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Continuity of Lyapunov exponents for cocycles with a single holonomy

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Dedicated to women in mathematics,
to their every day struggle
against harassment, isolation and mansplaning,
in particular to Karina.

Abstract

We study the continuity of the Lyapunov exponents for linear cocycles in $SL(2, \mathbb{R})$.

Backes, Brown and Butler have proved that the Lyapunov exponents are continuous when restricted to cocycles that admit uniform stable and uniform unstable holonomies.

A conjecture of Marcelo Viana states that this condition can be relaxed and a single uniform holonomy (stable or unstable) is sufficient to guarantee the continuity. We provide evidence that the conjecture is true by proving some partial results in this direction.

Contents

Chapter 1

Introduction

The notion of Lyapunov exponents goes back to the stability theory for differential equations developed on the work of A. M. Lyapunov in the late 19th century. It was extended to the field of ergodic theory by the results of Fustenberg-Kesten [11] and Oseledets [16] for linear cocycles. Lyapunov exponents also appear naturally in smooth dynamics through the concept of non-uniform hyperbolicity introduced by Pesin [17].

The theory of Lyapunov exponents for linear cocycles grew into a very broad area and active field. In this thesis, we are concerned with the continuity of the Lyapunov exponents for linear cocycles in $SL(2,\mathbb{R})$. That is, we study how the Lyapunov exponents vary as functions of the cocycle.

Discontinuity of Lyapunov exponents is typical for continuous $SL(2,\mathbb{R})$-valued cocycles over an invertible base. This has been proved in Theorem C of [5] as a particular case of Mañé-Bochi's Theorem. More precisely, in [5] was shown that the only C^0-continuity points for the Lyapunov exponents are the cocycles which are uniformly hyperbolic and those with zero Lyapunov exponents.

Even though, discontinuity is a common feature, there are some contexts where continuity has been established. Bocker and Viana [6] and Malheiro and Viana [15] proved continuity of Lyapunov exponents for random products of 2-dimensional matrices in the Bernoulli and in the Markov setting. In higher dimension, continuity of the Lyapunov exponents for i.i.d. random products of matrices has been announced by Avila, Eskin and Viana [1].

Still for 2-dimensional cocycles, Bocker and Viana [6] constructed an example of a locally constant cocycle with non-zero Lyapunov exponents that can be approximated in the Hölder topology by linear cocycles with zero Lyapunov exponents. Then, we cannot expect to have continuity of the Lyapunov exponents even if we consider higher regularity. Another counter-example in this setting has been constructed in [10].

A few years ago, Backes, Brown and Butler [4] proved that the continuity of Lyapunov exponents holds when restricted to the realm of fiber-bunched Hölder cocycles over any hyperbolic system with local product structure. The main feature that fiber-bunched cocycles exhibit is the existence of uniform invariant holonomies. In fact, the main theorem in [4] establishes that for cocycles that admit uniform stable and uniform unstable holonomies, denoted by H^s and H^u respectively, we have the following result:

$$\text{If } (\hat{A}_k, H^{s,k}, H^{u,k}) \xrightarrow{C^0} (\hat{A}, H^s, H^u), \text{ then } \lambda_+(\hat{A}_k) \to \lambda_+(\hat{A}),$$

for $\hat{A}, \hat{A}_k$ $SL(2,\mathbb{R})$-valued continuous cocycles. In particular, their theorem extends [6] and [15].

More recently, Viana and Yang [22] were able to prove the continuity of Lyapunov exponents in the C^0-topology for some type of linear cocycles when the dynamic in the base is an expanding map. This means that the Mañé-Bochi phenomenon cannot be generalized to the non-invertible setting. The main observation is that given a continuous uniformly expanding map, we can consider its natural extension and the lift of the linear cocycle. This new cocycle always admits a uniform stable holonomy. Therefore, the theorems in [22] suggest that the hypotheses in [4] can be relaxed: we may only need to ask for the existence of a single uniform holonomy.

Conjecture (Conjecture 6.3 of [21])**.**

$$\textit{If } (\hat{A}_k, H^{u,k}) \xrightarrow{C^0} (\hat{A}, H^u) \ \textit{ or } \ (\hat{A}_k, H^{s,k}) \xrightarrow{C^0} (\hat{A}, H^s),$$
$$\textit{then } \lambda_+(\hat{A}_k) \to \lambda_+(\hat{A}).$$

The results in the present work give a partial answer to this conjecture.

We use the concept of *non-uniform holonomy* defined in [22]. The main technique is to construct this kind of holonomy in order to compensate the fact that we have a single uniform holonomy.

In [22] are considered two different types of non-uniform holonomies: the first type is for non-uniformly fiber-bunched cocycles. This notion is weaker that the notion of fiber-bunched mentioned above, but still allows to prove the existence of holonomies in almost every point. This concept and its properties were first introduced in [19]. The second class of non-uniform holonomy is constructed using Pesin theory and therefore it is only possible to be obtained where the base and the cocycle are $C^{1+\epsilon}$.

Theorem A and Theorem B below are concerned with the setting of non-uniformly fiber-bunched cocycles. We are assuming that the base is an hyperbolic homeomorphism and we fix an ergodic measure with local product structure and fully supported. We refer the reader to Chapter 2 for the precise definitions and statements.

Theorem A. *Let $\hat{A}$ be a Hölder $SL(2,\mathbb{R})$-valued linear cocycle such that $\hat{A}$ is non-uniformly fiber-bunched and admits uniform stable holonomies. Consider a sequence $(\hat{A}_k, H^{s,k})$ such that $\hat{A}_k \to \hat{A}$ in the Hölder topology and $H^{s,k} \to H^s$ in the C^0 topology. Then, $\lambda_+(\hat{A}_k) \to \lambda_+(\hat{A})$.*

It is possible to obtain an analogous result for cocycles admitting only uniform unstable holonomies applying Theorem A to $\hat{A}^{-1}$.

This theorem is not a consequence of [4] because the non-uniform holonomies are not defined in every point and do not have the same properties that the uniform ones have. Several results need to be extended to our context in order to conclude Theorem A.

Observe that we ask for more regularity in the convergence of the sequence of cocycles than in the conjecture. We need to consider Hölder cocycles, because in this class non-uniform fiber-bunching implies existence of non-uniform holonomies. Moreover, in order to have some type of continuity of these holonomies as functions of the cocycles, the sequence also has to approximate the cocycle $\hat{A}$ in the Hölder topology.

Consider a continuous uniformly expanding map in the base and A a Hölder $SL(2,\mathbb{R})$ valued linear cocycle which is non-uniformly fiber-bunched. Then, for every sequence $A_k \to A$ in the Hölder topology, we have continuity of the Lyapunov exponents. This is a consequence of Theorem A and the argument mentioned above using the natural extension and the lift of the cocycles.

We remark that the hypothesis of existence of uniform stable holonomies in Theorem A cannot be removed. In fact, the example of discontinuity in [6] can be taken to be non-uniformly fiber-bunched. Therefore, we cannot expect continuity of the Lyapunov exponents to hold in the space of non-uniformly fiber-bunched cocycles without some extra hypotheses. In the next theorem, we remove the existence of uniform stable holonomies, but instead we impose certain conditions in the cocycle $\hat{A}$.

Theorem B. *Let $\hat{A}$ be a Hölder $SL(2,\mathbb{R})$-valued linear cocycle such that $\hat{A}$ is non-uniformly fiber-bunched, locally constant and irreducible. If $\hat{A}_k \to \hat{A}$ in the Hölder topology, then $\lambda_+(\hat{A}_k) \to \lambda_+(\hat{A})$.*

In the last part of this work, we consider the second type of non-uniform holonomies defined in [22]. In order to do this, we assume the map in the base to be a $C^{1+\epsilon}$ Anosov diffeomorphism and as before we consider an ergodic measure, fully supported and with local product structure, for example, the equilibrium state associated to a Hölder potential, see [9].

As in Theorem B, in the next theorem we ask for some type of irreducibility condition in the cocycle $\hat{A}$. Observe that this statement is for the C^0 topology. In particular, the next result extends the continuity part of Theorem B of [22].

Theorem C. *Let $\hat{A}$ be a $C^{1+\epsilon}$ $SL(2,\mathbb{R})$-valued linear cocycle that admits uniform stable holonomies. If $\hat{A}$ does not have an invariant section and $(\hat{A}_k, H^s_k) \to (\hat{A}, H^s)$ in the C^0 topology, then $\lambda_+(\hat{A}_k) \to \lambda_+(\hat{A})$.*

Chapter 2

Preliminaries and statements

In the present chapter we give the definitions needed to state the theorems. While in the Introduction we state the results in a more general way, with an hyperbolic homeomorphism in the base, the work done in [9] allow us to restrict the study to the case when we have a sub-shift of finite type in the base.

Let $Q = (q_{i,j})_{1\le i,j\le d}$ be a matrix with $q_{i,j} \in \{0, 1\}$. The sub-shift of finite type associated to the matrix Q is the subset of the bi-infinite sequences $\{1, ..., d\}^{\mathbb{Z}}$ satisfying

$$\hat{\Sigma} = \{(x_n)_{n\in\mathbb{Z}} \colon q_{x_n x_{n+1}} = 1 \text{ for every } n \in \mathbb{Z}\}.$$

We require that each row and column of Q contains at least one non-zero entry. For $\rho \in (0, 1)$, we consider in $\hat{\Sigma}$ the metric d_ρ, defined by

$$d_\rho(\hat{x}, \hat{y}) = \rho^{N(\hat{x},\hat{y})},$$

where $N(\hat{x}, \hat{y}) = \max\{N \ge 0; x_n = y_n \text{ for every } |n| < N\}$. Since, the topologies given by the distance d_ρ are equivalent for any $\rho \in (0, 1)$, from now on, we consider ρ fixed.

Let $\hat{f} \colon \hat{\Sigma} \to \hat{\Sigma}$ be the left-shift map, $\hat{f}(x_n)_{n\in\mathbb{Z}} = (x_{n+1})_{n\in\mathbb{Z}}$. The map $\hat{f}$ is an hyperbolic homeomorphism and for every $(x_n)_{n\in\mathbb{Z}} \in \hat{\Sigma}$ the local stable and unstable sets are given by,

$$\begin{aligned} W^s_{loc}(\hat{x}) =& \{(y_n)_{n\in\mathbb{Z}} \in \hat{\Sigma} : \; y_n = x_n \text{ with } n \ge 0\}, \\ W^u_{loc}(\hat{x}) =& \{(y_n)_{n\in\mathbb{Z}} \in \hat{\Sigma} : \; y_n = x_n \text{ with } n \le 0\}. \end{aligned}$$

If ρ is the constant given in the definition of the distance, then $\sigma = 1/\rho$ is the expansion rate of $\hat{f}$.

Define

$$\begin{aligned} \Sigma^u =& \{(x_n)_{n\in\mathbb{Z}} : q_{x_n x_{n+1}} = 1 \text{ for every } n \ge 0\} \\ \Sigma^s =& \{(x_n)_{n\in\mathbb{Z}} : q_{x_n x_{n+1}} = 1 \text{ for every } n \le -1\}. \end{aligned}$$

We denote as $P^u : \hat{\Sigma} \to \Sigma^u$ and $P^s : \hat{\Sigma} \to \Sigma^s$, the projections obtained by dropping all of the negative and positive coordinates, respectively, of a sequence in $\hat{\Sigma}$.

Consider the sets

$$\Omega^u = \{(\hat{x}, \hat{y}) \in \hat{\Sigma} \times \hat{\Sigma} \colon \hat{y} \in W^u_{loc}(\hat{x})\},$$
$$\Omega^s = \{(\hat{x}, \hat{y}) \in \hat{\Sigma} \times \hat{\Sigma} \colon \hat{y} \in W^s_{loc}(\hat{x})\}.$$

For each $i \in \{1 \dots d\}$ we define $[0; i] = \{\hat{x} \in \hat{\Sigma} : x_0 = i\}$. They can be expressed locally as the product of a cylinder in $\hat{\Sigma}$ with Σ^s and Σ^u.

Definition 2.1. *An $\hat{f}$-invariant measure $\hat{\mu}$ has local product structure if there exists a continuous function $\psi \colon \hat{\Sigma} \to (0, \infty)$ such that for each $i \in \{1, ..., d\}$,*

$$\hat{\mu}|_{[0;i]} = \psi \cdot \left(\mu^s|_{P^s([0;i])} \times \mu^u|_{P^u([0;i])} \right),$$

where $\mu^s = P^s_* \hat{\mu}$ and $\mu^u = P^u_* \hat{\mu}$.

2.1 Linear Cocycles

Let $\hat{A} \colon \hat{\Sigma} \to SL(2, \mathbb{R})$ be a continuous map, then $\hat{A}$ defines a linear cocycle $F_{\hat{A}} \colon \hat{\Sigma} \times \mathbb{R}^2 \to \hat{\Sigma} \times \mathbb{R}^2$ over $\hat{f}$ by:

$$F_{\hat{A}}(\hat{x}, v) = (\hat{f}(\hat{x}), \hat{A}(\hat{x})v).$$

By Furstenberg-Kesten [11], for every continuous $\hat{A} : \hat{\Sigma} \to SL(2, \mathbb{R})$ and every $\hat{f}$-invariant probability measure $\hat{\mu}$,

$$\lambda_+(\hat{A}, \hat{x}) = \lim_{n \to \infty} \frac{1}{n} \log \|\hat{A}^n(\hat{x})\|$$

and

$$\lambda_-(\hat{A}, \hat{x}) = \lim_{n \to \infty} \frac{1}{n} \log \|(\hat{A}^n(\hat{x}))^{-1}\|^{-1},$$

where $\hat{A}^n(\hat{x}) = \hat{A}(\hat{f}^{n-1}(\hat{x})) \dots \hat{A}(\hat{x})$, are well defined $\hat{\mu}$-almost every $\hat{x} \in \hat{\Sigma}$.

$\lambda_+(\hat{A}, \hat{x})$ and $\lambda_-(\hat{A}, \hat{x})$ are called extremal Lyapunov exponents of $\hat{A}$. They are $\hat{f}$-invariant and therefore if $\hat{\mu}$ is ergodic, they are constant $\hat{\mu}$-almost everywhere. In this case, we denote them $\lambda_+(\hat{A})$ and $\lambda_-(\hat{A})$.

Fixed $\hat{\mu}$ an ergodic measure, the maps $\hat{A} \mapsto \lambda_+(\hat{A})$ and $\hat{A} \mapsto \lambda_-(\hat{A})$ are, respectively, upper and lower semi-continuous with respect to the topology of uniform convergence for continuous cocycles $\hat{A}$ as a consequence of [11].

The set of α-Hölder maps $\hat{A} \colon \hat{\Sigma} \to SL(2, \mathbb{R})$ is denoted by $\mathcal{S}_\alpha(\hat{\Sigma}, 2)$. We equip this space with the α-Hölder topology given by the distance

$$D_\alpha(\hat{A}, \hat{B}) = \sup_{\hat{\Sigma}} \left\| \hat{A}(\hat{x}) - \hat{B}(\hat{x}) \right\| + H_\alpha(\hat{A} - \hat{B}),$$

where $H_\alpha(\hat{A})$ is the smallest constant $C > 0$ such that

$$\left\| \hat{A}(\hat{x}) - \hat{A}(\hat{y}) \right\| \leq C \, d_\rho(\hat{x}, \hat{y})^\alpha \text{ for any } \hat{x}, \hat{y} \in \hat{\Sigma} \text{ with } d_\rho(\hat{x}, \hat{y}) \leq 1.$$

Definition 2.2. *An* uniform stable holonomy *for $\hat{A}$ over $\hat{f}$ is a collection of linear isomorphisms $H^s_{\hat{x},\hat{y}}\colon \mathbb{R}^2 \to \mathbb{R}^2$, defined for $\hat{x}, \hat{y}$ in the same stable leaf, which satisfy the following properties,*

(a) $H^s_{\hat{y},\hat{z}} \circ H^s_{\hat{x},\hat{y}} = H^s_{\hat{x},\hat{z}}$ *and* $H^s_{\hat{x},\hat{x}} = Id$;

(b) $\hat{A}(\hat{y}) \circ H^s_{\hat{x},\hat{y}} = H^s_{\hat{f}(\hat{x}),\hat{f}(\hat{y})} \circ \hat{A}(\hat{x})$;

(c) $(\hat{x}, \hat{y}, \xi) \mapsto H^s_{\hat{x},\hat{y}}(\xi)$ *is continuous for every* $(\hat{x}, \hat{y}) \in \Omega^s$.

An uniform unstable holonomy *for $\hat{A}$ is defined analogously in Ω^u. We use the expression* uniform invariant holonomies *to refer to both uniform stable and uniform unstable holonomies.*

Let $\rho \in (0, 1)$ be the contraction rate of $\hat{f}$. We say that $\hat{A} \in \mathcal{S}_\alpha(\hat{\Sigma}, 2)$ is α- *fiber-bunched* if there is an $N > 0$ such that for every $\hat{x} \in \hat{\Sigma}$,

$$\|\hat{A}^N(\hat{x})\|\|(\hat{A}^N(\hat{x}))^{-1}\|^{-1}\rho^{\alpha N} < 1.$$

The main property of α-fiber-bunched cocycles is that they admit uniform invariant holonomies (see [7]).

The following definition generalizes the notion of fiber-bunched mentioned above. However, it still allow us to prove the existence of invariant holonomies in a non-uniform sense.

Definition 2.3. *Let $\hat{\mu}$ be an ergodic measure of $\hat{f}$. We say that an α-Hölder continuous map $\hat{A}\colon \hat{\Sigma} \to SL(2, \mathbb{R})$ is* non-uniformly fiber-bunched *if the extremal Lyapunov exponents of $\hat{A}$ satisfy*

$$\lambda_+(\hat{A}) < \frac{\alpha \log \sigma}{2},$$

or, equivalently

$$\limsup_n \|\hat{A}^n(\hat{x})\|\|\hat{A}^n(\hat{x})^{-1}\|\, \rho^{\alpha n} < 1 \text{ for } \hat{\mu}\text{-almost every } \hat{x}.$$

We observe that when we consider less regular cocycles, the bound in the Lyapunov exponent decreases. Since the Hölder case can be reduced to the Lipschitz case by replacing the metric in the base, from now on we restrict to the case $\alpha = 1$.

As the uniform invariant holonomies may not be unique (see, for example, Corollary 4.9 of [13]), we consider the cocycle and one of its holonomies in pairs. More precisely, $\mathcal{H}^s$ is the set of pairs $(\hat{A}, H^s)$ where $\hat{A} \in \mathcal{S}_1(\hat{\Sigma}, 2)$ and H^s is a uniform stable holonomy for $\hat{A}$.

