these points also by a line segment contained in $B(r_0)$; these curves enclose a compact set containing either β_1^- or β_2^-, hence one of the sets Ω_i^- is contained in this compact set, thus Ω_i^- is empty, since all components are unbounded, see Figure 3.8. Interchanging the roles of Ω_i^+ and Ω_i^-, we obtain a similar conclusion. Then, it is enough to prove that all the sets $\Omega_i^\pm$ are not empty. Suppose that $\Omega_1^+ = \varnothing$. In this case, the end E in a neighborhood of $\phi(\beta_1^+)$ is below the plane $q_0 y + m(\gamma_0)$. This situation is avoided by the geometric comparison principle. The same argument is valid for the other sets. $\square$

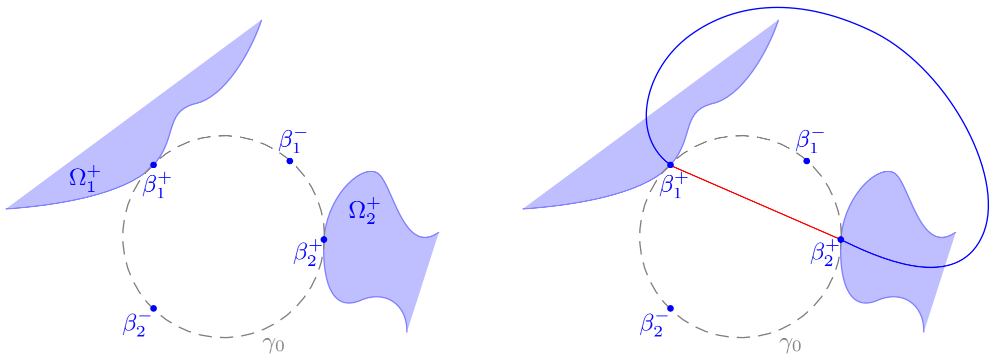

Figure 3.8: Assuming that $\Omega_1^+ = \Omega_2^+$.

Now we will consider the components of the sets

$$\left\{(x,y) \in \mathbb{R}^2 \setminus B(r_0) : \psi(x,y) > 0\right\} \quad \text{and} \quad \left\{(x,y) \in \mathbb{R}^2 \setminus B(r_0) : \psi(x,y) < 0\right\}.$$

Each of these sets have at least two components containing the sets $\Omega_i^\pm$. We will denote these components as $\Omega'^\pm_i$, then

$$\beta_i^+ \in \overline{\Omega_i^+} \subset \Omega'^+_i \ \text{ and } \ \beta_i^- \in \overline{\Omega_i^-} \subset \Omega'^-_i.$$

Claim E. *Each set $\Omega'^\pm_i$ contains a Jordan curve $J_i^\pm(t) = (x_i^\pm(t), y_i^\pm(t))$ contained in the set $\overline{G} \setminus B(r_0)$ such that $x_i^+(t) \to \infty$ as $t \to \infty$ and $x_i^-(t) \to -\infty$ as $t \to \infty$.*

Proof. If any of the sets $\Omega'^\pm_i$ contains a point of $L^\pm$, then by the connectedness it must contain the whole curve $L^\pm$. In this case the curve $L^\pm$ satisfies the condition required.

Now suppose that some set $\Omega'^\pm_i$ does not touch the curve $L^\pm$, this implies that $\Omega'^\pm_i \subset \overline{G} \setminus B(r_0)$. We now add the points $+\infty$ and $-\infty$ to the boundary of G and consider again the set $\Omega_i^\pm$, then clearly $\partial G \cap \partial \Omega_i^\pm \subset \{-\infty, +\infty\}$. Assume that $+\infty$ is not in the common boundary, thereby there exists a big enough positive constant C such that $\Omega_i^\pm \subset \{(x,y) \in \mathbb{R}^2 : x \leq C\}$. In this case we can easily find an angle less than π where the set $\Omega_i^\pm$ can be placed, since $\frac{\epsilon(r)}{q_0} \to 0$ as $r \to \infty$. The Figure 3.9 shows how to build the angle sector.

Applying Lemma 3.8 to the function $\psi - m(\gamma_0)$ on the set $\Omega_i^\pm$, we obtain that for r big enough and some (x,y) with $|(x,y)| = r$ we have

$$\psi(x,y) - m(\gamma_0) > cr.$$

However, ϕ has sublinear growth on $\Omega_i^\pm$, i.e.,

$$
\begin{aligned}
\psi(x,y) - m(\gamma_0) &= \phi(x,y) - q_0 y - m(\gamma_0) \\
&\leq r\epsilon(r) + q_0|y| - m(\gamma_0) \\
&\leq 2r\epsilon(r) - m(\gamma_0),
\end{aligned}
$$

but the above inequalities imply that $\epsilon(r) > \frac{c}{2}$, which is a contradiction since $\epsilon(r) \to 0$ as $r \to \infty$.

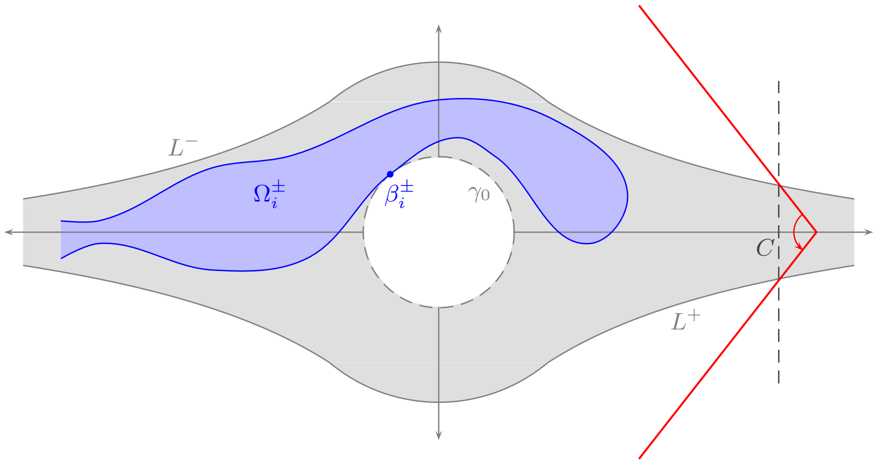

Figure 3.9: The set $\Omega_i^\pm$ placed in an angle sector.

Summarizing, we have proven that $\partial G \cap \partial \Omega_i^\pm = \{-\infty, +\infty\}$ for all sets $\Omega_i^\pm$; therefore Lemma 3.9 guaranties the existence of the Jordan curves $J_i^\pm$. $\square$

To finish the proof, consider for $i = 1, 2$ a curve in G that joins $+\infty$ and β_i^+, also consider the segment inside $B(r_0)$ joining β_1^+ and β_2^+. The closed curve formed by these three curves must enclose one of the points β_i^-, and, in consequence, they must enclose one of the sets Ω_i^-. This is a contradiction because now $-\infty$ is not accessible from within Ω_i^-, see Figure 3.10. This concludes the proof of the lemma.

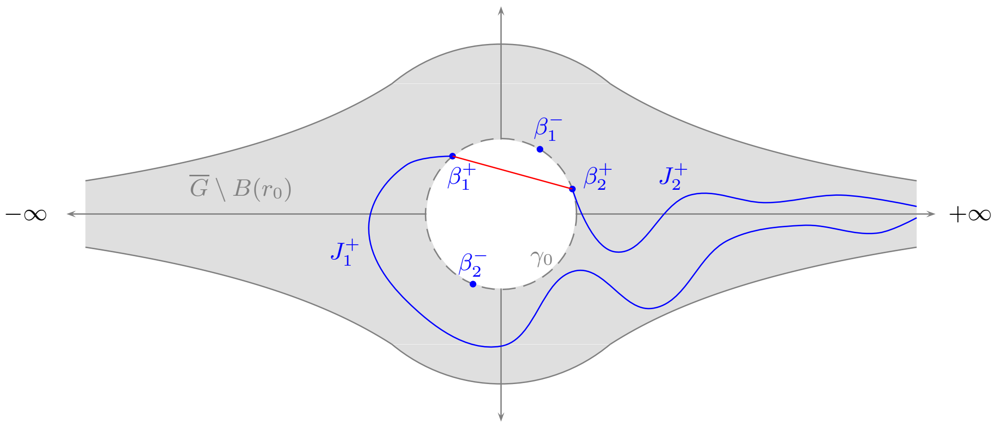

Figure 3.10: The point β_2^- cannot be connected to $-\infty$ by the curve J_2^-.

