On Mahler's U_m-numbers

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The genesis of transcendental number theory, took place in 1844 with Liouville's result on the "bad" approximation of algebraic numbers by rationals. More precisely, if α is an algebraic number of degree $n > 1$, then there exists a positive constant C, such that

$$\left|\alpha - \frac{p}{q}\right| > \frac{C}{q^n},$$

for all $p/q \in \mathbb{Q}^*$. Using this remarkable fact, he was able to build a non-enumerable set of transcendental numbers called *Liouville numbers*, denoted by $\mathbb{L}$. One curious fact about this set, proved by Paul Erdös, is that every real number can be writen as the sum and the product of two Liouville numbers.

Since then, several classifications of transcendental numbers have been developed, one of them proposed by Kurt Mahler in 1932. He splited the set of transcendental numbers on three disjoint sets: $S-$, $T-$ and $U-$numbers. In a certain sense, $U-$numbers generalize the idea of Liouville numbers. Yet, the set of $U-$numbers can be splited into U_m-numbers, which are those "rapidly" approximable by algebraic numbers of degree m.

On this work, made in cooperation with D. Marques, we prove the following result:

Theorem: Let $\vartheta : \mathbb{N} \to \mathbb{N}$, such that $\omega_n := \vartheta(n+1)/\vartheta(n) \to \infty$, as $n \to \infty$. Let $\xi \in \mathbb{R}$, such that there exists an infinite sequence of rational numbers $(p_n/q_n)_n$, satisfying

$$\left|\xi - \frac{p_n}{q_n}\right| < H\left(\frac{p_n}{q_n}\right)^{-\vartheta(n)},$$

where $H(p_{n+1}/q_{n+1}) \leq H(p_n/q_n)^{O(\vartheta(n))}$. Now take $\alpha_0, \ldots, \alpha_l, \beta_0, \ldots, \beta_r \in \overline{\mathbb{Q}}$, with $\beta_1 = 1$ and $\alpha_l \neq 0$, such that $[\mathbb{Q}(\alpha_0, \ldots, \alpha_l, \beta_0, \ldots, \beta_r) : \mathbb{Q}] = m$. Then, for $P(z), Q(z) \in \overline{\mathbb{Q}}[z]$, given by $P(z) = \alpha_0 +$

$\alpha_1 z + \cdots + \alpha_l z^l$ and $Q(z) = \beta_0 + \beta_1 z + \cdots + \beta_r z^r l$, $P(\xi)/Q(\xi)$ is a U_m-number.

In the previous statement, for $\alpha \in \overline{\mathbb{Q}}$, $H(\alpha)$ is the usual height of α, i.e., if $P_\alpha(z) \in \mathbb{Z}[z]$ is the minimal polynomial of α, given by $P_\alpha(z) = c_0 + c_1 z + \cdots + c_n z^n$, then $H(\alpha) = H(P) = \max_{0 \le i \le n}\{|c_i|\}$.

References

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