Consider $\mathcal{H}^s$ with the topology given by the inclusion

$$\mathcal{H}^s \hookrightarrow \mathcal{S}_1(\hat{\Sigma}, 2) \times C^0(\Omega^s, SL(2, \mathbb{R})). \tag{2.1}$$

This means that a sequence $\{(\hat{A}_k, H^{s,k})\}_{k\in\mathbb{N}}$ converges to $(\hat{A}, H^s)$ if $\hat{A}_k \to \hat{A}$ in the Lipschitz topology and the uniform stable holonomies converges uniformly in every local stable leaf.

2.2 Statement of the theorems

Now we are in conditions to give the precise statements of the theorems. In order to do that, we fix $\hat{\mu}$ as an ergodic $\hat{f}$-invariant measure with local product structure and $\operatorname{supp}\hat{\mu} = \hat{\Sigma}$.

Theorem A. *Let $\hat{A} \in \mathcal{S}_1(\hat{\Sigma}, 2)$ be such that $\hat{A}$ is non-uniformly fiber-bunched and admits uniform stable holonomies. Consider a sequence $(\hat{A}_k, H^{s,k}) \to (\hat{A}, H^s)$ in $\mathcal{H}^s$, then $\lambda_+(\hat{A}_k) \to \lambda_+(\hat{A})$.*

If we do not have the existence in every point of the holonomies $H^{s,k}$, we can not guarantee Theorem A to be true, see for example [6]. It is necessary to ask more conditions over the cocycle to be able to eliminate this hypothesis, for that, we work with a common class of cocycles, the locally constant cocycles defined as follow.

Let $\hat{M}$ be the full shift. That is, $\hat{M} = \hat{\Sigma}$ when the matrix Q in the definition has all its entries equal to 1. In this case, the measure $\hat{\mu} = \mu^{\mathbb{Z}}$ where μ is a measure in $\{1, ..., d\}$. We say that $\hat{A}\colon \hat{M} \to SL(2, \mathbb{R})$ is a *locally constant linear cocycle* if it only depends on the zeroth coordinate, that is, $\hat{A}(\hat{x}) = A(x_0)$ for some continuous function $A\colon \{1, ..., d\} \to SL(2, \mathbb{R})$.

A locally constant cocycle $\hat{A}$ is irreducible if there is no proper subspace of $\mathbb{R}^2$ invariant under $A(x_0)$ for μ-almost every $x_0 \in \{1 \dots d\}$.

Theorem B. *Let $\hat{A} \in \mathcal{S}_1(\hat{\Sigma}, 2)$ be non-uniformly fiber-bunched. If $\hat{A}$ is an irreducible locally constant cocycle, then for any $\hat{A}_k \to \hat{A}$ in $\mathcal{S}_1(\hat{\Sigma}, 2)$, we obtain that $\lambda_+(\hat{A}_k) \to \lambda_+(\hat{A})$.*

To be able to dispense the condition of being non-uniformly fiber-bunched, we need more regularity: let the base $\hat{f}\colon \hat{M} \to \hat{M}$ be $C^{1+\varepsilon}$ Anosov diffeomorphism on a compact manifold and $\hat{A}\colon \hat{M} \to SL(2, \mathbb{R})$ a $C^{1+\varepsilon}$ function.

Definition 2.4. An invariant s-section *for $(\hat{A}, H^s)$ is a continuous map $\xi\colon \hat{M} \to \mathbb{R}^2$ such that*

$$\hat{A}(\hat{x})\xi(\hat{x}) = \xi(\hat{f}(\hat{x})) \text{ for every } \hat{x} \in \hat{M}, \tag{2.2}$$

and $H^s_{\hat{x},\hat{y}}(\xi(\hat{x})) = \xi(\hat{y})$ for every $\hat{y} \in W^s_{loc}(\hat{x})$.

In this context the α-Hölder topology is no longer needed, the following result states continuity in the C^0 topology. We include the hypothesis of not having invariant s-section, which is generic, see [19].

Theorem C. *Let* $\hat{f}\colon \hat{M} \to \hat{M}$ *be a* $C^{1+\epsilon}$ *Anosov diffeomorphism on a compact manifold* $\hat{M}$ *and* $\hat{A}\colon \hat{M} \to SL(2,\mathbb{R})$ *a* $C^{1+\epsilon}$ *map that admits uniform stable holonomies. If* $\hat{A}$ *does not have an invariant section and* $(\hat{A}_k, H^s_k) \to (\hat{A}, H^s)$ *in the* C^0 *topology, then* $\lambda_+(\hat{A}_k) \to \lambda_+(\hat{A})$.

Chapter 3

Non-uniform invariant holonomies

In the following, we consider a weaker version of the holonomies introduced in Definition 2.2. In particular, for cocycles non-uniformly fiber-bunched we are able to construct this new type of holonomies with no other assumption. Our final objective is to prove continuity for this new type of invariant holonomies when the cocycle variates along $\mathcal{S}_1(\hat{\Sigma}, 2)$.

We fix the base dynamics $(\hat{f}, \hat{\mu})$, where $\hat{f}$ is a sub-shift of finite type and $\hat{\mu}$ is an ergodic $\hat{f}$-invariant measure with local product structure and $\operatorname{supp}\hat{\mu} = \hat{\Sigma}$.

Definition 3.1. *A* non-uniform stable holonomy *for $\hat{A}$ is given by a $\hat{\mu}$-full measure set M^s and a collection of maps such that for every $\hat{x} \in M^s$ and every $\hat{y}, \hat{z} \in W^s_{loc}(\hat{x})$, there exists a linear isomorphism $H^s_{\hat{y},\hat{z}} \colon \mathbb{R}^2 \to \mathbb{R}^2$ satisfying*

(a) $H^s_{\hat{y},\hat{z}} \circ H^s_{\hat{w},\hat{y}} = H^s_{\hat{w},\hat{z}}$ *and* $H^s_{\hat{y},\hat{y}} = Id$,

(b) $\hat{A}(\hat{z}) \circ H^s_{\hat{y},\hat{z}} = H^s_{\hat{f}(\hat{y}),\hat{f}(\hat{z})} \circ \hat{A}(\hat{y})$;

(c) $(\hat{y}, \hat{z}, \xi) \mapsto H^s_{\hat{y},\hat{z}}(\xi)$ *is measurable in M^s when $(\hat{y}, \hat{z}) \in \Omega^s$.*

Moreover, there exists an increasing sequence $\{\mathcal{D}_l\}_{l\in\mathbb{N}}$ of compact subsets such that

$\bigcup_l D_l = M^s$ *and for every $l \in \mathbb{N}$, $\mathcal{D}_l \subset \mathcal{D}_{l+1}$, $\hat{\mu}(\hat{\Sigma} \setminus \mathcal{D}_l) < \frac{1}{l}$ and,*

(d) *there exists $C_l > 0$ such that for every $\hat{x} \in \mathcal{D}_l$, $\hat{y}, \hat{z} \in W^s_{loc}(\hat{x})$,*

$$\left\|H^s_{\hat{y},\hat{z}} - id\right\| \leq C_l d(\hat{y}, \hat{z}).$$

A non-uniform unstable holonomy *for $\hat{A}$ is defined analogously for points in the same unstable leaf. We use the expression* non-uniform invariant holonomies *to refer to both non-uniform stable and non-uniform unstable holonomies.*

We call invariant holonomies both the uniform invariant holonomies as in Definition 2.2 and the non-uniform invariant holonomies as above.

We use the next definition to prove the existence of non-uniform holonomies.

Definition 3.2. *Given $N \in \mathbb{N}$ and $\theta > 0$, define $\mathcal{D}^s_{\hat{A}}(N, \theta)$ as the set of points $\hat{x}$ satisfying*

$$\prod_{j=0}^{k-1} \|\hat{A}^N(\hat{f}^{jN}(\hat{x})\|\|\hat{A}^N(\hat{f}^{jN}(\hat{x}))^{-1}\| \leq e^{kN\theta} \text{ for all } k \geq 1.$$

Analogously, we define $\mathcal{D}^u_{\hat{A}}(N, \theta)$ when the inequality is satisfied by $(\hat{f}^{-1}, \hat{A}^{-1})$ instead.

It is possible to construct the linear isomorphisms $H^s_{\hat{y},\hat{z}}$ as in Definition 3.1 for the elements in $\mathcal{D}^s_{\hat{A}}(N, \theta)$.

The following result was firstly introduced in [19], we expose the proof in here for completeness.

Proposition 3.3 (Proposition 2.5, [19])**.** *Let $\hat{A} \in \mathcal{S}_1(\hat{\Sigma}, 2)$. Given $N \in \mathbb{N}$ and $\theta > 0$ with $\theta < \log \sigma$, there exists $C = C(N, \theta) > 0$ such that for every $\hat{x} \in \mathcal{D}^s_{\hat{A}}(N, \theta)$ and $\hat{y}, \hat{z} \in W^s_{loc}(\hat{x})$, it holds*

$$H^s_{\hat{y},\hat{z}} = \lim_{n \to +\infty} \hat{A}^n(\hat{z})^{-1}\hat{A}^n(\hat{y})$$

exists and satisfies $\|H^s_{\hat{y},\hat{z}} - Id\| \leq Cd(\hat{y}, \hat{z})$.

Proof. Each difference

$$\|\hat{A}^{n+1}(\hat{z})^{-1}\hat{A}^{n+1}(\hat{y}) - \hat{A}^n(\hat{z})^{-1}\hat{A}^n(\hat{y})\|$$

is bounded by

$$\|\hat{A}^{n+1}(\hat{z})^{-1}\| \cdot \|\hat{A}(\hat{f}^n(\hat{z}))^{-1}\hat{A}(f^n(\hat{y})) - \mathrm{id}\|\|\hat{A}^n(\hat{y})\|.$$

Since $\hat{A}$ is Lipschitz continuous, the middle factor is bounded by

$$C_2 d(\hat{f}^n(\hat{y}), \hat{f}^n(\hat{z})) \leq C_2\sigma^{-n}d(\hat{y}, \hat{z}),$$

for some $C_2 > 0$ that depends only on $\hat{A}$.

On the other hand, we have that

$$\|\hat{A}^n(\hat{y})\|\|\hat{A}^n(\hat{z})^{-1}\| \leq L_1 \prod_{j=0}^{k-1} \|\hat{A}^N(\hat{f}^{jN}(\hat{y}))\|\|\hat{A}^N(\hat{f}^{jN}(\hat{z}))^{-1}\|,$$

where $k = [\frac{n}{N}]$ and the constant L_1 depends of $\hat{A}$ and N. It follows that

$$\prod_{j=0}^{k-1} \|\hat{A}^N(\hat{f}^{jN}(\hat{y}))\|\|\hat{A}^N(\hat{f}^{jN}(\hat{z}))^{-1}\| \leq L_2 \prod_{j=0}^{k-1} \|\hat{A}^N(\hat{f}^{jN}(\hat{x}))\|\|\hat{A}^N(\hat{f}^{jN}(\hat{x}))^{-1}\|,$$

where $L_2 = \exp(C_1 \sum_{j=0}^{\infty} \sigma^{-jN})$. The last therm is bounded by $L_2 \exp^{kN\theta} \leq L_2 \exp^{n\theta}$. Take $L = L_1 L_2$. Therefore, we have

$$\|\hat{A}^{n+1}(\hat{z})^{-1}\hat{A}^{n+1}(\hat{y}) - \hat{A}^n(\hat{z})^{-1}\hat{A}^n(\hat{y})\| \leq LC_2 \exp^{n(\theta-\log\sigma)} d(\hat{y}, \hat{x}).$$

Since $\theta < \log\sigma$, this proves that the sequence is Cauchy and the limit $H^s_{\hat{y},\hat{z}}$ satisfies

$$\|H^s_{\hat{y},\hat{z}} - Id\| \leq Cd(\hat{y}, \hat{z})$$

with $C = \sum_{n=0}^{\infty} LC_2 \exp^{n(\theta-\log\sigma)}$, proving the proposition. □

The next proposition gives sufficient conditions to guarantee the existence of non-uniform invariant holonomies. Its proof is a consequence of the results in [19].

Proposition 3.4. *If $\hat{A} \in \mathcal{S}_1(\hat{\Sigma}, 2)$ is non-uniformly fiber-bunched, then $\hat{A}$ admits non-uniform invariant holonomies.*

Proof. Corollary 2.4 in [19] states that if θ verifies

$$2\lambda_+(\hat{A}) < \theta < \log\sigma, \text{ then } \hat{\mu}\left(\bigcup_{N=1}^{\infty} \mathcal{D}^s_{\hat{A}}(N,\theta)\right) = 1.$$

Moreover, the subsets $\mathcal{D}^s_{\hat{A}}(N,\theta)$ satisfy:

(a) $\mathcal{D}^s_{\hat{A}}(N,\theta)$ is closed, then compact.

(b) $\mathcal{D}^s_{\hat{A}}(N,\theta) \subset \mathcal{D}^s_{\hat{A}}(lN,\theta)$ for each $l \geq 1$.

Therefore, we can define a sequence $\mathcal{D}^s_{\hat{A},l}$ of compact subsets such that $\mathcal{D}^s_{\hat{A},l} \subset \mathcal{D}^s_{\hat{A},l+1}$ and $\hat{\mu}(\mathcal{D}^s_{\hat{A},l}) \to 1$ when $l \to \infty$. In order to verify this, we observe that for each $l \in \mathbb{N}$ there exists k_l such that

$$\hat{\mu}\left(\bigcup_{N=1}^{k_l} \mathcal{D}^s_{\hat{A}}(N,\theta)\right) > 1 - \frac{1}{l}.$$

Take $N_l = k_l! N_{l-1}$, then consider $\mathcal{D}^s_{\hat{A},l} = \mathcal{D}^s_{\hat{A}}(N_l,\theta)$.

Taking $M^s = \bigcup_l \mathcal{D}^s_{\hat{A},l}$, then by Proposition 3.3, we conclude that the sets $\mathcal{D}^s_{\hat{A},l}$ verify all properties in Definition 3.1.

If we apply the same argument to $(\hat{f}^{-1}, \hat{A}^{-1})$, we obtain subsets $\mathcal{D}^u_{\hat{A},l}$ that allow us to conclude that there also exists a non-uniform unstable holonomy for $\hat{A}$. □

Remark 3.5. *Suppose $\hat{A}$ satisfies the hypotheses in Proposition 3.4 and $\{\hat{A}_k\}_{k\in\mathbb{N}}$ is a sequence of cocycles such that $\hat{A}_k \to \hat{A}$ in the Lipschitz topology. Since, λ_+ is upper semi-continuous, then*

$$\limsup_{k\to\infty} \lambda_+(\hat{A}_k) \leq \lambda_+(\hat{A}) < \frac{\log\sigma}{2}.$$

Therefore, we also have non-uniform invariant holonomies for every $\hat{A}_k$.

In the following lemma we show that the non-uniform invariant holonomies, given by Proposition 3.4, are continuous as a function of $\hat{A}$. The statement and proof are given for the non-uniform stable holonomy, but it is analogous for non-uniform unstable ones.

Proposition 3.6. *If $\hat{A}, \hat{A}_k \in \mathcal{S}_1(\hat{\Sigma}, 2)$, $\hat{x} \in \mathcal{D}^s_{\hat{A}}(N, \theta)$ and $\hat{A}_k \to \hat{A}$ in the Lipschitz topology, then there exists $k_N \in \mathbb{N}$ such that if $k \geq k_N$, $H^{s,k}_{\hat{y},\hat{z}}$ exists for all $\hat{y}, \hat{z} \in W^s_{loc}(\hat{x})$ and satisfies $H^{s,k}_{\hat{y},\hat{z}} \to H^s_{\hat{y},\hat{z}}$. Furthermore, the convergence is uniform in $\mathcal{D}^s_{\hat{A}}(N, \theta)$*

Proof. From Definition 3.2 we get that there exists an open neighbourhood $\mathcal{U}^N$ of $\hat{A}$ in the Lipschitz topology and a constant $\hat{\theta} > \theta$ such that if $\hat{x} \in \mathcal{D}^s_{\hat{A}}(N, \theta)$, then $\hat{x} \in \mathcal{D}^s_{\hat{B}}(N, \hat{\theta})$ for all $\hat{B} \in \mathcal{U}^N$.

For any sequence $\{\hat{A}_k\}_{k\in\mathbb{N}}$ converging to $\hat{A}$ in the Lipschitz topology, there exists k_N such that $\hat{A}_k \in \mathcal{U}^N$ with $k \geq k_N$. Therefore, by Proposition 3.3 the holonomies $H^{s,k}_{\hat{y},\hat{z}}$ are defined for every $\hat{A}_k$ with $k \geq k_N$ and for any $\hat{x} \in \mathcal{D}^s_{\hat{A}}(N, \theta)$.

Let $n \in \mathbb{N}$ and $k >> k_N$. Define

$$H^n_{\hat{y},\hat{z}} = \hat{A}^n(\hat{z})^{-1} \circ \hat{A}^n(\hat{y}) \quad \text{for each} \quad \hat{y}, \hat{z} \in W^s_{loc}(\hat{x}).$$

Analogously define $H^{n,k}_{\hat{y},\hat{z}}$ with $\hat{A}_k$ instead of $\hat{A}$.

