Instituto Nacional de Matemática Pura e Aplicada

Stable Intersections of Conformal Cantor sets in the complex plane and homoclinic bifurcations.

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Thesis presented to IMPA,
Instituto Nacional de Matemática Pura e Aplicada,
as partial fulfilment of requirements
for the degree of Doctor in Mathematics.

Rio de Janeiro, February of 2018

Abstract

The concept of stable intersection between Cantor sets in the real line is central in the study of arithmetic properties of Diophantine approximations of irrational numbers, via Markov and Lagrange spectrum; and in the study of the border of regions of stability, especially in regions with the presence of the Newhouse phenomenon. The techniques developed by C.G.T.A Moreira and J.C. Yoccoz establish a dichotomy for generic pairs of Cantor sets: the sum of their Hausdorff dimension is less than one, or else, less a translation, a stable intersection exists between them.

In this thesis we deal with the concept of stable intersections of conformal Cantor Sets dynamically defined in the complex plane. First, we show the relationship of this concept with horseshoes that appear in the context of automorphisms of two-dimensional complex space. Then, following the above work, we extend the concept of recurrent compact, proving that its existence is a sufficient criterion for the intersection of stability of certain relative positions of a pair of Cantor sets.

Finally, we use the techniques developed in two results. The first is that for certain generic families of automorphisms of the two-dimensional complex space displaying a horseshoe unfolding a tangency, if there is a stable intersection between the linearized versions of unstable and stable Cantor sets, then there is positive density of tangencies at the zero parameter. The second is to show that for a horseshoe example given by Buzzard, it is possible to construct a compact recurrent set for the pair of Cantor sets representing the unstable and stable Cantor sets.

Keywords: Cantor set; stable intersection; horseshoe

Resumo

O conceito de interseção estável de conjuntos de Cantor na linha real é central no estudo de propriedades aritméticas de aproximações diofantinas de números irracionais, via espectro de Markov e de Lagrange; e no estudo de regiões na fronteira de estabilidade, principalmente em relação a regiões com presença do Fenômeno de Newhouse. As técnicas desenvolvidas por C.G.T.A Moreira e J.C. Yoccoz permitiram estabelecer uma dicotomia: para pares genéricos de Conjuntos de Cantor, ou a soma de suas dimensões de Hausdorff é menor do que um ou, a menos de uma translação, existe interseção estável entre eles.

Nesta tese lidamos com o conceito de interseções estáveis de Conjuntos de Cantor dinamicamente definidos conformes no plano complexo. Primeiramente, demonstramos a relação desse conceito com ferraduras que aparecem no contexto de automorfismos do espaço complexo bidimensional. Depois, seguindo os trabalhos supracitados, estendemos o conceito de compacto recorrente, provando que a existência deste é um critério suficiente para a estabilidade de interseção de certas posições relativas de um par de conjuntos de Cantor.

Por fim, utilizamos as técnicas desenvolvidas em dois resultados. O primeiro é que para certas famílias genéricas de automorfismos do espaço complexo bidimensional exibindo uma ferradura que desdobram uma tangência, se existir interseção estável entre as versões linearizadas dos conjuntos de Cantor instável e estável, então há densidade positiva no parâmetro zero de tangências. O segundo é mostrar que para, um exemplo de ferradura dado por Buzzard, é possível construir um compacto recorrente para o par de conjuntos de Cantor que representa o Cantor instável e o estável.

Palavras-chave: conjunto de Cantor; interseção estável; ferradura

Acknowledgements

I would like to thank first my advisor Carlos "Gugu" Tamm de Araújo Moreira. His mathematical brilliance is only matched by his unending generosity and kindness, as well as his goal scoring abilities. I would like to thank him especially for the patience and the right words of encouragement when they were needed.

I would like to thank my family for the support they have always given me since the very beginning of my interest in mathematics, especially my mother Maria Angélica and my father Itamar. Also my sister Laís and my uncles Luiz Antonio da Fonseca Manso and João Edison Minnicelli.

Also my colleagues and friends from IMPA, Puc-Rio and life, which are many. So to name a few I should absolutely not forget Mateus Sousa, Mauricio Poletti, Alex Zamudio, Davi Lima, Renan Finder, Gregory Cosac, Leonardo Fontoura, Felipe Mello, Deborah Alves, Marina Lima, Bárbara Fortes, Leandro Cruz, Eduardo Garcez, Jaime Rocha, Wagner Ranter, Luiz Paulo, Daniel Ungaretti and Ravi Ramos. I achieved this degree with a little help from all of you (and some others).

Of course, all the teachers I had in my life have some part in this course I took through life, notably (and not cited yet) Luciano Castro, Yuri Lima, Samuel Barbosa, Nicolau Saldanha, Flávio Abdenur and Enrique Pujals.

I need to express my gratitude to the IMPA staff, they were always very helpful and attentive: Josenildo, Kênia, Isabel and Andréia.

Finally, I thank CNPq for the financial support. May CNPq and other similar institutions be able to continue their much needed work of fomenting scientific research in Brazil.

"Ce qui est admirable, ce n'est pas que le champ des étoiles soit si vaste, c'est que l'Homme l'ait mesuré." Anatole France

Contents

Chapter 1

Introduction

The theory of regular Cantor sets on the real line has played a central role in the study of homoclinic bifurcations. This interest began with the works of Newhouse [1], [2] and [3], where he showed that, in the space $Diff(\mathcal{M}^2)$, there are always regions of persistence of tangencies arbitrarily close to a C^2 diffeomorphism displaying a homoclinic tangency. There, he basically associated the presence of a tangency between the stable and unstable laminations of a horseshoe to a intersection between two Cantor sets and developed a sufficient criterion for such phenomenon: the *gap lemma*. Precisely, he defined $\tau(K)$, the thickness of a Cantor set K, a positive real number associated to the geometry of the gaps of K and showed that if the product $\tau(K_1) \cdot \tau(K_2)$ is larger than one for a pair of Cantor sets K_1 and K_2, then $K_1 + K_2$ has non-empty interior. These regions, of persistence of tangencies, display really complex dynamical behaviour, and in the case of the homoclinic tangency of a dissipative periodic point, it is possible to find a residual set of diffeomorphisms with coexistence of infinitely many sinks; the so called *Newhouse phenomenon*.

Later, Palis and Takens [4] proved a theorem that assured full density of hiperbolicity on a parameter family that generic unfolds a homoclinic tangency, provided that the Hausdorff dimension of the basic set is less than one. This hypothesis is related to the fact that if K_1 and K_2 are Cantor sets whose sum of Hausdoff dimensions is less than one then $K_1 - K_2$ has Lebesgue measure zero. These advances led Palis to conjecture that, for generic pairs of Cantor sets K_1 and K_2, $K_1 - K_2$ contains an interval or has Lebesgue measure zero.

Palis conjecture was proven true by Moreira and Yocozz [5], showing that for generic pairs K_1, K_2 of Cantor sets whose sum of Hausdoff dimension is larger than one there is a real number t such that K_1 and $K_2 + t$ have a stable intersection, a concept that was invented by Moreira [7], and which implies in particular that $K_2 - K_1$ contains an interval around t. To obtain this result, they introduced the recurrent compact criterion and limit geometries, two concepts that enable one to control the relative positioning of intervals of advanced steps of the constructions of K_1 and K_2.

In this thesis we extend these techniques to the complex plane setting and show that the recurrent compact criterion is a sufficient condition for the stability of intersection between two Cantor sets in $\mathbb{C}$ belonging to a specific class of Cantor sets, called dynamically defined Cantor sets. We later prove that it represents Cantor sets that arise naturally in the context of automorphisms of $\mathbb{C}^2$, precisely intersections between a horseshoe with a compact part of the stable or unstable manifold of a fixed (or periodic) point for a map $H \in Aut(\mathbb{C}^2)$.

This definition is inspired in the definition for regular Cantor sets in the real line and, mostly, in the definition of complex Cantor sets by Zamudio [9]. In fact, every complex Cantor set is a dynamically defined conformal Cantor set in the complex plane, but the converse is not true. The defining feature of a conformal Cantor set K is that the C^r, $r > 1$ map that defines it (as its maximal invariant set) is conformal on the Cantor set itself, whereas in [9] they are holomorphic in a neighbourhood of the Cantor set.

The concept of stable intersection is very similar to the one in the real line setting. We say that two Cantor sets K and K' are close to each other when the maps defining them are close to each other and so are the connected pieces of their domains $G(a)$ and $G'(a)$ (see the first section for details). Also, we define a configuration of a piece $G(a)$ as a $C^{1+\varepsilon}$ embedding $h : G(a) \to \mathbb{C}$ (see section 3.2 for details). That way, given a pair of configurations (h, h'), also referred to as a configuration pair or a relative positioning, we say it is stably intersecting whenever, for any pair $(\tilde{h}, \tilde{h}')$ close to (h, h') and any pair of Cantor sets $\tilde{K}, \tilde{K}'$ on a small neighbourhood of (K, K') the intersection between $\tilde{h}(\tilde{K})$ and $\tilde{h}'(\tilde{K}')$ is non-empty (see section 3.3 for further details).

The conformality is crucial to our interests, for if that was not the case, pieces of advanced steps of the construction of the Cantor set would be wildly distorted, something that would rule out our strategy to find stable intersections. Thus, we define renormalization operators that carry a configuration of a piece $G(a)$ to the corresponding configuration of an advanced step of the construction of the Cantor set. The presence of conformality causes quite the contrary: it allows us to construct maps called limit geometries, that work as a compact atractor (modulo affine transformations) in the space of configurations under the action of renormalization operators, implying that the shapes of advanced steps in the construction of the Cantor set are well behaved.

When these renormalization operators act on a configuration pair of Cantor sets K_1 and K_2, which can be seen as the relative positioning of K_1 and K_2 on $\mathbb{C}$, they are very tame with respect to the scale, a concept related to the ratio between the diameter of the images of configurations of pieces of both Cantor sets. However, they can be very wild on the translation part of it, a concept related to the distance between two base points on the configurations of K_1 and K_2. The presence of a *recurrent compact set of affine configurations of Cantor sets* $\mathcal{L}$ permits us to have a control over this last part, and guarantees stable intersection between pairs of configurations close to any pair of configurations representing an element $v \in \mathcal{L}$.

It is important to emphasize that some results are already known in the context of $Aut(\mathbb{C}^2)$. For example, a very important result [6] was the discovery of the Newhouse phenomenon close to a particular polynomial automorphism $H_0 \in Aut(\mathbb{C}^2)$ by Buzzard. His strategy was very similar to Newhouse's: finding a pair of Cantor sets related to the basic set of H_0 whose continuations still intersect in a neighbourhood of H_0. This result motivated us to find applications for the concepts developed, or more precisely extended in this thesis. Along these lines, the first application was naturally to check that the stable intersection of Buzzard's example could be obtained by the recurrent compact criterion. The other one, is an adaptation of a result of Moreira [7], inspired on the previously cited work of Palis and Takens [4]. It basically states that, for generic families of automorphisms, $\{H_\zeta\}_{\zeta \in \mathbb{D}}$, of $\mathbb{C}^2$ that unfold a homoclinic tangency of a fixed point p_0 of H_0, if there is a stable intersection between the linearized version of $W^u_{0,\mathrm{loc}}(p_0) \cap \Lambda_0$ and a translation of the linearized version of $W^s_{0,\mathrm{loc}}(p_0) \cap \Lambda_0$ then there is positive density of persistency of tangencies at the initial parameter. For the sake of clarification, Λ_ζ denotes the continuation of a horseshoe, as is usually denoted.

The text is organized as follows: in chapter one we define regular conformal Cantor sets and establish its relation to automorphisms of $\mathbb{C}^2$; while also proving some very basic properties and introducing the Buzzard's example of a horseshoe [6], in the second chapter we extend the concept of limit geometries and stable intersections to the context of conformal Cantor sets, and define the recurrent compact criterion, while also proving that it is a sufficient criterion for stability of the intersection; and in the final chapter we apply the techniques developed in this work to the two examples cited above.

Chapter 2

Dynamically defined conformal Cantor sets in the complex plane.

In this chapter we define the basic concept of a conformal regular Cantor set in the complex plane. The definition is made as closely as possibly to the definition of regular complex Cantor set by Zamudio in [9], but taking into consideration some limitations related to the maximum possible regularity of the unstable and stable foliations near Λ, a basic set of an automorphism of $\mathbb{C}^2$. The result is an extension of the ideas behind the construction in the real setting adapted to our context.

In the first section we give the definition itself and some basic properties closely related to it. Also, we define a topology for the space of all conformal Cantor sets of fixed type. In the second section we explore the properties of the unstable and stable foliations on a neighborhood of a basic set. The regularity of these foliations can be obtained using the theory of invariant manifolds developed by Hirsch, Pugh, Shub in ([10]). A C^1 - regularity result used by Buzzard in [6] can be found in Pixton [11] and the $C^{1+\varepsilon}$ regularity needed for this work was explained for surface diffeomorphisms by Palis and Takens in [15]. These proofs are complicated and left some details to the reader, so although very similar results may be found in the literature, we make a brief argumentation focusing on clarifying some details particular to our case. In the last section we show that for a basic set Λ with a saddle fixed point p the intersections between the basic set and compact parts of the stable or unstable manifolds of p can be seen as conformal Cantor sets, for some parametrization these manifolds, giving meaning to our definition.

Finally, we would like to observe that throughout this thesis we will identify a conformal operator over $\mathbb{R}^2$ with the operator over $\mathbb{C}$ given by a multiplication by a complex number. Precisely, if $A = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$, $\alpha, \beta \in \mathbb{R}$, we identify it with the operator over $\mathbb{C}$ given by the multiplication by $\alpha + \beta \cdot \mathrm{i}$.

2.1 Dynamically defined Conformal Cantor sets.

A map between two open sets of $\mathbb{C}$ is said to be C^r, $r \in \mathbb{R}$, when seen as a map between two open subsets of $\mathbb{R}^2$ it is C^r.

A dynamically defined Cantor set in $\mathbb{C}$ is given by the following data:

- A finite set $\mathbb{A}$ of letters and a set $B \subset \mathbb{A} \times \mathbb{A}$ of admissible pairs.
- For each $a \in \mathbb{A}$ a compact connected set $G(a) \subset \mathbb{C}$.
- A $C^{1+\varepsilon}$ map $g : V \to \mathbb{C}$ defined in an open neighbourhood V of $\bigsqcup_{a \in \mathbb{A}} G(a)$.

This data must verify the following assumptions:

- The sets $G(a)$, $a \in \mathbb{A}$ are pairwise disjoint
- $(a, b) \in B$ implies $G(b) \subset g(G(a))$, otherwise $G(b) \cap g(G(a)) = \emptyset$.
- For each $a \in \mathbb{A}$ the restriction $g|_{G(a)}$ can be extended to a $C^{1+\varepsilon}$ embedding (with $C^{1+\varepsilon}$ inverse) from an open neighborhood of $G(a)$ onto its image such that $m(Dg) > 1$, where $m(A) := \inf\limits_{v \in \mathbb{R}^2 \equiv \mathbb{C}} \frac{||Av||}{||v||}$, A being a linear operator on $\mathbb{R}^2$.
- The subshift (Σ, σ) induced by B, called the type of the Cantor set

$$\Sigma = \{\underline{a} = (a_0, a_1, a_2, \dots) \in \mathbb{A}^{\mathbb{N}} : (a_i, a_{i+1}) \in B, \forall i \geq 0\},$$

  $\sigma(a_0, a_1, a_2, \dots) = (a_1, a_2, a_3, \dots)$ is topologically mixing.

Once we have all this data we can define a Cantor set (i.e. a totally disconnected, perfect compact set) on the complex plane

$$K = \bigcap_{n \geq 0} g^{-n} \left(\bigsqcup_{a \in \mathbb{A}} G(a) \right)$$

We usually will write only K to represent all the data that defines a particular dynamically defined Cantor set. Of course, the compact set K can be described in multiple ways as a Cantor set constructed with the objects above, but whenever we say that K is a Cantor set we assume that one particular set of data as above is fixed. Also, when we are working with two Cantor sets K and K' we denote all the defining data related to the second accordingly. In other words, K' is given by a finite set $\mathbb{A}'$, a set B' of admissible pairs, a function g' defined on a neighbourhood of compact connected sets $G'(a')$, etc. We use the same convention for future objects that will be defined related to Cantor sets, such as limit geometries and configurations.

Main Defintion. *(Conformal regular Cantor set) We say that a regular Cantor set is conformal whenever the map g is conformal at the Cantor set K, that is, $\forall x \in K$, $Dg(x) : \mathbb{C} \equiv \mathbb{R}^2 \to \mathbb{C} \equiv \mathbb{R}^2$ is a linear transformation that preserves angles or, equivalently, a multiplication by a complex number.*

There is a natural topological conjugation between the dynamical systems $(K, g|_K)$ and (Σ, σ), the subshift Σ induced by B. It is given by a homeomorphism $h : K \to \Sigma$ that carries each point $x \in K$ to the sequence $\{a_n\}_{n\geq 0}$ that satisfies $g^n(x) \in G(a_n)$.

The definition above was made as closely as possible as the one in [9]. For example, all complex Cantor sets are conformal as described above. In interest of keeping this proximity, we also prove the following lemma:

Lemma 2.1.1. *Let K be a dynamically defined Cantor set, then there exist a family of sets $G^*(a) \subset \mathbb{C}$, for $a \in \mathbb{A}$, and $\mu > 1$ such that:*

1. *$G^*(a)$ is open and connected.*
2. *$G(a) \subset G^*(a)$ and $g|_{G(a)}$ can be extended to an open neighbourhood of $\overline{G^*(a)}$, such that it is a $C^{1+\varepsilon}$ embedding (with $C^{1+\varepsilon}$ inverse) from this neighbourhood to its image and $m(Dg) > \mu$.*
3. *The sets $\overline{G^*(a)}$, $a \in \mathbb{A}$ are pairwise disjoint*
4. *$(a, b) \in B$ implies $\overline{G^*(b)} \subset g(G^*(a))$, and $(a, b) \notin B$ implies $\overline{G^*(b)} \cap \overline{g(G^*(a))} = \emptyset$*

Proof. For a set G we define $V_\delta(G) = \{z \in \mathbb{C}, d(z, G) < \delta\}$. Choosing δ' small enough, $G^*(a) = V_{\delta'}(G(a))$ automatically satisfies $(i), (ii), (iii)$ and the second part of (iv). To get the first, fix $x \in G(a)$. Since $m(Dg) > \mu$ and g is invertible, there is a small δ_x such that $g(V_\delta(x)) \supset V_{\mu\delta}(g(x)), \forall \delta < \delta_x$. By compactness, there is a small δ, such that $V_\delta(G(a)) \subset \bigcup_{x\in G(a)}(V_{\delta_x}(x)$ for all $a \in \mathbb{A}$. Now, if $G(b) \subset g(G(a))$ (equivalently $(a,b) \in B$) and $x' \in \overline{V_\delta(G(b))}$ then there is $x \in g(G(a))$ with $d(x,x') \leq \delta$. But taking $\overline{x} = (g|_{G(a)})^{-1}(x)$ we have that $g(V_\delta(G(a))) \supset g(V_\delta(\overline{x})) \supset V_{\mu\delta}(x)) \ni x'$ so $\overline{G^*(b)} \subset g(G^*(a))$ as desired.

□

Associated to a Cantor set K we define the sets

$$\Sigma^{fin} = \{(a_0,\ldots,a_n) : (a_i,a_{i+1}) \in B\ \forall i, 0 \leq i < n\},$$

$$\Sigma^{-} = \{(\ldots,a_{-n},a_{-n+1},\ldots,a_{-1},a_0) : (a_{i-1},a_i) \in B\ \forall i \leq 0\}.$$

Given $\underline{a} = (a_0,\ldots,a_n)$, $\underline{b} = (b_0,\ldots,b_m)$, $\underline{\theta}^1 = (\ldots,\theta^1_{-2},\theta^1_{-1},\theta^1_0)$ and $\underline{\theta}^2 = (\ldots,\theta^2_{-2},\theta^2_{-1},\theta^2_0)$ we denote:

- if $a_n = b_0$, $\underline{ab} = (a_0,\ldots,a_n,b_1,\ldots,b_m)$
- if $\theta^1_0 = a_0$, $\underline{\theta^1 a} = (\ldots,\theta^1_{-2},\theta^1_{-1},a_0,\ldots,a_n)$
- if $\underline{\theta}^1 \neq \underline{\theta}^2$ and $\theta^1_0 = \theta^2_0$, $\underline{\theta}^1 \wedge \underline{\theta}^2 = (\theta_{-j},\theta_{-j+1},\ldots,\theta_0)$, in which $\theta_{-i} = \theta^1_{-i} = \theta^2_{-i}$ for all $i = 0,\ldots,j$ and $\theta^1_{-j-1} \neq \theta^2_{-j-1}$.
- if $\theta^1_0 = a_n$, $\underline{\theta}^1 \wedge \underline{a} = (a_{n-j},\ldots,a_n)$, in which $a_{n-i} = \theta^1_{-i}$ for all $i = 0,\ldots,j$ and $j \geq n$ or $a_{n-j-1} \neq \theta^1_{-j-1}$.

For $\underline{a} = (a_0,a_1,\ldots,a_n) \in \Sigma^{fin}$ we define:

$$G(\underline{a}) = \{x \in \bigsqcup_{a\in\mathbb{A}} G(a), g^j(x) \in G(a_j), j = 0,1,\ldots,n\}$$

and the function $f_{\underline{a}} : G(a_n) \to G(\underline{a})$ by:

$$f_{\underline{a}} = g|^{-1}_{G(a_0)} \circ g|^{-1}_{G(a_1)} \circ \ldots \circ (g|^{-1}_{G(a_{n-1})})|_{G(a_n)}.$$

Notice that, $f_{(a_i,a_{i+1})} = g$

With this notation we have the following lemma.

