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KNEADING SEQUENCES FOR TOY MODELS OF HÉNON MAPS

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Amar e mudar as coisas me interessa mais.
Alucinação - Belchior

To Sheila Cristina and José Ribamar

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Abstract

We present a study on how a certain type of combinatorial equivalence implies topological conjugacy. In this work, we present the concept of kneading sequences for a more general setting than one-dimensional dynamics. For this, we consider a two-dimensional family introduced by Benedicks and Carleson [BC91] as a toy model for Hénon maps and define the notion of kneading sequences of the critical line for a toy model. We show that these sequences are a complete invariant of the conjugacy class of the toy model. Furthermore, we show a version of Singer's Theorem for toy models and a combinatorial equivalence result for nonautonomous discrete dynamical systems.

Keywords: Toy models, kneading sequences, negative Schwarzian derivative, nonautonomous discrete dynamical systems

Resumo

Nós estudamos condições para que uma equivalêcia combinatória implique uma conjugação topológica. Neste trabalho, apresentamos a noção de sequências kneading para um contexto mais geral em relação a dinâmicas unidimensionais. Para este fim, consideramos uma família bidimensional definida por Benedicks e Carleson [BC91] como um “toy model” para aplicações de Hénon e definimos sequências kneading da linha crítica. Mostramos que tais sequências constituem um invariante completo para as classes de conjugação dos “toy models”. Além disso, mostramos um resultado análogo ao Teorema de Singer para nossa família bem como uma equivalência combinatória para sistemas dinâmicos não autônomos.

Palavras-chave: Toy models, sequências kneading, derivada Schwarziana negativa, sistemas dinâmicos não autônomos

Contents

CHAPTER 1

Introduction

One of the main questions in Dynamical Systems is whether two systems are 'the same', where by 'the same' we mean some type of equivalence between two systems.

In this sense, one of the most simple ways to say that two systems are equivalent is by obtaining an *orbit equivalence* map between them. Roughly speaking, we say that two topological dynamical systems are topologically orbit equivalent if there exists a homeomorphism between their phase spaces that preserves their structures and induces a one-to-one correspondence between their orbits. Formally, let X and Y be two topological spaces, and let $f : X \to X$ and $g : Y \to Y$ be functions. We say that f and g are orbit equivalent whenever there is a homeomorphism $h : X \to Y$ sending orbits to orbits, that is, $h(\mathscr{O}_f(x)) = \mathscr{O}_g(h(x))$ for every $x \in X$. When f and g are homeomorphisms, the definition means that there exist functions $\alpha, \beta : X \to \mathbb{Z}$ such that for all $x \in X$, $h \circ f(x) = g^{\alpha(x)} \circ h(x)$ and $h \circ f^{\beta(x)}(x) = g \circ h(x)$. So, given $f : X \to X$ a system we want to characterize the orbit equivalence class of f, that is, we want to determine the set $[f] := \{g : X \to X;\ g \text{ is orbit equivalent to } f\}$.

The notion of orbit equivalence was firstly studied in the context of probability measure preserving group actions, where the homeomorphism h is replaced by a measurable isomorphism. It follows from works of Dye [Dye59] and [Dye63], Ornstein and Weiss [OW80], and Connes, Feldman and Weiss

[CFW81] one of the most remarkable results: Any probability measure preserving action of an amenable group is orbit equivalent to a probability measure preserving action of $\mathbb{Z}$. This implies that in the measurable setting there is only one orbit equivalence class, at least when the action is made by an amenable group.

One notion stronger than orbit equivalence and which is the main problem addressed in the thesis is *topological conjugacy*. Let X and Y be topological spaces, and let $f : X \to X$ and $g : Y \to Y$ be continuous functions. We say that $f : X \to X$ and $g : Y \to Y$ are topologically conjugate (or *conjugate* for simple) if there exists a homeomorphism $h : X \to Y$ satisfying the conjugacy equation $h \circ f = g \circ h$. Note that if f is topologically conjugate to g, then f is orbit equivalent to g.

The space where the characterization of the topological conjugacy class was firstly constructed is the circle. In the late XIX century, Poincaré introduced the notion of *rotation number* for orientation preserving homeomorphisms of circle. He showed that if $f : \mathbb{S}^1 \to \mathbb{S}^1$ is an orientation preserving homeomorphism with irrational rotation number, then there exists a rigid rotation R with the same rotation number such that f is topologically semi-conjugate to R. Furthermore, if f contains a point x whose orbit is dense in $\mathbb{S}^1$, then f is topologically conjugate to R.

The next step is to consider an interval instead of the circle. However, if $f : I \to I$ is a homeomorphism, then the dynamic of f is trivial, because for all points $x \in I$ we get that either x is a periodic point or $f^n(x)$ converges to a periodic orbit. Therefore, for the interval the class of endomorphisms is more interesting to work with.

In this direction, Milnor and Thurston, in their famous paper [MT88], gave origin to the *Kneading Theory* for endomorphisms of the interval, which is an analogue to the Poincaré theory for homeomorphisms of the circle. More specifically, they considered maps $f : I \to I$, where I is an interval, with a finite number of turning points (a point of change of monotonicity) and defined the notion of *kneading sequences* (the itinerary of the turning points). They proved the following theorem:

Theorem 1.1. *Suppose that $f, g : I \to I$ are two l-modal maps with turning points $c_1^f < \cdots < c_l^f$ and $c_1^g < \cdots < c_l^g$. Assume that f and g have no wandering intervals, no intervals of periodic points and no attracting periodic points. If f and g have the same kneading sequences, then f and g are topologically conjugate.*

As our main result uses the same ideas, we will give the proof of Theorem 1.1 in the next section. The conjugacy is constructed by matching the orbits. We match "the inverse orbits" of the turning points. Besides this, the kneading theory plays an important role in one-dimensional dynamics, such as the continuity of the topological entropy and the monotonicity of the kneading sequence for the quadratic family. See de Melo and van Strien [dMvS93].

Over the last years, much research has been done attempting to construct a similar theory in dimension two. However, not much progress has been made. The great difficulty is the lack of critical points in the usual sense and the fact that the plane does not have a natural order like in dimension one.

In [PRH07], Pujals and Hertz with the goal to characterize the dynamical phenomenon that obstructs hyperbolicity, defined a notion of *critical point* for dissipative surface diffeomorphisms.

Based on the numerical results of [CGP88], Cvitanović introduced in [Cvi91] the concept of *pruning fronts* to homeomorphisms of the plane like Hénon family and Lozi family. The definition of pruning fronts is somewhat technical so we will not give it here. Cvitanović conjectured that every map $H_{a,b}$ in the Hénon family can be understood as a *pruned horseshoe*. That is, if $F : \mathbb{R}^2 \to \mathbb{R}^2$ is the Smale's horseshoe map, then after pruning (destroying) some orbits of F we have a map $\tilde{F}$ equivalent, in some sense, to $H_{a,b}$. This is known as the Pruning Front Conjecture. In this direction, Carvalho showed in [dC99] the following:

Theorem 1.2. *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a homeomorphism of the plane and $P \subset \mathbb{R}^2$ a pruning front of f. Then there exists an isotopy $H : \mathbb{R}^2 \times [0,1] \to \mathbb{R}^2$ with $\operatorname{supp}(H) \subset \bigcup_{k\in\mathbb{Z}} f^k(P)$, such that $H(\cdot,0) = f(\cdot)$ and $H(\cdot,1) = f_P(\cdot)$ is a homeomorphism under which every point of P is wandering.*

In other words, given a pruning front P of a homeomorphism f of the plane, up to isotopy, we can destroy all orbits of f which enter P (that is, transform these orbits into dynamically irrelevant orbits), while the dynamic outside P does not change. Although Carvalho did not prove the Pruning Front Conjecture, he constructed a family containing the Hénon and Lozi families of two-dimensional homeomorphisms going from trivial dynamics to a complex dynamics (that is, a horseshoe) like the full logistic family. Mendonza proved in [Men13] that the Pruning Front Conjecture holds in an open set of parameter space. More specifically,

Theorem 1.3. *There exists an open set A in the real parameter plane, such* *that if $(a, b) \in A$ then $H_{a,b}$ is topologically conjugate to a pruning homeomorphism of the horseshoe.*

In [Ish97], Ishii gave a solution of the Pruning Front Conjecture for the Lozi family. For more detail on pruning and its relationship with kneading theory, see [dCH02] and [dCH03].

