On the Dulac's Problem for Tangential Polycycles

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The study of piecewise smooth vector fields has become a common frontier between Mathematics, Physics, Engineering and Life Sciences. In short, a non-smooth vector field in an open set $U \subset \mathbb{R}^2$ is a vector field which is piecewise defined in disjoint open regions of U separated by codimension one curves, called switching manifolds, where the union of these regions and curves is equal to U, [5, 6]. The study of cycles for piecewise smooth systems has several open problems and interesting phenomena can happen even in the simplest cases: considering two open regions separated by a straight line.

The well known Hilbert's 16^{th} problem gave rise to a lot of works and it has motivated many researchers. A first step towards the solution of this problem is to to prove that a polynomial vector field on $\mathbb{R}^2$ has at most a finite number of limit cycles. This finiteness question can be reduced to the problem of non-accumulation of limit cycles for a polynomial vector field, called *Dulac's Problem* (Dulac gave an incomplete proof for it):

- *A elementary polycycle of an analytic vector field X cannot have limit cycles accumulating onto it.*

An elementary polycycle is a closed oriented curve formed by a finite union of regular orbits and elementary singular points of X. A correct proof for the finiteness question was given for quadratic vector fields by Bamon [3] and complete proofs of the finiteness result were obtained independently by Ecalle [4] and Il'Yashenko [7].

The main objective of this work is putting together the Dulac's problem and the study of typical tangential polycycles for piecewise analytic systems, a similar approach can be found in [1]. We consider planar piecewise analytic systems presenting connections between tangential singularities and investigate a version of the Dulac's Problem for this kind of tangential polycycles.

Consider a smooth embedded submanifold $\Sigma = h^{-1}(0)$ where $h : \mathbb{R}^2 \to \mathbb{R}$ is a smooth function for which 0 is a regular value. In this way, Σ splits $\mathbb{R}^2$ in two open regions

$$\Sigma^+ = \{p \in \mathbb{R}^2; h(p) > 0\} \quad \text{and} \quad \Sigma^- = \{p \in \mathbb{R}^2; h(p) < 0\}.$$

A piecewise analytic vector field in $\mathbb{R}^2$ is a vector field of the form

$$Z(p) = \begin{cases} X(p), & p \in \Sigma^+, \\ Y(p), & p \in \Sigma^-, \end{cases}$$

where X and Y are analytic vector fields in $\mathbb{R}^2$. Denote by Ω^ω the set of all piecewise analytic vector fields defined as above. In Σ the following regions are distinguished

- Crossing region: $\Sigma^c = \{p \in \Sigma;\ Xh \cdot Yh(p) > 0\}$,
- Sliding region: $\Sigma^s = \{p \in \Sigma;\ Xh(p) < 0 \text{ and } Yh(p) > 0\}$,
- Escaping region: $\Sigma^e = \{p \in \Sigma;\ Xh(p) > 0 \text{ and } Yh(p) < 0\}$,

where $Xh(p) = \langle X, \nabla h\rangle(p)$ is the Lie derivative of h, at p, in the direction of X, analogously for $Yh(p)$. Trajectories of Z follow the Filippov convention, see [5, 6]. Consider $Z = (X, Y) \in \Omega^\omega$, a point $p \in \Sigma$ is said to be a tangential singularity of order k (k a positive integer) of X (resp. of Y) if $Xh(p) = \cdots = X^{k-1}h(p) = 0$ and $X^k h(p) \neq 0$ (resp. $Yh(p) = \cdots = Y^{k-1}h(p) = 0$ and $Y^k h(p) \neq 0$). In this work, [2], we prove the following result:

Theorem A. *A tangential polycycle of a piecewise analytic vector field $Z = (X, Y) \in \Omega^\omega$ cannot have limit cycles accumulating onto it.*

References

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