Moving Planes Method

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Abstract

As a first beautiful illustration of the power of the Maximum Principle, we will prove the following symmetry theorem, which plays an important role in the study of nonlinear elliptic and parabolic PDE.

Introduction

The method of moving planes is used in proving symmetry, in fact, x_1 direction for solutions of nonlinear elliptic equation $F(x, u, Du, D^2u) = 0$ in a bounded domain Ω in $\mathbf{R}^n$ which is convex in the x_1 direction.

The essential ingredient in their use is the Maximum principle. In this method one point is the reflection of the other in a hyperplane $\{x_1 = \lambda\}$, and then, the plane is moved up to a critical position.

We choose a very simple example to illustrate such a method. The following result was first proved by Gidas, Ni and Nirenberg.

Theorem.1. Suppose $u \in C(B_1) \cap C^2(B_1)$ is a positive solution of

$$\begin{aligned} \triangle u + f(u) = 0 \quad &\text{in} \quad B_1 \\ u = 0 \quad &\text{on} \quad \partial B_1 \end{aligned}$$

where f is locally Lipschitz in $\mathbf{R}$. Then u is radially symmetric in B_1 and $\frac{\partial u}{\partial r}(x) < 0$ for $x \neq 0$.

The original proof requires that solutions be C^2 up to the boundary. Here we give a method which does not depend on the smoothness of domains nor the smoothness of solutions up to the boundary.

Lemma.1. Suppose that Ω is a bounded domain which is convex in the x_1 direction and symmetric which respect to the plane $\{x_1 = 0\}$. Suppose $u \in C(B_1) \cap C^2(B_1)$ is a positive solution of

$$\begin{aligned} \triangle u + f(u) &= 0 \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial\Omega \end{aligned}$$

where f is locally Lipschitz in $\mathbf{R}$. Then u is symmetric with respect ti x_1 and $D_{x_1}u(x) < 0$ for any $x \in \Omega$ with $x_1 > 0$.

Observation 1. This result is also valid if we replace the Laplacian by an operator invariant with respect to reflections in the x_1 direction.

Observation 2. Applying Lemma.1. in all directions we obtain Theorem 1.

References

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