$\square$

Theorem 3.11. *Let M be a complete connected ESWMT-surface in $\mathbb{R}^3$ with finite total second fundamental and one embedded end. Suppose that the function f associated to M is non-negative, Lipschitz at 0 and satisfies* (2.2)*. Then M is a plane.*

Proof. By Theorem 3.5 and the growth lemma, M lies in a half-space of $\mathbb{R}^3$. Since M has finite total second fundamental form, the skew curvature is bounded on M. Let q_M be the maximum value of this curvature and $\widetilde{f} : [0, \infty) \to \mathbb{R}$ be the function defined as

$$\widetilde{f}(t) = \begin{cases} f(t), & \text{if } 0 \leq t \leq q_M \\ a\sqrt{t} + b, & \text{if } t > q_M \end{cases},$$

where $a = 2\sqrt{q_M} f'(q_M)$ and $b = f(q_M) - 2q_M f'(q_M)$. On the one hand,

$$\widetilde{f}(q_M) = f(q_M) \quad \text{and} \quad \widetilde{f}'(q_M) = f'(q_M),$$

and on the other hand, if $t \geq q_M$, then

$$4t\big(\widetilde{f}'(t)\big)^2 = 4q_M\big(f'(q_M)\big)^2 < 1.$$

Summarizing, $\widetilde{f}$ is continuous in $[0, \infty)$, of class C^1 in $(0, \infty)$, non-negative, Lipschitz at 0 and satisfies (2.1). Moreover,

$$\lim_{t \to \infty} \Big((t - \widetilde{f}(t^2) \Big) = \lim_{t \to \infty} ((1 - a)t - b) = +\infty,$$

because $0 \leq a < 1$. Clearly M is a $\widetilde{f}$-surface and, by Corollary 2.9, M is a plane. $\square$

Corollary 3.12. *After a vertical translation of the end E, the function u in Section 3.2 is either non-negative or non-positive.*

Proof. If we write $\alpha = (x, y, u)$ for the function $\alpha : \mathbb{D}^\circ \to E \subset \mathbb{R}^3$ described in the last section, then we see that $u(z) = \phi(x(z), y(z))$. It follows from the growth lemma, Lemma 3.10, that either

$$\inf_{0<|z|\leq 1} u(z) > -\infty \;\; \text{or} \;\; \sup_{0<|z|\leq 1} u(z) < \infty,$$

therefore after a vertical translation of the end E, either $u \leq 0$ or $u \geq 0$ which is the desired conclusion. $\square$

3.4 The height near infinity

Summarizing, let M be a connected oriented ESWMT-surface with finite total second fundamental form, thereby M is of finite topology. Let E be one of its ends, E is conformally a punctured disk, that is, there exists a diffeomorphism $\alpha : \mathbb{D}^\circ \to E$ such that the parameter $z \in \mathbb{D}^\circ$ is conformal. Also, the Gauss map N extends continuously to the origin, where we assume, after a rigid motion in

$\mathbb{R}^3$, that $N(z)$ tends to $\nu = (0,0,1)$ as $|z| \to 0$. The function $u : \mathbb{D}^\circ \to \mathbb{R}$ defined by $u(z) = \langle \alpha(z), \nu \rangle$ satisfies the Poisson equation (3.3), which can be written as $\Delta u = 2\lambda\theta f(q)$ provided the Weingarten relation $H = f(q)$.

Henceforth, we will consider that $u \geq 0$, hence u and Δu have the same sign, by Corollary 3.12. Note that the case of $u \leq 0$ is completely analogous since we can redefine ν as $-\nu$, this will change the sign of the angle function θ and we will have the same conclusion.

From now on, we will consider f non-negative and Lipschitz. The Lipschitz condition will give that the Laplacian of the height function is integrable over the whole unit disc, this fact together with the non-negativity of the function will allow us to infer a first estimate on the growth of the end E near infinity, this is a first characterization of the function u around the origin (Lemma 3.15). Let us formalize these ideas.

We now turn to the Lipschitz hypothesis. Note that when $|z|$ goes to 0, the mean and Gaussian curvatures goes to zero, hence the skew curvature goes to zero as well. In conclusion, in a neighborhood of 0 we obtain that

$$\Delta u = 2\lambda\theta f(q) \leq 4\delta\lambda\theta q = \delta\lambda\theta(|II|^2 - 2K), \tag{3.6}$$

for some positive constant δ.

Recalling that $\theta \leq 1$ and integrating around the origin, we conclude, for some $0 < R < 1$ small enough, that

$$\int_{\mathbb{D}^\circ(R)} \Delta u \, |dz|^2 \leq \delta \int_{\mathbb{D}^\circ(R)} \lambda(|II|^2 - 2K) \, |dz|^2 \leq \delta \int_E (|II|^2 - 2K) \, dA < \infty, \tag{3.7}$$

Note that we have used the hypothesis of finite total second fundamental form and Theorem 1.16.

3.4.1 Isolated singularities of subharmonic functions

In this subsection we are going to work with the equation $\Delta u = \varphi$, where u and φ are non-negative functions defined in $\mathbb{D}^\circ$ and φ is integrable over the whole disk $\mathbb{D}$. Let us denote the sets $\mathbb{D}(r) = \{z \in \mathbb{C} : |z| \leq r\}$ and $\mathbb{D}(r)^\circ = \mathbb{D}(r) \setminus \{0\}$, where r is a positive real number. In the case $r = 1$ we just write $\mathbb{D}$ and $\mathbb{D}^\circ$ respectively as in the previous sections. For any $0 < r_1 \leq r_2$ we denote by $A(r_1, r_2)$ the set $\{z \in \mathbb{C} : r_1 \leq |z| \leq r_2\}$. The ideas and proofs presented here are based principally on some articles written by Taliaferro, specifically the references [Tal99; Tal06].

For $0 < r \leq 1$ we define the average of u on $\partial\mathbb{D}(r)$ as

$$\bar{u}(r) = \frac{1}{2\pi r} \int_{\partial\mathbb{D}(r)} u(x,y) \, ds.$$

Note that after a suitable change of coordinates we can rewrite this function as

$$\bar{u}(r) = \frac{1}{2\pi}\int_{\partial\mathbb{D}} u(rx, ry)\, ds.$$

Lemma 3.13. *If $\Delta u = \varphi$, then for any $0 < r \le r_1 \le 1$,*

$$r\bar{u}'(r) = r_1\bar{u}'(r_1) - \frac{1}{2\pi}\int_{A(r,r_1)} \varphi(z)\,|dz|^2. \tag{3.8}$$

Proof. We have

$$\begin{aligned}
\bar{u}'(r) &= \frac{1}{2\pi}\int_{\partial\mathbb{D}} \left(\frac{\partial u}{\partial x}(rx,ry)x + \frac{\partial u}{\partial y}(rx,ry)y\right) ds \\
&= \frac{1}{2\pi}\int_{\partial\mathbb{D}} \left(\nabla u(rx,ry)\cdot(x,y)\right) ds \\
&= \frac{1}{2\pi}\int_{\partial\mathbb{D}(r)} \left(\nabla u(x,y)\cdot\left(\frac{x}{r},\frac{y}{r}\right)\right) ds \\
&= \frac{1}{2\pi r}\int_{\partial\mathbb{D}(r)} \frac{\partial u}{\partial \eta}\, ds,
\end{aligned}$$

where $\frac{\partial u}{\partial \eta}$ is the normal derivative. This implies, for any $0 < r \le r_1 \le 1$, that

$$\begin{aligned}
r_1\bar{u}'(r_1) - r\bar{u}'(r) &= \frac{1}{2\pi}\left(\int_{\partial\mathbb{D}(r_1)} \frac{\partial u}{\partial \eta}\, ds - \int_{\partial\mathbb{D}(r)} \frac{\partial u}{\partial \eta}\, ds\right) \\
&= \frac{1}{2\pi}\int_{\partial A(r,r_1)} \frac{\partial u}{\partial \eta}\, ds \\
&= \frac{1}{2\pi}\int_{A(r,r_1)} \Delta u(z)\,|dz|^2.
\end{aligned}$$

