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BIRATIONAL GEOMETRY OF FOLIATIONS

Marco Brunella

Contents

The main aim of these notes is to develop a classification (or a beginning of classification) of holomorphic foliations on algebraic surfaces, which is in the same spirit as Enriques classification of complex algebraic surfaces. To start with, let us therefore remind some basic principles and results of Enriques classification [BPV], [Rei], [Fri].

Let X be a smooth, complex, compact, connected algebraic variety, of dimension n. The protagonist of the classification is the *canonical bundle*

$$K_X = \bigwedge^n T^*X$$

that is the line bundle on X whose sections are the n-forms on X. One defines the *plurigenera* of X as

$$P_m(X) = h^0(X, K_X^{\otimes m}) \qquad\qquad m \in \mathbf{N}^+$$

i.e. the dimensions of the spaces of global holomorphic sections of $K_X^{\otimes m}$. And one defines the *Kodaira dimension* of X, $kod(X)$, according to the behaviour of the sequence $\{P_m(X)\}$:

- $kod(X) = -\infty$ if $P_m(X) = 0$ for every $m \in \mathbf{N}^+$;
- $kod(X) = 0$ if $P_m(X) \in \{0,1\}$ for every m, but it is not always zero;
- $kod(X) = d \in \{1, 2, ..., n\}$ if $P_m(X)$ has a polynomial growth of degree d as m tends to $+\infty$.

In fact, one can prove that there are not other possibilities for the sequence of plurigenera. The Kodaira dimension turns out to be one of the most basic invariants of smooth algebraic varieties; it is, as well as plurigenera, a *birational* invariant, i.e. $kod(X) = kod(Y)$ if X and Y are birationally isomorphic.

Let us firstly consider the case of curves: $n = 1$. Then $kod(X) \in \{-\infty, 0, 1\}$, and quite elementary (but not too elementary) considerations show that $kod(X) = -\infty$ iff X is a rational curve (genus 0), $kod(X) = 0$ iff X is an elliptic curve (genus 1), $kod(X) = 1$ iff X is an hyperbolic curve (genus ≥ 2). Thus the Kodaira dimension faithfully reflects the fundamental trichotomy of algebraic curves.

Let us now pass to the case of surfaces, $n = 2$, which is the context of Enriques classification. Then $kod(X) \in \{-\infty, 0, 1, 2\}$, and it is a remarkable and deep fact that one can obtain a quite good description of algebraic surfaces whose Kodaira dimension is *not* equal to 2. Surfaces of Kodaira dimension 2 are called *of general type*, and in some sense they are left aside in Enriques classification; "most" surfaces are in fact surfaces of general type (as the name suggests), and so one cannot hope for a precise classification of these objects (however, one can obtain interesting and important results about their moduli spaces, their Chern numbers, their automorphisms,...).

Thus, a summary of Enriques classification looks as follows:

a) $kod(X) = -\infty$: the surface X is birational to $\mathbf{C}P^1 \times C$, for some algebraic curve C. Here the cornerstone is Castelnuovo's rationality criterion, giving a necessary and sufficient condition for the birationality to $\mathbf{C}P^2$ (i.e., to $\mathbf{C}P^1 \times \mathbf{C}P^1$). Note that, for easy reasons, the converse is also true: $kod(\mathbf{C}P^1 \times C) = -\infty$.

b) $kod(X) = 0$: the surface X is birational to an abelian surface or to a K3 surface or to an hyperelliptic surface (regular quotient of an abelian surface) or to an Enriques surface (regular quotient of a K3 surface). This is perhaps the most subtle point of Enriques classification. Note that all these surfaces have a *trivial* canonical bundle, up to a finite regular covering, and this obviously implies the vanishing of their Kodaira dimension. The subtlety of the classification consists in proving the converse of this implication. For instance, K_X could be (a priori) something like $\mathcal{O}_X(E)$, where $E \subset X$ is a smooth elliptic curve of zero selfintersection but whose normal bundle is not a torsion line bundle, so that $h^0(X, \mathcal{O}_X(mE)) = 1$ for every positive m. To exclude such a possibility requires to prove that if $K_X = \mathcal{O}_X(E)$, E smooth elliptic curve of zero selfintersection, then E actually belongs to an elliptic fibration on X, so that $kod(X) = 1$. The construction of such an elliptic fibration starting from a single elliptic curve is a delicate step [Rei] [Fri].

c) $kod(X) = 1$: the surface X is an elliptic fibration, i.e. there exists a morphism $\pi : X \to B$, B an algebraic curve, whose generic fibre is an elliptic curve (plus some conditions to exclude those special elliptic fibrations whose Kodaira dimension is 0 or $-\infty$). This is a quite easy part of the classification. More generally, n-dimensional varieties whose Kodaira dimension is strictly between 0 and n are easier to understand than varieties whose Kodaira dimension is $-\infty$, 0 or n.

d) $kod(X) = 2$: the surface X is, by definition, a general type surface, and nothing more is said by Enriques classification.

Many mathematicians contributed to these results: one has to mention, at least, Castelnuovo, Enriques, Zariski, Kodaira, Shafarevich, Mumford. Quite recently, starting from the eighties, a new perspective has been added to the subject. This perspective derives from the so-called Mori's Program [M-P], which was developed (and is developing) for the study of higher dimensional varieties, but which turns out to be useful also for the study of surfaces [Rei].

In Mori's program the protagonist is still the canonical bundle. However, instead of looking directly at Kodaira dimension, we firstly look at the "numerical properties" of K_X, that is at the degrees of K_X over curves (or divisors) on X. Let us recall that a line bundle L on an algebraic surface X is *pseudoeffective* if the intersection product $L \cdot H$ is nonnegative for every ample divisor H, and *nef* (*n*umerically *e*ventually *f*ree) if $L \cdot D \geq 0$ for every positive divisor D. Let us also observe the following fact: if K_X is pseudoeffective, then K_X can be canonically decomposed as $P \otimes N$, where P is a nef line bundle and N is a line bundle associated to a positive divisor whose support is a union of contractible curves, whose contraction produces a smooth surface with nef canonical bundle (it is an easy consequence of adjunction formula, which moreover may be almost generalized to any pseudoeffective line bundle under the name of Zariski decomposition). Being P nef, we have $P \cdot P \geq 0$. Then the *numerical Kodaira dimension* of X, $\nu(X)$, is defined as follows:

- $\nu(X) = -\infty$ if K_X is not pseudoeffective;
- $\nu(X) = 0$ if K_X is pseudoeffective and P is numerically trivial (i.e. $P \cdot D = 0$ for every D);
- $\nu(X) = 1$ if P is not numerically trivial, but $P \cdot P = 0$;
- $\nu(X) = 2$ if $P \cdot P > 0$.

As for $kod(X)$, one proves that $\nu(X)$ is a birational invariant. It is clearly related to $kod(X)$: for instance, if $\nu(X) = -\infty$ then there exists an ample H such that $K_X \cdot H < 0$, and this evidently implies $h^0(X, K_X^{\otimes m}) = 0$ for every positive m, i.e. $kod(X) = -\infty$. More in general, it is not difficult to prove the inequality

$$kod(X) \leq \nu(X)$$

for any algebraic surface X.

The first major result of Mori's approach is the classification of surfaces whose numerical Kodaira dimension is $-\infty$: they are birational to $\mathbf{C}P^1 \times C$, C an algebraic curve. Remark that, for the moment, this result is weaker that the classification of surfaces with Kodaira dimension $-\infty$: we (still) don't know if $kod(X) = -\infty$ implies $\nu(X) = -\infty$. This statement has however the advantage that it can be generalized to higher dimensions [M-P]. There are at least two possible proofs: either by a clever reduction to positive characteristic (and this works in any dimension) or by a slightly involved study of the so-called "cone of curves" [Rei] (but this works only in dimension 2).

Surfaces with $\nu(X) = 0$ are quite easy to classify: after contracting the support of N we obtain a smooth surface whose canonical bundle is numerically trivial. This is not yet the same as being analytically trivial (up to regular covering), however it is not really difficult to show that the surface is abelian or K3 or hyperelliptic or Enriques. In particular, $kod(X)$ is also 0, but as before the result is, for the moment, weaker than the classification of surfaces with vanishing Kodaira dimension.

Surfaces with $\nu(X) = 2$ are also easy to deal with, in the sense that Riemann-Roch theorem immediately shows that $kod(X)$ also is equal to 2, so that these surfaces are of general type and we say nothing more on them.

The really difficult case is the case $\nu(X) = 1$. By arguments close to those employed to classify surfaces with $kod(X) = 0$, one proves here the existence of an elliptic pencil on X, leading to $kod(X) = 1$. In particular, one completes here the proof of the so-called "abundance", asserting the validity of the equality

$$kod(X) = \nu(X).$$

In this way we recover Enriques classification, and in particular a classification of surfaces with $kod(X) = -\infty$ which follows a rather different approach than the "classical" one. Concerning surfaces with $kod(X) \geq 0$, the difference between the classical approach and Mori's approach is perhaps more philosophical that practical: in both cases, the difficult point is to prove the impossibility of $kod(X) = 0$ and $\nu(X) = 1$, and this can be equivalently seen as a problem about kod or a problem about ν.

After this digression on Enriques classification, let us turn to foliations. Given a holomorphic foliation $\mathcal{F}$, with isolated singularities, on a smooth algebraic surface X, we can define its *canonical bundle* $K_{\mathcal{F}}$: it is, for instance, the dual of the tangent bundle of $\mathcal{F}$ (well defined even if $\mathcal{F}$ has singularities). Its sections can be thought as 1-forms "along the leaves" of $\mathcal{F}$. It is therefore very natural to analyze the foliation $\mathcal{F}$ starting from analytical or numerical properties of $K_{\mathcal{F}}$ [Men] [Mc1]. More precisely, one can still define a *Kodaira dimension* of $\mathcal{F}$, $kod(\mathcal{F})$, and a *numerical Kodaira dimension* of $\mathcal{F}$, $\nu(\mathcal{F})$: the former by looking at global holomorphic sections of $K_{\mathcal{F}}^{\otimes m}$ for positive m, the latter by using the Zariski decomposition of $K_{\mathcal{F}}$. Both dimensions belong to $\{-\infty, 0, 1, 2\}$, and

$$kod(\mathcal{F}) \leq \nu(\mathcal{F}).$$

However, some precaution is needed here: we want that $kod(\mathcal{F})$ and $\nu(\mathcal{F})$ be birational invariants, and for that we have to limit ourselves to foliations with only special kinds of singularities. The natural choice is dictated by Seidenberg's reduction theorem [Sei]: we have to work with the so-called *reduced* singularities (another slightly different natural choice is proposed in [Mc1], the so-called *canonical* singularities, which moreover allow the surface X be singular; see chapter 8). From the birational point of view, this is not a serious restriction: every foliation is birational to a foliation all of whose singularities are reduced [Sei].

So, let $\mathcal{F}$ be a foliation with reduced singularities. If $kod(\mathcal{F}) = 2$, or equivalently $\nu(\mathcal{F}) = 2$, then $\mathcal{F}$ is said to be a *foliation of general type*. We may reasonably hope for a rather precise classification, up to birational isomorphism, of foliations *not* of general type. Let us resume what is known in this direction:

a) If $\nu(\mathcal{F}) = -\infty$, i.e. $K_{\mathcal{F}}$ is not pseudoeffective, then a very remarkable theorem of Miyaoka [Miy] (see also [ShB] and [B-M]) says that $\mathcal{F}$ is a foliation by rational curves, i.e. up to birational isomorphism $\mathcal{F}$ is a (trivial) $\mathbf{C}P^1$-bundle over a curve. This is parallel to the classification of surfaces X with $\nu(X) = -\infty$, cfr. above.

b) Foliations with $\nu(\mathcal{F}) = 0$ or 1 have been analyzed in detail by McQuillan in [Mc1] (see also [Men]). The case $\nu(\mathcal{F}) = 0$ is quite simple, and leads to foliations generated by a $\mathbf{C}$-action, up to birational isomorphisms and (very special) ramified coverings. Recall that $\mathbf{C}$-actions on algebraic surfaces are extremely well understood, see for example [Kob]. The case $\nu(\mathcal{F}) = 1$ is, not surprisingly, much more difficult. As in the surface case, the difficulty will be in the construction of an elliptic pencil on X, which contains an elliptic leaf as a fibre. However, a complete classification is here still lacking, unless we add the hypothesis $kod(\mathcal{F}) \geq 0$. In other words: if $\nu(\mathcal{F}) = 1$ and $kod(\mathcal{F}) \geq 0$ then McQuillan shows that $kod(\mathcal{F}) = 1$ and $\mathcal{F}$ is either a *Riccati foliation* or a *Turbulent foliation* or a special kind of fibration. Thus, as in the surface case, one is able to exclude the simultaneity of $\nu(\mathcal{F}) = 1$ and $kod(\mathcal{F}) = 0$ (but not $\nu(\mathcal{F}) = 1$ and $kod(\mathcal{F}) = -\infty$). In particular, one has a complete classification of foliations with $kod(\mathcal{F}) = 0$ ($\iff \nu(\mathcal{F}) = 0$) or $kod(\mathcal{F}) = 1$ ($\Rightarrow \nu(\mathcal{F}) = 1$).

c) To complete the picture, one has to deal with foliations with $\nu(\mathcal{F}) = 1$ and $kod(\mathcal{F}) = -\infty$. A remarkable fact is that foliations of this class really exist [Mc1] [Br1], thus in particular the "abundance" does not hold in the foliation case, contrary to the surface case. The classification of these foliations is the major open problem; a conjectural classification and some positive results can be found in [Mc1] (and in the last chapter of these notes).

We now briefly indicate the content of these notes.

Chapters 1 and 2 review some basic material on foliations, their singularities (especially Seidenberg's reduction theorem), their first global properties (adjunction-type formulae...). Chapter 3 concerns the index theorems of Baum-Bott and Camacho-Sad, as well as some generalisations and applications which will be useful later. Chapter 4 introduces and studies two important classes of foliations: *Riccati foliations* and *Turbulent foliations*. Roughly speaking, they play in foliation theory the same role as elliptic fibrations in surface theory. Chapter 5 is about a foliated version of Zariski's work on minimal models of surfaces: given any foliation, one looks for a special representative of that foliation in its birational class, hoping also for uniqueness of that representative except in few well understood cases. Chapter 6 concerns two other important classes of foliations, those generated by a global holomorphic 1-form and those generated by a global holomorphic vector field; this is a very classical and well known subject.

The core of these notes is represented by the last three chapters. In Chapter 7 Miyaoka's theorem is stated and proved in two ways: following Shepherd-Barron [ShB], by a very clever reduction to positive characteristic, and following Bogomolov and McQuillan [B-M], by a very elegant transcendental argument. We also discuss a third possible "metric" approach, which is however still incomplete. In Chapter 8 the "easy" step of the classification, i.e. the classification of foliations with vanishing numerical Kodaira dimension, is carried out. There is, however, some technical difficulty due to the fact that the Zariski decomposition of a pseudoeffective $K_{\mathcal{F}}$ is not so simple as the Zariski decomposition of a pseudoeffective K_X, and $\mathbf{Q}$-divisors and singularities naturally appear (just as in higherdimensional Mori's theory). In Chapter 9 the fundamental result of [Mc1], namely the abundance for foliations with $kod(\mathcal{F}) \geq 0$, is proved, so that one obtains a full classification of foliations with $kod(\mathcal{F}) \in \{0, 1\}$. A remarkable corollary, which was in fact one of the motivations for developing the classification, is the understanding of foliations possessing an entire transcendental leaf: by [Mc2] and [Br3] one knows that such a foliation has Kodaira dimension 0 or 1. We conclude the chapter with some speculations on the case $kod(\mathcal{F}) = -\infty$.

Prerequisites. We shall freely use the basic language and results of algebraic geometry, especially in the two-dimensional case. Useful references for this are the first chapters of [BPV] and [Rei]. Strictly speaking, we shall not need Enriques classification, except at few points (e.g. chapter 6), but of course some knowledge of that classification may be quite helpful. With the notable exception of chapter 7, we shall always work on the complex field, and we will mainly take a "transcendental" point of view; in fact, the first five chapters and a part of the sixth one are in the context of (compact) complex surfaces, with no algebraicity assumption. *Surface* will usually mean smooth surface; singular surfaces appear only at few points in the last two chapters. But *curve* will usually mean a curve which is possibly singular (and reduced, but possibly not irreducible). A notation like $A \cdot B$ denotes the intersection product (an integer or rational number) of A and B, where A and B may be curves, divisors, $\mathbf{Q}$-divisors, line bundles, cohomology classes,...

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In this chapter we recall some basic facts concerning singular points of holomorphic foliations on surfaces, and in particular Seidenberg's theorem [Sei] which will play a fundamental role also in the global theory. A good reference for this material is [CS2].

1. Reduced singularities and their separatrices

Let v be a holomorphic vector field defined on a complex surface U, for instance an open subset of $\mathbf{C}^2$. This vector fields defines, through its local integral curves, a foliation $\mathcal{F}$ on U, which has singularities in correspondence of the zeroes of v. We shall suppose, without a serious loss of generality, that these singularities are isolated, i.e. the zero set of v is a discrete subset of U. Our main aim, in this chapter, is to study the structure of $\mathcal{F}$ on a neighbourhood of a singular point. Of course, one may also ask about the local structure of $\mathcal{F}$ around a regular point, i.e. a point where v does not vanish, but in that case the rectification theorem says everything.

Often we shall adopt the "dual" point of view, and we shall see $\mathcal{F}$ as generated by (the kernel of) a holomorphic 1-form ω (still with isolated zeroes) instead of a vector field. Remark that the correspondence vector fields $\to$ foliations, or 1-forms $\to$ foliations, is not injective: two vector fields, or 1-forms, which differ by multiplication by a nowhere vanishing holomorphic function define the same foliation. In fact, this may be thought as the actual definition of foliation, which will be formally given in chapter 2.

Let $p \in U$ be a (isolated) singular point of $\mathcal{F}$. A *separatrix* of $\mathcal{F}$ at p is a holomorphic (possibly singular) irreducible curve C on a neighbourhood of p which passes through p and which is invariant by $\mathcal{F}$. That is, v is tangent to C, or the pull-back of ω to C is identically zero. An important result of Camacho and Sad [CS1], which we shall discuss in chapter 3, says that through each singular point there exists at least one separatrix. In some cases there may be infinitely many separatrices: for example, the foliation generated by $\omega = zdw - wdz$ has each line through $(0,0)$ as a separatrix through $(0,0)$. A singularity is called *dicritical* if there are infinitely many separatrices through it. We shall see later a characterization of dicritical singularities in terms of their resolution. One may also observe that the notion of separatrix at p makes sense even if p is a regular point. By the local rectification theorem (more precisely, the local existence and uniqueness theorem for solutions to O.D.E.), through a regular point there exists exactly one separatrix.

Given a separatrix C through a singular point p, we may suppose that C is defined on a small spherical neighbourhood of p, so that $C^* = C \setminus \{p\}$ is smooth, isomorphic to the punctured disc $\mathbf{D}^*$, and free of singularities of $\mathcal{F}$. We therefore may consider the holonomy of $\mathcal{F}$ along an oriented cycle $\gamma \subset C^*$ generating $\pi_1(C^*) = \mathbf{Z}$, called *holonomy of the separatrix*. We refer to [CS2], [M-M], or any book on foliations for the notion of holonomy.

In this section we will study separatrices and their holonomies for a special class of singularities, which will appear in Seidenberg's theorem.

If p is a singular point of v, the eigenvalues λ_1, λ_2 of the linear part $(Dv)(p)$ of v at p are well defined, because $v(p) = 0$. We can thus give the following definition.

Definition 1. The singularity p is called a *reduced singularity* if at least one of these eigenvalues (say, λ_2) is not zero and the quotient $\lambda = \frac{\lambda_1}{\lambda_2}$ is not a positive rational number. (Here and everywhere "positive" means "strictly positive").

Remark that the quotient λ is unchanged by multiplication of v by a nonvanishing holomorphic function, so that we may speak unambiguously of "reduced singularity of a foliation". If λ_1 also is not 0, we could consider $\lambda^{-1} = \frac{\lambda_2}{\lambda_1}$ instead of λ, but then $\lambda \notin \mathbf{Q}^+$ iff $\lambda^{-1} \notin \mathbf{Q}^+$. The complex number λ will be called *eigenvalue of $\mathcal{F}$ at p*, with an inessential abuse due to this exchange $\lambda \leftrightarrow \lambda^{-1}$.

A reduced singularity p is called *nondegenerate* if both λ_1 and λ_2 are not 0, i.e. $\lambda \neq 0$. Otherwise p is called a *saddle-node*.

We can now recall, from [CS2], [M-M] and [M-R], the results on separatrices of reduced singularities that we shall need in the following chapters. We distinguish several cases, depending on λ (and independing on the change $\lambda \to \lambda^{-1}$ if $\lambda \neq 0$).

1) $\lambda \notin \mathbf{R}^- \cup \{0\}$ ("Poincaré domain")

According to a classical result of Poincaré, the foliation is linearizable around p: in suitable coordinates (z, w) centered at p, the foliation $\mathcal{F}$ is generated by the vector field $z\frac{\partial}{\partial z} + \lambda w \frac{\partial}{\partial w}$ (or by the 1-form $zdw - \lambda w dz$), which can be explicitely integrated. There are exactly two separatrices: $\{z = 0\}$ and $\{w = 0\}$. The holonomy of each separatrix is conjugate (up to inversion) to the (germ of) diffeomorphism $h(x) = \mu x$, with $\mu = \exp(2\pi i \lambda^{\pm 1})$ and where the exponent of λ depends on which of the 2 separatrices has been choosen. If $\lambda \notin \mathbf{R}$ then $|\mu| \neq 1$ and we are in the so-called hyperbolic case: h or h^{-1} is a contraction. If $\lambda \in \mathbf{R}$, i.e. $\lambda \in \mathbf{R}^+ \setminus \mathbf{Q}^+$, then h is an irrational rotation, and its orbits are dense on the circles $\{|x| = \epsilon\}$. Note that, in any case, the germ h has infinite order.

2) $\lambda \in \mathbf{R}^-$ ("Siegel domain")

Here the situation is much more complicated, and the linearization is not always possible. However, as in the previous case we have exactly two separatrices, given by $\{z = 0\}$ and $\{w = 0\}$ in suitable coordinates. The holonomy is now of the type $h(x) = \mu x + o(1)$, $\mu = \exp(2\pi i \lambda^{\pm 1})$, not always linearizable. In fact, a result from [M-M] says that the holonomy of one separatrix determines uniquely the foliation $\mathcal{F}$ around p, up to biholomorphism. In particular, $\mathcal{F}$ is linearizable around p if and only if h is linearizable around 0. It is worth stressing that, as a consequence of this, the holonomy of *one* separatrix is linearizable if and only if the holonomy of the *other* separatrix is.

If $\lambda \in \mathbf{Q}^-$ then either h is linearizable (i.e. it is conjugate to a rational rotation, i.e. it has finite order), or h has a "flower-type" dynamics [CS2]: a neighbourhood of 0 is divided into $2k$ sectors $A_1, B_1, ..., A_k, B_k$, for a suitable $k \in \mathbf{N}^+$ which is more or less the order of tangency of h with a rational rotation, and $h^n(x) \to 0$ if $x \in A_j$ and $n \to +\infty$ or if $x \in B_j$ and $n \to -\infty$. If $\lambda \in \mathbf{R}^- \setminus \mathbf{Q}^-$ the dynamical behaviour of h is much more difficult to understand, and we only note the following obvious fact: h has always infinite order, because μ is not a root of 1.

3) $\lambda = 0$ (saddle-node)

According to Dulac, in suitable coordinates $\mathcal{F}$ is expressed by the vector field

$$[z(1 + \nu w^k) + wF(z, w)]\frac{\partial}{\partial z} + w^{k+1}\frac{\partial}{\partial w}$$

where $k \in \mathbf{N}^+$, $\nu \in \mathbf{C}$, and F is a holomorphic function which vanishes at $(0, 0)$ up to order k. The curve $\{w = 0\}$ is a separatrix, called *strong separatrix*. Its holonomy has the form $h(x) = x + x^{k+1} + o(k + 1)$, whose dynamics is of flower type, with $2k$ sectors (the reader may convince himself by analysing the case where $F \equiv 0$, which is explicitely integrable). A result of [M-R] affirms that this holonomy determines uniquely the foliation around p, up to biholomorphism. Sometimes, for example if $F \equiv 0$, there is a second separatrix through p, smooth and transverse to the first one. But "in general" such a second separatrix, called *weak separatrix*, does not exist, even if it always exists at the formal level (by a formal change of coordinates we can always obtain $F \equiv 0$). When the weak separatrix exists, its holonomy has the form $h(x) = \exp(2\pi i \nu)x + o(1)$ but it gives a relatively small amount of informations concerning the full structure of the foliation around p.

The positive integer $k + 1$ is called *multiplicity* of the saddle-node: under a generic perturbation, a saddle-node splits into $k + 1$ nondegenerate singularities. In chapter 2 we shall give a more general definition of multiplicity, appliable to any singularity. The complex number ν is called *formal invariant*, because (k,ν) uniquely characterizes the saddle-node up to formal conjugacy [M-R].

There are two important things to retain from the previous case-by-case analysis. The first one is that the union of all the separatrices at a reduced singularity is, on a sufficiently small neighbourhood, a normal crossing curve; this is not always true for nonreduced singularities, which may have separatrices with very degenerate singular points, or even infinitely many separatrices, so that their union is not a curve. The second thing to retain is that if p is a reduced singularity and C is a separatrix through p, strong if p is a saddle-node, whose holonomy has finite order, then p is nondegenerate, in the Siegel domain, with rational eigenvalue, and linearizable. In other words, $\mathcal{F}$ around p is given by the 1-form $mzdw + nwdz$, in suitable coordinates and for suitable $n, m \in \mathbf{N}^+$, that is by the level sets of the holomorphic function $z^n w^m$. One can say that "$\mathcal{F}$ has a holomorphic first integral at p".

2. Blowing-up and resolution

Let us again consider a foliation $\mathcal{F}$ on a surface U, generated by a vector field v or a 1-form ω with isolated zeroes. Let $p \in U$ be a singular point of $\mathcal{F}$, and let $\pi : \tilde{U} \to U$ be the blowing-up of U at p, with exceptional divisor $E = \pi^{-1}(p) \simeq \mathbf{C}P^1$. Let us explain how we can define a foliation $\tilde{\mathcal{F}}$ on $\tilde{U}$, also noted $\pi^*(\mathcal{F})$. The 1-form $\tilde{\omega} = \pi^*(\omega)$ is still holomorphic, but (as a simple computation shows) its zero set is no more discrete because it contains the full E. We shall denote by $l(p) \in \mathbf{N}^+$ the vanishing order of $\tilde{\omega}$ on E. Near each point of E we can divide $\tilde{\omega}$ by the $l(p)$-power of a local equation of E, thus obtaining a 1-form with isolated zeroes. On two overlapping neighbourhoods the two corresponding 1-forms differ by multiplication by a nowhere vanishing holomorphic function, and this means that we have a global foliation $\tilde{\mathcal{F}}$ on $\tilde{U}$. This is a particular instance of a more general construction explained in chapter 2, which can be applied to any rational map and not only to blowing-ups.

The number $l(p)$ can be algebraically computed as follows. Let $a(p) \in \mathbf{N}^+$ be the vanishing order of ω at p. At first sight one could think that $a(p) = l(p)$, but this is not exactly true. We have in fact two possibilities: either E is invariant by $\tilde{\mathcal{F}}$, and then we effectively have $a(p) = l(p)$, or E is not invariant by $\tilde{\mathcal{F}}$, and then we have $l(p) = a(p) + 1$. This can be checked by a direct and easy computation; the reason is that the non-invariant case corresponds to an additional degeneration of ω at p, and this forces $\tilde{\omega}$ to vanish on E to an higher order.

We can also start from v instead of ω: the pull-back $\pi^*(v) = \tilde{v}$ is defined, a priori, only on $\tilde{U} \setminus E$, but one can check that it has a holomorphic extension to E, whose vanishing order is equal to $l(p) - 1$.

The blowing-up of $\mathcal{F}$ at p can be done even if p is a regular point of $\mathcal{F}$, and in fact we shall do it very often. In that case $\tilde{\omega}$ does not vanish on E (i.e. $l(p) = 0$) and $\tilde{v}$ is not holomorphic, but it has a first order pole on E. The foliation $\tilde{\mathcal{F}}$ is tangent to E and it has there only one singularity, of the type $zdw + wdz = 0$; note that it is a reduced singularity.

The blowing-up construction can be iterated, and we arrive to one of the basic results:

<u>**Theorem 1**</u> [Sei]. *Given any singular point $p \in U$ of $\mathcal{F}$, there exists a sequence of blowing-ups $\hat{\pi} : \hat{U} \to U$ over p such that the foliation $\hat{\mathcal{F}} = \hat{\pi}^*(\mathcal{F})$ has only reduced singularities on $\hat{\pi}^{-1}(p)$.*

Proof.

We sketch a proof due to Van den Essen [M-M, appendix]. To any singular point p of a foliation we can associate a *multiplicity* $m(p) \in \mathbf{N}^+$, which is roughly speaking the number of singularities concentrated in p and which will be formally defined in the next chapter. Let $\pi : \tilde{U} \to U$ be the blowing-up at p and let $p_1, ... p_n$ be the singularities of $\tilde{\mathcal{F}}$ on E. Then we have the following relation, due to Van den Essen, between multiplicities on U and $\tilde{U}$:

$$\sum_{j=1}^{n} m(p_j) = m(p) - l(p)^2 + l(p) + 1.$$

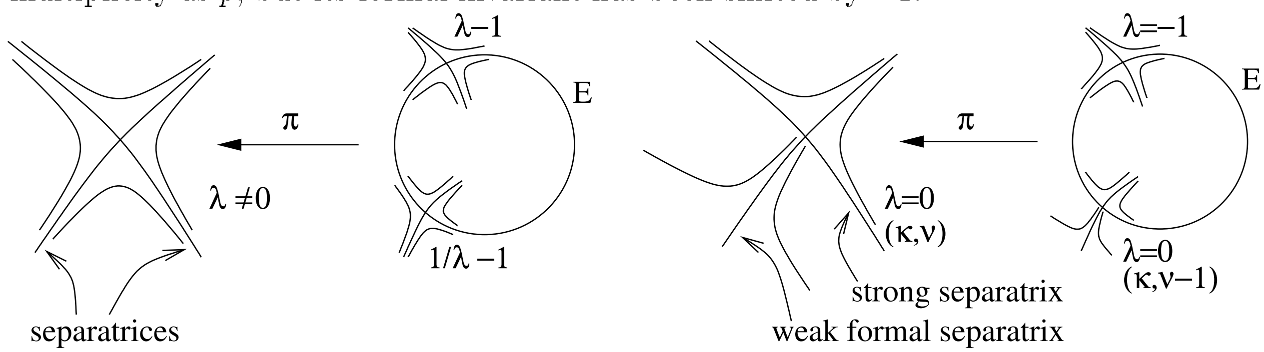
It follows that $m(p_j) < m(p)$ for every j if $l(p) \geq 2$, and in particular if $a(p) \geq 2$. In other words: if we blow-up a singular point with zero linear part (i.e. $a(p) \geq 2$) then we obtain new singular points with smaller multiplicity. However the multiplicity is a positive integer and it cannot decrease ad infinitum, hence after a finite number of blowing-ups we arrive to a situation where all the singularities have a nonzero linear part.

If such a linear part is not only nonzero but also nonnilpotent, then we have almost finished: the quotient of the two eigenvalues may be a positive rational, which prevents us to conclude, but it is easy to see that an additional sequence of blowing-ups eliminates this case. We shall return more explicitely on this point at the end of the chapter.

If the linear part is nonzero and nilpotent (conjugate to $w\frac{\partial}{\partial z}$), then we need a more subtle argument. If we blow-up, we obtain on the exceptional divisor, which is invariant, only one singular point, with bigger multiplicity (by Van den Essen formula, with $l(p) = a(p) = 1$) and order equal to 1 or 2. If the order is 1, the singularity has still a nilpotent linear part, but if we blow-up again we obtain a singularity with zero linear part, and a last blowing-up produces 3 singularities with nonnilpotent linear part, and thus we are done. If the order is 2, the singularity has a zero linear part and its blow-up generates 2 or 3 singular points, whose sum of the multiplicities has decreased and in fact is returned equal to the old one (again by Van den Essen formula, with $l(p) = a(p) = 2$). Hence each one of these 2 or 3 singularities has multiplicity strictly smaller than the initial one, and this allows to conclude by an inductive argument. $\triangle$

This result cannot be improved in a significant way. To see this, let us see what happens when we blow-up a reduced singularity p. If p is nondegenerate with eigenvalue $\lambda \neq 0$ then the exceptional divisor E is $\tilde{\mathcal{F}}$-invariant and contains two singularities p_1, p_2, both reduced, nondegenerate and with eigenvalues $\lambda - 1$ and $\frac{1}{\lambda} - 1$ (more precisely: if $(Dv)(p)$ has eigenvalues λ_1, λ_2 then $(D\tilde{v})(p_j)$ has eigenvalues $\lambda_1, \lambda_2 - \lambda_1$ or $\lambda_1 - \lambda_2, \lambda_2$). Remark that if p is in the Siegel domain then p_1 and p_2 also are in the Siegel domain, and if p is in the Poincaré domain with a nonreal eigenvalue then p_1 and p_2 have the same property; but if p is in the Poincaré domain with a real (positive, irrational) eigenvalue then one of the two singularities is in the Siegel domain whereas the other is in the Poincaré domain, with a still real eigenvalue.

If p is a saddle-node, then the exceptional divisor E is again $\tilde{\mathcal{F}}$-invariant and contains two singularities, a saddle-node p_1 and a reduced nondegenerate singularity p_2 with eigenvalue -1. The strong separatrix of p_1 is contained in E. The saddle-node p_1 has the same multiplicity as p, but its formal invariant has been shifted by -1.



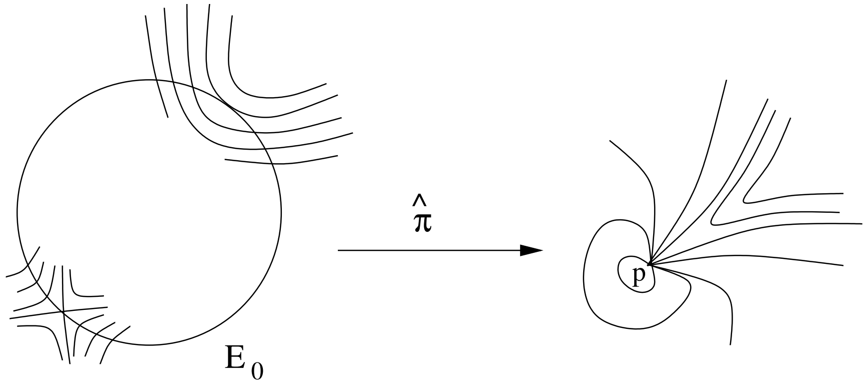
We see from this that one cannot further reduce the singularities of the foliation. Of course, one can say for example that the singularities with eigenvalue -1 can be further eliminated, by blowing-up them into singularities with eigenvalue -2, but this is not really interesting. Moreover, we want that the class of "reduced" singularities be invariant by blowing-ups, and so the blowing-up of a saddle-node or of a regular point shows that this class must contain singularities with eigenvalue -1, as well as singularities with any negative rational eigenvalue by a similar argument. There is also a more "global" reason to suggest that the class of reduced singularities is the good one to work with, but we shall explain this in the last chapters.

As a consequence of the resolution theorem, we have the following characterization of dicritical singularities.

Proposition 1. *A singularity p of a foliation $\mathcal{F}$ is dicritical if and only if there exists a sequence of blowing-ups $\hat{\pi} : \hat{U} \to U$ over p and an irreducible component E_0 of $\hat{\pi}^{-1}(p) = \hat{E}$ which is not invariant by $\hat{\mathcal{F}} = \hat{\pi}^*(\mathcal{F})$.*

Proof.

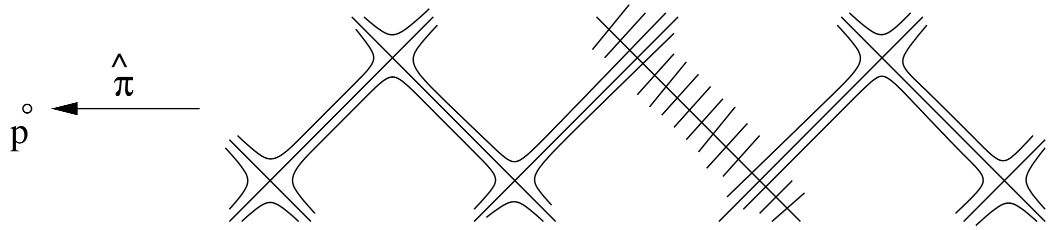
In one direction this is obvious: if there exists such a $\hat{\pi}$ and such a E_0 then a generic point of E_0 is regular for $\hat{\mathcal{F}}$ and the local leaf of $\hat{\mathcal{F}}$ through it is not contained in E_0, and so projects by $\hat{\pi}$ on U to a separatrix at p. In the other direction, we firstly note that if $\hat{\pi} : \hat{U} \to U$ is a sequence of blowing-ups over p and C is a separatrix at p, then the strict transform $\bar{C}$ of C is a separatrix of $\hat{\mathcal{F}}$ at a (possibly regular) point $\hat{p} \in \hat{E}$. If $\hat{E}$ is entirely $\hat{\mathcal{F}}$-invariant, then $\hat{p}$ is certainly a singular point for $\hat{\mathcal{F}}$. But we have seen that each reduced singularity has at most two separatrices, hence if we take as $\hat{\pi}$ the sequence given by Seidenberg's theorem we see that either p has a finite number of separatrices or $\hat{E}$ is not entirely $\hat{\mathcal{F}}$-invariant. $\triangle$



We see from the proof that, in the dicritical case, the infinitely many separatrices are organised into finitely many families. We shall see in chapter 6 a theorem of Jouanolou [Jou], which is in some sense a global counterpart to this phenomenon.

To conclude this chapter, we give the resolution of the singularities generated by a vector field whose linear part has eigenvalues λ_1, λ_2, both nonzero but with $\lambda = \frac{\lambda_1}{\lambda_2}$ in $\mathbf{Q}^+$. This is an edifying example.

If $\lambda \notin \mathbf{N}^+ \cup \frac{1}{\mathbf{N}^+}$ then we are in the so-called "nonresonant case", and Poincaré's linearization theorem, already used in section 1, is still valid: the foliation $\mathcal{F}$ is generated by the vector field $nz\frac{\partial}{\partial z} + mw\frac{\partial}{\partial w}$, in suitable coordinates, with $\frac{n}{m} = \lambda$. Equivalently, the foliation is given by the levels of the meromorphic function $\frac{z^m}{w^n}$. Blowing-up this singularity, we obtain two singularities p_1 and p_2 on the exceptional divisor, which is invariant. One of these singularities, say p_1, is reduced and in the Siegel domain, generated by a vector field of the type $nz\frac{\partial}{\partial z} + (m-n)w\frac{\partial}{\partial w}$ (where, to fix notations, we are supposing $n > m$), i.e. given by the levels of $z^{n-m}w^n$. The other singularity, p_2, is generated by a vector field of the form $(n-m)z\frac{\partial}{\partial z} + mw\frac{\partial}{\partial w}$, and we still have $\frac{n-m}{m} \in \mathbf{Q}^+$. We blow-up p_2, and so on. After a finite number of steps we arrive to a singularity p_l generated by a vector field of the form $z\frac{\partial}{\partial z} + w\frac{\partial}{\partial w}$. The blowing-up of p_l produces now an exceptional divisor which is not invariant by the foliation, more precisely it is free of singularities and everywhere transverse to the foliation. The resolution of p can therefore be represented as follows:

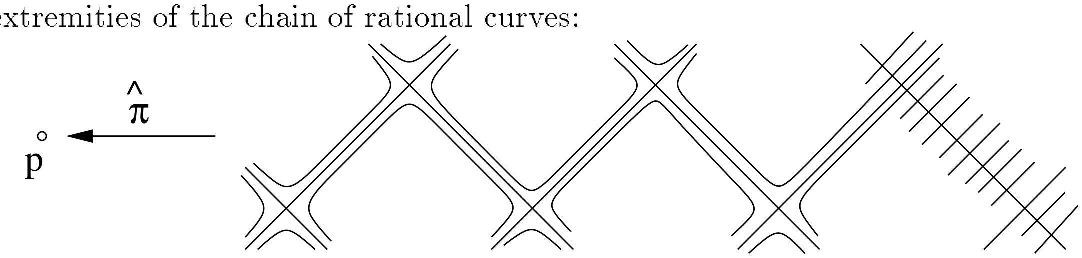


The divisor $\hat{\pi}^{-1}(p)$ is a chain of rational curves, which are all $\hat{\mathcal{F}}$-invariant except one (which is *never* one of the extremities), transverse to $\hat{\mathcal{F}}$. All the singularities on this divisor are in the Siegel domain, with a holomorphic first integral.

If $\lambda \in \mathbf{N}^+ \cup \frac{1}{\mathbf{N}^+}$ (the resonant case), Poincaré's linearization theorem must be replaced by Poincaré-Dulac normal form theorem, which asserts that $\mathcal{F}$ is generated by

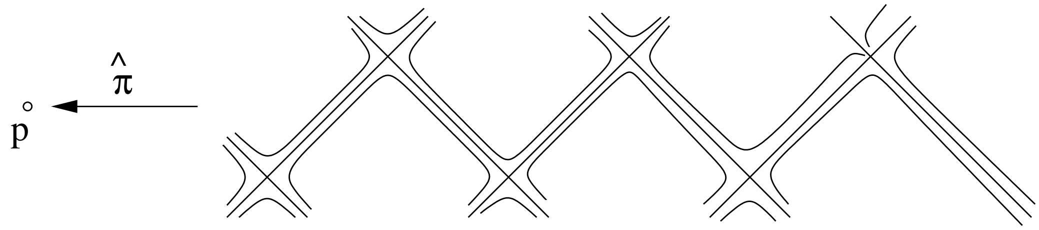
$$z\frac{\partial}{\partial z} + (nw + \epsilon z^n)\frac{\partial}{\partial w}$$

with $n = \lambda$ or λ^{-1} and $\epsilon \in \{0, 1\}$. If $\epsilon = 0$ we are in a situation very close to the previous one: the only difference is that now the non-invariant component of $\hat{\pi}^{-1}(p)$ is one of the two extremities of the chain of rational curves:



Note that in this case and in the previous one the singularity is dicritical; more in general, a foliation given by the levels of a meromorphic function is always dicritical where the function has indeterminacy points (but this sufficient condition for dicriticity is of course far from being a necessary one).

If $\epsilon = 1$, the essential difference is that the "last" singularity before complete resolution is not the radial one ($z\frac{\partial}{\partial z} + w\frac{\partial}{\partial w}$), but it is given by the nondiagonalizable vector field $z\frac{\partial}{\partial z} + (w+z)\frac{\partial}{\partial w}$, i.e. the Poincaré-Dulac normal form for $n = 1$ and $\epsilon = 1$. The blowing-up of this nonreduced singularity produces an exceptional divisor which is invariant by the foliation and contains only one singularity, which is a saddle-node of multiplicity 2, whose weak separatrix exists and is contained in the exceptional divisor:



The other singularities are in the Siegel domain, with rational eigenvalue but without holomorphic first integral. The singularity p has only one separatrix, the axis $\{z = 0\}$ in Poincaré-Dulac normal form, whose holonomy has a flower-type dynamics.

In this chapter we start the global study of foliations on complex surfaces. The most basic global invariants which may be associated to such a foliation are its normal and tangent bundles, and here we shall prove several formulae and study several examples concerning the calculation of these bundles. We shall mainly follow the presentation given in [Br1]; the book [G-O] may also be of valuable help.

1. Basic definitions

A *foliation* $\mathcal{F}$ on a (smooth) complex surface X may be defined in several ways. The possibly simplest one is the following, which was already used in chapter 1. We take an open covering $\{U_j\}_{j\in I}$ of X and on each U_j a holomorphic vector field v_j with isolated zeroes, and we require that on $U_j \cap U_i$ the vector fields v_j and v_i coincide up to multiplication by a nowhere vanishing holomorphic function:

$$v_i = g_{ij} v_j \qquad \text{on } U_i \cap U_j, \qquad g_{ij} \in \mathcal{O}_X^*(U_i \cap U_j).$$

This means that the local integral curves of v_i and v_j glue together, up to reparametrization, giving the so-called *leaves* of $\mathcal{F}$. Of course, the foliation $\mathcal{F}$ is not exactly the collection $\{U_j, v_j\}_{j\in I}$, but more precisely an equivalence class of collections of that type, where the equivalence relation is given by: $\{U_j, v_j\}_{j\in I} \sim \{U'_j, v'_j\}_{j\in I'}$ if v_j and v'_i coincide on $U_j \cap U'_i$ up to multiplication by a nowhere vanishing holomorphic function.

The *singular set* $Sing(\mathcal{F})$ of $\mathcal{F}$ is the discrete subset of X defined by

$$Sing(\mathcal{F}) \cap U_j = \text{zeroes of } v_j, \qquad \forall j \in I.$$

The functions $g_{ij} \in \mathcal{O}_X^*(U_i \cap U_j)$ form a multiplicative cocycle and hence give a cohomology class in $H^1(X, \mathcal{O}_X^*)$, that is a line bundle on X. One immediately verifies that this line bundle is intrinsecally defined by $\mathcal{F}$: if we change from $\{U_j, v_j\}_{j\in I}$ to an equivalent $\{U'_j, v'_j\}_{j\in I'}$, then we obtain a cocycle $\{g'_{ij}\}$ cohomologous to $\{g_{ij}\}$. It is called *cotangent bundle* of $\mathcal{F}$, and denoted by $T^*_\mathcal{F}$. Its dual $T_\mathcal{F}$, represented by the inverse cocycle $\{g_{ij}^{-1}\}$, is called *tangent bundle* of $\mathcal{F}$.

The relations $v_i = g_{ij}v_j$ on $U_i \cap U_j$ can be thought as defining relations of a global holomorphic section s of $T^*_\mathcal{F} \otimes TX$. Because each v_j has isolated zeroes, s also has isolated zeroes, more precisely over $Sing(\mathcal{F})$. However, s is not entirely intrinsecally defined by $\mathcal{F}$: if we change from $\{U_j, v_j\}_{j\in I}$ to $\{U_j, fv_j\}_{j\in I}$, where $f \in \mathcal{O}^*_X(X)$, then s will be replaced by fs. One easily sees that this is the only ambiguity. Conversely, any global holomorphic section s of $L \otimes TX$ with a discrete zero set Z defines a holomorphic foliation $\mathcal{F}$ with $Sing(\mathcal{F}) = Z$ and $T^*_\mathcal{F} = L$. We arrive in this way to an equivalent second definition of foliation: a section of $L \otimes TX$, for some line bundle L, with discrete zero set and modulo multiplication by a nowhere vanishing holomorphic function. Remark that if X is compact then this allows to identify the space of foliations on X with a given cotangent bundle L with a Zariski-open subset of the projective space $PH^0(X, L \otimes TX)$.

This second definition can also be reformulated as follows. The section s of $T^*_\mathcal{F} \otimes TX$ can be seen as a morphism $T_\mathcal{F} \to TX$, which is moreover injective over the regular points of $\mathcal{F}$ and trivial over $Sing(\mathcal{F})$. Hence, that morphism fits into an exact sequence

$$0 \to T_\mathcal{F} \to TX \to \mathcal{I}_Z \cdot N_\mathcal{F} \to 0$$

where $N_\mathcal{F}$ is a suitable line bundle on X and $\mathcal{I}_Z$ is an ideal sheaf supported on $Sing(\mathcal{F})$ (every point of $Sing(\mathcal{F})$ is affected by a multiplicity, on which we shall comment later). As usual, here we confuse between line bundles and their sheaves of sections.

A foliation on X can therefore be defined as an exact sequence as above, modulo multiplication by $\mathcal{O}^*_X(X)$. From a sheaf-theoretic point of view, we can say that a foliation on X is a coherent analytic rank 1 subsheaf $\mathcal{F}$ of TX such that the quotient sheaf $TX/\mathcal{F}$ is torsion free. Chapters 2 and 4 of [Fri] contain many useful facts about these concepts.

All these constructions can be done using 1-forms instead of vector fields: a foliation can be defined by a collection of 1-forms $\omega_j \in \Omega^1_X(U_j)$ with isolated zeroes and such that

$$\omega_i = f_{ij}\omega_j \qquad \text{on } U_i \cap U_j, \qquad f_{ij} \in \mathcal{O}^*_X(U_i \cap U_j).$$

The cocycle $\{f_{ij}\}$ defines a line bundle on X, called *normal bundle* of $\mathcal{F}$ and identifiable with the line bundle $N_\mathcal{F}$ which appears in the exact sequence above. To see this, note that a vector field w on U_j can be contracted with ω_j, giving a function which is zero if and only if w is tangent to $\mathcal{F}$, i.e. is a section of $T_\mathcal{F}$. The dual $N^*_\mathcal{F}$ is called *conormal bundle* of $\mathcal{F}$. The foliation now gives rise to a global holomorphic section of $N_\mathcal{F} \otimes T^*X$, with discrete zero set and modulo multiplication by $\mathcal{O}^*_X(X)$, and to an exact sequence

$$0 \to N^*_\mathcal{F} \to T^*X \to \mathcal{I}_Z \cdot T^*_\mathcal{F} \to 0.$$

These line bundles are related each other via the canonical bundle K_X of X:

$$K_X = T^*_{\mathcal{F}} \otimes N^*_{\mathcal{F}}.$$

This is implicitely expressed by the exact sequences above; anyway, any nowhere vanishing local holomorphic section of K_X, which is a 2-form, realizes (by contraction) an isomorphism between local vector fields and local 1-forms generating $\mathcal{F}$, so that $K_X \simeq Hom(T_{\mathcal{F}}, N^*_{\mathcal{F}}) = T^*_{\mathcal{F}} \otimes N^*_{\mathcal{F}}$.