Lemma 2.1.2. *let K be a dynamically defined Cantor set K and $G^*(a)$ the sets from lemma 1. Let $G^*(\underline{a})$ be defined is the same way as $G(\underline{a})$. There exist constants C and $\mu > 1$ such that:*

$$diam(G^*(\underline{a})) < C\mu^{-n}$$

Proof. The proof is essentialy the same as in [9]. Let $\mu > 1$ such that $m(Dg) > \mu$ in $\sqcup_{a\in \mathbb{A}} G^*(a)$. For $\underline{a} \in \Sigma^{fin}$ let $d_{\underline{a}}$ be the metric

$$d_{\underline{a}(x,y)} = \inf_{\underline{\alpha}} l(\alpha)$$

where α runs through all smooth curves inside $G^*(\underline{a})$ that connect x to y and $l(\alpha)$ denotes the lengths of such curves. Since g sends $G^*(a_0, a_1, \ldots, a_n)$ diffeomorphically onto $G^*(a_1, \ldots, a_n)$ and $m(Dg) > \mu$ then

$$d_{(a_1,\ldots,a_n)}(g(x), g(y)) \geq \mu \cdot d_{a_0,\ldots,a_n}(x, y).$$

for all $x, y \in G^*(a_0, \ldots a_n)$. Therefore,

$$\text{diam}_{(a_0,\ldots a_n)}(G^*(a_0, \ldots a_n)) \leq \mu^{-1} \cdot \text{diam}_{(a_1,\ldots a_n)}(G^*(a_1, \ldots a_n))$$

where $\text{diam}_{\underline{a}}$ is the diameter with respect to $d_{\underline{a}}$. We conclude that, by induction,

$$\text{diam}(G^*(\underline{a})) \leq \text{diam}_{\underline{a}}(G^*(\underline{a})) \leq \mu^{-n} \cdot \text{diam}_{a_n}(G^*(a_n)).$$

Taking any C larger than $\max_{a\in \mathbb{A}} \text{diam}(G^*(a))$ yields the result. □

As a consequence of this lemma we can see that

$$K = \bigcap_{n\geq 0} g^{-n} \left(\bigsqcup_{a\in\mathbb{A}} G^*(a) \right)$$

since $G(\underline{a}) \subset G^*(\underline{a})$ and $\text{diam}(G^*(\underline{a})) \to 0$.

In this manner, these lemmas show that the sets $G(a)$ can be substituted by the sets $\overline{G^*(a)}$ in the definition of K. So in what follows, additionally to the properties in the definition of Cantor sets we suppose that $G(a) = \overline{(\mathring{G}(a))}$ and that g can always be extended to a neighbourhood V_a of $G(a)$ such that it is a $C^{1+\varepsilon}$ embedding (with $C^{1+\varepsilon}$ inverse) and $m(Dg) > \mu$ over V_a, which by lemma 2 implies that $\text{diam}(G(\underline{a})) < C\mu^{-n}$, if $\underline{a} = (a_0, \ldots a_n)$. The most important examples of conformal Cantor sets come from intersections between compact parts of stable and unstable manifolds of periodic points and basic sets of saddle type of a automorphism of $\mathbb{C}^2$ and, as we will see, we can construct them from sets $G(a)$ with these properties already.

Finally, we have the definition:

Definition 2.1.1. (The space Ω_Σ .) The set of all conformal regular Cantor sets K with the type Σ is defined as the set of all conformal Cantor sets described as above whose set of data includes an alphabet A and the set B of admissible pairs used in the construction of Σ. We denote it by Ω_Σ.

This space can be seen as a topological space. The topology is generated by a basis of neighbourhoods $U_{K,\delta} \subset \Omega_\Sigma$, $K \in \Omega_\Sigma$, $\delta > 0$, the $U_{K,\delta}$ being the set of all conformal regular Cantor sets K' given by $g' : V' \to \mathbb{C}$, $V' \supset \bigsqcup_{a\in\mathbb{A}} G'(a)$ such that $G(a) \subset V_\delta(G'(a))$, $G'(a) \subset V_\delta(G(a))$ and the restrictions of g' and g to $V \cap V'$ are δ close in the C^r metric .

2.2 Semi-invariant foliations close to a basic set.

2.2.1 C^r section theorem

We begin with an adaptation of the C^r section theorem that can be found in Pugh and Shub [16]:

Theorem 2.2.1 (Adapted C^r section Theorem)**.** *Let $\Pi : E \to M$ be a C^m vector bundle over a manifold M, with an admissible metric on E, and D be the disc bundle in E of radius C, $C > 0$ a finite constant .*

Let $h : U \to M$ be an embedding map of class C^m (with a C^m inverse too), U a bounded open set such that $U \not\subset h(U)$ but $h(U) \cap U \neq \emptyset$, and $F : E|_U \to E|_{h(U)}$ a C^m map that covers h.

Let also $N \subset U$ an open neighborhood of $U \setminus h(U)$ and $s_0 : N \to D|_N$ a C^r invariant section ($\lceil r\rceil \leq m, r \in \mathbb{R}, m \in \mathbb{N}$) . By invariant we mean that whenever $x \in N$ and $h(x) \in N$ we have $s_0(h(x)) = F(s_0(x))$.

In this context, suppose that there is a constant k, $0 \leq k < 1$ such that the restriction of F to each fiber over $x \in U$, $F_x : D_x \to D_{h(x)}$ is Lipschitz of constant at most k, that h^{-1} is Lipschitz with constant μ, that $F^{(j)}$, $s^{(j)}$ and $h^{(j)}$ are bounded for $0 \leq j < \lceil k\rceil$, $j \in \mathbb{Z}$, and $k\mu^r < 1$. Then there is a unique invariant section $s : U \to D|_U$ (meaning that for $x \in U$ and $h(x) \in U$ we have $s_0(h(x)) = F(s_0(x))$) with $s|_N = s_0$ and such a section is C^r.

Before proving it, some observations. The loss of the overflowing condition on h and U is overcome by the presence of the invariant section s_0. The natural graph transform would carry sections over U to sections over $h(U)$ but since s_0 is invariant in $N \supset U \setminus h(U)$ given any section s that agrees with s_0 in N we are able to extend its graph transform from $h(U) \cap U$ back to whole open set U. This idea comes from Robinson [12]. Besides this, very little has to be changed or verified from the proof in Pugh and Shub.

Proof. The admissible hypothesis on the metric works the same way to allow us to work in the context of $E = M \times A$ and write a section as $s(x) = (x, \sigma(x))$.

Next we consider the complete metric space $\Gamma\left(U, D|_U; s_0\right)$ of local sections over U bounded by C that agree with s_0 on $N' \subset U$, an open set such that $N \supset \overline{N'} \cap U \supset N' \supset U \setminus h(U)$. Careful choice of N' allow us to use a C^∞ function λ on U that is equal to one on N' and zero outside of N, and thusly, taking $s = \lambda \cdot s_0$ yields a well-defined section that belongs to $\Gamma\left(U, D|_U; s_0\right)$; showing that it is not empty. Then consider $\Gamma_F : \Gamma\left(U, D|_U; s_0\right) \to \Gamma\left(U, D|_U; s_0\right)$ defined by:

$$\Gamma_F(s)(x) = \begin{cases} s(x), \text{ if } x \in N'. \\ F \circ s \circ h^{-1}(x), \text{ if } x \in h(U). \end{cases}$$

Since s in equal to s_0 over N', it is invariant in this open set and the definition above is coherent. Also, because $k < 1$ this transformation is a contraction, so there is a unique s in $\Gamma\left(U, D|_U; s_0\right)$ fixed by Γ_F. It is easy to verify that this is an invariant section over U that agrees with s_0 on N.

The verification of regularity of s has some minor technical differences. First, we need to verify that if $0 \le r < 1$ then s is r-Hölder in all U.

Since s agrees with s_0 on N it is r-Hölder on this set, that is, for $x, y \in N$ we have $d(\sigma(x), \sigma(y)) \le Hd(x, y)^r$. Now, $U \setminus h(U)$ and $U \setminus N$ have a positive distance ε between each other and the section s is bounded by C. So, if $x \in U \setminus h(U)$ and $y \in U \setminus N$ we have $d(\sigma(x), \sigma(y)) \le 2C \le H\varepsilon^r \le Hd(x, y)^r$ for some big enough constant H. This allow us to write $d(\sigma(x), \sigma(y)) \le Hd(x, y)^r$ for any pair $x \in U \setminus h(U)$ and $y \in U$.

As in the book we have the estimative:

$$d(\sigma(x), \sigma(y)) \le k^m d(\sigma(h^{-m}(x)), \sigma(h^{-m}(y))) + \tilde{H} \sum_{j=1}^{m} (\mu^r)^j k^{j-1} (d(x, y))^r$$

whenever $h^{-j}(x)\ , h^{-j}(y) \in U,\ \forall j = 0,\ 1,\ 2, \ldots,\ m$.

We are going to consider two cases:

If $x, y \in U$ are such that $h^{-j}(x)\ , h^{-j}(y) \in U,\ \forall j \in \mathbb{N}$, we let $m \to \infty$ in the inequality above, and, since $k\mu^r < 1$ and σ is bounded by C, the right hand side converges to $\tilde{H} \cdot\ \tilde{C} d(x, y)^r$.

If else, there is a finite maximal m, such that $h^{-j}(x)\ , h^{-j}(y) \in U,\ \forall j =$

$0,1,2,\dots,m$. In this case, we can assume without loss of generality that $h^{-m}(x) \in U \setminus h(U)$. But then, using again the estimative above we have:

$d(\sigma(x),\sigma(y)) \leq k^m d(\sigma(h^{-m}(x)),\sigma(h^{-m}(y))) + \tilde{H}\sum_{j=1}^{m}(\mu^r)^j k^{j-1}(d(x,y))^r \leq k^m \cdot H \cdot d((h^{-m}(x),(h^{-m}(y))^r + \tilde{H}\cdot\ \tilde{C}d(x,y)^r \leq H\cdot k^m\cdot \mu^{mr}d(x,y)^r + \tilde{H}\cdot \tilde{C}d(x,y)^r$,
and again since $k\mu^r < 1$ we finally have $d(\sigma(x),\sigma(y)) \leq (H+\tilde{H}\cdot\ \tilde{C})d(x,y)^r$ for any $x,y\in U$.

The smoothness is proved with the same argument as in the book adapted in some way as above. Using the same induction idea one can do as follows.

Consider $\tilde{D}$ the disc bundle of radius $\tilde{C}$ in the fiber bundle over M with each fiber being equal to $L(T_xM,A)$, $\tilde{C}$ being chosen so that $||\partial s|| < \tilde{C}$. This is admissible. Then the complete metric space $\Gamma(U,\tilde{D}\Big|_U;\partial s_0)$. of local sections that agree with ∂s_0 on N', ∂s is obtained by differentiating s on N and identifying the tangent plane to $(x,\sigma_0(x)$ with the graph of a linear transformation in $L(T_xM,A)$. The graph transform $\gamma_{DF}(\tau)$ is defined by:

$$\gamma_{DF}(\tau)(x) = \begin{cases}\partial s(x), \text{ if } x\in N' \\ \Gamma_{DF}\circ s\circ h^{-1}(x), \text{ if } x\in h(U))\end{cases},$$

where $\Gamma_{DF(L)} := (\Pi_2 DF_{x,\sigma(x)})\circ(Id,L)\circ Dh^{-1}_{h(x)}$, for any L a linear transformation in $L(T_xM,A)$, is a fiber contraction of constant $k\mu < 1$.

To show that the invariant section $\tilde{\tau}$ is indeed the tangent to $(x,\sigma(x)), x\in U$ we have to divide in cases as above:

If $x\in U\cap\bigcup_{n\in\mathbb{N}}h^n(N')$ then it is true by definition of ∂s and the fact that $\tilde{\tau}$ is invariant and equal to ∂s_0 on N' (remember that s_0 is C^r)

If not, then for any $n\in\mathbb{N}$ there is δ small enough such that if $d(x,y) < \delta$ then $h^{-j}(x),h^{-j}(y)\in U$ for $j=0,1,2,\dots,n$. This comes from the fact that $x\in\bigcap_{n\in\mathbb{N}}h^n(U)$ and $U\setminus\bigcup_{n\in\mathbb{N}}h^n(N)$ and $U\setminus\bigcup_{n=1}^{j}h(U)$ have a positive distance between each other. This is enough to show, by the same iteration argument, that $\mathrm{Lip}_0(\sigma(x+y),\sigma(x)+\tilde{\tau}(x)(y)) = 0$, which completes the proof. □

Remark 2.2.1. Observe that, from the argument above, if we just want to obtain an invariant section that is continuous we can just make $m=r=0$ and consider just the case in which M is a topological space rather than a manifold.

Remark 2.2.2. The proof above also shows that the invariant section varies continuously with the maps involved. More specifically, fixing h,F,s_0 and choosing any h', F', s'_0 such that h and h', and their inverses are C^m close; F and F' are C^m close (and F' covers h'); s_0 and s'_0 are invariant (by F and F' respectively) and C^r close; and $k\mu^r < 1$ then s and s' are both close in the C^r topology.

Actually, we need a more technical hypothesis on the maps involved: that h, F, s_0 and all their derivatives are uniformly continuous. For example, let us show that if $r = m = 0$ and all the maps involved are uniformly continuous then s' is C^0 close to s. We begin by observing that if s_0 is uniformly continuous then we can take $\tilde{s_1} \in \Gamma\left(U, D|_U; s_0\right)$ that is uniformly continuous and so are its iterates $\tilde{s}_n = \Gamma_F(\tilde{s}_{n-1})$ and s, since the convergence to the invariant section is uniform. Moreover, the whole family $\{s_n\} \cup s$ is uniformly equicontinuous.

We now proceed by showing that both fixed points, s, s' on the spaces $\Gamma\left(U, D|_U; s_0\right)$ and $\Gamma\left(U, D|_U; s'_0\right)$ are close to one another as described above. Observe that if h and h' are close enough then $N \supset U \setminus (h(U) \cup h'(U))$. Although those spaces are not the same the argument is almost the same as always. Given two close sections in the spaces above $\tilde{s} = \tilde{s}_1$ and $\tilde{s'}$ their iterates stay within a close distance. Indeed, by analysing the definition of Γ_F and using the triangle inequality, one can see that:

$$\begin{aligned}
||\Gamma_F(\tilde{s}) - \Gamma'_F(\tilde{s'})|| &\leq \max\Big(||\tilde{s} - \tilde{s'}||, ||F \circ \tilde{s} \circ h^{-1} - F' \circ \tilde{s'} \circ h'^{-1}||\Big) \\
&\leq \max\Big(||\tilde{s} - \tilde{s'}||, ||F' \circ \tilde{s} \circ h'^{-1} - F' \circ \tilde{s'} \circ h'^{-1}||+ \\
&||F' \circ \tilde{s} \circ h^{-1} - F' \circ \tilde{s} \circ h'^{-1}||+||F' \circ \tilde{s} \circ h^{-1} - F \circ \tilde{s} \circ h^{-1}||\Big) \\
&\leq \max\left(||\tilde{s} - \tilde{s'}||, (k+\varepsilon)||\tilde{s} - \tilde{s'}||+\right. \\
&\left.||F' \circ \tilde{s} \circ h^{-1} - F' \circ \tilde{s} \circ h'^{-1}||+d(F, F')\right) \\
&\leq \max\left(||\tilde{s} - \tilde{s'}||, (k+\varepsilon)||\tilde{s} - \tilde{s'}||+(k+\varepsilon)||\tilde{\sigma} \circ h^{-1} -\right. \\
&\quad \left.-\tilde{\sigma} \circ h'^{-1}||+d(h^{-1}, h'^{-1}) + d(F, F')\right) \\
&\leq \max\left(||\tilde{s} - \tilde{s'}||, (k+\varepsilon)||\tilde{s} - \tilde{s'}||+\varepsilon'\right)
\end{aligned}$$

in which ε and ε' can be taken arbitrarily small if we take $d(h^{-1}, h'^{-1})$ and $d(F, F')$ small enough. To control the size of $||\tilde{\sigma} \circ h^{-1} - \tilde{\sigma} \circ h'^{-1}||$ we used the uniformly continuity shown above.

Using the equicontinuity of the family $\{s_n\} \cup s$ and induction we have that $||\Gamma^n_F(\tilde{s}) - \Gamma^n_{F'}(\tilde{s'})|| \leq ||\tilde{s} - \tilde{s'}||$, provided that $d(h, h')$ and $d(F, F')$ are sufficiently small. Since, $\lim_{n\to\infty} \Gamma_F(\tilde{s}) = s$ and $\lim_{n\to\infty} \Gamma'_F(\tilde{s'}) = s'$ it is true that $||s - s'|| \leq \tilde{s} - \tilde{s'}||$. But we can choose $\tilde{s}$ agreeing with s_0 on N and $\tilde{s'}$

agreeing with s_0' on N arbitrarily close if s_0 and s_0' are close enough, seeing the construction used to show that both $\Gamma(U, D|_U; s_0)$ and $\Gamma(U, D|_U; s_0')$ are non-empty. The rest of the argument, namely the closeness of the Hölder constants and exponents of the $\lfloor r \rfloor$ derivatives and the induction step, comes from the estimatives on the proof above, proceeding as just detailed to contour the difficulty of uniform continuity.

2.2.2 The stable and unstable foliations

Before continuing into the next theorem we will remember some nomenclature. A basic set Λ for a automorphism G of $\mathbb{C}^2$ is a compact maximal invariant hyperbolic set with a dense orbit and a dense subset of periodic orbits. By maximal we mean that there is an open neighbourhood V of Λ, such that $\Lambda = \bigcap_{n\in\mathbb{Z}} G^n(V)$. We say that the basic set is of saddle type if $T_\Lambda\mathbb{C}^2 = E^s \oplus E^u$, both non-trivial and invariant, with $|DG|_{E^s}|< 1$ and $|(DG|_{E^u})^{-1}|< 1$. By using the structure of DG, the density of periodic orbits and the invariance of E^s and E^u it can be show that both can be seen as complex 1-dimensional subspaces of $\mathbb{C}^2$. This also implies that $DG|_{E^s,E^u}$ are linear complex transformations, or if we seen the vector spaces involved as real spaces a roto-homothety (a transformation given by a conformal matrix). Also, the classical theory of invariant manifolds shows that for any point $x \in \Lambda$ there are holomorphic embedded manifolds (curves) $W^s_\varepsilon(x)$ and $W^u_\varepsilon(x)$ that are invariant in the sense that $G(W^s_\varepsilon(x)) \subset W^s_\varepsilon(G(x))$ and $G^{-1}(W^u_\varepsilon(x)) \subset W^u_\varepsilon(G^{-1}(x))$. We also denote $W^{s,u}_G(p,\alpha) = \{q \in \mathbb{C}^2; d(G^{(+,-)\,n}(q), p) \leq \alpha\}$ and $W^{s,u}_G(\alpha) = \bigcup_{p\in\Lambda_G} W^{s,u}_G(p,\alpha)$ and refer to them as local stable or unstable manifolds. In this thesis we always assume that the basic set Λ is zero dimensional.

The next theorem is really close to the result by Pixton [11] used by Buzzard [6]. It uses the result above.

Theorem 2.2.2. *Let $U \subset \mathbb{C}^2$. Let $\Lambda \subseteq U$ be a basic set of saddle type for an injective holomorphism $G_0 : U \to M$, with $\Lambda = \bigcap_{n\in\mathbb{Z}} G_0^n(U)$ and let $E^s \oplus E^u$ be the associated splitting of $T_\Lambda\mathbb{C}^2$.*

Suppose that $||DG_0|_{E^s}||\cdot||DG_0|_{E^u}||^{-1}\cdot||DG_0|_{E^s}||^{-(1+\varepsilon)}< 1$.

Then, there is a compact set L and δ such that for any holomorphism $G : U \to \mathbb{C}^2$ with $||G - G_0||< \delta$ we can construct a $C^{1+\varepsilon}$ foliation $\mathcal{F}^u_G$ defined on a open set $V \subset U$ such that:

- *the basic set $\Lambda_G = \bigcap_{n\in\mathbb{Z}} G^n(U)$ satisfies $\Lambda_G \subset$ int $L \subset L \subset \mathcal{F}^u_G$*

- *if $p \in \Lambda_G$ then the leaf $\mathcal{L}^u_G(p)$ agrees with $W^u_{loc}(p)$*
- *if $p \in G(U) \cap U$ then $G(\mathcal{L}^u_G(p)) \supseteq \mathcal{L}^u_G(G(p))$.*
- *The association $G \to \mathcal{F}^u_G$ is continuous on the $C^{1+\varepsilon}$ topology.*

Proof. As in the previous theorem we will base or argument on the existent literature. The work of Pixton shows that we can construct a non necessarily smooth $\mathcal{F}^u_G$ for any G with the desired properties as above. The idea is described as follows.

We begin by constructing a transversal (not necessarily semi-invariant) foliation $\mathcal{F}_0$ to $W^s_G(\alpha)$, that cover an open set around $W^s_G(\alpha)$. This can be done locally and, in the case that $W^s_G(\alpha)$ is a zero dimensional transversal lamination, which is our case, it is possible to glue these constructions together by bump functions (check the original for details). We can restrict $\mathcal{F}_0$ to small neighbourhood V of $\overline{W^s_G(\alpha) \setminus G(W^s_G(\alpha))}$, in such a way that $G(V) \cap V = V'$ does not intersect $G^{-1}(V) \cap V = G^{-1}(V')$. We consider a new foliation $\mathcal{F}'_0$ on $V' \cup G^{-1}(V')$ defined being the same as $\mathcal{F}_0$ over $G^{-1}(V')$ and being equal to $G(\mathcal{F}_0)$ over V'. We can then, considering again that $W^s_G(\alpha)$ is transversely zero dimensional, construct a transversal foliation $\mathcal{F}_1$ to it that agrees with $\mathcal{F}'_0$ on $V \cup G^{-1}(V')$. Now we define recursively $\mathcal{F}_n = (G(\mathcal{F}_{n-1}) \cap U) \cup ((\mathcal{F}_{n-1} \cap V)$, which is possible because of the semi-invariance of $\mathcal{F}_1$. Notice that for any point $x \in U \setminus \bigcap_{n\in\mathbb{N}} G^n(U)$ for any integer n bigger than a integer n_x the leaf $\mathcal{L}_n(x)$ of $\mathcal{F}_n$ at x is the same so we can safely define in $U \setminus \bigcap_{n\in\mathbb{N}} G^n(U)$ the limit foliation $\mathcal{F}$. Finally, adding the submanifolds $W_u(x), x \in \Lambda$, yields a invariant foliation in an open subset of U (also see [10] for the idea of fundamental neighbourhood). Notice that we can chose L and δ small enough such that the items above are satisfied for any G, $||G - G_0|| < \delta$.

