Focused proof systems for geometric theories

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In the past years, there has been a great effort in order to systematically add non-logical axioms to first order logics, so that standard proof-theoretical results obtained for pure logic could be extended to the case of a certain class of mathematical theories. The main challenge is to determine a general procedure that guarantees that extensions are *smooth*, in the sense that good proof-theoretical properties, such as analyticity, are preserved.

Both semantic and syntactic approaches have been proposed for such a task. As an example for the first approach, in [4], Kracht establishes a correspondence between special rules and modal axioms in the setting of tense modal logic, result further generalized in [3] for logics with algebraic semantics based on bounded distributive lattices. On the other hand, Ciabattoni et al. [1] introduce a systematic procedure for transforming large classes of (Hilbert) axioms into equivalent structural inference rules in sequent and hypersequent substructural calculi. Analyticity for hypersequent calculi is proved semantically. Interestingly enough, both works use Ackermann lemma based algorithms in order to uniformly construct rules out of axioms. Considering syntactic methods only, in a series of works [7, 5, 6, 2, 8], Negri et al. introduce a general method for the representation of axioms as rules of inference of a suitable form. Their method is restricted to a certain form of axioms, called *geometric*, and it consists on converting simple conjuncts of formulas into their well known rule representations, so that that the logical content of the axiom is replaced by the meta-linguistic meaning of sequent rules. It should be noted that the method assumes some artificial conditions in order to assure the admissibility of contraction and, unlike Ciabattoni's work, it does not explore axioms falling outside the class of geometric axioms.

In this work, we strive at combining the rigor of the classification of axioms into a polarities' hierarchy with a systematic proof-theoretical construction of inference rules from axioms, using *focusing*. We show that, in fact, focusing *justifies* the inclusion of a class of axioms that *subsumes* the geometric ones, also shedding some light on the behavior of formulas falling outside this class. Moreover, our approach enables a uniform presentation for classical and intuitionistic first order systems.

References

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