With an approach different from the pruning techniques, Mendes and Sousa Ramos, in [MR04], developed a kneading theory for two-dimensional triangular maps.

Recently in [MŠ16], Misiurewicz and Štimac studied the Lozi family $L_{a,b}$ and their strange attractors $\Lambda_{a,b}$. They introduced a countable set of kneading sequences for the Lozi map and proved that all the dynamics in $\Lambda_{a,b}$ is characterized by this set.

In this work we will introduce the concept of *Kneading Sequences* for the two-dimensional family studied by Benedicks and Carleson, in [BC91], as a toy model for the Hénon maps. Namely, we will consider maps of the form $F(x, y) = (f(x, y), K(x, y))$ (*toy model*) acting on a two-dimensional rectangle, where f is a family of unimodal maps and K is an inverse branch of a Cantor map. In [MMP13], Matheus *et al* proved that Smale's Axiom A property is C^1-dense among the systems in this family and, on the C^2-topology, there exists an open subset where we have a type of Newhouse phenomenon. This indicates that this family may have other interesting properties. Now we can state the main result of this thesis.

Theorem A. Let F and G be two toy models. Assume that F and G have no wandering intervals, no interval of periodic points and no weakly attracting periodic points. If F and G have the same kneading sequences, then F and G are topologically conjugate.

Following this approach we obtain an analogue theorem for more general toy models. Besides that, the same ideas allow us to prove a combinatorial equivalence for nonautonomous discrete dynamical systems. We also prove a Singer's Theorem for toy models:

Theorem B. Let $F(x, y) = (f(y)(x), K^F_{sign(x)}(y))$ be a toy model. Suppose that $f(y) : [-1, 1] \to [-1, 1]$ has negative Schwarzian derivative for all $y \in [0, 1]$. Then the closure of the immediate basin of any strong attracting periodic orbit contains either a point of the critical line or a point of $\Lambda := \{-1, 1\} \times [0, 1]$.

This thesis is organized as follows. In the second chapter we recall the notions of kneading sequences for the one-dimensional case and prove Theorem 1.1 in a particular case. In the third chapter, we deal with the notion of kneading sequence for toy models and present our main results. In the fourth chapter, we prove our main results. In the fifth chapter, we present and proof Singer's Theorem for toy models. Finally, in the last chapter, we discuss some open problems related to the thesis.

CHAPTER 2

The one-dimensional case revisited

The purpose of this chapter is to revisit the notion of kneading sequence for unimodal maps. Besides that, we extend the results for discontinuous unimodal maps.

2.1 The continuous case

Let $I = [a, b]$ be a compact interval and $f : I \to I$ be a continuous map. We say that f is a unimodal map if $f(\partial I) \subset \partial I$ and there exists a unique point $c \in I \backslash \partial I$, the turning point, such that f is increasing to the left and decreasing to the right of c.

Consider the space $\Sigma_f = \{L, c, R\}^{\mathbb{N}}$, where $L = [a, c)$ and $R = (c, b]$. For each $x \in I$ define the itinerary $i_f(x) = (i_0(x), i_1(x), \ldots, i_n(x), \ldots) \in \Sigma_f$, where $i_n(x) = L$ if $f^n(x) \in L$, $i_n(x) = R$ if $f^n(x) \in R$ and $i_n(x) = c$ if $f^n(x) = c$. It is known that we can define an order structure $\preccurlyeq_f$ on Σ_f such that the map i_f is order preserving. The sequence

$$i_f(c) = (i_0(c), i_1(c), \ldots, i_n(c), \ldots)$$

is called the *kneading invariant* of f.

The importance of the sequence $i_f(c)$ is that it contains all information about the conjugacy class of f. Consider the set of pre-critical points

$$\mathscr{C}(f) = \{x \in [a, b];\ f^n(x) = c \text{ for some } n \geq 0\}.$$

The theorem below is well known and a particular case of Theorem 1.1, but, for the sake of completeness, we will proof it. This will give us intuition about how we will generalize it in the Theorem A. A similar proof of Theorem 2.1 can be found in [Ran78, Thm1].

Theorem 2.1. *Suppose that $f, g : I \to I$ are two unimodal maps with turning points c_f and c_g. Assume that $i_f(c_f) = i_g(c_g)$. Then there exists a strictly increasing bijection $h : \mathscr{C}(f) \to \mathscr{C}(g)$ such that $h \circ f = g \circ h$ on $\mathscr{C}(f)\backslash\{c_f\}$. Furthermore, if g has no wandering intervals, no intervals of periodic points and no attracting periodic points, then f and g are topologically semiconjugate.*

Proof. Let $\mathscr{C}_n(f) = \{x \in I; f^i(x) = c_f$ for some $0 \le i \le n-1\}$, $n \in \mathbb{N}$. Note that

$$\mathscr{C}(f) = \bigcup_{n\ge1} \mathscr{C}_n(f).$$

Since $\mathscr{C}_{n+1}(f) = \mathscr{C}_n(f) \cup f^{-n}(c_f)$, we have that $\mathscr{C}_n(f) \subset \mathscr{C}_{n+1}(f)$. Consider $\mathscr{P}_n(f) = \{I_i^f \subset I \ ; \ \partial I_i^f \subset \mathscr{C}_n(f) \cup \{a, b\}$ and $1 \le i \le k_n^f\}$, where the increasing order of the index of I_i^f is the same order as the intervals are placed in I and $k_n^f := \#\mathscr{P}_n(f)$. By definition, I_i^f is a maximal monotonicity closed interval of f^n for all $1 \le i \le k_n^f$.

Denote $f_{|_{[a,c_f]}}$ and $f_{|_{[c_f,b]}}$ by f_-, f_+ respectively. Furthermore, given a sequence $j = j_1j_2\ldots j_m \in \{-,+\}^m$ we define $f_j := f_{j_m} \circ \cdots \circ f_{j_2} \circ f_{j_1}$. Take $x \in \mathscr{C}_n(f)\backslash\{c_f\}$. Consider $j(x) = j_1j_2\ldots j_k \in \{-,+\}^k$, with $1 \le k \le n-1$, the minimal sequence such that $f_{j(x)}(x) = c_f$. We will denote x by $x_{j(x)}$.

Note that for each $I_i^f \in \mathscr{P}_n(f)$ there exists a unique sequence $j^i = j_1^ij_2^i\ldots j_n^i \in \{-,+\}^n$ such that f_{j^i} is strictly monotone on I_i^f. Thus we get

$$\partial I_i^f \in \{a, b, c_f, x_{j_1^ij_2^i\cdots j_k^i}\},$$

for some $1 \le k \le n-1$. For each $n \in \mathbb{N}$ we set

$$A_n(f) = \{j_1j_2\ldots j_k \in \{-,+\}^k; \exists\, x \in \mathscr{C}_n(f)\backslash\{c_f\} \text{ such that } x = x_{j_1j_2\cdots j_k} \text{ for some } 1 \le k \le n-1\}.$$

In the same way we can define $\mathscr{C}_n(g)$, $\mathscr{P}_n(g)$ and $A_n(g)$.