We can use the C^r section Theorem to show that this foliation is indeed $C^{1+\varepsilon}$. For this, we need only to show that the tangent directions to the leaves vary in a $C^{1+\varepsilon}$ fashion, and the same argument for the Fröbenius theorem gives the same regularity for the foliation itself, (remember that, because of Gronwall's inequality, a flow form a Hölder vector field is also Hölder).

We begin by extending the fibrate decomposition $E_{G_0} = E = E^s \oplus E^u$ over Λ to a close C^2 decomposition $E = E^s \oplus E^u$ over U such that the action of the derivative map $TG_x := E^s_x \oplus E^u_x \to E^s_{G(x)} \oplus E^u_{G(x)}$ can be written as a block matrix:

$$\begin{bmatrix} A_x & B_x \\ C_x & D_x \end{bmatrix}$$

in which $||A_x||< ||DG_0|_{E^s}||+\delta'$, $||D_x||> ||DG_0|_{E^u}||-\delta'$, and $||B_x||, ||C_x||< \delta'$ for some small δ' uniformly on U. Also, by possibly shrinking U we may assume that the tangent directions to $\mathcal{F}$ can be written as the graph of a linear map from E^u_x to E^s_x (bounded uniformly on U). Considering the C^2 bundle whose fibers are $L(E^u_x, E^s_x)$ we can consider the covering map :
$\Gamma_{DF}(x)(L)) = [B_x + A_x L)][D_x + C_x L]^{-1}$

in which $L \in L(E^u_x, E^s_x)$. By making δ and δ' sufficiently small we have that Γ_{DF} is a fiber contraction of constant at most $||DG_0|_{E^s}||\cdot||DG_0^{-1}|_{E^u}|+\delta''$. The Lipschitz constant of the base map G^{-1} is at most $||DG_0|_{E^s}||^{-1}+\delta'''$, and so there is an $r > 0$ such that

$$(||DG_0|_{E^s}||\cdot||DG_0^{-1}|_{E^u}||+\delta'') \cdot (||DG_0|_{E^s}||^{-1}+\delta''')^r < 1.$$

This is enough to show that the section $x, T_x(\mathcal{F} \cup W^u)$ is the unique invariant section of the C^r section Theorem that agrees with $\mathcal{F}$ on N, and so it is $C^{1+\varepsilon}$. The continuity in the $C^{1+\varepsilon}$ comes immediately from the construction and previous observations, we only require $\mathcal{F}_0$ and its derivatives to be uniformly continuous on V which is clearly possible to be done.

□

Corollary 2.2.2.1. *With the hypothesis* $||DG_0|_{E^s}||\cdot||DG_0|_{E^u}||< 1$ *the last theorem guarantees the existence of a* C^2 *foliation* $\mathcal{F}^u_G$ *for any* G *sufficiently close to* G_0.

This could be the case in the dissipative context, specially in the case of horseshoes arising from transversal homoclinic intersections.

Remark 2.2.3. Each leaf of the foliation obtained in theorem 2.2.2 can be chosen to be a holomorphic curve. This only depends on being able to consider the foliation $\mathcal{F}^1$ consisting of leaves that are holomorphic curves. The local construction of $\mathcal{F}^1$ in [11] involves only an isotopy and a bump function applied to create disk families along compact (and possibly very small) parts of W^s. Checking the details in the original, we observe that such construction can be done in a way that make those disk families be holomorphic embedded curves. This is mentioned in [6]; see the appendix of [13] for further details.

2.3 Holonomies

In this section we show how to describe $W^u(p) \cap \Lambda$ as a conformaly dynamically defined Cantor set K, where p is a fixed or periodic point. This can be done for a generic automorphism of $\mathbb{C}^2$, G, but we begin by describing how to make it in this case, using an example of Buzzard, [6].

Example 2.3.1. (Buzzard) Let $S(p;l) \subset \mathbb{C}$ denote the open square centered at p of sides parallel to the real and imaginary axis of side length equal to l. Consider the 9 points set $P = \{x + yi \in \mathbb{C}; (x, y) \in \{-1, 0, 1\}^2$ and a positive real number δ. Define $c_0 = 1 - \delta$ and:

$$K_0 = \bigcup_{a \in P} \overline{S(a; c_0)} \text{ and } K_1 = K_0 \times K_0 \subset \mathbb{C}^2.$$

We identify each connected component of K_1, $S(a;c_0) \times S(b;c_0)$ as the pair $(a, b) \in P^2$.

Consider now, some positive real number $c_1 \in (c_0 = 1 - \delta, \frac{3c_0}{2+c_0} = \frac{3-3\delta}{3-\delta})$ and the map $f : K_0 \to \mathbb{C}$ defined as,

$$f(w) := \sum_{a \in P} \frac{3}{c_1} \chi_{\overline{S(a;c_0)}}.$$

Notice that its image is composed of nine points as is P. Analogously, we can define $K_g = \bigcup_{a \in P} \overline{S(3a/c_1; 3)}$ and define,

$$g(z) := \sum_{a \in P} -a \cdot \chi_{\overline{S(\frac{3a}{c_1};3))}}.$$

Then defining the maps,

$$\begin{aligned} F_1(z, w) &:= (z + f(w), w) \\ F_2(z, w) &:= (z, w + g(z)) \\ F_3(z, w) &:= (\frac{c_1}{3} z, \frac{3}{c_1} w) \end{aligned}$$

and making $F : K_1 \to \mathbb{C}^2$; $F := F_3 \circ F_2 \circ F_1$ we have that in a connected component (a, b) of K_1

$$F(z, w) = (\frac{c_1}{3} z + b, \frac{3}{c_1}(w - b)).$$

The maximal invariant set of F over K_1, $\Lambda = \bigcap_{n \in \mathbb{Z}} F^n(K_1)$, is a hyperbolic set with 0 as a fixed saddle point. It is easy to see that $W^u_{F,\text{loc}}((0, 0)) := \{0\} \times \{S(0; c_0)\}$ is the connected component that contains $(0, 0)$ of the intersection between $W^u_F(0)$ and the connected component $(0, 0)$ of K_1. Also, the set $W^u_{F,\text{loc}}(0) \cap \Lambda$ can be seen as a conformal Cantor set K_F on the complex plane (in this case $0 \times \mathbb{C}$) given by the maps:

$$\begin{aligned} g_a : S(a; c_0) &\to S(0; 3) \\ z &\mapsto \frac{3}{c_1}(z - a) \end{aligned}$$

Likewise, we can write $W^s_{F,\text{loc}}(0) \cap \Lambda$ as the same cantor set K. The condition $c_1 < \frac{3c_0}{2+c_0}$ is necessary for the image of each g_a cover the union of their domains.

Now we work with automorphisms of $\mathbb{C}^2$ that are sufficiently close to this model F. First, we approximate f and g by polynomials p_f and p_g, obtaining a map $G_0 = F_1' \circ F_2' \circ F_3 \in Aut(\mathbb{C}^2)$, where $F_1'(z, w) := (z + p_f(w), w)$ and $F_2'(z, w) := (z, w + p_g(z))$. Then, we fix $K' \subset \overline{K'} \subset \text{int}(K_1)$ such that considering Λ_G the maximal invariant set by G of the open set U it is contained in K' whenever $||G - G_0|||_U$ is sufficiently small, where $U = S((0, 0); 3) \times S((0, 0); 3)$. Furthermore, there is a fixed point p_G that is the analytic continuation of the fixed point $(0, 0)$ of F. Since $||G - F||$ is small we can also show that the projection $\Pi : W^u(p_G; loc) \to S((0, 0); 3)$ is a biholomorphic map close to the identity, where $W^u(p_G; loc)$ is the connected component that contains p_G of $W^u(p_G) \cap U$.

Observe that $G(W^u(p_G; loc)) \cap K_0 \times S(0; \frac{3}{c_1}c_0)$ is made of nine different connected components, $W_1, W_2, W_3, \ldots, W_9$, each of them holomorphic curves close to being vertical, because of the continuity dependence of the foliations on G (so, as long as f and g are well approximated by p_f and p_g and $||G - G_0||$ is sufficiently small). Consider now $V_i = G^{-1}(W_i \cap K'), i = 1, 2, \ldots, 9$. Notice that all the V_i are disjoint subsets of $W^u(p_G; loc)$.

According to theorem 2.2.2 and remark 2.2.3, $\mathcal{F}^s_G$ can be defined whenever G is sufficiently close to F and we can consider its leaves to be holomorphic lines very close to the horizontal lines. However, its domain may be only a small neighbourhood of Λ_G. We now show a way of constructing it that cover a large subset of U. First, by iterating forwards from a neighbourhood of p_G a finite number of times we may consider $\mathcal{F}^s_G$ to be defined on an open set that contains $W^u(p_G; loc)$. Since G is very close to F we can write every leaf as the (translated) graph of a holomorphic map $(z + \Phi_z(w), w)$ for all $z \in W^u(p_G; loc)$ such that $||\Phi_z||$ is very small and w varies on a small neighbourhood of $0 \in \mathbb{C}$. Let $B_r(0)$ be contained in such a neighbourhood. We can now expand each of these leaves obtaining a new foliation that contains a neighbourhood of $U \setminus G(U)$ by carrying each leaf to $(z + \Phi_z(w), k' \cdot w)$, where

k' is a real number much larger than k^{-1}. Proceeding as in the proof of 2.2.2, bearing in mind remark 2.2.3 and the fact that backwards iteration expands in the horizontal direction, we can construct $\mathcal{F}^s$ defined on a neighbourhood of U.

In view of the continuous dependence of the foliation on G, and maybe by restricting the foliation to an open set, we can assume that the leaves of $\mathcal{F}^s_G$ are almost horizontal. Thusly, we can define the projections along stable leaves $\Pi_i : W_i \to W^u(p_G; loc)$.

Proposition 2.3.1. *We can express $K_G = \Pi(W^u(p_G; loc) \cap U \cap \Lambda_G)$ as a dynamically defined conformal Cantor set through the maps $f_i : \Pi(V_i) \to S((0,0); 3)$ where $f_i = \Pi \circ \Pi_i \circ G \circ \Pi^{-1}$.*

Proof. Let us show that K is the maximal invariant set of these maps. Take $x \in W^u(p_G; c_0) \cap U \cap \Lambda_G$. Thus, $G(x) \in \Lambda \subset U$, so there exists $i \in \{1, 2, 3, \ldots, 9\}$ such that $G(x) \in W_i$, which implies $x \in V_i$. Likewise, $y = \Pi_i(G(x)) \in \Lambda$. To show this, we see that $G^{-n}(y) \in W^u(p_G; c_0) \cap U$, for all $n \geq 0$, as this set is carried into itself by backwards iteration of G. Additionally, $G^n(y) \in U$ for all $n > 0$ because $y \in \mathcal{L}^s(G(x))$ and forward iterations of stable leaves always remain inside U by construction. So, $y \in \bigcap_{n \in \mathbb{Z} G^n(U)} = \Lambda_G$, and in particular $y \in W^u(p_G; c_0) \cap U$. Hence, as we have already shown, $y \in V_i$ for some $i \in \{1, 2, 3, \ldots, 9\}$. Repeating this argument inductively we obtain that the orbit of $\Pi(x)$ always remains on $\bigcup_{i=1}^{9} V_i$.

On the other hand, if $x \in W^u(p_G; c_0) \cap U$ is such that the forward orbit of $\Pi(x)$ by the maps f_i is always in $\bigcup_{i=1}^{9} V_i$, then, using that projections along the stable leaves commute with the map G and denoting by x_n the $n-$th term of the orbit of x, we can show that $G^n(x) = \Pi_s \circ \Pi^{-1}(x_n)$, $(n > 0)$, Π_s being a projection along stable leaves between two close components of $W^u(p_G) \cap U$. This implies that $G^n(x) \in U$, $\forall n > 0$, and as $G^n \in U, \forall n \leq 0$, then $x \in \Lambda_G$.

Lemma 2.3.1. *Let Λ_G be a basic set for a $G \in Aut(\mathbb{C}^2)$ as in the context of theorem 2.2.2 together with its stable foliation $\mathcal{F}^s_G$. Additionally, let N_1 and N_2 be two $C^{1+\varepsilon}$ transversal sections to $\mathcal{F}^s_G$. Suppose that for some point periodic point $p \in \Lambda_G$, the tangent planes of N_1 and N_2 to the points of intersection $N_1 \cap W^s_G(p) = q_1$ and respectively $N_2 \cap W^s_G(p) = q_2$ are complex lines of $\mathbb{C}^2$. Then, the derivative of the projection along stable leaves $\Pi_s : N_1 \to N_2$ is a $C^{1+\varepsilon}$ map conformal at q_1.*

Proof. Observe that, since $p \in \Lambda_G$ every forward iterate of the segment in $W^s_G(p)$ that connects q_1 and q_2 stays on the domain of the foliation. So, for every $n \in \mathbb{N}$ we can define a restriction $N^n_i \subset N_i$, $i = 1, 2.$ such that $G^n(N^n_i)$ is also on the domain of the foliation. Furthermore, this restriction can be done in such manner that, since p is periodic, we have, by the λ-lemma, that $G(N^n_1)$ and $G(N^n_2)$ are δ close to each other on the C^1 metric, for every $n > n_\delta$. Also, we can assume that their tangent directions at $q^n_i = G^n(q_i)$ are bounded away from $T_{q^n_i} W^s_G$, $i = 1, 2$. Let $\Pi^n_s : N^n_1 \to N^n_2$ be the projection along the stable foliation.

Looking at a small open set $\tilde{U}$, we can find a $C^{1+\varepsilon}$ map $f : \tilde{U} \to \mathbb{D} \times \mathbb{D}$ such that the stable leaves are taken into the horizontal levels $\mathbb{D} \times \{z\}$, $z \in \mathbb{D}$ and represent N^n_1 and N^n_2 as graphs $(h_1(z), z)$ and $(h_2(z), z)$ of $C^{1+\varepsilon}$ embeddings h_1 and h_2 with domain being a small disk $\mathbb{D}$ too. Under this identification, Π^n_s is a $C^{1+\varepsilon}$ map that carries $(h_1(z), z)$ to $(h_2(z), z)$, and, according to the previous paragraph conclusion, has a derivative δ close to the identity.

Now, the projection along stable foliations comute with G. Therefore, $\Pi_s = G^{-n} \circ \Pi^n_s \circ G^n$. Using the chain rule to calculate the derivative at q_1 we obtain a expression of the form

$$A_1 \cdot A_2 \cdots A_n \cdot D\Pi^n_s \cdot B_n \cdots B_1$$

where B_i represents the restriction of DG to $T_{q^n_1} N^n_1$ and A_i the restriction of $(DG)^{-1}$ to $T_{q^n_2} N^n_2$. But all of these tangent spaces are, by induction, complex lines in $\mathbb{C}^2$, so all the A_i and B_i are conformal.This way, the derivative of Π_s is at most δ distant from being conformal. Making $\delta \to 0$ (or equivalently, $n \to \infty$) we have the desired conformality.

□

It is clear that the manifolds $W^u_G(p_g, loc)$ and W_i satisfy the properties of transversal sections on the lemma just above. It is then clear that the maps f_i are $C^{1+\varepsilon}$ and conformal at K (notice that Π is a parametrization of a complex line). This finishes the proof.

□

One can also observe, that taking p_f and p_g sufficiently good approximations and requiring $||G - G_0|_U||$ to be sufficiently small, the Cantor set obtained above, identified as K_G is in a small open neighbourhood $\mathcal{V}$ of K_F in $\Omega_{P^{\mathbb{N}}}$. This will be important in section 4.2.

Remark 2.3.1. The general case of a generic basic saddle set can be done using Markov partitions, as described in [11]. The improvement from the work of Bowen [14] is that the rectangles are open sets of the ambient space filled with our stable and unstable foliations. Letting R_j, $j = 1, \dots, m$ be the Markov partition of Λ_G we consider $W(G)$, a large compact part of $W^u_G(p_G)$ of some fixed point G that has only one connected component intersection with R_j for all $j = 1, \dots, n$. Then, define the sets $G(i,j)$ as $G^{-1}(R_i \cap W(G))$ and the maps

$$\begin{aligned} g_{(i,j)} : G(i,j) &\to W(G) \\ q &\mapsto \Pi^s_j(G(q)) \end{aligned}$$

for all i, $j = 1, \dots, n$, where Π^u_j denotes the projection along the stable leaves *inside* R_j. Notice that in this case the previous need to extend the foliations disappears given the presence of the Markov partition.

Verifying that this set of data defines a dynamically defined Cantor set follows the arguments on the example above almost “*ipsis literis*”.

Chapter 3

A sufficient criterion for the stability of conformal Cantor sets.

In this chapter we explore the consequences of the conformality on the structure of the Cantor sets. The first result is the existence of limit geometries and some consequences of it. In the second section we define renormalization operators and verify that the limit geometries are an atractor respect to their actions. In the last section, we use this last fact to show that the concept of recurrent compact of relative pair of affine configurations of limit geometries is a sufficient criterion for the stability of intersections between Cantor sets. All of these concepts and techniques are natural extensions from the real case.

3.1 Limit Geometries

Given a conformal Cantor set K we define $K(a) = K \cap G(a)$ and fix $c(a) \in K(a)$ for all $a \in \mathbb{A}$. Additionally, given $\underline{\theta} = (\dots, \theta_{-n}, \dots, \theta_0) \in \Sigma^-$ we write $\underline{\theta}_n = (\theta_{-n}, \dots, \theta_0)$ and $r_{\underline{\theta}_n} := \operatorname{diam}(G^*(\underline{\theta_n}))$.

By lemma 1 we can extend g and its inverses to a neighbourhood of $\bigsqcup_{a\in\mathbb{A}} G(a)$, so we may consider, in the case that $(a_i, a_{i+1}) \in B$, $f_{(a_i,a_{i+1})}$ defined over a larger set, specifically from $G^*(a_{i+1})$ to $G^*(a_i)$; and hence also consider $f_{\underline{a}} : G^*(a_0) \to G^*(\underline{a})$ when $\underline{a} \in \Sigma^{fin}$. With this in mind we can define, for any $\underline{\theta} \in \Sigma^-$ and $n \geq 1$:

$$c_{\underline{\theta}_n} = f_{\underline{\theta}_n}(c_{\theta_0})$$
$$k_n^{\underline{\theta}} = \Phi_{\underline{\theta}_n} \circ f_{\underline{\theta}_n}$$

where $k_n^{\underline{\theta}} : G^*(\theta_0) \to \mathbb{C}$ and $\Phi_{\underline{\theta}_n}$ is the affine transformation over $\mathbb{C}$, $\Phi_{\underline{\theta}_n}(z) = \alpha \cdot z + \beta, \alpha \in \mathbb{C}^*, \beta \in \mathbb{C}$, such that $\Phi_{\underline{\theta}_n}(c_{\underline{\theta}_n}) = b$, b being a point outside of K and $D(\Phi_{\underline{\theta}_n} \circ f_{\underline{\theta}_n})(c_{\theta_0}) = 1 \in \mathbb{C}$. A transformation with these properties exists because the map g, and thusly its inverse branches, are conformal on the set K, so $Df_{\underline{\theta}_n}(c_{\theta_0})$ is a conformal matrix that can be seen as a linear operator over $\mathbb{C}$, or precisely, a multiplication by a complex number. We denote the space of affine transformations over $\mathbb{C}$ by $Aff(\mathbb{C})$.

Define $\Sigma_a^- = \{\underline{\theta} \in \Sigma^-, \underline{\theta}_0 = a\}$ and consider in this set the topology given by the metric $d(\underline{\theta}_1, \underline{\theta}_2) = \operatorname{diam}(G^*(\underline{\theta}_1 \wedge \underline{\theta}_2))$. Likewise, let the space $\operatorname{Emb}_{1+\varepsilon}(G^*(a), \mathbb{C})$ of $C^{1+\varepsilon}$ embeddings from $G^*(a)$ to $\mathbb{C}$ with $C^{1+\varepsilon}$ inverse have the topology given by the metric $d(g_1, g_2) = \max\{||g_1 - g_2||, \log||D(g_1 \circ (g_2)^{-1})||\}$. Notice that this metric is equivalent to the usual $C^{1+\varepsilon}$ metric. With these notations and considerations we have the following lemma.

Lemma 3.1.1. *(Limit Geometries) For each $\theta \in \Sigma^-$ the sequence of C^r embeddings $k_n^{\underline{\theta}} : G^*(\theta_0) \to \mathbb{C}$ converges in the C^r topology to an embedding $k^{\underline{\theta}} : G^*(\theta_0) \to \mathbb{C}$. Moreover, the convergence is uniform over all $\underline{\theta} \in \Sigma^-$ and in a small neighbourhood of g in Ω_Σ^r. The map $k : \Sigma_a^- \to Emb(G^*(a), \mathbb{C})$, $\underline{\theta} \mapsto k^{\underline{\theta}}$ is Hölder, if we consider the metrics described above for both spaces. The $k^{\underline{\theta}} : G^*(\theta_0) \to \mathbb{C}$ defined for any $\theta \in \Sigma^-$ are called the limit geometries of K.*

Proof. We are free to work with the slightly simpler case in which $b = 0$. Consider for each $n \geq 2$ (and $\underline{\theta} \in \Sigma^-$) the functions $\Psi_n^{\underline{\theta}} : \operatorname{Im}(k_{n-1}^{\underline{\theta}}) \to \mathbb{C}$:

$$\Psi_n^{\underline{\theta}} = \Phi_{\underline{\theta}_n} \circ f_{(\theta_{-n}, \theta-n+1)} \circ \Phi_{\underline{\theta}_{n-1}}^{-1}.$$

Notice that, then:

$$k_n^{\underline{\theta}} = \Psi_n^{\underline{\theta}} \circ \Psi_{n-1}^{\underline{\theta}} \circ \ldots \circ \Psi_2^{\underline{\theta}} \circ k_1^{\underline{\theta}} \tag{3.1}$$

We proceed by controlling the functions $\Psi_n^{\underline{\theta}}$ and showing that they are exponentially close to the identity.

