Instituto Nacional de Matemática Pura e Aplicada

Doctoral Thesis

SINGULAR GENUINE RIGIDITY AND A HARTMAN'S LIKE THEOREM

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Rio de Janeiro
Thursday 4th January, 2018

Instituto Nacional de Matemática Pura e Aplicada

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Thesis presented to the Post-graduate Program in Mathematics at Instituto Nacional de Matemática Pura e Aplicada as partial fulfillment of the requirements for the degree of Doctor in Mathematics.

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Rio de Janeiro
2018

CHAPTER 1

Acknowledgments

Every thesis demands a great deal of time, dedication and work. The journey is full of periods of joy and excitement, but it also teaches how to deal with frustration and disappointments. And these feelings are not unique to whoever is writing the thesis, but are to some degree shared with every friend.

I would especially like to thank my advisor Luis Florit for his patience and great help both in mathematics and in person. I am also grateful for the fruitful conversations with Professor Marcos Dajczer and my colleague Guilherme Machado.

I would like to thank the IMPA staff for being always available and whose efficiency has made it easier to develop this work.

Last but not least, I thank the CNPq for the financial support.

Abstract

This thesis addresses two independent problems of the theory of submanifolds in space forms.

The first part is devoted to add flexibility to modern notions of rigidity. It has been shown recently that singularities in isometric extensions must be allowed in order to obtain global rigidity results of Euclidean submanifolds outside the realm of hypersurfaces. Using this as motivation, we incorporate singularities in the theory of genuine rigidity of submanifolds. In the process, we found out that embracing singularities is quite natural and straightforward, even for local purposes, allowing us to deeply simplify the theory of genuine rigidity.

In the second part we show that a complete Euclidean submanifold with minimal index of relative nullity $\nu_0 > 0$ and Ricci curvature with a certain controlled decay must be a ν_0-cylinder. This is an extension of the classical Hartman cylindricity theorem.

Keywords: Genuine Rigidity, Singular Extensions, cylinders

Contents

CHAPTER 2

Singular genuine rigidity of submanifolds

2.1 Introduction

One of the fundamental questions in submanifold theory is the rigidity of isometric immersions in space forms. Satisfactory answers to the local version in low codimension were obtained under certain nondegeneracy hypothesis on the second fundamental form, like the ones in [13], [1], [7], [3] and [8]. Recently the concept of isometric rigidity was extended in [4] to the one of genuine rigidity in order to deal with isometric deformations that arise as deformations of larger submanifolds. This fruitful idea allowed to generalize the papers mentioned above, among others. The concept of genuine isometric rigidity has been extended to the conformal realm in [12].

Global rigidity results are more difficult to obtain. One of the most important and beautiful is the classical Sacksteder's theorem [16], in which it is proved that a compact Euclidean hypersurface is isometrically rigid provided its set of totally geodesic points does not disconnect it. For higher codimensions, it is shown in [6] that compact Euclidean submanifolds in codimension 2 are, along each connected component of an open dense subset, either genuinely rigid or a special kind of compositions. Although the authors did not have the tools to justify it at the time, they had to allow singularities in these compositions. The necessity to introduce singularities was justified only recently in [11], and this is precisely what motivated this work: to incorporate singular extensions into the genuine rigidity theory, mainly to apply this setting to get global results. In the process, we found out that introducing singular extensions is quite natural and straightforward, even for local purposes, allowing us to substantially simplify and unify the theory.

We say that a pair of isometric immersions $f : M^n \to \mathbb{R}^{n+p}$ and $\hat{f} : M^n \to \mathbb{R}^{n+q}$ *(possibly) singularly extends isometrically* when there are an embedding $j\colon M^n \hookrightarrow N^{n+r}$ into a manifold N^{n+r} with $r > 0$, and isometric maps $F\colon N^{n+r} \to \mathbb{R}^{n+p}$ and $\hat{F}\colon N^{n+r} \to \mathbb{R}^{n+q}$ such that $f = F \circ j$ and $\hat{f} = \hat{F} \circ j$, with the set of points where F and $\hat{F}$ fail to be immersions (that may even be empty) contained in $j(M)$. In other words, the isometric extensions F and $\hat{F}$ in the following commutative diagram are allowed to be singular, but only along $j(M)$:

$$\begin{array}{ccccc} & & & & \mathbb{R}^{n+p} \\ & \overset{f}{\nearrow} & & \nearrow{\scriptstyle F} & \\ M^n & \overset{j}{\hookrightarrow} & N^{n+r} & & \\ & \underset{\hat{f}}{\searrow} & & \searrow{\scriptstyle \hat{F}} & \\ & & & & \mathbb{R}^{n+q} \end{array} \qquad (1)$$

An isometric immersion $\hat{f} : M^n \to \mathbb{R}^{n+q}$ is a *strong genuine deformation* of a given isometric immersion $f : M^n \to \mathbb{R}^{n+p}$ if there is no open subset $U \subset M^n$ along which the restrictions $f|_U$ and $\hat{f}|_U$ singularly extend isometrically. Accordingly, the isometric immersion f is said to be *singularly genuinely rigid* in $\mathbb{R}^{n+q}$ for a fixed integer q if, for any given isometric immersion $\hat{f} : M^n \to \mathbb{R}^{n+q}$, there is an open dense subset $U \subset M^n$ such that $f|_U$ and $\hat{f}|_U$ singularly extend isometrically.

More geometrically, an isometric deformation of a Euclidean submanifold M^n is strongly genuine if no open subset of M^n is a submanifold of a higher dimensional (possibly singular) isometrically deformable submanifold, in such a way that the isometric deformation of the former is induced by an isometric deformation of the latter, while (possibly) including singularities along M^n. Since all our extensions are ruled, the singularities that eventually appear are quite easy to understand, as it is classically done for ruled surfaces in $\mathbb{R}^3$.

To state our main results we first need a few definitions. Consider a pair of isometric immersions $f : M^n \to \mathbb{R}^{n+p}$ and $\hat{f} : M^n \to \mathbb{R}^{n+q}$. Let

$$\tau : L^l \subset T_f^\perp M \to \hat{L}^l \subset T_{\hat{f}}^\perp M$$

be a vector bundle isometry and suppose that it preserves the second fundamental forms and the normal connections restricted to the rank l vector normal subbundles L and $\hat{L}$. Let $\phi_\tau : (TM \oplus L) \times TM \to L^\perp \times \hat{L}^\perp$ be the flat bilinear form given by

$$\phi_\tau(Y+\eta, X) = ((\tilde{\nabla}_X(Y+\eta))_{L^\perp}, (\tilde{\nabla}_X(Y+\tau\eta))_{\hat{L}^\perp})\ \forall X, Y \in TM\,,\ \eta \in L,$$

where $\tilde{\nabla}$ stands for the connection in Euclidean space and $L^\perp \times \hat{L}^\perp$ is endowed with the semi-Riemannian metric $\langle , \rangle = \langle , \rangle|_{L^\perp} - \langle , \rangle|_{\hat{L}^\perp}$. A subset $S \subset L^\perp \oplus \hat{L}^\perp$ is called *null* if $\langle \eta, \xi \rangle = 0$ for all $\eta, \xi \in S$. Given a distribution D on M^n, denote by $\mathcal{O}(D)$ the smallest totally geodesic distribution of M^n that contains D.

We can now state our main local result, which applies even to $l = 0$ and $\tau = 0$.