□

From the hypothesis on the function φ, we have

$$\lim_{r\to 0^+}\int_{A(r,r_1)} \varphi(z)\,|dz|^2 \le \int_{\mathbb{D}} \varphi(z)\,|dz|^2 < \infty,$$

therefore the right-hand side in equation (3.8) tends to some real value β as r goes to zero (which clearly does not depend of r_1). This observation allows us to write

$$r\bar{u}'(r) = \beta + o(1), \text{ as } r \to 0^+.$$

Thus $\bar{u}'(r) = \beta r^{-1} + o(r^{-1})$ as $r \to 0^+$, this implies by integration that $\frac{\bar{u}(r)}{\log r} = \beta + o(1)$ as $r \to 0^+$. Since $\lim_{r\to 0^+} r\log r = 0$, then

$$r\bar{u}(r) = \beta r \log r + o(1) \text{ as } r \to 0^+,$$

and we infer that

$$\int_{\mathbb{D}(\varepsilon)} u(z)\,|dz|^2 = \int_0^\varepsilon 2\pi r\bar{u}(r)\, dr = O\left(\varepsilon^2 \log\varepsilon\right), \text{ as } \varepsilon \to 0^+. \tag{3.9}$$

Consider the function $u_N : \mathbb{D} \to \mathbb{R}$ defined by

$$u_N(z) = \frac{1}{2\pi} \int_{\mathbb{D}} \log \frac{1}{|z-\xi|} \varphi(\xi)\, |d\xi|^2; \tag{3.10}$$

this function is called the Newtonian potential of φ in $\mathbb{D}$. It is well known that $\Delta u_N = \varphi$, nonetheless, a complete proof of this fact can be found in [GT01; Eva98]. Hence, repeating exactly the same steps as above, we conclude that

$$\int_{\mathbb{D}(\varepsilon)} u_N(z)\, |dz|^2 = O\left(\varepsilon^2 \log \varepsilon\right), \text{ as } \varepsilon \to 0^+. \tag{3.11}$$

Note that the function $u - u_N$ is a harmonic function on $\mathbb{D}^\circ$.

The next lemma shows the nature of the singularity of a harmonic function, imposing some condition on the integral of the absolute value of the function.

Lemma 3.14 (Taliaferro, [Tal99])**.** *Suppose v is harmonic in $\mathbb{D}^\circ$ and that*

$$\int_{\mathbb{D}(\varepsilon)} |v(z)|\, |dz|^2 = o(\varepsilon), \textit{ as } \varepsilon \to 0^+.$$

Then there exists a constant β such that $v(z) + \beta \log|z|$ has a harmonic extension to $\mathbb{D}$.

We would like to apply the last lemma to the function $u - u_N$ since, by (3.9) and (3.11), we have

$$\begin{aligned} \int_{\mathbb{D}(\varepsilon)} |u(z) - u_N(z)|\, |dz|^2 &\leq \int_{\mathbb{D}(\varepsilon)} |u(z)|\, |dz|^2 + \int_{\mathbb{D}(\varepsilon)} |u_N(z)|\, |dz|^2 \\ &= \int_{\mathbb{D}(\varepsilon)} u(z)\, |dz|^2 + \int_{\mathbb{D}(\varepsilon)} u_N(z)\, |dz|^2 \\ &= O\left(\varepsilon^2 \log \varepsilon\right). \end{aligned}$$

In conclusion, we have

Lemma 3.15 (Taliaferro, [Tal99])**.** *Let u and φ be non-negative functions defined on $\mathbb{D}^\circ$, such that $\Delta u = \varphi$ in $\mathbb{D}^\circ$ and φ is integrable over the whole disk $\mathbb{D}$. Then there exists a constant β and some continuous function $u_h : \mathbb{D} \to \mathbb{R}$ which is harmonic in $\mathbb{D}$, such that for any $z \in \mathbb{D}^\circ$ we have*

$$u(z) = \beta \log|z| + u_h(z) + u_N(z), \tag{3.12}$$

where the function u_N is defined by (3.10).

We finalize this section with the following lemma that shows the behaviour of the Newtonian potential of φ near the singularity.

Lemma 3.16 (Taliaferro, [Tal06]). *Let u and φ as in Lemma 3.15. If there exists a non-negative constant C and $0 < r \leq 1$ such that $\varphi(z) \leq (2|z|)^{-C}$ for $z \in \mathbb{D}(r)^\circ$, then the Newtonian potential of φ satisfies*

$$u_N(z) = o\left(\log \frac{1}{|z|}\right) \quad \textit{as} \quad |z| \to 0^+. \tag{3.13}$$

3.4.2 Asymptotic behaviour of the height

Here we are going to apply the results of the last subsection to the height function $u = \langle \alpha, \nu \rangle$. From Corollary 3.12, (3.3), (3.6) and (3.7) we have guarantied the hypothesis of Lemma 3.15. Then u can be written in the form (3.12).

As we have pointed out above, the mean curvature H goes to 0 as $|z| \to 0^+$. Then given a non-negative constant C and a positive constant A, for $|z|$ small enough, we can assume that $H(z) \leq 2^{-C+1} A^{-1}$. Therefore, by recalling that the angle function is bounded above by 1, from (3.3) we get

$$\Delta u \leq \frac{1}{2^C A} \lambda, \tag{3.14}$$

where λ is the conformal factor of the metric.

Now, in order to estimate λ, we have the following theorem that is an easy consequence of several theorems proven by Finn, see [Fin65; Fin64].

Theorem 3.17 (Finn, [Fin65]). *Let M be a complete Riemannian surface with finite total curvature in a neighborhood E of one of its ideal boundary components. Suppose that the region of positive curvature has compact support on the surface. Then E can be mapped conformally onto the punctured disk $\mathbb{D}^\circ$ in the complex plane. Let denote by $\lambda |dz|^2$ the conformal metric. Then there exist constants A and Φ_0 such that $\lambda(z) \leq A|z|^{-2\Phi_0}$ as $|z| \to 0^+$. Moreover $\Phi_0 \geq -1$.*

Therefore, using last theorem and (3.14), we conclude that for $|z|$ small enough $\Delta u \leq (2|z|)^{-C}$, where $C = \max\{0, 2\Phi_0\}$. This allows us to use Lemma 3.16 in our geometric context and infer that

$$\lim_{|z| \to 0^+} \frac{u(z)}{\log |z|} = \beta. \tag{3.15}$$

3.4.3 The Beltrami equation and quasiconformal maps

The Gauss map g of an immersed surface in $\mathbb{R}^3$ satisfies a very rich partial differential equation, the Beltrami equation. In general this equation can be written as

$$g_{\bar{z}} = \mu g_z. \tag{3.16}$$

Although there are a variety of equivalent definitions for a quasiconformal map, here we will consider the definition based on the Beltrami equation. This definition is taken from [Gut+12, p. 47].