First, the domain of $f_{(\theta_{-n}, \theta_{-n+1})}$ in the definition of $\Psi_n^{\underline{\theta}}$ is $G^*(\theta_n)$, which we know to have diameter less than $C \cdot \mu^{-n}$. But we can do better. If a map f is $C^{1+\varepsilon}$ on an open subset of $\mathbb{C}$, then, for any point $z \in \mathbb{C}$ and $h \in \mathbb{C}$ small enough:

$$||f(z+h) - (f(z) + Df(z) \cdot h)|| < C||h||^{1+\varepsilon}$$

Consequently, $f_{(\theta_{-n}, \theta_{-n+1})} : G^*(\underline{\theta}_{n-1}) \to G^*(\underline{\theta}_n)$ is $C_f \cdot r_{\underline{\theta}_{n-1}}^{1+\varepsilon}$ close to the map $A_{\underline{\theta}_n} \in Aff(\mathbb{C})$ described by $A_{\underline{\theta}_n}(c_{\underline{\theta}_{n-1}}) = c_{\underline{\theta}_n}$ and $DA_{\underline{\theta}_n}(c_{\underline{\theta}_{n-1}}) =$
$= Df_{(\theta_{-n}, \theta_{-n+1})}(c_{\underline{\theta}_{n-1}})$, and thus:

$$r_{\underline{\theta}_n} \leq |Df_{(\theta_{-n},\theta_{-n+1})}| \cdot r_{\underline{\theta}_{n-1}} + C_f \cdot r^{1+\varepsilon}_{\underline{\theta}_{n-1}} \leq r_{\underline{\theta}_{n-1}} \cdot \left(|Df_{(\theta_{-n},\theta_{-n+1})}| + C_1 \cdot \mu^{(-n+1)\varepsilon}\right)$$

Arguing by induction, we obtain:

$$\log r_{\underline{\theta}_n} \leq \log|Df_{\underline{\theta}_n}(c_{\theta_0})| + ||Dg|| \cdot \sum_{j=0}^{n-1} \left(C_1 \cdot \mu^{(-n+1)\varepsilon}\right) \leq \log|Df_{\underline{\theta}_n}(c_{\theta_0})| + \hat{C}, \text{so:}$$

$$r_{\underline{\theta}_n} \leq C' \cdot |Df_{\underline{\theta}_n}| \leq C' \cdot \mu^{-n}$$

In a completely analogous way we can show, maybe enlarging C', that:

$$C'^{-1} \cdot |Df_{\underline{\theta}_n}| \leq r_{\underline{\theta}_n} \leq C' \cdot |Df_{\underline{\theta}_n}| \tag{3.2}$$

and so the size of the $G^*(\underline{\theta}_n)$ is controlled.

This implies that $||f_{(\theta_{-n},\theta_{-n+1})} - A_{\underline{\theta}_n}|| \leq C \cdot |Df_{\underline{\theta}_n}(c_{\theta_0})|^{1+\varepsilon}$ for some constant C, for all $\underline{\theta} \in \Sigma^-$. On the other hand, by construction $\Phi_{\underline{\theta}_n} \circ A_{\underline{\theta}_n} \circ \Phi^{-1}_{\underline{\theta}_{n-1}} = Id$ and $D\Phi_{\underline{\theta}_n} = \left(Df_{\underline{\theta}_n}(c_{\theta_0})\right)^{-1}$, therefore:

$$\begin{aligned}
||\Psi^{\underline{\theta}}_n - Id|| &= ||\Psi^{\underline{\theta}}_n - \Phi_{\underline{\theta}_n} \circ A_{\underline{\theta}_n} \circ \Phi_{\underline{\theta}_{n-1}}|| \leq |D\Phi_{\underline{\theta}_n}| \cdot ||f_{(\theta_{-n},\theta_{-n+1})} - A_{\underline{\theta}_n}|| \\
&\leq |Df_{\underline{\theta}_n}(c_{\theta_0})|^{-1} \cdot C \cdot |Df_{\underline{\theta}_{n-1}}(c_{\theta_0})|^{1+\varepsilon} \\
&\leq \tilde{C} \cdot (\mu^{-\varepsilon})^n
\end{aligned}$$

as we wished to obtain.

This is enough to show that $\{\Psi^{\underline{\theta}}_n\}_{n\geq 0}$ is a Cauchy sequence, at least in C^0 metric. In fact, for $m, l \geq 1$,

$$\begin{aligned}
||\Psi^{\underline{\theta}}_{m+l} - \Psi^{\underline{\theta}}_m|| &= ||\Psi^{\underline{\theta}}_{m+l} \circ \ldots \circ \Psi^{\underline{\theta}}_2 \circ k^{\underline{\theta}}_1 - \Psi^{\underline{\theta}}_m \circ \ldots \circ \Psi^{\underline{\theta}}_2 \circ k^{\underline{\theta}}_1|| \\
&\leq \sum_{j=1}^{l} ||\Psi^{\underline{\theta}}_{m+j} \circ \ldots \circ \Psi^{\underline{\theta}}_2 \circ k^{\underline{\theta}}_1 - \Psi^{\underline{\theta}}_{m+j-1} \circ \ldots \circ \Psi^{\underline{\theta}}_2 \circ k^{\underline{\theta}}_1|| \\
&\leq \sum_{j=1}^{l} ||\Psi^{\underline{\theta}}_{m+j} - \mathrm{Id}|| \\
&\leq \sum_{j=1}^{l} \tilde{C} \cdot (\mu^{-\varepsilon})^{m+j} \leq \frac{\tilde{C} \cdot (\mu^{-\varepsilon})^m}{1 - \mu^{-\varepsilon}}
\end{aligned}$$

which implies that $||\Psi^{\underline{\theta}}_{m+l} - \Psi^{\underline{\theta}}_{m}|| \to 0$ as $m \to \infty$.

Further, for any point $z \in \mathrm{Im}(k^{\underline{\theta}}_{n-1})$ we can calculate $D\Psi^{\underline{\theta}}_{n}(z) = \Phi_{\underline{\theta}_n} \cdot Df_{(\theta_{-n},\theta_{-n+1})}(\Phi^{-1}_{\underline{\theta}_{n-1}}(z)) \cdot \Phi^{-1}_{\underline{\theta}_{n-1}}$. However, by hypothesis, we have that $D\Psi^{\underline{\theta}}_{n}(c_{\theta_0}) = \mathrm{Id}$ and $d\left(\Phi^{-1}_{\underline{\theta}_{n-1}}(z), \Phi^{-1}_{\underline{\theta}_{n-1}}(c_{\theta_0})\right) \leq \mathrm{diam} G^*(\underline{\theta}_{n-1})$. Then, using that Df is ε - Hölder, we conclude that

$$|D\Psi^{\underline{\theta}}_{n}(z) - \mathrm{Id}| \leq C_f \cdot |\Phi_{\underline{\theta}_n}| \cdot |\Phi_{\underline{\theta}_{n-1}}|^{-1} \cdot r^{\varepsilon}_{\underline{\theta}_{n-1}} \leq C'' \mu^{-n\varepsilon} \tag{3.3}$$

, because $|\Phi_{\underline{\theta}_n}|$ and $|\Phi_{\underline{\theta}_{n-1}}|$ are comparable (remember that $|\Phi_{\underline{\theta}_n}| = |Df_{\underline{\theta}_n}(c_{\theta_0})|^{-1}$).

This allows us to show that the sequence $\{||Dk^{\underline{\theta}}_{n}||\}_{n\geq 1}$ is bounded. Indeed, $||Dk^{\underline{\theta}}_{n}|| \leq \prod_{j\geq 2}^{n} ||\Psi^{\underline{\theta}}_{j}|| \cdot ||Dk^{\underline{\theta}}_{1}||$ implies that:

$$\begin{aligned}
\log\left(||Dk^{\underline{\theta}}_{n}||\right) &\leq \sum_{j=2}^{n} \log||\Psi^{\underline{\theta}}_{j}|| + C_0 \\
&\leq \sum_{j=2}^{n} \log|(||\mathrm{Id}|| + ||\Psi^{\underline{\theta}}_{j} - \mathrm{Id}||) + C_0 \\
&\leq \sum_{j=2}^{n} C'' \mu^{-j\varepsilon} + C_0 \\
&\leq \frac{C'''}{1-\mu^{\varepsilon}} = C_2
\end{aligned}$$

It follows that,

$$\begin{aligned}
||Dk^{\underline{\theta}}_{m+l} - Dk^{\underline{\theta}}_{m}|| &\leq \sum_{j=0}^{l-1} ||Dk^{\underline{\theta}}_{m+j+1} - Dk^{\underline{\theta}}_{m+j}|| \\
&\leq \sum_{j=0}^{l-1} ||D\Psi^{\underline{\theta}}_{m+j+1} - \mathrm{Id}|| \cdot ||Dk^{\underline{\theta}}_{m+j}|| \\
&\leq C_2 \cdot \sum_{j=0}^{l-1} C'' \mu^{-(m+j+1)\varepsilon} \leq C_3 \cdot \mu^{-m}
\end{aligned}$$

which shows that $\{\Psi^{\underline{\theta}}_{n}\}_{n\geq 0}$ is a Cauchy sequence also in the C^1 metric, and so it converges to a C^1 map $k^{\underline{\theta}}$.

We are left with showing that $k^{\underline{\theta}}$ is $C^{1+\varepsilon}$. This is true for $k^{\underline{\theta}}_{n}$ for all $n \geq 0$. For a given $\underline{\theta} \in \Sigma^-$ we write:

$$I_n(x,y) = |Dk_n^{\underline{\theta}}(x) - Dk_n^{\underline{\theta}}(y)| < H_n \cdot |x-y|^{\varepsilon}, n \geq 0, x, y \in G^*(\theta_0).$$

By equation 3.1, equation 3.3 and the fact that $Dk_n^{\underline{\theta}}$ are bounded we have that:

$$\begin{aligned}
I_n(x,y) &= |D(\Psi_n \circ k_{n-1}^{\underline{\theta}})(x) - D(\Psi_n \circ k_{n-1}^{\underline{\theta}})(y)| \leq \\
&\leq |D\Psi_n(k_{n-1}^{\underline{\theta}})(x)\left(Dk_{n-1}^{\underline{\theta}}(x) - Dk_{n-1}^{\underline{\theta}}(y)\right)| + |\Big(D\Psi_n(k_{n-1}^{\underline{\theta}})(x) - \\
&\quad D\Psi_n(k_{n-1}^{\underline{\theta}})(y)\Big) Dk_{n-1}^{\underline{\theta}}(y)| \\
&\leq (1 + C'' \cdot \mu^{(-n+1)\varepsilon}) \cdot I_{n-1}(x,y) + C_2 \cdot C_f \cdot ||D_g|| \cdot |k_{n-1}^{\underline{\theta}}(x) - k_{n-1}^{\underline{\theta}}(y)| \\
&\leq (1 + C'' \cdot \mu^{(-n+1)\varepsilon}) \cdot I_{n-1}(x,y) + C_2 \cdot C_f \cdot ||D_g|| \cdot \mu^{-n+1} \cdot |x-y|^{\varepsilon} \\
&\leq \left((1 + C'' \cdot \mu^{(-n+1)\varepsilon}) \cdot H_{n-1} + C_2 \cdot C_f \cdot ||D_g|| \cdot \mu^{-n+1}\right) \cdot |x-y|^{\varepsilon},
\end{aligned}$$

which inductively shows that these functions have Hölder continuous derivatives. Additionally, we can choose the Hölder constants satisfying the relation:

$$H_n \leq (1 + C'' \cdot \mu^{(-n+1)\varepsilon}) \cdot H_{n-1} + C_1' \cdot \mu^{-n+1} \tag{3.4}$$

and then the sequence $\{H_n\}_{n \geq 1}$ is bounded. Effectively, it is crescent and if $H_{n-1} > 1$ then $H_n \leq (1 + C'' \cdot \mu^{(-n+1)\varepsilon} + C_1' \cdot \mu^{-n+1}) \cdot H_{n-1} \leq (1 + C_2' \cdot \mu^{(-n+1)\varepsilon}) \cdot H_{n-1}$ and using the same strategy as above we have:

$$\begin{aligned}
\log H_n &\leq \log H_{n-1} + \log(1 + C_2' \cdot \mu^{(-n+1)\varepsilon}) \\
&\leq \sum_{j=1}^{n-1} \log(1 + C_2' \cdot \mu^{j\varepsilon}) \leq \sum_{j=1}^{n-1} C_2' \cdot \mu^{j\varepsilon} \leq H.
\end{aligned}$$

as stated.

Finally, for each pair $x, y \in G^*(\theta_0)$ there is $n \geq 0$ such that $Dk_n^{\underline{\theta}} - Dk^{\underline{\theta}}$ is less than $|x-y|^{\varepsilon}$ so, by triangle inequality we have $Dk^{\underline{\theta}}(x) - Dk^{\underline{\theta}}(y) < (H+2) \cdot |x - y^{\varepsilon}|$.

All the constants appearing in the estimatives above depend continuously (actually, they are simple functions) on the C^1 norm of g and the Hölder constant and exponent of Dg, and so, for any g' sufficiently close to g all of those estimatives would be the same except by a minor pre-fixed error. This implies that the convergence we just shown is uniform not only over Σ^- but also on a small neighborhood of g in the Hölder topology.

Now, for a given $\underline{\theta} \in \Sigma^-$, the norm of $D\left(k^{\underline{\theta}} \circ (k_n^{\underline{\theta}})^{-1}\right)$ is uniformly controlled by the diameter of $G^*(\underline{\theta}_n)$. Indeed,

$$D\left(k^{\underline{\theta}} \circ (k_n^{\underline{\theta}})^{-1}\right)(z) = \lim_{m\to\infty} D\Psi_m^{\underline{\theta}}(k_m^{\underline{\theta}} \circ (k_n^{\underline{\theta}})^{-1}(z)) \cdot \ldots \cdot D\Psi_{n+1}^{\underline{\theta}}(z)$$

and by using the same methods along this proof together with $|D\Psi_n^{\underline{\theta}}(z) - \mathrm{Id}| \leq C_f \cdot ||Dg|| \cdot r^{\varepsilon}_{\underline{\theta}_{n-1}}$ we can show that:

$$\log D\left(k^{\underline{\theta}} \circ (k_n^{\underline{\theta}})^{-1}\right) \leq C_4 \cdot r^{\varepsilon}_{\underline{\theta}_n}.$$

Also analogously we have that:

$$||k^{\underline{\theta}} - k_n^{\underline{\theta}}|| \leq C_4' \cdot r^{\varepsilon}_{\underline{\theta}_n}.$$

So for any $\underline{\theta}_1$ and $\underline{\theta}_2 \in \Sigma^-$ we have $||k^{\underline{\theta}_1} - k^{\underline{\theta}_2}|| \leq C_4' \cdot \mathrm{diam}(G^*(\underline{\theta}_1 \wedge \underline{\theta}_2))^{\varepsilon}$ and $||\log(|D\left(k^{\underline{\theta}_1} \circ (k^{\underline{\theta}_2})^{-1}\right)||| \leq C_4 \cdot \mathrm{diam}(G^*(\underline{\theta}_1 \wedge \underline{\theta}_2))^{\varepsilon}$. In this manner, in the metric described in the lemma, the association $\underline{\theta} \to k^{\underline{\theta}}$ is Hölder continuous as we wished to obtain.

□

Equation 3.2 can be written as the following result:

Corollary 3.1.0.1. *The diameter of the sets $G^*(\underline{\theta}_n)$ is of order $|Df_{\underline{\theta}_n}|$.*

As another immediate consequence of the equation 3.2 we have the following bounded distortion property:

Corollary 3.1.0.2. *There is a constant $C > 0$ such that for every pair of points c_1, $c_2 \in K(a)$ we have*

$$C^{-1} \leq \left|\frac{Df_{\underline{\theta}_n}(c_1)}{Df_{\underline{\theta}_n}(c_2)}\right| \leq C$$

for all $\underline{\theta}_n \in \Sigma^{fin}$ with $\theta_0 = a$.

Notice that the limit geometries depend on the choice of the base point c_{θ}, because the maps $\Psi_n^{\underline{\theta}}$ depend on it. However, the corollary just above shows for different choices of base point, the norm expansion factor of $\Psi_n^{\underline{\theta}}$ is bounded between $|C^{-1} \cdot Df_{\underline{\theta}_n}(c_{\theta_0})|$ and $|C \cdot Df_{\underline{\theta}_n}(c_{\theta_0})|$ for a fixed choice of c_{θ_0}. Since these maps also send the base point to 0 we have that different choices of base points c_1 and c_2 result in different limit geometries that are related by

$$k_1^{\underline{\theta}} = A \cdot k_2^{\underline{\theta}},$$

where A is a map in $Aff(\mathbb{C})$ bounded by some constant $C > 0$. So, less affine transformations, the limit geometries do not depend on the base point. Every time we mention the limit geometries of a Cantor set consider that a set of base points has been already fixed. The bounded distortion property can be improved:

Corollary 3.1.0.3. *There is a constant $C > 0$ such that for every pair of points x, $y \in G(\underline{\theta_0})$*

$$C^{-1} \leq \frac{Df_{\underline{a}}(x)}{Df_{\underline{a}}(y)} \leq C$$

for all $\underline{a} \in \Sigma^{fin}$. The larger n is and the closer x, y are to each other, the closer the ratio above is to 1.

Proof. Given any $\underline{\theta}^-$ whose final coincides with the word $\underline{a}$, by the proof of lemma 3.1.1, we have $\log D\left(k^{\underline{\theta}} \circ (k_n^{\underline{\theta}})^{-1}\right) \leq C_4 \cdot r_{\underline{\theta}_n}^{\varepsilon}|$ which implies

$$\exp(-C\mu^{-n\varepsilon}) \leq \frac{Df_{\underline{a}}(x)}{Df_{\underline{a}}(y)} \cdot \frac{Dk^{\underline{\theta}}(y)}{Dk^{\underline{\theta}}(x)} \leq \exp(C\mu^{-n\varepsilon}).$$

In view of the Hölder continuity of $\underline{\theta} \mapsto k^{\underline{\theta}}$, making n large and $|x - y|$ small the proof is complete.

□

Remark 3.1.1. There is some abuse of notation above, since the quotient does not really make sense. One can give this notation the proper meaning by considering that $\frac{A}{B}$, A, $B \in GL(\mathbb{R}^n)$ represents the interval defined by $[m(A \cdot B^{-1}), ||A \cdot B^{-1}||]$ and understand the inequalities $C^{-1} < \frac{A}{B} < C$ as $[m(A \cdot B^{-1}), ||A \cdot B^{-1}||] \subset [C^{-1}, C]$.

Remark 3.1.2. If the map g defining the cantor set is $C^r, r \in \mathbb{N}, r \geq 2$ than the convergence can be taken in the C^r metric. This happens because the composition with affine maps on the definition of $\Psi^{\underline{\theta}_n}$ "flattens" the derivatives of $f_{\theta_{-n},\theta_{-n+1}}$. As we have seen above the first order derivatives are close to the identity, or close (in norm) to $1 = r_{\underline{\theta}_n}^0$. Analogously, the derivative of r order has norm less than $r_{\underline{\theta}_n}^{r-1}$. This allow us to control the C^r norm of $k_n^{\underline{\theta}}$ by the C^1 metric and so the convergence is proved also in this metric.

3.2 Configurations and Renormalizations

Given a dynamically defined conformal Cantor set K, described by $(\mathrm{A}, \mathcal{B}, \Sigma, g)$ and a piece $G(a), a \in \mathrm{A}$ we say that a C^r, $r > 1$ diffeomorphism $h : G(a) \to U \subset \mathbb{C}$ is a *configuration* of the piece of Cantor set. In particular, if h is the restriction of a map $A \in Aff(\mathbb{C})$ to its domain then we say it is an affine configuration.

We write $\mathcal{P}(a)$ for the space of all configurations of the piece $\overline{G(a)}$ equipped with the topology of uniform convergence. The space $Aff(\mathbb{C})$ acts on $\mathcal{P}(a)$ by left composition and we denote the quotient space of this action $\overline{\mathcal{P}}(a)$. We also refer to $\mathcal{P}$ as the union $\bigcup_{a\in\mathrm{A}} \mathcal{P}(a)$ and $\overline{\mathcal{P}} = \bigcup_{a\in\mathrm{A}} \overline{\mathcal{P}}(a)$.

Configurations can be seen as the manner in which the Cantor set is embedded into the complex plane. For example, by using an affine configuration we can rotate, scale and translate a Cantor set that would be fixed in a certain region of the plane. Also, if $h : \bigsqcup_{a\in\mathrm{A}} G(a) \to U \subset \mathbb{C}$ is a C^r diffeomorphism such that Dh is conformal at the Cantor set K, then $h(K)$ can be seem as a Cantor set in the prescribed sense. To see this we need only to consider new sets $\tilde{G}(a) = h(G(a))$ and $\tilde{g} = h \circ g \circ h^{-1}$.