We know that $H^s_{\hat{y},\hat{z}} = \lim_{n\to\infty} H^n_{\hat{y},\hat{z}}$, in the same way $H^{s,k}_{\hat{y},\hat{z}} = \lim_{n\to\infty} H^{n,k}_{\hat{y},\hat{z}}$. Moreover,

$$\begin{aligned}
\|H^{n,k}_{\hat{y},\hat{z}} - H^{s,k}_{\hat{y},\hat{z}}\| &\leq \sum_{j=n}^{\infty} \|\hat{A}^{j+1}_k(\hat{z})^{-1} \circ \hat{A}^{j+1}_k(\hat{y}) - \hat{A}^j_k(\hat{z})^{-1} \circ \hat{A}^j_k(y)\| \\
&\leq \sum_{j=n}^{\infty} \|\hat{A}^j_k(\hat{z})^{-1}\| \|\hat{A}_k(\hat{f}^j(\hat{z}))^{-1} \hat{A}_k(\hat{f}^j(\hat{y})) - Id\| \|\hat{A}^j_k(\hat{y})\| \qquad (3.1) \\
&\leq Cd(\hat{y}, \hat{z}) \sum_{j=n}^{\infty} e^{j(\hat{\theta} - \log\sigma)} \leq Cd(\hat{y}, \hat{z}) e^{n(\hat{\theta} - \log\sigma)},
\end{aligned}$$

where C depends of N and the Lipschitz constant of $\hat{A}_k$, and then can be chosen to be uniform in $\mathcal{U}^N$. Then we get that $H^{n,k}_{\hat{y},\hat{z}}$ converges to $H^{s,k}_{\hat{y},\hat{z}}$ with a uniform rate independent of k. Therefore,

$$\begin{aligned}
\|H^{s,k}_{\hat{y},\hat{z}} - H^s_{\hat{y},\hat{z}}\| &\leq \|H^{s,k}_{\hat{y},\hat{z}} - H^{n,k}_{\hat{y},\hat{z}}\| + \|H^{n,k}_{\hat{y},\hat{z}} - H^n_{\hat{y},\hat{z}}\| + \|H^n_{\hat{y},\hat{z}} - H^s_{\hat{y},\hat{z}}\| \\
&\leq 2Cd(\hat{y}, \hat{z}) e^{n(\theta - \log\sigma)} + \|H^{n,k}_{\hat{y},\hat{z}} - H^n_{\hat{y},\hat{z}}\|.
\end{aligned}$$

Since $H^n_{\hat{y},\hat{z}}$ varies continuously with the cocycle, we get the result. □

Chapter 4

Invariance Principle

In this chapter we enunciate a new application of the Invariance Principle of Ledrappier [14] for the non-uniform invariant holonomies previously defined.

We fix the base dynamics $(\hat{f}, \hat{\mu})$, where $\hat{f}$ is a sub-shift of finite type and $\hat{\mu}$ is an ergodic $\hat{f}$-invariant measure with local product structure and $\operatorname{supp}\hat{\mu} = \hat{\Sigma}$.

Given $\hat{A} \in \mathcal{S}_1(\hat{\Sigma}, 2)$ define the projectivization of $F_{\hat{A}}$ by

$$\mathbb{P}(F_{\hat{A}}) \colon \hat{\Sigma} \times \mathbb{P}^1 \to \hat{\Sigma} \times \mathbb{P}^1,$$

where $\mathbb{P}(F_{\hat{A}})(\hat{x}, [v]) = (\hat{f}(\hat{x}), [\hat{A}(\hat{x})v])$.

Let $\pi \colon \hat{\Sigma} \times \mathbb{P}^1 \to \hat{\Sigma}$ be the canonical projection to the first coordinate. As $\pi \circ \mathbb{P}(F_{\hat{A}}) = \hat{f} \circ \pi$ and $\mathbb{P}(F_{\hat{A}})$ is a continuous map on a compact space, we have that there exists a $\mathbb{P}(F_{\hat{A}})$-invariant probability measure $\hat{m}$ such that $\pi_* \hat{m} = \hat{\mu}$.

If we know that there exist invariant holonomies for $\hat{A}$, denoted by H^s and H^u, then we can define invariant holonomies for $\mathbb{P}(F_{\hat{A}})$ as $h^s_{\hat{x},\hat{y}} = \mathbb{P}(H^s_{\hat{x},\hat{y}})$ and $h^u_{\hat{x},\hat{y}} = \mathbb{P}(H^u_{\hat{x},\hat{y}})$.

Definition 4.1. *Let h^u be a uniform unstable holonomy for $\mathbb{P}(F_{\hat{A}})$ and $\hat{m}$ be a $\mathbb{P}(F_{\hat{A}})$-invariant probability measure projecting to $\hat{\mu}$. We say that $\hat{m}$ is a* u-state *if there exist a disintegration $\{\hat{m}_{\hat{x}}\}_{\hat{x} \in \hat{\Sigma}}$ and a $\hat{\mu}$-full measure set M^u such that $(h^u_{\hat{x},\hat{y}})_* \hat{m}_{\hat{x}} = \hat{m}_{\hat{y}}$ for every $\hat{x}, \hat{y} \in M^u$ in the same unstable leaf.*

The definition of s-state *is stated analogously. If a measure is simultaneously a u-state and an s-state, we call it* su-state.

Since the fact of a measure being an s-state or a u-state depends only of a full measure set of $\hat{\Sigma}$, the same definition can be used in the case of non-uniform invariant holonomies.

The following version of the Invariance Principle was proven in [7] for uniform invariant holonomies as a consequence of the result of Ledrappier [14]. Here we adapt the proof to the general situation when the cocycle only admits non-uniform invariant holonomies. This context has been considered before in [19].

Recall the sets $\mathcal{D}^s_{\hat{A},l} = \mathcal{D}^s_{\hat{A}}(N_l, \theta)$, for a fixed $\theta < \frac{\log \sigma}{2}$, and an increasing sequence of integers N_l. They are considered in the proof of Proposition 3.4 and its elements admit non-uniform stable holonomies that vary continuously at a fixed rate depending on l.

Proposition 4.2 (Proposition 3.1, [19]). *Let $\hat{A} \in \mathcal{S}_1(\hat{\Sigma}, 2)$ be such that $\lambda_+(\hat{A}) = \lambda_-(\hat{A}) = 0$ and let $\hat{m}$ be any $\mathbb{P}(F_{\hat{A}})$-invariant probability measure that projects to $\hat{\mu}$. Then there exists a full $\hat{\mu}$-measure subset E^s_l of $\mathcal{D}^s_{\hat{A},l} \cap [0; i]$, for every $i \in \{1 \dots d\}$, such that the disintegration $\{\hat{m}_{\hat{z}}\}$ of $\hat{m}$ satisfies*

$$\hat{m}_{\hat{z}_2} = \left(h^s_{\hat{z}_1,\hat{z}_2}\right)_* \hat{m}_{\hat{z}_1}$$

for every $\hat{z}_1, \hat{z}_2 \in E^s_l$ in the same stable leaf.

Replacing $(\hat{f}, \hat{A})$ by $(\hat{f}^{-1}, \hat{A}^{-1})$ we get that the disintegration is also invariant under uniform unstable holonomy over a full $\hat{\mu}$-measure subset E^u_l in $\mathcal{D}^u_{\hat{A},l} \cap [0; i]$.

Theorem 4.3. *If $\hat{A} \in \mathcal{S}_1(\hat{\Sigma}, 2)$ and $\lambda_+(\hat{A}) = \lambda_-(\hat{A}) = 0$, then every $\mathbb{P}(F_{\hat{A}})$-invariant probability measure $\hat{m}$ projecting down to $\hat{\mu}$ is an su-state.*

Proof. As $\lambda_+(\hat{A}) = 0$, we can apply Proposition 3.4 to obtain a non-uniform stable holonomy for $\hat{A}$.

By Proposition 4.2, we get that for each $i \in \{1, .., d\}$ and $l \in \mathbb{N}$ there is a disintegration $\{\hat{m}_{\hat{z}}\}$ and a full measure subset E^s_l of $\mathcal{D}^s_{\hat{A},l} \cap [0; i]$, that satisfy

$$\hat{m}_{\hat{z}_2} = \left(h^s_{\hat{z}_1,\hat{z}_2}\right)_* \hat{m}_{\hat{z}_1}$$

for every $\hat{z}_1, \hat{z}_2 \in E^s_l$ in the same stable leaf.

We want to find a disintegration $\{\hat{m}_{\hat{z}}\}$ invariant by stable holonomies in a full measure set of $\hat{\Sigma}$.

Initially, we fix $[0; i]$ and denote $\{\hat{m}^l_{\hat{z}}\}$ the disintegration of $\hat{m}$ associated to $\mathcal{D}^s_{\hat{A},l} \cap [0; i]$. By the essential uniqueness of disintegrations, we get that $\hat{m}^l_{\hat{z}} = \hat{m}^{l+1}_{\hat{z}}$ for $\hat{\mu}$-almost every $\hat{z} \in \mathcal{D}^s_{\hat{A},l} \cap [0; i] \subset \mathcal{D}^s_{\hat{A},l+1} \cap [0; i]$. Then, for every $l \in \mathbb{N}$, we denote by E^s_l the full measure subset of $\mathcal{D}^s_{\hat{A},l} \cap [0; i]$, satisfying that $E^s_l \subset E^s_{l+1}$ and $\hat{m}^l_{\hat{z}} = \hat{m}^{l+1}_{\hat{z}}$ for $\hat{z} \in E^s_l$. Next, we define a disintegration of $\hat{m}$ on each $[0; i]$,

$$\hat{m}_{\hat{z}} = \begin{cases} \hat{m}^1_{\hat{z}} & \hat{z} \in E^s_1 \subset \mathcal{D}^s_{\hat{A},1}, \\ \hat{m}^{l+1}_{\hat{z}} & \hat{z} \in E^s_{l+1} \setminus E^s_l. \end{cases}$$

Let $E^s = \bigcup E^s_l$. In order to prove the invariance of the disintegration by the stable holonomies in E^s, it is enough to choose $\hat{z}_1, \hat{z}_2 \in E^s$ in the same stable leaf and verify $\hat{m}_{\hat{z}_2} = \left(h^s_{\hat{z}_1,\hat{z}_2}\right)_* \hat{m}_{\hat{z}_1}$. As $\{E^s_l\}$ is an increasing sequence of sets, there exists l such that $\hat{z}_1, \hat{z}_2 \in E^s_l$, and then, the definition of the disintegration implies

$$\hat{m}_{\hat{z}_2} = \hat{m}^l_{\hat{z}_2} = \left(h^s_{\hat{z}_1,\hat{z}_2}\right)_* \hat{m}^l_{\hat{z}_1} = \left(h^s_{\hat{z}_1,\hat{z}_2}\right)_* \hat{m}_{\hat{z}_1}.$$

Proceeding this way in every $[0; i]$, we obtain a disintegration in $\hat{\mu}$-almost every point in $\hat{\Sigma}$. We conclude that $\hat{m}$ is an s-state.

Applying the same argument to $(\hat{f}^{-1}, \hat{A}^{-1})$ we obtain that $\hat{m}$ is also a u-state. $\square$

Theorem 4.3 is also true if the cocycle $\hat{A}$ admits one uniform invariant holonomy and one non-uniform invariant holonomy. In this case, the proof follows from the argument above and Proposition 1.16 of [7].

The following proposition is a generalized version of the Proposition 4.8 in [2]. It is in this proof where the hypothesis of existence of a uniform unstable (stable) holonomy is needed. This results is essential to prove Theorem A. Both the non-uniform invariant holonomy and the local product structure of $\hat{\mu}$ allow us to use the uniform unstable (stable) holonomy to transport the disintegration of $\hat{m}$ in a stable (unstable) leaf to every point of the cylinder in a continuous way.

Proposition 4.4. *Suppose $\hat{A} \in \mathcal{S}_1(\hat{\Sigma}, 2)$ is non-uniformly fiber-bunched with a uniform unstable (stable) holonomy. If $\hat{m}$ is an su-state, then there exists a continuous disintegration $\{\hat{m}_{\hat{z}}\}$ that is invariant by the invariant holonomies.*

Proof. We start by considering the non-uniform stable holonomy given by the Proposition 3.4. By definition of *su*-state, there exist two disintegrations $\{\hat{m}^1_{\hat{x}}\}_{\hat{x}\in\hat{\Sigma}}$ and $\{\hat{m}^2_{\hat{x}}\}_{\hat{x}\in\hat{\Sigma}}$ of $\hat{m}$, and a $\hat{\mu}$-full measure subset $\hat{U}_i$ of $[0;i]$ for every $i \in \{1, ..., d\}$ such that

(i) $(h^s_{\hat{x},\hat{y}})_*\hat{m}^1_{\hat{x}} = \hat{m}^1_{\hat{y}}$ for each $\hat{y} \in W^s_{loc}(\hat{x})$ with $\hat{x} \in \hat{U}_i$ (*s*-state);

(ii) $(h^u_{\hat{x},\hat{y}})_*\hat{m}^2_{\hat{x}} = \hat{m}^2_{\hat{y}}$ for each $\hat{x}, \hat{y} \in \hat{U}_i$ with $\hat{y} \in W^u_{loc}(\hat{x})$ (*u*-state);

(iii) $\hat{m}^1_{\hat{x}} = \hat{m}^2_{\hat{x}}$ for each $\hat{x} \in \hat{U}_i$ (essential uniqueness of disintegrations).

We consider l large enough such that $\mathcal{D}^s_{\hat{A},l} \cap \hat{U}_i \neq \emptyset$ for each $i \in \{1, ..., d\}$ and fix $\hat{x} \in \mathcal{D}^s_{\hat{A},l} \cap \hat{U}_i$ such that $\mu^s_{\hat{x}}(W^s_{loc}(\hat{x}) \setminus \hat{U}_i) = 0$. Then define $\hat{m}_{\hat{x}} = \hat{m}^1_{\hat{x}}$ and

(a) $\hat{m}_{\hat{y}} = (h^s_{\hat{x},\hat{y}})_*\hat{m}_{\hat{x}} = (h^s_{\hat{x},\hat{y}})_*\hat{m}^1_{\hat{x}}$ for each $\hat{y} \in W^s_{loc}(\hat{x}) \cap [0;i]$;

(b) $\hat{m}_{\hat{z}} = (h^u_{\hat{y},\hat{z}})_*\hat{m}_{\hat{y}}$ for each $\hat{z} \in W^u_{loc}(\hat{y}) \cap [0;i]$ with $\hat{y} \in W^s_{loc}(\hat{x}) \cap [0;i]$.

By (i)-(iii), we have that $\hat{m}_{\hat{y}} = \hat{m}^1_{\hat{y}} = \hat{m}^2_{\hat{y}}$ for every $\hat{y} \in W^s_{loc}(\hat{x}) \cap \hat{U}_i$ and $\hat{m}_{\hat{z}} = \hat{m}^2_{\hat{z}}$ for every $\hat{z} \in W^u_{loc}(\hat{y}) \cap \hat{U}_i$ with $\hat{y} \in W^s_{loc}(\hat{x}) \cap \hat{U}_i$. By the choice of $\hat{x}$ and the fact that $\hat{\mu}$ has local product structure, the later corresponds to a full measure subset of points $\hat{z} \in [0;i]$. In particular, $\{\hat{m}_{\hat{z}}\}_{\hat{z}\in[0;i]}$ is a disintegration for $\hat{m}$.

The continuity of $\hat{m}_{\hat{x}}$ is a consequence of the fact that $\hat{z} \mapsto h^{su}_{\hat{x},\hat{z}}$ is a continuous map, where

$$h^{su}_{\hat{x},\hat{z}} = h^u_{\hat{y},\hat{z}} \circ h^s_{\hat{x},\hat{y}},$$

with $\hat{x}, \hat{y}, \hat{z}$ as in (a)-(b). This is true because $\hat{y} \mapsto h^s_{\hat{x},\hat{y}}$ is continuous in $W^s_{loc}(\hat{x})$ due to $x \in \mathcal{D}^s_{\hat{A},l}$, $\hat{z} \mapsto W^u_{loc}(\hat{z}) \cap W^s_{loc}(\hat{x})$ is continuous in $[0;i]$ and $(\hat{z}, \hat{y}) \mapsto h^u_{\hat{y},\hat{z}}$ is continuous in Ω^u.

By the definition of the disintegration it is clear that it is invariant by both holonomies. □

Another important relation between the Lyapunov exponents of $\hat{A}$ and $\mathbb{P}(F_{\hat{A}})$-invariant probabilities measures is the following classical lemma due to Furstenberg. Its proof can be found in Section 5.5.3 of [20]. This result is a key tool for Chapter 6 when we need to characterize the invariant measures of discontinuity points of the map $\lambda_+(\cdot)$.

Lemma 4.5. *Let $\hat{A} \in \mathcal{S}_1(\hat{\Sigma}, 2)$ and define $\Phi_{\hat{A}}(\hat{x}, v) = \log \left\| \hat{A}(\hat{x}) v \right\|$ for every $(\hat{x}, v) \in \hat{\Sigma} \times \mathbb{P}^1$. Then,*

$$\lambda_+(\hat{A}) = \max \left\{ \int \Phi_{\hat{A}}(\hat{x}, v)\, d\hat{m} : \hat{m} \text{ is a } u - \text{state for } \mathbb{P}(F_{\hat{A}}) \right\},$$
$$\lambda_-(\hat{A}) = \min \left\{ \int \Phi_{\hat{A}}(\hat{x}, v)\, d\hat{m} : \hat{m} \text{ is an } s - \text{state for } \mathbb{P}(F_{\hat{A}}) \right\}.$$

Chapter 5

Limit of su-states

In this chapter we prove that being a u-state is a closed property inside of the non-uniformly fiber-bunched class of cocycles. This is used in Chapter 6.

We fix the base dynamics $(\hat{f}, \hat{\mu})$, where $\hat{f}$ is a sub-shift of finite type and $\hat{\mu}$ is an ergodic $\hat{f}$-invariant measure with local product structure and $\operatorname{supp} \hat{\mu} = \hat{\Sigma}$.

The next proposition has been proved for uniform invariant holonomies in Lemma 4.3 of [4]. Since it is not possible to obtain uniform estimations for non-uniform invariant holonomies, we cannot adapt its proof to our context. The following argument is based on Appendix A of [18].

Proposition 5.1. *Let $\hat{A}_k, \hat{A} \in \mathcal{S}_1(\hat{\Sigma}, 2)$ such that $\hat{A}$ is non-uniformly fiber-bunched. Suppose that $\hat{A}_k$ converges in the Lipschitz topology to $\hat{A}$ and let $\hat{m}_k$ be an s-state (u-state) for $\mathbb{P}(F_{\hat{A}_k})$ for each $k \in \mathbb{N}$. If $\hat{m}_k$ converges to $\hat{m}$ in the weak-* topology, then $\hat{m}$ is an s-state (u-state) for $\mathbb{P}(F_{\hat{A}})$.*

Proof. By Remark 3.5, if $\hat{A}$ is non-uniformly fiber-bunched, then every $\hat{A}_k$ has a non-uniform stable holonomy defined in the same sets as $\hat{A}$ does (see proof of Lemma 3.6). We want to see that if $\hat{m}_k$ is an s-state for every k, then $\hat{m}$ is also an s-state. We begin by fixing $l_0 \in \mathbb{N}$ large enough such that $\mathcal{D}^s_{\hat{A},l_0} \cap [0;i] \neq \emptyset$ for each $i \in \{1, ..., d\}$. By the construction of the sets $\mathcal{D}^s_{\hat{A},l}$, the same property is true for every $l \geq l_0$.