Suppose that $f(c_f) \le c_f$, then $\mathscr{C}(f) = \mathscr{C}_n(f) = \{c_f\}$ for all $n \in \mathbb{N}$. Since $i_f(c_f) = i_g(c_g)$ the same happen with $\mathscr{C}(g)$ and the proof is complete. Note that this case does not happen if g has no wandering intervals, no intervals of periodic points and no periodic attractors.

From now on we make the assumption that $f(c_f) > c_f$. Since $i_f(c_f) = i_g(c_g)$ we have that $g(c_g) > c_g$. So $\mathscr{C}_1(f) = \{c_f\}$, $\mathscr{C}_1(g) = \{c_g\}$, $\mathscr{C}_2(f) = \{x^f_-, c_f, x^f_+\}$ and $\mathscr{C}_2(g) = \{x^g_-, c_g, x^g_+\}$, where $f_j(x^f_j) = c_f$ and $g_j(x^g_j) = c_g$, with $j = -, +$. By construction, $x^f_- < c_f < x^f_+$ and $x^g_- < c_g < x^g_+$. See Figure 2.1. This implies that there exist $h_i : \mathscr{C}_i(f) \to \mathscr{C}_i(g)$, $i = 1, 2$, such that h_i is a strictly increasing bijection and $h_{2|_{\mathscr{C}_1(f)}} = h_1$. Namely $h_1(c_f) := c_g$, $h_2(c_f) := c_g$ and $h_2(x^f_j) := x^g_j$, with $j = -, +$.

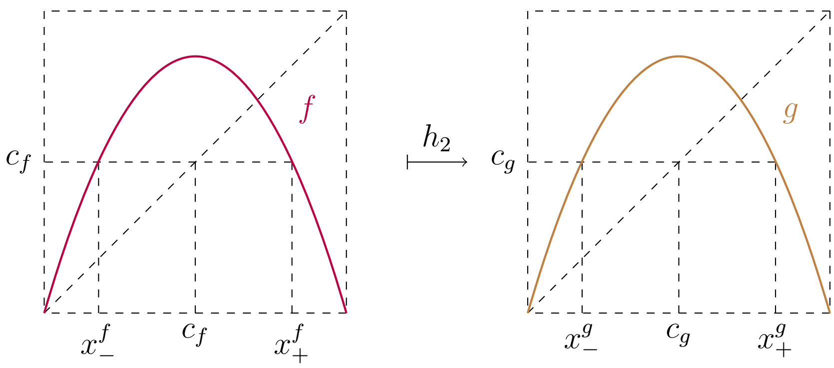

Figure 2.1: Construction of x^f_- and x^f_+. Similarly we can construct x^g_- and x^g_+.

We now proceed by induction on n. Let us suppose that there exists $h_n : \mathscr{C}_n(f) \to \mathscr{C}_n(g)$ strictly increasing bijection such that $h_n \circ f = g \circ h_n$ on $\mathscr{C}_n(f)\backslash\{c_f\}$ and $h_{n|_{\mathscr{C}_{n-1}(f)}} = h_{n-1}$. This implies that $\#\mathscr{P}_n(f) = \#\mathscr{P}_n(g)$ and $A_n(f) = A_n(g)$.

Let $I^f_i \in \mathscr{P}_n(f)$ and $j_1 j_2 \dots j_n \in \{-, +\}^n$ such that

$$I^f_i = [x^f_{j_1 j_2 \dots j_k}, x^f_{j_1 j_2 \dots j_l}],$$

with $1 \le k \ne l \le n-1$. By the manner that we order the intervals on $\mathscr{P}_n(f)$ and $\mathscr{P}_n(g)$ we have

$$h_n(x^f_{j_1 j_2 \dots j_k}) = x^g_{j_1 j_2 \dots j_k} \text{ and } h_n(x^f_{j_1 j_2 \dots j_l}) = x^g_{j_1 j_2 \dots j_l},$$

where

$$I^g_i = [x^g_{j_1 j_2 \dots j_k}, x^g_{j_1 j_2 \dots j_l}].$$

Furthermore,

$$f_{j_n} \circ \dots \circ f_{j_1}(I^f_i) = [f_{j_n} \circ \dots \circ f_{j_{k+1}}(c_f), f_{j_n} \circ \dots \circ f_{j_{l+1}}(c_f)] = [f^{n-k}(c_f), f^{n-l}(c_f)]$$

and

$$g_{j_n}\circ\cdots\circ g_{j_1}(I_i^g) = [g_{j_n}\circ\cdots\circ g_{j_{k+1}}(c_g), g_{j_n}\circ\cdots\circ g_{j_{l+1}}(c_g)] = [g^{n-k}(c_g), g^{n-l}(c_g)].$$

Since $i_f(c_f) = i_g(c_g)$, we get

$$[f^{n-k}(c_f), f^{n-l}(c_f)] \cap \{c_f\} \neq \emptyset \text{ if and only if } [g^{n-k}(c_g), g^{n-l}(c_g)] \cap \{c_g\} \neq \emptyset.$$

If $[f^{n-k}(c_f), f^{n-l}(c_f)] \cap \{c_f\} = \emptyset$, then $I_i^f \in \mathscr{P}_{n+1}(f)$ and, consequently, $I_i^g \in \mathscr{P}_{n+1}(g)$ too. If $\{f^{n-k}(c_f), f^{n-l}(c_f)\} \cap \{c_f\} \neq \emptyset$, then either $c_f = f^{n-k}(c_f)$ or $c_f = f^{n-l}(c_f)$. Thus $I_i^f \in \mathscr{P}_{n+1}(f)$. Hence, we also have $I_i^g \in \mathscr{P}_{n+1}(g)$. On the other hand, if $(f^{n-k}(c_f), f^{n-l}(c_f)) \cap \{c_f\} \neq \emptyset$ then there exists a unique

$$x^f_{j_1j_2\cdots j_n} := f_{j_1}^{-1}\circ f_{j_2}^{-1}\circ\cdots\circ f_{j_n}^{-1}(c_f) \in \operatorname{int}(I_i^f)\cap f^{-n}(c_f).$$

Hence, there also exists a unique

$$x^g_{j_1j_2\cdots j_n} := g_{j_1}^{-1}\circ g_{j_2}^{-1}\circ\cdots\circ g_{j_n}^{-1}(c_g) \in \operatorname{int}(I_i^g)\cap g^{-n}(c_g).$$

So we can define $h_{n+1}(x^f_{j_1j_2\cdots j_n}) := x^g_{j_1j_2\cdots j_n}$, where $x^f_{j_1j_2\cdots j_n} \in \mathscr{C}_{n+1}(f)\backslash\mathscr{C}_n(f)$ and $x^g_{j_1j_2\cdots j_n} \in \mathscr{C}_{n+1}(g)\backslash\mathscr{C}_n(g)$. Moreover, if $x \in \mathscr{C}_n(f)$, then we put $h_{n+1}(x) = h_n(x)$.

The cases where $I_i^f = [a, x_{j_1j_2\cdots j_k}]$, $I_i^f = [x_{j_1j_2\cdots j_k}, b]$, or $I_i^f = [x_{j_1j_2\cdots j_k}, c_f]$ are similar. Thus we have that $h_{n+1} : \mathscr{C}_{n+1}(f) \to \mathscr{C}_{n+1}(g)$ is a strictly increasing bijection such that $h_{n+1}\circ f = h_{n+1}\circ g$ on $\mathscr{C}_{n+1}(f)\backslash\{c_f\}$ and $h_{n+1|_{\mathscr{C}_n(f)}} = h_n$.