Theorem 1. *Let $\hat{f} : M^n \to \mathbb{R}^{n+q}$ be a strongly genuine deformation of $f : M^n \to \mathbb{R}^{n+p}$ and $\tau : L^l \subset T_f^\perp M \to \hat{L}^l \subset T_{\hat{f}}^\perp M$ a parallel vector bundle isometry that preserves second fundamental forms. Let $D \subset TM \oplus L$ be a subbundle such that $\phi_\tau(D, TM)$ is a null subset. Then $D \subset TM$ and, along each connected component of an open dense subset of M^n, f and $\hat{f}$ are mutually $\mathcal{O}(D)$-ruled.*

The key advantage of Theorem 1 is that it deals with easily to construct null subsets instead of nullity distributions of flat bilinear forms. A good example of this is the following singular version of Theorem 1 in [4] removing the assumptions on the codimensions. Recall that $X \in TM$ is a *(right) regular element* of ϕ_τ if rank $\phi_\tau^X = i(\tau)$, where $\phi_\tau^X = \phi_\tau(\cdot, X)$ and $i(\tau) = \max\{\text{rank } \phi_\tau^X : X \in TM\}$. Denote by $RE(\phi_\tau) \subset TM$ the open dense subset of regular elements of ϕ_τ.

Corollary 2. *Along each connected component of an open dense subset of M^n, f and $\hat{f}$ are mutually $\mathcal{O}(D^d)$-ruled, where $D^d = ker(\phi_\tau^Y) \subset TM$ for any $Y \in RE(\phi_\tau)$ and $d = n + l - i(\tau) \geq n - p - q + 3l$.*

As it is clear from the statements, the rulings in the above are larger and easier to compute than the ones in the main result in [4]. The bundles obtained in this work are also better suited for certain global applications as will be seen in Section 2.2.

The main idea in Sacksteder's theorem was to use the local rigidity of Beez-Killing's theorem [13] in certain subset, and from there spread to the whole manifold the information of the second fundamental form along the relative nullity leaves. This idea was also used by Dajczer and Gromoll in [6] to study global isometric rigidity in codimension 2. We will generalize these techniques in order to obtain the following global application of Theorem 1.

Theorem 3. *Let $f : M^n \to \mathbb{R}^{n+p}$ and $\hat{f} : M^n \to \mathbb{R}^{n+q}$ be isometric immersions of a compact submanifold with $min\{p,q\} \leq 5$ and $p+q<n$. Then, along each connected component of an open dense subset of M^n, either f and $\hat{f}$ singularly extend isometrically, or f and $\hat{f}$ are mutually D^d-ruled, with $d \geq n-p-q+3$.*

It turns out that, in low codimensions, when the pair $\{f, \hat{f}\}$ is mutually $(n-1)$-ruled we can also find singular extensions. Therefore, Theorem 3 unifies the main results in [6] and [16], provides a global version of the main theorem in [5], and gives a new global result for $\{p,q\} = \{1,3\}$. In fact, we prove:

Corollary 4. *Let $f : M^n \to \mathbb{R}^{n+p}$ be an isometric immersion of a compact Riemannian manifold M^n. Then, f is singularly genuinely rigid in $\mathbb{R}^{n+q}$ for $q < min\{5,n\} - p$.*

It is worth emphasizing that by allowing singular extensions we recover all the corollaries in [4], yet without certain restrictions on the codimensions required there. For example, applying Proposition 26 in [4] to Theorem 1 we obtain the following extension of Theorem 7 in [4]:

Corollary 5. *Let M^n be a compact manifold whose the k-th Pontrjagin class satisfies that $[p_k] \neq 0$ for some $k > \frac{3}{4}(p+q)$. Then, any analytic immersion $f : M^n \to \mathbb{R}^{n+p}$ (with the induced metric) is singularly genuinely rigid in $\mathbb{R}^{n+q}$ in the C^∞-category. The same conclusion holds for $k > \frac{3}{4}(p+q-3)$ if $min\{p,q\} \leq 5$.*

As we will see, the proof of our local Theorem 1 works for any simply connected space form. Moreover, our global Theorem 3 and Corollary 4 still hold for complete submanifolds, even if the ambient space is the hyperbolic space, as long as one of the immersions is bounded. For complete submanifolds in the sphere we show:

Theorem 6. *Let $f : M^n \to \mathbb{S}^{n+p}$ and $\hat{f} : M^n \to \mathbb{S}^{n+q}$ be isometric immersions of a complete submanifold with $min\{p,q\} \leq 5$ and $p+q < n - \nu_n$.*

Then, along each connected component of an open dense subset of M^n, either f and $\hat{f}$ singularly extend isometrically, or f and $\hat{f}$ are mutually D^d-ruled, with $d \geq n-p-q+3$.

In the above statement ν_n is defined as $\nu_n = \max\{k : \rho(n-k) \geq k+1\}$, where $\rho(n)$ is given by $\rho((odd)2^{4d+b}) = 8d + 2^b$, for any nonnegative integer d and $b \in \{0,1,2,3\}$. Some values of ν_n are: $\nu_n = n-$(highest power of $2 \leq n$) for $n \leq 24$, $\nu_n \leq 8d-1$ for $n < 16^d$ and $\nu_{2^d} = 0$. By Theorem 6, the proof of Corollary 4 gives us the analogous result for the sphere:

Corollary 7. *Let $f : M^n \to \mathbb{S}^{n+p}$ be an isometric immersion of a complete submanifold M^n. Then, f is singularly genuinely rigid in $\mathbb{S}^{n+q}$ for $q \leq 3-p$ if $4 \leq n \leq 7$, or $q \leq 4-p$ if $n \geq 8$.*

2.2 The structure of singular extensions

In this section we first provide the tools needed to build our extensions, and then we present the main result involving them. We finish the section with the proofs of our main local results.

Consider isometric immersions $f : M^n \to \mathbb{R}^{n+p}$ and $\hat{f} : M^n \to \mathbb{R}^{n+q}$ with second fundamental forms α and $\hat{\alpha}$, respectively. Endow $T_f^\perp M \times T_{\hat{f}}^\perp M$ with its natural semi-Riemannian metric of type (p,q) given by

$$\langle (\xi,\hat{\xi}),(\eta,\hat{\eta})\rangle = \langle \xi,\eta\rangle_{T_f^\perp M} - \langle \hat{\xi},\hat{\eta}\rangle_{T_{\hat{f}}^\perp M}.$$

Define the symmetric bilinear form $\beta : TM \times TM \to T_f^\perp M \times T_{\hat{f}}^\perp M$ by

$$\beta = \alpha \oplus \hat{\alpha}. \tag{2.1}$$

Gauss equation tells us that β is *flat*, that is,

$$\langle \beta(X,Y),\beta(Z,T)\rangle = \langle \beta(X,T),\beta(Z,Y)\rangle,\ \forall X,Y,Z,T \in TM.$$

The concept of flat bilinear forms was introduced by Moore in [15] to study isometric immersions of the round sphere in Euclidean space in low codimension, and was used afterwards in several papers about isometric rigidity, even implicitly, following a remark also in [15]. For example, it can be used in the classical Beez-Killing's theorem in [13], in which case the objective is to show that $\mathrm{Im}(\beta)$ is everywhere a null set. These definitions make sense even for nonsymmetric bilinear forms. For higher codimensions we also need to obtain information about the normal connections, so we need a different nonsymmetric flat bilinear form initially introduced in [4]. Yet, surprisingly, we will not make use of Theorem 3 in [3]. In particular, this will allow us to get rid of the usual constraints on the codimensions.