Let us consider now the case where X is *algebraic*. Then any line bundle on X has nontrivial meromorphic sections, so that any global holomorphic section s of $T^*_{\mathcal{F}} \otimes TX$ (or $N_{\mathcal{F}} \otimes T^*X$) can be thought as a meromorphic vector field (or meromorphic 1-form) on X. Because the section s has only isolated zeroes, that meromorphic vector field v has a zero divisor $(v)_0$ and a polar divisor $(v)_\infty$ which satisfy

$$T_{\mathcal{F}} = \mathcal{O}_X((v)_0 - (v)_\infty).$$

Similarly, that meromorphic 1-form ω has zero and polar divisors satisfying

$$N^*_{\mathcal{F}} = \mathcal{O}_X((\omega)_0 - (\omega)_\infty).$$

Hence, in the algebraic case, foliations can be thought as meromorphic vector fields or meromorphic 1-forms modulo multiplication by meromorphic functions, and $T_{\mathcal{F}}$, $N^*_{\mathcal{F}}$ are the line bundles associated to the zero-polar divisors of such vector fields or 1-forms.

If $T_{\mathcal{F}}$ or $N^*_{\mathcal{F}}$ is effective then we shall say that the foliation is generated by a global holomorphic vector field or 1-form; these foliations will be analysed in chapter 6. Note that (here and also at other places) we use the term "generated" in a somewhat improper way, because we do not require that the zeroes of such a global vector field or 1-form are isolated. It is perhaps opportune, at this point, to observe that the requirement of discreteness of $Sing(\mathcal{F})$ is not really indispensable: if $s \in H^0(X, L \otimes TX)$ vanishes on a divisor D, then we may consider s as defining a "foliation" $\mathcal{F}_0$ with $Sing(\mathcal{F}_0) \supset D$, but we may also look at s as a section of $L \otimes TX \otimes \mathcal{O}_X(-D)$ which vanishes only on a discrete set, and thus defining a foliation $\mathcal{F}$ with isolated singularities and with $T^*_{\mathcal{F}} = L \otimes \mathcal{O}_X(-D)$. The transfer from $\mathcal{F}_0$ to $\mathcal{F}$ is called *saturation of* $\mathcal{F}_0$. From the sheaf-theoretic point of view, it corresponds to replace $\mathcal{F}_0 \subset TX$ by its bidual $\mathcal{F}_0^{**} \subset TX$ (note that $TX/\mathcal{F}_0$ is not torsion free). *We shall always work with saturated foliations*, i.e. foliations with isolated singularities.

When X is *compact* (but not necessarily algebraic) an important information encoded into $T_{\mathcal{F}}$ or $N_{\mathcal{F}}$ is the number of singularities of $\mathcal{F}$. Let $p \in Sing(\mathcal{F})$ and let us choose a local generator $v = A(z,w)\frac{\partial}{\partial z} + B(z,w)\frac{\partial}{\partial w}$ of $\mathcal{F}$ around $p = (0,0)$. We define the *multiplicity* of p as

$$m(p) = \dim_{\mathbf{C}} \frac{\mathcal{O}_p}{< A, B >}$$

where $\mathcal{O}_p$ is the local algebra of X at p and $< A, B >$ is the ideal generated by A, B, as elements of $\mathcal{O}_p$. The name arises from the fact that by a generic local deformation of v the singularity p splits into $m(p)$ singularities, each one of multiplicity 1. We refer to [G-H], chapter 5, for this and other basic properties of $m(p)$. Note that if p is reduced nondegenerate then $m(p) = 1$, and if p is a saddle-node then $m(p)$ coincides with the multiplicity defined in the previous chapter.

Now we can set, if X is compact:

$$m(\mathcal{F}) = \sum_{p \in Sing(\mathcal{F})} m(p)$$

and we have the following formula.

Proposition 1. *Let $\mathcal{F}$ be a foliation on a compact surface X, then*

$$m(\mathcal{F}) = T_{\mathcal{F}} \cdot T_{\mathcal{F}} + T_{\mathcal{F}} \cdot K_X + c_2(X).$$

Proof.

Let us look at $\mathcal{F}$ as a section $s \in H^0(X, T^*_{\mathcal{F}} \otimes TX)$, with isolated zeroes. Thus, according to intersection theory, the second Chern class $c_2(T^*_{\mathcal{F}} \otimes TX)$ is equal to the intersection product between the graph of s and the graph of the null section. These two graphs intersect only over $Sing(\mathcal{F})$, and each singularity gives a contribution equal to $m(p)$. Hence

$$m(\mathcal{F}) = c_2(T^*_{\mathcal{F}} \otimes TX).$$

On the other hand (cfr. e.g. [Fri], chapter 2) one has

$$c_2(L \otimes TX) = c_2(TX) + c_1(TX) \cdot c_1(L) + c_1^2(L)$$

for any line bundle L, whence the conclusion. $\triangle$

By adjunction, or by looking at $c_2(N_{\mathcal{F}} \otimes T^*X)$, we also have

$$m(\mathcal{F}) = N_{\mathcal{F}} \cdot N_{\mathcal{F}} + N_{\mathcal{F}} \cdot K_X + c_2(X)$$

and

$$m(\mathcal{F}) = c_2(X) - T_{\mathcal{F}} \cdot N_{\mathcal{F}}.$$

This last formula follows also directly from the exact sequences associated to the foliation (where we now know that the multiplicity affecting Z is the multiplicity defined above).

Let us note the following nice consequence of these formulae: if X is a compact surface with odd $c_2(X)$ then any foliation on X has a nonempty singular set. The reason is that on any compact surface X and for any line bundle L on X the number $L \cdot L + L \cdot K_X$ is even, by Thom-Wu formula $c_1(X) \cdot L = w_2(X) \cdot L = L \cdot L \ mod\ 2$.

2. Degrees of the bundles on curves

In order to understand the geometric meaning of $T_{\mathcal{F}}$ and $N_{\mathcal{F}}$, we give here some formulae concerning the computation of their degrees on compact curves.

Let $\mathcal{F}$ be a foliation on the surface X, not necessarily compact, and let $C \subset X$ be a compact curve, possibly singular. We firstly consider the case where C is not invariant by $\mathcal{F}$, or more precisely each irreducible component of C is not $\mathcal{F}$-invariant. If p is any point of C, we can define an index $tang(\mathcal{F}, C, p)$, representing the tangency order of $\mathcal{F}$ to C at p, as follows: let $\{f = 0\}$ be a local (reduced) equation of C around p, let v be a local holomorphic vector field with isolated zeroes and generating $\mathcal{F}$ around p, and set

$$tang(\mathcal{F}, C, p) = \dim_{\mathbf{C}} \frac{\mathcal{O}_p}{< f, v(f) >}$$

where $\mathcal{O}_p$ is the local algebra of X at p, $v(f)$ is the Lie derivative of f along v, and $< f, v(f) >$ is the ideal generated by f and $v(f)$ in $\mathcal{O}_p$. Remark that $v(f)$ is not identically zero on C, because C is not $\mathcal{F}$-invariant, and so $< f, v(f) >$ has finite codimension in $\mathcal{O}_p$ and $tang(\mathcal{F}, C, p) < \infty$. In fact, $tang(\mathcal{F}, C, p) = 0$ except at those points, finite in number, where $\mathcal{F}$ is not transverse to C (if $\mathcal{F}$ is transverse to C at p, and only in that case, then $v(f)$ is a unit in $\mathcal{O}_p$). Hence we can set

$$tang(\mathcal{F}, C) = \sum_{p \in C} tang(\mathcal{F}, C, p).$$

We also introduce the *virtual* (or *arithmetic*) Euler characteristic $\chi(C)$ of C via the adjunction formula:

$$\chi(C) = -K_X \cdot C - C \cdot C.$$

When C is smooth then this coincides with the topological Euler characteristic, but when C has singularities this is strictly smaller, and equal to the topological Euler characteristic of a smoothing of C. Note that $\chi(C)$ depends only on the cohomology class of C.

Proposition 2. *Let $\mathcal{F}$ be a foliation on a complex surface X and let $C \subset X$ be a compact curve, each component of which is not invariant by $\mathcal{F}$. Then*

$$N_{\mathcal{F}} \cdot C = \chi(C) + tang(\mathcal{F}, C)$$

$$T_{\mathcal{F}} \cdot C = C \cdot C - tang(\mathcal{F}, C).$$

Proof.

Remark that the two formulae are equivalent, for $\chi(C) + C \cdot C = -K_X \cdot C$ and $K_X^* = T_{\mathcal{F}} \otimes N_{\mathcal{F}}$. Hence it is sufficient to prove the second one (a direct proof of the first one, based on Poincaré-Hopf formula, can be found in [Br1]). We choose an open covering $\{U_j\}$ of X, holomorphic vector field v_j on U_j generating $\mathcal{F}$, holomorphic functions f_j on U_j defining C. On the intersections $U_i \cap U_j$ we have

$$v_i = g_{ij} v_j$$

$$f_i = f_{ij} f_j$$

where $\{g_{ij}\}$ is a cocycle representing $T_{\mathcal{F}}^*$ and $\{f_{ij}\}$ is a cocycle representing $\mathcal{O}_X(C)$. Hence the functions $\{v_j(f_j)\}$ restricted to C give a section of $[T_{\mathcal{F}}^* \otimes \mathcal{O}_X(C)]|_C$, because by Leibniz's rule

$$v_i(f_i) = g_{ij} v_j(f_{ij} f_j) = g_{ij} f_{ij} v_j(f_j) + g_{ij} f_j v_j(f_{ij})$$

and $g_{ij} f_j v_j(f_{ij}) \equiv 0$ on C. This section vanishes at the points of C where $\mathcal{F}$ is not transverse to C, and the vanishing order is nothing but that $tang(\mathcal{F}, C, \cdot)$. Hence $tang(\mathcal{F}, C)$ is equal to the degree of $[T_{\mathcal{F}}^* \otimes \mathcal{O}_X(C)]|_C$, i.e. $T_{\mathcal{F}}^* \cdot C + C \cdot C$. $\triangle$

Note that when C is everywhere transverse to $\mathcal{F}$ we obtain the evident facts $T_{\mathcal{F}} \cdot C = C \cdot C$ and $N_{\mathcal{F}} \cdot C = \chi(C)$, evident because in that case $T_{\mathcal{F}}|_C$ (resp. $N_{\mathcal{F}}|_C$) is clearly isomorphic to the normal bundle N_C of C (resp. to its tangent bundle TC). More generally, one could define $tang(\mathcal{F}, C)$ as a *divisor* on C, not simply as a number, and then the proof above shows that $T_{\mathcal{F}}|_C$ is isomorphic to $N_C \otimes \mathcal{O}_C(-tang(\mathcal{F}, C))$ and $N_{\mathcal{F}}|_C$ is isomorphic to $TC \otimes \mathcal{O}_C(tang(\mathcal{F}, C))$. However, we shall not need these refinements of proposition 2, except perhaps in the transverse case.

Let us also note the following obvious consequence of proposition 2, which will be however of fundamental importance: if C is not $\mathcal{F}$-invariant then

$$T_{\mathcal{F}}^* \cdot C + C \cdot C \geq 0$$

with equality iff C is transverse to $\mathcal{F}$.

Let us consider now the case where C is invariant by $\mathcal{F}$, more precisely each irreducible component of C is $\mathcal{F}$-invariant. Here also we have to introduce a certain index $Z(\mathcal{F}, C, p)$, for any $p \in C$. Let $\{f = 0\}$ be a local equation of C around p, and let ω be a holomorphic 1-form generating $\mathcal{F}$ around p. Because C is $\mathcal{F}$-invariant, we can factorize ω around p as

$$g\omega = hdf + f\eta$$

where η is a holomorphic 1-form, g and h are holomorphic functions, and h, f (and therefore g, f) are relatively prime, that is h (and therefore g) does not vanish identically on each branch of C. See, for instance, [Suw], chapter V, for the existence of such a factorization, which is a consequence of standard local algebra arguments. The function $\frac{h}{g}|_C$ is a meromorphic function, not identically 0 nor ∞, which in fact does not depend on the choice of g, h, η (remark that $\frac{\omega}{f} = \frac{h}{g}\frac{df}{f} + \frac{1}{g}\eta$, so that $\frac{h}{g}|_C$ is, outside p, nothing but than the residue of $\frac{\omega}{f}$ along C). It does depend on the choice of ω and f, but changing ω and f has the effect that $\frac{h}{g}|_C$ change only by multiplication by a nowhere vanishing holomorphic function. Hence we can unambiguously set [Br2]:

$$Z(\mathcal{F}, C, p) = \text{ vanishing order of } \frac{h}{g}|_C \text{ at } p$$

$$= \sum_i \{\text{vanishing order of } \frac{h}{g}|_{C_i} \text{ at } p\}$$

where $C_i \subset C$ are the local irreducible components of C at p and where it is intended that if $\frac{h}{g}|_{C_i}$ has a pole at p then the "vanishing order" is minus the order of the pole. It can really happen that $\frac{h}{g}|_{C_i}$ has a pole at p, and hence that $Z(\mathcal{F}, C, p)$ is negative: for example, take $\omega = 2ydx - 3xdy$ and $f = x^2 - y^3$, and you will find $Z(Ker\ \omega, \{f = 0\}, 0) = -1$. However, it is proven in [Br2] that $Z(\mathcal{F}, C, p) \geq 0$ if p is a nondicritical singularity of $\mathcal{F}$.

When p is a regular point of C the computation of Z is quite easy: we can choose $g \equiv 1$ in the factorization above and then we find that $Z(\mathcal{F}, C, p)$ is equal to the vanishing order (i.e. the Poincaré-Hopf index) of $v|_C$ at p, where v is a holomorphic vector field generating $\mathcal{F}$ around p. In particular, $Z(\mathcal{F}, C, p) \geq 0$, with equality iff p is regular for $\mathcal{F}$. This Poincaré-Hopf interpretation of Z remains valid even if p is singular for C, provided one takes the definition of Poincaré-Hopf index given by Gomez-Mont, Seade and Verjovsky (see [Suw], chapter V, or [Br2]). However, in that case we have to "smooth" $v|_C$, and this can be done, in general, only in the smooth (real) category, not in the holomorphic one, whence the possibility of $Z(\mathcal{F}, C, p) < 0$.

Other properties of this index Z will be studied in section 2 of the next chapter.

Assuming now C compact, we can set

$$Z(\mathcal{F},C)=\sum_{p\in C} Z(\mathcal{F},C,p)$$

the sum being finite by the previous considerations.

Proposition 3 *Let $\mathcal{F}$ be a foliation on a complex surface X and let $C \subset X$ be a compact curve, each component of which is invariant by $\mathcal{F}$. Then*

$$N_{\mathcal{F}}\cdot C = C\cdot C + Z(\mathcal{F},C)$$

$$T_{\mathcal{F}}\cdot C = \chi(C) - Z(\mathcal{F},C).$$

Proof.

The two formulae are equivalent and it suffices to prove the first one (the second one is also directly proved in [Br1]). We can choose a covering $\{U_j\}$ of X, holomorphic functions f_j on U_j defining C, holomorphic 1-forms ω_j on U_j defining $\mathcal{F}$, and factorizations $g_j\omega_j = h_j df_j + f_j\eta_j$ on each U_j. On $U_i\cap U_j$ we have

$$\omega_i = g_{ij}\omega_j$$

$$f_i = f_{ij}f_j$$

where $\{g_{ij}\}$, resp. $\{f_{ij}\}$, is a cocycle representing $N_{\mathcal{F}}$, resp. $\mathcal{O}_X(C)$. It follows from

$$\frac{\omega_i}{f_i} = g_{ij}f_{ij}^{-1}\frac{\omega_j}{f_j}$$

that the functions $\{\theta_j = \frac{h_j}{g_j}|_C\}$ define a meromorphic section of $[N_{\mathcal{F}}\otimes\mathcal{O}_X(-C)]|_C$, which vanishes (or has a pole) at every $p\in C$ at order $Z(\mathcal{F},C,p)$. Hence $Z(\mathcal{F},C)$ is equal to $N_{\mathcal{F}}\cdot C - C\cdot C$. $\triangle$

Of course, the considerations which were done after proposition 2 can be repeated at this point, by considering $Z(\mathcal{F},C)$ as a divisor and not simply a number.

3. Some examples

1) *Blowing-up*

Let $\mathcal{F}$ be a foliation on a surface X, let $p\in X$, and let $\pi:\tilde{X}\to X$ be the blowing-up at p. We have seen in the previous chapter how to define a foliation $\tilde{\mathcal{F}}=\pi^*(\mathcal{F})$ on $\tilde{X}$.

Recall that we introduced there a nonnegative integer $l(p)$, which is zero iff p is regular for $\mathcal{F}$, and which is equal to the vanishing order along $E = \pi^{-1}(p)$ of $\tilde{\omega} = \pi^*(\omega)$, ω generating $\mathcal{F}$ around p. From the discussion of section 1 it follows that

$$N^*_{\tilde{\mathcal{F}}} = \pi^*(N^*_{\mathcal{F}}) \otimes \mathcal{O}_{\tilde{X}}(l(p)E).$$

In a similar way, or by adjunction because $K_{\tilde{X}} = \pi^*(K_X) \otimes \mathcal{O}_{\tilde{X}}(E)$, we have

$$T_{\tilde{\mathcal{F}}} = \pi^*(T_{\mathcal{F}}) \otimes \mathcal{O}_{\tilde{X}}((l(p)-1)E).$$

Let us see what happens by applying propositions 2 or 3 to the curve E. If E is $\tilde{\mathcal{F}}$-invariant, then from proposition 3 and $E \cdot E = -1$ we find

$$Z(\tilde{\mathcal{F}}, E) = N_{\tilde{\mathcal{F}}} \cdot E - E \cdot E = l(p) + 1.$$

Recall also that in this case $l(p) = a(p)$, the vanishing order of ω at p, and so

$$Z(\tilde{\mathcal{F}}, E) = a(p) + 1.$$

If E is not $\tilde{\mathcal{F}}$-invariant (and so $l(p) = a(p)+1$), we find from proposition 2

$$tang(\tilde{\mathcal{F}}, E) = T^*_{\tilde{\mathcal{F}}} \cdot E + E \cdot E = l(p) - 2 = a(p) - 1.$$

A consequence of these formulae is the following one, which was used in the proof of Seidenberg's theorem. Let $\tilde{p} \in E$ be a singular point of $\tilde{\mathcal{F}}$, then $a(\tilde{p})$ is certainly at most equal to $Z(\tilde{\mathcal{F}}, E, \tilde{p})$ if E is $\tilde{\mathcal{F}}$-invariant and to $tang(\tilde{\mathcal{F}}, E, \tilde{p})$ if E is not $\tilde{\mathcal{F}}$-invariant (easy computation, left to the reader). We therefore obtain the estimate

$$\sum_{\tilde{p} \in E} a(\tilde{p}) \leq a(p) \pm 1$$

($+1$ in the invariant case, -1 otherwise). Simple examples show that this estimate is optimal, and that the inequality is sometimes strict.

Let us suppose now that X is compact. Recall that $K_{\tilde{X}} = \pi^*(K_X) \otimes \mathcal{O}_{\tilde{X}}(E)$ and $c_2(\tilde{X}) = c_2(X) + 1$. From the above expressions for $N_{\tilde{\mathcal{F}}}$ and $T_{\tilde{\mathcal{F}}}$ and from proposition 1 we deduce

$$m(\tilde{\mathcal{F}}) = m(\mathcal{F}) - l(p)^2 + l(p) + 1$$

which is perfectly coherent with Van den Essen formula met in the proof of Seidenberg's theorem. In fact, these type of arguments can be localized around p, or around E, so that the compactness hypothesis on X becomes inessential, and so one is led to a full proof of Van den Essen formula.

Concerning blowing-ups, let us also observe the following fact, which will be used very frequently: given any foliation $\mathcal{G}$ on $\tilde{X}$, there exists a foliation $\mathcal{F}$ on X such that $\pi^*(\mathcal{F}) = \mathcal{G}$. This is, for instance, a consequence of Levi's extension theorem ($\mathcal{F}$ certainly exists on $X \setminus \{p\}$, and one only needs to extend to p).

2) $X = \mathbf{C}P^2$

Classically [G-O] given a foliation $\mathcal{F}$ on $\mathbf{C}P^2$ its *degree* $d(\mathcal{F})$ is defined as the number of tangency points with a generic line. In other words,

$$d(\mathcal{F}) = tang(\mathcal{F}, L)$$

where $L \subset \mathbf{C}P^2$ is any line which is not $\mathcal{F}$-invariant. It follows from proposition 2 that

$$T_{\mathcal{F}} = \mathcal{O}(1 - d(\mathcal{F}))$$

$$N_{\mathcal{F}} = \mathcal{O}(2 + d(\mathcal{F})).$$

The cases $d(\mathcal{F}) = 0, 1$ correspond to an effective $T_{\mathcal{F}}$, so that $\mathcal{F}$ is defined by a global holomorphic vector field (with a line of zeroes if $d(\mathcal{F}) = 0$, with isolated zeroes if $d(\mathcal{F}) = 1$). In these cases, our knowledge of the holomorphic automorphism group of $\mathbf{C}P^2$ answers to any possible question about $\mathcal{F}$. Let us observe, however, that the structure of such an $\mathcal{F}$ can be understood also by less analytic methods.

If $d(\mathcal{F}) = 0$ then any non-invariant line is in fact transverse to $\mathcal{F}$. Hence if $p \notin Sing(\mathcal{F})$ and L_p is the line through p and tangent to $\mathcal{F}$ at p, then L_p is necessarily $\mathcal{F}$-invariant. It clearly follows that $\mathcal{F}$ is a *radial* foliation, i.e. the foliation by lines through a point $p_0 \in \mathbf{C}P^2$. In suitable affine coordinates (such that $p_0 = (0,0)$) $\mathcal{F}$ is generated by $z\frac{\partial}{\partial z} + w\frac{\partial}{\partial w}$.

If $d(\mathcal{F}) = 1$ then any non-invariant line L has only one tangency point p, with $tang(\mathcal{F}, L, p) = 1$. But if we take a singular point p of $\mathcal{F}$ (which certainly exists by proposition 1) then an elementary argument shows that there is at least one line L through p such that the ideal appearing in the definition of $tang(\mathcal{F}, L, p)$ has codimension at least 2, i.e. $tang(\mathcal{F}, L, p) \geq 2$. It follows that such a line is in fact invariant by $\mathcal{F}$. If we choose affine coordinates in which L is at infinity, we find that $\mathcal{F}$ is generated by a vector field $A(z,w)\frac{\partial}{\partial z} + B(z,w)\frac{\partial}{\partial w}$, with A and B affine functions (or even linear, after a translation of the origin).

More generally [G-O], whatever is the degree of $\mathcal{F}$ one finds a generator of the type

$$[P(z,w) + zR(z,w)]\frac{\partial}{\partial z} + [Q(z,w) + wR(z,w)]\frac{\partial}{\partial w}$$

where P and Q are polynomials of degree $\leq d(\mathcal{F})$ and R is a homogeneous polynomial of degree $d(\mathcal{F})$ (plus some evident nondegeneracy condition to ensure that such a vector field has isolated zeroes on $\mathbf{C}^2$ and a pole of order $d(\mathcal{F}) - 1$ at infinity). The line at infinity is invariant if and only if $R \equiv 0$. If $d(\mathcal{F}) \leq 1$ we have seen that this can always be achieved, but if $d(\mathcal{F}) \geq 2$ then it may happen that the foliation has no invariant line.

Concerning the total number of singular points, it follows from proposition 1 that

$$m(\mathcal{F}) = d(\mathcal{F})^2 + d(\mathcal{F}) + 1.$$

3) $X = \tilde{\mathbf{C}P^2}$

Any foliation on X is the blowing-up of a foliation $\mathcal{F}_0$ on $\mathbf{C}P^2$, of degree $d = d(\mathcal{F}_0)$. Let $p \in \mathbf{C}P^2$ be the point which is blown-up, let $E \subset X$ be the exceptional divisor, and let $L \subset X$ be the (strict) transform of a line which does not pass through p. Let $l = l(p), a = a(p)$ be defined in the usual way. Then:

$$T_{\mathcal{F}} = \mathcal{O}_X((1-d)L) \otimes \mathcal{O}_X((l-1)E)$$

$$N_{\mathcal{F}} = \mathcal{O}_X((2+d)L) \otimes \mathcal{O}_X(-lE).$$

We obtain from proposition 1:

$$m(\mathcal{F}) = d^2 + d + 2 - l^2 + l.$$

Suppose now that $Sing(\mathcal{F}) = \emptyset$. Then $\mathcal{F}_0$ has only one singularity, at p, of multiplicity $d^2 + d + 1$. Moreover the exceptional divisor E is not $\mathcal{F}$-invariant (otherwise $Z(\mathcal{F}, E) = a + 1 > 0$ and $Sing(\mathcal{F}) \neq \emptyset$). Hence $l = a + 1$, and $m(\mathcal{F}) = 0$ translates into

$$a^2 + a = d^2 + d + 2.$$

The only admissible solution is $a = 1, d = 0$. This means that $\mathcal{F}_0$ is the radial foliation with (unique) singularity at p, and thus $\mathcal{F}$ is a $\mathbf{C}P^1$-bundle over $\mathbf{C}P^1$. We shall see in the next chapter a large generalisation of this fact, which will require however more sophisticated tools.

4) *Ramified coverings*

Let $\mathcal{F}_0$ be a foliation on a compact connected surface X_0 and let $r : X \to X_0$ be a ramified covering. By lifting 1-forms, we can construct a foliation $\mathcal{F} = \pi^*(\mathcal{F}_0)$ on X. Let us suppose that the ramification locus of r is a smooth connected curve $C \subset X_0$, which is not $\mathcal{F}_0$-invariant, which is disjoint from $Sing(\mathcal{F}_0)$, and which has only quadratic-type tangencies (i.e. $tang(\mathcal{F}_0, C, p) \leq 1$ for every p). If p is a generic point of C, that is a point where $\mathcal{F}_0$ is transverse to C, then in suitable local coordinates $C = \{z = 0\}$ and $\mathcal{F}_0 = Ker(dw)$. The pull-back of dw by r is a holomorphic 1-form still without zeroes. Hence

$$N^*_{\mathcal{F}} = r^*(N^*_{\mathcal{F}_0}).$$

On the other hand, if $\hat{C} = r^{-1}(C)$ (as a reduced curve) and k is the ramification order of r, then $r^*(dz \wedge dw)$ vanishes on $\hat{C}$ at order $k-1$, thus $K_X = r^*(K_{X_0}) \otimes \mathcal{O}_X((k-1)\hat{C})$ and therefore, by adjunction,

$$T^*_{\mathcal{F}} = r^*(T^*_{\mathcal{F}_0}) \otimes \mathcal{O}_X((k-1)\hat{C}),$$

which can be also derived from the analysis of $r^*(\frac{\partial}{\partial z})$.

The singular set of $\mathcal{F}$ consists of two subsets: over each $p \in Sing(\mathcal{F}_0)$, outside of C by hypothesis, $\mathcal{F}$ has k singular points of the same type as p; over each $p \in C$ which is a tangency point, $\mathcal{F}$ has one singular point of the type $d(y^2 + x^k) = 0$, whose multiplicity is $k-1$ (roughly speaking, in local coordinates we have $C = \{z = 0\}$, $\mathcal{F}_0 = Ker(d(z + w^2))$, $(z, w) = r(x, y) = (x^k, y)$). It follows that

$$m(\mathcal{F}) = km(\mathcal{F}_0) + (k-1)tang(\mathcal{F}_0, C).$$

Putting together these formulae and propositions 1 and 2, we find

$$c_2(X) = kc_2(X_0) + (1-k)\chi(C),$$

a well-known formula which, of course, can be proved also by a direct topological argument.

The case when C has more degenerate tangencies with $\mathcal{F}_0$ can be studied in a similar way and gives the same result. On the other side, if C is $\mathcal{F}_0$-invariant then one finds

$$N^*_{\mathcal{F}} = r^*(N^*_{\mathcal{F}_0}) \otimes \mathcal{O}_X((k-1)\hat{C})$$

$$T^*_{\mathcal{F}} = r^*(T^*_{\mathcal{F}_0}).$$

The first relation is perhaps more significantly written as

$$N^*_{\mathcal{F}} \otimes \mathcal{O}_X(\hat{C}) = r^*(N^*_{\mathcal{F}_0} \otimes \mathcal{O}_{X_0}(C))$$

which corresponds to the fact that a meromorphic section of $N^*_{\mathcal{F}_0}$ with first order poles on C lifts by r to a meromorphic section of $N^*_{\mathcal{F}}$ with first order poles on $\hat{C}$. This is a first example showing the naturality of logarithmic forms (cfr. chapter 6).

The case where C is not smooth is more complicated, because r is not finite-to-one over singularities of C (if we insists on the smoothness of X). However, if ω is any local holomorphic 1-form then its pull-back by r is certainly still holomorphic, and so we can at least say that

$$N^*_{\mathcal{F}} = r^*(N^*_{\mathcal{F}_0}) \otimes \mathcal{O}_X(E)$$

where E is an *effective* divisor on X.

5) *Fibrations*

Let X be a compact connected surface and let $\pi : X \to C$ be a *fibration*, that is a holomorphic surjective morphism onto a curve C. We shall suppose, without essential loss of generality, that the generic fibre of π is connected; it is therefore a curve of genus g, whose holomorphic structure may however be variable on C (except if $g = 0$, of course). In this context (see, e.g., [BPV, page 98]) one introduces the *relative canonical bundle* by the formula

$$K_{X/C} = K_X \otimes \pi^*(K_C^*).$$

The fibration π can be thought as a foliation $\mathcal{F}$ on X. At first sight one could think that $N^*_{\mathcal{F}}$ is the same as $\pi^*(K_C)$, and so $T^*_{\mathcal{F}}$ the same as $K_{X/C}$, but this is not always true due to the possible existence of multiple fibres, or more precisely multiple components of fibres. Given an irreducible component D of a fibre of π, let us denote by l_D the ramification order of π along D. In other words, $\sum(l_D - 1)D$ is the zero divisor of $d\pi$ (the sum is over all irreducible components of all fibres). Now, if η is a local nonvanishing 1-form on the base C, then $\pi^*(\eta)$ is (by definition) a local nonvanishing section of $\pi^*(K_C)$, but also a local section of $N^*_{\mathcal{F}}$ which vanishes on D at order $l_D - 1$. This means that

$$N^*_{\mathcal{F}} = \pi^*(K_C) \otimes \mathcal{O}_X(\sum(l_D - 1)D)$$

and consequently

$$T^*_{\mathcal{F}} = K_{X/C} \otimes \mathcal{O}_X(\sum(1 - l_D)D).$$

It is particularly interesting the case of elliptic fibrations ($g = 1$). If F is a smooth fibre then $T^*_{\mathcal{F}}|_F$ coincides with K_F and so it is trivial. Singular fibres are more complicated to work with: it may happen that $T^*_{\mathcal{F}}$ is not trivial on some irreducible component of a singular fibre, for instance if that singular fibre is the blow-up of a smooth one. However, in the *relatively minimal* case (i.e. no fibre contains a smooth rational curve of selfintersection

-1) one can obtain, thanks to Kodaira's work, a quite useful expression. Let us firstly recall Kodaira's canonical bundle formula [BPV, page 161]:

$$K_X = \pi^*(K_C \otimes (\pi_{*1}\mathcal{O}_X)^*) \otimes \mathcal{O}_X(\sum_j (m_j - 1)F_j)$$

where the sum is over all multiple fibres and $m_j \geq 2$ is the multiplicity of the multiple fibre m_jF_j. That sum is *not* the same as the one appearing in $N^*_{\mathcal{F}}$, because a nonmultiple fibre may have multiple components (this happens exactly for fibres of type I^*_0, I^*_b, II^*, III^*, IV^*). If $F = m_0F_0 = m_0 \sum a_iD_i$ is a fibre of π of multiplicity $m_0 \geq 1$, then F gives to K_X a contribution equal to $(m_0 - 1)F_0 = \sum(m_0 - 1)a_iD_i$ and to $N^*_{\mathcal{F}}$ a contribution equal to $\sum(m_0a_i - 1)D_i$. Therefore, its contribution to $T^*_{\mathcal{F}} = K_X \otimes N_{\mathcal{F}}$ is equal to $\sum(1 - a_i)D_i$. More precisely, by comparing the formulae above for K_X and $N^*_{\mathcal{F}}$ we see that

$$T^*_{\mathcal{F}} = \pi^*[(\pi_{*1}\mathcal{O}_X)^*] \otimes \mathcal{O}_X(\sum(F_{red} - F_{prime}))$$

where the sum is over all the fibres (multiple or not) and for every fibre $F = \sum b_iD_i$ we set $F_{red} = \sum D_i$ and $F_{prime} = \frac{1}{\mathrm{mcd}\{b_i\}} \sum b_iD_i$ (of course, $F_{red} \neq F_{prime}$ only for a finite set of fibres). Remark that multiplicities of fibres do not appear in that expression.

The non-relatively minimal case is more complex but it can be handled using the formula concerning blow-up foliations. In fact, elliptic fibrations which are not relatively minimal will play an important role in next chapters. We also note that [Ser] contains a rather detailed study of $T^*_{\mathcal{F}}$ for a fibration $\mathcal{F}$, in the spirit of what we shall do in the last two chapters.

INDEX THEOREMS

In this chapter we prove two important results of Baum-Bott [B-B] and Camacho-Sad [CS1], the first one concerning the computation of $N_{\mathcal{F}} \cdot N_{\mathcal{F}}$ for a foliation $\mathcal{F}$ on a compact surface, the second one concerning the computation of $C \cdot C$ for a compact curve C invariant by a foliation. These two results, which are in fact two manifestations of the same "vanishing principle", have a quite different nature than the easier formulae of the previous chapter: here the integrability of $\mathcal{F}$, that is the existence of leaves, plays a fundamental role. We shall also give several applications of these formulae. A comprehensive reference for these index theorems, and much beyond, is [Suw], especially chapter V; see also [Br2] and [Br3].

1. Baum-Bott formula

Let $\mathcal{F}$ be a foliation on a surface X, with singular set $Sing(\mathcal{F})$ and normal bundle $N_{\mathcal{F}}$. Our first aim is to construct a smooth closed 2-form Ω on X which represents, in the De Rham sense, the first Chern class of $N_{\mathcal{F}}$.

Let $p \in X \setminus Sing(\mathcal{F})$. Then in suitable coordinates (z, w) centered at p the foliation $\mathcal{F}$ is locally given by the 1-form $\omega = dz$, which is closed (it is in this form that the integrability of $\mathcal{F}$ appears). Instead of dz we can choose $\omega = f(z, w)dz$, for any nonvanishing holomorphic function f; it is no more closed, but at least we have $d\omega = \beta \wedge \omega$ where $\beta = \frac{df}{f}$ is a holomorphic 1-form.

If $p \in Sing(\mathcal{F})$ then things are not so simple. If ω is a local 1-form generating $\mathcal{F}$, then $d\omega(p)$ can be different from zero (this happens when the trace of the linear part of a vector field generating $\mathcal{F}$ is not zero), and then we certainly cannot factorize $d\omega$ as $\beta \wedge \omega$ for some holomorphic β, because $\omega(p) = 0$. However, we can find a smooth $(1, 0)$-form β on a neighbourhood of p (not holomorphic, in general) such that we have $d\omega = \beta \wedge \omega$ outside a smaller neighbourhood of p. More concretely, if

$$\omega = A(z, w)dw - B(z, w)dz$$

with A, B holomorphic and vanishing simultaneously only at $p = (0,0)$, then we set

$$\beta = F(z,w)\frac{A_z(z,w) + B_w(z,w)}{|A(z,w)|^2 + |B(z,w)|^2}(\bar{A}(z,w)dz + \bar{B}(z,w)dw)$$

where F is any smooth function equal to 1 outside a neighbourhood of p and equal to zero on some smaller neighbourhood of p.

Patching together these local constructions, we can find an open covering $\{U_j\}_{j\in I}$ of X, holomorphic 1-forms $\omega_j \in \Omega^1_X(U_j)$ with isolated zeroes and generating $\mathcal{F}$, smooth $(1,0)$-forms $\beta_j \in A^{1,0}(U_j)$, such that for every $j \in I$

$$d\omega_j = \beta_j \wedge \omega_j \qquad \text{on } U_j \setminus V_j$$

where $V_j \subset\subset U_j$ and $V_j \cap U_i = \emptyset$ for every $i \neq j$. In particular, $d\omega_j = \beta_j \wedge \omega_j$ on $U_j \cap U_i$ for every $i \neq j$.

On each $U_j \cap U_i$ we also have

$$\omega_i = g_{ij}\omega_j$$

where $\{g_{ij}\}$ is a $\mathcal{O}^*_X$-cocycle defining $N_{\mathcal{F}}$. Hence

$$\beta_i \wedge \omega_i = d\omega_i = dg_{ij} \wedge \omega_j + g_{ij}d\omega_j = (\frac{dg_{ij}}{g_{ij}} + \beta_j) \wedge \omega_i$$

i.e.

$$(\frac{dg_{ij}}{g_{ij}} + \beta_j - \beta_i) \wedge \omega_i = 0.$$

Therefore, the cocycle of $(1,0)$-forms $\{\frac{dg_{ij}}{g_{ij}} + \beta_j - \beta_i\}$ can be thought as a cocycle of smooth sections of $N^*_{\mathcal{F}}$, because it vanishes identically on $\mathcal{F}$. Now, the sheaf of smooth sections of $N^*_{\mathcal{F}}$ is certainly fine and so it has trivial cohomology. Hence we can find smooth $(1,0)$-forms $\gamma_j \in A^{1,0}(U_j)$ such that

$$\gamma_j \wedge \omega_j = 0 \qquad \text{on } U_j$$

$$\frac{dg_{ij}}{g_{ij}} = \beta_i - \beta_j + \gamma_i - \gamma_j \qquad \text{on } U_j \cap U_i.$$

Observe that we still have $d\omega_j = (\beta_j + \gamma_j) \wedge \omega_j$ on $U_j \setminus V_j$.

The 2-form defined by

$$\Omega = \frac{1}{2\pi i}d(\beta_j + \gamma_j) \qquad \text{on } U_j$$

is therefore a well-defined closed 2-form on X, which represents the first Chern class of $N_{\mathcal{F}}$ according to one of the possible definitions of Chern class [G-H]. Remark that it is not (in general) of type $(1,1)$, but only of type $(1,1)+(2,0)$, and thus Ω represents $N_{\mathcal{F}}$ in De Rham sense (as an element of H^2) but not in Dolbeault sense (as an element of $H^{1,1}$). Of course, we could take the $(1,1)$-component of Ω $(=\frac{1}{2\pi i}\bar{\partial}(\beta_j+\gamma_j))$ to represent $N_{\mathcal{F}}$ in $H^{1,1}$, but be careful that such a $(1,1)$-form is (in general) only $\bar{\partial}$-closed but not closed.

To each singularity $p \in Sing(\mathcal{F})$ we can associate the following residue-type index [B-B]:

$$BB(\mathcal{F},p) = Res_{(0,0)}\left\{\frac{[A_z(z,w)+B_w(z,w)]^2}{A(z,w)B(z,w)}dz\wedge dw\right\}$$

where $\omega = Adw - Bdz$ generates $\mathcal{F}$ around p. This index is well-defined, i.e. independent on the choice of the generator ω. We refer to [G-H], chapter V, for the basic properties of residues; one can find there also a proof of the equality

$$BB(\mathcal{F},p) = \frac{1}{(2\pi i)^2}\int_\Gamma \beta\wedge d\beta$$

where Γ is a (small) 3-sphere around p, oriented as the boundary of a (small) ball containing p, and β is any $(1,0)$-form around p which satisfies $d\omega = \beta\wedge\omega$ on a neighbourhood of Γ (in [G-H] the authors choose $\beta = \frac{A_z+B_w}{|A|^2+|B|^2}(\bar{A}dz+\bar{B}dw)$, but the specific choice is obviously inessential).

When p is nondegenerate, i.e. generated by a vector field whose linear part has eigenvalues λ_1, λ_2 both different from 0, then the computation of BB is particularly simple:

$$BB(\mathcal{F},p) = \frac{(\lambda_1+\lambda_2)^2}{\lambda_1\lambda_2} = \frac{\lambda_1}{\lambda_2}+\frac{\lambda_2}{\lambda_1}+2.$$

Remark that $BB(\mathcal{F},p)$ is a *complex* number, and not an integer one as previously introduced indices (which in fact can also be expressed as residues).

We can now state and prove Baum-Bott formula.

Theorem 1 [B-B]. *Let $\mathcal{F}$ be a foliation on a compact surface X. Then*

$$N_{\mathcal{F}}\cdot N_{\mathcal{F}} = \sum_{p\in Sing(\mathcal{F})} BB(\mathcal{F},p).$$

Proof.

Let Ω be the closed 2-form constructed above. Then

$$N_{\mathcal{F}}\cdot N_{\mathcal{F}} = \int_X \Omega\wedge\Omega.$$

Recall, however, that Ω is locally expressed as $\frac{1}{2\pi i}d\tilde{\beta}_j$, where $\tilde{\beta}_j$ satisfies $d\omega_j = \tilde{\beta}_j \wedge \omega_j$ outside certain small neighbourhoods V_j of the singular points of the foliation. We deduce that $\Omega \wedge \omega_j = \frac{1}{2\pi i}d\tilde{\beta}_j \wedge \omega_j = 0$ and therefore, outside the same V_j's, we have

$$\Omega \wedge \Omega \equiv 0.$$

This means that the integral above is localized around the singular points of $\mathcal{F}$. By Stokes theorem, each singular point gives a contribution equal to $\frac{1}{(2\pi i)^2}\int_{\partial V_j} \tilde{\beta}_j \wedge d\tilde{\beta}_j$, which is nothing but than Baum-Bott index. $\triangle$

This theorem is in fact a particular case of much more general results [B-B] [Suw] for foliations in any dimension, but the underlying philosophy is always the same: Chern numbers of the normal bundle (or sheaf) of a foliation can be localized at singular points (or sets) of the foliation, thanks to Frobenius theorem. Even in the surface case there is in fact a "second" Baum-Bott formula, concerning the localization of c_2 (instead of c_1^2) of the normal sheaf $TX/T_{\mathcal{F}}$ (which is not equal to $N_{\mathcal{F}}$ over $Sing(\mathcal{F})$, it is not a line bundle, and it has a nontrivial second Chern class), and giving proposition 1 of the previous chapter (computation of $m(\mathcal{F})$).

2. Camacho-Sad formula

Let again $\mathcal{F}$ be a foliation on X, and suppose moreover that $C \subset X$ is an $\mathcal{F}$-invariant curve (not necessarily smooth, nor irreducible). Near each $p \in C \cap Sing(\mathcal{F})$ we choose a local holomorphic 1-form ω generating $\mathcal{F}$ and a local equation f of C, and then, as in the previous chapter, we choose a decomposition

$$g\omega = hdf + f\eta$$

where g and h are holomorphic functions prime to f and η is a holomorphic 1-form. We can then define the following residue-type index [CS1] [Suw]:

$$CS(\mathcal{F}, C, p) = Res_p\{ -\frac{1}{h}\eta|_C \}$$

or, more explicitely,

$$CS(\mathcal{F}, C, p) = -\frac{1}{2\pi i}\int_\gamma \frac{1}{h}\eta$$

where $\gamma \subset C$ is a union of small circles around p, one for each local irreducible component C_i of C, oriented as the boundary of a small disc contained in C_i and containing p. As usual, one verifies that this index depends only on $\mathcal{F}$, C and p, and not on the choices of ω, f, g, h, η.

On the other hand, if β is a smooth $(1,0)$-form on a neighbourhood of p satisfying $d\omega = \beta \wedge \omega$ outside a smaller neighbourhood, we can define another index, called variation [Suw]:

$$Var(\mathcal{F}, C, p) = \frac{1}{2\pi i} \int_\gamma \beta,$$

still independing on the involved choices. Here, of course, γ is contained in the set where $d\omega = \beta \wedge \omega$ holds; in fact, it is sufficient that such a relation be satisfied only on a neighbourhood of γ.

These two indices are related to the index $Z(\mathcal{F}, C, p)$ introduced in the previous chapter by the following result.

Proposition 1 [Br2].

$$Var(\mathcal{F}, C, p) = Z(\mathcal{F}, C, p) + CS(\mathcal{F}, C, p).$$

Proof.

The index $Z(\mathcal{F}, C, p)$, which we recall is the vanishing order of $\frac{h}{g}|_C$ at p, can be expressed by the integral formula

$$Z(\mathcal{F}, C, p) = \frac{1}{2\pi i} \int_\gamma \frac{g}{h} d(\frac{h}{g}).$$

On a neighbourhood of γ (where, in particular, g and h have no zeroes) we can write

$$\omega = \frac{h}{g} df + f \frac{1}{g} \eta$$

and we can set

$$\beta_0 = \frac{g}{h} d(\frac{h}{g}) - \frac{1}{h} \eta.$$

Then we obtain

$$d\omega = \beta_0 \wedge \omega + f\Theta$$

for some 2-form Θ, holomorphic on a neighbourhood of γ. Hence, if β is a $(1,0)$-form satisfying $d\omega = \beta \wedge \omega$ on a neighbourhood of γ we forcely have $\beta|_\gamma = \beta_0|_\gamma$, from which it follows that

$$Var(\mathcal{F}, C, p) = \frac{1}{2\pi i} \int_\gamma \beta_0 = Z(\mathcal{F}, C, p) + CS(\mathcal{F}, C, p)$$

as desired. △

We can now state and prove Camacho-Sad formula, discovered in [CS1] in a particular case and later successively generalized by Lins Neto and Suwa [Suw].

Theorem 2 [CS1] [Suw]. *Let $\mathcal{F}$ be a foliation on a surface X and let $C \subset X$ be a compact $\mathcal{F}$-invariant curve. Then*

$$C \cdot C = \sum_{p \in C \cap Sing(\mathcal{F})} CS(\mathcal{F}, C, p).$$

Proof.

By proposition 3 of the previous chapter we have

$$C \cdot C = N_{\mathcal{F}} \cdot C - \sum_{p \in C \cap Sing(\mathcal{F})} Z(\mathcal{F}, C, p).$$

On the other hand, we also have

$$N_{\mathcal{F}} \cdot C = \int_C \Omega$$

where Ω is the smooth closed 2-form already used for the Baum-Bott formula. As observed in the proof of theorem 1, that 2-form vanishes when restricted to leaves of $\mathcal{F}$ ($\Omega \wedge \omega_j = 0$), and in particular when restricted to C, except on small neighbourhoods of $Sing(\mathcal{F})$. Therefore the integral $\int_C \Omega$ is again localized around singular points of $\mathcal{F}$ on C, and by Stokes theorem we find

$$\int_C \Omega = \sum_{p \in C \cap Sing(\mathcal{F})} Var(\mathcal{F}, C, p).$$

Now the conclusion follows from proposition 1: $CS = Var - Z$. △

Let us give another "interpretation" of this result. In the previous section we have constructed an explicit 2-form Ω representing the Chern class of $N_{\mathcal{F}}$. In an absolutely similar way, we can construct a 2-form $\tilde{\Omega}$ which represents $N_{\mathcal{F}} \otimes \mathcal{O}_X(-C)$: the only difference is that instead of using holomorphic 1-forms $\omega_j \in \Omega^1_X(U_j)$ defining $\mathcal{F}$ we use meromorphic 1-forms $\tilde{\omega}_j = \frac{1}{f_j}\omega_j$ having first order poles on $C = \{f_j = 0\}$, and the smooth $(1,0)$-forms β_j are replaced by smooth $(1,0)$-forms $\tilde{\beta}_j$ satisfying $d\tilde{\omega}_j = \tilde{\beta}_j \wedge \tilde{\omega}_j$ on $U_j \setminus V_j$. Then, as before, we have

$$[N_{\mathcal{F}} \otimes \mathcal{O}_X(-C)] \cdot C = \int_C \tilde{\Omega}$$

and this integral is localized around $Sing(\mathcal{F}) \cap C$, each singularity giving a contribution equal to $\tilde{Var}(\mathcal{F}, C, p) = \frac{1}{2\pi i} \int_\gamma \tilde{\beta}$. Hence Camacho-Sad formula is reduced to prove that $CS(\mathcal{F}, C, p) = Var(\mathcal{F}, C, p) - \tilde{Var}(\mathcal{F}, C, p)$ (because $C \cdot C = N_{\mathcal{F}} \cdot C - [N_{\mathcal{F}} \otimes \mathcal{O}_X(-C)] \cdot C$), and this follows from $d\tilde{\omega} = d(\frac{1}{f}\omega) = (\beta + \frac{1}{h}\eta) \wedge (\frac{1}{f}\omega)$, giving $\tilde{\beta}|_\gamma = \beta|_\gamma + \frac{1}{h}\eta|_\gamma$. Of course, one can prove theorem 2 also by a direct computation, not involving Var nor Z nor $\tilde{Var}$. We note, however, that the "logarithmic" point of view, which consists in working with meromorphic 1-forms with poles along invariant curves, is a quite natural and powerful point of view, as we shall see for instance in the next section 4 or in chapter 6.

