Spectral Representation of the Covariance Function

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We are interested in stationary random processes and their applications to modelling and predicting data collected sequentially in time. The initial aim is to understand rigorously some ingredients, like the spectral representation of the covariance function (Theorem Herglotz). Informally, the covariance function of every stationary (wide sense) random sequence can be represented by one distribution function F, called the spectral distribuition function, (up to normalization), whose support is concentrated on $[-\pi, \pi)$. We illustrate with some classical examples.

Now, let's see in more precise details what's the purpose of this work.

Definition 1. *A sequence of complex random variables $\xi = (\xi_n)_{n\in\mathbb{Z}}$ with $E(\xi_n)^2 < \infty, n \in \mathbb{Z}$, is stationary in wide sense, if, for all $n \in \mathbb{Z}$,*

$$E(\xi_n) = E(\xi_0) \text{ and } cov(\xi_{k+n}, \xi_k) = cov(\xi_n, \xi_0), k \in \mathbb{Z}$$

Definition 2. *We call the covariance function the number $R(n)$ defined below*

$$R(n) = cov(\xi_n, \xi_0), n \in \mathbb{Z}.$$

The main result of this work is the following.

Theorem of Herglotz 1. *Let $R(n)$ be the covariance function of a stationary random sequence with zero mean. Then, there is, on $([-\pi, \pi), \mathbb{B}([-\pi, \pi)))$, a finite measure $F = F(B), B \in \mathbb{B}([-\pi, \pi))$, such that for every $n \in \mathbb{Z}$,*

$$R(n) = \int_{\pi}^{-\pi} e^{i\lambda n} F(d\lambda).$$

These definitions and the theorem lead us to some classical examples which are those presented below.

Example 1. *(White noise) Let* $\xi = (\xi_n)$ *be an orthonormal sequence of random variables,* $E(\xi_n) = 0, E(\xi_i \overline{\xi_j}) = \delta_{ij}$. *Such a sequence is evidently stationary, and*

$$R(n) = \begin{cases} 1, n = 0 \\ 0, n \neq 0 \end{cases}$$

Observe that $R(n)$ *can be represented in the form* $R(n) = \int_{\pi}^{-\pi} e^{i\lambda n} dF(\lambda)$ *where*

$$F(\lambda) = \int_{\lambda}^{-\pi} f(v)dv;\ f(\lambda) = \tfrac{1}{2\pi},\ -\pi \leq \lambda < \pi.$$

A very wide class of stationary processes can be generated by using white noise as the forcing terms in a set of linear difference equations. This leads to the notion of an autoregressive-moving average (ARMA) process.

Example 2. *(ARMA) Let* $\xi = (\xi_n)$ *be the white noise introduced before. This random sequence is autoregressive model of order q if*

$$\xi_n + b_1\xi_{n-1} + \ldots + b_q\xi_{n-q} = \xi_n.$$

If, instead of that, we have

$$\xi_n = a_0\xi_n + a_1\xi_{n-1} + \ldots + a_p\xi_{n-p},$$

then $\xi = (\xi_n)$ *is a moving average of order p. Now, suppose we have*

$$\xi_n + b_1\xi_{n-1} + \ldots + b_q\xi_{n-q} = a_0\xi_n + a_1\xi_{n-1} + \ldots + a_p\xi_{n-p},$$

then we obtain a mixed model called autorregression and moving average of order (p,q). Under some hypothees, these sequence has the stationary solution $\xi = (\xi_n)$ *for which the covariance function is* $R(n) = \int_{\pi}^{-\pi} e^{i\lambda n} dF(\lambda)$ *with* $F(\lambda) = \int_{\lambda}^{-\pi} f(v)dv$, *where*

$$f(\lambda) = \tfrac{1}{2\pi} \left| \frac{P(e^{-i\lambda})}{Q(e^{-i\lambda})} \right|^2.$$

Given a sequence of random variables, the ARMA model is a tool for understanding and predicting the future values in this sequence. The autorregressive part involves regressing the variable on its own past values. The moving average part involves modeling the error term as a linear combination of error terms occurring contemporaneously and at various times in the past.

It's important to affirm that this work has further purposes in the study of time-series modelling. Our biggest goal is to rigorously justify the use of new estimators for time-series modelling. For example, we want to extend the understanding of properties and the asymptotic behavior of robust estimation quoted in [5].

References

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