The 2-dimensional Complex Jacobian Conjecture under the viewpoint of “pertinent variables”

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Let $F = (f, g) : \mathbb{C}^2 \to \mathbb{C}^2$ be a polynomial map. The Jacobian matrix of F at $(x, y) \in \mathbb{C}^2$ will be denoted by $JF(x, y)$. The 2-dimensional Complex Jacobian Conjecture, which is still open, can be expressed as follows: “if F satisfies the Non-Zero Condition $\det(JF(x, y)) = \text{constant} \neq 0, \forall (x, y) \in \mathbb{C}^2$, then F is non-proper”. A significant approach for the study of the Jacobian Conjecture is to remove the most possible polynomial maps that do not satisfy the Non-Zero Condition and work on the complementary set in the ring of polynomial maps. We define firstly in this paper *good polynomial maps* which satisfy the following property: if F satisfies the Non-Zero Condition, then F is a good polynomial map. Then, with the hypothesis “F is non-proper”, we define new variables called *pertinent variables* and we treat F under these variables. That allows us to define a class $\mathcal{C}_1$ of good, non-proper polynomial maps $F = (f, g) : \mathbb{C}^2 \to \mathbb{C}^2$ that are significant for the study of the Jacobian Conjecture. We continue to restrict the class $\mathcal{C}_1$ by expliciting a subclass $\mathcal{C}_2 \subset \mathcal{C}_1$ of polynomial maps which do not satisfy the Non-Zero Condition. Then, on the one side, we get a model of a counter-example for the Conjecture **if there exists**. On the other side, we provide a criterion for verifying the 2-dimensional Complex Jacobian Conjecture for the class of good, non-proper maps. Moreover, for verifying the 2-dimensional Complex Jacobian Conjecture, it is enough to verify it for the complementary set of the set $\mathcal{C}_2$ in the set of good maps. On another side, the class of dominant polynomial maps is also an important class for the study of the Jacobian Conjecture. The second part of this paper is to provide a criterion to verify the dominancy of a polynomial map in general. Applying this criterion, we prove that a polynomial map of the class $\mathcal{C}_1 \setminus \mathcal{C}_2$ is dominant and we describe its asymptotic set. We retrieve the Jelonek’s result: “the asymptotic set of a non-proper dominant polynomial map $F : \mathbb{C}^2 \to \mathbb{C}^2$ is a complex curve”.