It is worth noting some properties of the indices Z, CS and Var, see e.g. [Br2] or [Suw]. We consider a local situation, i.e. C is an invariant curve defined only on a (small) neighbourhood of $p \in Sing(\mathcal{F})$, and so a union of separatrices $C_1, ..., C_n$ of $\mathcal{F}$ at p, each one isomorphic to the disc. By additivity of the integral we certainly have

$$Var(\mathcal{F}, C, p) = \sum_{j=1}^{n} Var(\mathcal{F}, C_j, p)$$

but an analogous formula does *not* hold for Z and CS. The reason is that, whereas Var is defined by integrating a 1-form β which is independent on C, Z and CS are obtained through a decomposition of ω which *does* depend on C. Let us group the separatrices $C_1, ..., C_n$ into two disjoint groups C' and C''. Then the local (at p) intersection indices are related by

$$(C \cdot C)_p = (C' \cdot C')_p + (C'' \cdot C'')_p + 2(C' \cdot C'')_p$$

and this together Camacho-Sad formula suggests

$$CS(\mathcal{F}, C, p) = CS(\mathcal{F}, C', p) + CS(\mathcal{F}, C'', p) + 2(C' \cdot C'')_p$$

which is indeed true, as a simple computation shows [Suw]. From this and the additivity of Var it follows also

$$Z(\mathcal{F}, C, p) = Z(\mathcal{F}, C', p) + Z(\mathcal{F}, C'', p) - 2(C' \cdot C'')_p.$$

For instance, let us consider a radial singularity: $\mathcal{F}$ is given by $zdw - wdz = 0$. Any line L through the origin is $\mathcal{F}$-invariant, and $Z(\mathcal{F}, L, 0) = 1$. Hence if C is the union of k lines we find $Z(\mathcal{F}, C, 0) = -k^2 + 2k$. The negativity of Z, for $k \geq 3$, is due to the dicriticality of 0 [Br2].

Let us now see what happens by blowing-up X at p. To simplify, let us suppose that C is irreducible at p; anyway, the non-irreducible case can be treated with the help of the formulae above. The strict transform $\bar{C}$ of C is an $\tilde{\mathcal{F}}$-invariant curve, through a point $\tilde{p}$ on the exceptional divisor $E \subset \tilde{X}$. The large transform $\tilde{C}$ of C is, as a divisor, $\bar{C} + \mu E$, where $\mu \in \mathbf{N}^+$ is the so-called multiplicity of C at p, i.e. $\mu = (\bar{C} \cdot E)_{\tilde{p}}$. Hence

$$(C \cdot C)_p = (\bar{C} \cdot \bar{C})_{\tilde{p}} + \mu^2$$

and this relation together Camacho-Sad formula suggest

$$CS(\mathcal{F}, C, p) = CS(\tilde{\mathcal{F}}, \bar{C}, \tilde{p}) + \mu^2$$

which is indeed true as a direct computation shows [Suw]. Remark that $\tilde{p}$ may be an $\tilde{\mathcal{F}}$-regular point, in which case we set, consistently, $CS(\tilde{\mathcal{F}}, \bar{C}, \tilde{p}) = 0$. Concerning Z, one finds

$$Z(\mathcal{F}, C, p) = Z(\tilde{\mathcal{F}}, \bar{C}, \tilde{p}) - \mu^2 + \mu l(p)$$

where $l(p)$ is, as usual, the vanishing order of $\tilde{\omega}$ along E, that is $N^*_{\tilde{\mathcal{F}}} = \pi^*(N^*_{\mathcal{F}}) \otimes \mathcal{O}_{\tilde{X}}(l(p)E)$. Again, this can be checked by an easy computation, and is also suggested by a localization of the following argument. If C is compact, then $N_{\mathcal{F}} \cdot C$ is the sum of Var-indices, and $N_{\tilde{\mathcal{F}}} \cdot \bar{C} = N_{\mathcal{F}} \cdot C - \mu l(p)$. Hence $Var(\tilde{\mathcal{F}}, \bar{C}, \tilde{p})$ equals $Var(\mathcal{F}, C, p) - \mu l(p)$, and the corresponding formula for Z follows from proposition 1 and the formula for CS.

In the following we shall especially need the computation of these indices in the reduced case.

If p is reduced nondegenerate then we already know that there are exactly two separatrices, and a local generator of $\mathcal{F}$ has the form $\omega = z(1 + o(1))dw - \lambda w(1 + o(1))dz$, with $\lambda \notin \mathbf{Q}^+$, $\lambda \neq 0$. Hence:

$$CS(\mathcal{F}, \{z = 0\}, p) = Res_0\big\{ - \frac{1}{-\lambda w(1 + o(1))}(1 + o(1))dw|_{z=0}\big\} = \frac{1}{\lambda}$$

and

$$CS(\mathcal{F}, \{w = 0\}, p) = Res_0\big\{ - \frac{1}{z(1 + o(1))}(-\lambda)(1 + o(1))dz|_{w=0}\big\} = \lambda.$$

We also have $CS(\mathcal{F}, \{zw = 0\}, p) = \lambda + \frac{1}{\lambda} + 2$. Remark that this equals the Baum-Bott index $BB(\mathcal{F}, p)$. In [Br2] this is generalized to the case of the so-called "generalized curves" (nondicritical singularities without saddle-nodes in their resolution). Note also that $Z(\mathcal{F}, \{z = 0\}, p) = Z(\mathcal{F}, \{w = 0\}, p) = 1$ and $Z(\mathcal{F}, \{zw = 0\}, p) = 0$.

If p is a saddle-node, then a local generator of $\mathcal{F}$ can be put in Dulac's form $\omega = [z(1 + \nu w^k) + wo(k)]dw - w^{k+1}dz$, $\{w = 0\}$ being the strong separatrix. Hence

$$CS(\mathcal{F}, \{w = 0\}, p) = Res_0\big\{ - \frac{1}{z(1 + \nu w^k) + wo(k)}(-w^k dz)|_{w=0}\big\} = 0$$

$$Z(\mathcal{F}, \{w = 0\}, p) = 1.$$

Suppose that there exists a weak separatrix through p. Up to a change of coordinates we may obtain $\omega = z[1 + \nu w^k + o(k)]dw - w^{k+1}dz$ and so

$$CS(\mathcal{F}, \{z = 0\}, p) = \nu$$

$$Z(\mathcal{F}, \{z=0\}, p) = k+1.$$

In all these cases, we see that there exists a relation between $CS(\mathcal{F}, C, p)$ and the holonomy of $\mathcal{F}$ along a cycle $\gamma \subset C$ around p (oriented as usual): the linear part of that holonomy is $x \mapsto \exp(2\pi i CS(\mathcal{F}, C, p))x$. This is a general fact, which does not require that p is reduced [Suw]. This can also be seen via Seidenberg's theorem, because the holonomy does not change under blowing-up and CS changes only by an integer number, which does not affect $\exp(2\pi i CS)$.

3. The separatrix theorem and its singular generalization

The original motivation of Camacho-Sad formula was in the proof of the following result.

Theorem 3 [CS1]. *Through each singular point p of a foliation $\mathcal{F}$ on a surface X there exists at least one separatrix.*

This result was later generalized in [Cam], where the author allows the surface X have a normal singularity at p whose resolution diagram is a tree. By Seidenberg's resolution theorem, these results are consequences of the following one:

Theorem 4 [Cam]. *Let $\mathcal{F}$ be a foliation on a surface X and let $C \subset X$ be a connected compact $\mathcal{F}$-invariant curve such that:*

i) all the singularities of $\mathcal{F}$ on C are reduced (in particular, C has only normal crossing singularities);

ii) if $C_1, ..., C_n$ are the irreducible components of C then the intersection matrix $M = \{C_i \cdot C_j\}_{i,j}$ is definite negative and the dual graph Γ is a tree.

Then there exists at least one point $p \in C \cap Sing(\mathcal{F})$ and a separatrix through p not contained in C.

Theorem 3 follows from theorem 4, because by Seidenberg's theorem one can replace p by a tree of rational curves which satisfy all the required properties except, possibly, the $\mathcal{F}$-invariant condition, but in that exceptional situation $\mathcal{F}$ is dicritical at p and has even infinitely many separatrices. When the tree of rational curves is $\mathcal{F}$-invariant, the separatrix given by theorem 4 blows-down to a separatrix through the singularity of the initial foliation.

Proof of theorem 4.

It is a clever corollary to theorem 2. We shall follow the elegant proof of [Seb].

Let $V \simeq \mathbf{R}^n$ be the real vector space generated by the irreducible components $\{C_j\}_{j=1}^n$, equipped with the quadratic form $\mathcal{S}$ arising from the intersection form.

Let $p_1, ..., p_m$ be the nodes of C; each node p_j belongs to two irreducible components C_{j_1} and C_{j_2}, and each irreducible component C_j contains several nodes p_{j_k}, $k \in I_j$. To each p_j we associate the vector space $W_j \simeq \mathbf{R}^2$ generated by C_{j_1} and C_{j_2} and equipped with the quadratic form

$$\sigma_j = \begin{pmatrix} ReCS(\mathcal{F}, C_{j_1}, p_j) & 1 \\ 1 & ReCS(\mathcal{F}, C_{j_2}, p_j) \end{pmatrix}$$

in the basis $\{C_{j_1}, C_{j_2}\}$. Let $W = \oplus_{j=1}^m W_j$, equipped with the quadratic form $\sigma = \oplus_{j=1}^m \sigma_j$.

For every $j = 1, ..., n$ and every $k \in I_j$ we denote by (p_{j_k}, C_j) the element of W which belongs to W_{j_k} and is equal to C_j, which is one of the two generators of W_{j_k}. We can define a linear map $l : V \to W$ by setting

$$l(C_j) = \sum_{k \in I_j} (p_{j_k}, C_j) \qquad \forall j = 1, ..., n$$

and extending by linearity.

Suppose now, by contradiction, that the conclusion of the theorem does not hold. Then any singularity $p \in C \cap Sing(\mathcal{F})$ different from a node of C has only one separatrix, contained in C, thus it is a saddle-node whose strong separatrix is contained in C (and without weak separatrix). It follows that $CS(\mathcal{F}, C, p) = 0$ by the computations of the previous section. In particular, by Camacho-Sad formula any component C_j has self-intersection $C_j \cdot C_j$ equal to $\sum_{k \in I_j} CS(\mathcal{F}, C_j, p_{j_k})$. In this expression we can replace CS by its real part $ReCS$, because the sum is a real number, and therefore we see that

$$C_j \cdot C_j = \sigma(l(C_j), l(C_j)).$$

Also, if $i \neq j$ we clearly have

$$C_j \cdot C_i = \sigma(l(C_j), l(C_i)).$$

In other words, we have $l^* \sigma = \mathcal{S}$. But $\mathcal{S}$ is definite negative and $\dim V = n$, so that σ must have at least n negative eigenvalues.

On the other hand, each factor σ_j has at most 1 negative eigenvalue: if p_j is nondegenerate with eigenvalue λ then $\sigma_j = \begin{pmatrix} Re\lambda & 1 \\ 1 & Re\frac{1}{\lambda} \end{pmatrix}$, if p is a saddle-node with formal invariant ν then $\sigma_j = \begin{pmatrix} Re\nu & 1 \\ 1 & 0 \end{pmatrix}$, and in both cases σ_j is not definite negative because $\det(\sigma_j) \leq 0$.

Because $\sigma = \oplus_{j=1}^{m} \sigma_j$, we finally obtain

$$n \leq m$$

that is, the number of irreducible components of C is not greater than the number of its nodes. But this property contradicts the hypothesis that the dual graph Γ is a tree. $\triangle$

The interested reader may find in [Seb] an even more general result, which applies to the case of a non-arboreal dual graph. Note, however, that we cannot simply forget the hypothesis on Γ, as simple examples show.

4. An index theorem for invariant measures

The methods of sections 1 and 2 can be used also to prove a quite useful inequality concerning the intersection product between $N_{\mathcal{F}}$ and a closed positive current Φ invariant by $\mathcal{F}$ [Br3]. We refer to [Dem] for the basic facts (and much more) on closed positive currents.

Let $\mathcal{F}$ be a foliation on a surface X and let Φ be a closed positive $(1,1)$-current on X with compact support K. We shall say that Φ is *invariant* by $\mathcal{F}$ if $\Phi(\beta) = 0$ for every smooth 2-form β which vanishes on $\mathcal{F}$. Around a regular point of $\mathcal{F}$ we can choose local coordinates (z, w) such that $\mathcal{F}$ is given by $dz = 0$; then a 2-form β vanishes on $\mathcal{F}$ iff $\beta = \beta_1 \wedge dz + \beta_2 \wedge d\bar{z}$, for suitable 1-forms β_1 and β_2, and a current Φ is $\mathcal{F}$-invariant iff

$$\Phi = A \cdot dz \wedge d\bar{z},$$

where (with a standard abuse of notation) A is a distribution. Moreover, the closedness condition $d\Phi \equiv 0$ translates into the invariance of A along the leaves of $\mathcal{F}$, i.e. $A_w = A_{\bar{w}} = 0$. In other words, Φ is, locally around any regular point, the pull-back of a measure on the local leaf-space of $\mathcal{F}$. In dynamical literature, such an object is called a *transversely invariant measure* (for $\mathcal{F}$ and outside $Sing(\mathcal{F})$), see for instance [Sul] for more details on this correspondence between invariant currents and transversely invariant measures.

The simplest case is the following one: if $C \subset X$ is a compact $\mathcal{F}$-invariant curve, then the integration current δ_C on C is an $\mathcal{F}$-invariant closed positive current. It corresponds to an atomic transversely invariant measure. Conversely, if Φ is a closed positive $\mathcal{F}$-invariant current with compact support K, such that the corresponding transverse measure is atomic, then K is a countable union of compact curves and (consequently) Φ is a sum of integration currents on these compact curves (here we need to use the following standard removal of singularities argument: if $C \subset X \setminus Sing(\mathcal{F})$ is a (proper) curve then the closure $\bar{C} \subset X$ is still a curve).

Here we shall firstly consider the case where Φ has trivial atomic part, that is the corresponding transverse measure has no atoms at all. We shall say that Φ is *diffuse*. This condition is equivalent to say that the Lelong number $\nu(\Phi, p)$ of Φ at p is 0 for every $p \in X \setminus Sing(\mathcal{F})$ [Dem]. It may hovever happen that $\nu(\Phi, p) > 0$ for some $p \in Sing(\mathcal{F})$ (take, for example, any radial singularity in the support of Φ). We shall use the following results on diffuse currents, deduced from [Dem]:

i) the cohomology class $[\Phi]$ defined by Φ is *nef*, i.e. $[\Phi] \cdot C \geq 0$ for every curve $C \subset X$ (this is not exactly the usual definition of nef, when X is not algebraic, but for our purposes it doesn't matter);

ii) $[\Phi] \cdot [\Phi] \geq 0$, and $[\Phi] \cdot [\Phi] > 0$ if there exists $p \in X$ such that $\nu(\Phi, p) > 0$.

Theorem 5. *Let $\mathcal{F}$ be a foliation on a surface X, let Φ be a closed positive current on X with compact support K and invariant by $\mathcal{F}$, and suppose that Φ is diffuse. Then*

$$N_{\mathcal{F}} \cdot [\Phi] \geq 0$$

and moreover

$$N_{\mathcal{F}} \cdot [\Phi] = 0 \qquad \text{if } [\Phi] \cdot [\Phi] = 0.$$

Proof.

When $\mathcal{F}$ has reduced singularities this is proved in [Br2] and here we only sketch the main ideas:

1) in order to evaluate $N_{\mathcal{F}} \cdot [\Phi]$, one has to evaluate Φ over a 2-form representing the Chern class of $N_{\mathcal{F}}$, and using the special 2-form Ω already used in sections 1 and 2 one obtains that $N_{\mathcal{F}} \cdot [\Phi]$ is localized around $K \cap Sing(\mathcal{F})$.

2) using the fact that Φ is diffuse, one sees that $K \cap Sing(\mathcal{F})$ does not contain saddle-nodes.

3) if $p \in K \cap Sing(\mathcal{F})$ is in the Siegel domain then its contribution to $N_{\mathcal{F}} \cdot [\Phi]$ is 0.

4) if $p \in K \cap Sing(\mathcal{F})$ is in the Poincaré domain then its contribution to $N_{\mathcal{F}} \cdot [\Phi]$ is real and strictly positive.

From this it follows $N_{\mathcal{F}} \cdot [\Phi] \geq 0$. Moreover, if $[\Phi] \cdot [\Phi] = 0$ then $\nu(\Phi, p) = 0$ everywhere (by ii) above), hence K cannot contain singularities in the Poincaré domain (for which one verifies that the Lelong number would be positive), and therefore $N_{\mathcal{F}} \cdot [\Phi] = 0$.

Let us consider now the case where $\mathcal{F}$ has arbitrary singularities. If $\pi : \tilde{X} \to X$ is the blowing-up at $p \in Sing(\mathcal{F})$, with exceptional divisor E, then we know that

$$N_{\tilde{\mathcal{F}}} = \pi^*(N_{\mathcal{F}}) \otimes \mathcal{O}_{\tilde{X}}(-lE)$$

where $l > 0$. The current Φ also can be lifted to $\tilde{X}$, giving a closed positive current $\tilde{\Phi}$ which is $\tilde{\mathcal{F}}$-invariant and still diffuse (the lifting is defined through the lifting of a plurisubharmonic potential of Φ around p). From the cohomological point of view, we have

$$[\tilde{\Phi}] = \pi^*[\Phi] - \mu E$$

where $\mu = [\tilde{\Phi}] \cdot E \geq 0$ because $\tilde{\Phi}$ is nef. Hence

$$N_{\tilde{\mathcal{F}}} \cdot [\tilde{\Phi}] = N_{\mathcal{F}} \cdot [\Phi] - l\mu$$

and in particular $N_{\tilde{\mathcal{F}}} \cdot [\tilde{\Phi}] \leq N_{\mathcal{F}} \cdot [\Phi]$. Now the desired inequality follows from this inequality and Seidenberg's theorem.

Finally, observe that

$$0 \leq [\tilde{\Phi}] \cdot [\tilde{\Phi}] = [\Phi] \cdot [\Phi] - \mu^2$$

and so $[\Phi] \cdot [\Phi] = 0$ implies $\mu = 0$, that is $[\tilde{\Phi}] \cdot [\tilde{\Phi}] = 0$ and $N_{\mathcal{F}} \cdot [\Phi] = N_{\tilde{\mathcal{F}}} \cdot [\tilde{\Phi}]$. Again by Seidenberg's theorem, this reduces the second part of the theorem to the reduced case, proved above. $\triangle$

This result will be useful in the last chapter, in the analysis of foliations of negative Kodaira dimension.

An important class of closed positive currents invariant by a foliation arises from transcendental *entire curves* [Mc2] [Br3]. In that case the atomic part may be nontrivial, but anyway it is concentrated on a compact $\mathcal{F}$-invariant curve C, and the current Φ can be decomposed as

$$\Phi = \Phi_{diff} + \Phi_{alg}$$

where Φ_{diff} is diffuse (and still $\mathcal{F}$-invariant, obviously) and $\Phi_{alg} = \sum_{j=1}^{n} \lambda_j \delta_{C_j}$, C_j irreducible components of C and λ_j positive real numbers. Here the finiteness of the components of C (i.e. the fact that C is a curve, and not simply a countable union of curves) derives from a theorem of Jouanolou that we shall discuss in chapter 6. For these currents arising from entire curves one still has the nefness of the associated cohomology classes, even if the single atomic component may be not nef.

Motivated by this class of invariant currents we can now extend the first part of theorem 5. In order to get a reasonably simple statement, we shall also suppose that the support of the atomic component is a normal crossing curve.

Theorem 6. *Let $\mathcal{F}$ be a foliation on a surface X and let Φ be a closed positive current on X with compact support K and invariant by $\mathcal{F}$. Suppose that $\Phi = \Phi_{diff} + \Phi_{alg}$, where*

Φ_{diff} is diffuse and $\Phi_{alg} = \sum_{j=1}^{n} \lambda_j \delta_{C_j}$, and suppose moreover that $C = \cup_{j=1}^{n} C_j$ has only normal crossing singularities. Then

$$N_{\mathcal{F}} \cdot [\Phi] \geq [\Phi] \cdot C.$$

In particular, if moreover Φ is nef then

$$N_{\mathcal{F}} \cdot [\Phi] \geq 0.$$

Proof.

We shall prove the following two inequalities:

i) $(N_{\mathcal{F}} \otimes \mathcal{O}_X(-C)) \cdot [\Phi_{alg}] \geq 0$

ii) $(N_{\mathcal{F}} \otimes \mathcal{O}_X(-C)) \cdot [\Phi_{diff}] \geq 0$.

Concerning i), note that $N_{\mathcal{F}} \cdot C_j = C_j \cdot C_j + Z(\mathcal{F}, C_j)$ (chapter 2, proposition 3) and $Z(\mathcal{F}, C_j)$ is at least equal to the number of (normal) intersections of C_j with the other components of C. Hence $N_{\mathcal{F}} \cdot C_j \geq C \cdot C_j$, i.e. $(N_{\mathcal{F}} \otimes \mathcal{O}_X(-C)) \cdot C_j \geq 0$, from which i) follows.

Concerning ii), when $\mathcal{F}$ has reduced singularities the inequality is proved in [Br3], by a logarithmic variation of the arguments of theorem 5, and we simply refer to that paper. In the general case, one has to analyze what happens by blowing-up. So let $p \in Sing(\mathcal{F})$ and let $\pi : \tilde{X} \to X$ be the blowing-up at p, with exceptional divisor E. Let $\bar{C} \subset \tilde{X}$ be the strict transform of C.

If E is $\tilde{\mathcal{F}}$-invariant then $\hat{C} = \bar{C} \cup E$ is $\tilde{\mathcal{F}}$-invariant and we have

$$N_{\tilde{\mathcal{F}}} \otimes \mathcal{O}_{\tilde{X}}(-\hat{C}) = \pi^*(N_{\mathcal{F}} \otimes \mathcal{O}_X(-C)) \otimes \mathcal{O}_{\tilde{X}}((\epsilon - l)E)$$

where $l \geq 1$ and $\epsilon = 0$ (resp. $\epsilon = 1$) if p is a smooth (resp. nodal) point of C. Hence $l - \epsilon \geq 0$, and as in theorem 5 we obtain

$$(N_{\tilde{\mathcal{F}}} \otimes \mathcal{O}_{\tilde{X}}(-\hat{C})) \cdot [\tilde{\Phi}_{diff}] \leq (N_{\mathcal{F}} \otimes \mathcal{O}_X(-C)) \cdot [\Phi_{diff}].$$

The conclusion now follows by an induction and Seidenberg's theorem.

If E is not $\tilde{\mathcal{F}}$-invariant then we cannot choose $\hat{C}$ because it is no more $\tilde{\mathcal{F}}$-invariant. We have to work only with $\bar{C}$, and we have

$$N_{\tilde{\mathcal{F}}} \otimes \mathcal{O}_{\tilde{X}}(-\bar{C}) = \pi^*(N_{\mathcal{F}} \otimes \mathcal{O}_X(-C)) \otimes \mathcal{O}_{\tilde{X}}((\epsilon - l + 1)E).$$

But the non-invariance of E implies that $l \geq 2$, thus we still have $l - \epsilon - 1 \geq 0$ and we can conclude as before. $\triangle$

5. Regular foliations on rational surfaces

We give here, following more or less [Br1], the classification of regular foliations on rational surfaces; *regular* means that the singular set of the foliation is empty. The proof is based on theorem 1, which in the regular case says that $N_{\mathcal{F}} \cdot N_{\mathcal{F}} = 0$, on theorem 2, which says $C \cdot C = 0$ for any $\mathcal{F}$-invariant curve, and on standard tools of algebraic geometry: Riemann-Roch formula, Serre duality, Hodge index theorem, for which we refer to [Rei] or [BPV]. The proof can also be simplified by using Miyaoka's semipositivity theorem (chapter 7), whose proof however is far beyond the almost straigthforward arguments of this section.

Recall [BPV, page 191] that any rational surface is either $\mathbf{C}P^2$ or a $\mathbf{C}P^1$-bundle over $\mathbf{C}P^1$ (*Hirzebruch surface*) or a surface obtained by blowing-up an Hirzebruch surface. In the previous chapter we have already seen that $\mathbf{C}P^2$ cannot support a regular foliation. Concerning Hirzebruch surfaces we have:

Lemma 0. *Let X be an Hirzebruch surface and let $\mathcal{F}$ be a regular foliation on X. Then $\mathcal{F}$ is a $\mathbf{C}P^1$-fibration over $\mathbf{C}P^1$.*

Proof.

Recall that Hirzebruch surfaces form a contable family $\{X_k\}$, $k \in \mathbf{N}^+ \cup \{0\}$, with $X_0 = \mathbf{C}P^1 \times \mathbf{C}P^1$; for $k > 0$, X_k has a unique structure of $\mathbf{C}P^1$-bundle, and a unique section C with negative selfintersection, more precisely $C \cdot C = -k$. We only consider the case $k > 0$, leaving as an exercise for the reader the case $k = 0$. Then we have to prove that $\mathcal{F}$ coincides with the (unique) fibration.

We can choose as generators of $H^2(X_k, \mathbf{Q})$ the section C and a fibre F. Then we can write, at the cohomological level,

$$N_{\mathcal{F}} = aF + bC$$

with $a, b \in \mathbf{Q}$. The curve C is not $\mathcal{F}$-invariant, because its selfintersection is not 0, and hence by proposition 2 of chapter 2 we have

$$N_{\mathcal{F}} \cdot C = \chi(C) + tang(\mathcal{F}, C)$$

that is

$$a - kb = 2 + tang(\mathcal{F}, C) \geq 2.$$

If, by contradiction, $\mathcal{F}$ is not the $\mathbf{C}P^1$-bundle, then we can certainly choose the fibre F in such a way that it is not $\mathcal{F}$-invariant, and therefore, again by proposition 2 of chapter 2,

$$N_{\mathcal{F}} \cdot F = \chi(F) + tang(\mathcal{F}, F)$$

that is

$$b = 2 + tang(\mathcal{F}, F) \geq 2.$$

We thus obtain $a \geq 2 + kb \geq 2$. Let us now compute $N_{\mathcal{F}} \cdot N_{\mathcal{F}}$:

$$N_{\mathcal{F}} \cdot N_{\mathcal{F}} = 2ab - kb^2 = ab + b(a - kb) \geq 8.$$

This contradicts theorem 1, and proves the lemma. $\triangle$

Now we can state the general result.

Theorem 7 [Br1]. *Let X be a rational surface and let $\mathcal{F}$ be a regular foliation on X. Then X is an Hirzebruch surface and $\mathcal{F}$ is a $\mathbf{C}P^1$-fibration over $\mathbf{C}P^1$.*

Proof.

Recall that a line bundle L on an algebraic surface is called *pseudoeffective* if $L \cdot H \geq 0$ for every ample line bundle H; this is in fact equivalent to say that the class of L in $H^2(X, \mathbf{Q})$ is the limit of classes of positive $\mathbf{Q}$-divisors (see, e.g., [Fuj] and [Dem]).

Lemma 1. *The line bundle $T^*_{\mathcal{F}}$ is not pseudoeffective.*

Proof.

Suppose the contrary: $T^*_{\mathcal{F}} \cdot H \geq 0$ for every ample H. Let $C \subset X$ be any irreducible curve. If $C \cdot C \geq 0$ then $H + nC$ is an ample divisor for every positive n (Nakai-Moishezon criterion), thus $T^*_{\mathcal{F}} \cdot (H + nC) \geq 0$ and letting n going to $+\infty$ we obtain $T^*_{\mathcal{F}} \cdot C \geq 0$. If $C \cdot C < 0$ then C is certainly not $\mathcal{F}$-invariant and by proposition 2 of chapter 2 we find

$$T^*_{\mathcal{F}} \cdot C = -C \cdot C + tang(\mathcal{F}, C) > 0.$$

Hence $T^*_{\mathcal{F}}$ is not only pseudoeffective but also *nef*: $T^*_{\mathcal{F}} \cdot D \geq 0$ for every positive divisor D. Therefore, its selfintersection is nonnegative [Dem]:

$$T^*_{\mathcal{F}} \cdot T^*_{\mathcal{F}} \geq 0.$$

Being X a rational surface, we can certainly find a smooth rational curve R of positive selfintersection; a generic choice of such a curve is moreover not invariant by $\mathcal{F}$. Hence, again by proposition 2 of chapter 2, we have

$$N_{\mathcal{F}} \cdot R = \chi(R) + tang(\mathcal{F}, R) > 0$$

and also

$$T^*_{\mathcal{F}} \cdot R \geq 0$$

being $T^*_{\mathcal{F}}$ nef. Moreover, by Baum-Bott formula:

$$N_{\mathcal{F}} \cdot N_{\mathcal{F}} = 0.$$

From Hodge index theorem and these inequalities it now follows

$$N_{\mathcal{F}} \cdot T^*_{\mathcal{F}} \geq 0$$

but this contradicts proposition 1 of chapter 2: $N_{\mathcal{F}} \cdot T^*_{\mathcal{F}} = -c_2(X) < 0$.

(An alternative conclusion is the following: from $T^*_{\mathcal{F}} \cdot T^*_{\mathcal{F}} \geq 0$, $N_{\mathcal{F}} \cdot N_{\mathcal{F}} = 0$, and proposition 1 of chapter 2 one finds $c_1^2(X) \geq 2c_2(X)$, and this inequality implies that the rational surface X is in fact either $\mathbf{C}P^2$ or an Hirzebruch surface; but regular foliations on these surfaces have already been classified, and one verifies that their cotangent bundle is never nef). $\triangle$

In the following, which is the real heart of the proof, we shall use only the fact that $T^*_{\mathcal{F}}$ is not effective; we have however stated lemma 1 in a stroger form because of its relation with Miyaoka's semipositivity theorem (chapter 7).

Lemma 2. *The line bundle $N_{\mathcal{F}}$ is effective:*

$$h^0(X, N_{\mathcal{F}}) > 0.$$

Proof.

By Riemann-Roch formula, Serre duality, and $N_{\mathcal{F}} \cdot N_{\mathcal{F}} = 0$, $N_{\mathcal{F}} \cdot K_X = -c_2(X)$, we find

$$h^0(X, N_{\mathcal{F}}) + h^0(X, N^*_{\mathcal{F}} \otimes K_X) \geq \chi(\mathcal{O}_X) + \frac{1}{2}c_2(X) > 0$$

($\chi(\mathcal{O}_X) = 1$, $c_2(X) > 0$ because X is rational). Hence it is sufficient to prove that $h^0(X, N^*_{\mathcal{F}} \otimes K_X) = 0$.

As in the previous lemma, let $R \subset X$ be a smooth rational curve of positive selfintersection and not $\mathcal{F}$-invariant, so that $N_{\mathcal{F}} \cdot R > 0$. By adjunction formula, we also have $K_X \cdot R = -R \cdot R - \chi(R) < 0$. Hence $N^*_{\mathcal{F}} \otimes K_X$ has negative degree on R, and thus it cannot be effective being $R \cdot R > 0$. $\triangle$

Let now $\sum_{j=1}^k a_j D_j$ be an effective divisor representing $N_{\mathcal{F}}$ (D_j irreducible curves, a_j positive integers). Remark that it is not trivial, for $N_{\mathcal{F}} \cdot K_X \neq 0$ (for instance). Let us prove that each D_j is a smooth rational curve. For every j we have an exact sequence

$$0 \to \mathcal{O}_X(-D_j) \to \mathcal{O}_X \to \mathcal{O}_{D_j} \to 0$$

and from $h^1(X, \mathcal{O}_X) = h^2(X, \mathcal{O}_X) = 0$ (rationality of X) and Serre duality we deduce

$$h^1(D_j, \mathcal{O}_{D_j}) = h^0(X, K_X \otimes \mathcal{O}_X(D_j)).$$

However, $K_X \otimes \mathcal{O}_X(D_j) = T^*_{\mathcal{F}} \otimes N^*_{\mathcal{F}} \otimes \mathcal{O}_X(D_j)$ has no global nontrivial holomorphic sections: $N_{\mathcal{F}} \otimes \mathcal{O}_X(-D_j)$ is still effective, and so a nontrivial section of $K_X \otimes \mathcal{O}_X(D_j)$ would give a nontrivial section of $T^*_{\mathcal{F}}$, against lemma 1. Hence $h^1(D_j, \mathcal{O}_{D_j}) = 0$, which precisely means that D_j is a smooth rational curve.

If D_j is not $\mathcal{F}$-invariant then

$$N_{\mathcal{F}} \cdot D_j = \chi(D_j) + tang(\mathcal{F}, D_j) > 0$$

and if it is $\mathcal{F}$-invariant then

$$N_{\mathcal{F}} \cdot D_j = D_j^2 = 0.$$

From $N_{\mathcal{F}} \cdot N_{\mathcal{F}} = 0$ it now follows that each D_j is $\mathcal{F}$-invariant, and $D_j \cdot D_i = 0$ for every i, j. A smooth rational curve of zero selfintersection on a compact surface is always a regular fibre of a rational fibration [BPV, page 142], hence our $\{D_j\}$ are regular fibres of the same rational fibration of X. Any other regular fibre F of such a fibration is also $\mathcal{F}$-invariant, because $N_{\mathcal{F}} \cdot F = N_{\mathcal{F}} \cdot D_j = 0$. Thus $\mathcal{F}$ coincides with the rational fibration, and the theorem is proved. △

In this chapter we study two classes of ubiquitous foliations: Riccati foliations and turbulent foliations. A section will also be devoted to a very special foliation, which will play an important role in the minimal model theory.

1. Riccati foliations

A foliation $\mathcal{F}$ on a compact connected surface X is called *Riccati foliation* if there exists a rational fibration $\pi : X \to B$ (possibly with singular fibres) whose generic fibre is transverse to $\mathcal{F}$. We shall say that π is adapted to $\mathcal{F}$, or that $\mathcal{F}$ is Riccati with respect to π. It may happen that a Riccati foliation has more than one adapted fibration: for example, the foliation on $\mathbf{C}P^1 \times \mathbf{C}P^1$ defined by the vector field $z\frac{\partial}{\partial z} + \lambda w \frac{\partial}{\partial w}$, $\lambda \neq 0$, is Riccati w.r. to both the horizontal and the vertical fibration.

A simple but useful characterization of Riccati foliations is the following one. Let $\pi : X \to B$ be a rational fibration on X (compact, connected) and let F be any regular fibre of π: it is a smooth rational curve of zero selfintersection. Let $\mathcal{F}$ be a foliation on X, with tangent bundle $T_{\mathcal{F}}$ and normal bundle $N_{\mathcal{F}}$. Then $\mathcal{F}$ is Riccati w.r. to π if and only if

$$T_{\mathcal{F}} \cdot F = 0$$

or, equivalently by adjunction,

$$N_{\mathcal{F}} \cdot F = 2.$$

In fact, if $\mathcal{F}$ is Riccati w.r. to π then we choose as F a fibre transverse to $\mathcal{F}$, so that $T_{\mathcal{F}}|_F$ is the normal bundle of F and so $T_{\mathcal{F}} \cdot F = F \cdot F = 0$. The same equality holds for *any* regular fibre of π, because all regular fibres are in the same cohomology class. Conversely, if $T_{\mathcal{F}} \cdot F = 0$ then it follows from propositions 2 and 3 of chapter 2 that either F is not $\mathcal{F}$-invariant and $tang(\mathcal{F}, F) = 0$, i.e. F is transverse to $\mathcal{F}$, or F is $\mathcal{F}$-invariant and $Z(\mathcal{F}, F) = 2$. However, this second possibility can hold only for a finite number of regular fibres of π, because $Z(\mathcal{F}, F) > 0$ implies $Sing(\mathcal{F}) \cap F \neq \emptyset$ and $Sing(\mathcal{F})$ is finite. Hence $\mathcal{F}$ is Riccati with respect to π.

In a similar way, we see that $\mathcal{F}$ coincides with the rational fibration π if and only if

$$T_{\mathcal{F}} \cdot F = 2$$

or equivalently

$$N_{\mathcal{F}} \cdot F = 0.$$

In fact, $T_{\mathcal{F}} \cdot F = 2$ implies that every regular fibre is $\mathcal{F}$-invariant, otherwise we would obtain the absurdity $tang(\mathcal{F}, F) = -2 < 0$. By the same argument, we also see that the intermediate case $T_{\mathcal{F}} \cdot F = 1$ never happens.

Recall also that if F is a smooth rational curve of zero selfintersection on a compact connected surface X then F is a regular fibre of a rational fibration on X [BPV, page 142]. We can resume these considerations in the next proposition, for future reference.

Proposition 1. *Let $\mathcal{F}$ be a foliation on a compact connected surface X and let $F \subset X$ be a smooth rational curve of zero selfintersection and invariant by $\mathcal{F}$. Then:*

i) if $Z(\mathcal{F}, F) = 0$ then $\mathcal{F}$ is a rational fibration;

ii) if $Z(\mathcal{F}, F) = 2$ then $\mathcal{F}$ is a Riccati foliation;

iii) $Z(\mathcal{F}, F)$ cannot be equal to 1. $\triangle$

On a neighbourhood of each transverse fibre a Riccati foliation $\mathcal{F}$ defines, through its leaves, a local trivialization of the adapted fibration $\pi : X \to B$, and so a canonical identification between fibres close to that one. Glueing together these local identifications, we obtain a group representation called *monodromy* (or *global holonomy*) of $\mathcal{F}$:

$$\rho : \pi_1(B^*, b) \to Aut(\pi^{-1}(b))$$

where $B^* \subset B$ is the set of fibres transverse to $\mathcal{F}$, b is a point of B^*, $\pi_1(B^*, b)$ the fundamental group, and $Aut(\pi^{-1}(b))$ the group of holomorphic automorphisms of the fibre $\pi^{-1}(b) \simeq \mathbf{C}P^1$. In foliated terminology, one says that $\mathcal{F}|_{X^*}$, $X^* = \pi^{-1}(B^*)$, is the *suspension* of the representation ρ. The dynamics of $\mathcal{F}|_{X^*}$ is quite well represented by the dynamics of ρ, or more precisely of $Im\rho \subset Aut(\mathbf{C}P^1)$. For instance, dense leaves of $\mathcal{F}|_{X^*}$ correspond to dense orbits of ρ, transversely invariant measures for $\mathcal{F}|_{X^*}$ correspond to invariant measures for ρ, etc.. See for instance [G-M], and the references therein, on this aspect of Riccati foliations. When $B^* = B$, i.e. each fibre is transverse to the foliation, the monodromy ρ and the analytic type of the base B completely characterize the foliation, up to biholomorphism. However in that case the existence of a Riccati foliation is a subtle problem, which is related to stability properties of the rank 2 vector bundle on B whose projectivisation gives X (see [Fri], chapters 4 and 5).

Let us now analyze in some detail the fibres which are not $\mathcal{F}$-transverse. We can look at those fibres in two ways: we can suppose that every fibre of π is regular, up to blowing-down [BPV, page 142], or we can suppose that the singularities of $\mathcal{F}$ are reduced, up to blowing-up. Let us firstly take the former point of view.

If F is a regular fibre of π which is not $\mathcal{F}$-transverse then F is $\mathcal{F}$-invariant and $Z(\mathcal{F}, F) = 2$. Hence $\mathcal{F}$ has on F one or two singularities, and in the latter case each one of the two singularities is generated by a vector field whose linear part has at least one nonzero eigenvalue, the one along F. The monodromy of $\mathcal{F}$ around F gives informations on those singularities (or that singularity): for instance, suppose that $p \in Sing(\mathcal{F}) \cap F$ has a smooth separatrix transverse to F, then the monodromy $f \in Aut(\mathbf{C}P^1)$ of $\mathcal{F}$ around F has a fixed point corresponding to that separatrix, and the holonomy of that separatrix is the germ of f at that fixed point. Be careful, however, that the monodromy around F may be far from characterizing the structure of $\mathcal{F}$ on a neighbourhood of F (see below).

We can also write down some explicit formulae. On a neighbourhood of F we can choose π-trivializing coordinates $(z, w) \in \mathbf{D} \times \mathbf{C}P^1$, so that $\mathcal{F}$ will be generated by a meromorphic 1-form ω on $\mathbf{D} \times \mathbf{C}P^1$, rational in w. Because $N_{\mathcal{F}} \cdot F = 2$, we can even choose that 1-form with empty zero divisor and with polar divisor equal to two times the disc at infinity, and so we find

$$\omega = (a(z)w^2 + b(z)w + c(z))dz + d(z)dw$$

with a, b, c, d holomorphic functions on $\mathbf{D}$. Hence the leaves of $\mathcal{F}$ can be obtained by integrating (when possible!) a Riccati differential equation (whence the name "Riccati foliation"). The $\mathcal{F}$-invariance of $F = \{z = 0\}$ translates into $d(0) = 0$; then $Sing(\mathcal{F}) \cap F$ are obtained by solving the quadratic equation $a(0)w^2 + b(0)w + c(0) = 0$ (and taking into account the solution $w = \infty$ when $a(0) = 0$).

Let us firstly consider the *nondegenerate* case $d'(0) \neq 0$. If $b(0)^2 - 4a(0)c(0) \neq 0$ then F contains 2 singularities of $\mathcal{F}$, generated by vector fields whose linear parts have eigenvalues $(1, \lambda)$ and $(1, -\lambda)$ ($\pm\lambda$ along F), $\lambda \neq 0$. If $\lambda \notin \mathbf{Z}$, then both singularities have a separatrix transverse to F (see chapter 1: note that in the Poincaré-Dulac case there is only one separatrix, which is however tangent to the eigenspace whose eigenvalue has bigger modulus, hence that case does not appear in our context when $\lambda \in \frac{1}{\mathbf{Z}}$, $\lambda \neq \pm 1$, because F is already a separatrix tangent to the $\pm\lambda$-eigenspace and $|\lambda| < 1$). Up to a change of coordinates we may suppose that these two separatrices are $\{w = 0\}$ and $\{w = \infty\}$, i.e. the functions a and c are identically zero (and hence $b(0) \neq 0$). Then an additional change of the z coordinate reduces ω to the linear normal form

$$\omega = \lambda w dz - z dw \qquad (z, w) \in \mathbf{D} \times \mathbf{C}P^1.$$

The monodromy around F is $f(w) = \exp(2\pi i\lambda)w$. There are two fixed points, corresponding to the two separatrices above. When $\lambda \in \mathbf{Q}$ then f is periodic, and its periodic (non fixed) points correspond to the additional separatrices of the dicritical singularity of $\mathcal{F}$ on F.

If $\lambda \in \mathbf{Z}$ then there may be a singularity of $\mathcal{F}$ on F of Poincaré-Dulac type and without a separatrix transverse to F; but the other singularity certainly has such a separatrix, which can be put in $\{w = \infty\}$. In this way we obtain $a \equiv 0$. A change of coordinates (of the type $w \mapsto w + \beta(z)$, $z \mapsto \alpha(z)$) reduces $\mathcal{F}$ to the Poincaré-Dulac normal form

$$\omega = (nw + z^n)dz - zdw \qquad (z, w) \in \mathbf{D} \times \mathbf{C}P^1$$

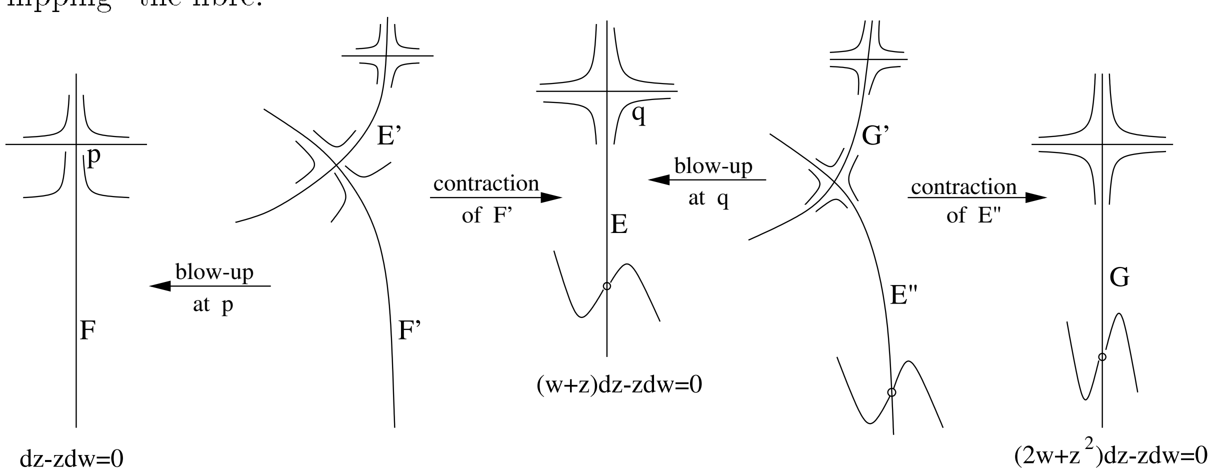
or still to the linear one $\omega = nwdz - zdw$. In the latter case the monodromy f around F is the identity. In the former case it is a parabolic automorphism (conjugate to $f(w) = w+1$).

If $b(0)^2 - 4a(0)c(0) = 0$ (but still $d'(0) \neq 0$) then F contains only one singularity of $\mathcal{F}$, which is a saddle-node of multiplicity 2, with strong separatrix transverse to F and weak separatrix in F. Now a change of coordinates reduces $\mathcal{F}$ to

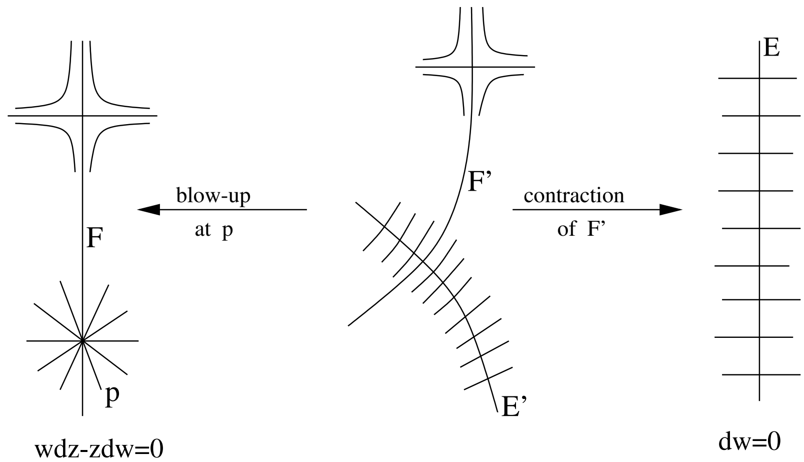
$$\omega = dz - zdw \qquad (z, w) \in \mathbf{D} \times \mathbf{C}P^1$$

(the saddle-node is at $(0, \infty)$). As in the Poincaré-Dulac case, the monodromy around F is parabolic.

We see from this case-by-case analysis that, when $d'(0) \neq 0$, the monodromy f around the fibre gives an almost complete description of the foliation on a neighbourhood of the fibre. More precisely, that monodromy gives a complete description of $\mathcal{F}$ modulo bimeromorphic isomorphism. For instance, when f is parabolic then $\mathcal{F}$ is either $(nw + z^n)dz - zdw = 0$ or $dz - zdw = 0$, but all these foliations are related each other by "flipping" the fibre:



Similarly, all foliations with trivial holonomy $nwdz - zdw = 0$ are bimeromorphic to $dw = 0$, i.e. to the foliation without invariant fibre:



This is a general fact, whatever is the monodromy. Remark that the flipping of the fibre does not change the monodromy around the fibre but it may change the holonomy of the fibre. For example, in the Poincaré-Dulac case that holonomy has linear part equal to $\exp\left(\frac{2\pi i}{n}\right)$ (but it is not linearizable), so it depends on n.

Let us now consider the *degenerate* case $d'(0) = 0$.

If $b(0)^2 - 4a(0)c(0) \neq 0$ then $Sing(\mathcal{F}) \cap F$ consists of two saddle-nodes of the same multiplicity, with strong separatrices contained in F. The situation is much more complicated than above, see [M-R] for a detailed study. The simplest case is when both saddle-nodes have a weak separatrix, which may be assumed to be $\{w = 0\}$ or $\{w = \infty\}$. In this way we obtain a generating 1-form $\omega = b(z)wdz + d(z)dw$. The meromorphic 1-form in one variable $-\frac{b(z)}{d(z)}dz$ is entirely determined, around 0, by the order of its pole and its residue [M-R], i.e. it can be transformed into $\frac{1+\nu z^k}{z^{k+1}}dz$ by a change of the z-coordinate, for some $k \in \mathbf{N}^+$ and $\nu \in \mathbf{C}$. It follows that the foliation around F is given by the Dulac normal form

$$\omega = w(1 + \nu z^k)dz - z^{k+1}dw \qquad (z, w) \in \mathbf{D} \times \mathbf{C}P^1.$$

The monodromy f is easily computed, because the leaves are the graphs of the multivalued functions $w = cz^\nu \exp\left(-\frac{k}{z^k}\right)$ and so $f(w) = \exp(2\pi i \nu)w$. By flipping the fibre, one finds also that ν can be changed to $\nu + n$, $n \in \mathbf{Z}$, so that the monodromy and the multiplicity of the fibre give a complete description of $\mathcal{F}$ up to bimeromorphism.