Throughout this work,

$$\tau : L^l \subset T_f^\perp M \to \hat{L}^l \subset T_{\hat{f}}^\perp M$$

will denote a vector bundle isometry that preserves the induced normal connections and the second fundamental forms in the rank l subbundles L and $\hat{L}$. That is, $\tau(\nabla_X^\perp \xi)_L = (\hat{\nabla}_X^\perp \tau\xi)_{\hat{L}}$ for every $X \in TM$, $\xi \in L$, and $\tau \circ \alpha_L = \hat{\alpha}_{\hat{L}}$, where we represent the orthogonal projection on L and $\hat{L}$ with the corresponding subindex. Equivalently, its natural extension $\overline{\tau} = Id \oplus \tau : TM \oplus L \to TM \oplus \hat{L}$, is a parallel vector bundle isometry. Let $\phi_\tau : (TM \oplus L) \times TM \to L^\perp \times \hat{L}^\perp \subset T_f^\perp M \times T_{\hat{f}}^\perp M$ be the bilinear form given by

$$\phi_\tau(v, X) = ((\tilde{\nabla}_X v)_{L^\perp}, (\tilde{\nabla}_X \overline{\tau} v)_{\hat{L}^\perp}), \tag{2.2}$$

where $\tilde{\nabla}$ denotes the connection of the Euclidean ambient spaces. Notice that, if $l = 0$, then $\overline{\tau} = Id$ is trivially parallel and $\phi_\tau = \beta$, that is, the bilinear forms ϕ_τ are natural extensions of β in (2.1). Since some algebraic tools used in this work apply to flat bilinear forms, we need to ensure first that ϕ_τ is flat. Indeed, for $X, Y \in TM$ and $v, w \in TM \oplus L$ we have that

$$\begin{aligned}
\langle \phi_\tau(v, X), \phi_\tau(w, Y) \rangle &= \langle (\tilde{\nabla}_X v)_{L^\perp}, (\tilde{\nabla}_Y w)_{L^\perp} \rangle - \langle (\tilde{\nabla}_X \overline{\tau} v)_{\hat{L}^\perp}, (\tilde{\nabla}_Y \overline{\tau} w)_{\hat{L}^\perp} \rangle \\
&= \langle \tilde{\nabla}_X v, \tilde{\nabla}_Y w \rangle - \langle \tilde{\nabla}_X \overline{\tau} v, \tilde{\nabla}_Y \overline{\tau} w \rangle \\
&= -\langle v, \tilde{\nabla}_Y \tilde{\nabla}_X w \rangle + \langle \overline{\tau} v, \tilde{\nabla}_Y \tilde{\nabla}_X \overline{\tau} w \rangle \\
&= -\langle v, \tilde{\nabla}_X \tilde{\nabla}_Y w \rangle + \langle \overline{\tau} v, \tilde{\nabla}_X \tilde{\nabla}_Y \overline{\tau} w \rangle \\
&= \langle \phi_\tau(v, Y), \phi_\tau(w, X) \rangle.
\end{aligned}$$

We now see how ϕ_τ allows to extend the pair $\{f, \hat{f}\}$ isometrically. For any smooth rank r subbundle $D^r \subset TM \oplus L$ such that $\phi_\tau(D^r, TM)$ is a null set, the maps $F = F_{D,f} : D^r \to \mathbb{R}^{n+p}$ and $\hat{F} = \hat{F}_{D,\hat{f}} : D^r \to \mathbb{R}^{n+q}$ defined as

$$F(v(p)) = f(p) + v(p), \quad \hat{F}(v(p)) = \hat{f}(p) + \overline{\tau} v(p), \ \forall v \in \Gamma(D^r), \tag{2.3}$$

are isometric. To see this, observe that for every $v \in \Gamma(D^r)$ and $Z \in TM$,

$$\begin{aligned}
\|(F \circ v)_* Z\|^2 = \|Z + (\tilde{\nabla}_Z v)\|^2 &= \|Z + (\tilde{\nabla}_Z v)_{TM \oplus L}\|^2 + \|(\tilde{\nabla}_Z v)_{L^\perp}\|^2 \\
&= \|Z + (\tilde{\nabla}_Z \overline{\tau} v)_{TM \oplus \hat{L}}\|^2 + \|(\tilde{\nabla}_Z \overline{\tau} v)_{\hat{L}^\perp}\|^2 \\
&= \|Z + (\tilde{\nabla}_Z \overline{\tau} v)\|^2 = \|(\hat{F} \circ v)_* Z\|^2,
\end{aligned}$$

since $\overline{\tau}$ is parallel. In particular, if $D^r \cap TM = 0$, both maps are immersions in a neighborhood N^{n+r} of the 0-section of D^r, and thus induce the same Riemannian metric on N^{n+r}. Therefore F and $\hat{F}$ are regular isometric extensions of f and $\hat{f}$.

We proceed to characterize singular ruled extensions, that occurs above when $D \subset TM$. Let D be a distribution in M^n considered as a subbundle of TM. We say that $F = F_{D,f}$ as above is a *singular extension of* f if it is an immersion in some open neighborhood of the 0-section of D, except of course along the 0-section itself. We say that F *nowhere induces a singular extension of* f if, for every open subset $U \subset M^n$ and every subbundle $D' \neq 0$ of $D|_U$, the restriction of F to D' is not a singular extension of $f|_U$. We show next that F nowhere induces a singular extension of f only when the latter is $\mathcal{O}(D)$-ruled, where $\mathcal{O}(D)$ is the smallest totally geodesic distribution of M^n that contains D.

Lemma 8. *Let $f : M^n \to \mathbb{R}^{n+p}$ be an isometric immersion and $D \subset TM$ a distribution in TM. Then, $F_{D,f}$ nowhere induces a singular extension of f if and only if f is $\mathcal{O}(D)$-ruled along each connected component of an open dense subset of M^n.*

Proof. Clearly, it is enough to give a proof for the direct statement and for a rank one distribution, i.e., $D = span\{X\}$ for some nonvanishing vector field X on M^n. Consider the map $F : D \cong M^n \times \mathbb{R} \to \mathbb{R}^{n+p}$ given by (2.3) for $D = \text{span}\{X\}$, that is, $F(p,t) = f(p) + tX(p)$. This map will be a singular extension in some open neighborhood of $p \in M^n$ if and only if it is an immersion in a neighborhood of $(p, 0)$, except at the points in $M^n \times \{0\}$. Therefore, for all $p \in M^n$ there exists a sequence $(p_m, t_m) \to (p, 0)$, with $t_m \neq 0$, such that $\text{rank}(F_{*(p_m,t_m)}) = n$. Define the tensors $K(Z) = \nabla_Z X$ and $H_t(Z) = Z + tK(Z)$ for $Z \in TM$. Thus, there is $Y_m \in T_{p_m}M$ such that $F_{*(p_m,t_m)}Y_m = X(p_m)$, since $H_t \to Id$ as $t \to 0$ and

$$F_*\partial_t = X,$$

$$F_*Z = H_t(Z) + t\alpha(X, Z)\ \forall Z \in TM.$$

Let S_X be the K-invariant subspace generated by X,

$$S_X = \text{span}\{X, K(X), K^2(X), K^3(X), ...\}.$$

Observe that $F_{*(p_m,t_m)}Y_m = X(p_m)$ is equivalent to $H_{t_m}Y_m = X(p_m)$ and $\alpha(X(p_m), Y_m) = 0$. In particular, if t_m is sufficiently small,

$$\alpha(X(p_m), H_{t_m}^{-1}(X(p_m))) = 0 \tag{2.4}$$

and $\lim_{m\to\infty} H_{t_m}^{-1}(X(p_m)) = X(p)$. Consider a precompact open neighborhood $U \subset M^n$ of p, so $\|\alpha\| < c$ and $\|K\| < c$ for some constant $c > 1$. Hence for $t \in I = (-\frac{1}{c^2}, \frac{1}{c^2})$ we have that H_t is invertible on U, and

$$H_t^{-1} = \sum_{i\geq 0}(-t)^i K^i,$$

since $H_t(\sum_{i=0}^{N}(-t)^i K^i) = Id - (-t)^{N+1}K^{N+1}$.