Therefore, we define

$$\begin{array}{rccl} h : & \mathscr{C}(f) & \longrightarrow & \mathscr{C}(g) \\ & y & \longmapsto & h(y) = h_n(y), \end{array}$$

where n is such that $y \in \mathscr{C}_n(f)$. Note that $h\circ f = g\circ h$ on $\mathscr{C}(f)\backslash\{c_f\}$

Now, we will assume that g has no wandering intervals, no intervals of periodic points and no attracting periodic points. We claim that we can extend continuously h to $\overline{\mathscr{C}(f)}$. In fact, take $y \in \overline{\mathscr{C}(f)}\backslash\mathscr{C}(f)$. Suppose that there exist $y_n^j \in \mathscr{C}(f)$, $j = 1, 2$, such that $y_n^1 \uparrow y$ and $y_n^2 \downarrow y$. Without loss of generality, we can assume that $y_n^j \in \mathscr{C}_n(f)$ for all $n \geq 1$, with $j = 1, 2$. From the definition of the $h_n's$, it follows that $h_n(y_n^1) < h_n(y_n^2)$, $h_n(y_n^1) < h_{n+1}(y_{n+1}^1)$ and $h_n(y_n^2) > h_{n+1}(y_{n+1}^2)$ for all $n \geq 1$. Since h_n is strictly increasing there are unique

$$h^1(y) := \lim_n h_n(y_n^1) \quad\text{and}\quad h^2(y) := \lim_n h_n(y_n^2).$$

Note that $h^1(y) \leq h^2(y)$. Besides that, $h^j(y)$ do not depend of the sequence y_n^j which converges to y. Now, it is enough to show that $h^1(y) = h^2(y)$. Assume that $h^1(y) < h^2(y)$. Let $J := [h^1(y), h^2(y)]$. Observe that $J \cap \mathscr{C}(g) = \emptyset$. Indeed, suppose that there exists $z^g \in J \cap \mathscr{C}(g)$. Consequently

$$h_n(y_n^1) < z^g < h_n(y_n^2) \text{ for all } n \geq 1.$$

Let $n_0 \geq 1$ and $z^f \in \mathscr{C}(f)$ such that $h_n(z^f) = z^g$ for all $n \geq n_0$. Hence $y_n^1 < z^f < y_n^2$ for all $n \geq n_0$. Thus we get $z^f = y$, which gives an absurd.

Since g has no wandering intervals, there exist $1 \leq l_1 < l_2$ such that $g^{l_1}(J) \cap g^{l_2}(J) \neq \emptyset$. As $J \cap \mathscr{C}(g) = \emptyset$, we may suppose, without loss of generality, that $l_1 = 0$, $l_2 = l$ and $l \geq 1$ is the smallest integer with such property. Put

$$L = \bigcup_{m \geq 0} g^{ml}(J).$$

It follows that L is a non-empty interval that contains no $\{c_g\}$ and $g^l(L) \subset L$. So g^l is strictly monotone in L. Thus either L contains an interval of periodic points for g^l, or some open interval in L converges to a single periodic point, which is a contradiction with the hypotheses on g. Therefore $h^1(y) = h^2(y)$. The cases where we have only $y_n \uparrow y$ or $y_n \downarrow y$, with $y_n \in \mathscr{C}(f)$ are similar. Hence, if $y \in \overline{\mathscr{C}(f)} \backslash \mathscr{C}(f)$ and $y_n \to y$, with $y_n \in \mathscr{C}(f)$, then

$$h(y) := \lim_n h_n(y_n)$$

defines a continuous extension of h to $\overline{\mathscr{C}(f)}$.

By a similar argument, if g has no wandering intervals, no intervals of periodic points and no attracting periodic points, then we also have that $\overline{\mathscr{C}(g)} = I$. Whence $h : \overline{\mathscr{C}(f)} \to I$ is a strictly increasing surjective map. From this, if $W = (\alpha, \beta)$ is a connected component of $I \backslash \overline{\mathscr{C}(f)}$, then we can extend h as $h(z) := h(\alpha)$ for all $z \in W$, once $h(\alpha) = h(\beta)$.

Claim: $h \circ f = g \circ h$.

It is clear that $h \circ f = g \circ h$ on $\overline{\mathscr{C}(f)}$. Note that for all $W \subset I \backslash \overline{\mathscr{C}(f)}$ and $n \geq 0$ there is $W_n' \subset I \backslash \overline{\mathscr{C}(f)}$ such that $f^n(W) \subset W_n'$. As f is injective on $f^{n-1}(W)$, we get $f^n(W) = W_n'$. Take now $z \in (\alpha, \beta) \subset I \backslash \overline{\mathscr{C}(f)}$. Thus $h(f(z)) = h(f(\alpha)) = g(h(\alpha)) = g(h(z))$. This proves the claim.

Therefore, h is a semiconjugacy on I between f and g. ■

The following corollary is immediate.

Corollary 2.1. *Suppose that $f, g : I \to I$ are two unimodal maps with turning points c_f and c_g. Assume that f and g have no wandering intervals, no intervals of periodic points and no attracting periodic points. If $i_f(c_f) = i_g(c_g)$, then f and g are topologically conjugate.*

2.2 The discontinuous case

In this section we prove a theorem analogue to Theorem 2.1 for discontinuous unimodal maps. In particular, we extend these maps to a continuous map. This argument will be useful to our main results.

Let $I = [a, b]$ be a compact interval and $c \in (a, b)$. Let $f : I \backslash \{c\} \to I$ be a map such that $f_{|_{[a,c)}}$ is strictly increasing, $f_{|_{(c,b]}}$ is strictly decreasing and $f(\{a, b\}) = a$. We call a map like this *discontinuous unimodal map*. See Figure 2.2.

We extend f to c by adding two points c^- and c^+ and taking $f(c^-) := \lim\limits_{x \nearrow c} f(x)$ and $f(c^+) := \lim\limits_{x \searrow c} f(x)$.

Let f_- and f_+ be two maps defined by

$$\begin{array}{rccl} f_- : & [a, c^-] & \longrightarrow & [-1, c^-] \cup (c^+, b] \\ & x & \longmapsto & f(x) \end{array}$$

and

$$\begin{array}{rccl} f_+ : & [c^+, b] & \longrightarrow & [-1, c^-) \cup [c^+, b] \\ & x & \longmapsto & f(x). \end{array}$$

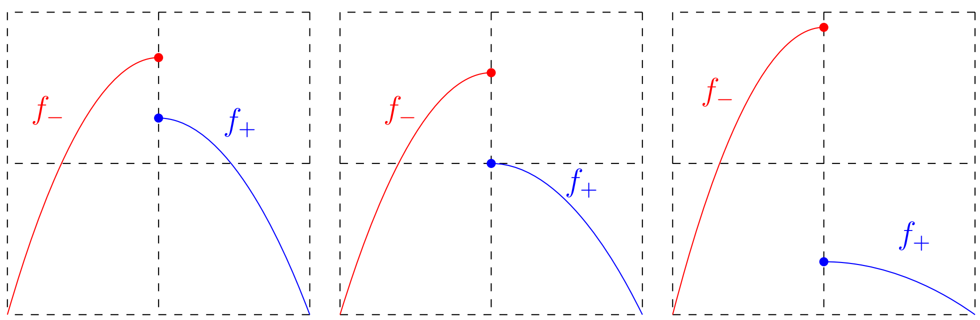

Figure 2.2: Some examples of discontinuous unimodal maps: (a), (b) and (c).

