Serrin type solvability criteria for Dirichlet problems for prescribed mean curvature equations in manifolds

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We establish new non-existence and existence results of solutions for the Dirichlet problem for the prescribed mean curvature vertical equation in $M \times \mathbb{R}$, where M is a complete Riemannian manifold of dimension n. It is well known that a necessary condition for the solvability of this problem in bounded domains of $\mathbb{R}^n$ with certain regularity is the *Serrin condition*. We investigated this fact in the product $M \times \mathbb{R}$ considering a large class of Riemannian manifolds within which are the Hadamard manifolds and manifolds whose sectional curvature is bounded above by a positive constant. Precisely, given a bounded domain Ω in M and a function $H = H(x, z)$ continuous in $\overline{\Omega} \times \mathbb{R}$ and non-decreasing in the variable z, we show that if the *strong Serrin condition* $(n-1)\mathcal{H}_{\partial\Omega}(y) \geq n \sup\limits_{z\in\mathbb{R}} |H(y, z)| \ \forall\, y \in \partial\Omega$ fails, then there exists boundary data for which the aforementioned problem has no possible solution. Considering that $H \in \mathcal{C}^{1,\alpha}(\overline{\Omega}\times\mathbb{R})$, we establish an existence result if H and its Riemannian gradient satisfy a relation involving the Ricci curvature of M, in addition to the strong Serrin condition. We also establish an existence result in the case where $M = \mathbb{H}^n$ and $\sup\limits_{\Omega\times\mathbb{R}} |H(x, z)| \leq \frac{n-1}{n}$. Our results generalize classical results of Jenkins-Serrin and of Serrin for the Euclidean ambient space, as well as Spruck's results in the $M \times \mathbb{R}$ setting. For instance, we conclude that *if* $\Omega \subset \mathbb{H}^n$ *is a bounded domain whose boundary is of class* $\mathcal{C}^{2,\alpha}$*, then the Dirichlet problem for the vertical mean curvature equation in* $\mathbb{H}^n \times \mathbb{R}$ *has a unique solution for every constant* H *and for arbitrary continuous boundary data if, and only if,* $(n-1)\mathcal{H}_{\partial\Omega} \geq n\,|H|$ *in* $\partial\Omega$.