Gauss-Manin Connection in Disguise for Dwork Family

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Gauss-Manin connection in disguise, GMCD for short, is an algebraic method introduced by Hossein Movasati.

The reason for preferring this method in our researches is that in GMCD it does not require to compute either the periods or the variation of the Hodge structure: all these objects become encoded in the Gauss-Manin connection. The base space of GMCD is a certain moduli space T of the pairs formed by Calabi-Yau n-folds of a definite family along with differential n-forms. More precisely, in this talk, the *enhanced moduli space* T is the moduli space of the pairs $(X, [\alpha_1, \alpha_2, \ldots, \alpha_{n+1}])$ where X is a Calabi-Yau n-fold arising from the so-called Dwork family and $\{\alpha_1, \alpha_2, \ldots, \alpha_{n+1}\}$ forms a basis of the n-th algebraic de Rham cohomology $H^n_{\text{dR}}(X)$ satisfying the following two conditions: (i) the basis must be compatible with the Hodge filtration of $H^n_{\text{dR}}(X)$, (ii) the intersection form matrix of the basis has to be constant.

One of the main objects of GMCD is the existence of a certain (unique) vector field R satisfying a definite equation involving the Gauss-Manin connection. In this talk we find the vector field R, which is called *modular vector field*, and state it explicitly in lower dimensions on the enhanced moduli space T. It turns out that the q-expansion of the solutions, up to multiplying by a constant rational number, has integer coefficients. In particular, in the case of elliptic curves and $K3$ surfaces, where $n = 1, 2$, the solutions can be written in terms of (quasi-)modular forms satisfying certain enumerative properties. Solutions of R are called *Calabi-Yau modular forms.* A very useful result of this work is that the modular vector field R together with the radial vector field and a degree zero vector field generates the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. It is worth to mention that one of the objectives of the project GMCD, considering the modularity of the Calabi-Yau varieties, is somehow to generalize the theory of modular forms. In this generalization the Calabi-Yau modular forms can be regarded as the suitable candidates to substitute the classical modular forms.