We claim that $\alpha(X, S_X) = 0$ on M^n. Otherwise, let $j = \min\{k \in \mathbb{N} : \alpha(X, K^k(X)) \neq 0\}$ and $p \in M^n$ such that $\alpha(X(p), K^j(X(p))) \neq 0$. By (2.4) it follows that

$$\sum_{i\geq j}(-t_m)^i \alpha(X(p_m), K^i(X(p_m))) = 0.$$

Dividing all terms in the above by t_m^j and taking $m \to \infty$ it follows that $\alpha(X(p), K^j(X(p))) = 0$, which is a contradiction.

Now, since $\alpha(X, S_X) = 0$ on M^n, for any $t \in I$ and $p \in U$ we have that $F_{*(p,t)}(H_t^{-1}(X)) = X$ since $H_t^{-1}(X) \in S_X$. It follows that $\text{rank}(F_*) = n$ in all $U \times I$, and therefore $F(U \times I) = f(U)$. Hence a segment of the line generated by X is contained in $f(U)$. In particular, f is $\mathcal{O}(\text{span}\{X\})$-ruled along each connected component of an open dense subset of M^n where $\mathcal{O}(\text{span}\{X\})$ has locally constant dimension. □

We are now able to prove our main local results.

Proof of Theorem 1. Locally, if $D \not\subset TM$ along some open set U then we have regular isometric extensions of $f|_U$ and $\hat{f}|_U$ by extending along any subbundle $\Lambda \subset D$ such that $D = (D \cap TM) \oplus \Lambda$. Hence, $D \subset TM$ and by Lemma 8 we conclude that f and $\hat{f}$ are mutually $\mathcal{O}(D)$-ruled almost everywhere. □

Proof of Corollary 2. Since ϕ_τ is flat, by Lemma 5 in [3] we have that

$$\phi_\tau(ker\,\phi_\tau^Y, TM) \subset \text{Im}(\phi_\tau^Y) \cap \text{Im}(\phi_\tau^Y)^\perp, \quad \forall Y \in RE(\phi_\tau). \tag{2.5}$$

In particular, $\phi_\tau(\ker\phi_\tau^Y, TM)$ is null and we apply Theorem 1 to τ and $D = \ker\phi_\tau^Y$. □

Remark 1. In most applications we have that $D = \mathcal{N}(\alpha_{L^\perp}) \cap \mathcal{N}(\hat{\alpha}_{\hat{L}^\perp})$. For example, this is the case if $d = n - p - q + 3l$ in Corollary 2, or if $l = \min\{p, q\}$, or if one of the codimensions is low enough. In this situation, $L_D = \alpha(D, TM) \subset L$, $\tau|_{L_D}$ is also parallel and preserves second fundamental forms, and therefore we recover the structure of the normal bundles in Theorem 1 in [4].

Remark 2. We say that τ is *maximal* if there is no open subset $U \subset M^n$ where there exists a vector bundle isometry $\sigma : E^{l+1} \to E^{l+1}$ such that σ is parallel and preserves the second fundamental forms, with $L^l \subset E^{l+1}$ and $\hat{L}^l \subset \hat{E}^{l+1}$. It is straightforward to check that, if τ is maximal, then $\mathrm{Im}(\phi_\tau)^\perp \subset L^\perp \oplus \hat{L}^\perp$ is Riemannian almost everywhere. In particular, ϕ_τ is nondegenerate.

All results obtained until now remain valid when the ambient space is the simply connected space form $\mathbb{Q}_c^m$ of constant sectional curvature c, just by using the exponential map of $\mathbb{Q}_c^m$ when constructing the extensions, e.g., as in (2.3).

2.3 Global applications

The purpose of this section is to give the proofs of Theorem 3, Corollary 4 and Corollary 5 in the Introduction. First, we establish some well-known properties of the splitting tensor adapted to our problem.

2.3.1 Intersection of relative nullities

Let M^n be a Riemannian manifold and D a smooth totally geodesic distribution on M^n. The *splitting tensor* C of D is the tensor $C : D \times D^\perp \to D^\perp$ defined by

$$C_Y X := C(Y, X) = -(\nabla_X Y)_{D^\perp}.$$

Let $f : M^n \to \mathbb{Q}_c^{n+p}$ be an isometric immersion of M^n with second fundamental form α and suppose now that $D \subset \Delta$, where

$$\Delta = \mathcal{N}(\alpha) = \{Y \in TM \,|\, \alpha(Y, Z) = 0\,,\, \forall Z \in TM\}$$

is the *relative nullity* of f. The *shape operator of* f in the direction $\xi \in T^\perp M$, denoted by A_ξ, is defined as $\langle A_\xi X, Y\rangle = \langle \alpha(X, Y), \xi\rangle$ for $X, Y \in TM$. The well-know equations below describe how they vary along D in terms of C.

For all $Y \in D$ and $\xi \in T_f^\perp M$ we have $\nabla_Y A_\xi = A_\xi C_Y + A_{\nabla_Y^\perp \xi}$. This follows directly from Codazzi equation, i.e., following exactly the same calculation done in [10] for the relative nullity. Let $\gamma : [0, b] \to M^n$ be a geodesic such that $\gamma([0,b))$ is contained in a leaf of D. If ξ is normal vector field parallel along γ, the above reduces to

$$A'_\xi = A_\xi C_{\gamma'}, \tag{2.6}$$

where we denote with a $'$ the covariant derivative with respect to the parameter of γ. Using the curvature tensor of $\mathbb{Q}_c^{n+p}$ we easily see that $C_{\gamma'}$ satisfies the Riccati type ODE

$$C'_{\gamma'} = C_{\gamma'}^2 + cI. \tag{2.7}$$

To prove Theorem 3, first we need to generalize the technique used in [16] and [6], that is, to use compactness to drag information of the second fundamental form throughout the whole manifold. In order to simplify the argument, we work with the intersection Δ_* of the relative nullities of the two immersions. However, since we use its associated splitting tensor we need the following three results for Δ_* that are well-know for relative nullity.