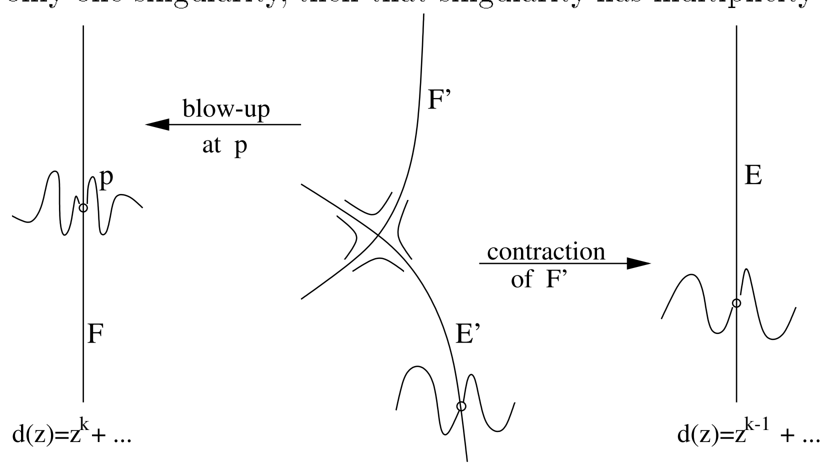
However, in the generic case the two saddle-nodes have no weak separatrix. The most classical example is Euler's equation

$$\omega = (w + z)dz - z^2 dw.$$

Again by a direct integration, one finds that the monodromy is a parabolic automorphism f, which therefore has only one fixed point; this fixed points is at $w = \infty$ and corresponds to the weak separatrix $\{w = \infty\}$ of $(0, \infty)$, whereas $(0, 0)$ has no weak separatrix. The reader may find in [M-R] examples where both saddle-nodes have no weak separatrix, with any prescribed monodromy.

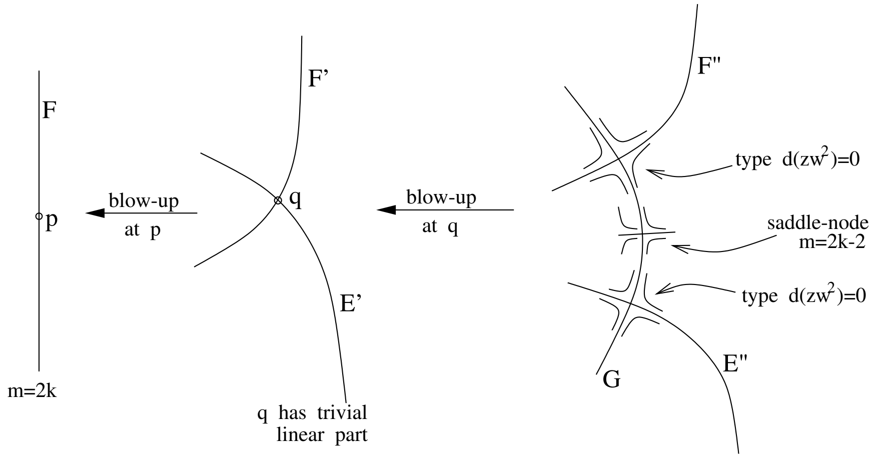
If $d'(0) = 0$ and $b(0)^2 - 4a(0)c(0) = 0$ then $\mathcal{F}$ has on F only one singularity p, with $Z(\mathcal{F}, F, p) = 2$. The singularity is generated by a vector field whose linear part is either trivial or nontrivial but nilpotent (if $c'(0) \neq 0$).

If the linear part is trivial, the blowing-up of the singularity has on $F' \cap E'$ (E' is the exceptional divisor, F' is the strict transform of F) a singularity of the type $d(zw) = 0$, and no more singularities on F'. If we collapse F', we obtain a new fibre E, invariant by a new Riccati foliation $\mathcal{F}'$, but now the multiplicity of that fibre (i.e. the vanishing order of $d(z)$ at 0) is strictly smaller than the previous one. On E there may be 1 or 2 singularities, but anyway the situation is less degenerate than before. All of this can be verified by a direct computation, or proved by using the usual formulae for the variation of the multiplicities and orders under blowing-ups. Note that if F is an invariant fibre of multiplicity $k \geq 2$ and containing only one singularity, then that singularity has multiplicity $2k$ and order 2.



This procedure can be iterated, if E contains only one singularity with trivial linear part, and hence we finally obtain either a nondegenerate fibre, or a degenerate fibre with two saddle-nodes, or a degenerate fibre with only one singularity with nontrivial nilpotent linear part.

Concerning this last case, its resolution was already described in the proof of Seidenberg's theorem in chapter 1. Here the situation is more special, and a direct computation shows that only two blowing-ups are sufficient to reduce the singularity. The situation is described in the following diagram:



Remark the consistency with Camacho-Sad formula ($G^2 = -1, (E'')^2 = (F'')^2 = -2$).

This completes the study of invariant fibres. Let us resume the essential features of this analysis.

Proposition 2. *Let $\mathcal{F}$ be a Riccati foliation on a compact connected surface X, with adapted fibration $\pi : X \to B$. Then there exists a birational map $f : X \dashrightarrow X'$ such that:*

i) f is biregular on $X^* = \pi^{-1}(B^*)$, where B^* is the set of fibres of π transverse to $\mathcal{F}$; in particular, the transform $\mathcal{F}'$ of $\mathcal{F}$ by f is still Riccati, with adapted fibration $\pi' = \pi \circ f^{-1} : X' \to B$;

ii) π' has no singular fibre;

iii) each $\mathcal{F}'$-invariant fibre of π' belongs to one of the following classes:

iii.1) nondegenerate *fibre: around the fibre, the foliation has equation*

$$\lambda w dz - z dw = 0 \qquad (z, w) \in \mathbf{D} \times \mathbf{C}P^1, \quad \lambda \notin \mathbf{Z}$$

or

$$dz - z dw = 0 \qquad (z, w) \in \mathbf{D} \times \mathbf{C}P^1.$$

iii.2) semidegenerate *fibre: the fibre contains two saddle-nodes, of the same multiplicity, whose strong separatrices are contained in the fibre;*

iii.3) nilpotent *fibre: the fibre contains only one singularity, generated by a vector field with nilpotent and nontrivial linear part, whose resolution has been described above.* △

Note that in *iii.1*, when $\lambda \in \mathbf{Q} \setminus \mathbf{Z}$ there is a dicritical singularity, whose resolution is very simple (see chapter 1). We shall say that $\mathcal{F}'$ is in *standard form* (this is not unique, except in few cases). Its minimal resolution $\mathcal{F}''$ will be called *reduced standard form.*

Let us explain how to compute the cotangent bundle $T^*_{\mathcal{F}}$ of a Riccati foliation, firstly when it is in standard form, with adapted fibration $\pi : X \to B$.

We are in a situation close to that concerning the computation of the conormal bundle of a fibration (chapter 2). If η is a local holomorphic 1-form on B, then the restriction of $\pi^*(\eta)$ to $\mathcal{F}$ (more precisely, the contraction of $\pi^*(\eta)$ with local vector fields defining $\mathcal{F}$) is a section of $T^*_{\mathcal{F}}$, which is however vanishing on tangency curves between $\mathcal{F}$ and the fibration, i.e. on $\mathcal{F}$-invariant fibres. If F is an invariant fibre of multiplicity k, then in local coordinates $\mathcal{F}$ is given by $z^k dw + A(z,w)dz = 0$, π is $(z,w) \mapsto z$, and η is something like dz, so that $\pi^*(\eta)|_{\mathcal{F}} = z^k$ vanishes on F at order k. Therefore, if $F_1, ..., F_n$ are the $\mathcal{F}$-invariant fibres, of multiplicity $k_1, ..., k_n$, we have

$$T^*_{\mathcal{F}} = \pi^*(K_B) \otimes \mathcal{O}_X(\sum_{j=1}^{n} k_j F_j).$$

Remark that $h^0(X, T^*_{\mathcal{F}}) > 0$ if B is not rational, or if B is rational and $\sum k_j \geq 2$ (which is equivalent to $\sum k_j \geq 1$, for holonomic reasons).

Let now $\tilde{\mathcal{F}}$ be the minimal resolution of $\mathcal{F}$, so that it is in reduced standard form. The relation between the cotangent bundles of $\mathcal{F}$ and $\tilde{\mathcal{F}}$ was already discussed in chapter 2. If $F'' \cup G \cup E''$ is the fibre arising from the resolution r of a nilpotent fibre F (see diagram above) then $T^*_{\tilde{\mathcal{F}}} = r^*(T^*_{\mathcal{F}}) \otimes \mathcal{O}_{\tilde{X}}(-G)$; on the other hand $r^*(\mathcal{O}_X(F)) = \mathcal{O}_{\tilde{X}}(F'' + 2G + E'')$, and so, denoting by k the multiplicity of F, we have around the fibre

$$T^*_{\tilde{\mathcal{F}}} = \tilde{\pi}^*(K_B) \otimes \mathcal{O}_{\tilde{X}}(kF'' + (2k-1)G + kE'').$$

If $\tilde{F}_0 \cup ... \cup \tilde{F}_l$ is the fibre arising from the resolution r of a nondegenerate fibre F with $\lambda \in \mathbf{Q} \setminus \mathbf{Z}$, then $T^*_{\tilde{\mathcal{F}}} = r^*(T^*_{\mathcal{F}}) \otimes \mathcal{O}_{\tilde{X}}(-\tilde{F}_m)$, where $\tilde{F}_m$ is the unique component of the chain which is not invariant by $\tilde{\mathcal{F}}$ (see chapter 1). Here $r^*(\mathcal{O}_X(F))$ is more difficult to compute, and it depends on the arithmetic of λ. However, by flipping the fibre F we may assume that $\lambda \in \frac{1}{\mathbf{Z}}$, more precisely $\lambda = \frac{1}{n}$ where n $(= l)$ is the period of the monodromy around F. In this case, in the chain $\tilde{F}_0 \cup ... \cup \tilde{F}_l$ the first component $\tilde{F}_0$ is the strict transform of F, the second one $\tilde{F}_1$ is transverse to $\tilde{\mathcal{F}}$, and $r^*(\mathcal{O}_X(F)) = \mathcal{O}_{\tilde{X}}(\tilde{F}_0 + l\tilde{F}_1 + (l-1)\tilde{F}_2 + ... + 2\tilde{F}_{l-1} + \tilde{F}_l)$. Hence, around the fibre, we have

$$T^*_{\tilde{\mathcal{F}}} = \tilde{\pi}^*(K_B) \otimes \mathcal{O}_{\tilde{X}}(\tilde{F}_0 + (l-1)\tilde{F}_1 + \sum_{j=2}^{l} (l+1-j)\tilde{F}_j).$$

2. A very special foliation

Let T be an automorphism of $\mathbf{C}P^2$ which cyclically permutes 3 noncollinear points p_1, p_2, p_3. In suitable projective coordinates $(s:t:u)$, T is expressed by $(s:t:u) \mapsto (u:s:t)$.

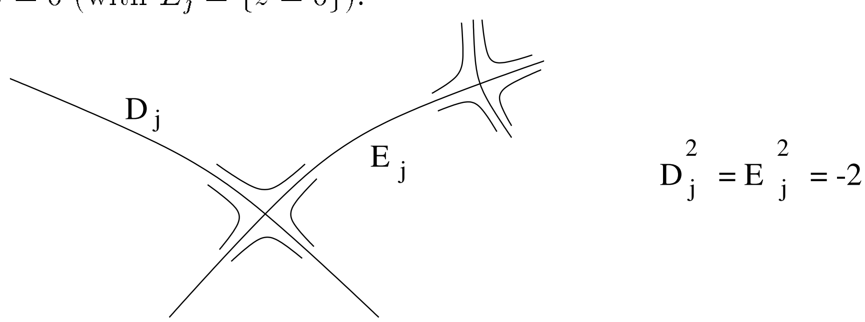
We see that $T^3 = id$, and that T has 3 fixed points, located at $(1 : \beta : \beta^2)$ with β a cubic root of 1. Set $Fix(T) = \{q_1, q_2, q_3\}$.

Let us look for a foliation on $\mathbf{C}P^2$ which is invariant by T. For instance, we may try with a foliation $\mathcal{L}$ generated by a linear vector field with zeroes in correspondence of p_1, p_2, p_3, that is $v = \lambda_1 z \frac{\partial}{\partial z} + \lambda_2 w \frac{\partial}{\partial w}$ (with $z = \frac{s}{u}$, $w = \frac{t}{u}$). This vector field transforms under T to $\hat{v} = -\lambda_2 z \frac{\partial}{\partial z} - (\lambda_2 - \lambda_1) w \frac{\partial}{\partial w}$. We see that $\hat{v}$ never equals v (except in the trivial case $v \equiv 0$); but the T-invariance of $\mathcal{L}$ is equivalent only to the weaker condition $\hat{v} \wedge v \equiv 0$, that is $\lambda_1^2 - \lambda_1\lambda_2 + \lambda_2^2 = 0$, and so we find that the only linear T-invariant foliations are those generated by

$$z\frac{\partial}{\partial z} + \frac{1}{2}(1 \pm i\sqrt{3})w\frac{\partial}{\partial w}.$$

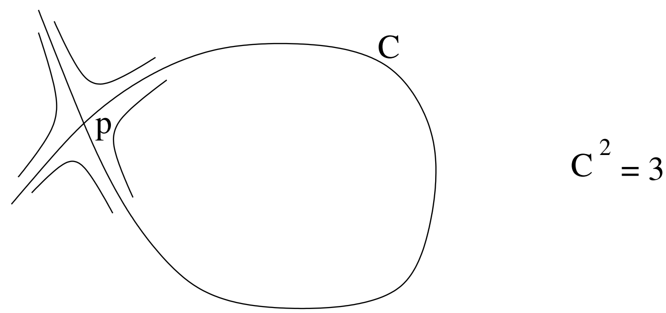
These two foliations are conjugate by the involution $(z, w) \mapsto (w, z)$, which also conjugates T to $T^2 = T^{-1}$. Hence we may say that, up to projective automorphisms, there exists only one pair $(\mathcal{L}, G)$ where $\mathcal{L}$ is a linear foliation on $\mathbf{C}P^2$ and G is a group of automorphisms of $\mathbf{C}P^2$ which preserves $\mathcal{L}$ and which is generated by an automorphism which permutes the three singularities of $\mathcal{L}$.

Consider now the quotient $Y_0 = \mathbf{C}P^2/T$. Because $\mathcal{L}$ is invariant by T, we naturally have on Y_0 a "quotient foliation" $\mathcal{H}_0 = \mathcal{L}/T$. However, Y_0 is a *singular* surface: it has 3 singular points $\hat{q}_1$, $\hat{q}_2$, $\hat{q}_3$ in correspondence with the 3 fixed points q_1, q_2, q_3 of T. Each one of these singular points $\hat{q}_j$ is of type $A_{3,2}$ (cfr. [BPV, page 84] for the terminology) and its minimal resolution consists of two (-2)-curves D_j, E_j intersecting at one point. Call Y the minimal resolution of Y_0, and $\mathcal{H}$ the lift of $\mathcal{H}_0$ on Y. The structure of $\mathcal{H}$ around the curves D_j, E_j is easily understood, because the original $\mathcal{L}$ is not singular at q_j: D_j and E_j are both $\mathcal{H}$-invariant, $E_j \cap D_j$ is a singular point of $\mathcal{H}$ of the type $d(z^2w) = 0$ (with $E_j = \{z = 0\}$, $D_j = \{w = 0\}$), and E_j contains an additional singular point of $\mathcal{H}$ of the type $d(z^2w^3) = 0$ (with $E_j = \{z = 0\}$).



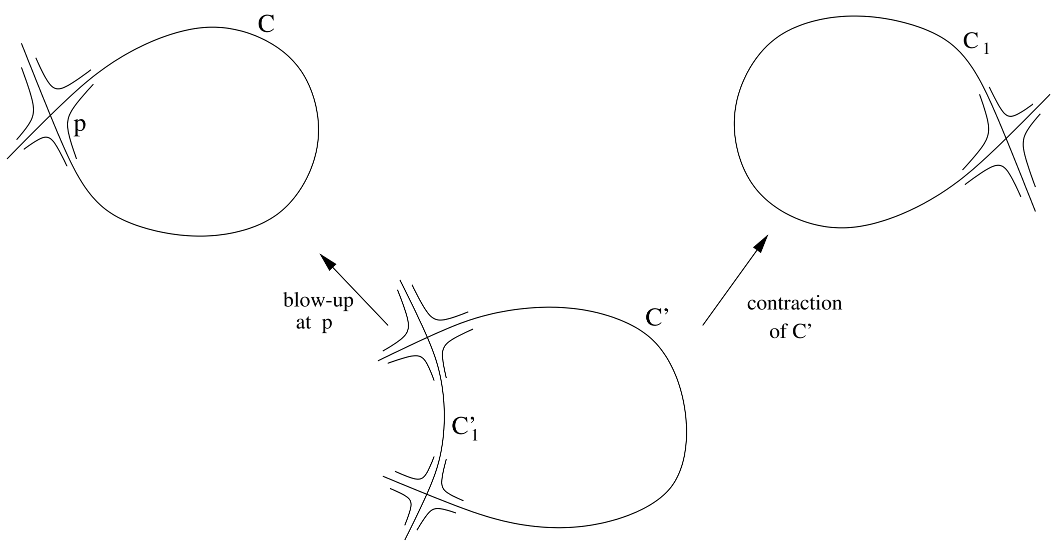
Beside the six curves $\{D_j, E_j\}_{j=1}^3$, the foliation $\mathcal{H}$ has another invariant curve, which arises from the quotient of the three lines of $\mathbf{C}P^2$ through p_1, p_2, p_3, invariant by $\mathcal{L}$. These three lines are cyclically permuted by T, and so their quotient on Y_0 is a rational curve with a node. This curve is disjoint from the singularities of Y_0, and its lift to Y will be noted C.

It is a rational curve with a node p, with $C \cdot C = 3$, invariant by $\mathcal{H}$. The singularity of $\mathcal{H}$ at p is of the same type as the singularities of $\mathcal{L}$ at p_j: a reduced nondegenerate singularity with eigenvalue $\lambda = \frac{1}{2}(1 + i\sqrt{3})$.



There are no more $\mathcal{H}$-invariant curves, for the three lines above are the unique curves invariant by $\mathcal{L}$ (for λ is not rational). And there are no more singularities beside the ones on C, E_j, D_j.

Remark that if we blow-up $p \in C$, the strict transform C' of C becomes a (-1)-curve, intersecting the exceptional divisor at two points; then we can contract C', and obtain a foliation $\mathcal{H}'$ which is in fact biholomorphic to $\mathcal{H}$:



Hence, $\mathcal{H}$ possesses some "nontrivial" birational automorphism, and it is exactly for this reason that we are interested in it: see the next chapter.

The following proposition provides a characterization of $\mathcal{H}$ in terms of the curve C; it can be compared with proposition 1. We give a proof which is intermediate between the original one [Br4] and results which will be developed in the subsequent chapters.

Proposition 3. *Let $\mathcal{F}$ be a foliation on a compact connected surface X and let $C \subset X$ be a rational curve with a node p, invariant by $\mathcal{F}$, and with $C \cdot C = 3$. Suppose that p is a reduced nondegenerate singularity of $\mathcal{F}$, and that it is the unique singularity of $\mathcal{F}$ on C. Then $\mathcal{F}$ is birational to $\mathcal{H}$, i.e. there exists a birational map from X to Y sending $\mathcal{F}$ to $\mathcal{H}$.*

Proof.

The first and main step consists in the construction of a ramified covering $Z \to X$, regular and of order 3 over C. This was done in [Br4] by analysing deformations of C, or more precisely of $\pi^{-1}(C)$ where π is the blowing-up at p. Here we give a different proof, which uses the presence of the foliation in a more substantial way and which is based on the following lemma.

Lemma 1. *There exists a neighbourhood U of C such that the line bundle $L = N^*_{\mathcal{F}} \otimes \mathcal{O}_X(C)$ has order 3 on U (i.e. $L^{\otimes 3}|_U \simeq \mathcal{O}_U$).*

Proof.

It is an application of a technique introduced in [CLS], which exploits the existence of almost-canonical coordinates in presence of nontrivial holonomy. Remark, first of all, that the eigenvalue of $\mathcal{F}$ at p can be calculated by Camacho-Sad formula:

$$3 = C \cdot C = CS(\mathcal{F}, C, p) = \lambda + 2 + \frac{1}{\lambda}$$

and so $\lambda = \frac{1 \pm i\sqrt{3}}{2}$. Note that it is nonreal.

Recall that [M-M] [CS2] given a point $q \in C \setminus \{p\}$ and a transversal T to $\mathcal{F}$ at q we have a corresponding *holonomy group* of $\mathcal{F}$, $Hol_{\mathcal{F}} \subset Aut(T, q)$. In our particular case, this group is very simple: it is infinite cyclic, and generated by an hyperbolic diffeomorphism, with linear part $\exp(2\pi i \lambda)$. Therefore, we may choose on T a $Hol_{\mathcal{F}}$-linearizing coordinate z, $z(q) = 0$. This coordinate can be extended to a full neighbourhood of q in X, constantly on the local leaves of $\mathcal{F}$. Then the logarithmic 1-form $\eta_q = \frac{dz}{z}$ defines $\mathcal{F}$, is closed, and $\eta_q|_T$ is $Hol_{\mathcal{F}}$-invariant.

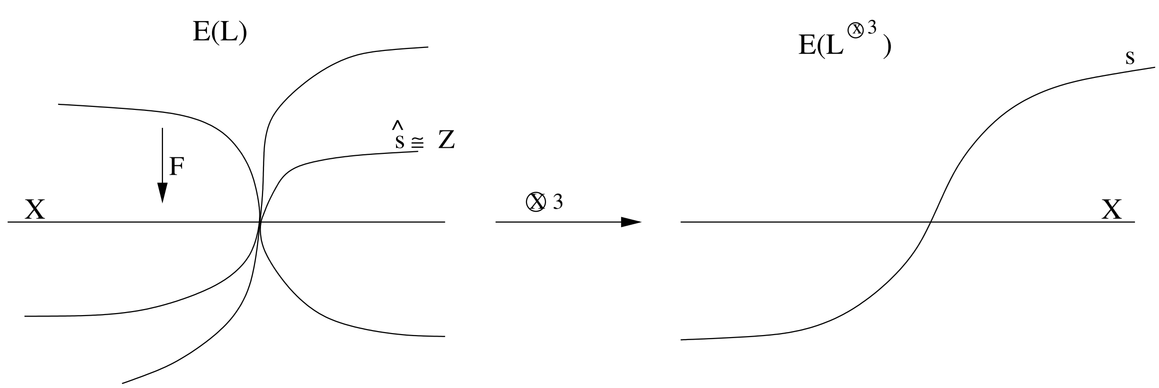
On a neighbourhood of p, by Poincaré linearization theorem $\mathcal{F}$ can still be defined by a closed logarithmic 1-form $\eta_p = \frac{dz}{z} - \lambda \frac{dw}{w}$. Clearly, if q is close to p and T is a transversal to $\mathcal{F}$ at q then $\eta_p|_T$ is $Hol_{\mathcal{F}}$-invariant.

In this way we cover a neighbourhood U of C by open sets U_j, such that in each U_j the foliation is defined by a logarithmic 1-form η_j, with poles on C, which is closed and $Hol_{\mathcal{F}}$-invariant (when restricted to transversals). On $U_i \cap U_j$ the 1-forms η_i and η_j differ only by a multiplicative constant: we have $\eta_i = f\eta_j$, $f \in \mathcal{O}^*$, the closedness of η_i and η_j implies that f is constant on the local leaves of $\mathcal{F}$, and moreover $f|_T$ is $Hol_{\mathcal{F}}$-invariant and hence constant because the holonomy is hyperbolic. We may treat $\{\eta_j\}$ as local nonvanishing sections of $N_{\mathcal{F}}^* \otimes \mathcal{O}_X(C) = L$. Then the previous property shows that $L|_U$ is definable by a locally constant cocycle. Hence, to prove that $L^{\otimes 3}|_U = \mathcal{O}_U$ is the same as to prove that $L^{\otimes 3}|_C = \mathcal{O}_C$. Now, taking the restriction of η_j to C, as a local section of L, is nothing but than taking its residue. Around $q \in C \setminus \{p\}$ we may choose η_q with any given residue, but around p we have a restriction on η_p: its residue on one separatrix is necessarily λ times the residue on the other separatrix. Because $\lambda^3 = 1$, it is now evident that $L^{\otimes 3}|_C = \mathcal{O}_C$ and therefore $L^{\otimes 3}|_U = \mathcal{O}_U$. $\triangle$

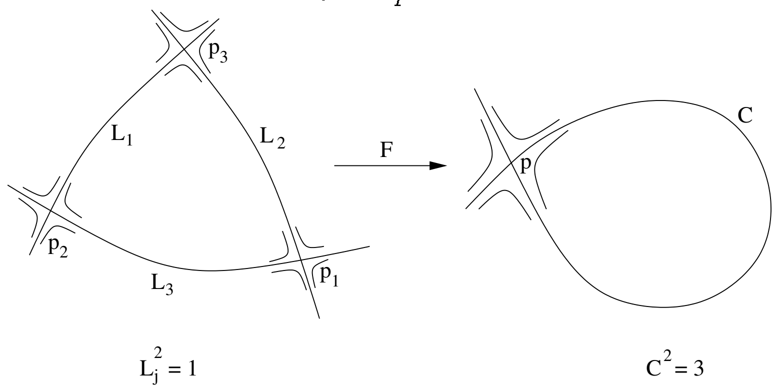
By this lemma, the line bundle $L^{\otimes 3}$ has a nontrivial section over U, which is moreover without zeroes. Because $C \cdot C = 3 > 0$, the open set $X \setminus C$ is strictly pseudoconvex and so that section can be extended to the full X as a global meromorphic section s of $L^{\otimes 3}$ ([Siu]; one can use also a more algebraic argument). This section defines a covering

$$F : Z \to X$$

by the following construction: we consider s as a subvariety (its "graph") in the compactified total space $E(L^{\otimes 3})$ of $L^{\otimes 3}$ (a $\mathbf{C}P^1$-bundle over X), and we take its preimage $\hat{s}$ by the map $E(L) \overset{\otimes 3}{\to} E(L^{\otimes 3})$ defined by the third tensor power. Then, up to desingularisation of $\hat{s}$ and elimination of the indeterminacies of its projection to X, we obtain Z and F (here we use the term "covering" in a somewhat improper way, because due to the desingularisation procedure F may contract some curve on Z). Remark that $Z \overset{F}{\to} X$ is regular and of order 3 over U. The deck transformations over U extend, by pseudoconvexity, to a birational transformations of Z.



We now lift $\mathcal{F}$ to Z via F, thus obtaining a foliation $\mathcal{G}$ which clearly leaves invariant three smooth rational curves L_1, L_2, L_3 forming a cycle over C. Moreover, from $C \cdot C = 3$ it follows $L_i \cdot L_i = 1$ for every i. The foliation $\mathcal{G}$ is singular over $L_1 \cup L_2 \cup L_3$ only at the crossing points $p_1 = L_2 \cap L_3$, $p_2 = L_1 \cap L_3$ and $p_3 = L_1 \cap L_2$, and the singularities of $\mathcal{G}$ at these points are the same as the one of $\mathcal{F}$ at p.



Remark that the three curves L_1, L_2, L_3 are linearly equivalent, i.e. they define the same line bundle on Z: this can be seen, for instance, by observing that if we blow-up p_1 then the strict transforms of L_2 and L_3 are smooth disjoint rational curves of zero selfintersection, and therefore they are regular fibres of the same rational fibration over $\mathbf{C}P^1$. Take now, for every $j = 1, 2, 3$, a section s_j of $\mathcal{O}_Z(L_j)$ vanishing on L_j: the three sections s_1, s_2, s_3 of the same line bundle define a rational map

$$(s_1 : s_2 : s_3) : Z \; - \; - \rightarrow \mathbf{C}P^2,$$

and one easily verifies that this map is birational, and moreover biregular on a neighbourhood of the cycle $L_1 \cup L_2 \cup L_3$. These three curves are mapped to three lines of $\mathbf{C}P^2$, and the foliation $\mathcal{G}$ is mapped to a foliation on $\mathbf{C}P^2$ which is clearly linear (because its tangent bundle has degree zero on the three lines) and equal to the foliation $\mathcal{L}$ introduced at the beginning of this section. The deck transformations of the covering $Z \rightarrow X$ are mapped to birational automorphisms of $\mathbf{C}P^2$ which are biregular around the three lines and hence everywhere; that is they are projective automorphisms of $\mathbf{C}P^2$. These automorphisms permute the three singularities of $\mathcal{L}$, and hence they coincide with the automorphisms T or T^2 introduced at the beginning of this section. Hence $\mathcal{F}$ is birational to $\mathcal{L}/T = \mathcal{H}_0$, that is to $\mathcal{H}$. $\triangle$

In the other direction, let us prove that $\mathcal{H}$ is not in the class of Riccati foliations. The proof that we give is not, perhaps, the shortest one, but it illustrates the use of dynamical considerations in the analysis of Riccati foliations.

Proposition 4. *$\mathcal{H}$ is not birational to a Riccati foliation.*

Proof.

Let us suppose the contrary, and let $\mathcal{F}$ be a Riccati foliation (on a surface X, with adapted fibration $\pi : X \rightarrow B$) birational to $\mathcal{H}$. We may suppose that $\mathcal{F}$ is in standard form (proposition 2). By the factorization theorem of birational maps [BPV, page 79] the birational map $\phi : X - - \to Y$ sending $\mathcal{F}$ to $\mathcal{H}$ can be factorized through a third surface Z which dominates both X and Y: $\phi = H \circ F^{-1}$, where $F : Z \to X$ and $H : Z \to Y$ are compositions of blowing-ups. Let $\mathcal{G} = F^*(\mathcal{F}) = H^*(\mathcal{H})$.

Looking at $\mathcal{G}$ as a blowing-up of $\mathcal{H}$, we see that $\mathcal{G}$ has no saddle-node singularities. Looking at $\mathcal{G}$ as a blowing-up of $\mathcal{F}$, this means that all invariant fibres of $\mathcal{F}$ are of non-degenerate type. More precisely, around each invariant fibre $\mathcal{F}$ is given by the 1-form $\mu w dz - z dw$ ($w \in \mathbf{C}P^2$, $z \in \mathbf{D}$) with either $\mu \notin \mathbf{R}$ or $\mu \in \mathbf{Q} \setminus \mathbf{Z}$. In the former case the monodromy of $\mathcal{F}$ around the fibre is an hyperbolic automorphism of $\mathbf{C}P^1$, in the latter case that monodromy is periodic. Because $\mathcal{H}$ has an hyperbolic singularity, certainly there exists at least one invariant fibre with $\mu \notin \mathbf{R}$. On the other hand, there cannot be two such fibres: otherwise, their preimages by F in Z would be two disjoint trees of rational $\mathcal{G}$-invariant curves R_1 and R_2, both projecting onto C by H, contradicting the birationality of $\mathcal{H}$ (note that R_1 and R_2 cannot be entirely contracted by H, because they come from rational curves of zero selfintersection, and so $H(R_1)$ and $H(R_2)$ can only be equal to the full C).

Hence, we exactly have one invariant fibre F_0 with hyperbolic monodromy ϕ_0, and several invariant fibres $F_1, ..., F_m$ with periodic monodromies $\phi_1, ..., \phi_m$. Note that the base B is rational, because X has to be a rational surface (as Y is); therefore the monodromy around F_0 is the product of the monodromies around the other invariant fibres: $\phi_0 = \phi_1 \circ ... \circ \phi_m$.

Observe now that the dynamics of $\mathcal{H}$ is very simple, in particular each transcendental leaf accumulates on C and only there. This obviously remains true for $\mathcal{F}$: each transcendental leaf accumulates only on one or more algebraic leaves. In term of monodromy, this means that the monodromy group $G \subset Aut(\mathbf{C}P^1)$ of $\mathcal{F}$ has a finite limit set. An easy and well-known argument shows that this limit set is composed only by the two fixed points of ϕ_0, and each ϕ_j, $j = 1, ..., m$, either has $Fix(\phi_j) = Fix(\phi_0)$ or has $Fix(\phi_j) \cap Fix(\phi_0) = \emptyset$ and it exchanges the two elements of $Fix(\phi_0)$ (in particular, it has order 2). In fact, there exists at least one ϕ_j (say, ϕ_1) with $Fix(\phi_j) \cap Fix(\phi_0) = \emptyset$, because otherwise $\phi_1 \circ ... \circ \phi_m$ would be still a periodic automorphism and not an hyperbolic one.

The two fixed points of ϕ_1 correspond to one or two leaves of $\mathcal{F}$ which are transcendental and which have an element of order 2 in their holonomy group. This fact is invariant by birational maps, so that $\mathcal{H}$ also has these type of leaves. However, this does not actually occur ($\mathcal{H}$ has transcendental leaves with holonomy of order 3, but not of order 2), and this contradiction completes the proof. $\triangle$

Having proved that $\mathcal{H}$ is not Riccati, it would be natural to look for other types of birational models of $\mathcal{H}$. For instance, Y is a rational surface and so $\mathcal{H}$ is birational to a foliation on $\mathbf{C}P^2$; it would be nice to obtain an explicit and simple equation, of lowest degree, for such a projective model of $\mathcal{H}$.

3. Turbulent foliations

A foliation $\mathcal{F}$ on a compact connected surface X is called *turbulent* if there exists an elliptic fibration $\pi : X \to B$ on X (called adapted to $\mathcal{F}$) whose generic fibre is transverse to $\mathcal{F}$. As in the Riccati case, there are exceptional cases where the adapted fibration is not unique (e.g. linear foliations on products of elliptic curves). Reasoning as in section 1, we see that if $F \subset X$ is any regular fibre of an elliptic fibration π and if $\mathcal{F}$ is any foliation on X, then the condition

$$T_{\mathcal{F}} \cdot F = 0 \qquad (\text{or } N_{\mathcal{F}} \cdot F = 0, \text{ by adjunction})$$

is equivalent to say that either $\mathcal{F}$ is turbulent with respect to π, or $\mathcal{F}$ coincides with π. We then have the following analogue of proposition 1.

Proposition 5. *Let $\mathcal{F}$ be a foliation on a compact connected surface X and let $F \subset X$ be a regular fibre of an elliptic fibration π on X, invariant by $\mathcal{F}$ and with $Z(\mathcal{F}, F) = 0$. Then either $\mathcal{F}$ is a turbulent foliation with respect to π, or it coincides with the elliptic fibration π.* △

Remark that the two alternatives are mutually exclusive, because a turbulent foliation with an invariant fibre is never a fibration (by the maximum principle, for instance). The difference with proposition 1 is that we have to assume explicitely that F is a fibre of an elliptic fibration: in general, a smooth elliptic curve of zero selfintersection may fail to have such a property. For example, let X be the quotient of $\mathbf{C}^* \times \mathbf{C}P^1$ by $(z, w) \sim (2z, w+1)$, and let $\mathcal{F}$ be the foliation on X given by the quotient of $\{dw = 0\}$. Then $\mathcal{F}$ is tangent to the elliptic curve $F \subset X$ corresponding to $\{w = \infty\}$, which has $F \cdot F = 0$ and even $\mathcal{O}_X(F)|_F = \mathcal{O}_F$, but which is not a fibre of an elliptic fibration. However, we shall see in the last chapter that this example is not far from being the unique exception.

As in the case of Riccati foliations, given a turbulent foliation $\mathcal{F}$, with adapted fibration $\pi : X \to B$, we may introduce its *monodromy*

$$\rho : \pi_1(B^*, b) \to Aut(\pi^{-1}(b))$$

where $B^* \subset B$ is the set of fibres transverse to $\mathcal{F}$ and $b \in B^*$. Remark that all the fibres $\pi^{-1}(b)$, $b \in B^*$, are isomorphic to the same elliptic curve E, the foliation itself giving such an isomorphisms. In other words, $X^* = \pi^{-1}(B^*)$ is not simply an elliptic fibration but also an elliptic fibre bundle, with fibre E. This means that an elliptic fibration adapted to a turbulent foliation is quite special: the so-called functional invariant J [BPV, page 151] is constant. One can convince himself that, conversely, any elliptic fibration with constant J is adapted to a suitable turbulent foliation (see e.g. [Sad] for some explicit constructions).

Recall that the automorphism group of an elliptic curve E is rather poor: it is a finite extension (by $\mathbf{Z}_2$ in most cases, by $\mathbf{Z}_4$ or $\mathbf{Z}_6$ in two exceptional cases) of the group of translations. Hence the turbulent foliations have a quite simple structure outside their invariant fibres, at least from the dynamical point of view. We also note that, as in the Riccati case, $\mathcal{F}|_{X^*}$ is entirely determined, up to biholomorphisms, by the analytic type of B^* and the conjugacy class of the monodromy ρ.

We now turn to the study of invariant fibres (or, better, fibres which are not transverse to $\mathcal{F}$). Let us firstly consider a *regular* fibre F. Then the foliation $\mathcal{F}$ is tangent to F and has no singularities on it. We may trivialize a neighbourhood U of F, because F is regular and the functional invariant of the fibration is constant: $U \simeq \mathbf{D} \times E$. Because $N_{\mathcal{F}}|_U$ is trivial (for $N_{\mathcal{F}}|_F \simeq \mathcal{O}_X(F)|_F \simeq \mathcal{O}_F$, $N_{\mathcal{F}}|_{F'} \simeq TF' \simeq \mathcal{O}_{F'}$ if F' is transverse to $\mathcal{F}$), $\mathcal{F}$ is given on U by a holomorphic 1-form ω without zeroes, and in coordinates $(z, w) \in \mathbf{D} \times E$ we find

$$\omega = dz - A(z)dw$$

where A is a holomorphic function on $\mathbf{D}$, vanishing at 0 (if $F = \{z = 0\}$). The vanishing order k of A at 0 is the *multiplicity* of the invariant fibre F.

Let us consider the meromorphic 1-form (on $\mathbf{D}$) $\eta = \frac{dz}{A(z)}$, which gives the "slope" of $\mathcal{F}$ and which has a pole of order k at 0. By a change of coordinate (see e.g. [M-R]) η can be put in the form $\lambda\frac{dz}{z}$ if $k = 1$ or $\frac{1+\lambda z^{k-1}}{z^k}dz$ if $k \geq 2$, where $\lambda \in \mathbf{C}^*$ if $k = 1$, $\lambda \in \mathbf{C}$ if $k \geq 2$. Hence, by a change of the z-coordinate the foliation $\mathcal{F}$ can be put in linear or Dulac normal form:

$$\omega = \lambda dz - z dw \qquad k = 1,\ \lambda \in \mathbf{C}^*$$

$$\omega = (1 + \lambda z^{k-1})dz - z^k dw \qquad k \geq 2,\ \lambda \in \mathbf{C}.$$

The *monodromy* around the fibre is easily computed: because $dw = \frac{dz}{A(z)}$, by integrating along a cycle in $\mathbf{D}^*$ around 0 we find that the monodromy f is $f(w) = w + 2\pi i\lambda$. In particular, $f = id$ iff $2\pi i\lambda$ belongs to the lattice $\Gamma \subset \mathbf{C}$ defining E.

The *holonomy* of F is more interesting. If $\gamma \subset F$ is a cycle, the holonomy of $\mathcal{F}$ along γ is computed by integrating the differential equation $\frac{dz}{dw} = A(z)$ along that cycle. If $k = 1$, one finds that the holonomy h is linear, $h(z) = \exp(\frac{1}{\lambda}\theta)z$, where $\theta \in \Gamma$ is the homology class of γ. If $k \geq 2$ then one finds $h(z) = z + z^k + o(k)$. Note, in particular, that the holonomy group of F is always infinite (on the other hand, if it were finite then the leaves of $\mathcal{F}$ around F would be compact, an evident absurdity).

In the case of Riccati foliations we used the flipping of fibres to reduce ulteriorly the list of normal forms. Here we can do the same by using instead the so-called *logarithmic transformations* ([BPV, page 164]; there, logarithmic transformation are introduced to transform multiple fibres into nonmultiple ones, but there are also, obviously, logarithmic transformations which transform a regular fibre into a still regular one). In a very imprecise form, this means that we do a "change of coordinates" of the type $(z, w) \mapsto (z, w + \frac{\theta}{2\pi i} \ln z)$, $\theta \in \Gamma$, which has the effect of changing λ into $\lambda - \frac{\theta}{2\pi i}$. In particular, in a neighbourhood of the fibre the foliation is uniquely determined by the multiplicity of the fibre and the monodromy around the fibre, up to logarithmic transformations. Remark that if $k = 1$ and $2\pi i\lambda \in \Gamma$ then we obtain, by a logarithmic transformation, a foliation transverse to F.

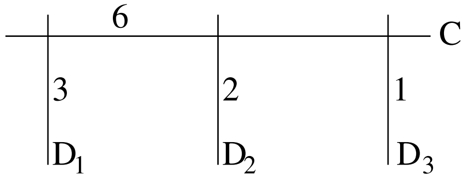
This parallel with Riccati foliations can also be explained as follows: we can lift $\mathcal{F}|_U$ on $\mathbf{D} \times \mathbf{C}^*$ by $(z, x) \mapsto (z, w = \frac{\theta}{2\pi i} \ln x)$, and then the lifted foliation can be extended to a Riccati foliation on $\mathbf{D} \times \mathbf{C}P^1$. It is however remarkable that all the "normalization problems" related to semidegenerate fibres of Riccati foliations [M-R] do not appear here, because the saddle-nodes appearing here have weak separatrices.

Let us now consider a *singular* fibre F of π, and to start with let us recall Kodaira's classification of singular fibres of elliptic fibrations. In [BPV, page 150] one finds such a classification in the relatively minimal case; however, we prefer to work with fibres having only nodes as singularities, and so we have to blow-up the types II, III and IV (see also [BPV, page 158]). The final result is that, up to blowing-up and blowing-down, each fibre of an elliptic fibration belongs to one of the classes described in the table of the next page. Except in cases I_0 and I_1, each irreducible component is a smooth rational curve, whose multiplicity in a nonmultiple fibre is indicated in the table (for example, if F is a nonmultiple fibre of type II then $F = 6C + 3D_1 + 2D_2 + D_3$). If F is a multiple fibre, then $F = mF_0$ for F_0 a nonmultiple one.

I_0 : a smooth elliptic curve

I_b , $b > 0$: a cycle of b (-2)-curves

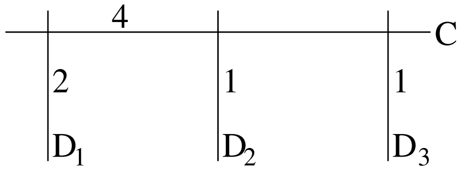
II :



$C^2 = -1$

$D_1^2 = -2 \quad D_2^2 = -3 \quad D_3^2 = -6$

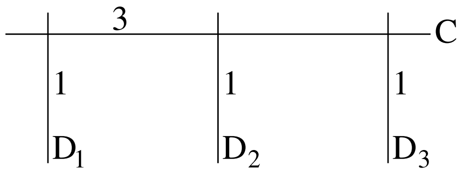
III :



$C^2 = -1$

$D_1^2 = -2 \quad D_2^2 = D_3^2 = -4$

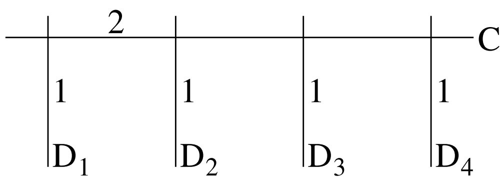
IV :



$C^2 = -1$

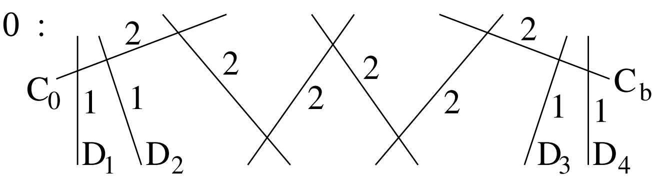
$D_1^2 = D_2^2 = D_3^2 = -3$

I_0^* :



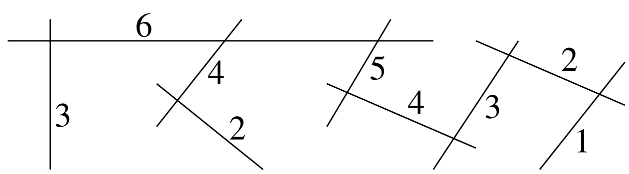
all (-2)-curves

I_b^* , $b > 0$:



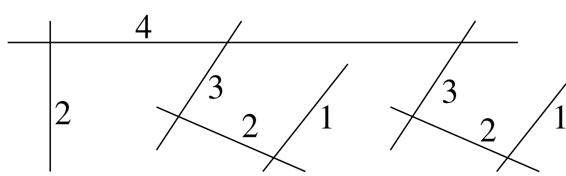
all (-2)-curves

II^* :



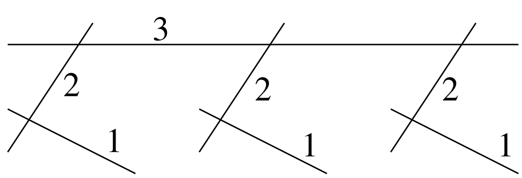
all (-2)-curves

III^* :



all (-2)-curves

IV^* :



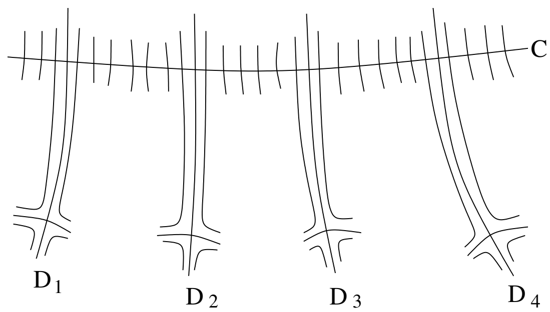
all (-2)-curves

A basic observation is now the following: if $\pi : X \to B$ is an elliptic fibration adapted to a turbulent foliation $\mathcal{F}$, then fibres of type I_b or I_b^*, $b \geq 1$, cannot occur. There are at least two (related) reasons for this:

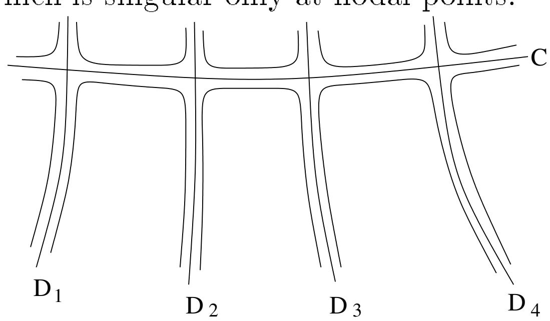
i) the functional invariant of π has always a pole of order b in correspondence of a fibre of type I_b or I_b^* [BPV, page 159], and so it cannot be constant;

ii) the homological monodromy of π around a fibre of type I_b or I_b^* is a nontrivial unipotent operator [BPV, page 159], and so it cannot be realized by a holomorphic automorphism of E.

All the other types can instead occur, as it is shown by the examples of [BPV, pages 156-157]. More precisely, we start with the trivial elliptic fibration $\mathbf{D} \times E$, equipped with the (turbulent) horizontal foliation $\mathcal{F}_0$. The group $\mathbf{Z}_2$ acts on $\mathbf{D} \times E$ by $(z,w) \mapsto (-z,-w)$, and the quotient of $\mathbf{D} \times E$ by this action has four A_1-singularities arising from the four fixed points of the action on the fibre $\{0\} \times E$. The minimal resolution of these singularities gives an elliptic fibration with a fibre of type I_0^* (the curves $D_1, ..., D_4$ come from the resolution, whereas C is the strict transform of the quotient of $\{0\} \times E$). The initial foliation $\mathcal{F}_0$ passes to this resolved quotient, giving a turbulent foliation $\mathcal{F}$ which is transverse to C and tangent to each D_j, and which has on each D_j a singularity of the type $d(z^2 w) = 0$ (being $D_j = \{w = 0\}$):



Of course, we can also start with any turbulent foliation on $\mathbf{D} \times E$ which is tangent to $\{0\} \times E$ and which is invariant by the $\mathbf{Z}_2$-action (that is, given by the equation $dz - A(z)dw = 0$ with $A(-z) = A(z)$). The final result is then a turbulent foliation which is tangent also to C, and which is singular only at nodal points:



The singularities here are of the type $d(z^2w) = 0$, being $D_j = \{w = 0\}$ and $C = \{z = 0\}$. However, the holonomy group of C is not finite, even if it is generated by involutions: it is an extension by a finite group of the infinite holonomy group of $\{0\} \times E$, and it contains elements of the type $x \mapsto x + x^2 + o(2)$.

This construction for the type I_0^* can be obviously done also for the types II, III, IV, II^*, III^*, IV^*, cfr. [BPV, page 157].

The important fact is now that *these examples are the only ones*. The reason is that the construction above can be reversed, so that any fibre of type I_0^* (for example) is the $\mathbf{Z}_2$-quotient of a regular fibre, and so a turbulent foliation around a I_0^*-fibre is the $\mathbf{Z}_2$-quotient of a turbulent foliation around a regular fibre. This is also a particular (and easy) instance of the so-called *stable reduction theorem* [BPV, page 95], asserting that any elliptic fibration can be reduced, up to "base changes" and birational transformations, to an elliptic fibration all of whose singular fibres are of type I_b; in our case, there are no singular fibres at all, for the reasons already explained above. Note that these arguments can be used also to eliminate, by a base change, all the multiple fibres. We can resume these facts in a proposition, which is in some sense the turbulent counterpart to proposition 2.

Proposition 6. *Let $\mathcal{F}$ be a turbulent foliation on a compact connected surface X, with adapted fibration $\pi : X \to B$. Then there exists a covering $p : Y \to X$, ramified along fibres, and a bimeromorphic map $q : Z - - \to Y$ such that $\pi \circ p \circ q : Z \to B$ is an elliptic fibre bundle and $(\pi \circ p \circ q)^*(\mathcal{F})$ is a nonsingular turbulent foliation w.r. to π.*

Let us also observe some properties of turbulent foliations around singular fibres, supposed belonging to the table above, which follow from the previous analysis. First of all, we always have a dichotomy similar to the one discovered above in the I_0^* case: either the turbulent foliation is tangent to each irreducible component of the fibre and its singularities are the same as the ones of the elliptic fibration (i.e. they are at the crossing points and they have holomorphic first integrals), or the turbulent foliation is transverse to one irreducible component and tangent to the others, and its singularities are where and of the type one expects to be. Note that in both cases all the singularities of the foliation are reduced, of the type $d(z^pw^q) = 0$. In other words: given a turbulent foliation with adapted fibration π, we may reduce (by blowing-downs and blowing-ups) the fibres of π to fibres in the table above (possibly with multiplicity), and after this operation the transformed foliation will be automatically in reduced form.