For any given configuration h we say that $h \circ f_{(\theta_0,\theta_1)}$ is the *renormalization* by $f_{(\theta_0,\theta_1)}$ of the given configuration and we write the renormalization operator as

$$T_{\theta_0,\theta_1} : \mathcal{P}(\theta_0) \to \mathcal{P}(\theta_1)$$

$$h \mapsto h \circ f_{(\theta_0,\theta_1)}.$$

Since this operator commutes with the action of affine maps over $\mathcal{P}$ it is well defined over $\overline{\mathcal{P}}$.

If we apply n consecutive renormalizations, by $f_{(\theta_0,\theta_1)}, \dots, f_{(\theta_{n-2},\theta_{n-1})}$, $f_{(\theta_{n-1},\theta_n)}$ we end up with $h \circ f_{\underline{\theta}_n}$, $\underline{\theta}_n = (\theta_0, \theta_1, \dots, \theta_n)$. Based on that, we define for any word $\underline{a} \in \Sigma_n$, $\underline{a} = (a_0, \dots, a_n)$ the renormalization operator operator as:

$$T_{\underline{a}} : G(a_0) \to G(a_n)$$

$$h \mapsto h \circ f_{\underline{a}}$$

This construction implies that $T_{\underline{a}} \circ T_{\underline{b}} = T_{\underline{ab}}$ for every pair of words a, $b \in \Sigma^{fin}$.

Notice that the image of $h \circ f_{\underline{\theta}_n}$ corresponds to the image by h of the set $G(\underline{\theta}_n)$, that is, the configuration of a piece of the n-th step in the definition of the Cantor set K, and, as seen in the lemma 3.1.1 and its proof, this map is close to $h \circ (\Phi_{\underline{\theta}_n})^{-1} \circ k^{\underline{\theta}}$. This observation indicates that the limit geometries work as an atractor in the space of configurations under the action of renormalizations (less affine transformations). The next two lemmas give a more precision statement of this fact.

Firstly, consider the space $\mathcal{A} = Aff(\mathbb{C}) \times \Sigma^-$. It represents the affine configurations of limit geometries and can be continuously associated with a subset of the space of configurations by:

$$I : Aff(\mathbb{C}) \times \Sigma^- \to \mathcal{P}$$

$$(A, \theta) \mapsto A \circ k^{\underline{\theta}}$$

Notice that this identification is continuous.

Lemma 3.2.1. *The action of the renormalization operator over $\mathcal{A}$ is given by:*

$$T_{\theta_1,\theta_0}(A, \underline{\theta}) = (A \circ F^{\underline{\theta}\theta_1}, \underline{\theta}\theta_1)$$

where $F^{\underline{\theta}\theta_1}$ is in $Aff(\mathbb{C})$.

Proof. From the lemma 3.1.1, in which we established the existence of limit geometries, we have that:

$$\begin{aligned} k^{\underline{\theta}\theta_1} \circ (k^{\underline{\theta}} \circ f_{\theta_0,\theta_1})^{-1} &= \lim_{n\to\infty} k^{\underline{\theta}\theta_1}_{n+1} \circ (k^{\underline{\theta}}_n \circ f_{\theta_0,\theta_1})^{-1} \\ &= \lim_{n\to\infty} \Phi_{(\underline{\theta}\theta_1)_{n+1}} \circ f_{(\underline{\theta}\theta_1)_{n+1}} \circ (\Phi_{\underline{\theta}_n} \circ f_{\underline{\theta}_n} \circ f_{\theta_0,\theta_1})^{-1} \\ &= \lim_{n\to\infty} \Phi_{(\underline{\theta}\theta_1)_{n+1}} \circ \Phi^{-1}_{\underline{\theta}_n} \end{aligned}$$

which implies that that the last limit exists and in particular belongs to $Aff(\mathbb{C})$ since this is a closed subset of the space of configurations. So, for any $\underline{\theta} = (\dots, \theta_{-1}, \theta_0) \in \Sigma^-$ and $(\theta_0, \theta_1) \in B$ we define $(F^{\underline{\theta}\theta_1})^{-1} = \lim_{n\to\infty} \Phi_{(\underline{\theta}\theta_1)_{n+1}} \circ \Phi^{-1}_{\underline{\theta}_n}$ and we have that $F^{\underline{\theta}\theta_1} \circ k^{\underline{\theta}\theta_1} = k^{\underline{\theta}} \circ f_{\theta_0,\theta_1}$ as we wanted to show.

□

For each configuration $h : G_{\theta_0} \to \mathbb{C}$ and any limit geometry $k^{\underline{\theta}}$, $\underline{\theta} \in \Sigma^-$, the map $h^{\underline{\theta}} : k^{\underline{\theta}}(G_{\theta_0}) \to \mathbb{C}$ is defined as $h^{\underline{\theta}} = h \circ (k^{\underline{\theta}})^{-1}$, which we call the *perturbation part of h relative to $\underline{\theta}$*. By definition, $h = h^{\underline{\theta}} \circ k^{\underline{\theta}}$. Also, for each configuration $h \in \mathcal{P}(a)$ we consider the *scaled* version of it the map $A_h \circ h$, where $A_h \in Aff(\mathbb{R}^2)$ is an affine transformation such that $A_h \circ h(c_a) = 0$ and $D(A_h \circ h)(c_a) = \mathrm{Id}$. For example, the scaled version of a limit geometry is the limit geometry itself.

Lemma 3.2.2. *Let K be a conformal Cantor set and $h \in \mathcal{P}(a)$ a configuration of a piece in K. Then, for every n large enough, if we consider a renormalization of h by any possible $f_{\underline{\theta}_n}$, $\underline{\theta}_n \in \Sigma^{fin}$, and represent it by h_n, the perturbation part of the scaled version of h_n relative to $\underline{\theta\theta}_n$ converges exponentially to the identity for any $\underline{\theta} \in \Sigma^-$. In other terms, $||A_{h_n} \circ h_n^{\underline{\theta\theta}_n} - Id|| < C \cdot diam(G(\underline{\theta}_n))^{-n\varepsilon} < C \cdot \mu^{-n\varepsilon}$, $C > 0$ a constant depending only on the Cantor set K and the initial configuration h.*

Proof. As seen in lemma 3.1.1, $h^{\underline{\theta}} \circ h^{\underline{\theta}} \circ f_{\theta_0,\theta_1} = h^{\underline{\theta}} \circ F^{\underline{\theta}\theta_1} \circ k^{\underline{\theta}\theta_1}$, and so concatenating all the renormalizations we have:

$$h_n^{\underline{\theta\theta}_n} = h^{\underline{\theta}} \circ F^{\underline{\theta}\theta_1} \circ F^{(\underline{\theta}\theta_1)\theta_2} \circ \cdots F^{(\underline{\theta\theta}_{n-1})\theta_n},$$

or equivalently, defining $F^{\underline{\theta\theta}_n} = F^{\underline{\theta}\theta_1} \circ F^{(\underline{\theta}\theta_1)\theta_2} \circ \cdots F^{(\underline{\theta\theta}_{n-1})\theta_n}$, $h_n = h \circ F^{\underline{\theta\theta}_n}$.

The expansion term of $F^{\underline{\theta\theta}_n}$ is equal to $\dfrac{\text{diam}(k^{\underline{\theta}}(G_{\underline{\theta}_n}))}{\text{diam}(k^{\underline{\theta\theta}_n}(G_{\theta_n}))}$, because of the relation $F^{\underline{\theta\theta}_n} \circ k^{\underline{\theta\theta}_n} = k^{\underline{\theta}} \circ f_{\underline{\theta}_n}$ applied to the domain $G(\theta_n)$. Since any of the maps $k^{\underline{\theta}}$, $\underline{\theta} \in \Sigma^-$, has a uniformly bounded derivative, there is a constant $C > 0$ such that the expansion term of $F^{\underline{\theta\theta}_n}$ is less than $C \cdot \text{diam}(G_{\underline{\theta}_n})$ and more than $C^{-1} \cdot \text{diam}(G_{\underline{\theta}_n})$.

On the other hand, by definition and the fact that $Dk^{\underline{\theta\theta}_n}(c_n) = \text{Id} \equiv 1$ and the derivative of $h^{\underline{\theta}}$ is bounded from bellow and above, $m(A_{h_n})$ and $|A_{h_n}|$ are also controlled by $\text{diam}(G_{\underline{\theta}_n})$ in the same way as $F^{\underline{\theta\theta}_n}$. Now, the domain of interest of h on the relation $h_n = h \circ F^{\underline{\theta\theta}_n}$ is the set $k^{\underline{\theta}}(G(\underline{\theta}_n))$ $(= F^{\underline{\theta\theta}_n} \circ k^{\underline{\theta\theta}_n}(G_{\theta_n}))$ whose size is also controlled by $\text{diam}(G_{\underline{\theta}_n})$. Then, arguing in the same way as in the analysis of the functions $\Psi_m^{\underline{\theta}}$ on lemma 3.1.1, that is, using a relation of the type:

$$|h(z+h) - (h(z) + Dh(z).h)| < C \cdot |h|^{1+\varepsilon},$$

and remembering that $A_{h_n} \circ {h_n}^{\underline{\theta\theta}_n}(0) = 0$ and $D(A_{h_n} \circ {h_n}^{\underline{\theta\theta}_n})(0) = \text{Id}$ we have that

$$||A_{h_n} \circ {h_n}^{\underline{\theta\theta}_n} - \text{Id}|| < C \cdot \text{diam}(G(\underline{\theta}_n))^{1+\varepsilon}$$

The exponential decay of ratio μ is a consequence of corollary 3.1.0.1.

□

3.3 Recurrent compact criterion

Given a pair of Cantor sets K and K' we are interested in finding configurations h and h'such that $h(K)$ intersects $h'(K')$. More importantly we want to find a criterion under which this intersection is stable, that is for small perturbations $\tilde{h}$, $\tilde{h}'$, $\tilde{K}$, $\tilde{K}'$ the sets $\tilde{h}(\tilde{K})$ and $\tilde{h}'(\tilde{K}')$ also have a non-empty intersection. We define, then, a topology for configurations

With these ideas in mind we consider, for any pair of configurations $(h_a, h'_{a'}) \in \mathcal{P}_a \times \mathcal{P}'_{a'}$ we say that it is:

- *linked* whenever $h_a(\overline{G(a)}) \cap h'_{a'}(\overline{G(a')}) \neq \emptyset$.
- *intersecting* whenever $h_a(K(a)) \cap h'_{a'}(K'(a')) \neq \emptyset$.
- *stably intersecting* whenever $\tilde{h}_a(\tilde{K}(a)) \cap \tilde{h}'_{a'}(\tilde{K}'(a') \neq \emptyset$ for any pairs of Cantor sets $(\tilde{K}, \tilde{K}') \in \Omega_\Sigma$ in a small neighbourhood of (K, K') and any configuration pair $(\tilde{h}_a, \tilde{h}'_{a'})$ that is sufficiently close to $(h_a, h'_{a'})$ in the $C^{1+\varepsilon}$ topology at $G(a) \cap \tilde{G}(a)$ and $G(a') \cap \tilde{G}'(a')$.

It is better to work with $\mathcal{Q}$, the quotient of $\mathcal{P} \times \mathcal{P}'$ by the diagonal action of $Aff(\mathbb{C})$. An element in $\mathcal{Q}$, represented by a pair (h, h') is called a relative configuration or as mentioned sometimes a relative positioning of the Cantor sets. Since the action of the affine group preserves the linking, intersecting or stably intersection of a pair of configurations, these notions are defined for relative configurations too. Also, we can define for any pair in $\mathcal{P} \times \mathcal{P}'$ and any pair of words $(\underline{a}, \underline{a}') \in \Sigma^{fin} \times \Sigma'^{fin}$ a renormalization operator

$$\tilde{\mathcal{T}}_{\underline{a},\underline{a}'}(h, h') := (T_{\underline{a}}(h), T_{\underline{a}'}(h')).$$

For the same reasons as above it can also be defined over $\mathcal{Q}$. Also, we can allow one of the words $\underline{a}$ or $\underline{a}'$ to be void. In that case, the operator only acts at the now trivial coordinate, for example

$$\tilde{\mathcal{T}}_{\emptyset,\underline{a}'}(h, h') = (h, T_{\underline{a}'}(h')).$$

Under this context, we have the following:

Lemma 3.3.1. *A pair of relative configurations (h_0, h'_0) is intersecting if, and only if, there is a relative compact sequence of relative configurations (h_n, h'_n) obtained inductively by applying a renormalization operator on one of the configurations on the pair.*

Proof. Indeed, if $h_0(K)$ and $h'_0(K')$ are intersecting at a point $q = h_0(p) = h'_0(p')$, ($p \in K$ and $p' \in K'$) consider the sequences $H(p) = (\theta_0, \theta_1, \dots) \in \Sigma$ and $H'(p) = (\theta'_0, \theta'_1, \dots) \in \Sigma'$ where H and H' are the homeomorphism defined in the section 2.1. We can construct a sequence of configurations

(h_n, h'_n), obtained by successively renormalizing by the functions $f_{\theta_{-i},\theta_{-i+1}}$ and $f'_{\theta_{-i},\theta_{-i+1}}$ chosen in a careful order such that the ratio of diameters of the images of the configurations are bounded away from infinite and zero. However, such pairs of configurations are always intersecting, since the point q belongs to both their images, and so, when seen in $\mathcal{Q}$ this sequence is relatively compact.

On the other hand, if such a relative compact sequence exists, choosing points $p_n \in h_n(K) \subset h_0(K)$ and $p'_n \in h'_n(K') \subset h'_0(K')$ we have that $\lim_{n\to\infty} p_n = p = \lim_{n\to\infty} p'_n$ and then $p \in h_0(K) \cap h'_0(K') \neq \emptyset$ as we wanted to show.

□

This lemma is really important in finding a criterion for stable intersection in Cantor sets. For it, we will work with the space of relative affine configurations of limit geometries, that is, $\mathcal{C}$, the quotient by the action of the affine group on the left of $\mathcal{A} \times \mathcal{A}'$. The concepts above were well defined for pairs of affine configurations of limit geometries, and again, since they are invariant by the action of $Aff(\mathbb{C})$ they are also defined for relative configurations in $\mathcal{C}$

Definition 3.3.1 (Recurrent compact criterion)**.** Let $\mathcal{L}$ be a compact set in $\mathcal{C}$. We say that $\mathcal{L}$ is *recurrent*, if for any relative affine configuration (of limit geometries) $v \in \mathcal{L}$, given by $(A, \underline{\theta})$, $(A', \underline{\theta}')$ there are words $\underline{a}$, $\underline{a}'$ such that the renormalized pair $T_{\underline{a}}(A, \underline{\theta})$, $T_{\underline{a}'}(A', \underline{\theta}')$ represents a relative affine configuration v, $v \in \text{int } \mathcal{L}$.

If such a renormalization can be done using words $\underline{a}$ and $\underline{a}'$ such that their total size combined is equal to one, we say that such a set is *immediately recurrent*.

Theorem 3.3.1. *The following properties are true:*

1. *Every recurrent compact set is contained on an immediately recurrent compact set.*

2. *Given a recurrent compact set $\mathcal{L}$ (resp. immediately recurrent) for g, g', for any $\tilde{g}$,$\tilde{g}'$ in a small neighbourhood of $(g, g') \in \Omega_\Sigma \times \Omega_{\Sigma'}$ we can choose points $\tilde{c}_a \in \tilde{G}(a) \subset \tilde{K}$ and $\tilde{c}_{a'} \in \tilde{G}(a') \subset \tilde{K}'$ respectively close to the pre-fixed c_a and $c_{a'}$ in a manner that $\mathcal{L}$ is also a recurrent compact set for $\tilde{g}$ and $\tilde{g}'$.*

3. *Any relative configuration contained in a recurrent compact set is stably intersecting.*

Proof. 1. We remember that $\mathcal{A}$ is a metric space, so for every point $v \in \mathcal{L}$, the recurrent compact set, there is a closed ball around v that is carried by a renormalization (given by a pair of words $\underline{a}^v$, $\underline{a}'v$) into the interior of $\mathcal{L}$. Since this set is compact, there is a finite number N, compact sets (balls) $\mathcal{L}^i$ and associated pair of words $(\underline{a}^i, \underline{a}'^i)$ for $1 \leq i \leq N$ such that $\mathcal{L}^i$ is carried into the interior of $\mathcal{L}$ by the renormalizations associated to the pair $(\underline{a}^i, \underline{a}'^i)$. Now considering for every such pair, all the pairs of words $(\underline{b}^{i,j}, (\underline{b}'^{i,j})$ that are contained on $(\underline{a}^i, \underline{a}'^i)$ we construct an immediately recurrent Cantor set $\mathcal{L}'$ on the following way.

First, we choose one of the pairs $(\underline{b}^{i,j_1}, (\underline{b}'^{i,j_1})$ with total length one and construct a Compact set $\mathcal{L}^{i,j_1}$ such that $T_{(\underline{b}^{i,j_1},(\underline{b}'^{i,j_1})} \subset \text{int } \mathcal{L}^{i,j_1}$ and $T_{(\underline{b}^{i,\overline{j_1}},(\underline{b}'^{i,\overline{j_1}})}\mathcal{L}^{i,j_1} \subset \text{int } \mathcal{L}$, where $(\underline{b}^{i,\overline{j_1}}, (\underline{b}'^{i,\overline{j_1}})$ is the word pair that need to be concatenated into $(\underline{b}^{i,j_1}, (\underline{b}'^{i,j_1})$ to result in $(\underline{a}^i, \underline{a}'^i)$. This can be done in the same manner the sets $\mathcal{L}^i$ were constructed just above.

Then we inductively construct a sequence of compact sets $\mathcal{L}^{i,j_k}$ $k = 1, \ldots, m$ such that for each $\mathcal{L}^{i,j_k}$ there is a renormalization by words of total size one that carries it into int $\mathcal{L}^{i,j_{k+1}}$ for $k = 1, \ldots, m-1$ and carries $\mathcal{L}^{i,j_m}$ into int $\mathcal{L}$. Then, taking $\mathcal{L}' = \bigcup_{i,j_k} \mathcal{L}^{i,j_k}$ we have an immediately recurrent compact set.

2. Using the decomposition $\mathcal{L} = \cup_{i=1}^N \mathcal{L}_i$, described in the previous argument, it is enough to show that for any $\varepsilon > 0$ there is δ such that for any pair $(\tilde{g}, \tilde{g}')$ in $U_{K,\delta} \times U_{K',\delta}$ the renormalizations operators associated to the words $\underline{a}^i$ and $\underline{a}'^i$, $i = 1, \ldots, N$, denoted by $\tilde{\mathcal{T}}_{\underline{a}^i,\underline{a}'^i} = (\tilde{T}_{\underline{a}^i}, \tilde{T}_{\underline{a}'^i})$ satisfy $||\tilde{\mathcal{T}} - \mathcal{T}|| < \varepsilon$. These operators are obtained by composition of a finite number of operators arising from pairs of words of total size one, and so we need only to that for any $\varepsilon' > 0$ a δ can be find such that $|T_{a,b} - \tilde{T}_{a,b}| < \varepsilon'$ and $|T_{a',b'} - \tilde{T}_{a',b'}|$ for all pairs $(a, b) \in B$ and $(a', b') \in B'$, or precisely, that for any $\underline{\theta} \in \Sigma^-$ and $\theta_1 \in \mathrm{A}$, $|F^{\underline{\theta}\theta_1} - \tilde{F^{\underline{\theta}\theta_1}}| < \varepsilon'$ and its analogous for K' and $\tilde{K}'$. However as seen in lemma 3.2.1 $F^{\underline{\theta}\theta_1} = k^{\underline{\theta}} \circ f_{\theta_0,\theta_1} \circ (k^{\underline{\theta}\theta_1})^{-1}$ an we need only to show that the values of all the function and their derivatives above at a fixed point x do not change much when considering its $\tilde{g}$ version. This is done the following Lemma.

Lemma 3.3.2. *For any Cantor set K given by a map $g \in \Omega_\Sigma$ and any $\varepsilon > 0$, there is a small $\delta > 0$ such that for any map $\tilde{g} \in U_{K,\delta}$ there is a choice of points $\tilde{c}_a \in \tilde{G}(a)$ sufficiently close to the points $c_a \in G(a)$ respectively in a manner that the resultant limit geometries* *satisfy* $||\tilde{k}^{\underline{\theta}} - k^{\underline{\theta}}||_{C^1} < \varepsilon$ *for all* $\underline{\theta} \in \Sigma^-$ *in the largest domain both these maps are defined.*

Proof. First we fix $c_a \in \mathrm{int}G(a)$. This can be done because of the additional hypothesis on the sets $G(a)$, described at the end of section 2.1. By the definition of $U_{K,\delta'}$, we can choose $\tilde{c}_a$ such that $H(c_a) = \tilde{H}(\tilde{c}_a)$, and that implies $|\tilde{c}_a - c_a| < \delta$ for every δ' sufficiently small. Let us analize the limit of $||\tilde{k}_n^{\underline{\theta}} - k_n^{\underline{\theta}}||_{C^0}$.

