Faithfully Quadratic Rings

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I'll describe the main features of an extension of the classical algebraic theory of quadratic from fields to a broad class of (commutative, unitary) rings; full development appears in [1]. This extension applies to:

- Diagonal forms with invertible coefficients [1].
- Rings where 2 is invertible, endowed with a proper preorder (in particular, -1 is not a sum of squares).

The extension is achieved by:

- Extending the notion of matrix isometry of quadratic forms to a suitable notion of T**-isometry**, where T is a preorder of the base ring A, or $T = A^2$, and
- Introducing in this context three axioms expressing simple properties of (value) representation of ring elements by quadratic forms, well-known to hold in the field case.

Under these axioms the ring-theoretic approach based on T-isometry coincides with the formal, abstract approach formulated in terms of reduced special groups, cf. [2]. Preordered rings $< A, T >$ satisfying these axioms are called T**-faithfully quadratic**.

In [1] we prove that the following classes of preordered rings (among others) are T-faithfully quadratic:

- Rings with many units satisfying a mild restriction (for all preorders T).
- Reduced f-rings (for any preorder T containing the underlying partial order of A). An outstanding class of examples of this type are the rings of continuous real-valued functions on a topological space.
- Strongly representable rings (i.e., bounded inversion preordered rings $< A, T >$ with an Archimedean preorder T).

For all these classes we can determine the reduced special group $G_T(A)$ canonically associated to $< A, T >$.

Most of the structural properties involving quadratic forms known to hold for fields extend to (T)-faithfully quadratic rings.

References

[1] M. Dickmann, F. Miraglia, *Faithfully Quadratic Rings*, Memoirs AMS 1128 (2015), xi + 129 pp.

[2] M. Dickmann, F. Miraglia, *Special Groups*, Memoirs AMS 689 (2000), xvi + 247 pp.

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[1] The case of diagonal forms with non-zero divisor coefficients will be treated by F. Miraglia later in this meeting.