Concerning the computation of $T^*_{\mathcal{F}}$, we can apply the same method as in the Riccati case. However, in the turbulent case the computation of $N^*_{\mathcal{F}}$ is perhaps more interesting. Let F be a fibre in the table above, and let F_{red} be the reduced divisor associated to

F. One can easily check that, whatever the foliation is, the line bundle $N^*_{\mathcal{F}} \otimes \mathcal{O}_X(F_{red})$ is topologically trivial on a neighbourhood of F (i.e. its degree is 0 on each irreducible component of F). And one can easily check an even stronger property: that line bundle is a torsion line bundle on a neigbourhood of F, and its order is equal to the order of the covering which reduces the singular fibre to a smooth one (in particular, that order is at most 6). In the last chapter, this property will be a guide for certain proofs.

MINIMAL MODELS

In this chapter we introduce, following [Br4], a notion of minimal model adapted to the bimeromorphic study of foliations, as well as a notion of relatively minimal model (implicit in [Br4]). We show that any foliation outside a list on notable exceptions has a minimal model, or equivalently a unique relatively minimal model. Be careful that in [Mc1] there is a slightly different notion of minimal model (which could be called "in Mori sense", whereas our definition is "in Zariski sense").

1. Minimal models and relatively minimal models

We shall use the notation $(X, \mathcal{F})$ to denote a foliation $\mathcal{F}$ on a connected compact surface X, and $(X, \mathcal{F}) \dashrightarrow (Y, \mathcal{G})$ to denote a bimeromorphic map from X to Y sending $\mathcal{F}$ to $\mathcal{G}$.

If one is interested in the study of foliations modulo bimeromorphisms, then it is quite natural to work with *reduced* foliations, i.e. foliations all of whose singularities are reduced: by Seidenberg's theorem (chapter 1) any foliation is bimeromorphic to a reduced one. However, such a foliation is never unique: for instance, we may blow-up any point, regular or singular, and obtain a new reduced foliation, still bimeromorphic to the initial one. In order to cut down this redundancy, one is naturally led to the following definitions.

Definition 1. Given a foliation $(X, \mathcal{F})$, we say that a curve $C \subset X$ is *$\mathcal{F}$-exceptional* if:

i) C is an exceptional curve, i.e. a smooth rational curve of selfintersection -1;

ii) the contraction of C to a point p produces a new foliation $(X', \mathcal{F}')$ which has at p a regular point or a reduced singular point.

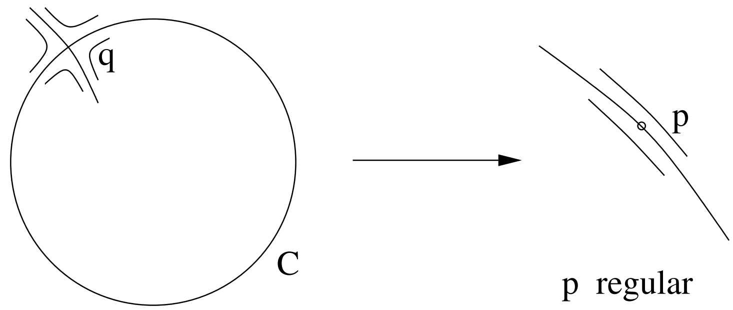
Definition 2. A foliation $(X, \mathcal{F})$ is called *relatively minimal* if:

i) it is reduced;

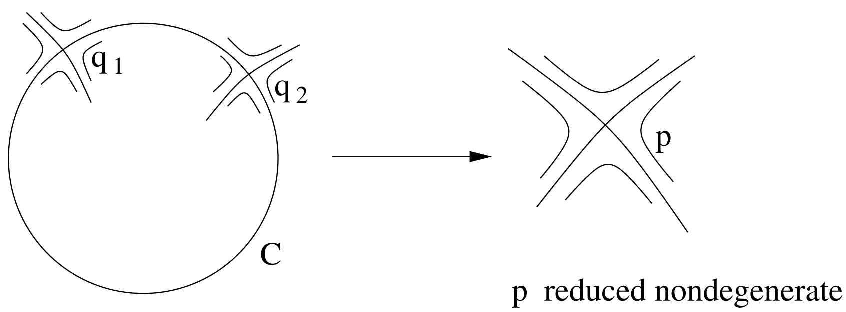
ii) there are no $\mathcal{F}$-exceptional curves.

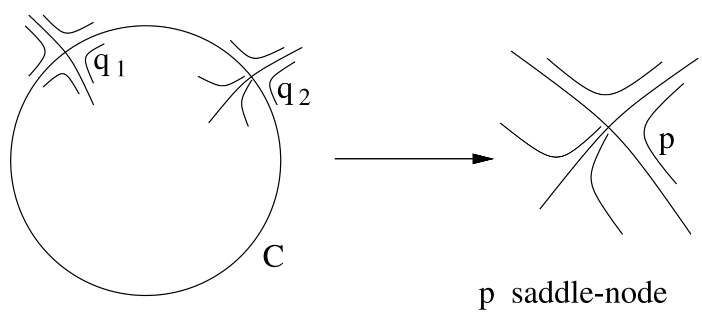
Let us explicitely see what an $\mathcal{F}$-exceptional curve is. We have in fact already analyzed the situation in chapter 1, §2, and chapter 2, §3. If C is $\mathcal{F}$-exceptional, there are two possibilities:

1) C is $\mathcal{F}$-invariant and contains only one singularity q of $\mathcal{F}$, with $Z(\mathcal{F}, C, q) = 1$: by contracting C we obtain a regular point p of the contracted foliation $\mathcal{F}'$ (because its order $a(p)$ is 0):

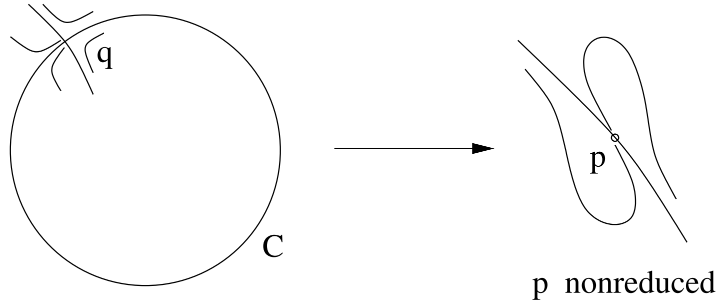


2) C is $\mathcal{F}$-invariant and contains only two singularities q_1, q_2 of $\mathcal{F}$, which are reduced and satisfy $Z(\mathcal{F}, C, q_1) = Z(\mathcal{F}, C, q_2) = 1$: by contracting C we obtain a singular point p of the contracted foliation $\mathcal{F}'$, which is a reduced singularity (because its order is 1 and the eigenvalues of its linear part have no positive rational quotient, as a simple computation shows):





Remark that in case 2) the curve C cannot contain two saddle-nodes (with strong separatrices inside C, for $Z(\mathcal{F}, C, q_j) = 1$), by an easy direct computation or by Camacho-Sad formula. Remark also that the condition on Z in 2) cannot be replaced by the weaker one "$Z(\mathcal{F}, C) = 2$", as the blowing-up of the nonreduced singularity $(z + w)dw - wdz = 0$ shows:



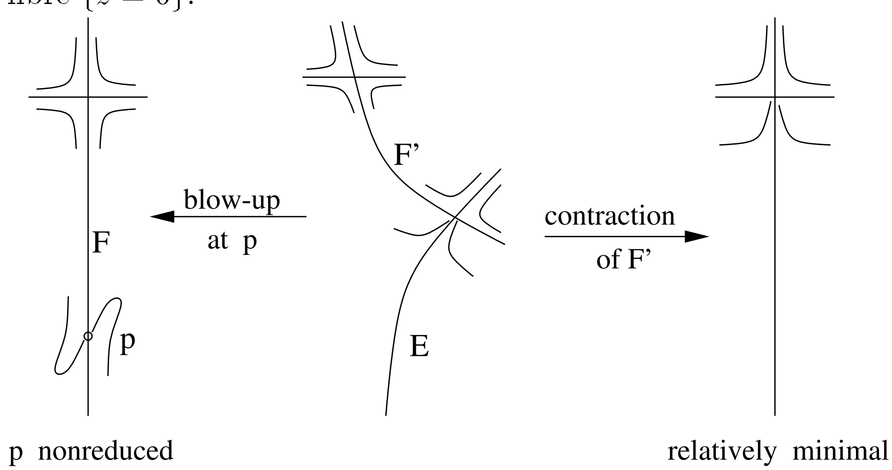
The existence of relatively minimal *models* of a given foliation (that is, relatively minimal foliations bimeromorphic to the given one) is easily established:

Proposition 1. *Any foliation* $(X, \mathcal{F})$ *has a relatively minimal model.*

Proof.

Firstly, by Seidenberg's theorem there exists a reduced foliation $(\tilde{X}, \tilde{\mathcal{F}})$ and a bimeromorphic morphism $(\tilde{X}, \tilde{\mathcal{F}}) \to (X, \mathcal{F})$. If $(\tilde{X}, \tilde{\mathcal{F}})$ is not relatively minimal, then it contains an $\tilde{\mathcal{F}}$-exceptional curve, whose contraction produces a new foliation $(\bar{X}, \bar{\mathcal{F}})$ which is still reduced. We can iterate this contraction procedure, but at some moment we must stop because each contraction reduces by 1 the rank of the second homology group of the surface. At this moment we have a relatively minimal foliation, bimeromorphic to $(X, \mathcal{F})$. $\triangle$

As an example, take the Riccati foliation $(w+z)dz - zdw = 0$ in $\mathbf{C}P^1 \times \mathbf{C}P^1$, which has at $(0,0)$ a nonreduced singularity. Then a relatively minimal model is obtained by flipping the fibre $\{z = 0\}$:



The uniqueness of relatively minimal models is a more difficult problem, and in fact there are examples where this uniqueness fails. But let us firstly precise what is meant by "uniqueness".

Definition 3. A foliation $(X, \mathcal{F})$ is called *minimal* if:

i) it is relatively minimal;

ii) if $(Y, \mathcal{G})$ is any relatively minimal foliation and if $f : (X, \mathcal{F}) \dashrightarrow (Y, \mathcal{G})$ is any bimeromorphic map sending $\mathcal{F}$ to $\mathcal{G}$, then f is in fact a biholomorphic map.

Observe that in ii) it is not excluded that $(Y, \mathcal{G}) = (X, \mathcal{F})$: in other words, a minimal foliation has the property that any bimeromorphic self-map is automatically a biholomorphic one. Thus, if a foliation $(X_0, \mathcal{F}_0)$ has a minimal model $(X, \mathcal{F})$, then such a minimal model is unique in the strongest sense: not only $(X, \mathcal{F})$ is the unique relatively minimal foliation bimeromorphic to $(X_0, \mathcal{F}_0)$, but also the bimeromorphic map $(X_0, \mathcal{F}_0) - - \rightarrow (X, \mathcal{F})$ is unique, up to composition with a self-biholomorphism of $(X, \mathcal{F})$.

Observe also that if $(X_0, \mathcal{F}_0)$ has a minimal model $(X, \mathcal{F})$ then the study of bimeromorphic self-maps of $(X_0, \mathcal{F}_0)$ is reduced to the study of biholomorphic self-maps of $(X, \mathcal{F})$, a much more tractable problem. An application of this fact to the study of polynomial diffeomorphisms of $\mathbf{C}^2$ can be found in [Br4].

We can now give examples of relatively minimal foliation which are not minimal (equivalently: foliations without a minimal model).

Example 1. Take $\mathcal{F}$ given by the fibres of a rational fibration $\pi : X \rightarrow B$. Then $\mathcal{F}$ is always reduced, and it is relatively minimal iff π has no singular fibres (i.e. it is a $\mathbf{C}P^1$-bundle over B). However $\mathcal{F}$ is never minimal, as shown by the flipping of a fibre (compare with [BPV, page 201]).

Example 2. Take $\mathcal{F}$ a Riccati foliation. Suppose that there exists a regular fibre F of the adapted fibration which is invariant by $\mathcal{F}$ and which contains two reduced distinct singularities (that is, either F is semidegenerate or it is nondegenerate of the type $\lambda w dz - z dw = 0$, $(z, w) \in \mathbf{D} \times \mathbf{C}P^1$, $\lambda \notin \mathbf{Q}$). We shall say that $\mathcal{F}$ is a *nontrivial* Riccati foliation (even if a Riccati foliation which is not nontrivial may be quite complicated). If $\mathcal{F}$ is relatively minimal, then a flipping of F shows that $\mathcal{F}$ is certainly not minimal. If $\mathcal{F}$ is not relatively minimal, then we can take a bimeromorphic map onto a relatively minimal model $\mathcal{F}_0$ which has no indeterminacy points over F (see the proof of proposition 1), so that the image of F will be an $\mathcal{F}_0$-invariant curve F_0, rational and smooth or with a node. Then the flipping of F induces a "flipping" of F_0, showing that $\mathcal{F}_0$ is not minimal.

Example 3. Take the very special foliation $\mathcal{H}$ of the previous chapter (section 2). We have already remarked there that $\mathcal{H}$ has nontrivial bimeromorphic self-maps, so that $\mathcal{H}$ is relatively minimal but not minimal.

We shall see in the next section that these examples are in fact the only ones.

2. Existence of minimal models

We firstly reformulate the definitions of minimality and relative minimality.

Proposition 2. *Let $(X, \mathcal{F})$ be a reduced foliation, then:*

a) $(X, \mathcal{F})$ is relatively minimal iff any bimeromorphic morphism $(X, \mathcal{F}) \to (Y, \mathcal{G})$ onto a reduced foliation $(Y, \mathcal{G})$ is in fact a biholomorphism;

b) $(X, \mathcal{F})$ is minimal iff any bimeromorphic map $(Y, \mathcal{G}) - - \to (X, \mathcal{F})$ from a reduced foliation $(Y, \mathcal{G})$ is in fact a morphism.

Proof.

a) This follows from the fact that any bimeromorphic morphism factorizes through blowing-ups [BPV, page 79], and that the blowing-up of a reduced foliation is still reduced.

b) Let $(X, \mathcal{F})$ be minimal and let $f : (Y, \mathcal{G}) - - \to (X, \mathcal{F})$ be as in the statement. Because $(Y, \mathcal{G})$ is reduced, we may find a relatively minimal model $(Y_0, \mathcal{G}_0)$ and a bimeromorphic morphism $g : (Y, \mathcal{G}) \to (Y_0, \mathcal{G}_0)$ (see the proof of proposition 1). Thus f factorizes as $h \circ g$, for some bimeromorphic map $h : (Y_0, \mathcal{G}_0) - - \to (X, \mathcal{F})$. By minimality of $(X, \mathcal{F})$, it follows that h is a biholomorphism, and therefore f is a morphism.

Conversely, let $(X, \mathcal{F})$ satisfy the property of the statement. Clearly X cannot contain $\mathcal{F}$-exceptional curves, so $(X, \mathcal{F})$ is relatively minimal. Let $(Y, \mathcal{G})$ be relatively minimal and $f : (Y, \mathcal{G}) - - \to (X, \mathcal{F})$ a bimeromorphic map. Then f must be a morphism by the property above, and thus a biholomorphism by a). Hence $(X, \mathcal{F})$ is minimal. △

This provides a better explanation of the terminology: the set of foliations bimeromorphic to a given one is partially ordered by the existence-of-morphism relation, and so relative minimal models and minimal models correspond respectively to relative minima and absolute minima for that relation.

We can now classify foliations without minimal model.

Theorem 1. *Let $(X, \mathcal{F})$ be a foliation without minimal model. Then $(X, \mathcal{F})$ is bimeromorphic to a foliation in the following list:*

1) rational fibrations;

2) nontrivial Riccati foliations;

3) the very special foliation $(Y, \mathcal{H})$ of chapter 4.

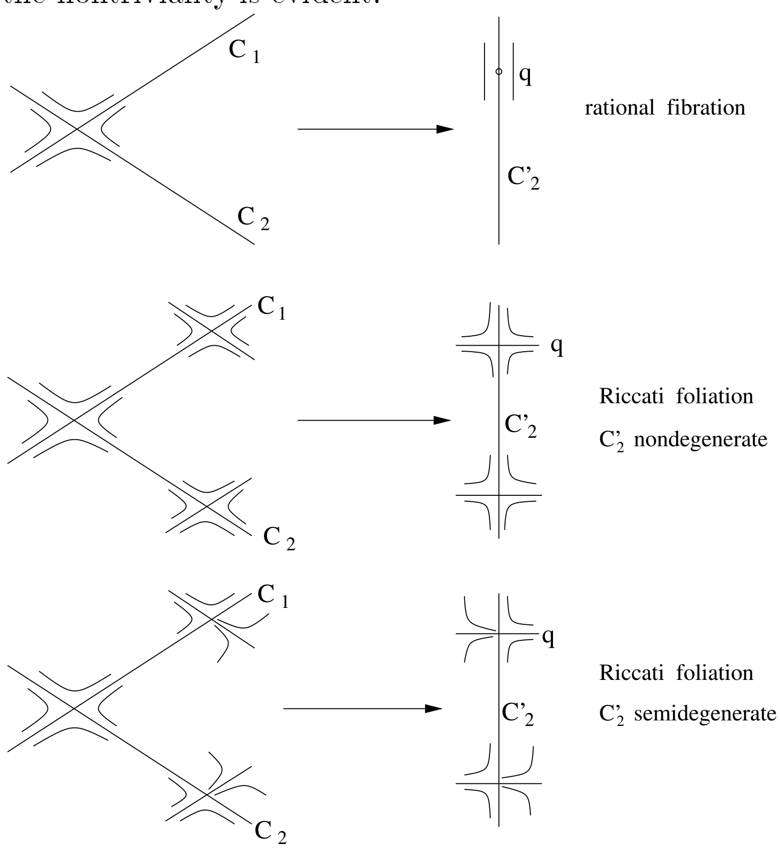
Proof.

Without loss of generality, we may suppose that $(X, \mathcal{F})$ is relatively minimal. Being not minimal, there exists a bimeromorphic map $f : (Y, \mathcal{G}) - - \to (X, \mathcal{F})$, $(Y, \mathcal{G})$ reduced, which is not a morphism. Let $\tilde{Y} \xrightarrow{\pi} Y$ be the minimal resolution of the indeterminacies of f, and let $\tilde{f} : (\tilde{Y}, \tilde{\mathcal{G}}) \to (X, \mathcal{F})$ be the induced morphism. Then $\tilde{Y}$ contains an exceptional curve C (the "last" exceptional divisor) which is not contracted by $\tilde{f}$ but is contracted by π, and obviously C is $\tilde{\mathcal{G}}$-exceptional because $\mathcal{G}$ is reduced. Remark that $\tilde{f}$ is not a biholomorphism on a neighbourhood of C, otherwise $\tilde{f}(C)$ would be an $\mathcal{F}$-exceptional curve, contrary to the relative minimality of $(X,\mathcal{F})$. On the other hand, $\tilde{f}$ is a composition of blowing-ups of X. Hence $\tilde{f}$ can be factorized as $(\tilde{Y},\tilde{\mathcal{G}}) \xrightarrow{g} (\tilde{X},\tilde{\mathcal{F}}) \xrightarrow{h} (X,\mathcal{F})$ in such a way that g is a biholomorphism on a neighbourhood of C and $C_1 = g(C)$ intersects an exceptional curve C_2 of $\tilde{X}$ contracted by h (it may happen $\tilde{X} = \tilde{Y}$, $g = id$). These two curves C_1 and C_2 are $\tilde{\mathcal{F}}$-exceptional, because $\mathcal{F}$ is reduced.

To summarize: if $(X,\mathcal{F})$ has no minimal model, then it is bimeromorphic to a reduced foliation $(\tilde{X},\tilde{\mathcal{F}})$ which contains two $\tilde{\mathcal{F}}$-exceptional curves C_1, C_2 with $C_1 \cap C_2 \neq \emptyset$.

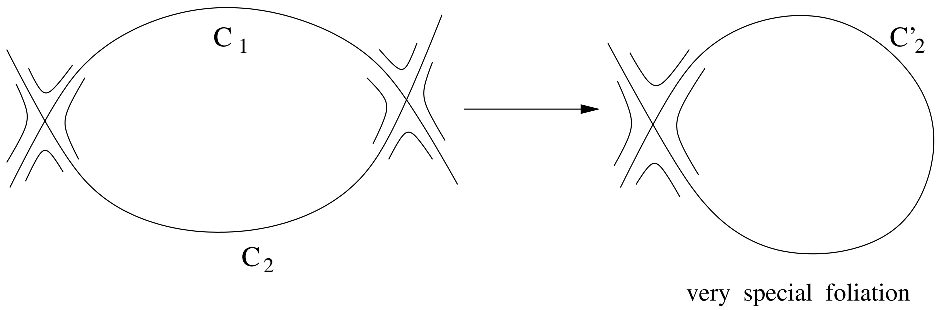
If $p \in C_1 \cap C_2$ then p is reduced nondegenerate (for $Z(\tilde{\mathcal{F}}, C_j, p) = 1$, $j = 1,2$). In particular, the intersections between C_1 and C_2 are transverse and their number is at most 2.

If $\sharp(C_1 \cap C_2) = 1$ then by contracting C_1 the curve C_2 is transformed into a smooth rational curve C_2' of zero selfintersection, invariant by the (still reduced) induced foliation $\bar{\mathcal{F}}$. The point $q \in C_2'$ under C_1 is either regular for $\bar{\mathcal{F}}$ or singular with $Z(\bar{\mathcal{F}}, C_2', q) = 1$, depending on the number of singularities of $\tilde{\mathcal{F}}$ on C_1. Hence $Z(\bar{\mathcal{F}}, C_2') \leq 2$ and proposition 1 of chapter 4 allows to conclude that $\bar{\mathcal{F}}$ is either a rational fibration or a Riccati foliation. In this last case the nontriviality is evident.



(The reader may also amuse himself by showing that if C_1 contains only one singularity then C_2 also, or that if C_1 contains a saddle-node then C_2 also; but this is not really indispensable for the proof).

If $\sharp(C_1 \cap C_2) = 2$ then by contracting C_1 the curve C_2 is transformed into a rational curve with a node C_2', of selfintersection 3. The induced foliation $\bar{\mathcal{F}}$ satisfies the hypotheses of proposition 3 of chapter 4, and therefore it is bimeromorphic to the very special foliation $(Y, \mathcal{H})$.



This completes the proof. △

The examples of the previous section show that the result is optimal.

In this chapter we recall some fundamental facts concerning holomorphic 1-forms on compact surfaces: Albanese morphism, Castelnuovo-De Franchis lemma, Bogomolov lemma. We also discuss the logarithmic case, which is extremely useful in the study of foliations with an invariant curve. Finally we recall the classification of holomorphic vector fields on compact surfaces. All of this is very classical and can be found, for instance, in [BPV, chapter IV], [ReB], [Kob],...

1. Holomorphic and logarithmic 1-forms

Let X be a connected compact surface. The basic property of holomorphic 1-forms on X is the following consequence of Stokes theorem: any such 1-form is *closed*. In fact, if $\omega \in \Omega^1_X(X)$ were not closed then $d\omega \wedge d\bar{\omega}$ would be an exact (smooth) 4-form on X, nonnegative everywhere and positive somewhere, contradicting Stokes theorem. On the other hand, such a global 1-form is never exact, unless identically zero: if $\omega = df$ then f is holomorphic and thus constant. Hence we have a natural (De Rham) map of $\Omega^1_X(X)$ into $H^1(X, \mathbf{C})$. If $\gamma \in H_1(X, \mathbf{Z})/\text{torsion}$, then γ defines a period map $\Omega^1_X(X) \to \mathbf{C}$ (integration of 1-forms along γ) and can be seen as an element of the dual space $\Omega^1_X(X)^*$. Thus $H_1(X, \mathbf{Z})/\text{torsion}$ may be naturally mapped to a subgroup $\Gamma \subset \Omega^1_X(X)^*$. It is a fundamental and nontrivial fact that Γ is discrete in $\Omega^1_X(X)^*$, and the quotient $\Omega^1_X(X)^*/\Gamma$ is a compact torus $Alb(X)$, called *Albanese torus* of X. If X is Kähler, or more generally (a priori) if the first Betti number of X is even, then this is a consequence of Hodge theory. If X is not Kähler, then this is a rather subtle ingredient of Kodaira's classification: see [BPV, pages 198-200]. Note also that in the Kähler case the map from $H_1(X, \mathbf{Z})/\text{torsion}$ to Γ is injective, whereas it has a nontrivial Kernel (infinite cyclic) in the non-Kähler case. All of this becomes false in higher dimension (one can still define an Albanese torus, following Blanchard, but only after restriction to a suitable, and possibly trivial, subspace of $\Omega^1_X(X)$).

If we fix a point $x_0 \in X$, then the integration of 1-forms along paths in X defines a morphism

$$alb_X : X \to Alb(X)$$

called *Albanese morphism.* By construction, it has the property that

$$alb_X^* : \Omega^1_{Alb(X)}(Alb(X)) \to \Omega^1_X(X)$$

is an isomorphism. Note that the only holomorphic 1-forms on $Alb(X)$ are the linear ones, and the foliations that they generate are those (called *Kronecker foliations*) whose leaves are translations of a codimension one subgroup of $Alb(X)$. In this way we have a pretty good description of foliations on X generated by a global holomorphic 1-form (or, more precisely, foliations with effective conormal bundle: the 1-form may have a nonempty zero divisor): any such foliation is the pull-back by alb_X of a Kronecker foliation on $Alb(X)$.

If the image of alb_X is one-dimensional, that is a curve of genus ≥ 1, then alb_X defines a fibration over that curve, called *Albanese fibration.* In this case, the only foliation on X with effective conormal bundle is the same Albanese fibration. We shall frequently use this fact in the following form: if X is a surface with $\dim \Omega^1_X(X) > 0$ and $\mathcal{F}$ is a foliation on X with $h^0(X, T^*_{\mathcal{F}}) = 0$, then $\mathcal{F}$ is the Albanese fibration (the absence of global sections of $T^*_{\mathcal{F}}$ implies that any 1-form on X vanishes on the leaves of $\mathcal{F}$, and so defines a nontrivial section of $N^*_{\mathcal{F}}$; moreover the image of alb_X is one-dimensional because all 1-forms vanish on the leaves of the same foliation). This simple fact is the reason for which all the problems concerning the Kodaira dimension of $T^*_{\mathcal{F}}$ occur on surfaces without holomorphic 1-forms.

If the image of alb_X is two-dimensional, then there are on X foliations with effective conormal bundle which are not fibrations; this is in fact the generic case. However, there is the following useful criterion to conclude that the foliation is a fibration.

Proposition 1 (Castelnuovo-De Franchis). *Let X be a connected compact surface and let $\mathcal{F}$ be a foliation on X with $h^0(X, N^*_{\mathcal{F}}) \geq 2$. Then $\mathcal{F}$ is a fibration over a curve of genus ≥ 2.*

Proof.

It is very simple, see e.g. [ReB]: if ω_1, ω_2 are two linearly independent global sections of $N^*_{\mathcal{F}}$, then $\omega_1 = f\omega_2$ for some nonconstant meromorphic function f on X, and the closedness of ω_1, ω_2 shows that f is a first integral of $\mathcal{F}$, i.e. it is constant on the leaves at a generic point. It is then easy to see that f has no indeterminacy points, and that its Stein factorization is a fibration over a curve of genus ≥ 2 (or, more precisely, $\geq h^0(N^*_{\mathcal{F}})$).
△

We shall say that a connected compact curve $C \subset X$ is *contractible* if there exists a bimeromorphic morphism $X \to Y$ onto a normal space Y which sends C to a point p and is a biholomorphism between $X \setminus C$ and $Y \setminus \{p\}$. According to a classical result of Grauert, this is equivalent to the negative definiteness of the intersection matrix associated to the irreducible components of C (contractible curves are usually called "exceptional", but we already used the term exceptional to denote (-1)-curves, which in turn are usually called "exceptional of the first kind").

With this definition in mind, we have the following slight generalization of Castelnuovo-De Franchi lemma.

Proposition 2. *Let X be a connected compact surface and let $\mathcal{F}$ be a foliation on X with $h^0(X, N^*_{\mathcal{F}}) \geq 1$. Suppose that there exists a connected compact curve $C \subset X$ which is not contractible and all of whose irreducible components are invariant by $\mathcal{F}$. Then $\mathcal{F}$ is a fibration over a curve of genus ≥ 1.*

Proof.

Let ω be a nontrivial global section of $N^*_{\mathcal{F}}$. Then the $\mathcal{F}$-invariance of C implies that $\omega|_C$ is identically zero, and thus the De Rham cohomology class of $\omega|_U$, with U a tubular neighbourhood of C, is zero. Thus $\omega|_U = df$, for some $f \in \mathcal{O}(U)$ vanishing (at least) on C. The non contractibility of C implies that f vanishes only on C, and thus it defines a proper fibration around C. It follows that $\mathcal{F}$ is, around C and therefore everywhere, a fibration. Clearly the base cannot be rational, because ω projects to a nontrivial holomorphic 1-form on it. $\triangle$

We now turn to logarithmic 1-forms. Let X be a connected surface, and let $C \subset X$ be a curve. A *logarithmic 1-form* on X with poles on C is a meromorphic 1-form, on some open subset $U \subset X$, with the following property:
for any $p \in U$ there exists a neighbourhood V of p in U such that $\omega|_V$ can be written as

$$\omega_0 + \sum_{j=1}^{n} g_j \frac{df_j}{f_j}$$

where ω_0 is a holomorphic 1-form on V, f_j and g_j are holomorphic functions on V for every j, and $\{f_j = 0\}$, $j = 1, ..., n$, are the reduced equations of the irreducible components of $C \cap V$. This definition is more general than the usual one, because we do not require that C has only normal crossing singularities. Note that the polar set $(\omega)_\infty$ is contained in C and it is a first order polar set; the same for $(d\omega)_\infty$. If C is a normal crossing curve, then these two properties ($(\omega)_\infty \subset C$, $(d\omega)_\infty \subset C$, both first order poles) provide a characterization of logarithmic 1-forms. But this is not true for more general curves.

Logarithmic 1-forms on X with poles on C form a coherent analytic sheaf $\Omega^1_X(\log C)$, which is moreover locally free and of rank 2 outside the points of X where C is not normal crossing. The basic property of this sheaf is that it fits into an exact sequence

$$0 \to \Omega^1_X \to \Omega^1_X(\log C) \stackrel{Res}{\to} \mathcal{O}_{\hat{C}} \to 0$$

where $\hat{C}$ is the normalization of C and the *residue map* Res is defined as follows: if, locally, $\omega = \omega_0 + \sum_{j=1}^n g_j \frac{df_j}{f_j}$ then

$$Res(\omega)|_{\hat{C}_j} = q^*(g_j|_{\{f_j=0\}})$$

where $q : \hat{C} \to C$ is the normalizing map and $\hat{C}_j$ is the local component of $\hat{C}$ over $\{f_j = 0\}$. The reader may easily verify that this is a well-posed definition (it does not depend on the choice of f_j, g_j, ω_0, etc.), and that the sequence above is really exact.

A more intrinsic and interesting definition of the residue map is the following one [Nog]. A logarithmic 1-form ω on $U \subset X$ defines a $(1,2)$-current T_ω on U by the formula

$$T_\omega(\phi) = \int_U \omega \wedge \phi \qquad\qquad \forall \phi \in A^{1,2}_{cpt}(U)$$

(the integral is well defined because ω is logarithmic). Then we can consider the differential of T_ω in the sense of currents. A simple computation, based on $\partial\bar{\partial} \log |z|^2 = -4\pi i \delta_0$, gives

$$\bar{\partial} T_\omega = 2\pi i Res(\omega) \delta_{\hat{C}}$$

where $\delta_{\hat{C}}$ is the integration $(1,1)$-current over $\hat{C}$, and its product with $Res(\omega) \in \mathcal{O}_{\hat{C}}$ is defined componentwise.

This interpretation of a logarithmic 1-form as a current immediately implies that any (global) logarithmic 1-form on any compact surface is *closed* [Nog]. In fact, the compactness of X (and hence of $\hat{C}$) implies that $Res(\omega)$ is locally constant on $\hat{C}$, and thus $d\omega$ is a *holomorphic* 2-form on X. By Stokes theorem (and $0 \le d\omega \wedge d\bar{\omega} = d(d\omega \wedge \bar{\omega})$, where $d\omega \wedge \bar{\omega}$ is a well defined current) we then find $d\omega \equiv 0$, that is ω is closed.

Many other properties of holomorphic 1-forms can be translated into the logarithmic context: Hodge theory (Deligne), Albanese morphism, etc.

Let now $\mathcal{F}$ be a foliation on X and suppose that the curve C is $\mathcal{F}$-invariant. On a neighbourhood V of a point $p \in C$, take a 1-form ω (holomorphic and with isolated zeroes) generating $\mathcal{F}$ on V, and a reduced equation f of $C \cap V$. Then the meromorphic 1-form $\frac{1}{f}\omega$ has first order poles on C, as well as its differential ($fd(\frac{1}{f}\omega) = d\omega - \frac{1}{f} df \wedge \omega$ is holomorphic, because $df \wedge \omega$ vanishes on C). However, we have already noted that this is not always sufficient for the logarithmicity of $\frac{1}{f}\omega$, except when p is a smooth point or a normal crossing point of C. If $\frac{1}{f}\omega$ is logarithmic around p, then we shall say that the foliation $\mathcal{F}$ is *logarithmic* at p along C. And if this property holds at every point of C

then we shall say that $\mathcal{F}$ is a *logarithmic foliation* on (X, C). Conversely, note that any logarithmic 1-form (on a surface X, with empty zero divisor and polar divisor equal to C) defines a logarithmic foliation on (X, C).

If C has only normal crossing singularities then *any* foliation on X and tangent to C is logarithmic on (X, C). Otherwise, one has to impose some conditions on the singularities of $\mathcal{F}$ at the points of C where C is not normal crossing. For instance, one may require that these singularities are nondicritical (cfr. [Br2], where the logarithmicity of nondicritical singularities is essentially shown in the proof of proposition 6). Looking at the discussion in chapter 2 around the Z-index, one can see also that a necessary condition for the logarithmicity of $\mathcal{F}$ at $p \in C$ is $Z(\mathcal{F}, C, p) \geq 0$; this necessary condition is also sufficient when C has only one branch at p.

If $\mathcal{F}$ is a logarithmic foliation on (X, C), then the usual exact sequence

$$0 \to N^*_{\mathcal{F}} \to \Omega^1_X \to \mathcal{I}_Z \cdot T^*_{\mathcal{F}} \to 0$$

can be turned into

$$0 \to N^*_{\mathcal{F}} \otimes \mathcal{O}_X(C) \to \Omega^1_X(\log C) \to \mathcal{I}_{Z'} \cdot T^*_{\mathcal{F}} \to 0$$

where Z' may be smaller than Z: the map from $\Omega^1_X(\log C)$ to $T^*_{\mathcal{F}}$ is given by contracting a logarithmic 1-form with a holomorphic vector field generating $\mathcal{F}$, and the result is a holomorphic function which need not vanish at singular points of $\mathcal{F}$ (example: if $v = z\frac{\partial}{\partial z} + \lambda w \frac{\partial}{\partial w}$, $C = \{z = 0\}$, then $\frac{dz}{z}(v) = 1$). In other words: the "singularities" of a logarithmic foliation are, in general, smaller than the singularities of the associated (non-logarithmic) foliation. We will profit of this, for instance, in theorem 3 of chapter 9.

The line bundle $N^*_{\mathcal{F}} \otimes \mathcal{O}_X(C)$ is called *logarithmic conormal bundle* of $\mathcal{F}$. The residue map induces an exact sequence

$$0 \to N^*_{\mathcal{F}} \to N^*_{\mathcal{F}} \otimes \mathcal{O}_X(C) \stackrel{Res}{\to} \mathcal{O}_{\hat{C}}$$

but the last map is *not* surjective, in general: for example, if $\mathcal{F}$ is given by $\frac{dz}{z} + \lambda \frac{dw}{w}$ and C is $\{zw = 0\}$ then for any $\omega \in H^0(N^*_{\mathcal{F}} \otimes \mathcal{O}_X(C))$ we have $Res(\omega)|_{\{w=0\}} = \lambda Res(\omega)|_{\{z=0\}}$. This fact was already met in lemma 1 of chapter 4.

For logarithmic foliations we have a logarithmic version of Castelnuovo-De Franchis lemma:

Proposition 3 (logarithmic Castelnuovo-De Franchis). *Let X be a connected compact surface, let $C \subset X$ be a curve, and let $\mathcal{F}$ be a logarithmic foliation on (X, C) with $h^0(X, N^*_{\mathcal{F}} \otimes \mathcal{O}_X(C)) \geq 2$. Then $\mathcal{F}$ is a fibration over a curve.*

Proof.

Take $\omega_1, \omega_2 \in H^0(X, N^*_{\mathcal{F}} \otimes \mathcal{O}_X(C))$ linearly independent. Then $\omega_1 = F\omega_2$, and F is a nonconstant meromorphic function on X, constant along the leaves of $\mathcal{F}$ because the logarithmic 1-forms ω_1, ω_2 are closed. To prove that F is a fibration (modulo Stein factorization) is the same as to prove that F has no indeterminacy point, i.e. $\mathcal{F}$ has no dicritical singularity (cfr. chapter 1).

Take $p \in Sing(\mathcal{F})$. Around p, $\mathcal{F}$ is generated by a logarithmic 1-form of the type

$$\omega = \omega_0 + \sum_j g_j \frac{df_j}{f_j}$$

(ω_0, g_j, f_j holomorphic), and for some holomorphic functions F_1, F_2 we have

$$\omega_1 = F_1\omega \qquad \omega_2 = F_2\omega$$

so that $F = \frac{F_1}{F_2}$. From $d\omega_2 \equiv 0$ it follows that the residue of ω_2 on $\{f_j = 0\}$ is constant. That residue is equal to $F_2 g_j|_{\{f_j=0\}}$, hence we have two possibilities:
i) If $F_2(p) = 0$ then the residue of ω_2 along any $\{f_j = 0\}$ is 0, thus ω_2 is holomorphic around p and a primitive of ω_2 around p gives a holomorphic first integral of $\mathcal{F}$; therefore p is certainly not dicritical.
ii) If $F_2(p) \neq 0$ then $F = \frac{F_1}{F_2}$ is holomorphic around p and, as before, p is not dicritical. $\triangle$

Note that in this case we cannot conclude that the base of the fibration has genus ≥ 2: logarithmic 1-forms exist also on rational or elliptic curves.

By the "branched covering trick" the lemma of Castelnuovo-De Franchis (holomorphic or logarithmic) remains true if $N^*_{\mathcal{F}}$ or $N^*_{\mathcal{F}} \otimes \mathcal{O}_X(C)$ is replaced by one of its tensor powers. Remark that if $Y \stackrel{\pi}{\to} X$ is a ramified covering and ω is a logarithmic 1-form on X with poles on C, then $\pi^*(\omega)$ is logarithmic on Y with poles on $\pi^{-1}(C)$. In this way one obtains a useful lemma of Bogomolov (see [ReB] for more details on this covering argument).

Proposition 4 (Bogomolov). *Let X be a connected compact surface, let $C \subset X$ be a curve (possibly empty), and let $\mathcal{F}$ be a logarithmic foliation on (X, C). Then*

$$kod(N^*_{\mathcal{F}} \otimes \mathcal{O}_X(C)) \leq 1$$

and if $kod(N^*_{\mathcal{F}} \otimes \mathcal{O}_X(C)) = 1$ then $\mathcal{F}$ is a fibration. $\triangle$

We shall recall in chapter 9 the definition and the basic properties of the Kodaira dimension kod of a line bundle, which appears in this statement. For the moment, we only recall that $kod(L) \geq 1$ is equivalent to say that some positive tensor power of L has two linearly independent global sections.

2. A theorem of Jouanolou

An application of the logarithmic formalism developed in the previous section is the following useful result of Jouanolou.

Theorem 1 [Jou]. *Let X be a compact connected surface and let $\mathcal{F}$ be a foliation on X. Suppose that $\mathcal{F}$ has at least $N = h^0(X, T^*_{\mathcal{F}}) + h^{1,1}(X) - h^{1,0}(X) + 2$ irreducible compact invariant curves. Then $\mathcal{F}$ has a meromorphic first integral.*

Proof.

Let $C_1, ..., C_N$ be irreducible $\mathcal{F}$-invariant curves, and set $C = \cup_{j=1}^N C_j$. From the exact sequence

$$0 \to \Omega^1_X \to \Omega^1_X(\log C) \stackrel{Res}{\to} \mathcal{O}_{\hat{C}} \to 0$$

and $h^0(\hat{C}, \mathcal{O}_{\hat{C}}) = N$ it follows that

$$h^0(X, \Omega^1_X(\log C)) \geq N - h^{1,1}(X) + h^{1,0}(X) = h^0(X, T^*_{\mathcal{F}}) + 2.$$

Because each C_j is $\mathcal{F}$-invariant, we still have a restriction map $\Omega^1_X(\log C) \to T^*_{\mathcal{F}}$, whose kernel is a subsheaf of $N^*_{\mathcal{F}} \otimes \mathcal{O}_X(C)$ (which is not necessarily full, because $\mathcal{F}$ is not supposed to be C-logarithmic). From the inequality above it follows that there exist two linearly independent logarithmic 1-forms ω_1, ω_2 which vanish on the leaves of $\mathcal{F}$. Being ω_1, ω_2 closed, we see that their quotient is a meromorphic first integral of $\mathcal{F}$. △

Corollary 1. *Let X be a compact connected surface and $\mathcal{F}$ a foliation on X with infinitely many compact invariant curves. Then $\mathcal{F}$ has a meromorphic first integral.* △

3. Holomorphic vector fields

There are several approaches to the study of global holomorphic vector fields on compact surfaces. For instance, using some Kähler geometry one can obtain various useful results in a rather direct way, in any dimension [Kob]. There are also many results on the structure of the full Lie algebra of holomorphic vector fields on a (Kähler) compact surface [Kob], but here we will concentrate only on a single generator of that algebra. We shall adopt a complex-analytic point of view, avoiding Kählerian arguments and stressing the relations with previous material.

Let X be a compact connected surface and let v be a (nontrivial) holomorphic vector field on X. If D denotes its zero divisor (possibly empty, if v has only isolated zeroes) and if $\mathcal{F}$ is the foliation generated by v, then $T_{\mathcal{F}} = \mathcal{O}_X(D)$ and thus $N^*_{\mathcal{F}} = K_X \otimes \mathcal{O}_X(D)$.

Hence X cannot be a surface of general type: $kod(X) = 2$ would imply $kod(N^*_{\mathcal{F}}) = 2$, contradicting Bogomolov's lemma (proposition 4).

Moreover, if $kod(X) = 1$ then $kod(N^*_{\mathcal{F}}) = 1$ and so, by the same lemma, $\mathcal{F}$ is a fibration over a curve B. The generic fibres are curves with a nontrivial vector field, hence rational or elliptic, but $kod(X) = 1$ excludes the rational case. Thus $\mathcal{F}$ is an elliptic fibration, and of course it is nothing but than the canonical elliptic fibration of X. We now observe the following general fact, holding whatever $kod(X)$ is and whose proof mimics (and examplifies) some arguments about the "period map" [BPV, chapter III].

Lemma 1. *Let $\pi : X \to B$ be an elliptic fibration, and let v be a nontrivial holomorphic vector field on X tangent to the fibres of π. Then every fibre of π is a smooth elliptic curve (but possibly multiple), and outside multiple fibres π is a locally trivial fibration. Moreover v has empty zero set.*

Proof.

Let $B^* \subset B$ be the set of smooth fibres (multiple or not) over which v does not vanish identically, i.e. has no zero at all. For every $t \in B^*$ set $F_t = \pi^{-1}(t)$, m_t = multiplicity of F_t, ω_t = the holomorphic 1-form on F_t dual to $v|_{F_t}$ (i.e. $\omega_t(v|_{F_t}) \equiv 1$). Then the real function

$$f(t) = \log \big(m_t \int_{F_t} \omega_t \wedge \bar{\omega}_t\big)$$

is superharmonic. In fact, a neighbourhood of a regular fibre can be represented as $\mathbf{D} \times \mathbf{C}/(z, w) \sim (z, w + 1) \sim (z, w + \tau(z))$ for some holomorphic function τ with positive imaginary part (the modulus of the fibre), and in such a representation the vector field v becomes $a(z)\frac{\partial}{\partial w}$ for some holomorphic and nonvanishing function a. Therefore we obtain, locally, $f(z) = \log \frac{Im\tau(z)}{|a(z)|^2} = \log Im\tau(z) - \log |a(z)|^2$, which is superharmonic. The case of a neighbourhood of a multiple (smooth) fibre is completely similar. Observe now that $f(t) \to +\infty$ as $t \to t_0 \in B \setminus B^*$: the fibre F_{t_0} contains some zero of v, which forces the dual 1-form ω_{t_0} to acquire some pole and so to loss the square-integrability. By the maximum principle we deduce that $B = B^*$ and that f is a constant. The first property means that every fibre of π is smooth and v has no zero. The second property implies that regular fibres are isomorphic, and thus π is locally trivial: $\log Im\tau(z)$ and $-\log |a(z)|^2$ are both superharmonic, their sum being a constant we see that they are both harmonic, and the harmonicity of $Im\tau(z)$ and $\log Im\tau(z)$ implies the constancy of $Im\tau(z)$, i.e. of $\tau(z)$. $\triangle$

We shall resume the conclusion of this proposition by saying that π (or $\mathcal{F}$) is an *almost elliptic fibre bundle.*

Suppose now that $kod(X) = 0$. We firstly consider the minimal case, i.e. X is free of (-1)-curves: any (-1)-curve is preserved by an holomorphic vector field, and after contracting it to a point p we obtain a new holomorphic vector field, vanishing at p. Then $K_X^{\otimes m}$ is trivial for some positive m [BPV, chapter VI], and so there is a regular covering $Y \to X$ such that $K_Y = \mathcal{O}_Y$. Vector fields on X lift to Y, and on Y the study of vector fields is equivalent to the study of 1-forms, because $TY \simeq T^*Y$. Thus, if alb_Y has one-dimensional image then any holomorphic vector field on Y is tangent to the fibres of the Albanese fibration $Y \to B$, which will be an almost elliptic fibre bundle (and even more, because $K_Y = \mathcal{O}_Y$). If alb_Y has two-dimensional image then there are on Y two holomorphic vector fields v_1, v_2 with $v_1 \wedge v_2 \not\equiv 0$; being K_Y^* trivial, we infer that v_1 and v_2 are everywhere linearly independent, and moreover they commute (this corresponds to the closedness of holomorphic 1-forms). We deduce that Y is a torus, and that the only holomorphic vector fields on Y are the linear ones, generating Kronecker foliations.

Returning to X, regular quotient of Y, we note that the quotient of an almost elliptic fibre bundle is still an almost elliptic fibre bundle, and that a nontrivial quotient of a Kronecker foliation is also an almost elliptic fibre bundle (i.e. the initial Kronecker foliation is an elliptic fibre bundle). This last fact is an exercise for the reader, who may also look at [BPV, page 148] for the descrition of regular quotients of tori (hint: these quotients are nonprincipal elliptic bundles).

All these vector fields are free of zeroes, hence they cannot blowed-up. We thus obtain the following resuming description.

Proposition 5. *Let X be a connected compact surface of nonnegative Kodaira dimension and let v be a nontrivial holomorphic vector field on X. Then:*

i) the zero set of v is empty;

ii) either v generates an almost elliptic fibre bundle, or it generates a Kronecker foliation on a torus. $\triangle$

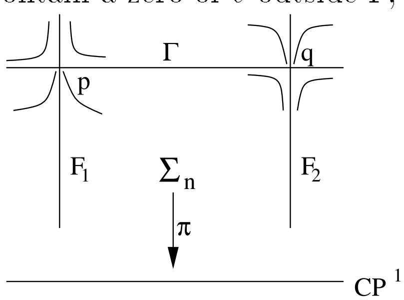
When X is Kähler, the first statement can be proved (in any dimension) by a differential geometric argument [Kob]. Note that we tried to use as less as possible the classification of surfaces.

Suppose now that $kod(X) = -\infty$ and, moreover, that X is algebraic. The Albanese morphism is either trivial (if $h^{1,0}(X) = 0$) or a rational fibration over a curve of genus $g \geq 1$ (if $h^{1,0}(X) = g$). The flow of an holomorphic vector field v on X preserves the Albanese fibration, and thus v projects to $alb_X(X)$ to an holomorphic vector field v_0. If $g \geq 2$ then $alb_X(X)$ is hyperbolic and $v_0 \equiv 0$, i.e. v is tangent to the fibres of alb_X. Remark that X need not to be minimal, and the zero set of v contains one or two sections of the Albanese fibration. If $g = 1$ then either $v_0 \equiv 0$ (and so v is again tangent to the fibres of alb_X) or v_0 (and hence v) is everywhere nonvanishing. In this last case alb_X has no singular fibre, i.e. X is minimal, and v generates a Riccati foliation without invariant fibres. If

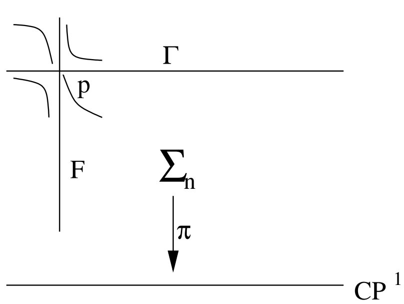
$$\rho : \pi_1(Alb(X)) \to Aut(\mathbf{C}P^1)$$

is the corresponding monodromy, we shall say that v is the *suspension* of ρ. Note that, conversely, any foliation transverse to the fibres of a $\mathbf{C}P^1$-bundle over an elliptic curve is generated by a global holomorphic vector field.