We fix $\hat{x}_i \in [0;i]$ for each $i \in \{1, ..., d\}$. For every $l \geq l_0$, define

$$\mathcal{D}_{i,l} = \{\hat{z} \in W^u_{loc}(\hat{x}_i) : W^s_{loc}(\hat{z}) \cap \mathcal{D}^s_{\hat{A},l} \neq \emptyset\}.$$

As a result, $\mathcal{D}^s_{\hat{A},l} \cap [0;i] \subset \bigcup\{W^s_{loc}(\hat{z}) : \hat{z} \in \mathcal{D}_{i,l}\}$.

Let $\mathcal{B}^l \subset \mathcal{M}$ be the sub σ-algebra generated by

$$\{W^s_{loc}(\hat{z}) : \hat{z} \in \mathcal{D}_{i,l} \text{ and } \ i \in \{1, ..., d\}\}.$$

Therefore, the elements in $\mathcal{B}^l$ are the measurable sets E such that for each $\hat{z}$ and j, either E contains $W^s_{loc}(\hat{z})$ or it is disjoint from it.

For each $k \in \mathbb{N}$, define the map

$$\overline{h}^k_{\hat{y}} = \begin{cases} h^{s,k}_{\hat{y},\hat{z}} & \text{if } \hat{y} \in W^s_{loc}(\hat{z}) \text{ with } \hat{z} \in \mathcal{D}_{i,l} \\ Id & \text{otherwise.} \end{cases}$$

Here $h^{s,k}$ is the projectivization of $H^{s,k}$ and hence it is an stable holonomy for $\mathbb{P}(F_{\hat{A}_k})$. By the definition of the sets $\mathcal{D}_{i,l}$ and the Proposition 3.4, we know that $\overline{h}^k$ is well defined.

Analogously, we define $\overline{h}$. By Lemma 3.6, we get that $\overline{h}^k_{\hat{y}}$ converge to $\overline{h}_{\hat{y}}$ uniformly on $\hat{\Sigma}$.

Consider $\overline{m}^k_{\hat{y}} = (\overline{h}^k_{\hat{y}})_* \hat{m}^k_{\hat{y}}$, and $\overline{m}$ with a similar construction.

We want to proof that $\overline{m}_k \to \overline{m}$ in the weak-* topology, for that we have to show that $\int \varphi \, d\overline{m}_k \to \int \varphi \, d\overline{m}$, for every continuous and bounded function $\varphi : \hat{\Sigma} \times \mathbb{P}^1 \to \mathbb{R}$.

By the definition of $\overline{m}_k$, we get that

$$\int \varphi \, d\overline{m}_k = \int \varphi(\hat{y}, \overline{h}^k_{\hat{y}}(v)) \, d\hat{m}^k_{\hat{y}}(v) d\hat{\mu}.$$

Then, denoting as $\overline{\varphi}_k(\hat{y}, v) = \varphi(\hat{y}, \overline{h}^k_{\hat{y}}(v))$, we obtain that

$$\left| \int \overline{\varphi}_k \, d\hat{m}_k - \int \overline{\varphi} \, d\hat{m} \right| \leq \int |\overline{\varphi}_k - \overline{\varphi}| \, d\hat{m}_k + \left| \int \overline{\varphi} \, d\hat{m}_k - \int \overline{\varphi} \, d\hat{m} \right|. \tag{5.1}$$

In order to bound the first term in the right side, it is enough to observe that for every $k \in \mathbb{N}$,

$$\overline{\varphi}_k(\hat{y}, v) \leq \sup |\varphi_k(\hat{x}, w)| \leq C,$$

and also that $\overline{\varphi}_k$ converges to $\overline{\varphi}$, because $\overline{h}^k_{\hat{y}} \to \overline{h}_{\hat{y}}$ uniformly. Therefore, given $\varepsilon > 0$, there exist $k_0 > 0$ such that for every $k > k_0$,

$$\int |\overline{\varphi}_k - \overline{\varphi}| \, d\hat{m}_k < \frac{\varepsilon}{2}.$$

Finally, to bound the second term in (5.1), we observe that $\overline{\varphi}(\hat{y}, v)$ is measurable as a function of $\hat{y}$, thus there exists a continuous function

$$\psi : \hat{\Sigma} \times \mathbb{P}^1 \to \mathbb{R},$$

and a compact set K with $\mu(\hat{\Sigma} \setminus K) < \frac{\varepsilon}{4 \sup |\varphi|}$ such that $\psi(\hat{y}, v) = \overline{\varphi}(\hat{y}, v)$ for $(\hat{y}, v) \in K \times \mathbb{P}^1$ and $\sup |\psi| \leq \sup |\varphi|$. As $\hat{m}_k \to \hat{m}$ in the weak-* topology, for $k \in \mathbb{N}$ large enough, we get that

$$\left| \int \overline{\varphi} \, d\hat{m}_k - \int \overline{\varphi} \, d\hat{m} \right| \leq \left| \int_K \psi \, d\hat{m}_k - \int_K \psi \, d\hat{m} \right| + \frac{\varepsilon}{4} < \frac{\varepsilon}{2},$$

which concludes the proof of $\overline{m}_k \to \overline{m}$.

Next we show that $\hat{y} \mapsto \overline{m}^k_{\hat{y}}$ is $\mathcal{B}^l$-measurable for every l large enough, and as $\overline{m}_k \to \overline{m}$ weakly$-^*$, this would imply that $\hat{y} \mapsto \overline{m}$ is also $\mathcal{B}^l$-measurable.

Claim 1. *If $\hat{m}_k$ is an s-state for all $k \in \mathbb{N}$, then $\hat{y} \mapsto \overline{m}^k_{\hat{y}}$ is $\mathcal{B}^l$-measurable mod 0 for all $l \geq l_0$.*

Proof. We need to proof that given any continuous and bounded function $\varphi : \mathbb{P}^1 \to \mathbb{R}$, the map $\Phi_k : \hat{\Sigma} \to \mathbb{R}$, defined as $\hat{y} \mapsto \Phi_k(\hat{y}) = \int \varphi \, d\overline{m}^k_{\hat{y}}$, is $\mathcal{B}^l$-measurable mod 0.

Fix any $l \geq l_0$. Because $\hat{m}_k$ is an s-state, the definition of $\overline{m}_k$ guarantees that

$$\Phi_k(\hat{y}) = \int \varphi \, d\overline{m}^k_{\hat{y}} = \int \varphi \, d\hat{m}^k_{\hat{z}}$$

is constant $\hat{\mu}$-almost every $\hat{y} \in W^s_{loc}(\hat{z})$.

Let $E \subset \mathbb{R}$ be a measurable set. If there exists $\hat{y} \in W^s_{loc}(\hat{z}) \cap \Phi_k^{-1}(E)$, then $\hat{\mu}$-almost every $\hat{w} \in W^s_{loc}(\hat{z})$, $\Phi_k(\hat{w}) = \int \varphi \, d\overline{m}^k_{\hat{w}} = \int \varphi \, d\overline{m}^k_{\hat{y}} = \Phi_k(\hat{y}) \in E$, concluding that $W^s_{loc}(\hat{z}) \subset \Phi_k^{-1}(E) \mod 0$.

We have proved that Φ_k is $\mathcal{B}^l$-measurable, and then $\hat{y} \mapsto \overline{m}^k_{\hat{y}}$ is also $\mathcal{B}^l$-measurable. □

Claim 2. *If $\overline{m}_k \to \overline{m}$ in the weak-* topology and $\hat{y} \mapsto \overline{m}^k_{\hat{y}}$ is $\mathcal{B}^l$-measurable mod 0, then $\hat{y} \mapsto \overline{m}_{\hat{y}}$ is $\mathcal{B}^l$-measurable mod 0 for all $l \geq l_0$.*

Proof. Firstly, lets see that $\hat{y} \mapsto \Phi_k(\hat{y}) = \int \varphi(v) \, d\overline{m}^k_{\hat{y}}$ converges in the weak topology of $L^2(\hat{\mu})$ to $\Phi(\hat{y}) = \int \varphi \, d\overline{m}_{\hat{y}}$. With that purpose, let $\psi : \hat{\Sigma} \to \mathbb{R}$ be a continuous bounded function and observe that

$$\int \psi \Phi_k \, d\hat{\mu} = \int \psi(\hat{y}) \int \varphi(v) \, d\overline{m}^k_{\hat{y}} \, d\hat{\mu} = \int \int \psi(\hat{y}) \varphi(v) \, d\overline{m}^k_{\hat{y}} \, d\hat{\mu}.$$

Because $\psi(\hat{y})\varphi(v)$ is also a continuous function and $\overline{m}_k \to \overline{m}$ in the weak-* topology, we get that

$$\int \psi(\hat{y})\varphi(v) \, d\overline{m}_k \to \int \psi(\hat{y})\varphi(v) \, d\overline{m} = \int \psi \Phi \, d\hat{\mu}.$$

Then, as continuous bounded function are dense in $L^2(\hat{\mu})$, Φ_k converges weakly to Φ.

By hypothesis, we have that Φ_k is $\mathcal{B}^l$ measurable mod 0 for each l, to get the conclusion we have to prove that Φ also is.

We know that the space $K \subset L^2(\hat{\mu})$ of $\mathcal{B}^l-$measurable functions is convex and close. Then, if $\Phi \notin K$, by the Hahn-Banach theorem, there exist $\xi \in L^2(\hat{\mu})$ such that $\int \xi \psi \, d\overline{m} = 0$ for all $\psi \in K$ and $\int \xi \Phi \, d\hat{\mu} > 0$. Since, $0 = \int \xi \Phi_k \, d\hat{\mu} \to \int \xi \Phi \, d\hat{\mu}$, we get a contradiction.

Finally, we conclude that $\hat{y} \mapsto \Phi(\hat{y}) = \int \varphi \, d\overline{m}_{\hat{y}}$ is $\mathcal{B}^l$-measurable mod 0 for each $l \geq l_0$. □

To conclude, we proof that if $\hat{y} \mapsto \overline{m}_{\hat{y}}$ is $\mathcal{B}^l$-measurable for l large, this implies that $\hat{m}$ is an s-state.

Since $\hat{y} \to \overline{m}_{\hat{y}}$ is $\mathcal{B}^l$-measurable mod 0, then for each $l \geq l_0$, there exists a full measure set E^s_l of $\bigcup \{W^s_{loc}(\hat{z}) : \hat{z} \in \mathcal{D}_{i,l}\}$ that verifies

$$\hat{z}_1, \hat{z}_2 \in E^s_l \cap W^s_{loc}(\hat{z}) \Rightarrow \overline{m}_{\hat{z}_1} = \overline{m}_{\hat{z}_2},$$

with $\hat{z} \in \mathcal{D}_{i,l}$, and thus

$$(h^s_{\hat{z}_1,\hat{z}})_*\hat{m}_{\hat{z}_1} = (h^s_{\hat{z}_2,\hat{z}})_*\hat{m}_{\hat{z}_2} \Leftrightarrow (h^s_{\hat{z}_1,\hat{z}_2})_*\hat{m}_{\hat{z}_1} = \hat{m}_{\hat{z}_2}.$$

Then, there is a full measure set $E^s = \bigcup E^s_l$, satisfying that if $\hat{y} \in E^s$, then $\hat{y} \in E^s_l$ for some l, and as $W^s_{loc}(\hat{y}) \subset E^s_l \mod 0$, we obtain that $(h^s_{\hat{z}_1,\hat{z}_2})_*\hat{m}_{\hat{z}_1} = \hat{m}_{\hat{z}_2}$ with $\hat{z}_1, \hat{z}_2 \in W^s_{loc}(\hat{y}) \mod 0$.

Applying the previous proceeding to $(\hat{f}^{-1}, \hat{A}^{-1})$, we obtain that the limit of u-states is also a u-state.

$\square$

Chapter 6

Measures induced by a u-state

In general, when we have an $\mathbb{P}(F_{\hat{A}})$-invariant probability measure it is not easy to find a continuous disintegration of it. Due to Proposition 4.4, when $\hat{m}$ is an su-state with a single uniform holonomy it has such disintegration and this allows us to discover more properties of it, what makes it easier to work with. In this chapter, we introduce other type of measures that needs not necessary to be su-states and that admits a continuous disintegration in some modify sense.

We fix the base dynamics $(\hat{f}, \hat{\mu})$, where $\hat{f}$ is a sub-shift of finite type and $\hat{\mu}$ is an ergodic $\hat{f}$-invariant measure with local product structure and $\operatorname{supp} \hat{\mu} = \hat{\Sigma}$. Recall that

$$\Sigma^u = \{(x_n)_{n \in \mathbb{Z}} : q_{x_n x_{n+1}} = 1 \text{ for every } n \geq 0\}$$

and $P^u : \hat{\Sigma} \to \Sigma^u$ is the projection defined by dropping all of the negative part of a sequence in $\hat{\Sigma}$.

Given $x, y \in \Sigma^u \cap [0; i]$, we define the *unstable holonomy map*

$$h_{x,y} : W^s_{loc}(x) \to W^s_{loc}(y),$$

by assigning to each $\hat{x} \in W^s_{loc}(x)$ the unique element $\hat{y} = h_{x,y}(\hat{x}) \in W^s_{loc}(y)$ such that $\hat{y} \in W^u_{loc}(\hat{x})$. Here $W^s_{loc}(x) = (P^u)^{-1}(x)$.

The partition of $(\hat{\Sigma}, \hat{\mu})$ into local stable manifolds is a measurable partition and this induces a disintegration into a family of conditional measures $\{\hat{\mu}_x\}_{x \in \Sigma^u}$, with every $\hat{\mu}_x$ supported on $W^s_{loc}(x)$.

The lemma below gives a consequence of the existence of local product structure for $\hat{\mu}$ which is well known, see for instance Lemma 2.2 of [8] or Lemma 2.5 of [4]. We reproduce the proof here to indicate explicitly how the local product structure of $\hat{\mu}$ is used.

Lemma 6.1. *Assume $\hat{\mu}$ has local product structure. Then, the measure $\hat{\mu}$ has a disintegration into conditional measures $\{\hat{\mu}_x\}_{x \in \Sigma^u}$ that vary continuously with x in the weak-* topology. In fact, for every $x, y \in \Sigma^u$ in the same cylinder $[0; i]$,*

$$h_{x,y} : (W^s_{loc}(x), \hat{\mu}_x) \to (W^s_{loc}(y), \hat{\mu}_y)$$

is absolutely continuous, with Jacobian $R_{x,y}$ depending continuously on (x, y).

Proof. For each $i \in \{1 \dots d\}$, the local product structure of $\hat{\mu}$ allows us to express $\hat{\mu}|_{[0;i]}$ as $\psi \cdot (\mu^s|_{P^s([0;i])} \times \mu^u|_{P^u([0;i])})$ for a positive continuous function ψ. Since $\mu^u = P^u_* \hat{\mu}$ we have that $\int_{W^s_{loc}(x)} \psi(\hat{x}) d\mu^s(\hat{x}) = 1$ on every local stable manifold and thus $\hat{\mu}_x = \psi(\hat{x})\mu^s$ and $R_{x,y}(\hat{x}) = \frac{\psi(h_{x,y})(\hat{x})}{\psi(\hat{x})}$ define a disintegration of $\hat{\mu}$ and a Jacobian for $h_{x,y}$ as we wanted. □

The following work is based in [4], with the difference that in our context the unstable holonomy is non-uniform, instead of the uniform ones considered on [4].

In the sequel, we are interested in families of measures on $\Sigma^u \times \mathbb{P}^1$ that are induced by measures in $\hat{\Sigma} \times \mathbb{P}^1$. We are able to identify some geometric properties that are enunciated next.

Definition 6.2. *A probability measure m on $\Sigma^u \times \mathbb{P}^1$ is induced by a u-state if there exist*

(i) *a linear cocycle $\hat{A} : \hat{\Sigma} \to SL(2, \mathbb{R})$ that is constant along local stable manifolds and admits unstable holonomies H^u,*

(ii) *and a $\mathbb{P}(F_{\hat{A}})$-invariant measure $\hat{m}$ on $\hat{\Sigma} \times \mathbb{P}^1$ projecting to $\hat{\mu}$ such that $\hat{m}$ is a u-state for $h^u = \mathbb{P}(H^u)$ and $m = (P^u \times Id)_* \hat{m}$.*

Note that $m = (P^u \times Id)_* \hat{m}$ is a $\mathbb{P}(F_A)$-invariant measure where A is such that $\hat{A} = A \circ P^u$.

If $\{\hat{m}_{\hat{x}}\}_{\hat{x} \in \hat{\Sigma}}$ is a disintegration of $\hat{m}$ and $\{\hat{\mu}_x\}_{x \in \Sigma^u}$ is a disintegration of $\hat{\mu}$ as in Lemma 6.1, then for $x \in \Sigma^u$

$$m_x = \int_{W^s_{loc}(x)} \hat{m}_{\hat{x}} d\hat{\mu}_x(\hat{x})$$

is a disintegration of m relative to $\{\pi^{-1}(x); x \in \Sigma^u\}$, where $\pi : \Sigma^u \times \mathbb{P}^1 \to \Sigma^u$ is the canonical projection.

Proposition 6.3. *Any probability measure m induced by a u-state admits a disintegration into conditional measures $\{m_x\}_{x \in \Sigma^u}$ that are defined for every $x \in \Sigma^u$ and vary continuously with x in the weak-* topology.*

Proof. If we have H^u as being uniform unstable holonomies, the result follows from Proposition 4.5 in [4]. Next we focus in the case when the unstable holonomies are non-uniform.

Recall the sets $\mathcal{D}^u_{\hat{A},l} = \mathcal{D}^u_{\hat{A}}(N_l, \theta)$, for a fixed $\theta < \frac{\log \sigma}{2}$, and an increasing sequence of integers N_l. They are considered in the proof of Proposition 3.4 and its elements admit non-uniform unstable holonomies that variate continuously at a fixed rate depending on l.

So, when we have non-uniform unstable holonomies, we fix $\hat{x}_i \in [0;i]$, for $i \in \{1,\dots,d\}$, and l large enough such that $\mathcal{D}^u_{\hat{A},l} \cap [0;i] \neq \emptyset$ for every i. We define

$$\mathcal{D}_{i,l} = \{\hat{z} \in W^s_{loc}(\hat{x}_i) :\ W^u_{loc}(\hat{z}) \cap \mathcal{D}^u_{\hat{A},l} \neq \emptyset\}.$$

With this, the set

$$\mathcal{C}_{i,l} = \cup\{W^u_{loc}(\hat{z}) :\ \hat{z} \in \mathcal{D}_{i,l}\},$$

does not depend of the $\hat{x}_i$ chosen and $\mathcal{D}^u_{\hat{A},l} \cap [0;i] \subset \mathcal{C}_{i,l}$.