Using the results and the same notation as in 3.1.1, we observe that:

$$\begin{aligned}
|\tilde{k}_{n+1}^{\underline{\theta}}(x) - k_{n+1}^{\underline{\theta}}(x)| &= |(\tilde{\Psi}_{n+1}^{\underline{\theta}} \circ \tilde{k}_n^{\underline{\theta}})(x) - (\Psi_{n+1}^{\underline{\theta}} \circ k_n^{\underline{\theta}})(x)| \\
&\leq |\Psi_{n+1}^{\underline{\theta}}(\tilde{k}_n^{\underline{\theta}}(x)) - \Psi_{n+1}^{\underline{\theta}}(k_n^{\underline{\theta}}(x))| + |(\tilde{\Psi}_{n+1}^{\underline{\theta}} - \Psi_{n+1}^{\underline{\theta}})(\tilde{k}_n^{\underline{\theta}}(x))| \\
&\leq ||D\Psi_{n+1}^{\underline{\theta}}|| \cdot |\tilde{k}_n^{\underline{\theta}}(x) - k_n^{\underline{\theta}}(x)| + ||\tilde{\Psi}_{n+1}^{\underline{\theta}} - \Psi_{n+1}^{\underline{\theta}}|| \\
&\leq (1 + C''\mu^{-n\varepsilon}) \cdot |\tilde{k}_n^{\underline{\theta}}(x) - k_n^{\underline{\theta}}(x)| + 2\tilde{C}\mu^{-n\varepsilon}
\end{aligned}$$

and then, passing to the limit and using induction as in 3.1.1, we show that

$$||\tilde{k}^{\underline{\theta}} - k^{\underline{\theta}}||_{C^0} < \varepsilon$$

provided that the distance between $k_1^{\underline{\theta}}$ and $\tilde{k}_1^{\underline{\theta}}$ is small enough, but this is controlled by the difference between f_{θ_{-1},θ_0} and $\tilde{f}_{\theta_{-1},\theta_0}$, which can be make small enough by choosing $\delta > 0$ sufficiently small.

There is a little detail that needs to be explained: it is not true that we can always apply $\Psi_{n+1}^{\underline{\theta}}$ to the image of $\tilde{k}_n^{\underline{\theta}}$; at least no considering the same domain used in the definition 2.1. However, the domain can be extended to a larger set, simply because

The argument for the C^1 norm is similar:

$$\begin{aligned}
|D\tilde{k}_{n+1}^{\underline{\theta}}(x) - Dk_{n+1}^{\underline{\theta}}(x)| &= |D(\tilde{\Psi}_{n+1}^{\underline{\theta}} \circ \tilde{k}_n^{\underline{\theta}})(x) - D(\Psi_{n+1}^{\underline{\theta}} \circ k_n^{\underline{\theta}})(x)| \\
&\leq |D(\tilde{\Psi}_{n+1}^{\underline{\theta}} \circ \tilde{k}_n^{\underline{\theta}})(x) - D(\tilde{\Psi}_{n+1}^{\underline{\theta}} \circ k_n^{\underline{\theta}})(x)| + \\
&|D(\tilde{\Psi}_{n+1}^{\underline{\theta}} \circ k_n^{\underline{\theta}})(x) - D(\Psi_{n+1}^{\underline{\theta}} \circ k_n^{\underline{\theta}})(x)| \\
&\leq ||D\Psi_{n+1}^{\underline{\theta}}|| \cdot |D\tilde{k}_n^{\underline{\theta}}(x) - Dk_n^{\underline{\theta}}(x)| + \\
&||D\tilde{\Psi}_{n+1}^{\underline{\theta}} - D\Psi_{n+1}^{\underline{\theta}}|| \cdot |Dk_n^{\underline{\theta}}(x)| \\
&\leq (1 + C''\mu^{-n\varepsilon}) \cdot |D\tilde{k}_n^{\underline{\theta}}(x) - Dk_n^{\underline{\theta}}(x)| + 2C''C_2\mu^{-n\varepsilon}
\end{aligned}$$

and proceeding as above the proof is complete $\square$

3. Given a recurrent compact set $\mathcal{L}$ relative to a pair of Cantor sets K and K' its image under

$$I : \mathcal{C} \to \mathcal{Q}$$

$$[(A, \theta), (A', \theta')] \mapsto [A \circ k^{\underline{\theta}}, A' \circ k^{\underline{\theta}'}]$$

is also a compact set, because this association is continuous. In what follows, whenever we work with the set $\mathcal{L}$ in the context of $\mathcal{Q}$ we are referencing the set $I(\mathcal{L})$. In this sense, following lemma 3.2.2, any pair $(A \circ k^{\underline{\theta}}, A' \circ k^{\underline{\theta}'})$ representing a relative affine configuration of limit geometries $v \in \mathcal{L}$ is intersecting. In light of the previous item, this implies that $A \circ \tilde{k}^{\underline{\theta}}$ and $A' \circ \tilde{k}^{\underline{\theta}'}$ also represent intersecting configurations for a pair of Cantor sets $(K', \tilde{K'})$ sufficiently close to (K, K').

Thus, it is enough to show that, for any pair of Cantor sets (K, K') that has a recurrent compact set $\mathcal{L}$ and a configuration pair $v \in \mathcal{L}$ represented by $[(A, \underline{\theta}), (A', \underline{\theta}')]$, if $h : \mathrm{Im}(A \circ k^{\underline{\theta}}) \to \mathbb{C}$ and $h' : \mathrm{Im}(A \circ k^{\underline{\theta}'}) \to \mathbb{C}$ are embeddings $C^{1+\varepsilon}$ close to the identity, then $h \circ A \circ k^{\underline{\theta}}$ and $h' \circ A' \circ k^{\underline{\theta}'}$ are also intersecting.

Again, it is a simple observation from real analysis that:

$$|h(z+\delta) - (h(z) + Dh(z) \cdot \delta)| < C \cdot |\delta|^{1+\varepsilon};\ z,\ z+\delta \in \mathrm{Dom}(h)$$

for some constant $C > 0$ that depends only on the map h. Without loss of generality we can assume the same for the map h'. Also again, as seen in the analysis of the functions $\Psi^{\underline{\theta}_n}$, if $\mathfrak{h} = A \circ h|_X \circ B$, A, $B \in Aff(\mathbb{R}^2)$ are maps chosen such that $\mathfrak{h}$ has a fixed point p with $D\mathfrak{h}(p) = \mathrm{Id}$ it is $C \cdot \mathrm{diam}(X)^{\varepsilon}$ - C^1 close to the identity, and the analogous result is valid for h'.

Now consider a decomposition of the recurrent compact set $\mathcal{L} = \cup \mathcal{L}^i$ as done in item 1 and fix a set of renormalizations $T_{\underline{a}^i, \underline{a}'^i}$ that carries each $\mathcal{L}^i$ to the interior of $\mathcal{L}$. We can find a $\delta > 0$ such that the distance of any $T_{\underline{a}^i, \underline{a}'^i}(v)$, $v \in \mathcal{L}$, to the frontier of $\mathcal{L}$ is bigger than δ.

We can construct a sequence of pairs of relative configurations $[h_n, h'_n]$, $n \geq 0$, obtained by inductively applying one of the renormalization operators $T_{\underline{a}^i, \underline{a}'^i}$ to the previous element, that starts with $[h \circ A \circ k^{\underline{\theta}}, h' \circ A' \circ k^{\underline{\theta}'})]$ and is relatively compact. We do as follows.

First, we obtain $[h_1, h_1']$ from $[h_0, h_0']$ by one of the renormalizations above that carries $[A\circ k^{\underline{\theta}}, A'\circ k^{\underline{\theta}'}]$ to the interior of $\mathcal{L}$. We choose to look at any of the pairs $[h_n, h_n']$ by its scaled representant $(A_{h_n}\circ h_n, A_{h_n}\circ h_n')$. In this manner we can write:

$$\begin{aligned}[h_1, h_1'] &= (A_{h_1}\circ h\circ A_{n1}\circ k^{\underline{\theta\theta}_{n1}}, A_{h_1}\circ h'\circ A'_{n'1}\circ k^{\underline{\theta'\theta'}_{n'1}}) \\ &= (\mathfrak{h}_1\circ k^{\underline{\theta\theta}_{n1}}, \mathfrak{h}_1'\circ B_1\circ k^{\underline{\theta'\theta'}_{n'1}})\end{aligned}$$

in which $\mathfrak{h}_1 = A_{h_1}\circ h\circ A_{n1}$, $\mathfrak{h}_1' = A_{h_1}\circ h'\circ A'_{n'1}\circ B_1^{-1}$ and $B_1 \in Aff(\mathbb{C})$ is constructed below.

First, we decompose $T = DA_{h_1}\cdot Dh'(A'_{n'1}(0)) = P\cdot B$; B, $P \in \mathrm{GL}_2(\mathbb{R})$ but B conformal. This decomposition can be done in this systematic way:

- If T is conformal, then $B = T$.
- If not, then if λ_1 and λ_2 are the eigenvalues of T (with repetition), we choose $B = \sqrt{\lambda_1\lambda_2}\cdot\mathrm{Id}$.

Under this notation $B_1 \in Aff(\mathbb{C})$ is equal to

$$B\cdot DA'_{n'1}\cdot(z-(A_{h_1}\circ h'\circ A'_{n'1})(0)) + (A_{h_1}\circ h'\circ A'_{n'1})(0)$$

Notice that if h and h' are chosen sufficiently C^1 close to the identity $(k^{\underline{\theta\theta}_{n1}}, B_1\circ k^{\underline{\theta'\theta'}_{n'1}}))$ represents a relative pair of configurations that is still in $\mathcal{L}$. In fact, by hypothesis $(k^{\underline{\theta\theta}_{n1}}, A_{A_{n1}\circ k^{\underline{\theta\theta}_{n1}}}\circ A'_{n'1}\circ k^{\underline{\theta'\theta'}_{n'1}}) \in \mathrm{int}(\mathcal{L})$, and then, since A_{h_1} is close to $A_{A_{n1}\circ k^{\underline{\theta\theta}_{n1}}}$, B_1 can be shown to be close to $A_{A_{n1}\circ k^{\underline{\theta\theta}_{n1}}}\circ A'_{n'1}$.

In light of a previous observation, the definition of A_{h_1}implies that

$$||\mathfrak{h}_1 - \mathrm{Id}||_{C^1} < C\cdot\mathrm{diam}(\mathrm{Im}(A_{n1}\circ k^{\underline{\theta\theta}_{n1}}))^{\varepsilon}.$$

Also, in a similar manner, if we construct $P_1 \in Aff(\mathbb{R}^2)$ as

$$P_1 = P\cdot(z-(A_{h_1}\circ h'\circ A'_{n'1})(0)) + (A_{h_1}\circ h'\circ A'_{n'1})(0),$$

then the definition of B_1 implies that:

$$||\mathfrak{h}_1' - P_1||_{C^1} < C\cdot\mathrm{diam}(\mathrm{Im}(A'_{n'1}\circ(B_1)^{-1}\circ k^{\underline{\theta'\theta'}_{n'1}}))^{\varepsilon}$$

Inductively, to obtain $[h_m, h'_m]$ we now renormalize by one of the operators $T_{\underline{a}^i,\underline{a}'^i}$ that carries $(k^{\underline{\theta\theta}_{n(m-1)}}, B_{m-1}\circ k^{\underline{\theta'\theta'}_{n'(m-1)}})$ into the interior of $\mathcal{L}$. The result can analogously be represented as

$$(\mathfrak{h}_m \circ k^{\underline{\theta\theta}_{n(m)}}, \mathfrak{h}'_m \circ B_m \circ k^{\underline{\theta'\theta'}_{n'(m)}})$$

with

$$||\mathfrak{h}_m - \mathrm{Id}||_{C^1} < C\cdot \mathrm{diam}(X_m)^{\varepsilon}, \text{and}$$
$$||\mathfrak{h}'_m - P_m||_{C^1} < C\cdot \mathrm{diam}(X'_m)^{\varepsilon}.$$

where X_m and X'_m are domains obtained by an affine transformation that is are least contracting to the domain of $\mathrm{Im}(k^{\underline{\theta\theta}_{n(m-1)}})$ and $\mathrm{Im}(k^{\underline{\theta'\theta'}_{n'(m-1)}})$ respectively.

However, there is a little difficulty on doing the induction step: the estimative we have on $\mathfrak{h}_{m-1}$ and $\mathfrak{h}'{}_{m-1}$ does not guarantee that the maps involved in the definition of B_m are going to be sufficiently close to the identity. It only guarantees this for a large fixed m_0, since at least one of the diameters of X_m or X'_m diminishes by at least a factor $r<1$ and the affine maps P_m are within a controlled distance of the identity (to see this observe that $||P_m - \mathrm{Id}||$ depends only on the distortion of Dh and Dh' at some specific points, which can be seem by opening the expressions that define then). Even so, all this only works if the induction steps work for all the m_0 steps that came before this one.

Nonetheless, we can always guarantee that $||\mathfrak{h}_m - Id||_{C^1} < R\cdot ||\mathfrak{h}_{m-1} - Id||_{C^1}$ for some constant $R>0$ and analogously for the $\mathfrak{h}'_m$ and $\mathfrak{h}'_{m-1}$. So, if the initial distance to the identity of h and h' is sufficiently small, we can work the first m_0 steps of the induction with this new estimative and proceed as previously for $m > m_0$.

Being the desire sequence constructed, we observe that it is in bounded neighbourhood of $\mathcal{L}$, and in light of lemma 3.2.2, the proof is completed. □

Chapter 4

Applications

4.1 Positive density of tangencies in a parametrized family

In this section we work with a specific context. Consider a family $\{H_\zeta\}_{\zeta\in\mathbb{D}}$ of automorphisms of $\mathbb{C}^2$, $\{H_\zeta\}_{\zeta\in\mathbb{D}}$. We assume this family

1. to be smooth, that is,

$$\begin{aligned} H : \mathbb{D}\times\mathbb{C}^2 &\to \mathbb{D}\times\mathbb{C}^2 \\ (\zeta,(z,w)) &\mapsto (\zeta, H_\zeta(z,w)) \end{aligned}$$

   is a C^r diffeomorphism map.

2. and that H_0 is an automorphism that has a fixed saddle point $p = p_0$ belonging to a horseshoe Λ_0. Also, that the stable and unstable manifolds of said point $W^s(p)$ and $W^u(p)$, that are complex manifolds of complex dimension one, are tangent to each other at a point of intersection q, or in other words, there is $q \in W^s(p)\cap W^u(p)$ such that $T_qW^s(p) \equiv T_qW^u(p)$.

We denote the continuation of the fixed saddle point p_0, obtained by the implicit function theorem, for the maps H_ζ by p_ζ ; the continuation of the basic set by by Λ_ζ; and the continuation of the stable and unstable manifolds $W^s_\zeta(p)$ and $W^u_\zeta(p)$. All these objects depend C^r on ζ. Of course, these continuations may be defined only for some ζ belonging to a smaller subset of $\mathbb{D}$. In this section, in all occasions in which we may need to restrict the domain of ζ to a smaller ball for the affirmations to stay valid, we consider the domain to be still the same, $\mathbb{D}$, since in all cases, we could restrict to a smaller ball and just re-scale the variable ζ, without having to change the properties of the family.

According to the theory exposed in section 2.2 we can fix a neighbourhood U of the basic set Λ_0 and construct L a compact neighbourhood of Λ_0 and a foliation $\mathcal{F}^u_\zeta$ with the following properties:

- the basic set $\Lambda_\zeta = \bigcap_{n\in\mathbb{Z}} H^n_\zeta(U)$ satisfies $\Lambda_\zeta \subset \text{int } L \subset L \subset \mathcal{F}^u_\zeta$
- if $p \in \Lambda_\zeta$ then the leaf $\mathcal{L}^u_\zeta(p)$ agrees with $W^u_{\zeta,\text{loc}}(p)$
- if $p \in H_\zeta(U) \cap U$ then $G(\mathcal{L}^u_\zeta(p)) \supseteq \mathcal{L}^u_\zeta(G(p))$.
- The association $\zeta \to \mathcal{F}^u_\zeta$ is C^r,

By iteration we can extend all of these foliations to a neighbourhood of q. In particular, maybe after performing a change of coordinates in $\text{GL}_2(\mathbb{C})$, we can assume that in a neighbourhood V of the point q the foliations $\mathcal{F}^u_\zeta$ are described by a C^r map

$$\begin{aligned} \mathrm{h}^u : \mathbb{D} \times \mathbb{D} \times \mathbb{D} &\to V \\ (\zeta, z, w) &\mapsto \mathrm{h}^u(\zeta, z, w) = (z, h^u_{\zeta,w}(z)) \end{aligned}$$

with the properties:

- for each fixed $\zeta \in \mathbb{D}$ the map $\mathbb{h}^u_\zeta = \mathbb{h}^u(\zeta, \cdot)$ is a C^r diffeomorphism map from $\mathbb{D} \times \mathbb{D}$ to an open subset of V that contains a small set $Q = B_\delta(q_1) \times B_\delta(q_2)$, where $q = (q_1, q_2)$.
- The image of any of the functions $\mathbb{h}^u_{\zeta,w} = \mathbb{h}^u(\zeta, w, \cdot)$ is contained on a leaf $\mathcal{L}^u_\zeta$ of the foliation $\mathcal{F}^u_\zeta$

As already discussed, we can assume that the maps $\mathbb{h}^u_{\zeta,w}$ are holomorphic on z. Likewise, we can define a family of foliation F^s_ζ with the same properties, except that if $p \in H_\zeta(U) \cap U$ then $G(\mathcal{L}^u_\zeta(p)) \subseteq \mathcal{L}^u_\zeta(G(p))$.

We can, without loss of generality, choose these parametrizations in a way so we can represent the tangency between the stable and unstable manifolds of the fixed point p of H_0 as

$$q = \mathbb{h}^u(0,0,0) = \mathbb{h}^s(0,0,0),$$
$$W^u_{\zeta,\mathrm{loc}}(q) = \mathrm{Im}(\mathbb{h}^u_{\zeta,0}),\ W^s_{\zeta,\mathrm{loc}}(q) = \mathrm{Im}(\mathbb{h}^s_{\zeta,0})$$
$$\frac{\partial h^u_{0,0}}{\partial z}(0) = \frac{\partial h^s_{0,0}}{\partial z}(0)$$

Under this convention we add new hypothesis on the family, specifically about the tangency:

3. $\frac{\partial^2 \mathbb{h}^u_{0,0}}{\partial z^2}(0) \neq \frac{\partial^2 \mathbb{h}^s_{0,0}}{\partial z^2}(0)$, meaning that the tangency is quadratic.

Under this setting, we have the following construction:

Lemma 4.1.1. *(Line of tangencies) Let $\{H_\zeta\}_{\zeta\in\mathbb{D}}$ be a family of automorphisms of $\mathbb{C}^2$ as in the context described above. For every $\zeta \in \mathbb{D}$ there is a C^r embedded disk $D_\zeta \subset V$ such that the leaves $\mathcal{L}^u_\zeta(x)$ and $\mathcal{L}^s_\zeta(x)$ are tangent with each other at x for evey $x \in D_\zeta$. Further, the embeddings depend C^r on ζ.*

Proof. First we obtain the line of tangencies for $\zeta = 0$. Consider the map

$$f(z,w) = \frac{\partial}{\partial z}\Pi_2((\mathbb{h}^u_0)^{-1} \circ \mathbb{h}^s_0)(z,w).$$

Expanding this expression and denoting $A(z,w) = \frac{\partial \Pi_2(\mathbb{h}^u_0)}{\partial w}((\mathbb{h}^u_0)^{-1} \circ \mathbb{h}^s_0(z,w)) \in \mathrm{GL}_2(\mathbb{R})$, we have:

$$f(z,w) = A(z,w)^{-1} \cdot \left(\frac{\partial(\Pi_2 \circ \mathbb{h}^u_0)}{\partial z} \circ ((\mathbb{h}^u_0)^{-1} \circ \mathbb{h}^s_0(z,w)) - \frac{\partial(\Pi_2 \circ \mathbb{h}^s_0)}{\partial z} \right)(z,w).$$

Observe that the matrix difference in the parenthesis above comprises two conformal matrices, since, by hypothesis on the foliations, both maps $\Pi_2 \circ \mathbb{h}^u_0$, $\Pi_2 \circ \mathbb{h}^u_0$ are holomorphic. Then, we can identify such difference with a complex number and, identifying $\mathbb{C}$ with $\mathbb{R}^2$, we can consider that f carries $\mathbb{R}^4$ to $\mathbb{R}^2$. Also, one can verify that the pre-image of $\mathrm{Id} \equiv 1 \in \mathbb{C}$, D'_0 is carried by $\mathbb{h}^s_0$) to point in which the foliations $\mathcal{F}^s_0$ and $\mathcal{F}^u_0$ are tangent to each other. By hypothesis, $(0,0) \in D'_0$ and we can calculate:

$$\frac{\partial f}{\partial z}(0,0) = A(z,w)^{-1} \cdot \left(\frac{\partial^2 \Pi_2 \mathbb{h}^u_0}{\partial z^2}(0,0) - \frac{\partial^2 \Pi_2 \mathbb{h}^s_0}{\partial z^2}(0,0) \right)$$

which by hypothesis (quadratic tangency) is invertible, since both matrices are conformal (and different), because the stable and unstable manifolds are holomorphic. So, by the implicit function theorem, there is a function α such that $D_0' \cap B_\delta(0,0)$ is the graph $(\alpha(w), w)$ for some small δ. By applying $\mathbb{h}_0^s$ to this set we have the line of tangencies for $\zeta = 0$.

It is very similar to construct the line of tangencies for other values of ζ. We consider this time the map

$$f(z,w,\zeta) = \frac{\partial}{\partial z}\Pi_2((\mathbb{h}_\zeta^u)^{-1} \circ \mathbb{h}_\zeta^s)(z,w).$$

and as previously we are interested in the pre-image of Id.

and as previously it is easy to show that f is a submersion in a neighbourhood of $(0,0,0)$. So the inverse image of Id is a C^r manifold, showing that the tangency line varies C^r with ζ.