If $h^{1,0}(X) = 0$ then X is a rational surface. Here we shall describe v modulo birational transformations which preserve the holomorphicity of v. As usual, by contracting the (-1)-curves, which are forcely invariant by v, we firstly reduce X to a minimal surface, i.e. $\mathbf{C}P^2$ or an Hirzebruch surface Σ_n, $n \geq 0$, $n \neq 1$. A holomorphic vector field on $\mathbf{C}P^2$ has always at least one zero, which can be blowed-up giving a holomorphic vector field on Σ_1; we may thus replace $\mathbf{C}P^2$ by Σ_1. On Σ_n, $n \geq 1$, there is a (unique) special section Γ of selfintersection $-n$. It is obviously preserved by the flow of v. If $v|_\Gamma \equiv 0$ then v is vertical and it generates the rational fibration of Σ_n; note that the zero set of v has the form $\Gamma \cup \Gamma'$, where Γ' is a second section, of selfintersection n. If $v|_\Gamma \not\equiv 0$ then $v|_\Gamma$ has 1 or 2 zeroes, corresponding to fibres which are either invariant fibres of the Riccati foliation $\mathcal{F}$ generated by v, or fibres over which v identically vanishes. At least one of these fibres must contain a zero of v outside Γ, otherwise we would contradict Camacho-Sad formula:

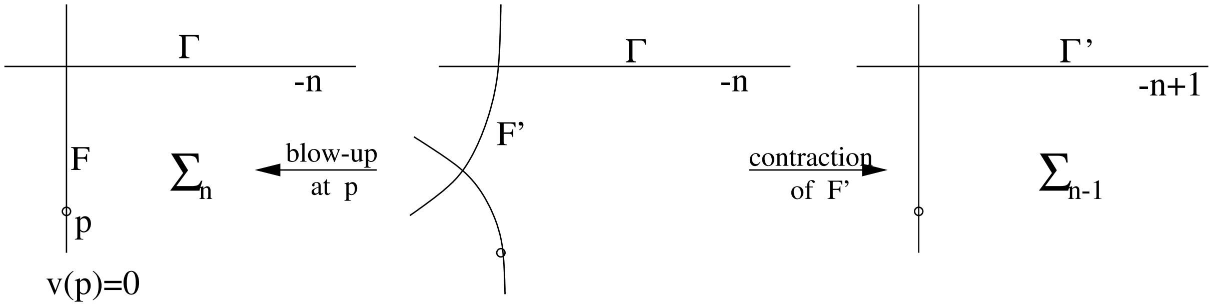


$$\Gamma^2 = CS(\mathcal{F},\Gamma,p) + CS(\mathcal{F},\Gamma,q) = 0 + 0 = 0$$



$$\Gamma^2 = CS(\mathcal{F},\Gamma,p) = 0$$

Hence, we can flip that fibre to obtain a new holomorphic vector field on the Hirzebruch surface Σ_{n-1}:



Iterating this procedure we finally arrive to a holomorphic vector field on $\Sigma_0 = \mathbf{C}P^1 \times \mathbf{C}P^1$, which is necessarily of the type $v_1 \oplus v_2$, v_1 and v_2 holomorphic vector fields on $\mathbf{C}P^1$. All of this works even if $v|_\Gamma \equiv 0$.

We thus obtain:

__**Proposition 6**__. *Let X be an algebraic surface of negative Kodaira dimension and let v be a nontrivial holomorphic vector field on X.*

i) If $h^{1,0}(X) \geq 2$ then v is tangent to the Albanese fibration.

ii) If $h^{1,0}(X) = 1$ then either v is tangent to the Albanese fibration or it is the suspension of a representation

$$\rho : \pi_1(Alb(X)) \to Aut(\mathbf{C}P^1) \qquad .$$

iii) If $h^{1,0}(X) = 0$ then there exists a birational map $X \,-\, - \to \mathbf{C}P^1 \times \mathbf{C}P^1$ sending v to a holomorphic vector field $v_1 \oplus v_2$ on $\mathbf{C}P^1 \times \mathbf{C}P^1$.

If $kod(X) = -\infty$ but X is not algebraic, then the situation is still not entirely understood, except when $b_2(X) = 0$ [In] (this paper gives a complete classification of foliations on surfaces of class VII vith $b_2 = 0$). We refer to [DOT] for significant results in the case $b_2 > 0$.

THE RATIONALITY CRITERION

In this chapter we explain a remarkable theorem of Miyaoka [Miy] which asserts that a foliation whose cotangent bundle is not pseudoeffective is a foliation by rational curves. The original Miyaoka's proof can be thought as a foliated version of Mori's technique of construction of rational curves by deformations of morphisms in positive characteristic [M-P]. Here, however, we shall firstly follow a slightly different proof, given by Shepherd-Barron [ShB] (see also [Eke]) and based on a more substantial use of the special features of foliations in positive characteristic. As in the other chapters, we limit ourselves to foliations on surfaces, but one has to say that these results can be (almost straightforwardly) adapted to a higher dimensional context, where they were in fact originated. In the special case of foliations on surfaces, or even of one dimensional foliations on higher dimensional varieties, there is also a relatively simple proof by Bogomolov and McQuillan [B-M] which stays entirely inside characteristic zero. In the last section we shall also explain another possible transcendental approach, based on results from [Suz] and [Kiz].

1. Statement and first consequencies

Theorem 1 [Miy]. *Let X be a smooth, compact, complex algebraic surface and let $\mathcal{F}$ be a foliation on X. Suppose that there exists an ample divisor H on X such that*

$$T_{\mathcal{F}} \cdot H > 0.$$

Then $\mathcal{F}$ is a foliation by rational curves: through each $x \in X$ there is a (nontrivial) rational curve tangent to $\mathcal{F}$. In other words, up to blowing-up $\mathcal{F}$ is a rational fibration.

Remark that one does not need to blow-up if the singularities of $\mathcal{F}$ are nondicritical (for instance, reduced). On the other hand, if $\mathcal{F}$ is a rational fibration then certainly there exists an ample H with $T_{\mathcal{F}} \cdot H > 0$: just take $H = nF + G$ where F is any regular fibre, G is any ample divisor and $n >> 0$, and observe that $T_{\mathcal{F}} \cdot F = 2$. Thus Miyaoka's theorem gives a *complete numerical characterization* of rational fibrations among foliations with nondicritical singularities.

In the following two chapters, theorem 1 will be used in the following form: *if $\mathcal{F}$ is not a foliation by rational curves then $T^*_{\mathcal{F}}$ is pseudoeffective*, i.e. $T^*_{\mathcal{F}} \cdot H \geq 0$ for every ample H. In this form ("generic semipositivity of the cotangent bundle of a nonrational foliation"), Miyaoka's theorem is the first (and most basic) step toward a "Mori theory" for foliations. The second step is the analysis of the negative part of $T^*_{\mathcal{F}}$ (when pseudoeffective), which will be carried out in the next chapter. The third step is the abundance conjecture, which will be treated in the last chapter.

We also note the following very particular case, related to the previous chapter: if $T_{\mathcal{F}}$ is effective and nontrivial (i.e. $\mathcal{F}$ is generated by a global holomorphic vector field v whose zero divisor $(v)_0$ is not empty) then $\mathcal{F}$ is a foliation by rational curves. In fact, in that case one has $T_{\mathcal{F}} \cdot H > 0$ for *any* ample divisor H. The power of theorem 1 is that it suffices to require $T_{\mathcal{F}} \cdot H > 0$ only for *some* ample H. Of course, when $T_{\mathcal{F}}$ is effective and nontrivial then one can directly prove the conclusion of Miyaoka's theorem by using the results of the previous chapter. A less trivial but still quite easy (?) case is when $T_{\mathcal{F}}^{\otimes n}$ is effective and nontrivial for some positive n.

As a first application of theorem 1, let us observe that it gives a quite short proof of the classification of nonsingular foliations on rational surfaces (chapter 3): once we know that the cotangent bundle of such a foliation is not pseudoeffective (lemma 1 of chapter 3) then Miyaoka's theorem immediately gives that $\mathcal{F}$ is a (regular) rational fibration.

The idea of Miyaoka's theorem is the following. We may suppose H very ample, and supported on the image of an embedding $f : C \to X$, for some algebraic curve C. The positivity of $T_{\mathcal{F}}$ on H "suggests" that $T_{\mathcal{F}}$ has nontrivial sections over H, which means that we may infinitesimally deform f along $\mathcal{F}$ (without changing C and fixing a point of C). Then we may hope that this infinitesimal deformation integrates to a true deformation, parametrized by a small disc and hence by a quasi-projective curve T. By compactifying, we obtain a map $F : \bar{T} \times C \dashrightarrow X$, and Mori's bend-and-break lemma gives rational curves on X as images of indeterminacy points of F. These rational curves are tangent to $\mathcal{F}$, because f has been deformed along $\mathcal{F}$ (see the lectures by Miyaoka in [M-P] for all these techniques). Of course, there are many gaps in such a proof: the positivity of a line bundle on a curve is not sufficient to construct sections, infinitesimal deformation is not the same as actual deformation, etc.. However, by reducing to positive characteristic these gaps can be filled, so that one can construct rational curves in positive characteristic which can be later lifted to zero characteristic (provided their degrees are uniformly bounded, which is actually the case). We refer to [Miy] for more details on this approach. In the next sections we will give proofs of theorem 1 following [ShB], still based on positive characteristic arguments, and [B-M], based on transcendental arguments, and we will also discuss a "metric" approach.

Let us finally note that it would be extremely useful to prove a "dual" version of theorem 1, that is to understand the structure of foliations $\mathcal{F}$ such that $N^*_{\mathcal{F}} \cdot H > 0$ for some ample H. By analogy with theorem 1, one could suspect that foliations with such a property are "close" to foliations generated by a global holomorphic 1-form. A very special case can be found in [Br1], where the pseudoeffectivity of $N^*_{\mathcal{F}}$ and the absence of singularities of the foliation are used to construct a transversely invariant hermitian metric. The absence of singularities is perhaps a minor obstacle to bypass, but the pseudoeffectivity of $N^*_{\mathcal{F}}$ is (a priori) much stronger than the positivity of $N^*_{\mathcal{F}}$ on some ample divisor.

2. Foliations in positive characteristic

Foliations on algebraic surfaces over a field k of positive characteristic p can be defined in exactly the same way as foliations on complex algebraic surfaces, by taking local regular vector fields or 1-forms with isolated zeroes which differ by multiplicative factors on overlapping open sets. The new, and very useful, fact is the following one. Let v be a regular vector field on a (Zariski) open set $U \subset X$. We may consider v as a derivation of the algebra $\mathcal{O}_X(U)$, and so we may take, for each $n \in \mathbf{N}^+$, its n-power $v^n = v \circ v \circ ... \circ v$ (n times), a differential operator of order n. For arbitrary n, v^n is no more a derivation, because Leibniz rule is violated; but for $n = p$ it turns out that v^p does satisfy Leibniz rule, and thus it corresponds to a regular vector field on U. Moreover, if $f \in \mathcal{O}_X(U)$ then a computation gives

$$(fv)^p = f^p v^p + v^{p-1}(f)v.$$

We see from that formula that a statement like "v^p is parallel to v" (i.e. $v^p \wedge v \equiv 0$) is invariant by multiplication of v by a function f. In other words, the validity of such a statement depends only on the foliation generated by v, and so the following definition is well posed.

Definition 1. A foliation $\mathcal{F}$ on an algebraic surface X over a field k of characteristic $p > 0$ is said to be *p-closed* if for every (or some) local regular vector field v tangent to $\mathcal{F}$ the vector field v^p is also tangent to $\mathcal{F}$.

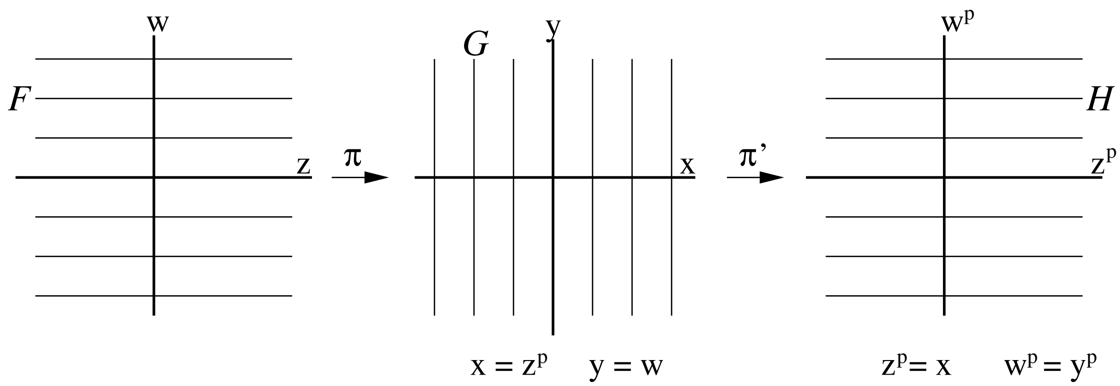
Of course, instead of taking local regular vector fields we may also take global rational ones, which are derivations of the algebra of rational functions.

The usefulness of p-closed foliations comes from the fact that they always have a good quotient space [Ses] [M-P]. Let us firstly recall that the field of rational functions on X, $k(X)$, contains a remarkable integrally closed subfield: $k(X)^p$, the field of p-powers of rational functions. This field corresponds to a surface $X^{(1)}$, and the inclusion $k(X)^p \subset k(X)$ corresponds to a projection $Frob : X \to X^{(1)}$, called *Frobenius morphism.* In local coordinates one has $Frob(z,w) = (z^p, w^p)$. Remark that the differential of $Frob$ is identically zero, even if $Frob$ is far from being a constant (it is a set-theoretic bijection!). It is a *purely inseparable morphism* of degree p^2: for each $x \in X^{(1)}$, $Frob^{-1}(x)$ is a single point of X, with multiplicity p^2.

Let now $\mathcal{F}$ be a p-closed foliation on X, and let $k(\mathcal{F})$ be the field of *first integrals* of $\mathcal{F}$, i.e. rational functions on X which are annihilated by vector fields tangent to $\mathcal{F}$ (one cannot say "constant on the leaves", because there are no leaves in positive characteristic...). It is an integrally closed subfield of $k(X)$, and moreover it contains $k(X)^p$: each p-power f^p has zero differential and so it is annihilated by *any* vector field. Thus $k(\mathcal{F})$ corresponds to a normal surface (possibly singular) $X/\mathcal{F}$, which is between X and $X^{(1)}$ and through which $Frob$ factorizes:

$$Frob : X \stackrel{\pi}{\longrightarrow} X/\mathcal{F} \stackrel{\pi'}{\longrightarrow} X^{(1)}.$$

In suitable local coordinates (z,w) on X and (x,y) on $X/\mathcal{F}$ (around a generic point), one has $(x,y) = \pi(z,w) = (z^p, w)$, and $\mathcal{F}$ is generated by $\frac{\partial}{\partial z}$. That is, the differential $D\pi$ has, at a generic point, a one-dimensional kernel, and $\mathcal{F}$ is generated by that kernel. On $X/\mathcal{F}$ also we have a p-closed foliation $\mathcal{G}$, equivalently described as the kernel of $D\pi'$ or the image of $D\pi$ (because $(D\pi') \circ (D\pi) = DFrob = 0$). In the previous local coordinates (x,y), $\mathcal{G}$ is generated by $\frac{\partial}{\partial y}$. Finally, on $X^{(1)}$ we have a foliation $\mathcal{H}$, image of $D\pi'$ and expressed by $\frac{\partial}{\partial z^p}$ in the local coordinates (z^p, w^p).



The two morphisms π and π' are purely inseparable of degree p. Conversely, given a factorization of $Frob$ by two purely inseparable morphisms π and π' of degree p we obtain a p-closed foliation on X by taking the kernel of $D\pi$.

Let us now consider the canonical bundle $K_{X/\mathcal{F}}$ of $X/\mathcal{F}$, or more precisely its lift to X by π. We may decompose $K_{X/\mathcal{F}}$ as $T^*_{\mathcal{G}} \otimes N^*_{\mathcal{G}}$. From $\pi^*(dy) = dw$ we see that $\pi^*(T^*_{\mathcal{G}}) = N^*_{\mathcal{F}}$. Similarly, we have $\pi'^*(T^*_{\mathcal{H}}) = N^*_{\mathcal{G}}$, and hence $Frob^*(T^*_{\mathcal{H}}) = \pi^*(N^*_{\mathcal{G}})$. But we also have $Frob^*(T^*_{\mathcal{H}}) = (T^*_{\mathcal{F}})^{\otimes p}$ (transition functions of $T^*_{\mathcal{H}}$ are p-powers of those of $T^*_{\mathcal{F}}$) and therefore we finally obtain [M-P] [ShB] [Eke]:

$$\pi^*(K_{X/\mathcal{F}}) = (T^*_{\mathcal{F}})^{\otimes p} \otimes N^*_{\mathcal{F}}.$$

This formula is particularly important: it says that for p large numerical properties of $K_{X/\mathcal{F}}$ are principally reflected by numerical properties of $T^*_{\mathcal{F}}$, and so numerical problems about p-closed foliations are translated into (possibly easier) numerical problems about surfaces. In some sense, that formula allows to reduce Miyaoka's theorem to Mori's theorem.

3. Proof of theorem 1

As the reader may imagine, the first step is to prove that the foliation $\mathcal{F}$ is p-closed, when reduced mod p for p large (see [M-P] for generalities about reduction mod p).

Lemma 1. *When reduced mod p for p sufficiently large, the foliation $\mathcal{F}$ is p-closed.*

Proof.

Let v_j be local regular vector fields with isolated zeroes and generating $\mathcal{F}$, so that

$$v_i = g_{ij} v_j$$

and $\{g_{ij}\} \in H^1(X, \mathcal{O}^*_X)$ represents $T^*_{\mathcal{F}}$. Let also ω_j be local regular 1-forms with isolated zeroes generating $\mathcal{F}$:

$$\omega_i = f_{ij} \omega_j$$

with $\{f_{ij}\}$ representing $N_{\mathcal{F}}$. Hence $\omega_j(v_j) \equiv 0$ for every j. After reducing mod p (via the projective embedding given by the ample divisor H) we have

$$v_i^p = (g_{ij} v_j)^p = g_{ij}^p v_j^p + v_j^{p-1}(g_{ij}) v_j$$

and therefore

$$\omega_i(v_i^p) = f_{ij} g_{ij}^p \omega_j(v_j^p).$$

Thus the functions $h_j = \omega_j(v_j^p)$ define a global regular section of $(T^*_{\mathcal{F}})^{\otimes p} \otimes N_{\mathcal{F}}$. But, for $p >> 0$, this line bundle has negative degree on the ample divisor H, and therefore such a regular section must vanish identically. That is

$$\omega_j(v_j^p) \equiv 0 \qquad \text{for every } j$$

which exactly means that v_j^p is tangent to $\mathcal{F}$. $\triangle$

By this lemma, we may consider for each $p >> 0$ the quotient space of X by $\mathcal{F}$:

$$X \stackrel{\pi}{\longrightarrow} X/\mathcal{F} \stackrel{\pi'}{\longrightarrow} X^{(1)}.$$

Without loss of generality, we may suppose H very ample. Take a generic point $x \in X$ and an irreducible curve $C \subset X$ through x and linearly equivalent to H. We may choose C not tangent to $\mathcal{F} = Ker\ D\pi$, so that if $C' = \pi(C)$ then $\pi|_C : C \to C'$ is separable, i.e. birational (recall that π is bijective). Therefore, from $\pi^*(K_{X/\mathcal{F}}) = (T^*_{\mathcal{F}})^{\otimes p} \otimes N^*_{\mathcal{F}}$ we find

$$K_{X/\mathcal{F}} \cdot C' = \pi^*(K_{X/\mathcal{F}}) \cdot C = p\ T^*_{\mathcal{F}} \cdot C + N^*_{\mathcal{F}} \cdot C$$

and in particular

$$K_{X/\mathcal{F}} \cdot C' < 0 \qquad \text{for } p >> 0.$$

Thanks to this inequality, we may apply Mori's deformation technique [M-P] to construct rational curves on $X/\mathcal{F}$. Let $H^{(1)}$ be the very ample divisor on $X^{(1)}$ derived by H (taking p-powers), and let $H' = (\pi')^*(H^{(1)})$, so that $\pi^*(H') = Frob^*(H^{(1)}) = pH$. Then [M-P, lecture II] there exists a rational curve $R' \subset X/\mathcal{F}$ passing through $x' = \pi(x) \in C'$ and whose degree $R' \cdot H'$ is bounded by

$$\frac{4C' \cdot H'}{-K_{X/\mathcal{F}} \cdot C'} = \frac{4p\ C \cdot H}{p\ T_{\mathcal{F}} \cdot C + N_{\mathcal{F}} \cdot C} \leq K$$

where K is a constant independent on p and x. Note that the fact that $X/\mathcal{F}$ is possibly singular gives no trouble here, for instance because we may assume C disjoint from $Sing(\mathcal{F})$ and consequently C' inside the smooth subset of $X/\mathcal{F}$.

Let now $R \subset X$ be the preimage of R' by π. It is still a rational curve, because π is bijective, and $x \in R$.

Lemma 2. *R is tangent to $\mathcal{F}$.*

Proof.

If not, then $\pi : R \to R'$ is birational and therefore

$$R' \cdot H' = R \cdot \pi^*(H') = pR \cdot H$$

and the bound above on $R' \cdot H'$ gives $R \cdot H \leq \frac{1}{p}K$, which is clearly absurd because for p large $\frac{1}{p}K < 1$ whereas $R \cdot H \in \mathbf{N}^+$. $\triangle$

Lemma 3. *$R \cdot H \leq K$.*

Proof.

Because $\pi : R \to R'$ is purely inseparable of degree p (being R tangent to $\mathcal{F}$ by the previous lemma), we have

$$pR \cdot H = R \cdot \pi^*(H') = pR' \cdot H' \leq pK$$

as required. $\triangle$

Thus, through a generic $x \in X$ we have found a rational curve on X mod p, tangent to $\mathcal{F}$ and whose H-degree is bounded by a constant independent on p and x. This uniform bound allows to lift these rational curves to characteristic zero, preserving the tangency to $\mathcal{F}$ and the bound on the degree [M-P]. Hence, on the original complex surface there is an open family of rational curves of bounded degree tangent to $\mathcal{F}$. A standard compactness argument (Chow scheme) permits to extend this family of rational curves to a rational fibration on the full X (with possibly some indeterminacy point) tangent to $\mathcal{F}$. This achieves the proof. $\triangle$

4. A proof by Bogomolov and McQuillan

In this section we present the elegant proof of theorem 1 discovered by Bogomolov and McQuillan [B-M]. That proof relies on two steps: firstly one proves that the leaves of the foliation are algebraic curves, and then one applies Arakelov's theorem [BPV, page 110] to conclude that these algebraic curves are rational ones.

Under the hypothesis of theorem 1, we have on X a smooth connected curve C (linearly equivalent to nH for some positive n) such that:
i) C is not $\mathcal{F}$-invariant, and $C \cap Sing(\mathcal{F}) = \emptyset$;
ii) $T_{\mathcal{F}} \cdot C > 0$.

Let us consider the threefold $Y = X \times C$. The foliation $\mathcal{F}$ lifts to Y as a one-dimensional foliation $\mathcal{G}$, tangent to the horizontal surfaces $X \times \{p\}$, $p \in C$. The curve C lifts to Y as the diagonal $D \subset C \times C \subset X \times C$. We have $D \cap Sing(\mathcal{G}) = \emptyset$, and moreover for every $p \in D$ the leaf of $\mathcal{G}$ through p is *not* tangent to D. Hence, for every $p \in D$ we may take a small disc, centered at p and inside the leaf of $\mathcal{G}$ through p, in such a way that the union of all these discs is a (transcendental) smooth surface $S \subset Y$ containing D.

This surface S can be also described as follows. We take a small tubular neighbourhood U of C in X, and we consider on U the (multivalued) map to C which associates to every $q \in U$ the intersection point(s) with C of the leaf of $\mathcal{F}|_U$ through q. Then S is the graph of that (multivalued) map. The possible multivaluedness comes from the tangency points of $\mathcal{F}$ with C.

Let us now compute the selfintersection of D in S. We clearly have $\mathcal{O}_S(D)|_D = T_{\mathcal{G}}|_D$, because $\mathcal{G}$ is nowhere tangent to D, and $T_{\mathcal{G}}|_D \simeq T_{\mathcal{F}}|_C$. Therefore,

$$D \cdot D = T_{\mathcal{F}} \cdot C > 0.$$

This implies that S is a pseudoconcave surface in the sense of Andreotti [And]. A theorem of [And] says that the algebraic dimension of S is, at most, equal to 2. Being Y an algebraic threefold, this means that the Zariski closure of S in Y is an algebraic surface; in other words, there exists a (possibly singular) algebraic surface $\hat{S} \subset Y$ which contains S.

Obviously, $\hat{S}$ is $\mathcal{G}$-invariant and $\mathcal{G}|_{\hat{S}}$ is nothing but than the foliation obtained by intersecting $\hat{S}$ with the horizontal surfaces $X \times \{p\}$, $p \in C$. That is, $\mathcal{G}|_{\hat{S}}$ is a fibration over C and its leaves are algebraic curves. The curve $D \subset \hat{S}$ is (after normalisation of $\hat{S}$) a section of that fibration, and a theorem of Arakelov [BPV, page 110] says that the fibration is a rational one because $D \cdot D > 0$. Therefore the leaves of $\mathcal{G}|_{\hat{S}}$, and consequently the leaves of $\mathcal{F}$, are rational curves.

5. Construction of special metrics

The pseudoeffectivity of a line bundle on an algebraic surface is equivalent to the existence on that bundle of a (singular) hermitian metric whose curvature is a closed positive current [Dem] [Fuj]. We may therefore try to prove Miyaoka's theorem by constructing such a metric on $T^*_{\mathcal{F}}$, when $\mathcal{F}$ is not a foliation by rational curves. The advantage of this point of view is that, potentially, it could be adapted to foliations on non-algebraic (but possibly Kählerian) surfaces, where the arguments of the two previous sections are not available (in Bogomolov-McQuillan proof the algebraicity of the threefold Y was essential). Moreover, this metric approach would give a result slightly stronger, because (see below) we would obtain a metric on $T^*_{\mathcal{F}}$ possessing also some regularity and homogeneity along the leaves, and this could be useful for several purposes (see, for instance, the last section of these notes).

Let us firstly explain a result of Kizuka [Kiz]. Let U be a Stein surface, equipped with a holomorphic surjective map $f : U \to \mathbf{D}$ without critical points. On each irreducible component of each fibre $f^{-1}(t)$ we put the *Poincaré metric*, which is (by definition) the unique hermitian metric of curvature -1 if the irreducible component is not $\mathbf{C}$ nor $\mathbf{C}^*$, and the zero metric otherwise. Then Kizuka proves that this family of metrics on the fibres of f has a "plurisubharmonic variation". In our terminology, this means precisely the following: looking at f as defining a foliation $\mathcal{G}$ on U, the Poincaré metric on fibres induces an hermitian metric on $T_{\mathcal{G}}$, thus by duality on $T^*_{\mathcal{G}}$, and then this last metric is a singular metric whose curvature is a closed positive current. We also note that the hypothesis on the absence of critical points of f is not really essential: everything works even if f has critical points, and even critical curves, provided that one gives the right definition of Poincaré metric in the singular case. Also, the Steinness hypothesis can be weakened to: U is holomorphically convex (which means, by a theorem of Remmert, that either U is a modification of a normal Stein space U_0, or f is proper).

Let now $\mathcal{F}$ be a foliation (not by rational curves) on a (smooth, compact) algebraic surface X. With no loss of generality, we may suppose that $\mathcal{F}$ is reduced (if $\tilde{\mathcal{F}}$ is the resolution of $\mathcal{F}$, then $T^*_{\tilde{\mathcal{F}}}$ pseudoeffective implies $T^*_{\mathcal{F}}$ pseudoeffective, by an easy argument). In order to apply Kizuka result, we are naturally led to a nice construction of Suzuki [Suz], the so-called *normal tubes*. Take a (small) transversal T to $\mathcal{F}$, isomorphic to the disc. On a neighbourhood of T we have a holomorphic first integral of $\mathcal{F}$, given by the projection on $T \simeq \mathbf{D} \subset \mathbf{C}$. Let $\mathcal{U}$ be the domain of holomorphy over X of this first integral. Then $\mathcal{U}$ contains an open subset U_T equipped with a projection $U_T \stackrel{f}{\to} T$ (the same first integral), such that for every $t \in T$ the fibre $f^{-1}(t)$ is a covering of the leaf of $\mathcal{F}$ through t. Here the leaves are those of $\mathcal{F}|_{X'}$, where $X' = (X \setminus Sing(\mathcal{F})) \cup Int(\mathcal{F})$, $Int(\mathcal{F})$ are the singular points around which $\mathcal{F}$ has a holomorphic first integral (of monomial type by the reducedness hypothesis), and if $q \in Int(\mathcal{F})$ then the local leaf through q is the union of the two separatrices through q. Roughly speaking, U_T is the "abstract" saturation of T by $\mathcal{F}$, obtained by glueing together the leaves of $\mathcal{F}$ through points of T but forgetting their (wild) inclusions into X. The projection f may have critical points (or curves), in correspondence with leaves through points of $Int(\mathcal{F})$. A fibre $f^{-1}(t)$ free of critical points corresponds to the holonomy covering of the leaf of $\mathcal{F}$ through t, leaf which does not pass through $Int(\mathcal{F})$. If $f^{-1}(t)$ contains critical points then the situation is a little more complicated, but easily understandable.

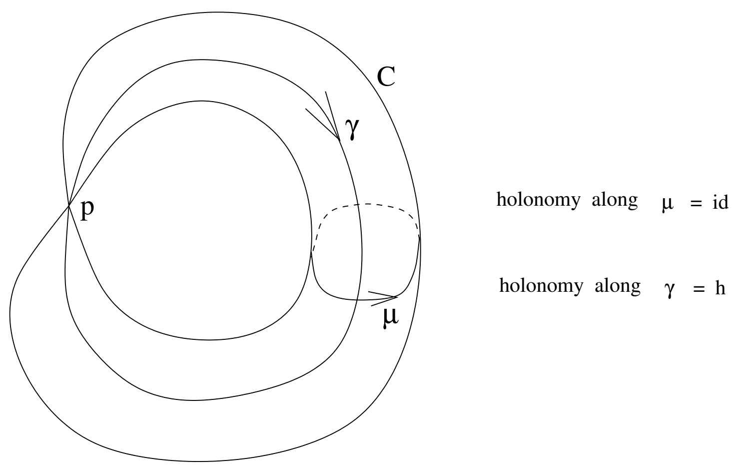
Now the fundamental question is the following: *is the normal tube U_T holomorphically convex?*

When the ambient surface X is Stein, instead of algebraic, then U_T also is Stein [Suz]; this is related to the fact that the Stein property is preserved by coverings. The same preservation by coverings does not hold, generally speaking, for the holomorphic convexity. A famous (and still open) conjecture by Shafarevich asserts however that the universal covering of an algebraic manifold should be holomorphically convex. We may thought at the question above as a "foliated" Shafarevich conjecture, in which the universal covering of the algebraic surface is replaced by the holonomy covering of the foliation, and we may hope for a positive answer to that question.

If for any transversal T the normal tube U_T is holomorphically convex then we can almost prove Miyaoka's theorem as follows. By Kizuka's result, the Poincaré metric on the fibres of $f : U_T \to T$ has a plurisubharmonic variation. This metric being canonical, it can be projected to X, and thus we obtain a metric on $T^*_{\mathcal{F}}$ (outside $Sing(\mathcal{F})$) whose curvature is a closed positive current (and which can therefore be extended to $Sing(\mathcal{F})$). There is, however, another problem: if all the fibres of f are isomorphic to $\mathbf{C}$ or $\mathbf{C}^*$ then the Poincaré metric becomes totally degenerate and we obtain on $T^*_{\mathcal{F}}$ the zero metric, which is useless. But one can hope that foliations all of whose leaves are entire are quite special and classifiable by other methods (these foliations will be actually classified in chapter 9, using Miyaoka's theorem but under much more weaker assumptions: a *single* entire transcendental leaf will suffice).

Anyway, the question on the holomorphic convexity seems interesting and its solution could be useful for other purposes. Let us conclude with an example showing (via Miyaoka's theorem) that a positive answer is not hopeless. This example illustrates also, in a particular case, the results which will be developed in the next chapters.

Example. Let $\mathcal{F}$ be a foliation on an algebraic surface X possessing an invariant curve C which is a rational curve with a node p (thus a virtually elliptic curve). Suppose that p is the only singularity of $\mathcal{F}$ on C, and that it is a singularity of the type $d(zw) = 0$, $C = \{zw = 0\}$. By Camacho-Sad formula we have $C \cdot C = 0$. The holonomy of $\mathcal{F}$ along $C^* = C \setminus \{p\}$ is clearly trivial. However, we may also consider the "holonomy" h of $\mathcal{F}$ along a cycle $\gamma \subset C$ which passes through p and which generates the fundamental group $\pi_1(C) = \mathbf{Z}$: this "holonomy" is well defined, because $\mathcal{F}$ has a first integral of the type zw around p.



It could happen that h is *not* the identity, and even that h has infinite order. In fact, in a *local* situation (i.e. X is not an algebraic surface, but only a tubular neighbourhood of such a curve) it is not difficult to construct examples where h is any prescribed germ, by doing some easy holomorphic surgery. Suppose that h has infinite order: then the normal tube U_T associated to a transversal T cutting C contains, as a fibre F of $f : U_T \to T$, an infinite chain of rational curves, each one of selfintersection -2. Such a chain is clearly an obstruction to the holomorphic convexity of U_T: any holomorphic function on U_T would be constant on the noncompact fibre F.

However, when X is algebraic the holonomy of h has finite order, and so that obstruction vanishes (which does not yet mean that U_T is holomorphically convex, however). This periodicity of h follows from the results of the next two chapters (see especially corollary 2 in chapter 9), which heavily rely on Miyaoka's theorem: we shall show that the virtually elliptic curve C is a (singular) fibre of an elliptic fibration $\pi : X \to B$, and therefore $\mathcal{F}$ either coincides with that fibration (i.e. h is periodic) or it is turbulent with respect to π. But this second possibility cannot occur, because we have already observed in chapter 4 that a turbulent foliation cannot have invariant fibres of type I_b, $b \geq 1$.

In this chapter we study, following [Mc1], the first properties of the Zariski decomposition of the cotangent bundle of a nonrational foliation. In particular, we shall give a detailed description of the negative part of that Zariski decomposition, and we shall obtain a detailed classification of foliations whose Zariski decomposition is reduced to its negative part (i.e. foliations of numerical Kodaira dimension 0). We shall also discuss the "singular" point of view adopted in [Mc1].

1. Zariski decomposition and numerical Kodaira dimension

From now on we shall be mainly concerned with the cotangent bundle of a foliation on an algebraic surface. It is convenient to introduce a new notation: we shall denote by

$$K_{\mathcal{F}}$$

the cotangent bundle $T^*_{\mathcal{F}}$ of a foliation $\mathcal{F}$, and we shall call it the *canonical bundle* of $\mathcal{F}$. This is also to stress the analogy with the classification of surfaces (cfr. the introduction).

Suppose that $\mathcal{F}$, foliation on a smooth algebraic surface X, is *not* a foliation by rational curves. Then Miyaoka's rationality criterion says that $K_{\mathcal{F}}$ is *pseudoeffective*, i.e.

$$K_{\mathcal{F}} \cdot H \geq 0$$

for every ample divisor H. Note that the same inequality holds even if H is only nef instead of ample, being the nef cone the closure of the ample cone (see, e.g., [Dem] and [Fuj]). A theorem of Fujita [Fuj] says that $K_{\mathcal{F}}$ can be *numerically* decomposed as

$$K_{\mathcal{F}} \stackrel{num}{=} P + N$$

where:

i) P is a *nef* $\mathbf{Q}$-divisor, i.e. $P = \sum_{j=1}^{p} a_j E_j$ with E_j irreducible curves, a_j rational numbers, and $P \cdot C \geq 0$ for every curve $C \subset X$;

ii) N is a *contractible* $\mathbf{Q}^+$-divisor, i.e. $N = \sum_{j=1}^n b_j D_j$ with D_j irreducible curves, b_j positive rational numbers, and the intersection matrix $(D_i \cdot D_j)_{i,j}$ is negative definite (i.e. each connected component of $\cup_j D_j$ can be contracted to a normal singularity);

iii) $P \cdot D_j = 0$ for every $j = 1, ..., n$.

Recall that numerical equivalence $\overset{num}{=}$ means $K_{\mathcal{F}} \cdot C = P \cdot C + N \cdot C$ for every curve $C \subset X$, or equivalently $K_{\mathcal{F}}$ and $P + N$ define the same class in $H^2(X, \mathbf{Q})$. Remark that P and N are only $\mathbf{Q}$-divisors, not $\mathbf{Z}$-divisors, and so (in general) they do not correspond to line bundles on X. This is a source of many technical and conceptual problems. Of course, for a suitable $l \in \mathbf{N}^+$ the $\mathbf{Q}$-divisors lP and lN are $\mathbf{Z}$-divisors, and so they do correspond to line bundles; however the line bundle $\mathcal{O}_X(lP) \otimes \mathcal{O}_X(lN)$, numerically equivalent to $K_{\mathcal{F}}^{\otimes l}$, is not necessarily equal to it: the difference is a numerically trivial line bundle (i.e. a flat line bundle, by Hodge theory) which may be nontrivial if $h^1(X, \mathcal{O}_X)$ is not zero or if $H^2(X, \mathbf{Z})$ has torsion.

The decomposition above is called *Zariski decomposition* of $K_{\mathcal{F}}$, P is called *positive* or *nef* part, and N is called *negative* or *contractible* part. One can easily prove that N (but not P) is uniquely defined, as a $\mathbf{Q}^+$-divisor. Note that if C is a curve such that $K_{\mathcal{F}} \cdot C < 0$ then necessarily $N \cdot C < 0$ (because P is nef) and so $C \subset Supp(N)$ (because N is positive). Be careful, however, that the converse is not true, in general: the negative part N of the Zariski decomposition of a line bundle may contain a curve D_j such that $N \cdot D_j > 0$ (exercise: find an example). It is however evident that N (if not empty) must contain at least one D_i such that $N \cdot D_i < 0$, for the negative definiteness of the intersection matrix.

If $K_{\mathcal{F}}$ is effective then P can be chosen with positive (but not integer, in general) coefficients, and then the choice of P also becomes almost unique, as a $\mathbf{Q}^+$-divisor with $Supp(P) \subset Supp(K_{\mathcal{F}})$. Note that the condition $P \cdot D_j = 0$ does not exclude that D_j belongs to $Supp(P)$: we shall see later several natural examples.

From the fact that P is nef it follows that $P \cdot P \geq 0$ [Dem] [Fuj]. We then define the *numerical Kodaira dimension* $\nu(\mathcal{F})$ of $\mathcal{F}$ as:

$\nu(\mathcal{F}) = 0$ if $P \overset{num}{=} 0$

$\nu(\mathcal{F}) = 1$ if $P \overset{num}{\neq} 0$ but $P \cdot P = 0$

$\nu(\mathcal{F}) = 2$ if $P \cdot P > 0$.

In order to be complete, we also set

$\nu(\mathcal{F}) = -\infty$ if $K_{\mathcal{F}}$ is not pseudoeffective.

There is no ambiguity in such a definition, because even if P is not uniquely defined, its numerical properties are obviously completely determined by those of $K_{\mathcal{F}}$ ($P \overset{num}{=} K_{\mathcal{F}} - N$, and N is unique). But from the birational point of view this is not a good definition, in the sense that if $(X, \mathcal{F})$ and $(Y, \mathcal{G})$ are birationally isomorphic then $\nu(\mathcal{F})$ and $\nu(\mathcal{G})$ are not necessarily equal (it may even happen that one is 2 and the other is $-\infty$). However, if $\mathcal{F}$ and $\mathcal{G}$ have only reduced singularities then $\nu(\mathcal{F}) = \nu(\mathcal{G})$, and thus the definition is a good one if we restrict our attention only to reduced foliations, as it was already done in chapter 5. We shall return on this point more in detail in the next chapter.

We now give some examples.

Example 1. Let $(Y, \mathcal{H})$ be the very special foliation of chapter 4, a resolved $\mathbf{Z}_3$-quotient of a linear foliation $(\mathbf{C}P^2, \mathcal{L})$. Recall that over each singular point $\hat{q}_j$ of $\mathbf{C}P^2/\mathbf{Z}_3$ we have a pair of $\mathcal{H}$-invariant smooth rational curves $D_j, E_j \subset Y$, each one of selfintersection -2, with $D_j \cdot E_j = 1$. Moreover, $Z(\mathcal{H}, D_j) = 1$ and $Z(\mathcal{H}, E_j) = 2$, that is $K_{\mathcal{H}} \cdot D_j = -1$ and $K_{\mathcal{H}} \cdot E_j = 0$. On a neighbourhood of $D_j \cup E_j$ we therefore have $K_{\mathcal{H}} \overset{num}{=} \frac{2}{3} D_j + \frac{1}{3} E_j$ (note that the coefficients of D_j, E_j are uniquely defined, for $D_j \cup E_j$ is a contractible curve). On the other hand, the linear foliation $\mathcal{L}$ has trivial canonical bundle, thus $K_{\mathcal{H}}$ is numerically concentrated over $\cup_{j=1}^{3} D_j \cup E_j$. Hence we have

$$K_{\mathcal{H}} \overset{num}{=} \sum_{j=1}^{3} (\frac{2}{3} D_j + \frac{1}{3} E_j) = N.$$

In this case, the positive part P of $K_{\mathcal{H}}$ is zero, so that

$$\nu(\mathcal{H}) = 0.$$

Note also that $h^0(Y, K_{\mathcal{H}}) = 0$ but $h^0(Y, K_{\mathcal{H}}^{\otimes 3}) = 1$ (recall the discussion in lemma 1, chapter 4). Finally, note that in this case the negative part N corresponds to a line bundle (the same $K_{\mathcal{H}}$!), even if N is not a $\mathbf{Z}$-divisor. The single $\frac{2}{3} D_j + \frac{1}{3} E_j$, however, does not correspond to a line bundle: its selfintersection is $-\frac{2}{3}$, a non-integer number.

Example 2. Let $(X, \mathcal{F})$ be a Riccati or a turbulent foliation, and suppose $K_{\mathcal{F}}$ pseudoeffective (which is not always the case). If $F \subset X$ is a generic fibre of the adapted fibration, then $K_{\mathcal{F}} \cdot F = 0$ and thus $P \cdot F = 0$, because $P \cdot F \geq 0$ being P nef and $N \cdot F \geq 0$ being $F \not\subset Supp(N)$ (this last is contractible and cannot contain a full fibre). By Hodge index theorem it follows that P is proportional to F: $P \overset{num}{=} qF$ for some $q \in \mathbf{Q}^+$ or $q = 0$. Hence

$$\nu(\mathcal{F}) = 0 \text{ if } q = 0 \,, \qquad \nu(\mathcal{F}) = 1 \text{ if} q > 0 \,.$$

The negative part has support disjoint from a generic fibre F, because $N \cdot F = 0$ and $F \not\subset Supp(N)$, and thus $Supp(N)$ is a union of certain curves inside singular fibres. When $\nu(\mathcal{F}) = 1$, we shall say that $\mathcal{F}$ is *properly* Riccati or turbulent.

For example, suppose that $\mathcal{F}$ is a turbulent foliation and let $\hat{F}$ be a singular fibre of type I_0^*: $\hat{F} = 2C + \sum_{j=1}^4 D_j$, all (-2)-curves, $C \cdot D_j = 1$ for every j, $D_i \cdot D_j = 0$ for every $i \neq j$. We have seen in chapter 4 that C may be $\mathcal{F}$-invariant or $\mathcal{F}$-transverse, but from the numerical point of view there is no difference: in any case, we have $K_{\mathcal{F}} \cdot C = 2$ and $K_{\mathcal{F}} \cdot D_j = -1$ for every j. On a neighbourhood of $\hat{F}$ we may therefore write $K_{\mathcal{F}} \stackrel{num}{=} \sum_{j=1}^4 \frac{1}{2} D_j$ (but we could also write $K_{\mathcal{F}} \stackrel{num}{=} r\hat{F} + \sum_{j=1}^4 \frac{1}{2} D_j$, because $\hat{F}$ is numerically trivial on such a neighbourhood). If, to fix ideas, $\hat{F}$ is the only singular fibre of the adapted fibration, we find globally

$$K_{\mathcal{F}} \stackrel{num}{=} qF + \sum_{j=1}^{4} \frac{1}{2} D_j$$

for a suitable $q \geq 0$ (which will depend on the genus of the base, the number of invariant fibres, etc.). But we may also write

$$K_{\mathcal{F}} \stackrel{num}{=} q\hat{F} + \sum_{j=1}^{4} \frac{1}{2} D_j$$

because $F \stackrel{num}{=} \hat{F}$. In this case, we have $Supp(P) \supset Supp(N)$.

Example 3. Let $(X, \mathcal{F})$ be a foliation which has an invariant curve $\hat{F}$ which is a "numerical" I_0^*-fibre: $\hat{F}$ is the union of five (-2)-curves $C, D_1, ..., D_4$ with the same intersection form as in the previous example and C of multiplicity 2, without being necessarily the fibre of an elliptic fibration (for instance, the normal bundle $\mathcal{O}_X(\hat{F})|_{\hat{F}}$ could be non-torsion). Suppose also that $K_{\mathcal{F}} \cdot \hat{F} = 0$, and as usual that $K_{\mathcal{F}}$ is pseudoeffective. Being $\hat{F}$ nef, we still find, by Hodge theorem as in the previous example, that $P = q\hat{F}$ and then

$$K_{\mathcal{F}} \stackrel{num}{=} q\hat{F} + \sum_{j=1}^{4} \frac{1}{2} D_j + N_0$$

where $q \geq 0$ and N_0 has support disjoint from $\hat{F}$. We therefore have $\nu(\mathcal{F}) = 0$ or 1, depending on $q = 0$ or > 0. In this case $Supp(P) \cap Supp(N) \neq \emptyset$, and (a priori) we cannot avoid this because we cannot choose, instead of $\hat{F}$, a "regular fibre" F as in the previous example.

Example 4. This is due to Lins Neto [L-N], in a different form. Let E be the elliptic curve which has an automorphism f of order 6 and with a fixed point. Then $T : E \times E \to E \times E$, $T(x, y) = (f^2(x), f^2(y))$, is an automorphism of order 3, with 9 fixed points arising from the 3 fixed points of f^2: $Fix(T) = Fix(f^2) \times Fix(f^2)$. The quotient $E \times E/T$ has 9 singularities of type $A_{3,1}$, and each singularity is resolved by a (-3)-curve L_j, $j = 1, ..., 9$ [BPV, page 84]. Call X the minimal resolution of $E \times E/T$. Any Kronecker foliation on $E \times E$ is T-invariant and gives rise to a foliation $\mathcal{F}$ on X, tangent to each L_j and having there a singularity of the type $d(z^3 w) = 0$ ($L_j = \{w = 0\}$). We are in a situation very close to that of example 1, and we see that

$$K_{\mathcal{F}} \overset{num}{=} \sum_{j=1}^{9} \frac{1}{3} L_j = N.$$

We have $\nu(\mathcal{F}) = 0$, $h^0(X, K_{\mathcal{F}}) = 0$, $h^0(X, K_{\mathcal{F}}^{\otimes 3}) = 1$ (a Kronecker foliation on $E \times E$ is T-invariant, but not the holomorphic vector field generating it).

To see the relation with [L-N], observe that on $E \times E$ there are 12 smooth elliptic curves (3 horizontals, 3 verticals, the diagonal and 2 of its translates, the graph of f^3 and 2 of its translates) which are individually T-invariant and which intersect only at the 9 fixed points of T, each fixed point belonging to 4 elliptic curves and each elliptic curve passing through 3 fixed points. After quotienting and resolving the singularities, these 12 elliptic curves become 12 smooth rational curves of selfintersection -1 and pairwise disjoint. They can be simultaneously contracted, and the result is the familiar $\mathbf{C}P^2$ (for instance, you may easily compute its Chern numbers). The 9 (-3)-curves L_j become 9 lines in $\mathbf{C}P^2$, intersecting at 12 points, each point belonging to 3 lines and each line passing through 4 points. The foliation $\mathcal{F}$ becomes a foliation $\mathcal{G}$ tangent to this configuration of lines. If the initial Kronecker foliation on $E \times E$ is not tangent to any of the 12 elliptic curves above, then $\mathcal{G}$ has 21 singularities: 12 radial type singularities in correspondence with the intersections of the lines plus 9 singularities of the type $d(z^3 w) = 0$. Therefore, its degree is 4. This is the original description of [L-N], and we refer to that paper for the consequencies of the existence of such a foliation on $\mathbf{C}P^2$.

Example 5. A similar construction as in example 4, but with involutions instead of automorphisms of order 3, gives foliations of numerical Kodaira dimension 0 on certain K3 surfaces (Kummer surfaces).

Example 6. If $\mathcal{F}$ is a fibration of genus $g \geq 1$ then the Zariski decomposition of $K_{\mathcal{F}}$ has been analyzed in [Ser], anticipating theorem 1 below. In particular, if the fibration (with reduced singularities) is not isotrivial then one has $\nu(\mathcal{F}) = g$, whereas if the fibration is isotrivial then $\nu(\mathcal{F}) = g - 1$.