Definition 3.18. *Let $D \subset \mathbb{C}$ be a domain. The function $g : D \to \mathbb{C}$ is called a quasiconformal map if it is a homomeomorphic solution of the Beltrami equation* (3.16) *with $||\mu||_\infty < 1$.*

Given a quasiconformal map g, at points $z \in D$ where g is differentiable we define the value

$$K_\mu(z) = \frac{1 + |\mu(z)|}{1 - |\mu(z)|} = \lim_{r\to 0} \frac{\max\limits_{|w-z|=r} |g(z) - g(w)|}{\min\limits_{|w-z|=r} |g(z) - g(w)|}.$$

This value is called the dilatation of g at z. The maximal dilatation of g is defined as $K_g = \sup\limits_{z\in D} K_\mu(z)$ and it satisfies

$$K_g = \frac{1 + ||\mu||_\infty}{1 - ||\mu||_\infty}. \tag{3.17}$$

It is very common to find the notation K_g-quasiconformal to indicate a quasiconformal map g with maximal dilatation K_g or parameter of quasiconformality K_g, see for example [Ahl66].

Note that 1-quasiconformal maps are exactly the conformal maps because $K_g = 1$ only when $\mu = 0$ which means, from the Beltrami equation, that $g_{\bar{z}} = 0$, i.e., g is holomorphic. This is precisely what happens when the function g is the Gauss map of a minimal immersion. In this case we say that g is antiholomorphic since we usually construct g by sterographic projection from $(0, 0, 1)$.

A posthumous paper by Mori [Mor56] studies quasiconformal maps from the unit ball into itself such that the origin is fixed. The main result of this paper is known nowadays as the Mori theorem and it establishes a Hölder condition for the function g.

Theorem 3.19 (Mori theorem). *Let $g : \mathbb{D} \to \mathbb{D}$ be a K_g-quasiconformal map with $g(0) = 0$. Then*

$$|g(z) - g(w)| \leq 16|z - w|^{1/K_g}, \tag{3.18}$$

for all $z, w \in \mathbb{D}$. Furthermore, the constant 16 in (3.18) *cannot be replaced by any smaller constant independent of K_g.*

Let us go back to our geometric context, where g is the Gauss map of an ESWMT-surface with finite total second fundamental form. According to Theorem 1.10, the Gauss map satisfies the Beltrami equation with $\mu = \frac{H\lambda}{Q}$. Using the Weingarten relation $H = f(q)$ and Lemma 1.2 we conclude that

$$|\mu| = \frac{f(q)}{\sqrt{q}}. \tag{3.19}$$

We can assert that

$$\begin{aligned}
0 \le \limsup_{t\to 0^+} \frac{f(t)}{\sqrt{t}} &\le \limsup_{t\to 0^+} \frac{f'(t)}{\frac{1}{2\sqrt{t}}} \\
&= \limsup_{t\to 0^+} 2\sqrt{t}f'(t) \\
&= \sqrt{\limsup_{t\to 0^+} 4t\big(f'(t)\big)^2}.
\end{aligned}$$

Assuming that f is Lipschitz, we conclude that $f'(t)$ is uniformly bounded, thus

$$\limsup_{t\to 0^+} 4t\big(f'(t)\big)^2 = 0.$$

And therefore,

$$\limsup_{r\to 0^+} \sup\{|\mu(z)| : 0 < |z| < r\} = \limsup_{t\to 0^+} \frac{f(t)}{\sqrt{t}} = 0.$$

This means that, redefining the end E if necessary, we can get $||\mu||_\infty$ as small as we want. In particular, $||\mu||_\infty \le \sigma < 1$ for some non-negative real constant σ.

Summarizing, we have proved that if the elliptic function f is Lipschitz, then the Gauss map is a K_g-quasiconformal map, where the parameter of quasiconformality satisfies

$$K_g \le \frac{1+\sigma}{1-\sigma}.$$

Moreover, we can redefine the end E in order to make the parameter of quasiconformality be as close to 1 as we wish.

In order to use the Mori theorem, we are going to distinguish for a moment between the Gauss map obtained by the stereographic projection from $(0,0,-1)$ and $(0,0,1)$ writing g for the first case (as it was being) and ϱ for the second one. Then we have that $\varrho(0) = 0$ and, by the Mori theorem, ϱ satisfies for all $z \in \mathbb{D}$ that

$$|\varrho(z)| \le 16|z|^{\frac{1-\sigma}{1+\sigma}}.$$

This means that $\varrho(z) = O\left(|z|^\epsilon\right)$ as $|z| \to 0^+$, where $1 \ge \epsilon = \frac{1-\sigma}{1+\sigma} > 0$. Since $g\bar{\varrho} = 1$ we deduce that

$$g(z) = O\left(|z|^{-\epsilon}\right) \text{ as } |z| \to 0^+. \tag{3.20}$$

Remark 3.20. *In order to guaranty* (3.20) *we just need that the function f satisfies that*

$$\limsup_{t\to 0^+} 4t\big(f'(t)\big)^2 < 1. \tag{3.21}$$

The Lipschitz condition for the function f, as we showed, is stronger than (3.21)*; we can also verify this through the function $f(t) = \frac{1}{2}t\sin^2\left(\frac{1}{\sqrt{t}}\right)$. On the other hand, the Lipschitz condition at zero for f, i.e., there exist a constant $C > 0$ such that $f(t) \le Ct$*

for any $t \geq 0$*, is not enough to reach* (3.21) *since we can construct an elliptic function Lipschitz at zero such that* $\limsup_{t \to 0^+} 4t\big(f'(t)\big)^2 = 1$.

3.4.4 Regularity at infinity for ESWMT-surfaces

Let E be an embedded end of an ESWMT-surface with finite total second fundamental form. According to Theorem 3.5, E can be described as a graph of a real valued function ϕ defined on the complement of a compact set in a plane, which is orthogonal to the limit of the Gauss map at infinity. We would like to understand the asymptotic behaviour of this function as in the case of minimal surfaces, we expect to find an expression for ϕ analogous to (3.1). Until now, we have described the end E by a conformal parametrization $\alpha : \mathbb{D}^\circ \to E$ such that the function $u = \langle \alpha, \nu \rangle$ is determined by (3.12) and satisfies $u(z) = O(\log |z|)$ as $|z| \to 0^+$.

In Lemma 3.15 we have shown that the function u_h is harmonic on $\mathbb{D}$. Using an homogeneous expansion for the function u_h at $0 \in \mathbb{D}$, see [Axl+01, p. 100], and Lemma 3.16, we can rewrite (3.12) for any $z = x + iy \in \mathbb{D}$ as

$$u(z) = \beta \log |z| + a_0 + a_1 x + a_2 y + O\left(|z|^2\right) + o\left(\log \frac{1}{|z|}\right) \text{ as } |z| \to 0^+, \qquad (3.22)$$

for some real constants β, a_0, a_1, a_2. Here, we have used that there are only two linearly independent homogeneous harmonic polynomials of degree m in two dimensions, namely, the real and the imaginary part of z^m.

In order to establish an easy notation, we will assume that the Gauss map takes the value $\nu = (0, 0, 1)$ at infinity and the function ϕ is defined on the complement of a compact subset of $\mathbb{R}^2 = \{(x_1, x_2, 0) : x_1, x_2 \in \mathbb{R}\} \subset \mathbb{R}^3$. Note that with this convention the function u is the third coordinate function of α; naming x_1 and x_2 the other coordinates of the diffeomorphism α we can write $\alpha = (x_1, x_2, u)$. From this, it is clear that the functions u and ϕ are essentially the same, in fact $\phi(x_1, x_2) = u(z(x_1, x_2))$. In order to know how the parameter z depends of x_1 and x_2 we will use the Kenmotsu representation, Theorem 1.12, to recover the coordinate functions x_1 and x_2 in terms of z.

From Theorem 1.9 we have the following equations

$$\begin{aligned}
\partial_z x_1 &= -\frac{\bar{g}_z(1+g^2)}{H(1+g\bar{g})^2}, \\
\partial_z x_2 &= -\frac{i\bar{g}_z(1-g^2)}{H(1+g\bar{g})^2}, \\
\partial_z u &= -\frac{2\bar{g}_z g}{H(1+g\bar{g})^2}.
\end{aligned}$$

Hence we conclude that $\partial_z(x_1 + ix_2) = -g\partial_z u$ and we can recover $x_1 + ix_2$ by integration from the Kenmotsu representation.