Lemma 9. *Let $f : M^n \to \mathbb{Q}_c^{n+p}$ and $\hat{f} : M^n \to \mathbb{Q}_c^{n+q}$ be isometric immersions of a Riemannian manifold M^n, and set $\Delta_* = \mathcal{N}(\beta)$. Then, on each connected component of an open dense subset of M^n, $\nu_* = \dim \Delta_*$ is constant and Δ_* is a totally geodesic distribution. In particular, there is a splitting tensor associated to Δ_*.*

Proof. It is straightforward to check that β in (2.1) is a Codazzi tensor with respect to the connection $\check{\nabla}^\perp = (\nabla^\perp, \hat{\nabla}^\perp)$, i.e.,

$$(\check{\nabla}^\perp_X \beta)(Y,Z) = (\check{\nabla}^\perp_Y \beta)(X,Z) \text{ for all } X,Y,Z \in TM.$$

Then taking $X, Z \in \Delta_*$ and $Y \in TM$ we get $(\check{\nabla}^\perp_X \beta)(Y,Z) = -\beta(Y, \nabla_X Z) = (\check{\nabla}^\perp_Y \beta)(X,Z) = 0$, and the lemma follows. □

Lemma 10. *Let $U \subset M^n$ be an open subset where ν_* is constant, $\gamma : [0,b] \to M^n$ a geodesic with $\gamma([0,b))$ contained in a leaf of Δ_* in U, and P_γ the parallel transport along γ beginning at $t = 0$. We have that $\Delta_*(\gamma(b)) = P_\gamma(\Delta_*(\gamma(0)))(b)$, and the splitting tensor $C_{\gamma'}$ of Δ_* smoothly extends to $t = b$. In particular, the ODE (2.6) holds at time $t = b$.*

Proof. Let $T : \Delta_*^\perp(\gamma) \to \Delta_*^\perp(\gamma)$ be the unique solution in $[0,b)$ of the ODE

$$T' + C_{\gamma'} \circ T = 0\,,\; T(0) = I. \tag{2.8}$$

From (2.7) it follows that T also satisfies the linear ODE with constant coefficients $T'' + cT = 0$, and hence it extends smoothly to $t = b$, where it is defined in $P_\gamma(\Delta_*^\perp(\gamma(0)))(b)$.

For any $Z \in TM$ and $0 \neq V \in \Delta_*^\perp$ parallel along γ in $[0, b)$ we have that

$$\check{\nabla}^\perp_{\frac{d}{dt}}(\beta(T(V), Z)) = \beta((T' + C_{\gamma'} \circ T)(V), Z) = 0.$$

Thus $\beta(T(V), Z)$ is parallel along γ. Since $Z(0)$ is arbitrary, we have that $T(V) \notin \Delta_*$ in $[0, b]$ and, in particular, T is invertible in $[0, b]$. Since $P_\gamma(\Delta_*(\gamma(0)))(b) \subset \Delta_*(\gamma(b))$ by continuity, it follows that $P_\gamma(\Delta_*^\perp(\gamma(0)))(b) = \Delta_*^\perp(\gamma(b))$. Therefore, $C_{\gamma'}$ extends to $[0, b]$ as $C_{\gamma'} = -T' \circ T^{-1}$ by (2.8). □

Corollary 11. *On the open subset of M^n where ν_* is minimal, the leaves of Δ_* are complete and totally geodesic in M^n, $\mathbb{Q}_c^{n+p}$ and $\mathbb{Q}_c^{n+q}$.*

2.3.2 Proof of Theorem 3 and Corollary 4

Although our next result is a natural generalization of the key idea in [6], the proof presented here is simpler and more conceptual.

Proposition 12. *Let $f : M^n \to \mathbb{R}^{n+p}$ and $\hat{f} : M^n \to \mathbb{R}^{n+q}$ be isometric immersions of a compact Riemannian manifold M^n with min$\{p, q\} \leq 5$ and $p + q < n$. Then, at any point of M^n, there are unit vectors $\xi \in T_f^\perp M$ and $\hat{\xi} \in T_{\hat{f}}^\perp M$ such that $\hat{A}_{\hat{\xi}} = A_\xi$.*

Proof. Let

$$M_0 = \{x \in M \, : \, \nu_*(x) \leq n - p - q - 1\}.$$

Observe first that there exists a point where $\nu_* = 0$, and in particular, $M_0 \neq \emptyset$. In fact, if there is no such a point then $\nu_* \geq 1$ in all M^n, and since the leaves of Δ_* are complete and unbounded in the Euclidean space where ν_* is minimal, then $f(M)$ and $\hat{f}(M)$ cannot be bounded. Hence, by Theorem 3 in [3], along $M_0 \neq \emptyset$ the proposition follows by the existence of a nontrivial null component of β. By continuity, the proposition stills holds on $\overline{M_0}$.

Let $M_1 = M \backslash \overline{M_0}$, $r_1 = \min\{\nu_*(x) \, : \, x \in M_1\} > 0$ and

$$U_1 = \{x \in M_1 \, : \, \nu_*(x) = r_1\}.$$

By the semi-continuity of ν_* and its minimality in M_1 it follows that U_1 is open. Fix $x_0 \in U_1$ and consider the maximal leaf $L \ni x_0$ of Δ_* in U_1. Since, as seen above, f does not admit complete geodesics in Δ_*, there is a geodesic $\gamma : [0, l] \to M^n$ such that $\gamma(0) = x_0$, $\gamma([0, l)) \subset L \subset U_1$ and $\gamma(l) = x_1 \notin U_1$. Moreover, by Lemma 10, $\Delta_*(x_1)$ is the parallel transport of $\Delta_*(x_0)$ along γ, $\nu_*(x_1) = r_1$ and $x_1 \in \overline{M}_0$. Let ξ, $\tilde{\xi}$ be the parallel transports along γ of the normal directions given by the proposition at x_1 and $C_{\gamma'}$ the splitting tensor associated to Δ_* in U_1. We know that A_ξ and $\hat{A}_{\tilde{\xi}}$ are solutions of the first order linear ODE (2.6) extended by Lemma 10 to $[0, l]$. Since $A_\xi(x_1) = \hat{A}_{\tilde{\xi}}(x_1)$, then $A_\xi = \hat{A}_{\tilde{\xi}}$ along γ. In particular the equality holds at x_0. Once x_0 was taken arbitrarily, the proposition holds in $\overline{U}_1$.

Assume now that the proposition holds in $U_i = \{x \in M_1 : \nu_* = r_i\}$, where $M_i = M_{i-1} \backslash \overline{U_{i-1}}$ and $r_i = \min\{\nu_*(x) : x \in M_i\} > 0$, for $0 \leq i < k \leq n$ for some k. With the same argument as for U_1 by induction the proposition holds in U_k. □

Remark 3. Observe that in the proof above we only needed that there is no complete half geodesic contained in Δ_*. In particular, Proposition 12, and thus Theorem 3 and Corollary 4, holds for complete manifolds if we require that either $f(M)$ or $\hat{f}(M)$ contains no complete half straight line.

We need to ensure that we can make a smooth choice of the sections in Proposition 12:

Lemma 13. *Let $\hat{f} : M^n \to \mathbb{R}^{n+q}$ and $f : M^n \to \mathbb{R}^{n+p}$ be isometric immersions of a Riemannian manifold M^n. Assume that at each point of M^n there are unit vectors $\xi \in T_f^\perp M$ and $\tilde{\xi} \in T_{\hat{f}}^\perp M$ such that $A_\xi = \hat{A}_{\tilde{\xi}}$. Then, there is an open dense subset of M^n where the vector fields ξ and $\tilde{\xi}$ can be taken smooth and parallel along the leaves of Δ_*.*

Proof. Let $V \subset M^n$ be an open dense subset where Δ_*, $\mathcal{S}(\beta) := \text{span Im}(\beta)$, $\mathcal{S}(\beta)^\perp$ and $\mathcal{S}(\beta) \cap \mathcal{S}(\beta)^\perp$ have locally constant dimensions. Set

$$W_1 = \{x \in V : \langle , \rangle|_{\mathcal{S}(\beta)\times\mathcal{S}(\beta)} \text{ is nondegenerate}\}.$$

Let $x \in W_1$ and U_x be an open neighborhood of x where the index of $\langle , \rangle|_{\mathcal{S}(\beta)\times\mathcal{S}(\beta)}$ is constant. At each point of U_x we have the nontrivial light-cone $\Lambda_y \subset \mathcal{S}(\beta)$, and hence $\Lambda = \cup_{y\in U_x}\Lambda_y$ is a smooth bundle over U_x. Then any smooth non-vanishing section of Λ satisfies the first claim, and so there is a open dense subset U_1 in W_1 which still satisfies it.