2. The structure of the negative part

Recall that a compact curve C in a surface X is called *Hirzebruch-Jung string* [BPV, page 73] if $C = \cup_{j=1}^{r} C_j$, each C_j is a smooth rational curve of selfintersection ≤ -2,

$C_j \cdot C_i = 1$ if $|i-j| = 1$ and 0 if $|i-j| \geq 2$. Such a curve can be contracted to a rational singularity, more precisely to a cyclic quotient singularity.

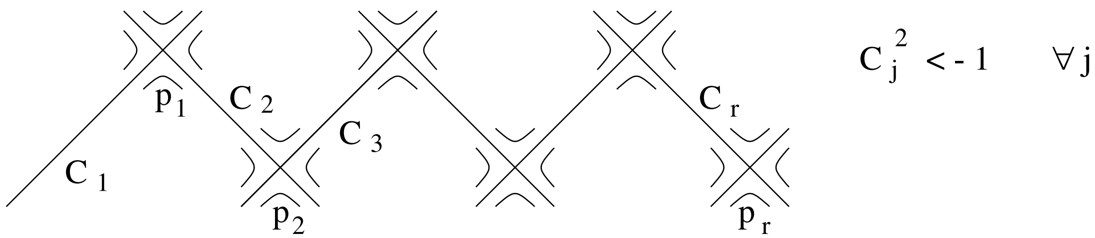
Definition 1. Given a foliation $\mathcal{F}$ on a surface X, we shall say that a curve $C \subset X$ is an $\mathcal{F}$-*chain* if:

i) C is an Hirzebruch-Jung string, $C = \cup_{j=1}^{r} C_j$;

ii) each irreducible component C_j is $\mathcal{F}$-invariant;

iii) $Sing(\mathcal{F}) \cap C$ are all reduced and nondegenerate;

iv) $Z(\mathcal{F}, C_1) = 1$, $Z(\mathcal{F}, C_j) = 2$ for every $j = 2, ..., r$.



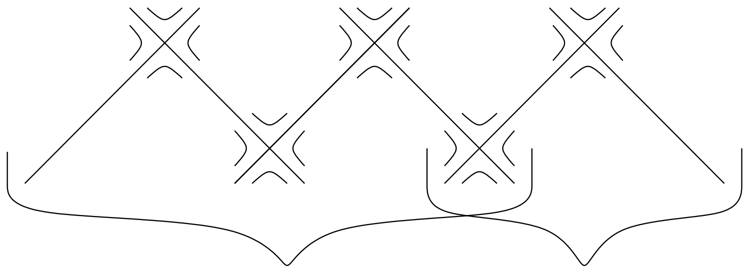
Set $p_j = C_j \cap C_{j+1}$ for every $j = 1, ..., r-1$. Then $Sing(\mathcal{F}) \cap C = \{p_1, ..., p_{r-1}, p_r\}$, where $p_r \in C_r$ is a smooth point of C. Look at p_1: it is a reduced nondegenerate singularity with a separatrix inside C_1 without holonomy (because $C_1 \setminus \{p_1\}$ is a simply connected leaf). Thus (cfr. chapter 1) p_1 is a singularity of the type $d(z^n w) = 0$, $\{w = 0\}$ being the local equation of C_1. Moreover, from Camacho-Sad formula (chapter 3) we see that $n = -C_1^2$. Look now at p_2: its separatrix C_2 has n-periodic holonomy (to turn in $C_2 \setminus Sing(\mathcal{F})$ around p_2 is the same as to turn around p_1), hence p_2 also is a singularity with a local holomorphic first integral, of the type $z^n w^l$ being $\{z = 0\} = C_2$. From Camacho-Sad formula we also find $l = -nC_2^2 - 1$. By iterating this procedure we can determine the structure of all the singularities of $\mathcal{F}$ on C, once the selfintersections C_j^2 are known. In particular, each singularity p_j, $j = 1, ..., r$, is in the Siegel domain with a holomorphic first integral.

Note that we have met already several times these $\mathcal{F}$-chains: in the examples of the previous section, in the resolution of the singularities of the type $nzdw - mwdz = 0$, $n, m \in \mathbf{N}^+$ (chapter 1), in the resolution of the nilpotent fibres of Riccati foliations (chapter 4),...

In the examples of the previous section the negative part of $K_{\mathcal{F}}$ was always supported on a union of $\mathcal{F}$-chains. Our aim is to prove that this is a general fact, provided that, of course, $\mathcal{F}$ is relatively minimal (recall that this means: $Sing(\mathcal{F})$ are reduced and there are no $\mathcal{F}$-exceptional curves, see chapter 5). Let us firstly make two remarks (assuming $K_{\mathcal{F}}$

pseudoeffective):

1) any $\mathcal{F}$-chain is clearly contained in a maximal one, and two maximal $\mathcal{F}$-chains are necessarily disjoint: otherwise their union would be a longer Hirzebruch-Jung string which would satisfy all the conditions of definition 1 except that $Z(\mathcal{F}, C_r) = 1$ instead of 2:



but, being an Hirzebruch-Jung string contractible, this would contradict Camacho's generalisation of the separatrix theorem (theorem 4, chapter 3; in this particular case the proof is particularly simple, and follows also from the previous analysis of the singularities of $\mathcal{F}$ on an $\mathcal{F}$-chain).

2) any maximal $\mathcal{F}$-chain is contained in the support of the negative part N of $K_{\mathcal{F}}$: the first component C_1 because $K_{\mathcal{F}} \cdot C_1 = -1 < 0$, thus the second one also because $K_{\mathcal{F}} \cdot C_2 = 0$ and $C_1 \cdot C_2 > 0$, and so on. Observe that $K_{\mathcal{F}} \cdot C_j = 0$ for every $j = 2, ..., r$.

Theorem 1 [Mc1]. *Let $\mathcal{F}$ be a relatively minimal foliation on an algebraic surface X. Suppose that $\mathcal{F}$ is not a rational fibration, and let $K_{\mathcal{F}} \overset{num}{=} P + N$ be the Zariski decomposition of its canonical bundle. Then $Supp(N)$ is a disjoint union of maximal $\mathcal{F}$-chains.*

Proof.

We may decompose the $\mathbf{Q}^+$-divisor N as

$$N = \sum_{j=1}^{l} N_j + N_0$$

where each

$$N_j = \sum_{i=1}^{r(j)} b_{j,i} D_{j,i} \qquad j > 0$$

has a maximal $\mathcal{F}$-chain as support (with $D_{j,1}$ the first curve, i.e. the one containing only one singularity) and each irreducible component of N_0 is not contained in a $\mathcal{F}$-chain. We have $Supp(N_i) \cap Supp(N_j) = \emptyset$ for $i, j > 0$, $i \neq j$, but $Supp(N_0)$ may intersect one or more $Supp(N_j)$. We have to prove that $N_0 \equiv 0$.

Let us firstly observe that for every irreducible $C \subset Supp(N)$ we have $N \cdot C \geq 0$ unless C is the first curve of an $\mathcal{F}$-chain, in which case $N \cdot C = -1$. In fact, if C is $\mathcal{F}$-invariant then

$$N \cdot C = K_{\mathcal{F}} \cdot C = -\chi(C) + Z(\mathcal{F}, C)$$

and this is ≥ 0 unless C is a smooth rational curve and $Z(\mathcal{F}, C) = 1$, that is the first curve of an $\mathcal{F}$-chain (note that the unique singularity of $\mathcal{F}$ on C is nondegenerate, for index type or holonomic reasons, and $C^2 \leq -2$ because $\mathcal{F}$ is relatively minimal). If C is not $\mathcal{F}$-invariant then

$$N \cdot C = K_{\mathcal{F}} \cdot C = -C^2 + tang(\mathcal{F}, C) \geq -C^2 > 0,$$

which can also be written as $(N + C) \cdot C \geq 0$. Because $Supp(N)$ certainly contains at least one curve C with $N \cdot C < 0$ (unless $N \equiv 0$), we see that N actually contains one or more maximal $\mathcal{F}$-chains (unless $N \equiv 0$).

Let now T be the union of all the irreducible components of $Supp(N_0)$ which are not $\mathcal{F}$-invariant (for a first reading, it may help the reader to assume that T is empty). For every $j = 1, ..., l$ set

$$R(j) = \max\{i | D_{j,i} \cap T = \emptyset\} \qquad (= 0 \ if \ \ D_{j,1} \cap T \neq \emptyset)$$

and then for every $i = 1, ..., R(j)$ set

$$\hat{b}_{j,i} = \min\{b_{j,i}, \frac{\delta_i}{-D_{j,i}^2}\}$$

where $\delta_i = 2$ for $i < R(j)$ and $\delta_i = 1$ for $i = R(j)$. Note that $\hat{b}_{j,i} \in (0, 1]$ for every j, i, and $\hat{b}_{j,R(j)} \in (0, \frac{1}{2}]$, because $D_{j,i}^2 \leq -2$ for every j, i. Finally, set

$$\bar{N}_j = N_j - \sum_{i=1}^{R(j)} \hat{b}_{j,i} D_{j,i}$$

$$\bar{N} = \sum_{j=1}^{l} \bar{N}_j + N_0 = N - \sum_{j=1}^{l} \sum_{i=1}^{R(j)} \hat{b}_{j,i} D_{j,i}.$$

Remark that $\bar{N}$ is still a $\mathbf{Q}^+$-divisor, because $\hat{b}_{j,i} \leq b_{j,i}$ for every j, i. We will prove that $\bar{N} \equiv 0$.

Let us compute $\bar{N} \cdot D_{j,i}$. If $i \leq R(j)$ we have two possibilities:
a) $\hat{b}_{j,i} = b_{j,i}$, and so $D_{j,i}$ is not contained in $Supp(\bar{N})$ and, therefore, $\bar{N} \cdot D_{j,i} \geq 0$;

b) $\hat{b}_{j,i} = \frac{\delta_i}{-D_{j,i}^2} < b_{j,i}$, and we still find $\bar{N} \cdot D_{j,i} \geq 0$:

$\bar{N} \cdot D_{j,i} = N \cdot D_{j,i} - \hat{b}_{j,i-1} + \delta_i - \hat{b}_{j,i+1} = (K_{\mathcal{F}} \cdot D_{j,i} - \hat{b}_{j,i-1}) + (\delta_i - \hat{b}_{j,i+1}) \geq -1 + 1 = 0.$

If $i = R(j) + 1$ we only obtain:

$$\bar{N} \cdot D_{j,i} = K_{\mathcal{F}} \cdot D_{j,i} - \hat{b}_{j,i-1} \geq -1$$

but we also have $D_{j,i} \cap T \neq \emptyset$ and thus $(\bar{N} + T) \cdot D_{j,i} \geq 0$. If $i \geq R(j) + 2$ then $\bar{N} \cdot D_{j,i} = N \cdot D_{j,i} = 0$.

Therefore, for every $j = 1, ..., l$ and every $i = 1, ..., r(j)$ we have

$$(\bar{N} + T) \cdot D_{j,i} \geq 0.$$

If $C \subset Supp(N_0)$ is not $\mathcal{F}$-invariant (i.e. $C \subset T$) we also have

$$(\bar{N} + T) \cdot C \geq 0$$

because $(\bar{N} + T) \cdot C \geq (\bar{N} + C) \cdot C = (N + C) \cdot C \geq 0$.

On the other hand, the contractibility of the support of $\bar{N} + T$ implies $(\bar{N} + T)^2 < 0$ (unless $\bar{N} + T \equiv 0$), and the fact that $\bar{N} + T$ is a positive divisor implies that there exists an irreducible component C_0 of that support such that

$$(\bar{N} + T) \cdot C_0 < 0.$$

By the previous computations, C_0 is in the support of N_0 and it is $\mathcal{F}$-invariant. Observe now that C_0 can intersect $Supp(N_j)$ only at the last singularity of $\mathcal{F}$ on $D_{j,r(j)}$, in a transverse way. If $R(j) < r(j)$ (i.e. T does intersect $Supp(N_j)$) then $\bar{N}_j \cdot C_0 = N_j \cdot C_0$, and if $R(j) = r(j)$ then $\bar{N}_j \cdot C_0 = N_j \cdot C_0 - \hat{b}_{j,r(j)}(D_{j,r(j)} \cdot C_0) \geq N_j \cdot C_0 - \frac{1}{2} D_{j,r(j)} \cdot C_0$. We thus obtain

$$\bar{N} \cdot C_0 \geq N \cdot C_0 - \frac{1}{2} \sum_{j \in I} D_{j,r(j)} \cdot C_0$$

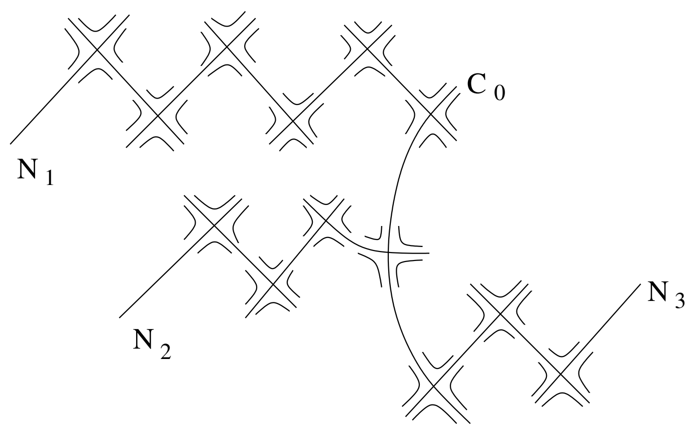
where $I = \{j | R(j) = r(j)\}$. The negativity of $(\bar{N} + T) \cdot C_0$ can be now written (using $N \cdot C_0 = K_{\mathcal{F}} \cdot C_0 = -\chi(C_0) + Z(\mathcal{F}, C_0))$ as

$$-\chi(C_0) + Z(\mathcal{F}, C_0) + T \cdot C_0 < \frac{1}{2} \sum_{j \in I} D_{j,r(j)} \cdot C_0.$$

An intersection point between C_0 and $D_{j,r(j)}$ corresponds necessarily to a singularity of $\mathcal{F}$ on C_0 and hence gives a contribution to $Z(\mathcal{F}, C_0)$ equal to 1. That is, we have $Z(\mathcal{F}, C_0) \geq \sum_{j\in I} D_{j,r(j)} \cdot C_0$, and the previous inequality becomes

$$\frac{1}{2}Z(\mathcal{F}, C_0) < \chi(C_0) - T \cdot C_0,$$

showing in particular $Z(\mathcal{F}, C_0) \leq 3$ (and ≥ 1, of course) and C_0 smooth and rational. We have already remarked at the beginning of the proof that $Z(\mathcal{F}, C_0) = 1$ corresponds to C_0 being the first component of an $\mathcal{F}$-chain, excluded by $C_0 \subset Supp(N_0)$. If $Z(\mathcal{F}, C_0) = 2$ then C_0 cuts one or two $\mathcal{F}$-chains (because $\frac{1}{2}\sum_{j\in I} D_{j,r(j)} \cdot C_0 > T \cdot C_0 \geq 0$), and in both cases we have a contradiction with the maximality of the $\mathcal{F}$-chains N_j. If $Z(\mathcal{F}, C_0) = 3$ then C_0 cuts three $\mathcal{F}$-chains (because $\frac{1}{2}\sum_{j\in I} D_{j,r(j)} \cdot C_0 > 1 + T \cdot C_0 \geq 1$), and we are in contradiction with Camacho's version of the separatrix theorem (theorem 4, chapter 3):



These contradictions leave only one possibility: $\bar{N} + T$ is empty. That is, $N_0 = \emptyset$ and for every $j = 1, ..., l$

$$N_j = \sum_{i=1}^{r(j)} \hat{b}_{j,i} D_{j,i}$$

which completes the proof. $\triangle$

Addendum: we have also proved that $\hat{b}_{j,i} = b_{j,i}$ for every j, i, that is

$$b_{j,i} \leq \frac{\delta_i}{-D_{j,i}^2}$$

and in particular

$$b_{j,i} \leq 1 \qquad , \qquad b_{j,r(j)} \leq \frac{1}{2}.$$

In fact, the strict inequality

$$b_{j,i} < 1$$

holds for every $i = 1, ..., r(j)$, as a simple computation proves, and this will be quite useful in the next section. Note that, once we know that $N_0 \equiv 0$, the coefficients $b_{j,i}$ are given by the systems

$$b_{j,1} D_{j,1}^2 + b_{j,2} = N \cdot D_{j,1} = -1$$

$$b_{j,i-1} + b_{j,i} D_{j,i}^2 + b_{j,i+1} = N \cdot D_{j,i} = 0 \qquad i = 2, ..., r(j)$$

for every $j = 1, ..., l$.

3. Foliations with vanishing numerical Kodaira dimension

Once the structure of the negative part of the Zariski decomposition of $K_{\mathcal{F}}$ has been understood, it is quite easy to classify those foliations whose canonical bundle is numerically reduced only to that negative part.

Theorem 2 [Mc1]. *Let $\mathcal{F}$ be a relatively minimal foliation on an algebraic surface X. Suppose that $\mathcal{F}$ is not a rational fibration, and that $\nu(\mathcal{F}) = 0$ (i.e. the positive part of the Zariski decomposition of $K_{\mathcal{F}}$ is numerically trivial). Then there exists a ramified covering $Y \xrightarrow{\pi} X$ and a birational morphism $Y \xrightarrow{p} Z$ such that $p_*(\pi^*(\mathcal{F}))$ is generated by a global holomorphic vector field with isolated zeroes.*

Proof.

By hypothesis, we have

$$K_{\mathcal{F}} \overset{num}{=} N$$

where N is a contractible $\mathbf{Q}^+$-divisor, whose support is a collection of disjoint maximal $\mathcal{F}$-chains by theorem 1.

Lemma 1. *If, moreover, $h^0(X, K_{\mathcal{F}}) > 0$ then $K_{\mathcal{F}}$ is trivial.*

Proof.

The zero divisor of a nontrivial section $s \in H^0(X, K_{\mathcal{F}})$ is forcely equal to N, by uniqueness of the negative part of the Zariski decomposition. But we have seen that the coefficients of N are all strictly smaller than 1, and so $(s)_0 = N$ actually implies $N \equiv 0$, that is $K_{\mathcal{F}} = \mathcal{O}_X$. $\triangle$

Lemma 2. *There exists $n \in \mathbf{N}^+$ such that $h^0(X, K_{\mathcal{F}}^{\otimes n}) > 0$.*

Proof.

If $h^1(X, \mathcal{O}_X) = 0$ then numerical equivalence of line bundles is the same as linear equivalence modulo torsion. Hence for a suitable $n \in \mathbf{N}^+$ the divisor nN is integral and defines a line bundle $\mathcal{O}_X(nN)$ isomorphic to $K_{\mathcal{F}}^{\otimes n}$, i.e. this last bundle is effective.

If $h^1(X, \mathcal{O}_X) \neq 0$ we have two possibilities. Either there exists a global holomorphic 1-form ω which does not vanish identically on $\mathcal{F}$ and thus which defines a nontrivial section of $K_{\mathcal{F}}$, or every 1-form vanishes on $\mathcal{F}$ and thus $\mathcal{F}$ coincides with the Albanese fibration $alb_X : X \to \Sigma \subset Alb(X)$.

In this last case, the Albanese fibration $\mathcal{F}$ is elliptic, because the support of N is contained in some fibres and so $K_{\mathcal{F}}$ has zero degree on a generic fibre, which is therefore elliptic. By [Ser], $\mathcal{F}$ is an isotrivial fibration, otherwise we would have $\nu(\mathcal{F}) = 1$. According to the stable reduction theorem [BPV, pages 93-96 and 155-158] there exists a covering $X_1 \stackrel{r}{\to} X$, ramified over certain fibres, and a birational morphism $X_1 \stackrel{q}{\to} X_2$ such that the induced elliptic fibration $\mathcal{F}_2 = q_* r^*(\mathcal{F})$ on X_2 has all its fibres of type I_b, $b \geq 0$. Of course, $\mathcal{F}_2$ is still isotrivial, and so all its fibres are in fact of type I_0, i.e. smooth. It follows [BPV, page 110] that $K_{\mathcal{F}_2}$ is a torsion line bundle: $K_{\mathcal{F}_2}^{\otimes m} = \mathcal{O}_{X_2}$ for some positive m. A nontrivial section f of $K_{\mathcal{F}_2}^{\otimes m}$ can be pulled-back to X_1 to a section g of $K_{\mathcal{F}_1}^{\otimes m}$ ($\mathcal{F}_1$ is obtained by blowing-up $\mathcal{F}_2$, which is reduced, and so $K_{\mathcal{F}_1} = q^*(K_{\mathcal{F}_2}) \otimes \mathcal{O}_{X_1}(D)$ where $D \geq 0$). This section g is possibly not equivariant, i.e. it cannot be projected to X to a section of $K_{\mathcal{F}}^{\otimes m}$; however, some power $g^{\otimes l}$ is certainly equivariant and projects to X to a nontrivial section of $K_{\mathcal{F}}^{\otimes lm}$. To see this, look at the explicit form of the covering r [BPV, pages 155-158] and note that the global triviality of $K_{\mathcal{F}_2}^{\otimes m}$ implies that after a local trivialization of $\mathcal{F}_2$ the section f becomes locally constant: $X_2 \stackrel{loc}{=} \mathbf{D} \times E$, $\mathcal{F}_2 = \{dz = 0\}$, $f = c(dw)^{\otimes m}$ (this can be proved, for instance, by the same arguments used in lemma 1 of chapter 6). It is in fact sufficient to take l so that lm is a multiple of 12.

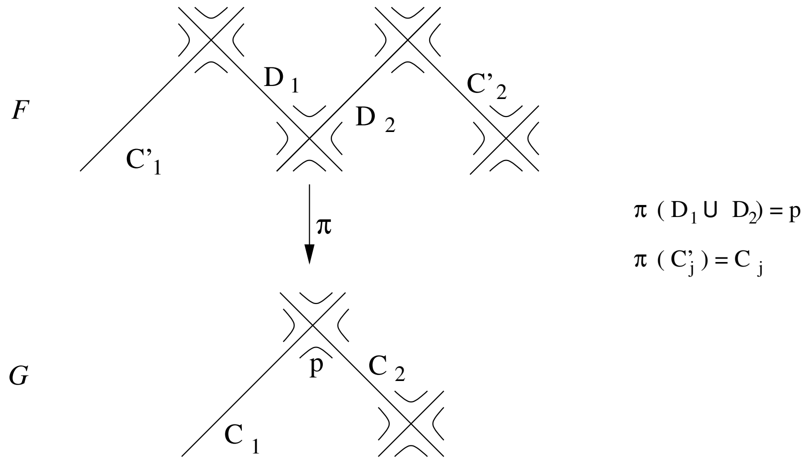
Remark that in this elliptic case we have already proved not only the lemma but also the full theorem, being $\mathcal{F}_2$ covered by an elliptic fibre bundle generated by a global holomorphic vector field. △

Let now s be a nontrivial section of $K_{\mathcal{F}}^{\otimes n}$, whose zero divisor $(s)_0$ forcely coincides with nN. We use a classical "ramified covering trick", which was already used in the proof of proposition 3, chapter 4, and in the proof of the previous lemma, under the label of stable reduction theorem. The section s defines a covering $Y \stackrel{\pi}{\to} X$ of order n, ramified over $Supp(N)$: here Y is the resolution of $Y_0 \subset E(K_{\mathcal{F}})$ = the total space of $K_{\mathcal{F}}$, Y_0 is the preimage of the graph of s in $E(K_{\mathcal{F}}^{\otimes n})$ by the map $E(K_{\mathcal{F}}) \stackrel{\otimes n}{\to} E(K_{\mathcal{F}}^{\otimes n})$, and π is the composition of the resolution $Y \to Y_0$ and the projection of Y_0 on the base X. Then the same s defines a holomorphic section $\hat{s}$ of $\pi^*(K_{\mathcal{F}})$ over Y, vanishing on $\pi^{-1}(Supp(N))$.

The fact that $(s)_0$ has only normal crossing singularities permits to understand very well the structure of $\pi : Y \to X$: the normalization $\hat{Y}_0$ of Y_0 has isolated singularities over the crossing points of $(s)_0$, and these singularities are resolved by Hirzebruch-Jung strings

[BPV, page 82]. Being $(s)_0$ itself a union of Hirzebruch-Jung strings, its preimage in Y is also a union of chains of rational curves, which however may fail to be Hirzebruch-Jung strings for the possible presence of (-1)-curves.

Let us consider the foliation $\mathcal{G} = \pi^*(\mathcal{F})$ on Y. Because $\mathcal{F}$ is given, around each crossing point of $(s)_0$, by the levels of a monomial function, we see that each chain in $\pi^{-1}((s)_0)$ is totally $\mathcal{G}$-invariant and is in fact like a $\mathcal{G}$-chain except for the possible presence of (-1)-curves:



Note that $K_{\mathcal{G}}$ is holomorphically trivial on a neighbourhood of $\pi^{-1}(p)$, p a crossing point of $(s)_0$: it is trivial on the Hirzebruch-Jung string $\pi^{-1}(p)$ and therefore on a neighbourhood of it. It follows that if v is a holomorphic vector field around p generating $\mathcal{F}$ then $\pi^*(v)$ (defined outside the ramification divisor of π) extends to a holomorphic vector field around $\pi^{-1}(p)$ generating $\mathcal{G}$. In other words: we have

$$\pi^*(K_{\mathcal{F}}) = K_{\mathcal{G}}.$$

Thus the section $\hat{s}$ of $\pi^*(K_{\mathcal{F}})$ is actually a section of $K_{\mathcal{G}}$, that is we have $h^0(Y, K_{\mathcal{G}}) > 0$. Note also that $\nu(\mathcal{G}) = 0$. However, $\mathcal{G}$ is not necessarily relatively minimal: the (-1)-curves in $\pi^{-1}((s)_0)$ are in fact $\mathcal{G}$-exceptional curves. When we contract them, $K_{\mathcal{G}}$ remains effective and $\nu(\mathcal{G})$ remains 0, and so after a finite number of contractions $Y \stackrel{p}{\to} Z$ we arrive to a foliation $\mathcal{H}$ which satisfies all the hypotheses of lemma 1. Hence $K_{\mathcal{H}}$ is trivial and $\mathcal{H}$ is generated by a global holomorphic vector field with only isolated zeroes. △

Foliations generated by global vector fields were studied in chapter 6. In the case of vector fields (on algebraic surfaces) with isolated zeroes, we recall the list:

1) almost elliptic fibre bundles;

2) Kronecker foliations on tori;

3) suspensions of representations $\rho : \pi_1(E) \to Aut(\mathbf{C}P^1)$, E an elliptic curve;

4) (up to birational maps) foliations on $\mathbf{C}P^1 \times \mathbf{C}P^1$ generated by $v_1 \oplus v_2$, v_j holomorphic vector fields on $\mathbf{C}P^1$.

It would be interesting to obtain an explicit list also for all the possible quotients of these foliations. This leads to study the birational automorphisms of these foliations, but we don't know if this would be sufficient, because we don't know if the ramified coverings appearing in theorem 2 are of Galois type.

As an application of this result, we may rediscover proposition 3 of chapter 4: in the notations used there, and assuming moreover that $\mathcal{F}$ is relatively minimal without loss of generality, we have $\nu(\mathcal{F}) = 0$, because $K_{\mathcal{F}} \cdot C = 0$ and $C \cdot C > 0$ (by Hodge index theorem, this implies that the positive part P of $K_{\mathcal{F}}$ is numerically trivial). Some work is still necessary, to show that $\mathcal{F}$ is really covered 3 times by a linear foliation $\mathcal{L}$, but these are easy details.

4. Contraction of the negative part and canonical singularities

It is suggested in [Mc1] and [Br4] that the negative part of the Zariski decomposition of $K_{\mathcal{F}}$ should be contracted in order to obtain a more coherent theory, as it is done in the context of Mori's minimal model program [M-P]. If each connected component of $Supp(N)$ is contracted to a point, we obtain a normal surface X_0 all of whose singularities are of Hirzebruch-Jung (or cyclic quotient) type. The foliation $\mathcal{F}$ becomes a foliation $\mathcal{F}_0$ which has the advantage that $K_{\mathcal{F}_0}$ is nef. The disadvantage, however, is that $K_{\mathcal{F}_0}$ is no more a line bundle: on a neighbourhood of $p \in Sing(X_0)$ the foliation $\mathcal{F}_0$ is *not* generated by a holomorphic vector field having an isolated zero at p (this is related to the nonintegrality of N). In other words, $K_{\mathcal{F}_0}$ exists as a sheaf, but it is not locally free at p. There is, fortunately, something to save: for some positive m, $K_{\mathcal{F}_0}^{\otimes m}$ is locally free (one could call $K_{\mathcal{F}_0}$ a $\mathbf{Q}$-*bundle*). This is a nontrivial property: if, instead of N, we contract something more complicated, for example an $\mathcal{F}$-invariant cycle of rational curves, we may obtain a foliation $\mathcal{F}_1$ whose $K_{\mathcal{F}_1}$ is not a $\mathbf{Q}$-bundle, i.e. $K_{\mathcal{F}_1}^{\otimes m}$ is never locally free, for any positive m. An explicit example is in [Mc1, IV.2.2]; the point is that Pic^0 of a cycle of rational curves contains non-torsion elements.

Given a normal surface X and a foliation $\mathcal{F}$ on X, we shall say that $p \in X$ is an *Hirzebruch-Jung singularity of* $\mathcal{F}$ if p is an Hirzebruch-Jung singularity of X and on the minimal resolution $\tilde{X} \xrightarrow{\pi} X$ the foliation $\mathcal{F}$ lifts to a foliation $\tilde{\mathcal{F}}$ for which $\pi^{-1}(p)$ is a $\tilde{\mathcal{F}}$-chain. It turns out (see the discussion at the beginning of section 2) that $\mathcal{F}$ on a neighbourhood of p is uniquely defined, up to biholomorphism, by the selfintersections of the Hirzebruch-Jung string resolving p, and by which of the two extremities of the chain contains only one singular point of $\tilde{\mathcal{F}}$. Note that $\mathcal{F}$ has a holomorphic first integral on a neighbourhood of p; we can describe $\mathcal{F}$ around p also as the cyclic quotient of a regular foliation.

At this point, we could repeat all the theory developed up to here in the context of foliations on normal surfaces all of whose singularities are either reduced (on smooth points of the surface) or Hirzebruch-Jung singularities (on singular points of the surface). For instance, the theory of minimal models developed in chapter 5 can be easily adapted to this more general context, and so we obtain the following result: any foliation on any normal algebraic surface, which is not birational to a rational fibration or a Riccati foliation or the very special foliation $\mathcal{H}$, is birational to a *unique* foliation $\mathcal{F}$ all of whose singularities are reduced or Hirzebruch-Jung, and whose canonical sheaf is nef.

Let us recall, once a time, the step-by-step construction of such an $\mathcal{F}$:

1) firstly, resolve the singularities of the surface;
2) secondly, resolve the singularities of the foliation;
3) thirdly, contract $\mathcal{F}$-exceptional curves;
4) lastly, contract maximal $\mathcal{F}$-chains.

If the initial foliation is not in the forbidden list, the program works without ambiguities ($\mathcal{F}$-exceptional curves are pairwise disjoint) and gives the desired and unique "minimal model". If the initial foliation is birational to a Riccati foliation or to $\mathcal{H}$, then there still exists a model with nef canonical sheaf, but uniqueness is lost.

This is the "minimalist" point of view, in which we choose the *smallest* class of singularities of the foliations in order to obtain a unique model with nef canonical sheaf, up to few "well-understood" exceptions. But there is also the "maximalist" point of view, in which we choose a larger class of singularities (but not too large, otherwise the list of exceptions becomes too long and no more "well-understood"...). This is the point of view of [Mc1], and it naturally leads to the following definition.

Definition 2. Given a foliation $\mathcal{F}$ on a normal algebraic surface X, we shall say that $\mathcal{F}$ has *canonical singularities* if for any birational morphism $\hat{X} \xrightarrow{\pi} X$ we have ($\hat{\mathcal{F}} = \pi^*(\mathcal{F})$):

$$K_{\hat{\mathcal{F}}} \overset{num}{=} \pi^*(K_{\mathcal{F}}) + R$$

where R is a $\mathbf{Q}^+$-divisor supported on the exceptional divisor of π.

If $p \notin Sing(X)$, then p regular or reduced (for $\mathcal{F}$) certainly implies that p is also canonical: when we blow-up p we obtain $K_{\hat{\mathcal{F}}} = \pi^*(K_{\mathcal{F}}) + \delta E$, with $\delta = 1$ if p is regular and $\delta = 0$ if p is singular, and similarly if we blow-up several times. The converse is not exactly true: a Poincaré-Dulac singularity $(nw + z^n)dz - zdw = 0$ is canonical (but not $nwdz - zdw = 0$). It is easy to see that, when p is a smooth point of X, then p is a canonical singularity of $\mathcal{F}$ if and only if p is either regular or reduced or Poincaré-Dulac. It is not a real problem to add Poincaré-Dulac singularities to the list of permitted singularities; we have already met them several times, for instance in the discussion of $\mathcal{F}$-exceptional curves (chapter 5), or in the study of Riccati foliations (chapter 4).

If $p \in Sing(X)$, then the situation is more complicated. Hirzebruch-Jung singularities are certainly canonical, but they are not the only ones. We refer to [Mc1] for a more detailed study of canonical singularities of foliations, and also for their use in the minimal model program. In fact, in [Mc1] canonical singularities are introduced also because the author wants a model having not only the property "$K_{\mathcal{F}}$ nef" but also the property "$K_{\mathcal{F}} \cdot C = 0 \Rightarrow C \cdot C \geq 0$", related to the abundance conjecture discussed in the next chapter.

In this chapter we classify, following [Mc1], foliations of Kodaira dimension 0 or 1. The main step is to prove that Kodaira dimension 0 actually implies that also the numerical Kodaira dimension is 0: this is a particular case of the so-called "abundance". We also discuss foliations of Kodaira dimension $-\infty$, but here the classification is still incomplete; the problem is that in this case the abundance does not hold, because there exist foliations of Kodaira dimension $-\infty$ and numerical Kodaira dimension 1. As an application of all of this, we give a complete description of foliations with an entire transcendental leaf [Mc2] [Br3].

1. Kodaira dimension of foliations

Recall that, given a line bundle L on an algebraic surface X, its *Kodaira dimension* (or Iitaka dimension) $kod(L)$ is defined as $-\infty$ if $h^0(X, L^{\otimes n}) = 0$ for every $n \in \mathbf{N}^+$, and as $trdeg(\sum_{n=0}^{\infty} H^0(X, L^{\otimes n})) - 1$ if $h^0(X, L^{\otimes n}) > 0$ for some $n \in \mathbf{N}^+$. Here $trdeg$ denotes the degree of transcendence over $\mathbf{C}$. In other words, $kod(L) \geq k$ if and only if for some $n \in \mathbf{N}^+$ there are $k+1$ nontrivial sections $s_j \in H^0(X, L^{\otimes n})$, $j = 0, ..., k$, which are algebraically independent (i.e. the meromorphic functions $f_j = \frac{s_j}{s_0}$, $j = 1, ..., k$, are algebraically independent). Geometrically, this means that $s_0, ..., s_k$ give a rational map of X onto $\mathbf{C}P^k$. We obviously have $kod(L) \in \{-\infty, 0, 1, 2\}$. This dimension can also be evaluated by looking at the asymptotic behaviour of $h^0(X, L^{\otimes n})$:

$$kod(L) = \limsup_{n \to +\infty} \frac{1}{\log n} \log h^0(X, L^{\otimes n}).$$

If L is pseudoeffective, we have at our disposal also its *numerical Kodaira dimension* $\nu(L)$, derived from its Zariski decomposition $L \overset{num}{=} P + N$. As in the previous chapter, we set $\nu(L) = -\infty$ if L is not pseudoeffective. We then have the general inequality (see, for instance, [M-P, page 81])

$$kod(L) \leq \nu(L)$$

and one speaks of "abundance" when $kod(L) = \nu(L)$, i.e. when $L^{\otimes n}$ has as many sections as its numerical properties let expect (strictly speaking, abundance usually denotes a somewhat stronger property, but this is a minor point in our context).

Let now $\mathcal{F}$ be a foliation on X. We define its Kodaira dimension $kod(\mathcal{F})$ as the Kodaira dimension of its canonical bundle $K_{\mathcal{F}}$. As already remarked in the previous chapter concerning $\nu(\mathcal{F})$, this is not a birational invariant of $\mathcal{F}$, unless $\mathcal{F}$ has reduced singularities, as was observed in [Men] and [Mc1].

Proposition 1. *If $(X, \mathcal{F})$ and $(Y, \mathcal{G})$ are foliations all of whose singularities are reduced and if there exists a birational map $X - - \rightarrow Y$ sending $\mathcal{F}$ to $\mathcal{G}$, then*

$$kod(\mathcal{F}) = kod(\mathcal{G}) \qquad \text{and} \qquad \nu(\mathcal{F}) = \nu(\mathcal{G}).$$

Proof.

By the factorization theorem of birational maps, it is sufficient to prove that if $(X, \mathcal{F})$ is a reduced foliation and $\pi : \tilde{X} \to X$ is the blowing-up at p then (setting $\tilde{\mathcal{F}} = \pi^*(\mathcal{F})$) we have $kod(\tilde{\mathcal{F}}) = kod(\mathcal{F})$ and $\nu(\tilde{\mathcal{F}}) = \nu(\mathcal{F})$. If $p \in Sing(\mathcal{F})$ then $K_{\tilde{\mathcal{F}}} = \pi^*(K_{\mathcal{F}})$ and everything is obvious. If $p \notin Sing(\mathcal{F})$ then $K_{\tilde{\mathcal{F}}} = \pi^*(K_{\mathcal{F}}) \otimes \mathcal{O}_{\tilde{X}}(E)$, being $E = \pi^{-1}(p)$ the exceptional divisor. The exact sequences ($n \in \mathbf{N}^+$)

$$0 \to \pi^*(K_{\mathcal{F}}^{\otimes n}) \to K_{\tilde{\mathcal{F}}}^{\otimes n} \to K_{\tilde{\mathcal{F}}}^{\otimes n}|_{nE} \to 0$$

and $h^0(nE, \mathcal{O}_{nE}(nE)) = 0$ show that $h^0(\tilde{X}, K_{\tilde{\mathcal{F}}}^{\otimes n}) = h^0(X, K_{\mathcal{F}}^{\otimes n})$ and thus $kod(K_{\tilde{\mathcal{F}}}) = kod(K_{\mathcal{F}})$. We also clearly have $K_{\tilde{\mathcal{F}}}$ pseudoeffective iff $K_{\mathcal{F}}$ pseudoeffective, and in that case the positive part of $K_{\tilde{\mathcal{F}}}$ is the pull-back by π of the positive part of $K_{\mathcal{F}}$, whence $\nu(\tilde{\mathcal{F}}) = \nu(\mathcal{F})$. $\triangle$

Remark that everything remains true in the slightly more general context of foliations on normal surfaces with canonical singularities (cfr. previous chapter). But when the singularities are arbitrary the proposition may fail in the most extreme way: the radial foliation $\mathcal{F}$ on $\mathbf{C}P^2$ has ample $K_{\mathcal{F}}^{-1}$ (so that its dimensions are $-\infty$) but it can be birationally transformed into a foliation $\mathcal{G}$ on $\mathbf{C}P^2$ of degree 2 or more, for which $K_{\mathcal{G}}$ is ample (so that its dimensions are 2). The philosophical meaning of the proposition above is that the class of reduced foliations (or canonical foliations...) is the good class to work with in order to obtain a classification in the spirit of Enriques. Note also that, as the proof shows, not only $kod(\mathcal{F})$ is a birational invariant of reduced foliations, but also the "plurigenera" $h^0(X, K_{\mathcal{F}}^{\otimes n})$ are.

We now note two easy particular cases of abundance.

Example 1. If $\nu(\mathcal{F}) = 2$ then $kod(\mathcal{F}) = 2$. This is in fact a general property of line bundles, which follows from Riemann-Roch formula, see e.g. [M-P, page 81].

Example 2. If $\nu(\mathcal{F}) = 0$ then $kod(\mathcal{F}) = 0$. This was proved in lemma 2 of the previous chapter (the hypothesis of relative minimality of the foliation in inessential for that lemma).

We therefore see that the difficult case, concerning abundance, is the case where $\nu(\mathcal{F}) = 1$. In the next section we classify foliations with $kod(\mathcal{F}) = 1$, which have also $\nu(\mathcal{F}) = 1$ by example 1; but later we shall see that there are also foliations with $\nu(\mathcal{F}) = 1$ and $kod(\mathcal{F}) = -\infty$ (but not $kod(\mathcal{F}) = 0$).

2. Foliations of Kodaira dimension 1

These foliations are easily classified [Men] [Mc1] using the natural fibration (Iitaka fibration) associated to any line bundle of Kodaira dimension 1.

Theorem 1 [Mc1] [Men]. *Let $\mathcal{F}$ be a reduced foliation on an algebraic surface X, and suppose that $kod(\mathcal{F}) = 1$. Then $\mathcal{F}$ belongs to one of the following classes:*

i) Riccati foliations

ii) Turbulent foliations

iii) Nonisotrivial elliptic fibrations

iv) Isotrivial fibrations of genus ≥ 2.

Proof.

We firstly recall the construction of the Iitaka fibration. From $kod(\mathcal{F}) = 1$ we infer $\nu(\mathcal{F}) = 1$, and moreover the positive part P of $K_{\mathcal{F}}$ moves in a pencil of curves. From $P \cdot P = 0$ we see that this pencil has no base point, and thus it gives a fibration $\pi : X \to B$.

If F is a generic fibre of π, we clearly have

$$K_{\mathcal{F}} \cdot F = 0$$

and we may also say that

$$K_{\mathcal{F}}^{\otimes m} = \mathcal{O}_X(nF + D)$$

for some positive integers m, n, where D is a positive integral divisor whose support is contractible and disjoint from a generic fibre of π.

If $\mathcal{F}$ coincides with the fibration π, then $K_{\mathcal{F}} \cdot F = 0$ means that the fibration is elliptic, and so we are in the class iii). The nonisotriviality follows from [Ser].

Otherwise a generic fibre of π is transverse to $\mathcal{F}$, because $tang(\mathcal{F}, F) = K_{\mathcal{F}} \cdot F + F \cdot F = 0$. If F is rational or elliptic we therefore obtain that $\mathcal{F}$ is Riccati or turbulent.

If F has genus ≥ 2 we claim that $\mathcal{F}$ is a fibration. To see this, observe that (as in the Riccati or turbulent case) we can define a monodromy $\rho : \pi_1(B^*) \to Aut(F)$, being $B^* \subset B$ the set of $\mathcal{F}$-transverse fibres, but now this monodromy is essentially trivial because $Aut(F)$ is a finite group. In other words, the leaves of $\mathcal{F}|_{X^*}$, $X^* = \pi^{-1}(B^*)$, are graphs of multivalued (but finitevalued) holomorphic maps $B^* \to F$, and the hyperbolicity of F allows to extend these maps to B. Note that the fibration π is isotrivial, and therefore standard results on fibrations and period maps [BPV, chapter III] show that π is, modulo a covering ramified along fibres and modulo birational transformations, a locally trivial fibre bundle. The extension property above means that the leaves of $\mathcal{F}$ (closures of the leaves of $\mathcal{F}|_{X^*}$) are algebraic curves of X, so that $\mathcal{F}$ has a meromorphic first integral, and the fact that $Sing(\mathcal{F})$ are reduced (in particular nondicritical) means that $\mathcal{F}$ is a fibration.

A generic fibre G of $\mathcal{F}$ is transverse to the (reduced) fibres of π (but $\pi|_G$ may have critical points, in correspondence of multiple components of fibres), and this proves the isotriviality of $\mathcal{F}$: the same fibration π gives to $\mathcal{F}$ the structure of a locally trivial fibre bundle, outside a finite number of fibres. Finally, we have $K_{\mathcal{F}} \cdot G \geq \frac{n}{m} F \cdot G > 0$, and therefore G has genus ≥ 2. Hence $\mathcal{F}$ belongs to the class iv). $\triangle$

Conversely, the foliations in the four classes above always have $kod \leq 1$, with equality realized in "most" cases. More precisely, by [Ser] any nonisotrivial elliptic fibration or any isotrivial fibration of genus ≥ 2 has $kod(\mathcal{F}) = 1$. Concerning Riccati and turbulent foliations, the vanishing or even the negativity of the Kodaira dimension is certainly possible, but it appears as a quite exceptional case. Observe also that the four classes are not totally disjoint.

3. Foliations of Kodaira dimension 0

Theorem 2 [Mc1]. *Let $\mathcal{F}$ be a reduced foliation on an algebraic surface X, and suppose that $kod(\mathcal{F}) = 0$. Then $\nu(\mathcal{F}) = 0$.*

Proof.

Let

$$K_{\mathcal{F}} \overset{num}{=} P + N = \sum_{j=1}^{p} a_j E_j + \sum_{j=1}^{n} b_j D_j$$

be a Zariski decomposition of $K_{\mathcal{F}}$. Because $kod(\mathcal{F}) \geq 0$, we may assume that the positive part P, and not only N, is a $\mathbf{Q}^+$-divisor: $a_j \geq 0$ for every j. More precisely, we may suppose that, for some positive integer l, $l(P+N)$ is the zero divisor of a section s of $K_{\mathcal{F}}^{\otimes l}$. We therefore have to prove that P is in fact empty.

We may assume, by proposition 1, that $\mathcal{F}$ is relatively minimal. Hence (theorem 1 of chapter 8) $Supp(N)$ is a disjoint union of maximal $\mathcal{F}$-chains. We shall denote by $N = \sum_{j=1}^{m} N_j$ the decomposition of N into divisors supported on maximal $\mathcal{F}$-chains. Each N_j is of the form $\sum_{i=1}^{k(j)} b_{j,i} D_{j,i}$, $b_{j,i} \in (0,1)$, being $D_{j,1}$ the first curve of the $\mathcal{F}$-chain (the one with only one singularity of $\mathcal{F}$).

From $P \cdot P = 0$ ($P \cdot P > 0$ would imply $kod(\mathcal{F}) = 2$) and $P \geq 0$, P nef, we derive $P \cdot E_j = 0$ for every $j = 1, ..., p$. Moreover $P \cdot D_j = 0$ for every $j = 1, ..., n$, by definition of Zariski decomposition.

Lemma 1. *Each irreducible component $E_j \subset Supp(P)$ is $\mathcal{F}$-invariant.*

Proof.

Suppose by contradiction that one component, say E_1, is not $\mathcal{F}$-invariant. For every $j = 1, ..., m$ let

$$h(j) = \min\{i = 1, ..., k(j) | D_{j,i} \cap E_1 \neq \emptyset\}$$

($= k(j) + 1$ if E_1 does not intersect $Supp(N_j)$) and

$$\bar{N}_j = \sum_{i=h(j)}^{k(j)} b_{j,i} D_{j,i}$$

$$Q = E_1 + \sum_{j=1}^{m} \bar{N}_j .$$

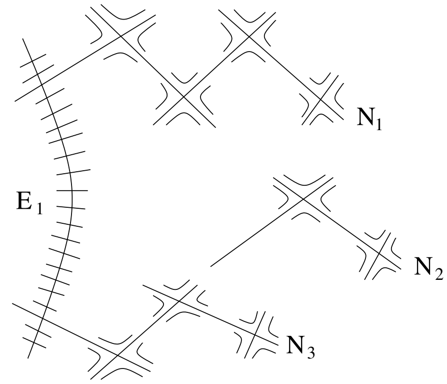
Then, for every $D_{j,i} \subset Supp(Q)$:

i) $Q \cdot D_{j,h(j)} = E_1 \cdot D_{j,h(j)} + N_j \cdot D_{j,h(j)} - b_{j,h(j)-1} \geq 0$, and > 0 unless $h(j) = 1$ and $E_1 \cdot D_{j,1} = 1$ (because $E_1 \cap D_{j,h(j)} \neq \emptyset$);

ii) for $i > h(j)$, $Q \cdot D_{j,i} = E_1 \cdot D_{j,i} + N_j \cdot D_{j,i} \geq 0$, and > 0 unless $E_1 \cap D_{j,i} = \emptyset$;

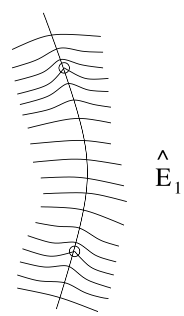
iii) $Q \cdot E_1 = (E_1 + N) \cdot E_1 = (E_1 + K_{\mathcal{F}}) \cdot E_1 = tang(\mathcal{F}, E_1) \geq 0$, and > 0 unless E_1 is transverse to $\mathcal{F}$.

The intersection form is definite seminegative on the span of $\{E_j, D_i\}$, with kernel generated by P. Thus $Q \cdot Q \leq 0$, and being $Q \geq 0$ we deduce by the previous inequalities that $Q \cdot D_{j,i} = Q \cdot E_1 = 0$ for every j, i. That is: E_1 is transverse to $\mathcal{F}$ and it cuts each N_j in at most one point, on $D_{j,1}$:



Moreover, $Q \cdot Q = 0$ and thus P is proportional to Q; in particular $E_1 \cup Supp(\sum_j N_j)$, the sum being restricted to those j such that $N_j \cap E_1 \neq \emptyset$, is a connected component of $Supp(P)$ (which could contain other connected components, "parallel" to that one).

At this point, it is convenient to contract each maximal $\mathcal{F}$-chain intersecting E_1, and to work on the singular surface $\hat{X}$ obtained in this way (this is not absolutely indispensable, everything can be done on X).