□

As seen in the section 2.2, there are holomorphic parametrizations $a_\zeta^s(z) = (a_{\zeta,1}^s(z), a_{\zeta,2}^s(z))$, $a_\zeta^s : W \subset \mathbb{C} \to \mathbb{C}^2$ and $a_\zeta^u(z) = (a_{\zeta,1}^u(z), a_{\zeta,2}^u(z))$, $a_\zeta^u : W \subset \mathbb{C} \to \mathbb{C}^2$ of $W_{\zeta,\text{loc}}^s(p_\zeta)$ and $W_{\zeta,\text{loc}}^u(p_\zeta)$ such that $K_\zeta^s = (a_\zeta^u)^{-1}(W_{\zeta,\text{loc}}^u(p_\zeta) \cap \Lambda_\zeta)$ and $K_\zeta^u = (a_\zeta^s)^{-1}(W_{\zeta,\text{loc}}^s(p_\zeta) \cap \Lambda_\zeta)$ can be seen as conformal Cantor sets. They vary C^r with ζ and we can assume that $(a_\zeta^s)(0) = p_\zeta$ and $(a_\zeta^u)(0) = p_\zeta$. Similarly, considering parametrizations of the lines of tangencies, D_ζ, $b_\zeta(z) = (b_{\zeta,1}(z), b_{\zeta,2}(z))$, and projecting along the stable and unstable foliations, we obtain $(b_\zeta)^{-1} \circ \Pi_\zeta^s \circ a_\zeta^s(K_\zeta^s)$ and $(b_\zeta)^{-1} \circ \Pi_\zeta^u \circ a_\zeta^u(K_\zeta^u)$ are configurations of the given conformal Cantor sets in $\mathbb{C}$ such that $h_\zeta^s = (b_\zeta)^{-1} \circ \Pi_\zeta^s \circ a_\zeta^s$ and $h_\zeta^u = (b_\zeta)^{-1} \circ \Pi_\zeta^u \circ a_\zeta^u$ depend C^r on ζ. If these intersect, for some $\zeta > 0$ this implies that there is a tangency between $W^s(\Lambda_\zeta)$ and $W^u(\Lambda_\zeta)$.

Before going into the main result of this section we need one more result related to these foliations:

Theorem 4.1.1. *Let λ^s and λ^u be the eigenvalues of DH_0 at the fixed point p_0. There is a set $S \subset \mathbb{C}^2$ of total Lebesgue measure such that if $(\lambda^u, \lambda^u) \in S$, then $D((h^u)^{-1} \circ h^s)(0) \in GL_2(\mathbb{R})$ is a conformal matrix.*

Proof. The set S above is know as a Siegel domain and has this name because of the following theorem due to Siegel:

Theorem. *(Siegel) Let F be a germ of holomorphic diffeomorphism fixing* 0 *and denote by $\lambda_1, \ldots, \lambda_n$ the eigenvalues of $DF(0)$ (that we assume diagonalizable). If there exist $C > 0$ and $v \in \mathbb{N}$ such that for all $l = 1, \ldots, n$ and $m_1, \ldots, m_n \in \mathbb{N}$ such that $\sum m_j \geq 2$ it holds*

$$|\lambda_l - \lambda_1^{m_1} \cdots \lambda_1^{m_1}| \geq \frac{C}{(\sum_{j=1}^n m_j)^v}$$

then F is holomorphically linearizable.

The condition above is of full Lebesgue measure for large enough v and in, our case, S is the set of all pairs $(\lambda_1, \lambda_2) \in \mathbb{C}^2$ that satisfy it for $n = 2$. By holomorphically linearizable we mean that there are neighbourhoods of U and V of 0 and a holomorphic map $G : U \to V$ such that $T = G \circ F \circ G^{-1}(z_1, \dots, z_n) = (\lambda_1 \cdot z_1, \dots, \lambda_n \cdot z_n)$. See [17] and [18] for further details.

The domains of linearization U and V may be extend to $\cup_{n=1}^{k} F^n(U)$ and $\cup_{n=1}^{k} T^n(V)$ respectively via step by step making $G|F^n(U) = T \circ G \circ F^{-1}|F^n(U)$. They can also be extended by iterating F and T backwards equivalently. Getting back to our case, that is, making $F = H_0$ and $T(z, w) = (\lambda^s \cdot z, \lambda^u \cdot w)$, we can extend U to U^u, an open set that contain the whole segment of $W^u(p)$ that contains p and the point of tangency q and V to V^u. Now, we analyse the properties of the unstable foliation near q under these linearizing coordinates.

Let $q^u = G(q) \in V$, where G was extended as described above. Given the analicity of each leaf, we can represent any $G(\mathcal{L}^u)$ near q^u by a parametrized disk $D(z)$ for some fixed z:

$$(q^u + w, z + \alpha(z) \cdot w + \beta(z) \cdot w^2 + r(z, w)),\ w \in \delta \cdot \mathbb{D}, \text{ for some small } \delta > 0,$$

where $\alpha(z)$ and $\beta(z)$ are complex numbers that vary C^r in z, $r(z, 0)) = 0$ and $\lim_{z \to 0} \frac{r(z,w))}{|w|^2} = 0$.

If $|z| < C \cdot (\lambda^s)^n$ for some constant C, then $T^{-n}(D(z)) \in V$. On the other hand, since $\mathcal{F}^u$ is invariant by backwards iteration of H_0, $G^{-1}((T^{-n}(D(w)))$ is a small piece of an unstable leaf, that can be represented by $(w, (\lambda^s)^{-n} z + \alpha \cdot w + r(w))$ with $\lim_{z \to 0} \frac{r(w))}{|w|} = 0$ and z taking domain over a small disk.

But $T^{-n}(D(w))$ can be parametrized by

$$(q^u_n + (\lambda^u)^{-n}(w), (\lambda^s)^{-n}(z + \alpha(z) \cdot w + \beta(z) \cdot w^2 + r(z, w))),$$

and by substituting $(\lambda^u)^{-n}(w) \mapsto w$ we get

$$(q^u_n + w, z_n + (\alpha(w) \cdot (\lambda^s)^{-n} \cdot (\lambda^u)^n) \cdot w + (\beta(z) \cdot (\lambda^s)^{-n} \cdot (\lambda^u)^n) \cdot w^2 + r(z, w)).$$

Differentiating with respect to w at 0 we get that $\alpha(z) = \alpha \cdot (\lambda^s)^n \cdot (\lambda^u)^{-n}$. That way, there is a constant A such that if $|z| < C \cdot (\lambda^s)^n$ then $|\alpha(z)| <$

$A \cdot |z| \cdot (\lambda^u)^{-n}$. That is enough to show that $D\alpha(0) = 0$. But $\alpha(z)$ represents the inclination of the foliation $G^{-1}(\mathcal{F}^u)$ along the line (z, q^u) and, since its derivative along the z line is conformal, the map that associates for each pair $(z, w) \in V^u$ the inclination of $G^{-1}(\mathcal{F}^u)$ at (z, w) has a derivative at q^u that is conformal along the z and w axis. Now, since G is holomorphical, this implies that the maps that associates to any point $(z, w) \in U^u$ the inclination of $(\mathcal{F}^u)$ at (z, w) has a derivative at q that is conformal along the z and w axis.

The same can be done for $W^s(p)$, obtaining U^s and V^s, but we have to highlight that the domains U^s and U^u are not the same, so neither would be the respective holomorphic conjugating maps G. Nonetheless the derivative of the inclination of $(\mathcal{F}^s)$ would still have conformal derivative along the z and w axis at q. Using the formula for the derivative of implicit function map this is enough to show that at $\zeta = 0$ the disk of tangencies has $T_q D^T = L$, where L can be seen as a complex subspace of $\mathbb{C}^2$. As we have already seen 2.3.1, this is enough to guarantee that the projection along the stable and unstable leaves from $W^u(p)$ and $W^s(p)$ to D^T are conformal and the result follows by application of the chain rule.

□

Finally, we need a final generic hypothesis on the family. Let $c^s(\zeta) = h^s_\zeta(0)$ and $c^u(\zeta) = h^u_\zeta(0)$. We can choose b_ζ in a way such that $c^s(\zeta) = 0$ for all ζ small, we need only to compose it with a translation if necessary. Under this notation, it is true that generically the speed of $c^u(\zeta)$ is non-zero in relation to any direction that ζ may move, that is:

4. $\frac{\partial c^u}{\partial \zeta}(0) = B \in \mathrm{GL}_2(\mathbb{R})$.

Theorem 4.1.2. *Let H_ζ be a family of automorphisms of $\mathbb{C}^2$ as in the context above. Let θ^u (resp. θ^s) represent the piece that contains the fixed point p_0 in the construction of K^u_0 (resp. K^s_0). We write $\underline{\theta}^u = (\ldots, \theta^u, \theta^u, \ldots, \theta^u) \in \Sigma^-$ and respectively $\underline{\theta}^s = (\ldots, \theta^s, \theta^s, \ldots, \theta^s) \in \Sigma^-$. Fix $c_{\theta^u} = c_{\theta^s} = p_0$ and suppose there is a complex number ν such that*

$$(k^{\underline{\theta}^u}(K^u_0),\ (k^{\underline{\theta}^s}(K^s_0)) + \nu)$$

is a stably intersecting pair of configurations.

Then, under generic conditions on the values of λ^u_0 and λ^s_0, the set of tangencies,

$$C_{tan} = \{\zeta \in \mathbb{D};\ \textit{there is a point of tangency between } W^s(\Lambda_\zeta) \textit{ and } W^u(\Lambda_\zeta)\}$$

has interior with positive inferior density. We mean by that,

$$\liminf_{\varepsilon\to 0}\frac{m(int(C_{tan})\cap\varepsilon\cdot\mathbb{D})}{m(\varepsilon\cdot\mathbb{D})}>0$$

where m denotes the Lebesgue measure on the disk.

Proof. As already discussed,

$$C_{\tan}=\{\zeta\in\mathbb{D};\ h^u_\zeta(K^u_\zeta)\cap h^s_\zeta(K^s_\zeta)\neq\emptyset\}.$$

Before we begin, let λ^s_ζ and $\lambda^s_\zeta\in\mathbb{C}$ be the eigenvalues of the derivative of the fixed point p_ζ.

Let us have a look at $K^{u,m}_\zeta=K^u_\zeta(\underline{\theta}^u_m)=K^u_\zeta\cap G^u_\zeta(\underline{\theta}^u_m)$, where $\underline{\theta}^u_m=(a_0,\dots,a_m)$; $a_i=\theta^u$, $\forall i=0,\dots,m$.

If we consider the configuration $h^u_\zeta:G(\theta^u)\to\mathbb{C}$, then $h^u_\zeta(K^{u,m}_\zeta)$ is the image of $(K^u_\zeta(\theta^u))$ under the configuration $i_{\zeta,m}=T_{\zeta,\underline{\theta}^u_m}(h^u_\zeta)=h^u_\zeta\circ f_{\zeta,\underline{\theta}^u_m}$. As seen in lemma 3.2.2, the scaled configuration $\mathfrak{h}^u_{\zeta,m}=A_{(h^u_\zeta)\circ i_{\zeta,m}}\circ(h^u_\zeta)\circ i_{\zeta,m}$ is at least $C\cdot(\lambda^u_\zeta)^{-m(r-1)}$ close to the limit geometry $k^{\underline{\theta}^u}_\zeta$. One can see this writing $h^u_\zeta=h^{u,\underline{\theta}^u}_\zeta\circ k^{\underline{\theta}^u}_\zeta$ and observe that $\underline{\theta}^u\underline{\theta}^u_m=\underline{\theta}^u$. The same can be done for K^s_ζ to show that if $i'_{\zeta,n}=T_{\underline{\theta}'^s_n}(i'_0)=\mathrm{Id}\circ f_{\underline{\theta}'^s_n}$, the the scaled version of $T_{\zeta,\underline{\theta}^s_m}(h^u_\zeta)=(h^s_\zeta)\circ i'_{\zeta,n}$, $\mathfrak{h}^s_{\zeta,n}$, is $C\cdot(\lambda^s_\zeta)^{n(r-1)}$ close to the limit geometry $k^{\underline{\theta}'^s}_\zeta$.

This way, we can estimate the renormalization:

$$(T_{\zeta,\underline{\theta}^u_m}(h^u_\zeta),T_{\zeta,\underline{\theta}^s_m}(h^s_\zeta))=(\mathfrak{h}^u_{\zeta,m},A_{(h^u_\zeta)\circ i_{\zeta,m}}\circ A^{-1}_{(h^s_\zeta)\circ i'_{\zeta,n}}\circ\mathfrak{h}^s_{\zeta,n})$$

as follows.

First, the map $A_{(h^u_\zeta)\circ i_{\zeta,m}}\circ A^{-1}_{(h^s_\zeta)\circ i'_{\zeta,n}}$ is not necessarily in $Aff(\mathbb{C}^2)$. However, its formula is easy to calculate as

$$Dh^u_\zeta(0)\cdot(\lambda^u_\zeta)^m\cdot(\lambda^s_\zeta)^n\cdot(Dh^s_\zeta(0))^{-1}(z)-Dh^u_\zeta(0)\cdot(\lambda^u_\zeta)^m\cdot c^u(\zeta),$$

considering the definitions of A_h for any configuration h and the fact that $c^s(\zeta)\equiv 0$. But, by the C^r dependence on the parameter, we can approximate it by

$$A''_{m,n}=(\lambda^u_\zeta)^m\cdot(\lambda^s_\zeta)^n\cdot D((h^u)^{-1}\circ h^s)(0)(z)+Dh^u(0)\cdot(\lambda^u_\zeta)^m\cdot c^u(\zeta)\in Aff(\mathbb{C}^2),$$

with an error of δ on the expansion factor and an error of $(\lambda^u_\zeta)^m \cdot \delta$ on the translation part, provided that ζ is sufficiently small and n, m sufficiently large. On the other hand, by the generic hypothesis on the tangency, $\dfrac{\partial c^u}{\partial \zeta}(0) = B \in \mathrm{GL}_2(\mathbb{R})$, we have $c^u(\zeta) = B \cdot \zeta + r(\zeta)$, where $r(\zeta) < C \cdot |\zeta|^r$. So, perhaps by considering, smaller ζ the approximation

$$A'_{m,n} = (\lambda^u_\zeta)^m \cdot (\lambda^s_\zeta)^n \cdot D((h^u)^{-1} \circ h^s)(0)(z) + Dh^u(0) \cdot B \cdot (\lambda^u_\zeta)^m \cdot \zeta \in Aff(\mathbb{C}^2),$$

is δ close in the expansion part and $\delta \cdot (\lambda^u_\zeta)^m$ on the translation part.

Now, suppose that for $m, n \in \mathbb{N}$ $|(\lambda^u_0)^m \cdot (\lambda^s_0)^n - |D((h^u)^{-1} \circ h^s)(0)| < \delta$. By the C^r dependence on ζ,

$$|\lambda^u_\zeta - \lambda^u_0| < C' \cdot |\zeta|$$

So $\frac{(\lambda^u_\zeta)^m}{(\lambda^u_0)^m}$ is close to $(1 \pm C' \cdot \frac{\zeta}{\lambda^u_0})^m$ and if ζ is of order $|(\lambda^u_0)^{-m}|$ this is close to $(1 \pm C''(\lambda^u_0)^{1-m})^m$, a function of m that converges to 1 as $m \to \infty$. This way, remembering that by hypothesis $|(\lambda^s_0)^n|$ is of order $|(\lambda^u_0)^{-m}|$, we can show that for m, and consequently also n, large enough $|\frac{\lambda^u_\zeta}{\lambda^u_0} - 1| < \delta$ and $|\frac{\lambda^s_\zeta}{\lambda^s_0} - 1| < \delta$, for every ζ of order $|(\lambda^u_0)^{-m}|$. This allow us to finally approximate $A_{(h^u_\zeta) \circ i_{\zeta,m}} \circ A^{-1}_{(h^s_\zeta) \circ i'_{\zeta,n}}$ by

$$A_{m,n} = (\lambda^u_0)^m \cdot (\lambda^s_0)^n \cdot D((h^u)^{-1} \circ h^s)(0)(z) + Dh^u(0) \cdot B \cdot (\lambda^u_0)^m \cdot \zeta \in Aff(\mathbb{C}^2),$$

for every ζ of order $|(\lambda^u_0)^{-m}|$ keeping the estimative of the error as it was for $A'_{m,n}$.

But, by hypothesis, if ζ is sufficiently small, there is $\varepsilon > 0$ such that for all $\nu' \in B_\varepsilon(\nu)$ and every pair (h, h') ε-close to $(k^{\theta^u}_\zeta, k^{\theta'^s}_\zeta)$, there is a stable intersection between $h(K^u_\zeta)$ and $h'(K^s_\zeta)) + \nu'$. Hence, choosing $\zeta \in (\lambda^u_0)^{-m} \cdot (Dh^u(0) \circ B)^{-1}(B_\delta(\nu)))$, and n, m large enough, so that all the approximations above have error $\delta << \varepsilon$, we have that $h^u_\zeta(K^u_\zeta) \cap h^s_\zeta(K^s_\zeta)$have stable intersection. The proof is completed if we show that for generic values of λ^u_0 and λ^s_0 there are infinitely many values of $m, n \in \mathbb{N}$ (with some periodicity on m) such that $|(\lambda^u_0)^m \cdot (\lambda^s_0)^n - D((h^u)^{-1} \circ h^s)(0)^{-1}| < \delta$. This is a consequence of this lemma

Lemma 4.1.2. *For generic complex numbers $z = R \cdot e^{ia}$ and $w = r \cdot e^{ib}$; $a, b, R, r \in \mathbb{R}$, the set $X = \{u = z^m \cdot w^n \in \mathbb{C}; m, n \in \mathbb{Z}\}$ is dense on $\mathbb{C}$. Moreover, for each $v \in \mathbb{C}$ and $\delta > 0$ there is $M = M_{v,\delta} > 0$ such that,* *if* $z^m \cdot w^n \in B_\delta(v)$*, then there is a pair* (m', n') *with* $m < m' < m + M$*,* $n < n' < n + N$ *and* $z^{(m+m')} \cdot w^{(n+n')} \in B_\delta(v)$*.*

Proof. Applying logarithms we need to show that $X' = \{(m \cdot \log R - n \cdot \log r, ma - nb + 2k \cdot \pi) \in \mathbb{R}^2; m, n, k \in \mathbb{Z}\}$ is dense in $\mathbb{R}^2$ for generic choices of a, b, r, R. By Kronecker's theorem we know that if $e_1 = (1, 0)$, $e_2 = (0, 1)$ and $v = (c, d)$ are vectors such that 1, c, d are linearly independent over $\mathbb{Q}$, then the set $Y = \{m \cdot e_1 + n \cdot e_2 + k \cdot v;\ m,\ n,\ k \in \mathbb{Z}\}$ is dense on $\mathbb{R}^2$. Consequently, consider the linear operator $T = \begin{bmatrix} \log R & -\log r \\ a & b \end{bmatrix}$,

$c = \frac{2\pi \log r}{a \log r - b \log R}$ and $d = \frac{2\pi \log R}{a \log r - b \log R}$. Under these choices, $(m \cdot \log R - n \cdot \log r, ma - nb + 2k \cdot \pi) = T(m \cdot e_1 + n \cdot e_2 + k \cdot v)$, so, $X' = T(Y)$ is dense if $a \log r - b \log R$, $\pi \log R$ and $\pi \log r$ are linearly independent over $\mathbb{Q}$. This is clearly a generic condition.

Under this hypothesis fix $\varepsilon > 0$ and consider a finite set $S \subset B_\varepsilon(0)$ such that

$$\bigcup_{s \in S} B_\varepsilon(s) \supset \overline{B_\varepsilon(0)}$$

(in the case of the plane this can be done with three points). As a consequence, we can choose $\delta > 0$ with

$$\bigcup_{s \in S} B_{\varepsilon - \delta'}(s) \supset \overline{B_\varepsilon(0)}.$$

Now, defining $Y_M = \{m \cdot e_1 + n \cdot e_2 + k \cdot v;\ m,\ n,\ k \in \mathbb{Z},\ |m|,\ |n|,\ |k| < M\}$ there is $M > 0$ such that $V_{\delta'}(Y_M) \supset S$. If $p = m' \cdot e_1 + n' \cdot e_2 + k' \cdot v \in B_\varepsilon(0)$ there is an $s \in S$ with $p \in B_{\varepsilon - \delta'}(s)$ and a point $-p_s = m \cdot e_1 + n \cdot e_2 + k \cdot v \in Y_M \cap B_{\delta'}(s)$, which implies $p + p_s \in B_\varepsilon(0)$, or equivalently $(m + m') \cdot e_1 + (n + n') \cdot e_2 + (k + k') \cdot v \in B_\varepsilon(0)$. Since all the transformations used above are continuous this concludes the lemma.

□

Finally, making $v = D((h^u)^{-1} \circ h^s)(0)^{-1}$, $z = \lambda_0^u$ and $w = \lambda_0^s$ in the previous lemma, for each $\varepsilon > 0$ we can find a sequence of pairs of integers $(m_i, n_i)_{i \geq 0}$ with $0 < m_0 < \cdots < m_i < \ldots$ and $m_{i+1} - m_i < M$ such that

$$|(\lambda_0^u)^{m_i} \cdot (\lambda_0^s)^{n_i} - D((h^u)^{-1} \circ h^s)(0)^{-1}| < \delta$$

and

$$(\lambda_0^u)^{-m_i} \cdot (Dh^u(0) \circ B)^{-1}(B_\delta(\nu))) \subset \varepsilon \cdot \mathbb{D}$$

for all $i \geq 0$. Also from the previous lemma, we can suppose that $(\lambda_0^u)^{-m_0} \geq \tilde{C} \cdot \varepsilon^{-1}$ for some constant $\tilde{C} > 0$ and ε sufficiently small. Choosing a subsequence of the m_i and enlarging M if necessary we can assume all $(\lambda_0^u)^{-m_i} \cdot (Dh^u(0) \circ B)^{-1}(B_\delta(\nu)))$ are disjoint. This implies that

$$m(int(C_{\tan}) \cap \varepsilon \cdot \mathbb{D}) \geq m((Dh^u(0) \circ B)^{-1}(B_\delta(\nu)))) \sum_{j \geq 0} (\lambda_0^u)^2(-m_0 - Mj)$$
$$\gtrsim (\lambda_0^u)^{-2m_0} \gtrsim \varepsilon^2$$

and this concludes the proof.

□

4.2 Buzzard's example

In the article [6], Buzzard found an open set in $U \subset Aut(\mathbb{C}^2)$ with a residual subset $\mathcal{N} \subset U$ with coexistence of infinitely many sinks, thus establishing the existence of Newhouse phenomenon on the complex two dimensional context. The strategy was very similar to the one by Newhouse in his works [1], [2] and [3]. Consider the example 2.3.1 in section 2.3.