Since $u(z) = O(\log|z|)$ as $|z| \to 0^+$ by (3.15), we have that $\partial_z u(z) = O\left(|z|^{-1}\right)$ as $|z| \to 0^+$. On the other hand, if the elliptic function f is Lipschitz in $[0, \infty)$, then by (3.20), $g(z) = O\left(|z|^{-\epsilon}\right)$ as $|z| \to 0^+$, for some $\epsilon > 0$. We reach upon integration that $x_1 + ix_2 = O\left(|z|^{-\epsilon}\right)$ as $|z| \to 0^+$. Defining $r = |x_1 + ix_2| = \sqrt{x_1^2 + x_2^2}$, we have that $|z| = O\left(r^{-1/\epsilon}\right)$ as $r \to \infty$. Again, rewriting the function u, from the equation (3.22), we achieve the following description of the asymptotic behaviour of the function ϕ,

$$\phi(x_1, x_2) = \beta \log r + a_0 + \frac{a_1 x_1}{r^{2/\epsilon}} + \frac{a_2 x_2}{r^{2/\epsilon}} + O\left(r^{-2/\epsilon}\right) + o\left(\log r\right) \text{ as } r \to \infty,$$

for some real constants $\epsilon, \beta, a_0, a_1, a_2$ (not necessarily the same constants of (3.22)) where $\epsilon > 0$.

Summarizing all the results achieved in this chapter, we get

Theorem 3.21. *Let M be a complete ESWMT-surface in $\mathbb{R}^3$ with finite total second fundamental form and embedded ends. Suppose that the function f associated to M is non-negative and Lipschitz. Then each end E_i is the graph of a function ϕ_i over the exterior of a bounded region in some plane Π_i. Moreover, if x_1, x_2 are the coordinates in Π_i, then the function ϕ_i have the following asymptotic behaviour for $r = \sqrt{x_1^2 + x_2^2}$ large*

$$\phi_i(x_1, x_2) = \beta \log r + a_0 + \frac{a_1 x_1}{r^{2/\epsilon}} + \frac{a_2 x_2}{r^{2/\epsilon}} + O\left(r^{-2/\epsilon}\right) + o\left(\log r\right), \tag{3.23}$$

where $\epsilon, \beta, a_0, a_1, a_2$ are real constants depending on i and $0 < \epsilon \leq 1$.

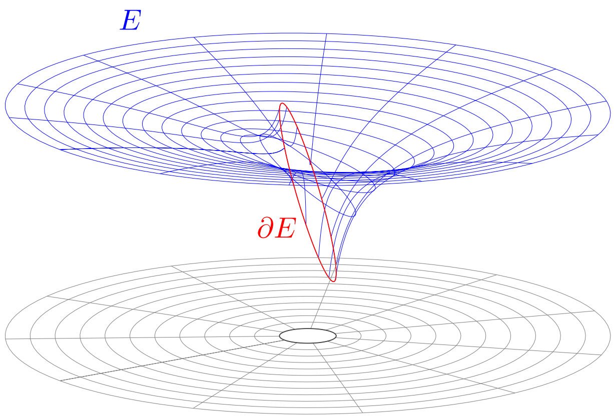

Figure 3.11: Representation of an end E of an ESWMT-surface satisfying the hypothesis of Theorem 3.21.

We would like to finish this section by noting that the equality (3.23) can be seen as a generalization of the concept of regular at infinity given by Schoen in [Sch83], Definition 3.1.

Characterization of ESWMT-surfaces with two ends

4

> *Mathematics is the greatest adventure of thought. In other activities we also think, obviously, but we also have the guidance and control of empirical observation. In pure mathematics we navigate through a sea of abstract ideas, with no compass other than logic.*
>
> — **Jesús Mosterín**
> Los lógicos

In this chapter we prove that a connected, oriented ESWMT-surface M embedded in $\mathbb{R}^3$ with two ends, non-negative mean curvature and finite total second fundamental form is rotationally symmetric with respect to a fixed line, which means that the surface M must be one of the rotational examples M_τ (Theorem 2.7) obtained by Sa Earp and Toubiana in [ST95]. This result will be a consequence of the asymptotic behaviour of the ends obtained in Chapter 3 and the Aleksandrov reflection method presented in Subsection 1.4.3.

4.1 Embedded ESWMT-surfaces with two ends

Let M be a complete ESWMT-surface in $\mathbb{R}^3$ with finite total second fundamental form and embedded ends. Then, for each end E_i there exists a unit vector ν_i such that E_i is the graph of a function ϕ_i defined over the exterior of a compact set in the plane $\Pi_{\nu_i} = \{x \in \mathbb{R}^3 : \langle x, \nu_i \rangle = 0\}$. If we also assume that f is non-negative and Lipschitz, then each function ϕ_i has logarithmic growth by Theorem 3.21. This implies that, if the planes Π_{ν_i} are not parallel, then the ends must intersected, which means that the surface M is not embedded.

In particular, when the surface M is embedded and has two ends, we have that these ends can be described as graphs of functions defined over the same plane Π. Moreover, the surface M must grow infinitely in both sides of the plane Π, since it cannot be contained in a half-space by the half-space theorem, Corollary 2.9.

Recall that the vectors ν_i are the continuous extension of the Gauss map to the ideal boundary $\{p_i\}$, Theorem 1.17. When the vectors ν_i are parallel, we will say that the corresponding ends E_i are parallel.

We summarize these facts in the next lemma.

Lemma 4.1. *Let M be a complete ESWMT-surface embedded in $\mathbb{R}^3$ with finite total second fundamental form and two ends. Suppose that the function f associated to M is non-negative and Lipschitz, then the ends of M are parallel and the surface must grow infinitely in opposite directions.*

4.2 The Aleksandrov function and tilted planes

In this section we are going to prove that, given an end E of an ESWMT-surface as a graph of a function of the form (3.23) and a vertical plane $\mathcal{P}$, the maximum value of the Aleksandrov function Λ_1 of E respect to the plane $\mathcal{P}$ is achieved at ∂E.

Without loss of generality we can assume that $\partial E \subset \Pi_\nu$ since we can translate vertically the end E and also redefining the end cutting off the compact part that is at one side of the plane Π_ν after the translation. Besides, we can also consider that the function ϕ is non-negative, after a reflection through the plane Π_ν, if necessary.

In order to define the Aleksandrov function, recall Subsection 1.4.3. Let W be the connected component of the upper half-space of Π_ν that contains the bounded domain limited by $E \cap \Pi_\nu$. Consider a plane $\mathcal{P}$ parallel to the vector $\nu = (0,0,1)$ and the height function respect to the plane Π_ν defined by $h(p) = \langle p, \nu \rangle$. We define the associated Aleksandrov function $\Lambda : [0, \infty) \to \{-\infty\} \cup \mathbb{R}$ on E as

$$\Lambda(\rho) = \max \{\Lambda_1(p) : p \in \mathcal{D},\, h(p) = \rho\}, \tag{4.1}$$

where $\mathcal{D}$ is the domain of the Aleksandrov function Λ_1, (1.24), associated to E.

Note that the assumptions made above give us that, for any $\rho \geq 0$, the level set $E_\rho = \{q \in E : h(q) = \rho\}$ is a non-empty compact set. Then the function Λ is finite valued, since the set $\{p \in \mathcal{D} : h(p) = \rho\}$ is non-empty and compact. This situation changes severely if we tilt the plane $\mathcal{P}$ slightly. We will study the consequences, on the associated Aleksandrov function, caused by the tilting.