The curve E_1 becomes a curve $\hat{E}_1$ (which is not Cartier, but $\mathbf{Q}$-Cartier, at singular points of $\hat{X}$), and the foliation $\mathcal{F}$ becomes a foliation $\hat{\mathcal{F}}$ the leaves of which are, on a neighbourhood of $\hat{E}_1$, discs intersecting $\hat{E}_1$ at only one point. The section $s \in H^0(X, K_{\mathcal{F}}^{\otimes l})$ becomes a section $\hat{s}$ of the sheaf $K_{\hat{\mathcal{F}}}^{\otimes l}$ vanishing on $\hat{E}_1$, and only there on a neighbourhood of $\hat{E}_1$. This section can be integrated to a holomorphic function on a neighbourhood of $\hat{E}_1$ vanishing on $\hat{E}_1$ and only there. This is clear if $l = 1$: in that case, $\hat{s}$ gives on each leaf a holomorphic 1-form, and being the leaf a disc such a 1-form is the differential of an unique holomorphic function on the leaf, vanishing at the intersection of the leaf with $\hat{E}_1$. But the same can be done even if $l > 1$: use simply the fact that if $\omega \in \Omega^1(\mathbf{D})^{\otimes l}$ vanishes only at the origin then there exists a unique $f \in \mathcal{O}(\mathbf{D})$ such that $f(0) = 0$ and $\omega = f^{1-l}(df)^{\otimes l}$ $(= [d(lf^{\frac{1}{l}})]^{\otimes l}$, in a less precise but more suggestive form). Of course, this leaf-by-leaf construction produces a function F on a neighbourhood of $\hat{E}_1$ which is fully holomorphic, and not only along the leaves. It is also clear that the zero set of F is exactly $\hat{E}_1$, with some multiplicity $m \geq 2$. Thus the level sets of F give a proper fibration on a neighbourhood of $\hat{E}_1$, containing $\hat{E}_1$ as a multiple fibre. Standard facts (compactness of the Hilbert scheme) show that this local fibration extends to a global fibration of the full $\hat{X}$.

Returning to X, we therefore see that a multiple of P is a multiple fibre of a fibration $\pi : X \to B$. But this implies that $kod(\mathcal{F}) = 1$, because $kod(\mathcal{F}) = kod(\mathcal{O}_X(lP))$ and $kod(\mathcal{O}_X(lP)) = 1$ if a multiple of lP is a fibre. This contradiction with our assumption $kod(\mathcal{F}) = 0$ proves the lemma. $\triangle$

From now on we shall suppose, to simplify notations, that $Supp(P)$ is connected; the non-connected case is absolutely similar (and even simpler, as the reader may imagine).

Recall [Rei] that a connected $\mathbf{Q}^+$-divisor $P = \sum_{j=1}^{p} a_j E_j$ is said to be *of elliptic fibre type* if $P \cdot E_j = 0$ for every $j = 1, ..., p$ and $K_X \cdot P = 0$. One can easily see that Kodaira analysis of singular fibres of elliptic fibrations can be applied also to $\mathbf{Q}^+$-divisors of elliptic fibre type: that is, if such a divisor does not contain (-1)-curves in its support then it will be a rational multiple of one of the divisors appearing in Kodaira's table [BPV, page 150].

Lemma 2. *The positive part P of $K_{\mathcal{F}}$ is a divisor of elliptic fibre type, and more precisely it is a rational multiple of one of the divisors appearing in the table of section 3, chapter 4. Moreover, $Sing(\mathcal{F}) \cap Supp(P) = Sing(Supp(P))$, and all these singularities are nondegenerate.*

Proof.

We have already noted that $P \cdot E_j = 0$ for every $j = 1, ..., p$. Let us consider the normal bundle $N_{\mathcal{F}}$ of $\mathcal{F}$. We have

$$N_{\mathcal{F}} \cdot E_j = E_j \cdot E_j + Z(\mathcal{F}, E_j)$$

and $Z(\mathcal{F}, E_j) \geq (\cup_{k \neq j} E_k) \cdot E_j$ (remark that $Supp(P)$ has only normal crossing singularities, being entirely $\mathcal{F}$-invariant, and each $p \in E_k \cap E_j$ has $Z(\mathcal{F}, E_j, p) \geq 1$). Thus

$$N_{\mathcal{F}} \cdot E_j \geq (\cup_k E_k) \cdot E_j$$

and so

$$N_{\mathcal{F}} \cdot P \geq (\cup_k E_k) \cdot P = 0.$$

On the other hand,

$$K_X \cdot P = (N^*_{\mathcal{F}} + P + N) \cdot P = N^*_{\mathcal{F}} \cdot P.$$

The strict inequality $N_{\mathcal{F}} \cdot P > 0$ leads to a contradiction: by Riemann-Roch (and $P \cdot P = 0$, $K_X \cdot P < 0$) we would obtain $kod(\mathcal{O}_X(lP)) = 1$ and thus $kod(\mathcal{F}) = 1$. Hence $N_{\mathcal{F}} \cdot P = 0 = K_X \cdot P$, and P is of elliptic fibre type. Moreover, the same vanishing of $N_{\mathcal{F}} \cdot P$ implies the statement about $Sing(\mathcal{F})$, because the inequalities above have to become equalities.

Being P a normal crossing divisor, it is either in the table of chapter 4 or it is a blowing-up of one in that table; but the relative minimality excludes the latter case. $\triangle$

Our aim is to prove that a multiple of P, supposed nonempty by contradiction, is actually a (singular) fibre of an elliptic fibration: as in lemma 1, this would give a contradiction with $kod(\mathcal{F}) = 0$.

The case where $h^1(X, \mathcal{O}_X) > 0$ is particularly simple. In fact, take $\omega \in \Omega^1_X(X)$, $\omega \not\equiv 0$. If $\omega|_{\mathcal{F}} \equiv 0$ then $h^0(X, N^*_{\mathcal{F}}) > 0$ and from proposition 2 of chapter 6 we deduce that $\mathcal{F}$ itself is an elliptic fibration, containing P as a fibre. If $\omega|_{\mathcal{F}} \not\equiv 0$ for any holomorphic 1-form ω, then $h^1(X, \mathcal{O}_X) = 1$ (for in that case this dimension is bounded by $h^0(X, K_{\mathcal{F}}) \leq 1$) and so $alb_X : X \to B$ is a fibration over an elliptic curve; moreover either P is a fibre of alb_X (and so we finish) or P is a smooth elliptic curve transverse to alb_X; but in this last case any nontrivial 1-form ω restricted to $\mathcal{F}$ defines a section of $K_{\mathcal{F}}$ not vanishing on P, an evident contradiction with $kod(\mathcal{F}) = 0$ and $K_{\mathcal{F}}^{\otimes l} = \mathcal{O}_X(l(P + N))$.

For these reasons, from now on we shall suppose that

$$h^1(X, \mathcal{O}_X) = 0.$$

Let $Q = qP$ be the smallest $\mathbf{Z}^+$-divisor proportional to P. To prove that a multiple of Q is a fibre is the same as to prove that its normal bundle $\mathcal{O}_X(Q)|_Q$ is a torsion line bundle (see e.g. [Sad]: if m is the order of $\mathcal{O}_X(Q)|_Q$, from the exact sequences $0 \to \mathcal{O}_X((n-1)Q) \to \mathcal{O}_X(nQ) \to \mathcal{O}_X(nQ)|_Q \to 0$ it follows, by recurrence on $n = 1, ..., m-1$, that $h^1(\mathcal{O}_X((m-1)Q)) = h^1(\mathcal{O}_X) = 0$, and thus $h^0(\mathcal{O}_X(mQ)) = 2$). We are therefore faced with a problem which is mainly a local problem.

As a first reduction, observe that the branched covering trick (already used in the proof of theorem 2, chapter 8) and the particular structure of the foliation around $Supp(P + N)$ given by lemma 2 allows to reduce everything to the case "$l = 1$", i.e. $K_{\mathcal{F}}$ is effective, $K_{\mathcal{F}} = \mathcal{O}_X(P + N)$. This does not mean, however, that $N = \emptyset$ (as it did in theorem 2, chapter 8): $P + N$ is an integral divisor, but not the single N. Anyway, we must have $Supp(N) \subset Supp(P)$.

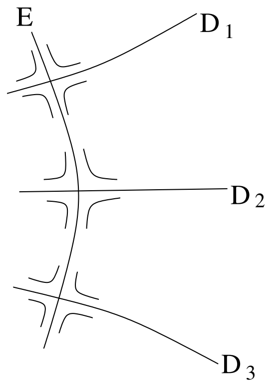
Lemma 3. *On a neighbourhood of $Supp(P)$ and for some $k \in \mathbf{N}^+$ we have*

$$N^*_{\mathcal{F}} \otimes \mathcal{O}_X(Q_{red}) = \mathcal{O}_X(-kQ)$$

(Q_{red} is the reduced divisor associated to Q).

Proof.

Recall that Q belongs to the table of section 3, chapter 4. Instead of giving a "general" proof, we prefere to give a concrete proof in a particular case: we shall suppose that Q is of type II:



$E^2 = -1 \qquad D_1^2 = -2 \qquad D_2^2 = -3 \qquad D_3^2 = -6$

$Q = 6E + 3D_1 + 2D_2 + D_3$

(the other types are treated in an analogous way).

Each D_j is a maximal $\mathcal{F}$-chain, and clearly $N = \frac{1}{2}D_1 + \frac{1}{3}D_2 + \frac{1}{6}D_3$, $K_{\mathcal{F}} = \mathcal{O}_X(\lambda Q + N)$ for some positive λ. The integrality of $\lambda Q + N$ means that $\lambda = k - \frac{1}{6}$, for some $k \in \mathbf{N}^+$. That is

$$K_{\mathcal{F}} = \mathcal{O}_X(kQ - E).$$

On the other hand, let us compute K_X. We can contract Q to an irreducible rational curve with a cusp Q_0 on a smooth surface X_0 (this is the type II in Kodaira's table). From the exact sequence

$$0 \to K_{X_0} \to K_{X_0} \otimes \mathcal{O}_{X_0}(Q_0) \to K_{X_0} \otimes \mathcal{O}_{X_0}(Q_0)|_{Q_0} \to 0,$$

from $h^1(K_{X_0}) = h^1(\mathcal{O}_{X_0}) = 0$ and from $K_{X_0} \otimes \mathcal{O}_{X_0}(Q_0)|_{Q_0} = K_{Q_0} = \mathcal{O}_{Q_0}$ we deduce that $K_{X_0} \otimes \mathcal{O}_{X_0}(Q_0)$ has a section vanishing at no point of Q_0, i.e. K_{X_0} has a meromorphic section with a first order pole on Q_0 and no zero around Q_0. Returning to X, by blowing-up Q_0, we find that K_X has a meromorphic section with a first order pole on $D_1 \cup D_2 \cup D_3$ and a second order pole on E, i.e. on a neighbourhood of Q we have

$$K_X = \mathcal{O}_X(-Q_{red} - E).$$

From $N_{\mathcal{F}}^* = K_X \otimes K_{\mathcal{F}}^*$ it now follows that

$$N_{\mathcal{F}}^* = \mathcal{O}_X(-kQ - Q_{red})$$

on a neighbourhood of Q, as required. △

Let us note, once a time, the natural appearance of logarithmic 1-forms (sections of $N_{\mathcal{F}}^* \otimes \mathcal{O}_X(Q_{red})$).

If Q is a smooth elliptic curve (and so $Q_{red} = Q$) the previous lemma is already sufficient to conclude: in that case we obviously have $N_{\mathcal{F}}|_Q = \mathcal{O}_X(Q)|_Q$, and thus by the lemma $\mathcal{O}_X(-kQ)|_Q = \mathcal{O}_Q$, showing that $\mathcal{O}_X(Q)|_Q$ is a torsion line bundle, as desired.

The same happens if Q is a cycle of (-2)-curves (and so $Q_{red} = Q$ again). A simple computation via Camacho-Sad formula shows in that case that each singularity of $\mathcal{F}$ on Q is of the type $z(1+...)dw + w(1+...)dz = 0$, and using this fact one easily proves that $N_{\mathcal{F}}|_Q = \mathcal{O}_X(Q)|_Q$ as in the smooth elliptic case (cfr. also the proof of lemma 1 in chapter 4, for similar computations).

The general case is almost reduced to the two previous ones by the familiar branched covering trick. To fix ideas, let us consider the case in which Q is of type II, as in the previous lemma; the other cases are absolutely similar. Even if Q is not (yet) a fibre, we may apply to it the stable reduction theorem [BPV, page 95]. Namely, the line bundle $\mathcal{O}_X(Q)$ has, on a neighbourhood U of Q, a root of order 6: there exists $L \in Pic(U)$ such that $L^{\otimes 6} = \mathcal{O}_X(Q)$ (proof: $\mathcal{O}_X(Q)|_{Q_{red}}$ is trivial, being numerically trivial and being Q_{red} a tree of rational curves, thus $\mathcal{O}_X(Q)$ is definable by a cocycle $\{g_{ij}\}$ with $g_{ij}|_{Q_{red}} = 1$ for every i,j, and so we may unambiguously extract the 6-root of g_{ij}, by requiring $g_{ij}^{\frac{1}{6}}|_{Q_{red}} = 1$ for every i,j). Then the natural section of $\mathcal{O}_X(Q)$ can be pulled-back to the total space of L, giving a 6-fold covering of U, etc.. More precisely, we obtain in this way a proper covering $V \stackrel{\pi}{\to} U$ and a proper bimeromorphic morphism $V \stackrel{r}{\to} W$ such that:
i) $C = r(\pi^{-1}(Q_{red}))$, as a reduced curve, is a smooth elliptic curve of zero selfintersection, and $r_*\pi^*(\mathcal{O}_U(Q)) = \mathcal{O}_W(6C)$;
ii) $\mathcal{G} = r_*\pi^*(\mathcal{F})$ is tangent to C and free of singularities;
iii) $N_{\mathcal{G}}^* \otimes \mathcal{O}_W(C) = r_*\pi^*(N_{\mathcal{F}}^* \otimes \mathcal{O}_U(Q_{red})) = \mathcal{O}_W(-6kC)$ (by i) and lemma 3).

The new difficulty is, however, that to prove the torsionness of $\mathcal{O}_X(Q)|_Q$ is equivalent to prove the torsionness of $\mathcal{O}_W(6C)|_{6C}$, that is of $\mathcal{O}_W(C)|_{6C}$, and the evident torsionnes of $\mathcal{O}_W(C)|_C$ (arising from iii) above and $N_{\mathcal{G}}|_C = \mathcal{O}_W(C)|_C$) is no more sufficient. We need a more sophisticated analysis, provided by the next lemma.

<u>**Lemma 4**</u>. *On a neighbourhood of C we have*

$$N_{\mathcal{G}}^* \otimes \mathcal{O}_W(C) = \mathcal{O}_W(-hC)$$

for some $h \in \mathbf{N}^+ \setminus 6\mathbf{N}^+$.

Proof.

It is a variation on the method of [CLS], already used in the proof of lemma 1, chapter 4.

Returning for a moment to $Q = 6E + 3D_1 + 2D_2 + D_3$, let us observe that the holonomy of $\mathcal{F}$ along E is generated by three (germs of) diffeomorphisms of $(\mathbf{C}, 0)$, f, g, h, satisfying

$$f^2 = g^3 = h^6 = id \quad , \qquad fgh = id$$

(these are the holonomies around the three singularities of $\mathcal{F}$ on E, which have holomorphic first integrals). This holonomy group is therefore a solvable group, but it cannot be an abelian one: otherwise it would be a finite group and so the leaves of $\mathcal{F}$ around Q would be compact and would define an elliptic fibration, contradicting $kod(\mathcal{F}) = 0$. The commutator of this holonomy group is therefore a nontrivial abelian group, which embeds into the flow of the vector field $z^{h+1}\frac{\partial}{\partial z}$ around 0, for a suitable $h \in \mathbf{N}^+ \setminus 6\mathbf{N}^+$ [L-M] (more precisely, $h \notin 2\mathbf{N}^+ \cup 3\mathbf{N}^+$, but $h \notin 6\mathbf{N}^+$ is sufficient for our purposes). Such a vector field is uniquely defined if we normalize it by requiring that its time-one-flow is one of the (two) generators of the commutator group. Remark that the dual meromorphic 1-form $\frac{1}{z^{h+1}}dz$ is invariant by that commutator group.

A little thought shows that the holonomy of $\mathcal{G}$ along C is nothing but than the commutator of the holonomy of $\mathcal{F}$ along E. Therefore, on each transversal to C we may construct a meromorphic 1-form which is holonomy invariant and which has a pole of order $h + 1$. Such a 1-form can be extended to a neighbourhood of the transversal in W to a closed meromorphic 1-form which defines $\mathcal{G}$. On two intersecting neighbourhoods, the corresponding 1-forms differ by a multiplicative constant (they differ by a multiplicative function which is constant on the leaves and holonomy invariant, thus a constant), and the normalization above gives that such a constant is 1. That is: these closed meromorphic 1-forms glue together to a meromorphic section of $N^*_{\mathcal{G}}$ over a full neighbourhood of C and having a pole of order $h + 1$ along C. This exactly means that $N^*_{\mathcal{G}} \otimes \mathcal{O}_W((h + 1)C)$ is trivial on such a neighbourhood. $\triangle$

Now the proof can be achieved: by lemma 4 and property iii) above, we have on a neighbourhood of C

$$\mathcal{O}_W(-6kC) = \mathcal{O}_W(-hC)$$

and so (being $h \in \mathbf{N}^+ \setminus 6\mathbf{N}^+$) $\mathcal{O}_W(C)$ is a torsion line bundle. This implies that $\mathcal{O}_W(C)|_{6C}$, and hence $\mathcal{O}_X(Q)|_Q$, is a torsion line bundle, as desired.

Let us conclude with few remarks:

a) it is not strictly necessary to do a ramified covering and to reduce the (virtual) fibre to a stable one: as the proof of lemma 4 suggests, everything can be done working always on X, at the price of a minor elegance;

b) it is not strictly indispensable to use the precise normal form result of [L-M]: after all, we need only to prove the torsionness of $\mathcal{O}_W(C)|_{6C}$, and not of $\mathcal{O}_W(C)$, and so we need only to construct (in lemma 4) a meromorphic 1-form which is holonomy invariant "up to order 6", etc. (this allows to forget the difficult convergence problems related to normal form theorems);

c) the case of a fibre of type I_b^*, leading to a cycle of (-2)-curves, requires some easy and inessential modifications of the arguments above, due to singularities without first integral. $\triangle$

A first corollary to this theorem and theorem 2 of the previous chapter is:

Corollary 1. *Let $\mathcal{F}$ be a reduced foliation on an algebraic surface X, and suppose that $kod(\mathcal{F}) = 0$. Then, up to a ramified covering and up to birational maps, $\mathcal{F}$ is generated by a global holomorphic vector field with isolated zeroes.* $\triangle$

A second corollary to the theorem is the following answer to some problems posed by [DPS] and [Sad]:

Corollary 2. *Let $\mathcal{F}$ be a foliation on an algebraic surface X and suppose that $\mathcal{F}$ is tangent to a smooth elliptic curve E, free of singularities of $\mathcal{F}$. Then either E is a (multiple) fibre of an elliptic fibration (and therefore $\mathcal{F}$ coincides with that fibration or it is turbulent with respect to it) or, up to ramified coverings and birational maps, $\mathcal{F}$ is the suspension of a representation $\rho : \pi_1(\hat{E}) \to Aut(\mathbf{C}P^1)$, $\hat{E}$ an elliptic curve.*

Proof.

We may assume $\mathcal{F}$ reduced and even relatively minimal, without loss of generality. From $K_{\mathcal{F}} \cdot E = 0$ and $E \cdot E = 0$ it follows, by Hodge index theorem, that the positive part P of $K_{\mathcal{F}}$ is proportional to E: $P = qE$, $q \in \mathbf{Q}^+ \cup \{0\}$. We have $kod(\mathcal{F}) \geq 0$: this is evident if $h^1(X, \mathcal{O}_X) = 0$, and it is proved in the usual way, using the Albanese morphism and results on elliptic fibrations, if $h^1(X, \mathcal{O}_X) > 0$. If $q > 0$ then $\nu(\mathcal{F}) = 1$, therefore $kod(\mathcal{F}) = 1$ by theorem 2, and E is a (possibly multiple) fibre of an elliptic fibration. Remark that in this case the proof of theorem 2 considerably simplifies, because we already know that $Supp(P)$ is a smooth $\mathcal{F}$-invariant elliptic curve. If $q = 0$ then $\nu(\mathcal{F}) = 0$ and $\mathcal{F}$, up to ramified coverings and birational maps, is generated by a global holomorphic vector field with isolated zeroes. Looking at the list of these vector fields, we deduce that either E is again an elliptic fibre or the vector field is given by the suspension of a (quite special) representation $\rho : \pi_1(\hat{E}) \to Aut(\mathbf{C}P^1)$. $\triangle$

One can easily generalize this corollary to the case where E is not smooth, but it is a divisor of elliptic fibre type.

Finally, concerning the abundance of $K_{\mathcal{F}}$ we may resume as follows all the previous results:

Proposition 2. *Let $\mathcal{F}$ be a reduced foliation on an algebraic surface with $kod(\mathcal{F}) \neq \nu(\mathcal{F})$. Then $kod(\mathcal{F}) = -\infty$ and $\nu(\mathcal{F}) = 1$.*

In the last section we shall see that this can really occur.

4. Foliations with an entire leaf

A remarkable application of the classification of foliations with Kodaira dimension 0 or 1 is the classification of foliations which have an entire transcendental leaf.

Let us firstly recall some important facts related to entire curves on complex manifolds [Mc2] [Voj] [Br3]. If X is a compact complex manifold, of dimension n, and $f : \mathbf{C} \to X$ a (nonconstant) entire curve, then we can associate to f a (nontrivial) closed positive current Φ of bidimension $(1,1)$, which is a normalized Nevanlinna type integration current over $f(\mathbf{C})$. Denoting $[\Phi] \in H^{n-1,n-1}(X, \mathbf{R})$ its cohomology class, we have the following easy but fundamental property: $[\Phi] \cdot Z \geq 0$ for every hypersurface $Z \subset X$ such that $f(\mathbf{C}) \not\subset Z$. In particular, if $f(\mathbf{C})$ is Zariski dense then the class $[\Phi]$ is nef.

We can lift $f : \mathbf{C} \to X$ to $f' : \mathbf{C} \to PTX$, taking its (projectivized) derivative, and also to f' we can associate a current Φ'. Let $\mathcal{O}_{PTX}(-1)$ be the tautological line bundle of PTX. Recall that, by definition, if $C \subset X$ is a smooth compact curve and $C' \subset PTX$ its natural lift to PTX, then $\mathcal{O}_{PTX}(-1)|_{C'}$ is the tangent bundle of C and thus $\mathcal{O}_{PTX}(-1) \cdot C' = \chi(C)$. Hence the product $\mathcal{O}_{PTX}(-1) \cdot [\Phi']$ can be thought as the "Euler characteristic" of $[\Phi]$. Being Φ "uniformized" by $\mathbf{C}$, one expects that this product is nonnegative, and this is indeed the case as McQuillan proved in [Mc2]:

$$\mathcal{O}_{PTX}(-1) \cdot [\Phi'] \geq 0.$$

This is the so-called *tautological inequality.* There exists also a logarithmic version of such an inequality [Voj], implicit in [Mc2]. Suppose that $D \subset X$ is a normal crossing divisor and that $f(\mathbf{C}) \cap D = \emptyset$. Instead of TX we may consider its logarithmic version $V = TX(\log D)$ (= the dual of $\Omega^1_X(\log D)$; a local section of V is therefore a local holomorphic vector field tangent to D). On PV we still have a tautological line bundle $\mathcal{O}_{PV}(-1)$, and we still can lift f to $f' : \mathbf{C} \to PV$ and associate a current Φ' to f'. Then we have [Voj]:

$$\mathcal{O}_{PV}(-1) \cdot [\Phi'] \geq 0.$$

This is the *logarithmic tautological inequality.*

Suppose now that $\dim X = 2$ and that $f(\mathbf{C})$ is Zariski dense, so that in particular $[\Phi]$ is nef. Suppose also that f is tangent to a foliation $\mathcal{F}$ all of whose singularities are reduced. This means that Φ is an $\mathcal{F}$-invariant current, and we have seen in chapter 3, section 4, theorem 6, that the following inequality

$$N_{\mathcal{F}} \cdot [\Phi] \geq 0$$

holds. On the other side, the (logarithmic) tautological inequality gives a similar inequality for $T_{\mathcal{F}}$:

Theorem 3 [Mc2]. *We have*

$$T_{\mathcal{F}} \cdot [\Phi] \geq 0.$$

Proof.

We only sketch the proof, refering to [Mc2] for more details; see also [Br3].

First of all, observe that if we blow-up $p \in Sing(\mathcal{F})$, $\pi : X^{(1)} \to X$, we can lift f to $f^{(1)} : \mathbf{C} \to X^{(1)}$ and construct a current $\Phi^{(1)}$ with $\pi_* \Phi^{(1)} = \Phi$. Setting $\nu(\Phi, p) = [\Phi^{(1)}] \cdot E^{(1)}$ ($E^{(1)} = \pi^{-1}(p)$) we have

$$[\Phi^{(1)}] = \pi^*[\Phi] - \nu(\Phi, p) E^{(1)}$$

and thus

$$[\Phi^{(1)}]^2 = [\Phi]^2 - \nu(\Phi, p)^2.$$

Note that $\nu(\Phi, p) \geq 0$, being $[\Phi^{(1)}]$ nef (or $f^{(1)}(\mathbf{C}) \not\subset E^{(1)}$). This number is in fact related to the Lelong number of Φ at p.

On $E^{(1)}$ we have two singularities $p^{(1)}$, $q^{(1)}$ of $\mathcal{F}^{(1)} = \pi^*(\mathcal{F})$. We blow-up them, $\pi^{(1)} : X^{(2)} \to X^{(1)}$, obtaining a new nef current $\Phi^{(2)}$ satisfying

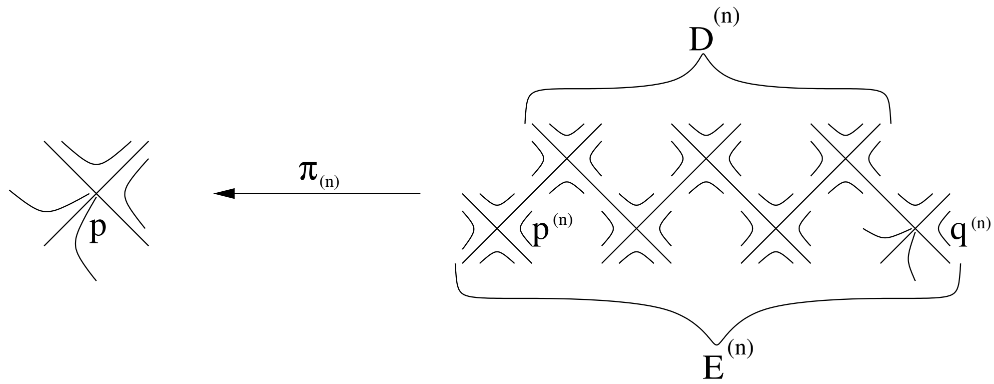
$$[\Phi^{(2)}]^2 = [\Phi^{(1)}]^2 - \nu(\Phi^{(1)}, p^{(1)})^2 - \nu(\Phi^{(1)}, q^{(1)})^2 =$$

$$= [\Phi]^2 - \nu(\Phi, p)^2 - \nu(\Phi^{(1)}, p^{(1)})^2 - \nu(\Phi^{(1)}, q^{(1)})^2.$$

On $E^{(2)} = (\pi^{(1)})^{-1}(E^{(1)})$ we have four singularities of $\mathcal{F}^{(2)} = (\pi^{(1)})^*(\mathcal{F}^{(1)})$: two at the crossings of $E^{(2)}$, which we don't touch, and two at smooth points of $E^{(2)}$, which we blow-up. Iterating this construction n times, each time blowing-up the two "extreme" singularities, we finally obtain a nef current $\Phi^{(n)}$ on $X^{(n)}$ satisfying (with evident notations)

$$[\Phi^{(n)}]^2 = [\Phi]^2 - \nu(\Phi, p)^2 - \sum_{j=1}^{n-1} [\nu(\Phi^{(j)}, p^{(j)})^2 + \nu(\Phi^{(j)}, q^{(j)})^2].$$

Remark now that $[\Phi^{(n)}]^2 \geq 0$, being $\Phi^{(n)}$ nef, for any $n \in \mathbf{N}^+$. Thus $\nu(\Phi^{(n)}, p^{(n)})$ and $\nu(\Phi^{(n)}, q^{(n)})$ must tend to 0 as $n \to +\infty$. Remark also that $\mathcal{F}^{(n)}$ is tangent to the exceptional divisor $E^{(n)}$ of $X^{(n)} \to X$ (a chain of rational curves), and $Sing(\mathcal{F}^{(n)}) \cap E^{(n)} = Sing(E^{(n)}) \cup \{p^{(n)}, q^{(n)}\}$. Moreover, the crossing points of $E^{(n)}$ are nondegenerate for $\mathcal{F}^{(n)}$.



To simplify notations, suppose that p is the only singularity of $\mathcal{F}$. Suppose also, to avoid technicalities, that p, and hence $p^{(n)}, q^{(n)}$, are not saddle-nodes. Let us apply the logarithmic tautological inequality to $f^{(n)} : \mathbf{C} \to X^{(n)}$, with normal crossing divisor $D^{(n)} =$ the closure of $\{E^{(n)} \setminus \{$ the two extreme rational curves$\}\}$ (of course, we have $f^{(n)}(\mathbf{C}) \cap D^{(n)} = \emptyset$). To this end, consider the graph Γ of $\mathcal{F}^{(n)}$ in PV, $V = TX^{(n)}(\log D^{(n)})$. This graph is a section of PV over $X^{(n)} \setminus \{p^{(n)}, q^{(n)}\}$, and contains the fibres F_p, F_q over $p^{(n)}, q^{(n)}$: note that the other singularities of $\mathcal{F}^{(n)}$ on $E^{(n)}$ are *not* singular from the logarithmic point of view. Because $p^{(n)}$ and $q^{(n)}$ are nondegenerate, Γ is smooth, and the projection $\Gamma \stackrel{pr}{\to} X^{(n)}$ is the blowing-up of $p^{(n)}$ and $q^{(n)}$. The restriction of the tautological bundle $\mathcal{O}_{PV}(-1)$ to Γ is easily computed, by tautology:

$$\mathcal{O}_{PV}(-1)|_{\Gamma} = pr^*(T_{\mathcal{F}^{(n)}}) \otimes \mathcal{O}_{\Gamma}(F_p + F_q).$$

The knowledge of this restriction is sufficient to compute the product $\mathcal{O}_{PV}(-1) \cdot [\Phi^{(n)\prime}]$, because $f^{(n)}$ is tangent to $\mathcal{F}^{(n)}$ and so $f^{(n)\prime}$ has values into $\Gamma \subset PV$. Therefore the logarithmic tautological inequality asserts

$$pr^*(T_{\mathcal{F}^{(n)}}) \cdot [\Phi^{(n)\prime}] + [\Phi^{(n)\prime}] \cdot F_p + [\Phi^{(n)\prime}] \cdot F_q \geq 0$$

and this can be rewritten as

$$T_{\mathcal{F}^{(n)}} \cdot [\Phi^{(n)}] \geq -\nu(\Phi^{(n)}, p^{(n)}) - \nu(\Phi^{(n)}, q^{(n)}).$$

On the other side, because $\pi_{(n)} : X^{(n)} \to X$ is a composition of blowing-ups at reduced singular points of the foliations, we also have $T_{\mathcal{F}^{(n)}} = \pi_{(n)}^*(T_{\mathcal{F}})$ and thus $T_{\mathcal{F}^{(n)}} \cdot [\Phi^{(n)}] = T_{\mathcal{F}} \cdot [\Phi]$, whence

$$T_{\mathcal{F}} \cdot [\Phi] \geq -\nu(\Phi^{(n)}, p^{(n)}) - \nu(\Phi^{(n)}, q^{(n)}).$$

The proof is now complete, because the right hand side tends to 0 as $n \to +\infty$. △

With these results we can now give the classification.

Theorem 4. *Let $\mathcal{F}$ be a reduced foliation on an algebraic surface X and suppose that there exists an entire curve $f : \mathbf{C} \to X$ which is tangent to $\mathcal{F}$ and whose image is Zariski dense. Then $kod(\mathcal{F}) \in \{0, 1\}$.*

Proof.

Obviously $\mathcal{F}$ is not a rational fibration, and so $K_{\mathcal{F}}$ is pseudoeffective and has a Zariski decomposition

$$K_{\mathcal{F}} \overset{num}{=} P + N.$$

Let Φ be a current associated to f. Then, by theorem 3, we have $K_{\mathcal{F}} \cdot [\Phi] \leq 0$ and thus $P \cdot [\Phi] \leq 0$, being $[\Phi]$ nef. By Hodge index theorem, we deduce $P \cdot [\Phi] = 0 = P \cdot P$, and more precisely $P \overset{num}{=} \lambda[\Phi]$ for some $\lambda \in \mathbf{R}^+ \cup \{0\}$. If $\lambda = 0$ then $\nu(\mathcal{F}) = 0$ and hence $kod(\mathcal{F}) = 0$. If $\lambda > 0$ then $\nu(\mathcal{F}) = 1$ and $kod(\mathcal{F}) \in \{-\infty, 1\}$; we have to exclude the case $-\infty$. Remark that, by theorem 6 of chapter 3,

$$K_X \cdot P = N^*_{\mathcal{F}} \cdot P = \lambda N^*_{\mathcal{F}} \cdot [\Phi] \leq 0.$$

If $h^1(X, \mathcal{O}_X) = 0$ then, by Riemann-Roch formula and Serre duality, we obtain the inequality $h^0(X, \mathcal{O}_X(lP)) \geq \chi(\mathcal{O}_X) > 0$ for l large, so that $kod(\mathcal{F}) \geq 0$. If $h^1(X, \mathcal{O}_X) \neq 0$ then either $h^0(X, K_{\mathcal{F}}) > 0$, so that $kod(\mathcal{F}) \geq 0$, or $\mathcal{F}$ is the Albanese fibration; but this last possibility is excluded by the existence of a Zariski dense leaf. △

From theorems 1 and 2 we therefore obtain that $\mathcal{F}$ as in theorem 4 is either covered by a foliation generated by a global holomorphic vector field, or it is a Riccati or turbulent foliation. Of course, a Riccati or turbulent foliation having a transcendental entire leaf is very special, and the classification could be ulteriorly refined. We only note the following

Corollary 3. *Let $\mathcal{F}$, X, f be as in theorem 4 and suppose that $kod(\mathcal{F}) = 1$, so that $\mathcal{F}$ is Riccati or turbulent. Then any current Φ associated to f is of the type $\sum_{j=1}^{m} \lambda_j \delta_{F_j}$, $\lambda_j > 0$, F_j totally $\mathcal{F}$-invariant fibres (possibly singular).*

Proof.

The positive part P of $K_{\mathcal{F}}$ is nontrivial and numerically proportional to a fibre of the adapted fibration, and $P \cdot [\Phi] = 0$. If F is a generic fibre, transverse to $\mathcal{F}$, then $F \cdot [\Phi] = 0$ and this actually implies that $Supp\Phi$ does not intersect F (Φ induces a measure on F, and $F \cdot [\Phi]$ is the total mass of that measure). Being $Supp\Phi$ $\mathcal{F}$-invariant and $[\Phi] \cdot [\Phi] = 0$, the conclusion follows easily. △

In fact, it should be possible to prove that the "limit set" of the leaf containing the image of f is a union of (one or two) $\mathcal{F}$-invariant fibres.

5. Foliations of negative Kodaira dimension

Example 1 [Br1]. Let X be a compact surface uniformised by the bidisc $\mathbf{D}\times\mathbf{D}$. The automorphism group of $\mathbf{D}\times\mathbf{D}$ preserves the vertical and the horizontal foliations, so that X is naturally equipped of two (nonsingular) foliations $\mathcal{F}$ and $\mathcal{G}$, one transverse to the other. If $\pi_1(X) = \Gamma \subset Aut(\mathbf{D}\times\mathbf{D})$ is not, up to finite index, a product $\Gamma_1 \times \Gamma_2$, $\Gamma_i \subset Aut(\mathbf{D})$, then both $\mathcal{F}$ and $\mathcal{G}$ have dense leaves in X. In that case, we have $\nu(\mathcal{F}) = \nu(\mathcal{G}) = 1$ but $kod(\mathcal{F}) = kod(\mathcal{G}) = -\infty$. The computation of ν follows from the fact that $\mathcal{F}$ is certainly relatively minimal, being nonsingular, and the negative part of $K_{\mathcal{F}}$ is certainly empty, so that $K_{\mathcal{F}} = P$; but from the formulae of chapters 2 and 3 we have $P^2 = 0$ (Baum-Bott) and $P \cdot K_X = c_2(X) > 0$, so that $\nu(\mathcal{F}) = 1$. The computation of kod follows from the evident isomorphism $K_{\mathcal{F}} \simeq N^*_{\mathcal{G}}$: Bogomolov's lemma (chapter 6) excludes $kod(\mathcal{F}) \geq 1$, and $kod(\mathcal{F}) = 0$ is excluded by the previous classification results, in particular "$kod = 0 \Rightarrow \nu = 0$". Note also that if Γ is a product $\Gamma_1 \times \Gamma_2$ (up to finite index) then $\mathcal{F}$ and $\mathcal{G}$ are both isotrivial fibrations of genus ≥ 2.

Example 2 [Mc1]. Let X be a Hilbert modular surface [BPV, page 177]: it is a smooth compactification of an open surface $\tilde{X}_0$ which is the (minimal) resolution of $X_0 = \mathbf{D}\times\mathbf{D}/\Gamma$, where $\Gamma \subset Aut(\mathbf{D}\times\mathbf{D})$ is a special subgroup, of arithmetic nature. Singularities of X_0 arise from the torsion of Γ, and they are quotient singularities, resolved by Hirzebruch-Jung strings. The compactification of $\tilde{X}_0$ is done by adding cycles of rational curves. As in the previous example, we have on X two natural foliations $\mathcal{F}$ and $\mathcal{G}$, originated by the two fibrations of $\mathbf{D}\times\mathbf{D}$. These foliations have some compact leaves: the Hirzebruch-Jung strings, which are moreover maximal $\mathcal{F}$-chains and $\mathcal{G}$-chains, and the cycles of rational curves, on which $\mathcal{F}$ and $\mathcal{G}$ are singular only at crossing points, the singularities being reduced and nondegenerate. Outside these compact leaves, $\mathcal{F}$ and $\mathcal{G}$ are nonsingular and transverse each other. Hence we have $K_{\mathcal{F}} \otimes N_{\mathcal{G}} = \mathcal{O}_X(D)$, where D is a positive integral divisor (the so-called "tangency divisor" between $\mathcal{F}$ and $\mathcal{G}$, locally defined by $\omega(v) = 0$ being ω a local 1-form generating $\mathcal{G}$ and v a local vector field generating $\mathcal{F}$) whose support is equal to the union of the compact leaves. If P is the positive part of $K_{\mathcal{F}}$, we also see that $P = N^*_{\mathcal{G}} \otimes \mathcal{O}_X(Z)$, where Z is the (reduced) union of the cycles of rational curves. We can now apply the same arguments of the previous example, in their logarithmic version, to conclude that $kod(\mathcal{F}) = -\infty$ and $\nu(\mathcal{F}) = 1$, and similarly for $\mathcal{G}$.

It seems quite possible that these two classes of examples, which we shall both call *Hilbert modular foliations*, are the only examples of foliations of Kodaira dimension $-\infty$ and numerical Kodaira dimension 1. In other words: any reduced foliation of Kodaira dimension $-\infty$ is either a rational fibration ($\nu = -\infty$) or a Hilbert modular foliation

($\nu = 1$). We shall explain in this section an approach to this conjecture, proposed in [Mc1].

Let $\mathcal{F}$ be a relatively minimal foliation on the algebraic surface X, with

$$kod(\mathcal{F}) = -\infty \qquad \nu(\mathcal{F}) = 1.$$

As usual, we have $h^1(X, \mathcal{O}_X) = 0$: otherwise we easily reach a contradiction using the Albanese morphism. If we want to prove that $\mathcal{F}$ is Hilbert modular, a natural thing to do is to look for its companion $\mathcal{G}$. Let us suppose, to simplify, that the negative part of $K_{\mathcal{F}}$ is empty, i.e. $K_{\mathcal{F}}$ is nef (anyway, this can always be realized if we allow to work on a singular surface, with quotient singularities; alternatively, the following arguments can be applied to P instead of $K_{\mathcal{F}}$). In particular, we have $K_{\mathcal{F}} \cdot K_{\mathcal{F}} = 0$. Then [Dem] there exists on $K_{\mathcal{F}}$ a singular hermitian metric whose curvature Ω is a closed positive current; as in chapter 7, "singular" means that the metric is not smooth, but only L^1_{loc}. That current Ω, which numerically represents $K_{\mathcal{F}}$, cannot be the integration current over a $\mathbf{Q}^+$-divisor, otherwise we would have $kod(K_{\mathcal{F}}) \geq 0$. But actually a little more can be said: its Lelong numbers are zero everywhere.

To see this, use Siu's decomposition formula [Dem]:

$$\Omega = \sum_{j=1}^{n} \lambda_j \delta_{C_j} + \Omega_{res}$$

where $\lambda_j > 0$, δ_{C_j} is the integration current over the algebraic curve C_j, $n \in \mathbf{N} \cup \{\infty\}$, and Ω_{res} is a closed positive current whose Lelong numbers are zero outside a countable set. From $[\Omega]$ nef and $[\Omega]^2 = 0$ it follows that $[\Omega] \cdot C_j = 0$ for every j and $[\Omega] \cdot [\Omega_{res}] = 0$. Being also $[\Omega_{res}]$ nef, by Hodge index theorem we have $[\Omega_{res}] = \lambda[\Omega]$, for some $\lambda \in [0, 1]$. If $\lambda \neq 1$ then we obtain that $[\Omega]$ is the class of a positive rational divisor $(\frac{1}{1-\lambda} \sum \lambda_j C_j)$, which is impossible for $kod(\mathcal{F}) = -\infty$ (and $h^1(X, \mathcal{O}_X) = 0$). Hence $\lambda = 1$, that is $\Omega = \Omega_{res}$. Finally, $[\Omega]^2 = 0$ implies that the Lelong numbers of Ω are zero everywhere [Dem].

This property of Ω and the fact $[\Omega]^2 = 0$, *suggest* that Ω defines, through its "kernel", a foliation $\mathcal{G}$ on X (a so-called *Monge-Ampère foliation*): if Ω has some mild regularity then $[\Omega]^2 = 0$ translates into $\Omega \wedge \Omega \equiv 0$, which means that the kernel of the $(1, 1)$-form Ω_x is a one-dimensional complex line $L_x \subset T_x X$ (where $\Omega_x \not\equiv 0$), and then this distribution of lines is integrable due to the closedness of Ω. There are, unfortunately, many obstacles to pursuit this construction:

1) it is not clear that our current Ω is sufficiently regular to be able to speak of "kernel of Ω_x", or at least to be able to define the wedge product $\Omega \wedge \Omega$ (however, the vanishing of Lelong numbers, which is a sort of regularity property, leaves open some hope);

2) even if Ω is sufficiently regular, say C^1, the foliation $\mathcal{G}$ is defined only where Ω is not zero, and it is not clear that such an open set is large (e.g., dense in X, or better equal to X minus a finite set);
3) the foliation $\mathcal{G}$ so defined has complex leaves, but it may certainly fail to be holomorphic (this is a quite common feature of Monge-Ampère foliations).

Anyway, in spite of these difficulties let us suppose that $\mathcal{G}$ exists and is holomorphic. More precisely, and slightly more generally, we do the following

Hypothesis: there exists on X a holomorphic foliation $\mathcal{G}$ such that the closed positive current Ω is $\mathcal{G}$-invariant.

Being Ω diffuse and $[\Omega]^2 = 0$, we obtain from theorem 5 of chapter 3:

$$N_{\mathcal{G}} \cdot [\Omega] = 0,$$

that is $N_{\mathcal{G}} \cdot K_{\mathcal{F}} = 0$. On the other hand, $N_{\mathcal{F}} \cdot K_{\mathcal{F}} = -K_X \cdot K_{\mathcal{F}} < 0$, otherwise Riemann-Roch and $h^1(X, \mathcal{O}_X) = 0$ would give $kod(\mathcal{F}) \geq 0$. In particular, $\mathcal{G}$ is different from $\mathcal{F}$. We can therefore analyse their *tangency divisor* D, which satisfies

$$\mathcal{O}_X(D) = K_{\mathcal{F}} \otimes N_{\mathcal{G}}.$$

From $K_{\mathcal{F}} \cdot K_{\mathcal{F}} = K_{\mathcal{F}} \cdot N_{\mathcal{G}} = 0$ we obtain

$$K_{\mathcal{F}} \cdot D = 0$$

and if $D = \sum_{j=1}^{n} m_j D_j$ then the nefness of $K_{\mathcal{F}}$ gives

$$K_{\mathcal{F}} \cdot D_j = 0 \qquad \forall j.$$

The intersection form on $\{D_j\}$ must be negative definite, otherwise these relations and Hodge index theorem would give $kod(\mathcal{F}) \geq 0$. In particular, $D_j^2 < 0$ for every j, and therefore each D_j is $\mathcal{F}$-invariant (otherwise $K_{\mathcal{F}} \cdot D_j = -D_j^2 + tang(\mathcal{F}, D_j) > 0$), more precisely each D_j is either a smooth rational curve with $Z(\mathcal{F}, D_j) = 2$ or a rational curve with a node and with $Z(\mathcal{F}, D_j) = 0$ (the smooth elliptic case with $Z(\mathcal{F}, D_j) = 0$ is excluded by $D_j^2 < 0$).

The $\mathcal{F}$-invariance of each D_j implies also its $\mathcal{G}$-invariance, being D_j the curves along which $\mathcal{F}$ and $\mathcal{G}$ are not transverse. Hence $N_{\mathcal{G}} \cdot D_j = D_j^2 + Z(\mathcal{G}, D_j)$, that is $N_{\mathcal{G}} \otimes \mathcal{O}_X(-D_{red}) \cdot D_j = Z(\mathcal{G}, D_j) - (\cup_{k \neq j} D_k) \cdot D_j$. Each intersection of D_j with D_k, $k \neq j$, gives a positive contribution to $Z(\mathcal{G}, D_j)$ (observe that D has only normal crossing singularities, being $\mathcal{F}$-invariant and being $\mathcal{F}$ reduced), hence we obtain

$$N_{\mathcal{G}} \otimes \mathcal{O}_X(-D_{red}) \cdot D_j \geq 0 \qquad \forall j.$$

We have already seen that $K_{\mathcal{F}}$ is numerically trivial on a neighbourhood of the support of D. Thus $N_{\mathcal{G}} \stackrel{num}{=} \mathcal{O}_X(D)$ on the same neighbourhood, and so $N_{\mathcal{G}} \otimes \mathcal{O}_X(-D_{red}) \geq 0$, numerically. This positivity and the inequalities above, together with the negativity of the intersection form on the subspace spanned by the curves D_j, implies that $N_{\mathcal{G}} \otimes \mathcal{O}_X(-D_{red})$ is actually numerically trivial, or equivalently $D = D_{red}$. Thus $Z(\mathcal{G}, D_j)$ must be equal to $(\cup_{k \neq j} D_k) \cdot D_j$, i.e. $Sing(\mathcal{G}) \cap D = Sing(D)$, all nondegenerate.

A connected component of D is a chain or a cycle of rational curves, but the case of a chain is excluded: the structure of $\mathcal{G}$ around such a connected component would rapidly lead to a contradiction (by the same arguments used in chapter 8 to prove that two maximal $\mathcal{F}$-chains cannot intersect).

Thus: each connected component of D is a cycle of rational curves, $Sing(\mathcal{F}) \cap D = Sing(\mathcal{G}) \cap D = Sing(D)$ (all reduced), $\mathcal{F}$ and $\mathcal{G}$ are transverse each other outside D and they are tangent at first order along D.

We clearly are in a situation very close to that of example 2, and indeed at this point one can actually prove that $\mathcal{F}$ and $\mathcal{G}$ are the two canonical foliations on a Hilbert modular surface. The above properties of the tangency divisor between $\mathcal{F}$ and $\mathcal{G}$ mean that $\Omega^1_X(\log D)$ splits as a direct sum $K_{\mathcal{F}} \oplus K_{\mathcal{G}}$. Moreover, $\det \Omega^1_X(\log D) = K_{\mathcal{F}} \otimes K_{\mathcal{G}} = L$ is nef (note that $K_{\mathcal{G}}$, as $K_{\mathcal{F}}$, has trivial negative part and thus it is nef) and big (i.e. $L^2 > 0$, which follows from $K_{\mathcal{G}} \cdot K_{\mathcal{F}} = (K_X + N_{\mathcal{G}}) \cdot K_{\mathcal{F}} > 0$). Also, if C is a curve not contained in D then $K_{\mathcal{F}} \cdot C > 0$ and so $L \cdot C > 0$ (observe that C cannot be $\mathcal{F}$-invariant). The uniformization theorem of [T-Y] now shows that X is a Hilbert modular surface and $\mathcal{F}$, $\mathcal{G}$ are Hilbert modular foliations.

Let us conclude with the following remark. We have seen that a foliation $\mathcal{F}$ with $kod(\mathcal{F}) = -\infty$ and $\nu(\mathcal{F}) = 1$ satisfies the inequality $N_{\mathcal{F}} \cdot K_{\mathcal{F}} < 0$ (if $K_{\mathcal{F}}$ is nef, otherwise we replace it by its positive part). Hence there exists an ample H such that $N_{\mathcal{F}} \cdot H < 0$, i.e. $N_{\mathcal{F}}$ is not pseudoeffective. As observed in chapter 7, it is therefore of paramount importance to study the structure of foliations whose normal bundle is not pseudoeffective.

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