From now on f will be the map with $c^{\pm}$ glued into its domain. We say that f is continuous if the functions f_- and f_+ are continuous. For this, we need define a topology into $[a, c^-] \cup [c^+, b]$.

2.2.1 One-side topology

First, let us define a usual neighborhood of a point $x \in [a, c^-] \cup [c^+, b]$. If $x \in [a, c^-) \cup (c^+, b]$, then a neighborhood of x is the standard Euclidean neighborhood. On the other hand, if $x = c^{\pm}$ then a neighborhood of x is obtained from the intersection of $[a, c]$ or $[c, b]$ with a standard Euclidean neighborhood of x. This we call *Euclidean Topology* on $[a, c^-] \cup [c^+, b]$.

Now we will refine that neighborhood basis in each point. Set

$$\mathscr{C}(f) = \{x \in [a, c^-] \cup [c^+, b]; f^l(x) \in \{c^-, c^+\} \text{ for some } l \geq 0\}.$$

For $x \notin \mathscr{C}(f)$ we do not add any open neighborhood. Take now $x \in \mathscr{C}(f) \backslash \{c^-, c^+\}$ and $l > 0$ such that $f^l(x) \in \{c^-, c^+\}$. Suppose that $f^l(x) = c^-$ (The other case is similar). Let $\varepsilon > 0$. Then there is $\delta > 0$ such that either $(x - \delta, x]$ satisfies $f^l((x - \delta, x]) \subset (c^- - \varepsilon, c^-]$ or $[x, x + \delta)$ satisfies $f^l([x, x+\delta)) \subset (c^- - \varepsilon, c^-]$. On the first case, we add the set $\{(x-\delta, x]; \delta > 0\}$ to the neighborhood base of x and on the second case we add the set $\{[x, x + \delta); \delta > 0\}$.

Let $\mathcal{B}_f$ be the family of sets constructed above. Note that $\mathcal{B}_f$ is a basis for a topology on $[a, c^-] \cup [c^+, b]$. We call *One-side topology* the topology $\mathfrak{T}_f$ generated by $\mathcal{B}_f$ and the euclidean topology. So f is continuous with this topology.

Remark 1. In some cases, we get ask that a point should be itself an open set. That happen, for example, in the case (b) on Figure 2.2 where the point c^+ is an open set on the one-side topology.

2.2.2 Kneading sequence for discontinuous unimodal maps

Let f be a (dis)continuous unimodal map under the topological space $([a, c^-] \cup [c^+, b], \mathfrak{T}_f)$.

Consider the space $\Sigma_f = \{L, c^-, c^+, R\}^{\mathbb{N}}$, where $L = [a, c^-)$ and $R = (c^+, b]$. So for each $x \in I$,its itinerary is defined by

$$i_f(x) = (i_0(x), i_1(x), \dots, i_n(x), \dots) \in \Sigma_f,$$

where $i_n(x) = L$ if $f^n(x) \in L$, $i_n(x) = R$ if $f^n(x) \in R$ and $i_n(x) = c^\pm$ if $f^n(x) = c^\pm$. Now, the *kneading invariants* of f are the sequences

$$i_f(c^\pm) = (i_0(c^\pm), i_1(c^\pm), \ldots, i_n(c^\pm), \ldots).$$

The proof of the theorem below is similar to Theorem 2.1. So, it will be omitted.

Theorem 1. *Let f and g be two discontinuous unimodal maps. Assume that g has no wandering intervals, no intervals of periodic points and no attracting periodic points. If $i_f(c_f^\pm) = i_g(c_g^\pm)$, then f and g are topologically semiconjugate.*

CHAPTER 3

Toy models and kneading sequences

Once the conjugacy class of one-dimensional maps is well understood, it is natural to look for approaches analogous for higher dimensional maps. In this chapter we define the class of maps with which we shall to work and its kneading sequences.

Following [MMP13], we study a two-dimensional class of maps defined as follows. Consider a one parameter family

$$f(y) : [-1, 1] \to [-1, 1], \text{ with } y \in [0, 1],$$

depending continuously on y, such that $f(y)$ is a unimodal map verifying that 0 is the turning point and $f(y)(-1) = f(y)(1) = -1$ for all $y \in [0, 1]$.

Let $k : [0, a] \cup [b, 1] \to [0, 1]$ be a differentiable function such that $k(0) = k(1) = 0$, $k(a) = k(b) = 1$ and $|k'| > \gamma > 1$. We put

$$K(x, y) = \begin{cases} K_+(y), & \text{if} \quad x > 0 \\ K_-(y), & \text{if} \quad x < 0, \end{cases}$$

where $K_+ = (k|_{[0,a]})^{-1}$ and $K_- = (k|_{[b,1]})^{-1}$.

We will study the map F, called *Toy Model*, defined by

$$\begin{array}{rccl} F : & ([-1, 1]\backslash\{0\}) \times [0, 1] & \longrightarrow & [-1, 1] \times [0, 1] \\ & (x, y) & \longmapsto & (f(y)(x), K_{sign(x)}(y)). \end{array}$$

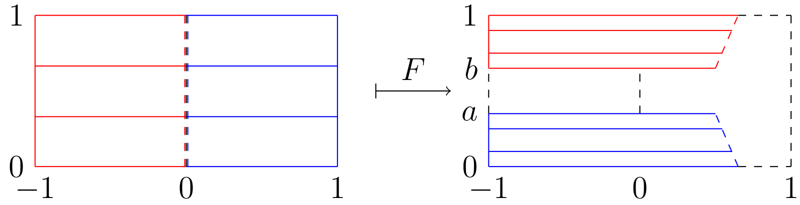

Figure 3.1: Dynamics of F.

Given a toy model F and its Cantor map $k : [0, a] \cup [b, 1] \to [0, 1]$ we can consider the Cantor set, denoted by $\mathscr{K}^F$, induced by k, that is,

$$\mathscr{K}^F = \bigcap_{n \geq 0} k^{-n}([0, a] \cup [b, 1]).$$

In order to make the domain of F compact we introduce the points $(0^{\pm}, y)$ and extend F to them via the formula $F(0^{\pm}, y) = (f(y)(0), K_{\pm}(y))$. These points will be called the *turning points* of the Toy Model F and the set

$$\mathscr{L}_c(F) = \{(0^{\pm}, y); y \in [0, 1]\}$$

is called the *critical line* of F.

Let us define the topology on

$$\mathrm{Dom}(F) := (([-1, 1]\backslash\{0\}) \times [0, 1]) \cup \mathscr{L}_c(F).$$

If $(x, y) \in \mathrm{Dom}(F)\backslash\mathscr{L}_c(F)$, then a neighborhood of (x, y) is the standard Euclidean neighborhood. On the other hand, if $(x, y) \in \mathscr{L}_c(F)$ then a neighborhood of (x, y) is obtained from the intersection of $[-1, 0] \times [0, 1]$ or $[0, 1] \times [0, 1]$ with a standard Euclidean neighborhood of (x, y).

Remark 2. Notice that F is now defined on a compact set, but we have to pay a price. The map F is not continuous.

3.1 Notations and preliminaries

In this section we establish some notations and define some notions that will be used throughout the Thesis.

For each $y \in [0, 1]$ we will use the notation $f_{-}(y)$ and $f_{+}(y)$ for $f(y)|_{[-1,0^-]}$, $f(y)|_{[0^+,1]}$ respectively. See Figure 3.2.