Let n be the normal vector of the plane $\mathcal{P}$ and consider the vectors $\nu^\varepsilon = \nu + \varepsilon n$ and $n^\varepsilon = n - \varepsilon \nu$ for a small $\varepsilon > 0$. Note that $\langle \nu^\varepsilon, n^\varepsilon \rangle = 0$. We will define the tilted planes $\Pi_{\nu^\varepsilon} = \{p \in \mathbb{R}^3 : \langle p, \nu^\varepsilon \rangle = 0\}$ and $\mathcal{P}^\varepsilon$ being the plane passing through the origin with normal vector n^ε. The height function from the plane Π_{ν^ε} is h^ε defined by $h^\varepsilon(p) = \langle p, \nu^\varepsilon \rangle$. Clearly $\mathcal{P}^\varepsilon \to \mathcal{P}$, $\Pi_{\nu^\varepsilon} \to \Pi_\nu$, and $h^\varepsilon \to h$ as $\varepsilon \to 0$.

Since the end E has logarithmic growth, we will obtain two important facts when $\varepsilon > 0$ is small. The first one is that the height function h^ε restricted to $\mathcal{D}^\varepsilon$ must attain a minimum value h_0^ε. The second one is that the Aleksandrov function Λ_1^ε is valued as $-\infty$ outside a non-empty compact set of $\mathcal{D}^\varepsilon$. More precisely,

Lemma 4.2. *Let $\varepsilon > 0$. There exists a non-empty compact set $B \subset \mathcal{P}^\varepsilon$ such that for any $p \in \mathcal{D}^\varepsilon \cap (\mathcal{P}^\varepsilon \setminus B)$ there exists just first contact point of the set $\{p + tn^\varepsilon : t \in \mathbb{R}\} \cap \bar{W}$ as t decreases from $+\infty$, i.e., $\Lambda_1^\varepsilon(p) = -\infty$.*

Proof. We are going to prove this lemma in the case that $n = (1,0,0)$. This is sufficient, since the expansion for the function ϕ in Theorem 3.21 is invariant under rotation of the (x_1, x_2)-coordinates. Let us consider a new coordinate system for $\mathbb{R}^3$ defined by

$$(y_1, y_2, y_3) = (x_1 - \varepsilon x_3, x_2, x_3 + \varepsilon x_1). \tag{4.2}$$

Using these coordinates we get the planes $\mathcal{P}^\varepsilon = \{(y_1, y_2, y_3) \in \mathbb{R}^3 : y_1 = 0\}$ and $\Pi_{\nu^\varepsilon} = \{(y_1, y_2, y_3) \in \mathbb{R}^3 : y_3 = 0\}$. The intersection of the end E with the plane parallel to Π_{ν^ε} at height $\tau > 0$ is given by the curve

$$\Gamma_\tau = \left\{(y_1, y_2, \tau) \in \mathbb{R}^3 : \phi\left(\frac{y_1}{1+\varepsilon^2} + \frac{\varepsilon\tau}{1+\varepsilon^2}, y_2\right) = \frac{\tau}{1+\varepsilon^2} - \frac{\varepsilon y_1}{1+\varepsilon^2}\right\}.$$

Let Φ be the function defined as

$$\Phi(y_1, y_2) = \phi\left(\frac{y_1}{1+\varepsilon^2} + \frac{\varepsilon\tau}{1+\varepsilon^2}, y_2\right) - \frac{\tau}{1+\varepsilon^2} + \frac{\varepsilon y_1}{1+\varepsilon^2}.$$

Hence,

$$\frac{\partial \Phi}{\partial y_1}(y_1, y_2) = \frac{1}{1+\varepsilon^2}\frac{\partial \phi}{\partial x_1}(x_1, x_2) + \frac{\varepsilon}{1+\varepsilon^2}, \tag{4.3}$$

where $(x_1, x_2) = \left(\frac{y_1}{1+\varepsilon^2} + \frac{\varepsilon\tau}{1+\varepsilon^2}, y_2\right)$. Since $|\nabla\phi| \to 0$ as $r \to \infty$, there exists $R > 0$, big enough, such that $\frac{\partial\phi}{\partial x_1} > -\varepsilon$ for all $r > R$. On the one hand, let E_R be the set of points in E such that $r \leq R$, i.e.,

$$E_R = \{(x_1, x_2, x_3) \in E : r = \sqrt{x_1^2 + x_2^2} \leq R\}.$$

Note that E_R is a non-empty compact set, since it is the graph of a continuous function defined over a compact set. Let ρ be the maximum value of the function h^ε on the set E_R, then for $\tau > \rho$ we have that all points of Γ_τ satisfy $r > R$. On the other hand, without any assumption on τ, if $(y_1, y_2, \tau) \in \Gamma_\tau$ satisfies $|y_2| > R$, then $r > R$. Summarizing, by (4.3), if $(y_1, y_2, \tau) \in \Gamma_\tau$ is a point satisfying either $\tau > \rho$ or $|y_2| > R$, then $\frac{\partial\Phi}{\partial y_1}(y_1, y_2) > 0$. This means that if $\tau > \rho$, the entire curve Γ_τ, in the (y_1, y_2)-plane is the graph of a function defined in the variable y_2 and, consequently, (in the (y_1, y_2, y_3)-coordinates) for all $y_2 \in \mathbb{R}$, the line $\{(t, y_2, \tau) \in \mathbb{R}^3 : t \in \mathbb{R}\}$ intersects the end E exactly once. We have the same conclusion in the case that $|y_2| > R$, but only on the part of the curve satisfying the hypothesis. We define the non-empty compact set

$$B = \{(0, y_2, y_3) \in \mathcal{P}^\varepsilon : |y_2| \leq R \text{ and } h_0^\varepsilon \leq y_3 \leq \rho\}.$$

Thus, if $p \in \mathcal{D}^\varepsilon \cap (\mathcal{P}^\varepsilon \setminus B)$, then $\Lambda_1^\varepsilon(p) = -\infty$. $\square$

Let us define the sets $\mathcal{D}_0^\varepsilon = \{p \in \mathcal{D}^\varepsilon : h^\varepsilon(p) = h_0^\varepsilon\}$, $E_0^\varepsilon = \{p \in E : h^\varepsilon(p) \geq h_0^\varepsilon\}$ and $W_0^\varepsilon = \{p \in W : h^\varepsilon(p) \geq h_0^\varepsilon\}$, see Figure 4.1.

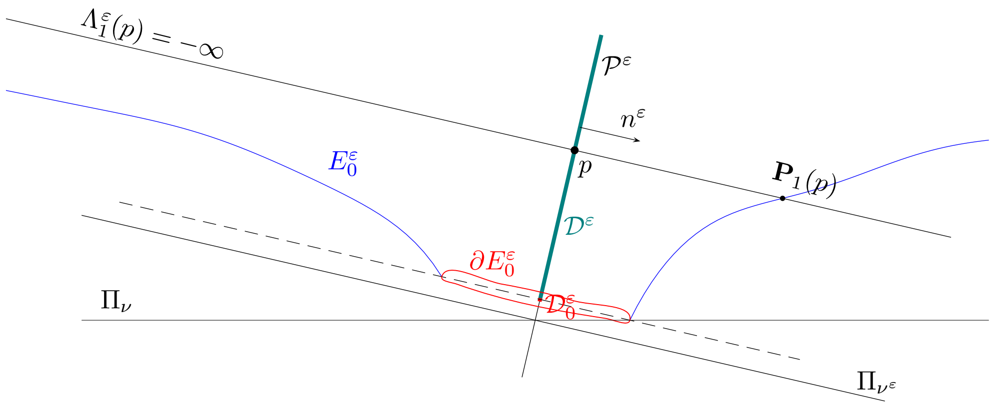

Figure 4.1: The Aleksandrov function Λ_1 is valued as $-\infty$ outside of a compact set.

Lemma 4.3. *For the end E (with all the assumptions made in this section) and any plane $\mathcal{P}$ parallel to $\nu = (0, 0, 1)$, either the function Λ is strictly decreasing, or else M has a plane of reflection parallel to $\mathcal{P}$.*

Proof. We will begin proving that the function Λ is non-increasing. For this it suffices to show that $\Lambda(\rho) \leq \Lambda(0)$ for all $\rho > 0$, since we can translate vertically E and redefine the end to choose the level $\rho = 0$ arbitrarily.

