On Mahler's U_m-numbers

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The genesis of transcendental number theory, took place in 1844 with Liouville's result on the "bad" approximation of algebraic numbers by rationals. More precisely, if α is an algebraic number of degree $n > 1$, then there exists a positive constant C, such that $|\alpha - p/q| > Cq^{-n}$, for all $p/q \in \mathbb{Q}^*$. Using this remarkable fact, he was able to build a non-enumerable set of transcendental numbers called *Liouville numbers*. Since then, several classifications of transcendental numbers have been developed, one of them proposed by Kurt Mahler in 1932. He splited the set of transcendental numbers on three disjoint sets:$S-$,$T-$ and $U-$numbers. In a certain sense, $U-$numbers generalize the concept of Liouville numbers. Yet, the set of $U-$numbers can be splited into U_m-numbers, that are numbers "rapidly" approximable by algebraic numbers of degree m.

On this series of lectures, the following result, made in cooperation with D. Marques, will be proved:

Theorem: Let $\vartheta : \mathbb{N} \to \mathbb{N}$, such that $\omega_n := \vartheta(n+1)/\vartheta(n) \to \infty$, as $n \to \infty$. Let $\xi \in \mathbb{R}$, such that there exists an infinite sequence of rational numbers $(p_n/q_n)_n$, satisfying

$$\left|\xi - \frac{p_n}{q_n}\right| < H\left(\frac{p_n}{q_n}\right)^{-\vartheta(n)},$$

where $H(p_{n+1}/q_{n+1}) \leq H(p_n/q_n)^{O(\vartheta(n))}$.

Now, take $\alpha_0, \ldots, \alpha_l, \beta_0, \ldots, \beta_r \in \overline{\mathbb{Q}}$, with $\beta_r = 1$ and $\alpha_l \neq 0$, such that $[\mathbb{Q}(\alpha_0, \ldots, \alpha_l, \beta_0, \ldots, \beta_r) : \mathbb{Q}] = m$. Then, for $P(z), Q(z) \in \overline{\mathbb{Q}}[z]$, given by $P(z) = \alpha_0 + \alpha_1 z + \cdots + \alpha_l z^l$ and $Q(z) = \beta_0 + \beta_1 z + \cdots + \beta_r z^r$, $P(\xi)/Q(\xi)$ is a U_m-number.

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