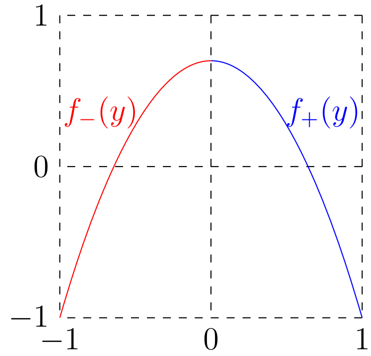

Figure 3.2: Dynamics of $f(y)$.

Consider the maps F_- and F_+ defined as follow

$$\begin{array}{rcl} F_- \;:\; [-1,0^-]\times[0,1] & \longrightarrow & ([-1,0^-]\cup(0^+,1])\times[0,1] \\ (x,y) & \longmapsto & (f_-(y)(x),K_-(y)). \end{array}$$

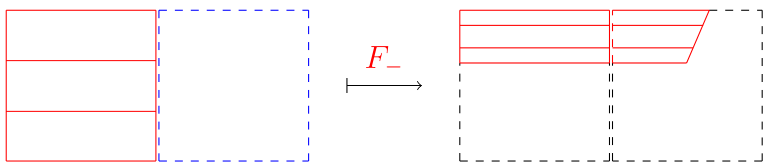

Figure 3.3: Dynamics of F_-.

and

$$\begin{array}{rcl} F_+ \;:\; [0^+,1]\times[0,1] & \longrightarrow & ([-1,0^-)\cup[0^+,1])\times[0,1] \\ (x,y) & \longmapsto & (f_+(y)(x),K_+(y)). \end{array}$$

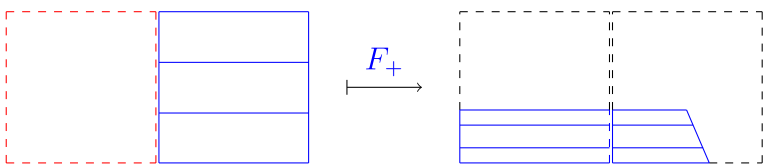

Figure 3.4: Dynamics of F_+.

This allows us to see the orbit of a point $(x,y)\in\mathrm{Dom}(F)$ by compositions of the functions $F_\pm$. More specifically, for each $(x,y)\in\mathrm{Dom}(F)$ there is a sequence $j(x,y)=(j_1j_2\cdots j_m\cdots)\in\{-,+\}^{\mathbb{N}}$ such that

$$F^{m+1}(x,y)=F_{j_{m+1}}(x_m,y_m)$$

for all $m \geq 0$, where $x_m = f_{j_m}(y_{j_{m-1}\cdots j_1}) \circ \cdots \circ f_{j_2}(y_{j_1}) \circ f_{j_1}(y)(x)$ and $y_m = K_{j_m} \circ \cdots \circ K_{j_1}(y) := y_{j_m \cdots j_1}$.

We denote the set $([-1, 0^-] \cup [0^+, 1]) \times \{y\}$ by $I(y)$. In addition, a subset $J \subset I(y)$ is called an *interval* if there are $x_1, x_2 \in [-1, 0^-]$(or $[0^+, 1]$) such that

$$J = \{(x, y) \in [-1, 0^-] \cup [0^+, 1] \times \{y\} \ ; \ x_1 < x < x_2\}.$$

Sometimes we will denote an interval J by $[(x_1, x_2), y]$. If

$$J = \{(x, y) \in [-1, 0^-] \cup [0^+, 1] \times \{y\} \ ; \ x_1 \leq x \leq x_2\},$$

then we denote it by $[[x_1, x_2], y]$.

Let (p, q) be a periodic point of F and set

$$\mathscr{B}(p, q) = \{(x, y); \ F^l(x, y) \to \mathscr{O}(p, q) \text{ as } l \to \infty\}.$$

We say that (p, q) is an *weakly attracting periodic point* if $\mathscr{B}(p, q)$ contains an interval. If $\mathscr{B}(p, q)$ contains an open subset, then we say that (p, q) is a *strong attracting periodic point.* When we have $\mathscr{O}_F(p, q) \cap \mathscr{L}_c(F) = \emptyset$, then the immediate basin $\mathscr{B}_0(p, q)$ of $\mathscr{O}(p, q)$ is the union of the connected components of $\mathscr{B}(p, q)$ which contain points from $\{(p, q), F(p, q), \ldots, F^{n-1}(p, q)\}$ so that $\mathscr{B}_0(p, q) \cap \mathscr{L}_c(F) = \emptyset$, where n is the period of (p, q). In this case, there is a sequence $j_1 \cdots j_n \in \{-, +\}^n$ such that for each $(x, y) \in \mathscr{B}_0(p, q)$ we get

$$\left(f_{j_m}(y_{j_{m-1}\cdots j_1}) \circ \cdots \circ f_{j_2}(y_{j_1}) \circ f_{j_1}(y)(x), y_{j_m \cdots j_1}\right) \in \mathscr{B}_0(p, q),$$

for all $0 \leq m \leq n - 1$.

Furthermore, if $B(p, q) \subset \mathscr{B}_0(p, q)$ is the component containing (p, q) then

$$\left(f_{j_n}(y_{j_{n-1}\cdots j_1}) \circ \cdots \circ f_{j_2}(y_{j_1}) \circ f_{j_1}(y)(x), y_{j_n \cdots j_1}\right) \in U(p, q),$$

for all $(x, y) \in B(p, q)$. Note that if (p, q) is a strong attracting periodic point, then (p, q) is a weakly attracting periodic point.

Given $y \in [0, 1]$, an interval $J \subset I(y)$ is called *interval of periodic points* if all $(x, y) \in J$ are periodic points for F.

Definition 1. Let F be a toy model and let $J \in \mathrm{Dom}(F)$ be an interval. We say that J is **wandering** if

1. $F^n(J) \cap F^m(J) = \emptyset$ for all $n \neq m$;

2. For all $n \geq 0$ we get $F^n(J) \cap \mathscr{L}_c(F) = \emptyset$.

It is well known that if $f : I \to I$ is an unimodal map and f has no intervals of periodic points and no attracting periodic points, then the set $\mathscr{C}(f)$ is dense on the non-wandering set of f. The same holds for toy models. For this, let $\Omega(F)$ be the non-wandering set of F and set

$$\mathscr{C}(F) = \left\{(x, y) \in \mathrm{Dom}(F);\ \exists\, l \geq 0 \text{ such that } F^l(x, y) \in \mathscr{L}_c(F)\right\}.$$

We get

Proposition 1. *Let F be a toy model and let $(x, y) \in \Omega(F)$. Suppose that F has no weakly attracting periodic point and no interval of periodic points. Then $(x, y) \in \overline{\mathscr{C}(F)}$.*