Set $W_2 = V \backslash \overline{U}_1$. Thus $\dim(\mathcal{S}(\beta) \cap \mathcal{S}(\beta)^\perp) \geq 1$ along W_2, and hence any local non-vanishing smooth section of this vector subbundle satisfies our first claim.

It remains to show that we can replace the vector fields, if necessary, in order to obtain their parallelism along Δ_*. Locally, let $N^{n-\nu_*}$ be a submanifold of M^n transversal to Δ_*. If ξ_0 and $\hat{\xi}_0$ are smooth vector fields such that $A_{\xi_0} = \hat{A}_{\hat{\xi}_0}$, then we can redefine ξ and $\hat{\xi}$ on a small tubular neighborhood W of $N^{n-\nu_*}$ as the parallel transport of $\xi_0|_N$ and $\hat{\xi}_0|_N$ along the leaves of Δ_*. Using the same arguments as in Proposition 12 we obtain that $A_\xi = \hat{A}_{\hat{\xi}}$ on W, and we can choose an open dense subset which is a union of such neighborhoods W. □

Proof of Theorem 3. By Proposition 12 and Lemma 13 there is a trivially parallel isometry of line bundles $\tau : \text{span}\{\xi\} \to \text{span}\{\hat{\xi}\}$ defined in a dense open subset $U \subset M^n$ that preserves second fundamental forms. The result now follows from Corollary 2 applied to each connected component of U. □

Proof of Corollary 4. By Theorem 3 and Corollary 2 the only case to deal with is the $(n-1)$-ruled one, that is, when the pair $\{f, \hat{f}\}$ is mutually ruled by $D^{n-1} \subset TM$, $l = 1$, $p + q = 4$ and $i(\tau) = 2$. By Remark 1 and (2.5), $D = \mathcal{N}(\phi_\tau)$, i.e., $D = \mathcal{N}(\alpha_{L^\perp}) \cap \mathcal{N}(\hat{\alpha}_{\hat{L}^\perp})$.

We claim that L and $\hat{L}$ are parallel along D. It follows immediately if $D \subset \Delta_*$. If $D \not\subset \Delta_*$, for every $X, Y \in D$ and $Z \in TM$ we have $\check{\nabla}^\perp_X \beta(Z, Y) \in L \oplus \hat{L}$ by Codazzi equation, and the claim follows.

Since $D = \mathcal{N}(\alpha_{L^\perp}) \cap \mathcal{N}(\hat{\alpha}_{\hat{L}^\perp})$, for one of the immersions, say f, there is a unit vector field $\eta \in L^\perp \subset T^\perp_f M$ parallel along D such that $A_\eta \neq 0$. Then, locally f is a composition by Proposition 8 in [3], i.e., there is $g : M^n \to W \subset \mathbb{R}^{n+p-1}$ and $h : W \subset \mathbb{R}^{n+p-1} \to \mathbb{R}^{n+p}$ isometric immersions such that $f = h \circ g$, with $L \subset T^\perp_g M$. Indeed, it is easy to check that $\alpha_{\langle\eta\rangle^\perp}$ together with the induced connection $\nabla^\perp|_{\langle\eta\rangle^\perp}$ satisfy the fundamental equations of submanifolds in $\mathbb{R}^{n+p-1}$. Therefore, we apply Corollary 2 to the pair $\{g, \hat{f}\}$, and the corollary follows easily. □

Remark 4. Corollary 4 for $p = q = 2$ reduces to the main result in [6], except for the fact that singular flat extensions can occur in the former. This is a consequence of a gap in [6], whose long and involved case by case proof did not cover all possibilities.

Proof of Corollary 5. By Proposition 26 in [4] and Corollary 2, if an isometric immersion $\hat{f} : M^n \to \mathbb{R}^{n+q}$ is a strong genuine deformation of $f : M^n \to \mathbb{R}^{n+p}$, then the k-th Pontrjagin form p_k of M^n vanishes for any k such that $4k > 3(p+q)$. When $\min\{p, q\} \leq 5$ we use Theorem 3 instead of Corollary 2. □

2.4 The space forms case

As we pointed out, Theorem 3 and Corollary 4 hold for compact manifolds when the ambient space is the hyperbolic space following the same proofs. In this section we show that they also hold for *complete* submanifolds in the sphere.

In the following we use the fact that $\rho(k) - 1$ is the maximum number of continuous linearly-independent vector fields on $\mathbb{S}^{k-1}$.

Lemma 14. *Let $f : M^n \to \mathbb{S}^{n+p}$ be an isometric immersion and $D^d \subset \Delta$ a nontrivial totally geodesic distribution. If there exists a nonconstant geodesic $\sigma : [0, \infty) \to M^n$ in D^d, then the splitting tensor $C_{\sigma'}$ associated to D^d has no real eigenvalues. In particular, such a geodesic cannot exist if $\rho(n-d) < d+1$.*

Proof. By (2.7) $C_{\sigma'}$ is given by

$$(P_\sigma^{-1} \circ C_{\sigma'} \circ P_\sigma)(t) = (\sin(t)I + \cos(t)C_{\sigma'}(0))(\cos(t)I - \sin(t)C_{\sigma'}(0))^{-1},$$

where P_σ is the parallel transport along σ. Since $C_{\sigma'}$ is defined for all $t \geq 0$, we easily conclude that $C_{\sigma'(0)}$ has no real eigenvalues.

For the last assertion, if $T_1, ..., T_d$ is a basis of $D(x)$, then by the first assertion, for any unit vector $Z \in D^\perp(x)$ and $a, a_1, ..., a_d \in \mathbb{R}$, the equation

$$0 = aZ + \sum_{i=1}^{d} a_i C_{T_i} Z = aZ + C_{\sum_{i=1}^{d} a_i T_i} Z$$

implies $a = a_i = 0$. Hence $Z, C_{T_1}Z, ..., C_{T_d}Z$ are linearly independent in $D^\perp(x)$. Since this holds for any unit vector $Z \in D^\perp(x)$, considering Z as the position vector of the unit sphere $\mathbb{S}^{n-d-1} \subset D^\perp(x)$ we get d nonvanishing linearly independent vector fields in $\mathbb{S}^{n-d-1}$. Hence, $d \leq \rho(n-d) - 1$. □

Proof of Theorem 6. By Lemma 14, geodesics in Δ_* cannot be defined for arbitrary large time if $\nu_n \geq n - p - q$ when the ambient space is the sphere. Thus, by Remark 3, Proposition 12 holds for complete manifolds when the ambient spaces are spheres with $p + q < n - \nu_n$, and the theorem follows using Corollary 2. □

Proof of Corollary 7. It is analogous to the one for Corollary 4 using Theorem 6 instead of Theorem 3, just observing that for small codimensions we can simplify the assumptions on ν_n. □

The above Corollary generalizes the unpublished work [9] since if $n > 2p+2q$, then $k > \rho(n-k) - 1$ for all $k \geq n - p - q$.