Let $\varphi : \mathbb{P}^1 \to \mathbb{R}$ be a continuous function and consider $x, y \in \Sigma^u$ in the same cylinder $[0;i]$, then

$$\begin{aligned}
&\int_{\mathbb{P}^1} \varphi(v) dm_y = \int_{W^s_{loc}(y)} \int_{\mathbb{P}^1} \varphi d\hat{m}_{\hat{y}} d\hat{\mu}_y(\hat{y}) \\
&= \int_{W^s_{loc}(x)\cap\mathcal{C}_{i,l}} \left(\int_{\mathbb{P}^1} \varphi \circ h^u_{\hat{x},\hat{y}} d\hat{m}_{\hat{x}} \right) R_{x,y}(\hat{x}) d\hat{\mu}_x(\hat{x}) + \int_{W^s_{loc}(y)\cap\mathcal{C}^c_{i,l}} \int_{\mathbb{P}^1} \varphi d\hat{m}_{\hat{y}} d\hat{\mu}_y(\hat{y}).
\end{aligned}$$

As

$$\hat{\mu}_y(W^s_{loc}(y) \cap \mathcal{C}^c_{i,l}) = \int_{W^s_{loc}(y)\cap\mathcal{C}^c_{i,l}} \psi(\hat{y}) d\hat{\mu}^s = \hat{\mu}(\mathcal{C}^c_{i,l}) < \frac{1}{l},$$

we get

$$\begin{aligned}
\left| \int_{\mathbb{P}^1} \varphi dm_y - \int_{\mathbb{P}^1} \varphi dm_x \right| &= \int_{W^s_{loc}(x)\cap\mathcal{C}_{i,l}} \int_{\mathbb{P}^1} \left| \varphi \circ h^u_{\hat{x},\hat{y}} \cdot R_{x,y}(\hat{x}) - \varphi \right| d\hat{m}_{\hat{x}} d\hat{\mu}_x(\hat{x}) \\
&\quad + \int_{W^s_{loc}(x)\cap\mathcal{C}^c_{i,l}} \int_{\mathbb{P}^1} |\varphi| d\hat{m}_{\hat{x}} d\hat{\mu}_x(\hat{x}) \\
&\quad + \int_{W^s_{loc}(y)\cap\mathcal{C}^c_{i,l}} \int_{\mathbb{P}^1} |\varphi| d\hat{m}_{\hat{y}} d\hat{\mu}_y(\hat{y}). \\
&\leq \int_{W^s_{loc}(x)\cap\mathcal{C}_{i,l}} \int_{\mathbb{P}^1} \left| \varphi \circ h^u_{\hat{x},\hat{y}} \cdot R_{x,y}(\hat{x}) - \varphi \right| d\hat{m}_{\hat{x}} d\hat{\mu}_x(\hat{x}) + \frac{2C}{l},
\end{aligned}$$

where $C = \sup|\varphi|$. By Proposition 3.3, we have that $\|h^u_{\hat{x},\hat{y}} - Id\|$ is uniformly small when $\hat{x}, \hat{y}$ are close and $\hat{y} \in W^u_{loc}(\hat{x}) \subset \mathcal{C}_{i,l}$. Then, let $\varepsilon > 0$ and set l such that $\frac{4\sup|\varphi|}{\varepsilon} < l$, we can choose $\delta > 0$, such that if $d(x,y) < \delta$, then the expression $\|R_{x,y} - 1\|_{L^1}$ is small, by Lemma 6.1, and thus $\|\varphi \circ h^u_{x,y} \cdot R_{x,y}(\hat{x}) - \varphi\| < \frac{\varepsilon}{2}$, concluding that $\left|\int \varphi dm_x - \int \varphi dm_y\right| < \varepsilon$. □

Because of this last statement, it is possible to extend Proposition 4.7 and Proposition 4.8 of [4] to our context. We enunciate them next for completeness, but the proof is the same as in [4].

Proposition 6.4 (Proposition 4.7, [4])**.** *Let $\hat{m}_k$ and $\hat{m}$ be $\mathbb{P}(F_{\hat{A}_k})$ and $\mathbb{P}(F_{\hat{A}})$-invariant probability measures induced by u-states such that $\hat{m}_k \to \hat{m}$ in the weak$-^*$ topology.*

Then they admit disintegrations $\{m_x^k\}_{x\in\Sigma^u}$ *and* $\{m_x\}_{x\in\Sigma^u}$*, respectively, which are defined for every* $x \in \Sigma^u$ *and such that the family* $\{\{m_x^k\}_{x\in\Sigma^u}, \{m_x\}_{x\in\Sigma^u}\}_k$ *is equicontinuous. More precisely, for every continuous function* $\varphi : \mathbb{P}^1 \to \mathbb{R}$ *and* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that* $d(x, y) < \delta$ *implies* $|\int \varphi dm_x - \int \varphi dm_y| < \varepsilon$ *and* $|\int \varphi dm_x^k - \int \varphi dm_y^k| < \varepsilon$ *for every* $k \in \mathbb{N}$*.*

Under the same hypothesis as before, we state the following.

Proposition 6.5 (Proposition 4.8, [4])**.** *For any* $x \in \Sigma^u$*,* $m_x^k \to m_x$*. Moreover, the convergence is uniform in* x*.*

Proof. Let $\varphi \colon \mathbb{P}^1 \to \mathbb{R}$ be continuous and $\varepsilon > 0$. By Proposition 6.4, there exists $\delta > 0$ such that, if $d_\rho(\hat{x}, \hat{y}) \leq \delta$ then

$$\left|\int_{\mathbb{P}^1} \varphi dm_x - \int_{\mathbb{P}^1} \varphi dm_y\right| < \frac{\varepsilon}{10}$$

and

$$\left|\int_{\mathbb{P}^1} \varphi dm_x^k - \int_{\mathbb{P}^1} \varphi dm_y^k\right| < \frac{\varepsilon}{10}$$

for every $k \in \mathbb{N}$. Cover Σ^u with finitely many clopen sets V_i such that $\operatorname{diam}(V_i) < \delta$. As m_k converges to m there exists $k_0 \in \mathbb{N}$ such that

$$\left|\int_{V_i} \left(\int_{\mathbb{P}^1} \varphi dm_x^k\right) d\mu(x) - \int_{V_i} \left(\int_{\mathbb{P}^1} \varphi dm_x\right) d\mu(x)\right| < \frac{\varepsilon\mu(V_i)}{10},$$

where $\mu = \hat{\mu}^u$. Given $x \in \Sigma^u$, take V_i with $x \in V_i$. Then

$$\begin{aligned}
\left|\int_{\mathbb{P}^1} \varphi dm_x^k - \int_{\mathbb{P}^1} \varphi dm_x\right| =& \frac{1}{\mu(V_i)} \left|\int_{V_i} \left(\int_{\mathbb{P}^1} \varphi dm_x^k\right) d\mu(y) - \int_{V_i} \left(\int_{\mathbb{P}^1} \varphi dm_x\right) d\mu(y)\right| \\
\leq& \frac{1}{\mu(V_i)} \left|\int_{V_i} \int_{\mathbb{P}^1} \varphi dm_x^k d\mu(y) - \int_{V_i} \int_{\mathbb{P}^1} \varphi dm_y^k d\mu(y)\right| \\
&+ \frac{1}{\mu(V_i)} \left|\int_{V_i} \int_{\mathbb{P}^1} \varphi dm_y^k d\mu(y) - \int_{V_i} \int_{\mathbb{P}^1} \varphi dm_y d\mu(y)\right| \\
&+ \frac{1}{\mu(V_i)} \left|\int_{V_i} \int_{\mathbb{P}^1} \varphi dm_y d\mu(y) - \int_{V_i} \int_{\mathbb{P}^1} \varphi dm_x d\mu(y)\right| \\
\leq& \frac{1}{\mu(V_i)} \left(\int_{V_i} \int_{\mathbb{P}^1} |\varphi dm_x^k - \varphi dm_y^k| d\mu(y)\right) \\
&+ \frac{1}{\mu(V_i)} \left|\int_{V_i} \int_{\mathbb{P}^1} \varphi dm_y^k d\mu(y) - \int_{V_i} \int_{\mathbb{P}^1} \varphi dm_y d\mu(y)\right| \\
&+ \frac{1}{\mu(V_i)} \left(\int_{V_i} \int_{\mathbb{P}^1} |\varphi dm_y - \varphi dm_x| d\mu(y)\right) \\
\leq& \frac{3\varepsilon}{10}
\end{aligned}$$

□

Chapter 7

Characterization of discontinuity points

For every cocycle satisfying $\lambda_-(\hat{A}) < 0 < \lambda_+(\hat{A})$, in particular this apply to discontinuity points of the Lyapunov exponents, every $\mathbb{P}(F_{\hat{A}})$-invariant measure have a particular condition: it is a convex combination of the extremal points of the set

$$\left\{ \int \Phi_{\hat{A}}(\hat{x}, v)\, d\hat{m} : \hat{m} \text{ is a } su - \text{state for } \mathbb{P}(F_{\hat{A}}) \right\}.$$

By this procedure, every $\mathbb{P}(F_{\hat{A}})$-invariant measure inherits the property of being an su-state from the extremal measures.

We fix the base dynamics $(\hat{f}, \hat{\mu})$, where $\hat{f}$ is a sub-shift of finite type and $\hat{\mu}$ is an ergodic $\hat{f}$-invariant measure with local product structure and $\operatorname{supp} \hat{\mu} = \hat{\Sigma}$. By the semi-continuity of $\lambda_+(\cdot)$ and $\lambda_-(\cdot)$, we have that if $\hat{A} \in \mathcal{S}_1(\hat{\Sigma}, 2)$ is a discontinuity point of $\lambda_+(\cdot)$, then $\lambda_-(\hat{A}) < 0 < \lambda_+(\hat{A})$. Let $\mathbb{R}^2 = E^{s,\hat{A}}_{\hat{x}} \oplus E^{u,\hat{A}}_{\hat{x}}$ be the Oseledets decomposition associated to $\hat{A}$ at the point $\hat{x} \in \hat{\Sigma}$. Consider the measures in $\hat{\Sigma} \times \mathbb{P}^1$ defined by

$$\hat{m}^s = \int_{\hat{\Sigma}} \delta_{(\hat{x}, \mathbb{P}(E^{s,\hat{A}}_{\hat{x}}))}\, d\hat{\mu} \quad \text{and} \quad \hat{m}^u = \int_{\hat{\Sigma}} \delta_{(\hat{x}, \mathbb{P}(E^{u,\hat{A}}_{\hat{x}}))}\, d\hat{\mu}. \tag{7.1}$$

By construction, they are both $\mathbb{P}(F_{\hat{A}})$-invariant probability measures which project to $\hat{\mu}$ and

$$\lambda_-(\hat{A}) = \int_{\hat{\Sigma}} \Phi_{\hat{A}}(\hat{x}, v)\, d\hat{m}^s \quad \text{and} \quad \lambda_+(\hat{A}) = \int_{\hat{\Sigma}} \Phi_{\hat{A}}(\hat{x}, v)\, d\hat{m}^u,$$

where $\Phi_{\hat{A}} = \log \|\hat{A}(\hat{x})v\|$. That is, those are the measures that achieve the minimum and maximum in Lemma 4.5. Moreover, if $\hat{A}$ admits invariant holonomies, then $\hat{m}^s$ is an s-state and $\hat{m}^u$ is a u-state.

Proposition 7.1. *Let $\hat{A} \in \mathcal{S}_1(\hat{\Sigma}, 2)$ such that $\lambda_-(\hat{A}) < 0 < \lambda_+(\hat{A})$ and let $\hat{m}$ be a probability measure in $\hat{\Sigma} \times \mathbb{P}^1$ projecting to $\hat{\mu}$. Then $\hat{m}$ is $\mathbb{P}(F_{\hat{A}})$-invariant if and only if it is a convex combination of $\hat{m}^s$ and $\hat{m}^u$.*

Indeed, one only has to note that every compact subset of $\mathbb{P}^1$ disjoint from $\{E^s, E^u\}$ accumulates on E^u in the future and on E^s in the past. That α is constant (independent of $\hat{x} \in \hat{\Sigma}$) follows from ergodicity, see [2] for the full proof.

The following result gives a characterization of discontinuity points of $\lambda_+(\cdot)$. Note that in our statement it is only needed that the cocycle is non-uniformly fiber-bunched, in particular we are not asking for convergence of the holonomies because in this case that is a consequence of 3.6. It is in this proof that we need Proposition 5.1.

Proposition 7.2. *Let $\hat{A} \in \mathcal{S}_1(\hat{\Sigma}, 2)$ be non-uniformly fiber-bunched. If $\hat{A}$ is a discontinuity point of $\lambda_+(\cdot)$ for the Lipschitz topology, then every $\mathbb{P}(F_{\hat{A}})$-invariant probability measure $\hat{m}$ projecting to $\hat{\mu}$ is an su-state.*

Proof. By the upper semi-continuity of $\lambda_+(\cdot)$, passing to a subsequence we may assume $\lim_{k\to\infty} \lambda_+(\hat{A}_k) < \lambda_+(\hat{A})$. By Lemma 4.5, for each $k \in \mathbb{N}$, there exists an ergodic $\mathbb{P}(F_{\hat{A}_k})$-invariant probability measure $\hat{m}_k$ projecting to $\hat{\mu}$ such that

$$\lambda_+(\hat{A}_k) = \int_{\hat{\Sigma}\times\mathbb{P}^1} \Phi_{\hat{A}_k}(\hat{x}, v)\, d\hat{m}_k. \tag{7.2}$$

If $\lambda_+(\hat{A}_k) = 0$, then any $\mathbb{P}(F_{\hat{A}_k})$-invariant probability measure $\hat{m}_k$ projecting to $\hat{\mu}$ satisfies (7.2) and by Theorem 4.3 is be an *su*-state. If $\lambda_+(\hat{A}_k) > 0$, we take $\hat{m}_k = \int_{\hat{\Sigma}} \delta_{(\hat{x}, \mathbb{P}(E^{u,k}_{\hat{x}}))}\, d\hat{\mu}$ that satisfies (7.2) and it is a *u*-state. Consequently,

$$\lim_{k\to\infty} \int_{\hat{\Sigma}\times\mathbb{P}^1} \Phi_{\hat{A}_k}(\hat{x}, v)\, d\hat{m}_k < \lambda_+(\hat{A}).$$

Taking sub-sequences again, we may assume that $(\hat{m}_k)_k$ converges weak-* to a $\mathbb{P}(F_{\hat{A}})$-invariant probability measure $\hat{m}$. By Proposition 5.1, $\hat{m}$ is a *u*-state and by Proposition 7.1, there exists $\alpha \in [0, 1]$ such that

$$\hat{m} = \alpha \hat{m}^u + (1 - \alpha)\hat{m}^s.$$

Finally, by the uniform convergence of $\Phi_{\hat{A}_k} \to \Phi_{\hat{A}}$, we have

$$\int_{\hat{\Sigma}\times\mathbb{P}^1} \Phi_{\hat{A}}(\hat{x}, v)\, d\hat{m} = \lim_{k\to\infty} \int_{\hat{\Sigma}\times\mathbb{P}^1} \Phi_{\hat{A}_k}(\hat{x}, v)\, d\hat{m}_k < \lambda_+(A) = \int_{\hat{\Sigma}\times\mathbb{P}^1} \Phi_{\hat{A}}(\hat{x}, v)\, d\hat{m}^u,$$

hence $\hat{m} \neq \hat{m}^u$. It follows that $\alpha \neq 1$ and

$$\hat{m}^s = \frac{1}{1-\alpha}\left(\hat{m} - \alpha \hat{m}^u\right)$$

is a *u*-state.

Analogously, using Lemma 4.5 for $\lambda_-(\hat{A})$, we conclude that $\hat{m}^u$ is an *s*-state. Then, Proposition 7.1 concludes the statement. □

Remark 7.3. *By the proof of Proposition 7.2 it is evident that when we have uniform holonomies such that $H^{s,k} \to H^s$ uniformly, every $\mathbb{P}(F_{\hat{A}})$-invariant probability measure is an s-state. This is the context of Proposition 5.2 of [4] or [2].*

Chapter 8

Holonomies without bunching

In this chapter we introduce a new type of holonomies, that is weaker than Definition 3.1. While the definition for non-uniform invariant holonomies determines this holonomies as linear isomorphisms defined in all the projective space, in the following results we determine other type of holonomies that do not necessary satisfy this condition.

The following results are fully developed in [22], we only include them for completness, as we are going to need them for the proof of Theorem C.

We fix the base dynamics $(\hat{f}, \hat{\mu})$, where $\hat{f}$ is a sub-shift of finite type and $\hat{\mu}$ is an ergodic $\hat{f}$-invariant measure with local product structure and $\operatorname{supp} \hat{\mu} = \hat{\Sigma}$. Consider $\mathbb{P}(F_{\hat{A}})$ as a differential cocycle, with $\hat{A} \in \mathcal{S}_1(\hat{\Sigma}, 2)$, and we restrict ourselves to the subclass of $\mathbb{P}(F_{\hat{A}})$- invariant measures satisfying

$$\lim_n \frac{1}{n} \|D\mathbb{P}(\hat{A})^n(\hat{x})v\| \le c < \frac{\log \sigma}{2} \text{ for } \hat{m} - \text{almost every } (\hat{x}, v) \in \hat{M} \times \mathbb{P}^1, \tag{8.1}$$

where $D\mathbb{P}(\hat{A})(\hat{x})v$ denotes the derivative of $\mathbb{P}(\hat{A}) : \mathbb{P}^1 \to \mathbb{P}^1$.