Proof. Take $(x, y) \in \Omega(F)$ and suppose that $(x, y) \notin \overline{\mathscr{C}(F)}$. Hence, there exists V neighborhood of (x, y) such that $F^n(V) \cap \mathscr{L}_c(F) = \emptyset$ for all $n \geq 0$. We will assume that V is of the form $I \times J$, where $x \in I$ and $y \in J$ and I, J are open intervals. Since $(x, y) \in \Omega(F)$, there exists $m \geq 1$ such that $F^m(V) \cap V \neq \emptyset$. Put

$$\widehat{L} = \bigcup_{n \geq 0} F^n(V).$$

From choice of m, it follows that $\bigcup_{j \geq 0} F^{jm+r}(V)$ are connected sets for all $r = 0, \ldots, m-1$. This forces that $\widehat{L}$ has finitely many connected components. Set $\widehat{L} = L_1 \sqcup \cdots \sqcup L_s$, where $L_1, \ldots, L_s$ are connected. Since $F(\widehat{L}) \subset \widehat{L}$, we get for all i there is j such that $F(L_i) \subset L_j$. Let L be the connected component which contains (x, y). Thus we have $F^m(L) \cap L \neq \emptyset$ and consequently $F^m(L) \subset L$. Consider $l = \min\{d \geq 1;\ F^d(L) \subset L\}$. This implies that L is a domain and $F^l(L) \subset L$. In addition, $F^i(L) \cap \mathscr{L}_c(F) = \emptyset$ and $F^i(L) \cap F^j(L) = \emptyset$ for all $i, j \in \{0, \ldots l-1\}$ with $i \neq j$. Thus, there exists a sequence $j_0 j_1 \cdots j_{l-1} \in \{-, +\}^l$ such that, for any $(a, b) \in L$ and $n \geq 0$ we have $F^{nl}(a, b) = (a_{nl}, b_{(j_{l-1} \cdots j_1 j_0)^n})$, where

$$b_{(j_{l-1} \cdots j_1 j_0)^n} = \left(K_{j_{l-1}} \circ \cdots \circ K_{j_0}\right)^n (b)$$

and

$$a_{nl} = f_{j_{l-1}}(b_{j_{l-2} \cdots j_0 (j_{l-1} \cdots j_0)^{n-1}}) \circ \cdots \circ f_{j_{l-1}}(b_{j_{l-2} \cdots j_0}) \circ \cdots \circ f_{j_1}(b_{j_0}) \circ f_{j_0}(b)(a).$$

Let π_2 be the projection on the second coordinate. Since $K_{j_{l-1}} \circ \cdots \circ K_{j_0}$ is a contraction, there exists $w \in \pi_2(\overline{L})$ such that $K_{j_{l-1}} \circ \cdots \circ K_{j_0}(w) = w$. We claim that $w = y$. In fact, for each $n \geq 1$ take $V_n := I_n \times J_n \subsetneqq I \times J$

neighborhood of (x, y) such that $I_{n+1} \times J_{n+1} \subsetneq I_n \times J_n$ with $|I_n| \to 0$ and $|J_n| \to 0$ as $n \to \infty$. As before, for each $n \geq 1$ there are $(x_n, y_n) \in V_n$ and $k_n \geq 1$ such that $F^{k_n}(x_n, y_n) \in V_n$ and $F^{k_n}(x_n, y_n) \to (x, y)$ as $n \to \infty$. We may suppose that $k_n \to \infty$ as $n \to \infty$. Furthermore, by construction of L, we have $k_n = r_n l$ for some $r_n \geq 1$. As $K := K_{j_{l-1}} \circ \cdots \circ K_{j_0}$ is a contraction, there exists $\lambda \leq 1$ such that

$$|K(x) - K(y)| \leq \lambda |x - y|,$$

for all $x, y \in [0, 1]$. Note that

$$\begin{aligned} |K^{r_n}(y_n) - w| &\leq |K^{r_n}(y_n) - K^{r_n}(y)| + |K^{r_n}(y) - w| \\ &\leq \lambda^{r_n}|y_n - y| + |K^{r_n}(y) - w| \to 0. \end{aligned}$$

Therefore

$$\pi_2(F^{k_n}(x_n, y_n)) = K^{r_n}(y_n) \to w.$$

As $F^{k_n}(x_n, y_n) \to (x, y)$, we have that $w = y$. This proves the claim.

For every $(v, y) \in L \cap I(y)$ and $n \geq 1$ we get

$$F^{ml}(v, y) = \left(\left(f_{(j_{l-1}\cdots j_0)}(y)\right)^m(v), y\right) \in L \cap I(y),$$

where $f_{(j_{l-1}\cdots j_0)}(y) := f_{j_{l-1}}(y_{j_{l-2}\cdots j_0}) \circ \cdots \circ f_{j_0}(y)$ is a strictly monotone map. Without loss of generality, assume that $f_{(j_{l-1}\cdots j_0)}(y)$ is strictly increasing. Let $\{I_i\}_i$ be the connected components of $L \cap I(y)$.

Claim: There exists i_0 such that $F^l(I_{i_0}) \subset I_{i_0}$.

Assume that the claim does not hold. Since $f_{(j_{l-1}\cdots j_0)}(y)$ is strictly increasing, for every i there exists $\sigma(i)$ such that $F^l(I_i) \subset I_{\sigma(i)}$ and $\sup I_i < \inf I_{\sigma(i)}$. Fixed any k, this implies that the sequence $\{F^{ln}(I_k)\}_{n\geq 0}$ is formed by disjoint intervals satisfying $\sup F^{ln}(I_k) < \inf F^{l(n+1)}(I_k)$ and $|F^{ln}(I_k)| \to 0$ as n goes to infinity. Thus, there is (v_0, y) such that $F^l(v_0, y) = (v_0, y)$ and $F^{ln}(I_k) \to (v_0, y)$, contradicting the nonexistence of weakly attracting periodic point. This proves the claim.

Let i_0 given by claim. So, $F^l : I_{i_0} \to I_{i_0}$ is a strictly increasing interval map. Therefore, either I_{i_0} contains an interval of periodic points for F^l , or some open interval in I_{i_0} converges to a single periodic point, which is again a contradiction with the hypotheses on F. The proof of the proposition is finished.

■

3.2 Kneading sequences and main results

In this section we define sequences of symbols and show that these sequences completely determine the combinatorial type of the toy models.

Let π_1 be the projection on first coordinate and let F be a toy model. Consider the alphabet $\mathcal{A} = \{L, 0^-, 0^+, R\}$ the address $t_F(x,y)$ of a point $(x,y) \in \mathrm{Dom}(F)$ is defined by

$$t_F(x,y) = \begin{cases} L & \text{if} \quad \pi_1(x,y) < 0 \\ 0^- & \text{if} \quad \pi_1(x,y) = 0^- \\ 0^+ & \text{if} \quad \pi_1(x,y) = 0^+ \\ R & \text{if} \quad \pi_1(x,y) > 0 \end{cases}$$

Applying this map to an orbit of a given point $(x,y) \in \mathrm{Dom}(F)$, we associate to that orbit one sequence of symbols.

Definition 2. Consider the sequence of symbols in $\mathcal{A}$

$$T_F(x,y) = (t_F(x,y), t_F(F(x,y)), \ldots, t_F(F^n(x,y)), \ldots).$$

This infinite sequence is called itinerary of $(x,y) \in \mathrm{Dom}(F)$.

This definition allows us to define the map

$$\begin{array}{rccl} T_F & : & \mathrm{Dom}(F) & \longrightarrow & \Sigma \\ & & (x,y) & \longmapsto & T_F(x,y), \end{array}$$

where $\Sigma := \{L, 0^-, 0^+, R\}^{\mathbb{N}}$. Let $\Phi : \Sigma \to \Sigma$ be the one-side shift, we get $T_F \circ F = \Phi \circ T_F$.

We call *Kneading Sequences* of F, denoted by $\mathcal{K}_F(\mathscr{L}_c)$, the set of all sequences associated to elements of $\mathscr{L}_c(F)$. So, it is reasonable to ask whether the Theorem 2.1 still holds for Toy Models family. Indeed, the Theorem A answers positively this question.

The following proposition is clear, but we will present a proof for completeness.