Checking the section 5 of [6], we find a very favourable construction of a tangency between $W_F^s(0)$ and $W_F^u(0)$; the disk of tangencies D_T is equal to a small vertical plane $\{q\} \times \rho_2 \cdot \mathbb{D}$ and, choosing a suitable parametrization of D_T we can assume that the projections Π_s and Π_u along the stable and unstable foliations from $W_{F,\text{loc}}^u$ and $W_{F,\text{loc}}^s$ to D_T are the identity (considering also the obvious inclusion of such sets to $\mathbb{C}$).

Moreover, we remember that, as already discussed in section 2.3, for any $G \in Aut(\mathbb{C}^2)$ such that $||G|_{K_1} - F||$ is sufficiently small, both foliations are also defined and can be taken C^r, $r > 1$, very close to their originals. That means, denoting by p_G the continuation of the fixed point 0 for F, there are continuations $W_{G,\text{loc}}^{u,s}(p_G)$, parametrized by $a^u(w) = (\alpha^u(w), w)$, $w \in S(0; c_0)$ and $a^s(z) = (z, \alpha^u(z)), z \in S(0; c_0)$ respectively, with α^s and α^u very close to zero, such that the sets $(a^u)^{-1}(W_{G,\text{loc}}^u(p_G \cap \Lambda_G)) = K_G^s$ and $(a^s)^{-1}(W_{G,\text{loc}}^s(p_G) \cap \Lambda_G) = K_G^u$ are Cantor sets very close to K in the topology we consider for $\Omega_{P^{\mathbb{N}}}$. Further, the line of tangencies D_T^G is also well-defined and can be parametrized by something close to the parametrization of D_T. Therefore, the projections Π_s from $W_{G,\text{loc}}^u$ to D_T^G and the projection Π_u from $W_{G,\text{loc}}^s$ to D_T^G can be seen, under these parametrizations, as diffeomorphisms h^s and h^u very close to the identity.

The existence of a tangency between between $W^u(\Lambda_G)$ and $W^u(\Lambda_G)$ corresponds to a intersection between $h^s(k^s_G)$ and $h^u(K^u_G)$. Consequently, if we can show that the configuration pair $(\mathrm{Id}, \mathrm{Id})$ is stably intersecting for the pair (K, K) of conformal Cantor sets, then, for every $G \in Aut(\mathbb{C}^2)$ such that $||G|_{K_1} - F||$ is sufficiently small there is a homoclinic tangency at G. We show that this is the case:

Theorem 4.2.1. *There is δ sufficiently small for which the pair of Cantor sets (K, K) defined above has a recurrent compact set of affine configurations of limit geometries $\mathcal{L}$ such that $[Id, Id] \in \mathcal{L}$.*

Proof. The first observation is that the maps defining K

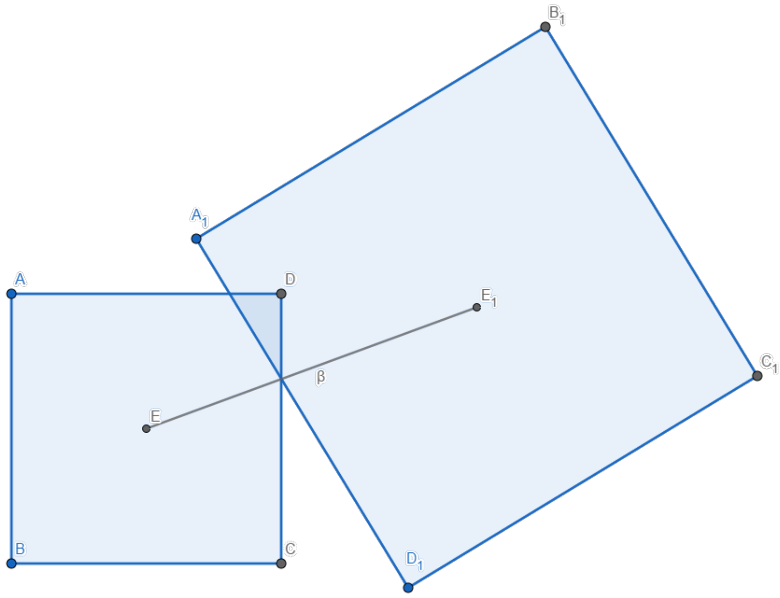
$$\begin{aligned} g_a : S(a; c_0) &\to S(0; 3) \\ z &\mapsto \frac{3}{c_1}(z - a) \end{aligned}$$

are all affine. Hence, if $\underline{\theta} \in (P^{\mathbb{N}})^-$ has $\theta^0 = a$ then $k^{\underline{\theta}}$ is an affine transformation with derivative $\mathrm{Id} \equiv 1 \in \mathbb{C}$ that carry a base point to 0. So, choosing for any of the pieces $S(a; c_0)$ the base point $c_a = a$ we have $k^{\underline{\theta}}(z) = z - a$, whose image is always the set $S(0; c_0)$.

It is also easy to verify that, under our notation, for any a, $b \in P$, $f_{a,b}(z) = \frac{c_1}{3}z + a$. We can them verify that the action of the renormalization operators is described by

$$F^{\underline{\theta}ab}(z) = \frac{c_1}{3}(z + b).$$

As already discussed, we denote every configuration pair $[h, h'] \in \mathcal{Q}$ by its representative that is scaled in the first coordinate $(A_h \circ h, A_h \circ h')$. Similarly, any configuration pair $[(A, \underline{\theta}), (A', \underline{\theta}')] \in \mathcal{C}$ will be represented by the "triple" $(\underline{\theta}, \underline{\theta}', \mathrm{Id}, A^{-1} \circ A)$. However, proceeding by algebraically calculating the renormalization operator under this identification makes it hard to construct a recurrent compact set, so we also choose a geometric interpretation. For this we may, identify any map $B \in Aff(\mathbb{C})$ with $B(S(0; c_0))$, which is a square embedded on $\mathbb{C}$, considering the orientation of its vertices. The figure below exemplifies this idea for our identification.



The square $ABCD$ represents $\mathrm{Id}(S(0;c_0))$ with $E = 0$ as its center. The square $A_1B_1C_1D_1$ represents the image of $S(0;c_0)$ by $A^{-1} \circ A$. If $A^{-1} \circ A = \alpha \cdot z + \beta$, α, $\beta \in \mathbb{C}$, then the vector $\vec{EE_1}$ represents β and $\alpha = \frac{A_1 - b}{A}$, when A, A_1, β are seen as complex numbers. In this figure, one can easily see that $\alpha = R \cdot \exp i\omega$, with $R > 1$ and $\omega \in (0, \pi/4)$.

For each $\kappa \in (0, 1]$ and each complex number α define X^κ_α as the set of all $B \in Aff(\mathbb{C})$ that are equal to $\alpha \cdot z + \beta$, for some $\beta \in \mathbb{C}$, such that the area of $S(0;c_0) \cap B(S(0;c_0))$ is at least $\kappa \cdot c_0^2 \cdot (1 + |\alpha|^2)$, meaning that their intersection has an area of at least a small percentage of the sum of their areas. For $\kappa = 0$ we write X^κ_α for the set of all $B \in Aff(\mathbb{C})$ that are equal to $\alpha \cdot z + \beta$, for some $\beta \in \mathbb{C}$, such that the area of $S(0;c_0) \cap B(S(0;c_0)) \neq \emptyset$. For example, if we consider the affine map identified with the figure just above, it is true that, for κ very close to 0, it is in X^κ_α. Also for any real number $c \in (0, 1]$ define the set $R_c = \{z \in \mathbb{C};\, c^{1/2} \leq |z| \leq c^{-1/2}\}$.

Let $\frac{1}{3} < c < 1$. If $\kappa_1 < \frac{c}{36(1+c)}$ then, for any $A \in X^0_\alpha \setminus X^{\kappa_1}_\alpha$ with $\alpha \in R_c$ the intersection $S(0;c_0) \cap A(S(0;c_0))$ is contained in one of the four strips $S_1 = \{z;\, \mathrm{Re}(z) > \frac{c_0}{3}\} \cap S(0;c_0)$, $S_2 = \{z;\, \mathrm{Re}(z) < -\frac{c_0}{3}\} \cap S(0;c_0)$, $S_3 = \{z;\, \mathrm{Im}(z) > \frac{c_0}{3}\} \cap S(0;c_0)$ and $S_4 = \{z;\, \mathrm{Im}(z) < -\frac{c_0}{3}\} \cap S(0;c_0)$. It is also contained in one of the for strips $A(S_i)$, $i = 1, 2, 3, 4$. This estimative comes form the fact that if that would not be the case, then $1/3 \cdot S(0;c_0) \cap A(S(0;c_0))$

or $S(0;c_0)\cap A(1/3\cdot S(0;c_0))$ would be non-empty. But in any of these cases, the area of $S(0;c_0)\cap A(S(0;c_0))$ would be larger than $\frac{c}{36}$.

Now, let $0<\kappa<\kappa_1$. We show that if c_1 is really close to 1 the for any $v\in\mathcal{C}$ identified by $(\underline{\theta},\underline{\theta}',\mathrm{Id},A)$ with $A\in X_\alpha^\kappa\setminus X_\alpha^{\kappa_1}$ and $\alpha\in R_{\frac{c_1}{3}}$ we can find a pair of letters $(a,a')\in P^2$ such that the renormalization $\tilde{\mathcal{T}}_{\theta_0 a,\theta_0' a'}$ carries v to $(\underline{\theta}a,\underline{\theta}'a',\mathrm{Id},A')$ with $A'\in X_\alpha^{\kappa'}$ with $\kappa'>\kappa\cdot\lambda$ where $\lambda>1$ is a constant to be determined.

Any of these renormalization operators has a very simple visual description when we consider the graphical identification we defined in this section. Precisely, it carries the square that represents A to an inner square $A_{A(a')}$, that is centered at the point $A(\frac{c_1}{3}a')$ and whose side is equal to $\alpha\frac{c_1}{3}c_0$. The square that represents the identity is carried into $S(\frac{c_1}{3}a;\frac{c_1}{3}c_0)$.

We begin by observing that, since $A\in X_\alpha^0\setminus X_\alpha^{\kappa_1}$, we can assume, without loss of generality for our next calculation, that $S(0;c_0)\cap A(S(0;c_0))\subset S_1\cap A(S_1)$. Dividing then S_1 into 3 squares of side $\frac{c_0}{3}$, Q_1, Q_2, Q_3, we observe that it is impossible for $(Q_i)\cap A(Q_j)\neq\emptyset$, $\forall i,j=1,2,3$. On the other hand, the area of

$$\bigcup_{a\in P}S(\frac{c_1}{3}a;\cdot c_0\cdot\frac{c_1}{3})\cap\bigcup_{a'\in P}A(S(\frac{c_1}{3}a';c_0\cdot\frac{c_1}{3})$$

is at least c_0^2 times $c_1^2(1+|\alpha|^2)-(1-\kappa)(1+|\alpha^2|))$ divide along at most 8 intersections of the type $S(\frac{c_1}{3}a;c_0\cdot\frac{c_1}{3})\cap A(S(\frac{c_1}{3}a';c_0\cdot\frac{c_1}{3})\neq\emptyset$ for $(a,a')\in P^2$. Since the areas of these squares are c_0^2 times $(\frac{c_1}{3})^2$ and $(\frac{c_1}{3})^2\cdot|\alpha|^2$ respectively we need only to show that for c_1 big enough:

$$(1+|\alpha|^2)\frac{(c_1^2-(1-\kappa))}{8}\geq(1+|\alpha|^2)\frac{c_1^2\cdot\lambda\cdot\kappa}{9}$$

and it is clear that we can choose c_1, λ and κ respecting all the previously fixed constraints in a way that the inequality above is true. Notice that the closer c_1 gets to 1 the closer δ has to be to 0. A simple calculation shows that for $\kappa>\kappa_0$ we can take any c_1 such that $c_1^2>\frac{9-9\kappa_0}{9-8\kappa_0}$.

On the other hand, if $A\in X_\alpha^{\kappa_1}$ with $\kappa_1>\frac{c}{36(1+c)}$ and $\alpha\in R_{\frac{c_1}{3}}$ we have that the area of $S(0;c_0)\cap A(S(0;c_0))$ is larger than $\frac{1}{4\cdot 36}(1+\|\alpha|^2)$. That way, given $\kappa>0$, the intersection

$$\bigcup_{a\in P}S(\frac{c_1}{3}a;\kappa_0\frac{c_1}{3}c_0)\cap\bigcup_{a'\in P}A(S(\frac{c_1}{3}a';\kappa_0\frac{c_1}{3}c_0))$$

is empty, if and only if,

$$c_1^2 \cdot \kappa_0^2 < (1 - \frac{c}{36(1+c)}).$$

On the other hand, if such a intersection is non-empty, then the area of the intersection $S(\frac{c_1}{3}a; \frac{c_1}{3}c_0) \cap A(S(\frac{c_1}{3}a'; \frac{c_1}{3}c_0))$ for some pair $(a, a') \in P^2$ is larger than $\frac{(1-\kappa_0)^2}{4}(\frac{c_1^2}{9})(1+|\alpha|^2)$. That way, if $c > 1/3$ and $v \in \mathcal{C}$ identified by $(\underline{\theta}, \underline{\theta}', \mathrm{Id}, A)$ with $A \in X_\alpha^{\kappa_1}$ and $\alpha \in R_{\frac{c_1}{3}}$, we can find a pair of letters $(a, a') \in P^2$ such that the renormalization $\tilde{\mathcal{T}}_{\theta_0 a, \theta_0' a'}$ carries v to $(\underline{\theta}a, \underline{\theta}'a', \mathrm{Id}, A')$ with $A' \in X_\alpha^{\kappa_0}$, whenever $c_1^2 \geq \frac{(1-2\sqrt{\kappa_0})^2}{4 \cdot 36}$.

We are almost able to construct the recurrent compact set. Before that, fix $\frac{c_1}{3} > c' > \frac{1}{3}$, let $\kappa_2 < \frac{c'}{36(1+c')}$ and $\alpha \in R_{c'} \setminus R_{\frac{c_1}{3}}$. We divide into cases:

1. $|\alpha|^2 > \frac{3}{c_1}$

   In this case, if we consider any $v \equiv (\underline{\theta}, \underline{\theta}', \mathrm{Id}, A) \in \mathcal{C}$ such that $A = \alpha \cdot z + \beta \in X_\alpha^{\kappa_2}$ then, by choosing c_1 sufficiently close to 1 and κ_0 close to 0 we can find a renormalization operator $\tilde{\mathcal{T}}_{\emptyset, \theta'_0 a}$ that sends v to $(\underline{\theta}, \underline{\theta}'a', \mathrm{Id}, \tilde{A}')$ with $A' = \alpha' \cdot z + \beta'$, $\alpha' \in R_{\frac{c_1}{3}}$, $\beta' \in \mathbb{C}$ and $A' \in X_{\alpha'}^{\kappa_0}$. Checking the formula for the renormalization operator we have that $\alpha' = \alpha \cdot \frac{c_1}{3} \in R_{\frac{c_1}{3}}$ by definition, so the hard part is to control the translation part of A'.

   Now, we know that the area of $S(0; c_0) \cap A(S(0; c_0))$ is at most $c_0^2(1 + |\alpha|^2)(1 - \kappa_2)$. This implies that, for c_1 really close to one the area of $S(0; c_0) \cap \bigcup_{a \in P} A(S(\frac{c_1}{3}a; c_0 \cdot \frac{c_1}{3})$ is at least $c_0^2(1 + c_1^2|\alpha^2| - (1+|\alpha|^2)(1-\kappa_2))$. By pigeonhole principle we want to find a choice of c_1 and κ_0 that satisfies:

   $$c_0^2 \frac{(1 + c_1^2|\alpha^2| - (1+|\alpha|^2)(1-\kappa_2))}{9} \geq c_0^2 \cdot \kappa_0 (1 + (\frac{c_1}{3})^2 |\alpha|^2).$$

   But after some manipulation and applications of the hypothesis $|\alpha|^2 > \frac{3}{c_1}$ it is enough to guarantee that:

   $$c_1^2 \geq 1 + \frac{12\kappa_0 - 4\kappa_2}{3}.$$

   Which is always possible if κ_0 is sufficiently small and c_1 close to 1.

2. $|\alpha|^2 < \frac{c_1}{3}$

   This case is very similar to the previous one; the difference being that for $v \equiv (\underline{\theta}, \underline{\theta}', \mathrm{Id}, A) \in \mathcal{C}$ such that $A = \alpha \cdot z + \beta \in X_\alpha^{\kappa_2}$ $a \in P$ we find a renormalization operator $\tilde{\mathcal{T}}_{\theta_0 a, \emptyset}$ that sends v to $(\underline{\theta}a, \underline{\theta}', \mathrm{Id}, \tilde{A}')$ with $A' = \alpha' \cdot z + \beta'$, $\alpha' \in R_{\frac{c_1}{3}}$, $\beta' \in \mathbb{C}$ and $A' \in X_{\alpha'}^{\kappa_0}$. Once again, $\alpha' = \alpha \cdot \frac{3}{c_1} \in R_{\frac{c_1}{3}}$, so we proceed to check the translation part.

   Again, the area of $S(0; c_0) \cap A(S(0; c_0))$ is at most $c_0^2(1 + |\alpha|^2)(1 - \kappa_2)$. By definition $A' = (F^{\underline{\theta}a})^{-1} \circ A$, but since $F^{\underline{\theta}a}$ is affine, $A' \in X_{\alpha'}^{\kappa_0}$ if, and only if, the area of $(F^{\underline{\theta}a}(S(0; c_0) = S(\frac{c_1}{3}a; \cdot c_0 \cdot \frac{c_1}{3})) \cap A(S(0; c_0)$ is larger than

   $$c_0^2 \cdot \kappa_0 \left((\frac{c_1}{3})^2 + |\alpha|^2 \right).$$

   By the same logic as the previous item we are left with the inequality

   $$c_0^2 \frac{(c_1^2 + |\alpha|^2)(1 - \kappa_2)}{9} \geq c_0^2 \cdot \kappa_0^2((\frac{c_1}{3})^2 + |\alpha|^2).$$

   After some manipulation and applications of the hypothesis $|\alpha|^2 < \frac{c_1}{3}$ it is enough to guarantee that:

   $$c_1^2 \geq 1 + \frac{12\kappa_0 - 4\kappa_2}{3}.$$

   It is no surprise that the quota would be the same given the geometry of the problem.

Thusly, we can construct a *recurrent compact set* $\mathcal{L} \subset \mathcal{C}$ as a union $\mathcal{L} = \mathcal{L}^{-1} \cup \mathcal{L}^0 \cup \mathcal{L}^1$, $\mathcal{L}^i = P^{\mathbb{N}^-} \times P^{\mathbb{N}^-} \times \mathrm{Id} \times L^i$, for $i = -1, 0, 1$, where:

- $L^1 = \bigcup_{\alpha \in R_{c'}^1} X_\alpha^{\kappa_2}$; $R_{c'}^1 = \{\alpha \in R_{c'},\ |\alpha| > \sqrt{\frac{3}{c_1}}\}$
- $L^{-1} = \bigcup_{\alpha \in R_{c'}^{-1}} X_\alpha^{\kappa_2}$; $R_{c'}^{-1} = \{\alpha \in R_{c'},\ |\alpha| < \sqrt{\frac{c_1}{3}}\}$
- $L^0 = \bigcup_{\alpha \in R_{c'} \setminus (R_{c'}^1 \cup R_{c'}^{-1})} X_\alpha^{\kappa_0}$,

and κ_0, κ_2, and c_1 are chosen respecting the constraints we have already fixed. As we have already shown, for almost all $v \in \mathcal{L}$ one of the renormalization operators T we already found above makes $T(v) \in \mathrm{int}(\mathcal{L})$. The exceptions are the $v = (\underline{\theta}, \underline{\theta}', \mathrm{Id}, A)$ where $A = \alpha \cdot z + \beta$ with $|\alpha|^2 = \frac{c_1}{3}$ or $\frac{c_1}{3}$ and $A \in X_\alpha^{\kappa_0} \setminus A \in X_\alpha^{\kappa_2}$. Yet, in this case we can repetitively apply the renormalization operators previously described appropriate to this case to obtain a sequence $v_n = T_{\underline{\theta}^n a_n, \underline{\theta}'^n a'_n}(v_{n-1})$ for which

- $v_0 = v$
- $(\underline{\theta}^n, \underline{\theta}'^n) = (\underline{\theta}^{n-1} a_{n-1}, \underline{\theta}'^{n-1} a'_{n-1})$ and
- $v_n \in X_\alpha^{\lambda^n \kappa_0}$

hence, if n is large enough $\lambda^n \kappa_0 > \kappa_2 \implies v_n \in \mathrm{int}(\mathcal{L})$ as we wished to obtain.

□

Remark 4.2.1. Given the constraints on the proof above, more importantly $\kappa_2 < \frac{\sqrt{c'}}{36(1+c')}$, we can calculate that for

$$c_1^2 \geq \min_{\kappa_0 \in [0,1]} \max \left\{ \frac{9 - 9\kappa_0}{9 - 8\kappa_0}, 1 + \frac{4\kappa_0 - \frac{1}{36}}{3}, \frac{(1 - 2\sqrt{\kappa_0})^2}{4 \cdot 36} \right\} = x^2$$

κ_0 can be chosen such that the construction above works at all steps. A computational calculation (although it can be done by hand) show that $x < 0.9997$ and so, since by definition $c_1 < \frac{3c_0}{2+c_0}$ we can estimate that for all $\delta < 0.0004$ the Cantor set defined in this section has a stable intersection with itself (provided we choose c_1 sufficiently close to $\frac{3c_0}{2+c_0}$). This quota may be improved by adaptations in the argument. Any quota of this sort is also absent in [6].

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