Proposition 8.1 (Proposition 5.2, [22])**.** *If $\hat{m}$ satisfies (8.1) then for every $(\hat{x}, v)$ in a full $\hat{m}-$measure subset Λ of $\hat{M} \times \mathbb{P}^1$ there are C^1 functions $\phi_{\hat{x},v} \colon W^u_{loc}(\hat{x}) \to \mathbb{P}^1$ such that*

$$(\hat{x}, v) \mapsto \phi_{\hat{x},v} \text{ is measurable, and } \phi_{\hat{x},v}(\hat{x}) = v.$$

Finally the graphs $\mathcal{W}^u_{loc}(\hat{x}, v) = \{(\hat{y}, \phi_{\hat{x},v}(y)) \colon \hat{y} \in W^u_{loc}(\hat{x})\}$ satisfy

(a) $\mathbb{P}(F_{\hat{A}})^{-1}(\mathcal{W}^u_{loc}(\hat{x}, v)) \subset \mathcal{W}^u_{loc}(\mathbb{P}(F_{\hat{A}})^{-1}(\hat{x}, v))$ *for every* $(\hat{x}, v) \in \Lambda$,

(b) *for any* $(\hat{y}, w) \in \mathcal{W}^u_{loc}(\hat{x}, v)$,

$$d(\mathbb{P}(F_{\hat{A}})^{-n}(\hat{x}, v), \mathbb{P}(F_{\hat{A}})^{-n}(\hat{y}, w)) \to 0 \text{ exponentially fast.}$$

Proof. The assumption (8.1) ensures that there exists $\hat{m}$-almost everywhere an Oseledets strong-unstable subspace $E^u_{(\hat{x},v)} \subset T_{\hat{x}}\hat{M} \times \mathbb{R}^2$ that is a graph over the unstable direction $E^u_{\hat{x}}$ of $\hat{f}$. Then, by Pesin theory, there exists $\hat{m}$-almost everywhere a C^1 embedded disk $\widetilde{W}^u(\hat{x},v)$ tangent to $E^u_{(\hat{x},v)}$ and such that

$$F_{\hat{A}}^{-n}(\hat{y},w) \in \widetilde{W}^u_{loc}(\hat{x},v) \text{ and } d(F_{\hat{A}}^{-n}(\hat{x},v), F_{\hat{A}}^{-n}(\hat{y},w)) \leq \sigma^{-n}$$

for every $n \geq 0$ and $(\hat{y},w) \in \widetilde{W}^u(\hat{x},v)$. This also implies that $\widetilde{W}^u(\hat{x},v)$ is a C^1 graph over a neighbourhood of $\hat{x}$ inside $W^u(\hat{x})$. While the radius $r(\hat{x})$ of this neighbourhood needs not be bounded from zero, in principle, Pesin theory also gives that it decreases sub-exponentially along the orbits:

$$\lim_n \frac{1}{n} \log r(\hat{f}^{-n}(\hat{x})) = 0.$$

On the other hand, the size of $\hat{f}^{-n}(W^u_{loc}(\hat{x}))$ decreases exponentially fast (faster than σ^{-n}). Thus, the projection of $\widetilde{W}^u(F_{\hat{A}}^{-n}(\hat{x},v))$ contains $\hat{f}^{-n}(W^u_{loc}(\hat{x}))$ for any large n. So we can obtain that $F^n_{\hat{A}}(\widetilde{W}^u(F_{\hat{A}}^{-n}(\hat{x},v)))$ is a C^1 graph whose projection contains $W^u_{loc}(\hat{x})$. Finally, we take $\mathcal{W}^u_{loc}(\hat{x},v)$ as the part of that graph that lies over $W^u_{loc}(\hat{x})$. $\square$

Denote $\Lambda_{\hat{x}} = \Lambda \cap (\{\hat{x}\} \times \mathbb{P}^1)$ for each $\hat{x} \in \hat{M}$, note that $\hat{m}_{\hat{x}}(\Lambda_{\hat{x}}) = 1$ for $\hat{\mu}$-almost every $\hat{x}$, because $\hat{m}(\Lambda) = 1$. With this we can introduce a different type of invariant holonomies along the unstable leaf for m.

Definition 8.2. *We say that $\hat{m}$ is a u-state if it satisfies (8.1) and admits a disintegration $\{\hat{m}_{\hat{x}}\}$ along the fibers $\{\hat{x}\} \times \mathbb{P}^1$ such that*

$$(h^u_{\hat{x},\hat{y}})_* \hat{m}_{\hat{x}} = \hat{m}_{\hat{y}} \textit{ for } \hat{\mu}\textit{-almost every } \hat{x} \textit{ and any } \hat{y} \in W^u_{loc}(\hat{x}),$$

where $h^u_{\hat{x},\hat{y}} : \Lambda_{\hat{x}} \to \{\hat{y}\} \times \mathbb{P}^1$ is defined by $h^u_{\hat{x},\hat{y}}(\hat{x},v) = (\hat{y}, \phi_{(\hat{x},v)}(\hat{y}))$.

This definition does not imply that the cocycle has non-uniform invariant holonomies as in Chapter 3. The principal difference between these two definitions is that the second type might not to be defined in all $\mathbb{P}^1$, they are only defined in full measure subsets $\{\Lambda_{\hat{x}}\}_{\hat{x}}$ of $\mathbb{P}^1$. If the cocycle does have non-uniform invariant holonomies, this two have to coincide in a full measure subset. This last type of invariant holonomies is useful when we do not have a bunching condition, as in Theorem C.

For these holonomies we need a new version of the Invariance Principle.

Theorem 8.3 (Theorem 6.1, [22]). *Every $\mathbb{P}(F_{\hat{A}})$-invariant probability measure $\hat{m}$ satisfying*

$$\lim_n \frac{1}{n} \|D\hat{A}^n(\hat{x})v\| \leq 0 \textit{ for } \hat{m}\textit{-almost every } (\hat{x},v) \in \hat{M} \times \mathbb{P}^1,$$

is a u-state.

Chapter 9

Proof of the Theorems

9.1 Proof of Theorem A

From now on we suppose that there exists a sequence $\hat{A}_k \in \mathcal{S}_1(\hat{\Sigma}, 2)$ satisfying $\hat{A}_k \to \hat{A}$ in $\mathcal{H}^s$, but $\lambda_+(\hat{A}_k)$ does not converge to $\lambda_+(\hat{A})$. That is, $\hat{A}$ is a discontinuity point of $\lambda_+(\cdot)$ and in particular $\lambda_-(\hat{A}) < 0 < \lambda_+(\hat{A})$.

For every $k \in \mathbb{N}$, we denote by $\hat{m}_k$ the ergodic measure obtained in Lemma 4.5 that achieves $\lambda_+(\hat{A}_k)$. Proposition 4.4 gives us a property about su-states which allows us to understand better the nature of these measures. Thus, the proof of Theorem A is divided into two cases: firstly, we suppose that there exists a subsequence j_k such that $\hat{m}_{j_k}$ is an su-state, and in the second case we suppose that every $\hat{m}_k$ is not an su-state. In order to simplify the notation in the first case, we suppose that every $\hat{m}_k$ is an su-state.

The argument in Case I turns out to be more efficient than others that are known for the fact that leaves in evidence which hypotheses are needed, and which can be weakened. For example, in the first case it is showed that we do not need a full measure set of points with non-uniform unstable holonomies, it is enough to have some particular points.

Case I: $\hat{m}_k$ are su-states

By Kalinin [12], we know that there exists a periodic point $\hat{p}$ of $\hat{f}$ such that $\hat{A}^{n_p}(\hat{p})$ is hyperbolic, where $n_p = per(\hat{p})$.

Let $i_{\hat{p}}$ be the element in $\{1, ..., d\}$ such that $\hat{p} \in [0; i_{\hat{p}}]$ and let a and r be the attractive and repellent directions of $\mathbb{P}(\hat{A}^{n_p}(\hat{p}))$.

By Proposition 4.4 and Proposition 7.2, the measures $\hat{m}^u$ and $\hat{m}^s$ defined by Equation (7.1) have a continuous disintegration satisfying that $\hat{m}^u_{\hat{z}} = \delta_{a_{\hat{z}}}$ and $\hat{m}^s_{\hat{z}} = \delta_{r_{\hat{z}}}$, which implies that the maps $\hat{z} \mapsto a_{\hat{z}}$ and $\hat{z} \mapsto r_{\hat{z}}$ are also continuous. In particular, $\hat{m}^u_{\hat{p}} = \delta_a$ and $\hat{m}^s_{\hat{p}} = \delta_r$. Therefore, the sets

$$M^+ = \{(\hat{z}, a_{\hat{z}})\}_{\hat{z} \in \hat{\Sigma}} \text{ and } M^- = \{(\hat{z}, r_{\hat{z}})\}_{\hat{z} \in \hat{\Sigma}}$$

are compact and disjoint. In particular, there exists an $\varepsilon > 0$ such that

$$d(M^-, M^+) > \varepsilon.$$

Because hyperbolicity is an open condition and $\hat{A}_k$ converges to $\hat{A}$, for a k large enough, $\hat{A}_k^{n_p}(\hat{p})$ is also hyperbolic. We denote as $\{a_k, r_k\}$ at the attractive and repellent directions of $\mathbb{P}(\hat{A}_k^{n_p}(\hat{p}))$. Then, we have that $a_k \to a$ and $r_k \to r$.

Due to Proposition 4.4, we know that each $\hat{m}_k$ has a disintegration $\{\hat{m}^k_{\hat{z}}\}_{\hat{z}\in\hat{\Sigma}}$ such that $\hat{z} \mapsto \hat{m}^k_{\hat{z}}$ is continuous and invariant by the holonomies. In the following steps we express the supp $\hat{m}^k_{\hat{z}}$ at every point as a transformation of the elements $\{a_k, r_k\}$.

Because $\hat{\mu}$ is an ergodic measure, we know that for each $i \in \{1, ..., d\} \setminus \{i_{\hat{p}}\}$ there exists $j_i > 0$ such that $\hat{f}^{j_i}([0; i_{\hat{p}}]) \cap [0; i]$ is a positive measure set, we choose j_i as the smaller integer with this property. If $i = i_{\hat{p}}$, define $j_i = 0$.

We use the sets $\mathcal{D}^u_{\hat{A},l}$ defined at the end of the proof of Proposition 3.4 where the non-uniform unstable holonomies $h^u_{\hat{x},\hat{y}} = \mathbb{P}(H^u_{\hat{x},\hat{y}})$ are defined for every $\hat{x} \in \mathcal{D}^u_{\hat{A},l}$ and every $\hat{y} \in W^u_{loc}(\hat{x})$.

We choose l large enough such that $\hat{f}^{j_i}([0; i_{\hat{p}}]) \cap [0; i] \cap \mathcal{D}^u_{\hat{A},l} \neq \emptyset$ for every $i \in \{1, ..., d\}$ and fix an $\hat{x}_i \in \hat{f}^{-j_i}([0; i] \cap \mathcal{D}^u_{\hat{A},l}) \cap [0; i_{\hat{p}}]$.

Because of the election of the $\hat{x}_i$, it is possible to define for each $\hat{z} \in [0; i_{\hat{p}}]$,

$$a^k_{\hat{z}} = h^k_{\hat{z}} a_k,$$

where

$$\hat{y_1} \in W^s_{loc}(\hat{p}) \cap W^u_{loc}(\hat{x}_{i_{\hat{p}}}) \text{ and } \hat{y_2} \in W^s_{loc}(\hat{z}) \cap W^u_{loc}(\hat{x}_{i_{\hat{p}}}),$$

and

$$h^k_{\hat{z}} = h^{s,k}_{\hat{y_2},\hat{z}} \circ h^{u,k}_{\hat{y_1},\hat{y_2}} \circ h^{s,k}_{\hat{p},\hat{y_1}}.$$

On the other hand, if $\hat{z} \in [0; i]$ with $i \neq i_{\hat{p}}$, we define $a^k_{\hat{z}}$ as following

$$a^k_{\hat{z}} = h^{su,k}_{\hat{f}^{j_i}(\hat{x}_i),\hat{z}} \circ \mathbb{P}(\hat{A}_k^{j_i}(\hat{x}_i)) a^k_{\hat{x}_i},$$

where

$$y \in W^u_{loc}(\hat{f}^{j_i}(\hat{x}_i)) \cap W^s_{loc}(\hat{z}),$$

and

$$h^{su,k}_{\hat{f}^{j_i}(\hat{x}_i),\hat{z}} = h^{s,k}_{\hat{y},\hat{z}} \circ h^{u,k}_{\hat{f}^{j_i}(\hat{x}_i),\hat{y}}.$$

We define $r^k_{\hat{z}}$ analogously using r_k instead of a_k.

With these definitions the maps $\hat{z} \mapsto a^k_{\hat{z}}$ and $\hat{z} \mapsto r^k_{\hat{z}}$ are continuous, due to the continuity of $h^k_{\hat{z}}$, $\hat{A}_k$ and $h^{su,k}$. Also, the topology in $\mathcal{H}^s$ and Lemma 3.6 allow us to obtain that $a^k_{\hat{z}} \to a_{\hat{z}}$ and $r^k_{\hat{z}} \to r_{\hat{z}}$ uniformly on $\hat{\Sigma}$.

Since $\hat{m}_k$ admits a continuous disintegration $\{\hat{m}^k_{\hat{z}}\}$ invariant by the holonomies, then $\#\text{supp } \hat{m}^k_{\hat{z}} = \#\text{supp } \hat{m}^k_{\hat{p}}$ for every $\hat{z}$. In particular, since supp $\hat{m}^k_{\hat{p}} \subset \{a_k, r_k\}$ we have that supp $\hat{m}^k_{\hat{z}} \subset \{a^k_{\hat{z}}, r^k_{\hat{z}}\}$. This last statement implies that $\#\text{supp } \hat{m}^k_{\hat{z}} \in \{1, 2\}$.

First, we suppose that $\#\mathrm{supp}\ \hat{m}^k_{\hat{z}} = 2$ and we prove that both $\hat{z} \mapsto a^k_{\hat{z}}$ and $\hat{z} \mapsto r^k_{\hat{z}}$ are $\mathbb{P}(F_{\hat{A}_k})$-invariant sections. We prove this by contradiction and we start by supposing that they are not invariant sections.

As $\mathbb{P}(\hat{A}_k(\hat{z}))_* \hat{m}^k_{\hat{z}} = \hat{m}^k_{\hat{f}(\hat{z})}$ we get that

$$\mathbb{P}(\hat{A}_k(\hat{z}))(\{a^k_{\hat{z}}, r^k_{\hat{z}}\}) = \{a^k_{\hat{f}(\hat{z})}, r^k_{\hat{f}(\hat{z})}\}$$

for each $\hat{z} \in \hat{\Sigma}$. Then, for each k there exist a $j_k > k$ and a $\hat{z}_{j_k}$ such that

$$\mathbb{P}(\hat{A}_{j_k}(\hat{z}_{j_k}))(a^{j_k}_{z\hat{j}_k}) = r^{j_k}_{\hat{f}(\hat{z}_{j_k})}.$$

By compactness, there exists a $\hat{z}_0$ where the sequence $\{\hat{z}_{j_k}\}$ accumulates. In order to simplify the notation we suppose that $j_k = k$.

Next, taking ε as in (9.1), take k large enough such that

$$d(a^k_{\hat{z}_0}, a_{\hat{z}_0}) < \frac{\varepsilon}{6} \quad \text{and} \quad d(r^k_{\hat{z}_0}, r_{\hat{z}_0}) < \frac{\varepsilon}{6}.$$

Therefore,

$$\|r_{\hat{f}(\hat{z}_0)} - a_{\hat{f}(\hat{z}_0)}\| \leq \|r_{\hat{f}(\hat{z}_0)} - r^k_{\hat{f}(\hat{z}_k)}\| + \|\mathbb{P}(\hat{A}_k(\hat{z}_k))(a^k_{\hat{z}_k}) - \mathbb{P}(\hat{A}(\hat{z}_0))(a_{\hat{z}_0})\| < \frac{\varepsilon}{3},$$

which is a contradiction to the fact that M^+ and M^- are separated sets.

Thus, there exists a $k_0 \in \mathbb{N}$ such that for every $k \geq k_0$, $\hat{z} \mapsto a^k_{\hat{z}}$ and $\hat{z} \mapsto r^k_{\hat{z}}$ are $\mathbb{P}(F_{\hat{A}_k})$-invariant sections. Then, it is possible to define two $\mathbb{P}(F_{\hat{A}_k})$-invariant measures,

$$\hat{m}^s_k = \int \delta_{a^k_{\hat{z}}}\, d\hat{\mu} \quad \text{and} \quad \hat{m}^u_k = \int \delta_{r^k_{\hat{z}}}\, d\hat{\mu}.$$

Therefore, we have that $\hat{m}_k = \alpha_k \hat{m}^s_k + (1-\alpha_k)\hat{m}^u_k$ with $\alpha_k \neq 0$, which contradicts the ergodicity of $\hat{m}_k$. This contradiction arises from the supposition that $\#\mathrm{supp}\ \hat{m}^k_{\hat{z}} = 2$. Assume now that $\#\mathrm{supp}\ = 1$, then either $\hat{m}_k = \hat{m}^s_k$ or $\hat{m}_k = \hat{m}^u_k$.

We suppose that $\hat{m}_k = \hat{m}^s_k$ and show that $\lambda_+(\hat{A}_k) \to \lambda_+(\hat{A})$, which contradicts the initial hypothesis. The case $\hat{m}_k = \hat{m}^u_k$ is analogous.

In the following, we prove that $\hat{m}_k \to \hat{m}^s$ in the weak-* topology. Take any continuous bounded function $\varphi : \hat{\Sigma} \times \mathbb{P}^1 \to \mathbb{R}$ and define $\psi_k(\hat{z}) := \varphi(\hat{z}, r^k_{\hat{z}})$ for every k, analogously define ψ using $r_{\hat{z}}$ instead of $r^k_{\hat{z}}$. Thus, we obtain a collection of continuous functions that satisfy that for each $\hat{z} \in \hat{\Sigma}$, $\mid \psi_k(\hat{z}) \mid < \sup_{(\hat{x},v)\in\hat{\Sigma}\times\mathbb{P}^1} \mid \varphi(\hat{x}, v) \mid$. As $r^k_{\hat{z}} \to r_{\hat{z}}$ uniformly in $\hat{\Sigma}$, we get that $\psi_k(\hat{z}) \to \psi(\hat{z})$, then, the dominated convergence theorem implies that

$$\int \varphi\, d\hat{m}_k = \int \psi_k\, d\hat{\mu} \to \int \psi\, d\hat{\mu} = \int \varphi\, d\hat{m}^s.$$

Finally,

$$\lambda_-(\hat{A}_k) = \int \Phi_{\hat{A}_k}(\hat{x}, v)\, d\hat{m}_k \to \int \Phi_{\hat{A}}(\hat{x}, v)\, d\hat{m}^s = \lambda_-(\hat{A}),$$

and as $\lambda_-(\hat{A}) = -\lambda_+(\hat{A})$, then $\lambda_+(\hat{A}_k) \to \lambda_+(\hat{A})$ which is a contradiction.

Case II: $\hat{m}_k$ are not su-states

To prove the result for the case when the $\hat{m}_k$ are not su-states, we are based in the work done in [4] for the context when both H^s and H^u are uniform invariant holonomies. In out case this is possible because of the results in Chapter 6.

Firstly, we apply the results of Chapter 6 for the ergodic measures $\hat{m}_k$ obtained in Lemma 4.5 that achieves $\lambda_+(\hat{A}_k)$. We have to take them to the same language, firstly we need to find a cocycle with the same Lyapunov exponents as $\hat{A}$ and that is constant along stable manifolds.