Recall that $W = \mathcal{C}(N) \cap \{p \in \mathbb{R}^3 : h(p) \geq 0\}$. We claim:

Claim A. $\Lambda(\rho) \leq \Lambda(0)$ *for all* $\rho > 0$ *if, and only if,*

$$E_{t+}^* \cap \{p \in \mathbb{R}^3 : h(p) > \rho\} \subset \bar{W} \text{ for all } t > \Lambda(0).$$

Proof. If $\Lambda(\rho) \leq \Lambda(0)$ for all $\rho > 0$, then for an arbitrary $t > \Lambda(0)$ we have that $\Lambda(\rho) \leq t$ for all $\rho > 0$. By the definition of the Aleksandrov function, we infer that $E_{t+}^* \cap \{p \in \mathbb{R}^3 : h(p) > \rho\} \subset \bar{W}$.

If $E_{t+}^* \cap \{p \in \mathbb{R}^3 : h(p) > \rho\} \subset \bar{W}$ for all $t > \Lambda(0)$, we will show that $\Lambda(\rho) \leq \Lambda(0)$ for all $\rho > 0$ reasoning by contradiction. Suppose there exist $\rho_0 > 0$ and $p \in \mathcal{D}$ such that $h(p) = \rho_0$ and

$$\Lambda_1(p) = \Lambda(\rho_0) > \Lambda(0).$$

Considering $t = \Lambda(\rho_0)$ we have that if any neighborhood of E_{t+}^* containing the reflection of the point $\mathbf{P}_1(p)$ through the plane $\mathcal{P}_t$, where contained in $\bar{W}$, the maximum principle would yield that $\mathcal{P}_t$ is a plane of symmetry for M (as in the proof of Lemma 1.26). But this is impossible since $\Lambda(0) \neq t$. $\square$

To prove that $E^*_{t+} \cap \{p \in \mathbb{R}^3 : h(p) > \rho\} \subset \bar{W}$ for all $t > \Lambda(0)$ we will use tilted planes and Lemma 1.27.

The semicontinuity of the Aleksandrov function, Lemma 1.27, and Lemma 4.2 imply that the maximum value of the function Λ_1^ε must be achieved at some point at the compact set $\mathcal{D}^\varepsilon \cap B$, where B is the set obtained from Lemma 4.2. We are going to prove that the height of this point from the plane Π_{ν^ε} is exactly h_0^ε.

Claim B. *The function Λ_1^ε must attain its maximum at some point in $\mathcal{D}_0^\varepsilon$.*

Proof. Suppose that the function Λ_1^ε attains its maximum value t at some point $q \in \mathcal{D}^\varepsilon \cap B$. Note that if $p \in \mathcal{D}^\varepsilon \cap (\mathcal{P}^\varepsilon \setminus B)$ and $\mathbf{P}_1(p) \in E_{t+}$ then, by Lemma 4.2, the reflection, $\mathbf{P}_1(p)^*$, of $\mathbf{P}_1(p)$ through the plane $\mathcal{P}_t^\varepsilon$ is in $\bar{W}_0^\varepsilon$. Then we just have to pay attention on the points $p \in \mathcal{D}^\varepsilon \cap B$ when we reflect through the plain $\mathcal{P}_t^\varepsilon$. Suppose that $h^\varepsilon(q) > h_0^\varepsilon$, then the points $\mathbf{P}_1(q)$ and $\mathbf{P}_2(q)$ are far away from ∂E_0^ε, i.e., q is either an interior tangent point or a boundary tangent point, see Figure 4.2. In any case, by Lemma 1.26, the set E_0^ε has a plane of symmetry parallel to $\mathcal{P}^\varepsilon$, which is a contradiction, thereby $q \in \mathcal{D}_0^\varepsilon$. $\square$

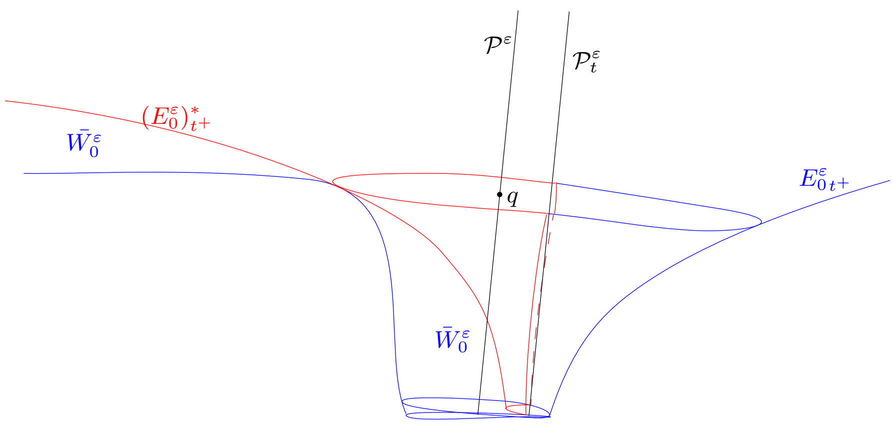

Figure 4.2: Λ_1^ε attains its maximum value t at $q \in \mathcal{D}^\varepsilon \cap B$ such that $h^\varepsilon(q) > h_0^\varepsilon$.

Writing z^ε for the maximum value of Λ_1^ε, it follows that

$$(E_0^\varepsilon)^*_{t+} \subset \bar{W}_0^\varepsilon \text{ for all } t \geq z^\varepsilon. \tag{4.4}$$

By letting $\varepsilon \to 0$, from (4.4) we get

$$E^*_{t+} \subset \bar{W} \text{ for all } t \geq \limsup_{\varepsilon \to 0} z^\varepsilon. \tag{4.5}$$

The semicontinuity of the Aleksandrov function, Lemma 1.27, and Claim B implies that

$$\limsup_{\varepsilon \to 0} z^\varepsilon \leq \Lambda(0).$$

Given $t \geq \Lambda(0)$, then $t \geq \limsup_{\varepsilon \to 0} z^{\varepsilon}$ by the inequality above, thus (4.5) says that

$$E^*_{t^+} \cap \{p \in \mathbb{R}^3 : h(p) > \rho\} \subset \bar{W}.$$

Hence, using Claim A, we have proven that the function Λ is non-increasing.

To finish the proof of the lemma, note that the function Λ is either strictly decreasing or constant in some interval. In the second case, the Aleksandrov function Λ_1 has a local interior maximum and, Lemma 1.26 determines the existence of a plane of symmetry parallel to $\mathcal{P}$. ☐

4.3 A Schoen type theorem for ESWMT-surfaces

In this section we are going to present the main result of this work. We will establish a Schoen type theorem for embedded ESWMT-surfaces.

Theorem 4.4 (Main Theorem). *Let M be a complete connected ESWMT-surface embedded in $\mathbb{R}^3$ with finite total second fundamental form and two ends. Suppose that the function f associated to M is non-negative and Lipschitz. Then M must be rotationally symmetric. In fact, there exists $\tau > 0$ such that M is the surface of revolution M_τ determined by Sa Earp and Toubiana in [ST95].*

Proof. Let E_1 and E_2 be the ends of M. By Lemma 4.1 we know that E_1 and E_2 are parallel ends and M grows infinitely in opposite directions. Let ν be the continuous extension of the Gauss map of one of the ends (observe that the Gauss map goes to $-\nu$ at the other end) and $\mathcal{P}$ a plane parallel to ν; up to a rigid motion we can assume $\nu = (0, 0, 1)$. In order to apply Lemma 4.3 we will consider the Aleksandrov functions associated to each end E_1 and E_2 denoted by Λ^1 and Λ^2 respectively. Then either both of these Aleksandrov functions are strictly decreasing or at least one of the ends has a plane of symmetry parallel to $\mathcal{P}$. In the second case it is clear that the geometric comparison principle, Theorem 1.22, implies that M has a plane of symmetry parallel to $\mathcal{P}$. For the first case we will consider the Aleksandrov function associated to the entire surface M. This function is clearly defined over $\mathbb{R}$ since M grows infinitely in opposite directions. This function is now strictly increasing on some interval $(-\infty, a]$ and strictly decreasing on some interval $[b, \infty)$, which means that the maximum value is attained on the compact set $[a, b]$; which is a global maximum, see Figure 4.3. This maximum value must be attained at an interior point and Lemma 1.26 guaranties that M has a plane of symmetry parallel to $\mathcal{P}$.