CHAPTER 3

A Hartman's like Theorem

3.1 Introduction

The simplest examples of isometric immersions $f : M^n \to \mathbb{R}^m$ such that the index of relative nullity is positive everywhere are the s-cylinders. The isometric immersion f is said to be an *s-cylinder* if there exists a Riemannian manifold N^{n-s} such that M^n, $\mathbb{R}^m$ and f have factorizations

$$M^n = \mathbb{R}^s \times N^{n-s}, \ \mathbb{R}^m = \mathbb{R}^s \times \mathbb{R}^{m-s} \text{ and } f = I \times h,$$

where $h : N^{n-s} \to \mathbb{R}^{m-s}$ is an isometric immersion and $I : \mathbb{R}^s \to \mathbb{R}^s$ is the identity map. Clearly, in this case the minimal index of relative nullity ν_0 of f is precisely s, as long as that of h is zero.

The classical Hartman theorem states that these are the only possible complete examples with nonnegative Ricci curvature.

Theorem 15 (Maltz [14])**.** *Let M^n be a complete manifold with nonnegative Ricci curvature and let $f : M^n \to \mathbb{R}^m$ be an isometric immersion with minimal index of relative nullity $\nu_0 > 0$. Then f is a ν_0-cylinder.*

The main purpose of this part of the Thesis is to extend the above result to submanifolds with Ricci curvature having a certain controlled decay.

Theorem 16. *Let M^n be a complete manifold with*

$$\mathrm{Ric} \geq -\left(\mathrm{Hess}\,\psi + \frac{d\psi \otimes d\psi}{n-1}\right) \tag{3.1}$$

for some function ψ bounded from above on M^n and let $f : M^n \to \mathbb{R}^m$ be an isometric immersion with minimal index of relative nullity $\nu_0 > 0$. Then f is a ν_0-cylinder.

Note that we recover Theorem 15 from the above by simply taking ψ to be constant. In Wylie [17], such a Riemannian manifold satisfying (3.1) was said to be $CD(0,1)$ with respect to the *potential function* ψ.

We actually prove a version of Theorem 16 that is more general in two ways. The first is that we can weaken the upper bound on ψ assumption to an integral condition along geodesics, the so-called bounded energy distortion. Secondly the function ψ can be replaced with a vector field. We discuss this result in Section 3.4.

3.2 Preliminaries

The main step in the proof of Theorem 16 is Lemma 17 below (see Maltz [14]).

Lemma 17. *Suppose $M^n = \mathbb{R} \times N^{n-1}$ is the Riemannian product of $\mathbb{R}$ and a connected Riemannian manifold N^{n-1}, and suppose $f : M^n \to \mathbb{R}^m$ is an isometric immersion mapping a geodesic of the form $\mathbb{R} \times \{q\}$ onto a straight line in $\mathbb{R}^m$. Then f is a 1-cylinder.*

Our result also relies on the fundamental fact that the leaves of the minimum relative nullity distribution of a complete submanifold of $\mathbb{R}^m$ are also complete (cf. Dajczer [2]).

Lemma 18. *Let $f : M^n \to \mathbb{R}^m$ be an isometric immersion of a complete Riemannian manifold M^n with $\nu > 0$ everywhere. Then, the leaves of the relative nullity distribution are complete on the open subset where $\nu = \nu_0$ is minimal.*

Theorem 15 follows easily from Lemmas 17 and 18 above together with the Cheeger-Gromoll splitting theorem. Indeed, under the assumptions of Theorem 15, Lemma 18 yields that M^n contains ν_0 linearly independent lines through each point where the index of relative nullity is minimal. By the splitting theorem of Cheeger-Gromoll, M^n is isometric to a Riemannian product $\mathbb{R}^{\nu_0} \times N^{n-\nu_0}$, and Theorem 15 then follows inductively from Lemma 17.

The proof of our Theorem 16 uses the same ideas above, taking advantage of a recent warped product version of the splitting theorem by Wylie [17]. According to this latter result, estimate (3.1) is sufficient to split a complete Riemannian manifold M^n that admits a line into a warped product $\mathbb{R}\times_\rho N^{n-1}$ over $\mathbb{R}$. But since this splitting comes from a line of relative nullity, our goal is to show that the warping function ρ must be constant, and thus $\mathbb{R}\times_\rho N^{n-1}$ is actually a Riemannian product, so that Lemma 17 can be applied to conclude the proof. To do this we need to collect geometric information on the behavior of a warped product as above along the line $\mathbb{R}$. For later use, we carry out this study within the broader class of *twisted products* $M^n = \mathbb{R}\times_\rho N^{n-1}$ over $\mathbb{R}$, where (N, h) is a Riemannian manifold, $\rho : M^n \to \mathbb{R}_+$ the *twisting function*, and M^n is endowed with the metric $g = dr^2 + \rho^2 h$. If ρ is a function of r only, then we have a *warped product* over $\mathbb{R}$. The following lemma describes how vector fields vary along $\mathbb{R}$.

Lemma 19. *Let $M^n = \mathbb{R}\times_\rho N^{n-1}$ be a twisted product over $\mathbb{R}$. Then*

$$\nabla_{\partial_r}\partial_r = 0 \tag{3.2}$$

and

$$\nabla_{\partial_r}X = \nabla_X\partial_r = \frac{1}{\rho}\frac{\partial\rho}{\partial r}X \tag{3.3}$$

for all $X \in \mathfrak{X}(N)$.

Proof. Let us write $\rho_r = \rho(r,\cdot)$ and denote by N_{ρ_r} the Riemannian manifold N endowed with the conformal metric rescaled by ρ_r^2. It is straightforward to check that ∇ given by (3.2), (3.3) and

$$\nabla_X Y = \nabla_X^{N_{\rho_r}} Y - \langle X, Y\rangle \frac{1}{\rho}\frac{\partial\rho}{\partial r}\partial_r$$

for all X, $Y \in \mathfrak{X}(N)$ defines a compatible symmetric connection on TM, hence it coincides with the Levi-Civita connection of M^n. □

Next, we use Lemma 19 to compute the sectional curvatures K along planes containing ∂_r.

Lemma 20. *Let $M^n = \mathbb{R}\times_\rho N^{n-1}$ be a twisted product over $\mathbb{R}$. Then*

$$K(\partial_r, X) = -\frac{1}{\rho}\frac{\partial^2\rho}{\partial r^2} \tag{3.4}$$

for all unit vector $X \in T_xN$ and all $x \in N^{n-1}$.

Proof. Differentiating $\langle X, X\rangle = \rho^2$ twice with respect to r gives

$$\langle \nabla_{\partial_r}\nabla_{\partial_r}X, X\rangle + \|\nabla_{\partial_r}X\|^2 = \rho\frac{\partial^2\rho}{\partial r^2} + \left(\frac{\partial\rho}{\partial r}\right)^2.$$

Using (3.2) and (3.3), we conclude that

$$\langle R(\partial_r, X)\,\partial_r, X\rangle = \rho\frac{\partial^2\rho}{\partial r^2},$$

from which the result follows. □

We are now in a position to state and prove our main lemma, in which by a *line of nullity* of a Riemannian manifold M^n we mean a curve $\gamma : \mathbb{R} \to M^n$ such that $\gamma'(t) \in \Gamma(\gamma(t))$ for all $t \in \mathbb{R}$, where

$$\Gamma(x) = \{X \in T_xM : R(X, Y) = 0 \text{ for all } Y \in T_xM\}$$

is the nullity subspace at $x \in M^n$.