Claim 3. *Let $\hat{A} \in \mathcal{S}_1(\hat{\Sigma}, 2)$ be the cocycle with uniform stable holonomy. Then there exists a cocycle $\tilde{A}$ constant along stable manifolds such that $\lambda_+(\hat{A}) = \lambda_+(\tilde{A})$.*

Proof. Fix d points $\hat{z}_1, \dots \hat{z}_d$ with $\hat{z}_i \in [0; i]$. For each $\hat{x} \in [0; i]$, let $g(\hat{x})$ be the unique point in the intersection $W^u_{loc}(\hat{z}_i) \cap W^s_{loc}(\hat{x})$. Note that $g(\hat{x}) = g(\hat{y})$ if $\hat{y} \in W^s_{loc}(\hat{x})$. Define

$$\tilde{A}(\hat{y}) = H^s_{f(\hat{y}),g(f(\hat{y}))} \circ \hat{A}(\hat{y}) \circ H^s_{g(\hat{y}),\hat{y}}.$$

Next, we verify that $\lambda_+(\hat{A}) = \lambda_+(\tilde{A})$. As $\tilde{A}^n(\hat{x}) = H^s_{f^n(\hat{x}),g(f^n(\hat{x}))}\hat{A}^n(\hat{x})H^s_{g(\hat{x}),\hat{x}}$ and the norm of $H^s_{f^n(\hat{x}),g(f^n(\hat{x}))}$ is bounded and does not depend on n, we get that $\lambda_+(\tilde{A}) \leq \lambda_+(\hat{A})$. The converse inequality is analogous. $\square$

Next we define

$$\tilde{m}_{\hat{x}} = \left(h^s_{\hat{x},g(\hat{x})}\right)_* \hat{m}_{\hat{x}}, \tag{9.1}$$

where $h^s = \mathbb{P}(H^s)$, and let $\tilde{m}$ be the probability measure on $\hat{\Sigma} \times \mathbb{P}^1$ projecting to $\hat{\mu}$ with this disintegration along the $\mathbb{P}^1$ fibers, that is $\tilde{m} = \int \tilde{m}_{\hat{x}} d\hat{\mu}$. With this definition $\tilde{m}$ is a $\mathbb{P}(F_{\tilde{A}})$- invariant probability measure projecting down to $\hat{\mu}$.

Claim 4. *If $\hat{x} \mapsto \tilde{m}_{\hat{x}}$ is constant along stable manifolds* $\bmod 0$*, then $\hat{m}$ is an s-state.*

Proof. Let $\{\hat{m}_{\hat{x}}\}_{\hat{x}\in\hat{\Sigma}}$ be a disintegration of $\hat{m}$ defined in a $\hat{\mu}$-full measure set E. We may assume that E is $\hat{f}$-invariant. Suppose that $\tilde{m}_{\hat{x}}$ is constant along $W^s_{loc}(\hat{x}) \mod 0$ for $\hat{x} \in E$, this means that for $\hat{\mu}$-almost every $\hat{y} \in W^s_{loc}(\hat{x})$, $\tilde{m}_{\hat{x}} = \tilde{m}_{\hat{y}}$. Therefore,

$$\left(h^s_{\hat{y},g(\hat{y})}\right)_* \hat{m}_{\hat{y}} = \tilde{m}_{\hat{y}} = \tilde{m}_{\hat{x}} = \left(h^s_{\hat{x},g(\hat{x})}\right)_* \hat{m}_{\hat{x}}.$$

Using that $g(\hat{x}) = g(\hat{y})$ we obtain that $\hat{m}_{\hat{y}} = \left(h^s_{g(\hat{x}),\hat{y}} \circ h^s_{\hat{x},g(\hat{x})}\right)_* \hat{m}_{\hat{y}} = (h^s_{\hat{x},\hat{y}})_* \hat{m}_{\hat{x}}$. $\square$

Let $\tilde{A}_k$ be as in Claim 3, and let $\tilde{m}_k$ be the invariant measures associated to them defined as in Equation (9.1). With this definition we still have that $\tilde{A}_k \to \tilde{A}$.

As well as in Chapter 6, we can now define $m_k = (P^u \times Id)_* \tilde{m}_k$ as being a $\mathbb{P}(F_{A_k})$-invariant measure with A_k such that $\tilde{A}_k = A_k \circ P^u$.

We use Claim 4 to prove that if there is a disintegration $\{m^k_x\}$ of m_k such that every m^k_x is atomic then $\hat{x} \to \tilde{m}^k_{\hat{x}}$ should be constant along stable manifolds, and thus $\hat{m}_k$ is a su-state, which is a contradiction to our initial hypothesis.

Claim 5. *If there is a x such that m_x^k are atomic measures in $\mathbb{P}^1$ then $\hat{m}_k$ are su-states.*

Proof. We prove it by contradiction. Suppose that the measures $\hat{m}_k$ are not su-states. By hypothesis, for every k there exist $x \in \Sigma^u$ such that the conditional measure m_x^k contains an atom. Lemma 6.1 in [4] states that this implies that in fact for every element $x \in \Sigma^u$ there exists a $v_x^k \in \mathbb{P}^1$ such that the measure m_x^k satisfies $\gamma_k = m_x^k(v_x^k) > 0$.

In addition, Lemma 6.2 of [4] announces that if for a $x \in \Sigma^u$ we have that $\gamma_k = m_x^k(v_x^k) > 0$, for some $v_x^k \in \mathbb{P}^1$, then $\hat{\mu}_{\hat{x}}$-almost every $\hat{x} \in W_{loc}^s(x)$ satisfies that $\gamma_k = \tilde{m}_{\hat{x}}^k(v_x^k) > 0$.

On the other hand, as $\hat{m}_k$ is not an *su*-state, then $\lambda_+(\tilde{A}_k) = \lambda_+(\hat{A}_k) > 0$, which implies that $\tilde{m}_k = \int \delta_{\mathbb{P}(\tilde{E}_{\hat{x}}^u)} d\hat{\mu}$. We conclude that the only option is that $\gamma_k = 1$ and $v_x^k = \mathbb{P}(\tilde{E}_{\hat{x}}^u)$ for $\hat{\mu}_x$-almost every $\hat{x} \in W_{loc}^s(x)$.

Finally, we obtain that $\hat{x} \mapsto \tilde{m}_{\hat{x}}^k$ is a constant map along every stable manifold mod 0, thus, Claim 4 gives us that $\hat{m}_k$ is an *s*-state. As the measures $\hat{m}_k$ are also u-states, we get the contradiction and the Claim 5 is proved. □

As a consequence of the previous argumentation we conclude that m_x^k can not be atomic for any $x \in \Sigma^u$.

Because Proposition 4.4 there are continuous functions $\xi^* \colon \hat{\Sigma} \to \mathbb{P}^1$ such that $\mathbb{P}(E_{\hat{x}}^*) = \xi^*(\hat{x})$ with $* \in \{s, u\}$. Using ξ^*, we perform a final continuous change of coordinates, that is projective in each fiber, such that for $\hat{x} \in \hat{\Sigma}$

$$\xi^s(\hat{x}) = [1:0] := q, \text{ and } \xi^u(\hat{x}) = [0:1] := p.$$

In particular, after this coordinate change the projective cocycle $\mathbb{P}(\hat{A})$ leaves q and p invariant for every $\hat{x}$. As the change of coordinates is constant on local stable manifolds, the cocycle $\hat{A}$ is still of the form $\hat{A} = A \circ P^u$ for some $A \colon \Sigma^u \to SL(2, \mathbb{R})$.

Because the results in Chapter 7, every m_k has a continuous disintegration $\{m_x^k\}_{x \in \Sigma^u}$ of non-atomic measures. Moreover, with this last change of coordinates m_k converges to $\hat{\mu}^u \times (\alpha\delta_p + (1-\alpha)\delta_q)$ and the conditional measures m_x^k converge uniformly to $\alpha\delta_p + (1-\alpha)\delta_q$ for every $x \in \Sigma^u$. Those are the fundamental conditions needed to implement the following argument. The next section is fully done in [4] and an idea of its proof is included here only for completeness.

The energy argument.

In this section we are going to apply the energy argument to a suitable family of measures which is going to allow us to conclude the result. This argument consist in finding a family of symmetric self-coupling for this measures with finite energy. For fixed parameters, when A_k approach close enough of A, the energy of this family of self-coupling gets a reduction and iterating this process is that we arrive to a contradiction.

The energy argument was first introduced by [1] and [20] as part of a project to extend, to dimension higher than two, the continuity of Lyapunov exponents for locally constant cocycles. In this setting, the condition of being a u-state is translated to the notion of stationary measure and therefore they need to study the limit measure η of a sequence of non-atomic stationary measures η_k. The conclusion follows since the energy argument allow them to prove that the expanding point of $\mathbb{P}(A)$ defined by the stable subspace associated to $\lambda_-(A)$ is invisible for η.

The generalization of that argument to the context of [4] is possible because of an analogous version of Proposition 6.5: $m_x^k \to m_x$ uniformly. Then, they can consider a suitable family of sets U_x and apply the energy argument to the measures $\{m_x^k|_{U_x}\}_{x\in\Sigma}$. Their approach is closer to the higher dimensional version in [1], since they consider additive Margulis functions to conclude their result, for a closer approach see [21].

We begin by recalling that, given $B \in SL(2,\mathbb{R})$ and $v \in \mathbb{P}^1$, the derivative at the point v of $\mathbb{P}(B)$ in the projective space is given explicitly by

$$D_v\mathbb{P}(B)(\dot{v}) = \frac{\mathrm{proj}_{B(v)}B(\dot{v})}{\|B(v)\|} \text{ for every } \dot{v} \in T_v\mathbb{P}^1 = \{v\}^{\perp}$$

where $\mathrm{proj}_v : w \to w - v\frac{<w,v>}{<v,v>}$ denotes the orthogonal projection to the hyperplane orthogonal to v.

Claim 6. *For $\hat{\mu}^u$-almost every $x \in \Sigma^u$ we have*

$$\lim_{n\to\infty} \log\left(\|D_q\mathbb{P}(A)^n(x)\|^{\frac{1}{n}}\right) = \lambda_+(\hat{A}) - \lambda_-(\hat{A}) =: c > 0.$$

Because of this claim, we get also that

$$\lim_{n\to\infty}\frac{1}{n}\int \log\left(\|D_q\mathbb{P}(A)^n(\hat{x})\|\right)d\hat{\mu} > 0.$$

Fix $N \in \mathbb{N}$ big enough such that, if $\kappa : \Sigma^u \to \mathbb{R}$ defined as

$$\kappa(x) := \log\left(\|D_q\mathbb{P}(A)^N(\hat{x})\|\right)$$

then

$$\int \kappa(x)d\hat{\mu} > 4.$$

Definition 9.1. *Consider a Borel probability measure μ' on Σ^u and a μ'-measurable family $\{\nu_x\}_{x\in\Sigma^u}$ of finite Borel measures on $\mathbb{P}^1$. The measures ν_x are not assumed to be probability ones nor are they assumed to have the same mass. For $j \in \{1,2\}$, let $\pi_j : \mathbb{P}^1\times\mathbb{P}^1 \to \mathbb{P}^1$ be the projection on the j-th factor. For $x \in \Sigma^u$, let ξ_x be a measure on $\mathbb{P}^1\times\mathbb{P}^1$. We say that a parametrized family of measures $\{\xi_x\}_{x\in\Sigma^u}$ on $\mathbb{P}^1\times\mathbb{P}^1$ is a (measurable)* family of symmetric self-couplings *of $\{\nu_x\}_{x\in\Sigma^u}$ if*

1. *$x \mapsto \xi_x$ is μ'-measurable,*
2. *$(\pi_j)_*\xi_x = \nu_x$ for $j \in \{1,2\}$, and*
3. *$\iota_*\xi_x = \xi_x$ where $\iota : \mathbb{P}^1\times\mathbb{P}^1 \to \mathbb{P}^1\times\mathbb{P}^1$ is the involution $\iota : (u,v) \mapsto (v,u)$.*

We note that we always have one family of symmetric self-couplings constructed by taking for every x the measure

$$\xi_x = \frac{1}{\|\nu_x\|}\nu_x \times \nu_x$$

for all x with $\|\nu_x\| \neq 0$ where $\|\nu_x\| = \nu_x(\mathbb{P}^1)$ denotes the mass of the measure ν_x. We define the function $\varphi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{R}$ by

$$\varphi(u, v) = -\log d(u, v),$$

where d is a normalized distance on $\mathbb{P}^1$. Note that φ is non-negative.

Definition 9.2. *For a family of symmetric self-couplings $\{\xi_x\}_{x\in\Sigma^u}$ of $\{\nu_x\}_{x\in\Sigma^u}$ we define the (additive)* energy *of $\{\xi_x\}_{x\in\Sigma^u}$ to be*

$$\int_{\Sigma^u}\int_{\mathbb{P}^1} \varphi(u, v) d\xi_x(u, v) d\mu'(x).$$

With the propose of giving an outline of the proof, we choose some fixed parameters for the remainder. Recall the N fixed above and the function κ.

1. Let $U_0 \subset \mathbb{P}^1$ be an open ball centred at q with $r \notin \overline{U}_0$.

2. Let $U_1 \subset \overline{U}_1 \subset U_0$ be an open neighbourhood of q such that for every $x \in \Sigma^u$ and every sufficiently large k we have

   (a) $\mathbb{P}(A_k^N)(x)(\overline{U}_1) \subset U_0$;

   (b) $\overline{U}_1 \subset \mathbb{P}(A_k^N)(x)(U_0)$;

   (c) Fixing $\delta > 0$ small and k sufficiently large, we have that $m_x^k(U_1) > \alpha - \delta$ for every $x \in \Sigma^u$.

   (d) for every $u, v \in U_1$

   $$d(\mathbb{P}(A_k^N)(x)(u), \mathbb{P}(A_k^N)(x)(v)) \geq e^{-\alpha} e^{\kappa(x)} d(u, v).$$

   It follows

   $$\varphi(\mathbb{P}(A_k^N)(x)(u), \mathbb{P}(A_k^N)(x)(v)) \leq \varphi(u, v) - \kappa(x) + \alpha.$$

3. Continuity of the disintegration $\{m_x^k\}$ and punctual convergence $m_x^k \to m_x$ give us the possibility to choose subsets U_x that variate continuously with x and whose mass $m_x^k(U_x)$ is a constant near α for each x.

The construction of the symmetric self-coupling for $\{m_x^k|_{U_x}\}_{x\in\Sigma^u}$ is based on several modifications of an initial family.

We do not include the proof of the following Lemma, it can be found in [4].

Lemma 9.3. *There exist a family of symmetric self-couplings* $\{\xi_x\}_{x\in\Sigma^u}$ *of* $\{m_x^k|_{U_x}\}_{x\in\Sigma^u}$ *with finite energy.*

The final contradiction comes from an iteration of the next Proposition, in this case we are going to give only a sketch of the full proof. For the full argument see [4].

Proposition 9.4. *Let* $\{\xi_x\}_{x\in\Sigma^u}$ *be a family of symmetric self-couplings of* $\{m_x^k|_{U_x}\}_{x\in\Sigma^u}$ *with finite energy. Then there exists a family of symmetric self-couplings* $\{\dot{\xi}_x\}_{x\in\Sigma^u}$ *of* $\{m_x^k|_{U_x}\}_{x\in\Sigma^u}$ *such that*

$$\int\int \varphi d\dot{\xi}_x d\hat{\mu} \leq \int\int \varphi d\xi_x d\hat{\mu}(x) - \alpha.$$

For an outline of the proof, we start defining

$$\hat{\xi}_x = \sum_{y\in f^{-1}(x)} \frac{1}{J_{\hat{\mu}} f^N(y)} \left(\mathbb{P}(A_k^N)(y) \times \mathbb{P}(A_k^N)(y)\right)_* \xi_x.$$

However the restriction of $\hat{\xi}$ to $U_x \times U_x$ is not a self-coupling of $\{m_x^k\}_{x\in\Sigma^u}$. Indeed the difference between $(\pi_1)_*\left(\hat{\xi}|_{U_x\times U_x}\right)$ and $m_x^k|_{U_x}$ has a mass sufficiently small.

Therefore we construct $\dot{\xi}_x$ by adding those terms that are needed for $\hat{\xi}_x$ being a symmetric self coupling. So, we obtain an inequality as following

$$\int\int \varphi d\dot{\xi}_x d\hat{\mu} \leq \int\int \varphi d\hat{\xi}_x|_{U_x\times U_x} d\hat{\mu} + C_0\delta,$$

where δ is a small constant that depends on the previous fixed parameters.

So, we get

$$\begin{aligned}\int\int \varphi d\hat{\xi}_x|_{U_x\times U_x} d\hat{\mu} \leq &\int \sum_{y\in f^{-1}(x)} \frac{1}{J_{\hat{\mu}} f^N(y)} \int_{U_1\times U_1} \varphi(\mathbb{P}(A_k^N)(y)(u), \mathbb{P}(A_k^N)(y)(v)) d\xi_x d\hat{\mu}\\ &+ C_1\delta.\end{aligned}$$

The term $C_1\delta$ comes from the difference between $U_x \times U_x$ and $U_1 \times U_1$ having an small mass respect to ξ_x.

To conclude, using the properties of the set U_1, we have

$$\begin{aligned}&\int \sum_{y\in f^{-1}(x)} \frac{1}{J_{\hat{\mu}} f^N(y)} \int_{U_1\times U_1} \varphi(\mathbb{P}(A_k^N)(y)(u), \mathbb{P}(A_k^N)(y)(v)) d\xi_x d\hat{\mu}\\ &\leq \int\int_{U_1\times U_1} \varphi(u,v) d\xi_x d\hat{\mu} - \int \xi_x(U_1\times U_1)\kappa(x) d\hat{\mu}(x) + \alpha\end{aligned}$$

As $m_x^k(U_1) > \alpha - \delta$, we obtain that $\int \xi_x(U_1 \times U_1)\kappa(x)d\hat{\mu}(x) > 3\alpha$.

Finally

$$\int\int \varphi d\hat{\xi}_x|_{U_x\times U_x} d\hat{\mu} \leq \int\int_{\mathbb{P}^1\times\mathbb{P}^1} \varphi(u,v)d\xi_x d\hat{\mu} - 3\alpha + C_2\delta + \alpha$$
$$\leq \int\int_{\mathbb{P}^1\times\mathbb{P}^1} \varphi(u,v)d\xi_x d\hat{\mu} - \alpha,$$

as δ is small. Iterating this procedure we get the contradiction.