Therefore, for any vertical plane, $\mathcal{P}$, M is symmetric respect to some plane parallel to $\mathcal{P}$. Since the center of mass of any cross section must be contained in any plane of symmetry, then all vertical planes of symmetry intersect in a line parallel to ν. Hence, M is rotationally symmetric respect to this line.

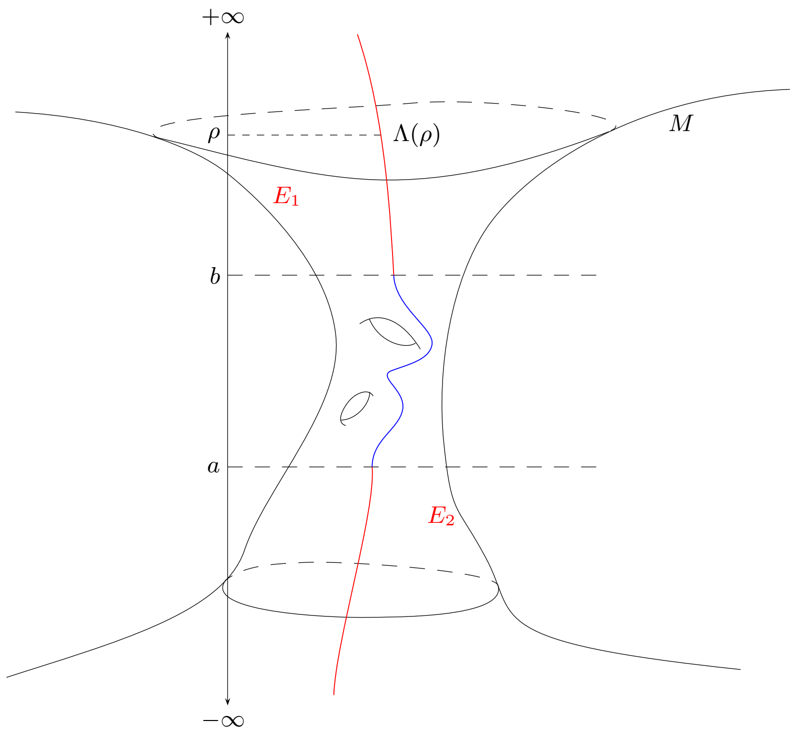

Figure 4.3: The associated Aleksandrov function, Λ, on M.

Now we are going to conclude the last part of the theorem. Let f be the elliptic function associated to M and observe that f satisfies trivially (2.2). The ellipticity condition (2.1) implies that $1-2tf'(t^2) > 0$ for all $t > 0$. Then, the function $t-f(t^2)$ in non-decreasing, hence the following limit exists, possibly $+\infty$,

$$\kappa = \lim_{t\to\infty} \left(t - f(t^2)\right).$$

By Theorem 2.6 there is a unique rotationally symmetric f-surface M_τ for each $\tau > 0$ satisfying $\frac{1}{\tau} < \kappa$. By the uniqueness of the ESWMT-surfaces rotationally symmetric, Theorem 2.10, there exists $\tau \in (\frac{1}{\kappa}, \infty)$ such that $M = M_\tau$. $\square$

Remark 4.5. *By Theorem 2.6, the surface has a horizontal plane of symmetry, which means that the growth of the two ends are the same; in other words, the constant β in Theorem 3.21 is the same for both ends up to sign. However, this fact can be proved using the maximum principle. Let suppose, without loss of generality, that the growth of the end E_1 is bigger than the growth of the end E_2, this is, the coefficients of $\log r$ for each end in Theorem 3.21 satisfy $|\beta_1| > |\beta_2|$, see Figure 4.4. Consider a horizontal plane low enough and reflect, through this plane, the part of the surface that remains below the plane; this reflection intersects the surface just on the plane since $|\beta_1| - |\beta_2| > 0$, see Figure 4.5. Now we move up the plane until obtain a boundary tangent point, see Figure 4.6; this point is attained exactly when the Gauss map is horizontal. By the geometric comparison principle, Theorem 1.22, the horizontal plane is a plane of symmetry, therefore $|\beta_1| = |\beta_2|$.*

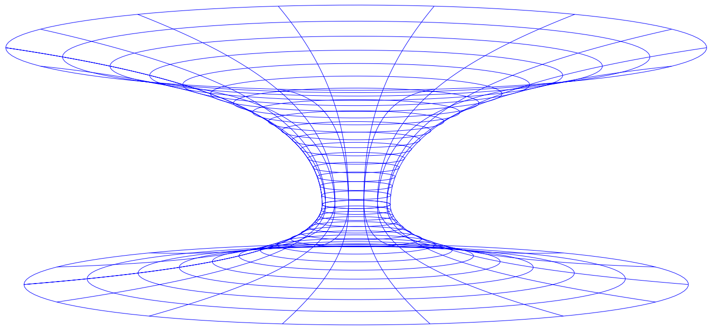

Figure 4.4: A rotational ESWMT-surface with different growth ends.

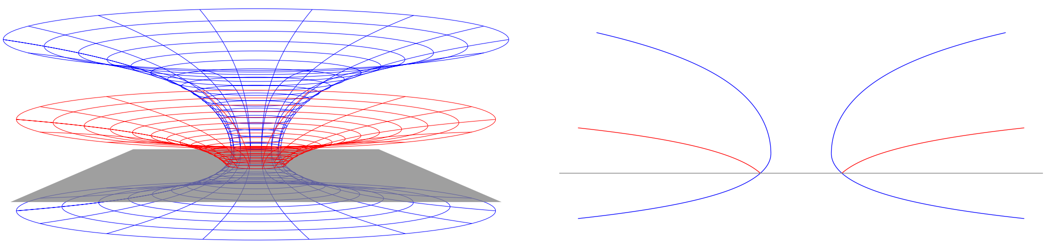

Figure 4.5: Aleksandrov reflection using a horizontal plane.

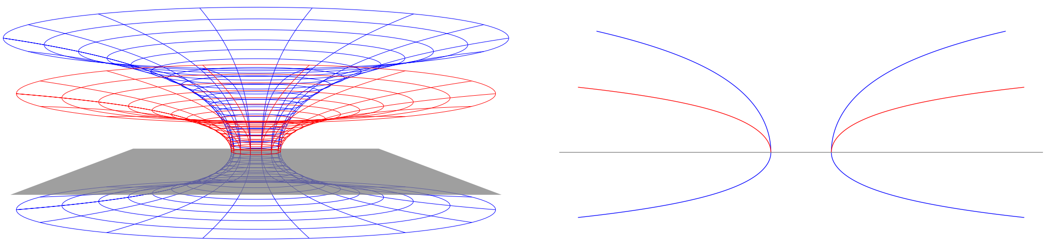

Figure 4.6: Finding a boundary tangent point.

As corollary of the main theorem, we can recover the original Schoen theorem, namely

Corollary 4.6. *Let M be a complete connected embedded minimal surface in $\mathbb{R}^3$ with finite total curvature and two ends. Then M is a catenoid.*

Proof. Since M is minimal, it has finite total curvature if, and only if, it has finite total second fundamental form. Then, we can apply Theorem 4.4 to the particular case $f \equiv 0$. The conclusion is obtained from the uniqueness of the catenoid as the only non-planar minimal surface rotationally symmetric. □

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Declaration

I, Héber Mesa Palomino, confirm that the work presented here was solely undertaken by myself and that no help was provided from other sources as those allowed. All sections of the paper that use quotes or describe an argument or concept developed by another author have been referenced, including all secondary literature used, to show that this material has been adopted to support my thesis.

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