Lemma 21. *Let $M^n = \mathbb{R} \times_\rho N^{n-1}$ be a twisted product over $\mathbb{R}$. If $\mathbb{R} \times \{q\}$ is a line of nullity of M^n for some $q \in N^{n-1}$, then $\rho_r = \rho_0$ does not depend on r, and hence M^n is actually the Riemannian product $\mathbb{R} \times N^{n-1}_{\rho_0}$.*

Proof. It follows from (3.4) that $\frac{\partial^2\rho}{\partial r^2} \equiv 0$, but since the twisting function ρ is positive on the whole real line it must be constant. □

3.3 Proof of Theorem 16

As previously discussed, Lemma 17 is at the core of the proof of Theorem 16, whereas Lemma 21 is the principle behind its use.

Proof of Theorem 16. We can assume that $\nu_0 = 1$, since the general case follows easily by induction on ν_0. Take a point $p \in M^n$ where $\nu = 1$. It follows from Lemma 18 that M^n contains a line l through p. By the warped product version of the splitting theorem of Cheeger-Gromoll due to Wylie [17], the Riemannian manifold M^n is isometric to a warped product $\mathbb{R}\times_\rho N^{n-1}$ over $\mathbb{R}$, the line l corresponding to $\mathbb{R} \times \{q\}$ for some $q \in N^{n-1}$. Since l is a leaf of the relative nullity foliation, we have in particular that $\mathbb{R} \times \{q\}$ is a line of nullity of $\mathbb{R} \times_\rho N^{n-1}$, and thus, by Lemma 21, $\rho_r = \rho_0$ does not depend on r and $\mathbb{R} \times_\rho N^{n-1}$ is actually the Riemannian product $\mathbb{R} \times N^{n-1}_{\rho_0}$. Hence, we may consider $f : \mathbb{R} \times N^{n-1}_{\rho_0} \to \mathbb{R}^m$, and as f maps $\mathbb{R} \times \{q\}$ onto a straight line in $\mathbb{R}^m$, the result then follows from Lemma 17. □

3.4 A slight generalization

In this section we explain how the result above also has a version for non-gradient potential fields. Curvature inequality (3.1) has a natural extension to vector fields X and can be regarded as the special case where $X = \nabla\psi$.

Our result in the gradient case assumes boundness of the potential function ψ. While there is no potential function for a non-gradient field X on a Riemannian manifold M^n, we can still make sense of bounds by integrating X along geodesics. Let $\gamma : (a, b) \to M^n$ be a geodesic that is parametrized by arc-length. Define

$$\psi_\gamma(t) = \int_a^t \langle \gamma'(s), X(\gamma(s)) \rangle \, ds,$$

which is a real valued function on the interval (a, b) with the property that $\psi'_\gamma(t) = \langle \gamma'(t), X(\gamma(t)) \rangle$. When $X = \nabla\psi$ is a gradient field then $\psi_\gamma(t) = \psi(\gamma(t)) - \psi(\gamma(a))$. In the non-gradient case, we can think of ψ_γ as being the anti-derivative of X along the geodesic γ.

We now recall the notion of 'bounded energy distortion', introduced by Wylie [17].

Definition 1. Let M^n be a non-compact complete Riemannian manifold and $X \in \mathfrak{X}(M)$ a vector field. Then we say X has *bounded energy distortion* if, for every point $x \in M^n$,

$$\limsup_{r\to\infty} \inf_{l(\gamma)=r} \left\{ \int_0^r e^{-\frac{2\psi_\gamma(s)}{n-1}} ds \right\} = \infty,$$

where the infimum is taken over all minimizing unit speed geodesics γ with $\gamma(0) = x$.

Note that if a vector field X has the property that ψ_γ is bounded for all unit speed minimizing geodesics then it has bounded energy distortion. However, even in the gradient case, bounded energy distortion is a weaker condition than ψ bounded above.

Our most general cylindricity theorem is the following, where we replace the Hessian of ψ by the Lie derivative $L_X g$ and the 1-form $d\psi$ by the dual of X.

Theorem 22. *Let (M^n, g) be a complete manifold with*

$$\mathrm{Ric} \geq -\left(\frac{1}{2} L_X g + \frac{X^\sharp \otimes X^\sharp}{n-1} \right) \tag{3.5}$$

for some vector field X with bounded energy distortion and let $f : M^n \to \mathbb{R}^m$ be an isometric immersion with minimal index of relative nullity $\nu_0 > 0$. Then f is a ν_0-cylinder.

In particular, when $X = \nabla\psi$, we conclude that Theorem 16 still holds under the weaker condition that ψ has bounded energy distortion rather than being bounded from above.

By Wylie [17], inequality (3.5) allows to split M^n as a twisted product $\mathbb{R} \times_\rho N^{n-1}$ over $\mathbb{R}$, provided there is a line. But since Lemma 21 actually holds for twisted products, the proof of Theorem 22 then follows by the same arguments as in Section 3.3.

Bibliography

[1] C. B. Allendoerfer. Rigidity for spaces of class greater than one. *Amer. J. Math.*, 61:633–644, 1939.

[2] M. Dajczer. *Submanifolds and isometric immersions*, volume 13 of *Mathematics Lecture Series*. Publish or Perish, Inc., Houston, TX, 1990.

[3] M. Dajczer and L. A. Florit. Erratum: "Compositions of isometric immersions in higher codimension" [Manuscripta Math. **105** (2001), no. 4, 507–517; MR1858501 (2002m:53096)]. *Manuscripta Math.*, 110(1):135, 2003.

[4] M. Dajczer and L. A. Florit. Genuine deformations of submanifolds. *Comm. Anal. Geom.*, 12(5):1105–1129, 2004.

[5] M. Dajczer, L. A. Florit, and R. Tojeiro. Euclidean hypersurfaces with genuine deformations in codimension two. *Manuscripta Math.*, 140:621–643, 2013.

[6] M. Dajczer and D. Gromoll. Isometric deformations of compact Euclidean submanifolds in codimension 2. *Duke Math. J.*, 79(3):605–618, 1995.

[7] M. Dajczer and R. Tojeiro. On compositions of isometric immersions. *J. Differential Geom.*, 36(1):1–18, 1992.

[8] M. do Carmo and M. Dajczer. A rigidity theorem for higher codimensions. *Math. Ann.*, 274(4):577–583, 1986.

[9] D. Ferus. The rigidity of complete hypersurfaces. *Unpublished*.

[10] D. Ferus. On the completeness of nullity foliations. *Michigan Math. J.*, 18:61–64, 1971.

[11] L. A. Florit and G. M. de Freitas. Classification of codimension two deformations of rank two riemannian manifolds. *Comm. Anal. Geom.*, 25(4):751–797, 2017.

[12] L. A. Florit and R. Tojeiro. Genuine deformations of submanifolds II: the conformal case. *Comm. Anal. Geom.*, 18(2):397–419, 2010.

[13] W. Killing. Die nicht-euklidischen raumformen in analytische behandlung. *Teubner, Leipzig*, 1885.

[14] R. Maltz. Cylindricity of isometric immersions into Euclidean space. *Proc. Amer. Math. Soc.*, 53(2):428–432, 1975.

[15] J. D. Moore. Submanifolds of constant positive curvature. I. *Duke Math. J.*, 44(2):449–484, 1977.

[16] R. Sacksteder. The rigidity of hypersurfaces. *J. Math. Mech.*, 11:929–939, 1962.

[17] W. Wylie. A warped product version of the Cheeger-Gromoll splitting theorem. *Trans. Amer. Math. Soc.*, 369(9):6661–6681, 2017.