9.2 Proof of Theorem B

Firstly, we observe that because $\hat{A}$ is a locally constant cocycle, it has both uniform invariant holonomies which are in fact the identity. We suppose that, as before, $\hat{A}$ is a discontinuity point for the map $\hat{A} \mapsto \lambda_+(\hat{A})$, so we have that Equation (7.1) holds.

Proposition 7.2 states that every $\mathbb{P}(F_{\hat{A}})$-invariant measure is now an su-state, therefore $\hat{m}^u$ is also an su-state, and because $\hat{A}$ has both uniform invariant holonomies, Proposition 4.4 gives us that $\xi : \hat{\Sigma} \to \mathbb{P}^1$, defined as $\xi(\hat{x}) = \operatorname{supp} \hat{m}^u_{\hat{x}}$ is an invariant section.

Because the uniform invariant holonomies are both the identity and ξ is invariant by them, we have in fact that $\xi(\hat{x}) = E$ where $E \in \mathbb{P}^1$ for every $\hat{x} \in \hat{\Sigma}$.

Finally, as $\hat{m}^u$ is a $\mathbb{P}(F_{\hat{A}})$-invariant measure, we have that $(\mathbb{P}(F_{\hat{A}}))_*\hat{m}^u_{\hat{x}} = \hat{m}^u_{\hat{f}(\hat{x})}$ for $\hat{\mu}$-almost every point, which means that $\mathbb{P}(\hat{A})(\hat{x})E = E$ for $\hat{\mu}$-almost every point. If the projectivization of $\hat{A}$ has a fixed point, then $\hat{A}$ has an invariant subspace, which contradicts the hypothesis of irreducibility and concludes the proof of the Theorem B.

□

9.3 Proof of Theorem C

This proof is also done by contradiction. We suppose again that $\hat{A}$ is a discontinuity point for the map $\hat{A} \mapsto \lambda_+(\hat{A})$, thus $\lambda_-(\hat{A}) < 0 < \lambda_+(\hat{A})$ and there exists $\hat{m}^u$ as in Equation (7.1). By Proposition 5.1, we know that $\hat{m}^u$ is an s-state.

Case $\lambda_+(\hat{A}) < \frac{\log\sigma}{2}$

The case of $\lambda_+(\hat{A}) < \frac{\log\sigma}{2}$ is a conclusion of the previous results. By Proposition 7.2 every $\mathbb{P}(F_{\hat{A}})$-invariant measure is an su-state, in particular $\hat{m}^u$ is an *su*-state. Because we have uniform stable holonomies, Proposition 4.4 gives us that there exists $\{\hat{m}^u_{\hat{z}}\}$ a continuous disintegration for $\hat{m}^u$. On the other hand, $\{\delta_{\mathbb{P}(E^u_{\hat{z}})}\}$ is also a disintegration for $\hat{m}^u$. By essential uniqueness of disintegrations, we get $\hat{m}^u_{\hat{z}} = \delta_{\mathbb{P}(E^u_{\hat{z}})}$ for $\hat{\mu}$-almost every point.

As $\hat{m}^u$ is a $\mathbb{P}(F_{\hat{A}})$-invariant measure and an s-state, the map $\hat{z} \mapsto \hat{m}^u_{\hat{z}}$ is an invariant section.

Case $\lambda_+(\hat{A}) \geq \frac{\log \sigma}{2}$

The proof in this case is based in Theorem 7.1 of [22], with the difference that in the following case the stable uniform holonomy might not to be the identity. The proof is similar to the one in Proposition 4.4, we find a continuous disintegration of the measure $\hat{m}^u$ using the holonomies defined in Chapter 8, while in this case there exists the problem of the holonomies not being defined in all the projective space.

Recall that we denote as r and a the attracting and repellent directions of $\mathbb{P}(\hat{A})$ at the point $\hat{p}$. Since $\hat{f}$ is an Anosov, there exists a neighbourhood of $\hat{p}$ such that $V_{\hat{p}} \approx W^u_{loc}(\hat{p}) \times W^s_{loc}(\hat{p})$.

Because the cocycle does not have the bunching condition that allows us to construct the non-uniform invariant holonomies, we are going to restrict to the class of measures that satisfy the Equation (8.1). With this measures we are able to construct holonomies as in Chapter 8.

In particular $\hat{m}^u$ satisfies (8.1) and according to the Invariance Principle of Theorem 8.3, we obtain that

Claim 7. *$\hat{m}^u$ is a u-state.*

Proof. The operator $D\mathbb{P}(\hat{A})(\hat{x})v$ is a projection satisfying

$$\|D\hat{A}(\hat{x})v\| = |\det \hat{A}(\hat{x})| \left(\frac{\|v\|}{\|\hat{A}(\hat{x})v\|}\right)^2,$$

where $|\det \hat{A}(\hat{x})| = 1$. So

$$\int \log \|D\hat{A}(\hat{x})v\| d\hat{m}^u = -2\int \log\left(\frac{\|\hat{A}(\hat{x})v\|}{\|v\|}\right) d\hat{m}^u = -2\lambda_+(\hat{A}).$$

Define the continuous function $\varphi : \hat{M} \times \mathbb{P}^1 \to \mathbb{R}$ by $\varphi(\hat{x}, v) = \log \|D\hat{A}(\hat{x})v\|$. By ergodicity of $\hat{m}^u$, we have that

$$\lim_n \frac{1}{n}\sum_{i=0}^{n-1} \varphi \circ \mathbb{P}(F_{\hat{A}})^i(\hat{x}, v) = \int \varphi(\hat{x}, v) dm^u = -2\lambda_+(\hat{A}),$$

which allow us to conclude that

$$\lim_n \frac{1}{n} \log \|D\hat{A}^n(\hat{x})v\| = \lim_n \frac{1}{n}\sum_{i=0}^{n-1} \varphi \circ \mathbb{P}(F_{\hat{A}})^i(\hat{x}, v) = -2\lambda_+(\hat{A}) \leq 0,$$

which satisfies the hypothesis in Theorem 8.3, hence $\hat{m}^u$ is a u-state. $\square$

Because of Remark 7.3 $\hat{m}^u$ is also an s-state, then it is an su-state.

Finally we prove that the map $\hat{x} \mapsto \operatorname{supp} \hat{m}^u_{\hat{x}}$ is an s-section for $(\hat{A}, H^s)$. We take $\hat{m} = \hat{m}^u$.

By definition, there exist two disintegrations $\{\hat{m}^1_{\hat{x}}\}_{\hat{x}\in\hat{\Sigma}}$ and $\{\hat{m}^2_{\hat{x}}\}_{\hat{x}\in\hat{\Sigma}}$ of $\hat{m}$, and a $\hat{\mu}$-full measure subset $U_{\hat{p}}$ of $V_{\hat{p}}$ such that

(i) $(h^u_{\hat{x},\hat{y}})_*\hat{m}^1_{\hat{x}} = \hat{m}^1_{\hat{y}}$ for each $\hat{x}, \hat{y} \in U_{\hat{p}}$ with $\hat{y} \in W^u_{loc}(\hat{x})$ (u-state);

(ii) $(h^s_{\hat{x},\hat{y}})_*\hat{m}^2_{\hat{x}} = \hat{m}^2_{\hat{y}}$ for each $\hat{y} \in W^s_{loc}(\hat{x})$ with $\hat{x} \in V_{\hat{p}}$ (s-state);

(iii) $\hat{m}^1_{\hat{x}} = \hat{m}^2_{\hat{x}}$ for each $\hat{x} \in U_{\hat{p}}$ (essential uniqueness of disintegrations).

Since the Pesin unstable manifolds $\mathcal{W}^u(\hat{z}, v)$ vary mensurable with the point, we may find compact sets $\Lambda_1 \subset \Lambda_2 \subset \ldots \subset \Lambda$ such that $\hat{m}(\Lambda_j) \to 1$ and $\mathcal{W}^u(\hat{z}, v)$ varies continuously on every Λ_j. We may choose these compact sets in such a way that $\mathbb{P}(F_{\hat{A}})(\Lambda_j) \subset \Lambda_{j+1}$ for every $j \geq 1$. Up to reducing $U_{\hat{p}}$ if necessary, $\hat{m}^2_{\hat{x}}(\Lambda_{j,\hat{x}}) \to 1$ for every $\hat{x} \in U_{\hat{p}}$.

Fix any $\hat{x} \in U_{\hat{p}}$ such that $\hat{\mu}^u_{\hat{x}}(W^u_{loc}(\hat{x}) \setminus U_{\hat{p}}) = 0$. Then define $\hat{m}_{\hat{x}} = \hat{m}^1_{\hat{x}}$ and

(a) $\hat{m}_{\hat{y}} = (h^u_{\hat{x},\hat{y}})_*\hat{m}_{\hat{x}}$ for each $\hat{y} \in W^u_{loc}(\hat{x})$.

(b) $\hat{m}_{\hat{z}} = (h^s_{\hat{y},\hat{z}})_*\hat{m}_{\hat{y}} = (h^s_{\hat{y},\hat{z}})_*\hat{m}^2_{\hat{y}}$ for each $\hat{z} \in W^s_{loc}(\hat{y}) \cap V_{\hat{p}}$ with $\hat{y} \in W^u_{loc}(\hat{x})$.

By (i)-(iii), we have that $\hat{m}_{\hat{y}} = \hat{m}^1_{\hat{y}} = \hat{m}^2_{\hat{y}}$ for every $\hat{y} \in W^u_{loc}(\hat{x}) \cap U_{\hat{p}}$ and $\hat{m}_{\hat{z}} = \hat{m}^2_{\hat{z}}$ for every $\hat{z} \in W^s_{loc}(\hat{y}) \cap \hat{V}_{\hat{p}}$ with $\hat{y} \in W^u_{loc}(\hat{x}) \cap U_{\hat{p}}$. By the choice of $\hat{x}$ and the fact that $\hat{\mu}$ has local product structure, the later corresponds to a full measure subset of points $\hat{z} \in V_{\hat{p}}$. In particular, $\{\hat{m}_{\hat{z}}\}$ is a disintegration for $\hat{m}$ in $V_{\hat{p}}$.

With these definitions we have that the map $\hat{z} \mapsto \tilde{\Lambda}_{j,\hat{z}} = h^s_{\hat{y},\hat{z}} \circ h^u_{\hat{x},\hat{y}}|_{\Lambda_{\hat{x}}}$ is continuous, so the disintegration $\hat{m}_{\hat{z}}|_{\tilde{\Lambda}_{j,\hat{z}}}$ is also continuous in $V_{\hat{p}}$ for every j.

Since $\{\hat{m}_{\hat{z}} : \hat{z} \in V_{\hat{p}}\}$ is a disintegration and $\hat{m}$ is $\mathbb{P}(F_{\hat{A}})$-invariant,

$$\left(\mathbb{P}(F_{\hat{A}})^{n_{\hat{p}}}\right)_* \hat{m}_{\hat{z}} = \hat{m}_{\hat{f}^{n_{\hat{p}}}(\hat{z})} \text{ for } \hat{\mu} - \text{almost every } \hat{z} \in \hat{f}^{-n_{\hat{p}}}(V_{\hat{p}}) \cap V_{\hat{p}}. \tag{9.2}$$

The identity may not hold for $\hat{z} = \hat{p}$, so we can not state as in the non-uniformly fiber-bunched case that the measure $\hat{m}_{\hat{p}}$ is a convex combination of not more that two Dirac masses, but the following result claim that in fact supp $\hat{m}_{\hat{p}} \subset \{a, r\}$. The full proof is done in [22].

Claim 8. *The support of $\hat{m}_{\hat{p}}$ is contained in $\{a, r\}$.*

Idea of the proof. Suppose that $\hat{m}_{\hat{p}}$ is not supported in $\{a, r\}$, so we can choose a compact set $K \subset \tilde{\Lambda}_{j,\hat{p}}$ in the fundamental domain of $\hat{A}^{n_{\hat{p}}}$ with positive $\hat{m}_{\hat{p}}$-measure. Choose a point $\hat{q} \in V_{\hat{p}}$ satisfying Equation (9.2) close enough to $\hat{p}$. We can transform K using the holonomies into a set $K_{\hat{q}}$ in the projective fiber of $\hat{q}$ such that $\hat{m}_{\hat{p}}(K) \approx \hat{m}_{\hat{q}}|_{\Lambda_{j,\hat{q}}}(K_{\hat{q}})$.

Taking $\hat{m}_{\hat{q}}(\tilde{\Lambda}_{k,\hat{q}}) = \alpha_k$ and $l \geq 1$. As

$$(\hat{m}_{\hat{p}}|_{\tilde{\Lambda}_{k,\hat{p}}})(\mathbb{P}(F_{\hat{A}})^{-n_{\hat{p}}l}(K)) \approx (\hat{m}_{\hat{q}}|_{\tilde{\Lambda}_{k,\hat{q}}})(\mathbb{P}(F_{\hat{A}})^{-n_{\hat{p}}l}(K_{\hat{q}})) \geq \hat{m}_{\hat{p}}(K) + \alpha_k - 1,$$

making k go to infinity, $\alpha_k \to 1$ and we obtain that the measure of K grows as we iterate $\mathbb{P}(F_{\hat{A}})^{n_{\hat{p}}l}$ backwards, that is $\hat{m}_{\hat{p}}(\mathbb{P}(F_{\hat{A}})^{-n_{\hat{p}}l}(K)) \geq \hat{m}_{\hat{p}}(K) > 0$. Since those sets are pairwise disjoint, it follows that $\hat{m}_{\hat{p}}$ is an infinite measure, which is a contradiction. $\square$

By Claim 8, $\hat{m}_{\hat{p}}$ is a convex combination of not more than two Dirac masses. Then, because of the previous definition of the disintegration, and by continuity of the map $\hat{z} \mapsto \hat{m}_{\hat{z}}|_{\Lambda_{j,\hat{z}}}$, the same is true about $\hat{m}_{\hat{z}}$ for every $\hat{z} \in V_{\hat{p}}$. As $\hat{m} = \hat{m}^u$ we have that $\hat{m}_{\hat{z}} = \delta_{\mathbb{P}(E^u_{\hat{z}})}$ for almost every $\hat{z} \in V_{\hat{p}}$, which implies that in fact $\hat{m}_{\hat{z}}$ is a Dirac mass for every $\hat{z}$.

The same argument shows that for any point $\hat{y} \in \hat{M}$ there exists a continuous disintegration $\{\hat{m}_{\hat{y},\hat{z}} \colon \hat{z} \in V_{\hat{y}}\}$ of $\hat{m}$. Since disintegrations are essentially unique and the neighbourhoods $V_{\hat{y}}$ overlap on positive $\hat{\mu}$-measure subsets, all these conditional measures most be also Dirac masses. Thus, the map $\xi(\hat{z}) = \operatorname{supp} \hat{m}_{\hat{z}}$ defines a continuous map.

Finally as $\hat{m} = \hat{m}^u$ is an s-state we obtain that ξ satisfies that $h^s_{\hat{x},\hat{y}}\xi(\hat{x}) = \xi(\hat{y})$, and then is an invariant section, concluding the proof of Theorem C. $\square$

Bibliography

[1] A. Avila, A. Eskin, M. Viana. Continuity of Lyapunov exponents of random matrix products. In preparation.

[2] A. Avila, M. Viana. Extremal Lyapunov exponents: an invariance principle and applications. *Inventiones mathematicae* **181**, (2010) 115-178.

[3] A. Avila, J. Santamaria, M. Viana. Holonomy invariance: rough regularity and applications to Lyapunov exponents. *Astérisque* **358**, (2013) 13-74.

[4] L. Backes, A. Brown, C. Butler. Continuity of Lyapunov exponents for cocycles with invariant holonomies. *Journal of Modern Dynamics* **12**, (2018) 223-260.

[5] J. Bochi. Genericity of zero Lyapunov exponents. *Ergodic Theory and Dynamical Systems* **22**, (2002) 1667-1696.

[6] C. Bocker-Neto, M. Viana. Continuity of Lyapunov exponents for random $2d$ matrices. *Ergodic Theory and Dynamical Systems* **37**, (2017) 1413-1442.

[7] C. Bonatti, X. Gómez-Mont, M. Viana. Généricité d' exposants de Lyapunov non-nuls pour des produits déterministes de matrices. *Annales de l'Institut Henri Poincaré C, Analyse Non Linéaire* **20**, (2003) 579-624.

[8] C. Bonatti, M. Viana. Lyapunov exponents with multiplicity 1 for deterministic products of matrices. *Ergodic Theory and Dynamical Systems* **24**, (2004) 1295-1330.

[9] R. Bowen. Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms. *Lectures Notes in Mathematics* **470**, Springer (1975) 90-107.

[10] C. Butler. Discontinuity of Lyapunov exponents near fiber-bunched cocycles. *Ergodic Theory and Dynamical Systems* **38**, (2018) 523-539.

[11] H. Furstenberg, H. Kesten. Products of random matrices. *Annals of mathematics* **31**, (1960) 457-469.

[12] B. Kalinin. Livšic theorem for matrix cocycles. *Annals of mathematics* **173**, (2011) 1025-1042.

[13] B. Kalinin, V. Sadovskaya. Holonomy and cohomology for cocycles over partially hyperbolic diffeomorphisms. *Discrete and Continuous Dynamical Systems* **36**, (2016) 245-259.

[14] F. Ledrappier. Positivity of the exponent for stationary sequences of matrices. *Lyapunov exponents* **1186**, (1986) 56-73.

[15] E. C. Malheiro, M. Viana. Lyapunov exponents of linear cocycles over markov shifts. *Stochastics and Dynamics* **15**, (2015) 1-27.

[16] V. I. Oseledets. A multiplicative ergodic theorem: Lyapunov characteristic numbers for dynamical systems. *Transactions of the Moscow Mathematical Society* **19**, (1968) 197-231.

[17] Ya. Pesin. Characteristic Lyapunov exponents and smooth ergodic theory. *Uspekhi Matematicheskikh Nauk* **32**, (1977) 55-114.

[18] M. Poletti. Stably positive Lyapunov exponents for symplectic linear cocycles over partially hyperbolic diffeomorphisms. *Discrete and Continuous Dynamical Systems* **38**, (2018) 5163-5188.

[19] M. Viana. Almost all cocycles over any hyperbolic system have non-vanishing exponents. *Annals of mathematics* **167**, (2008) 643-680.

[20] M. Viana. Lectures on Lyapunov exponents. *Cambridge University Press* **145** (2014).

[21] M. Viana. (Dis)continuity of Lyapunov exponents. *Ergodic Theory and Dynamical Systems*, (2018) 1-35.

[22] M. Viana, J. Yang. Continuity of Lyapunov exponents in the C^0 topology. *Israel Journal of Mathematics* **229**, no.1, (2019) 461-485.