Proposition 2. *Let $F(x,y) = (f(y)(x), K^F_{sign(x)}(y))$ and $G(x,y) = (g(y)(x), K^G_{sign(x)}(y))$ be two toy models. There exists a strictly increasing continuous map $\psi : [0,1] \to [0,1]$ such that*

$$\psi \circ K^F_{\pm} = K^G_{\pm} \circ \psi.$$

Moreover, $\psi|_{\mathscr{K}^F}$ is independent of choice of ψ.

Proof. Let $k^F : [0, a^F] \cup [b^F, 1] \to [0, 1]$ and $k^G : [0, a^G] \cup [b^G, 1] \to [0, 1]$ the Cantor maps of F and G. If $\mathscr{K}^F$ and $\mathscr{K}^G$ are the Cantor sets generated by k^F, k^G respectively, then there is $\psi : \mathscr{K}^F \to \mathscr{K}^G$ strictly increasing map such that $\psi \circ k^F = k^G \circ \psi$. Now we will extend ψ continuously to gaps of $\mathscr{K}^F$. Take any strictly increasing homeomorphism $h : (a^F, b^F) \to (a^G, b^G)$. Denote $k^{F,G}_{|_{[0,a^{F,G}]}}$ and $k^{F,G}_{|_{[b^{F,G},1]}}$ by $k^{F,G}_+$, $k^{F,G}_-$ respectively. Observe that given a gap $J \subset [0, 1]$ of $\mathscr{K}^F$ there exist unique $n \geq 1$ and a sequence $j_1 \cdots j_n \in \{-, +\}^n$ such that $k^F_{j_n} \circ \cdots \circ k^F_{j_1}(J) = (a^F, b^F)$. Thus, for each $x \in J$ we define

$$\psi(x) := (k^G_{j_1})^{-1} \circ \cdots \circ (k^G_{j_n})^{-1} \circ h \circ k^F_{j_n} \circ \cdots \circ k^F_{j_1}(x).$$

It is clear that $\psi : [0, 1] \to [0, 1]$ is a strictly increasing continuous map which $\psi|_{\mathscr{K}^F}$ is independent of choice of ψ and $\psi \circ K^F_\pm = K^G_\pm \circ \psi$. ■

The following definition gives a notion of equality between two kneading sequences.

Definition 3. Let $F(x, y) = (f(y)(x), K^F_{sign(x)}(y))$ and $G(x, y) = (g(y)(x), K^G_{sign(x)}(y))$ be two toy models and let $\psi : [0, 1] \to [0, 1]$ be the map constructed on the Proposition 2. We say that $\mathcal{K}_F(\mathscr{L}_c) = \mathcal{K}_G(\mathscr{L}_c)$ if

$$T_F(0^\pm, y) = T_G(0^\pm, \psi(y))$$

for all $y \in [0, 1]$.

We are now ready to state the main step on the proof of Theorem A.

Theorem 2. *Let F and G be two toy models with kneading sequences $\mathcal{K}_F(\mathscr{L}_c)$ and $\mathcal{K}_G(\mathscr{L}_c)$. Assume that $\mathcal{K}_F(\mathscr{L}_c) = \mathcal{K}_G(\mathscr{L}_c)$. Then there exists a bijective map $H : \mathscr{C}(F) \to \mathscr{C}(G)$ such that $H \circ F = G \circ H$ on $\mathscr{C}(F)\backslash\mathscr{L}_c(F)$ and $H(\mathscr{C}(F) \cap I(y)) \subset \mathscr{C}(G) \cap I(\psi(y))$ for each $y \in [0, 1]$. Moreover, $H|_{\mathscr{C}(F)\cap I(y)}$ is strictly increasing for every $y \in [0, 1]$.*

The proof of this theorem will be given in the next chapter.

In order to classify toy models, we need a notion of equivalence between them.

Definition 4. Let $F(x, y) = (f(y)(x), K^F_{sign(x)}(y))$ and $G(x, y) = (g(y)(x), K^G_{sign(x)}(y))$ be two toy models. We say that F and G are **topologically conjugate** if there is a homeomorphism which is strictly increasing on intervals

$$H : \mathrm{Dom}(F) \to \mathrm{Dom}(G),$$

such that $H \circ F = G \circ H$ and $H(I(y)) \subset I(\psi(y))$, where $\psi : [0,1] \to [0,1]$ is the map constructed on the Proposition 2.

Note that if H conjugates F and G, then $H(\mathscr{L}_c(F)) = \mathscr{L}_c(G)$, that is, $H(0^{\pm}, y) = (0^{\pm}, \psi(y))$.

The proposition below is an immediate consequence of the Definition 4 and implies that *Kneading Sequences* are a topological invariant.

Proposition 3. *Let F and G be two toy models. Suppose that F and G are topologically conjugate. Then $\mathcal{K}_F(\mathscr{L}_c) = \mathcal{K}_G(\mathscr{L}_c)$.*

Proof. We will show that for all $y \in [0,1]$ we have $T_F(0^{\pm}, y) = T_G(0^{\pm}, \psi(y))$. Fix $y \in [0,1]$. It is clear that $t_F(0^{\pm}, y) = t_G(0^{\pm}, \psi(y))$. Now, as F and G are conjugate we get

$$H(F^n(0^{\pm}, y)) = G^n(0^{\pm}, \psi(y))$$

for all $n \geq 1$. Since H is strictly increasing on intervals, we have

$$t_F(F^n(0^{\pm}, y)) = t_G(G^n(0^{\pm}, \psi(y)))$$

for all $n \geq 1$. Therefore, $T_F(0^{\pm}, y) = T_G(0^{\pm}, \psi(y))$. ■

Remark 3. Notice that the Proposition above and the Theorem A imply that the *Kneading Sequences* are a 'complete' topological invariant.

Definition 5. Let F be a toy model and let $V \subset \mathrm{Dom}(F)$ be a domain. We say that V is **wandering** if

1. $F^n(V) \cap F^m(V) = \emptyset$ for all $n \neq m$;

2. For all $n \geq 0$ we have $F^n(V) \cap \mathscr{L}_c(F) = \emptyset$.

It is not clear that if we change the hypothesis of nonexistence of wandering intervals by nonexistence of wandering domains the Theorem A would still be true. On the other hand, if we add a technical hypothesis we can give another version of Theorem A.

Let F be a toy model. For each $y \in [0,1]$ and $n \geq 1$ we define

$$\mathscr{C}_n^F(y) := \left\{(x,y) \in \mathrm{Dom}(F) \cap I(y);\ F^l(x,y) \in \mathscr{L}_c(F) \text{ for some } 0 \leq l \leq n-1\right\}.$$

Consider the map $\phi_n^F : [0,1] \to 2^{\mathrm{Dom}(F)}$ defined by $\phi_n^F(y) := \mathscr{C}_n^F(y)$. We can state another formulation of the Theorem A.

Theorem C. Let F and G be two toy models. Assume that F and G have no wandering domain and $\mathcal{K}_F(\mathscr{L}_c) = \mathcal{K}_G(\mathscr{L}_c)$. If the family $\{\phi_n^G\}$ is equicontinuous, then F and G are topologically semiconjugate.

Remark 4. Here, the equicontinuity means that for all $\varepsilon > 0$ there is $\delta > 0$ such that $|y_1 - y_2| < \delta$ implies $d_{\mathcal{H}}(\mathscr{C}_n^G(y_1), \mathscr{C}_n^G(y_2)) < \varepsilon$ for all $n \geq 1$, where $d_{\mathcal{H}}$ is the Hausdorff distance.

Remark 5. Note that if G has no wandering domain and $\{\phi_n^G\}$ is equicontinuous, then G has no wandering intervals. Whence the Theorem C follows from Theorem A. However, we will give an independent proof.

The proof of Theorem C will be given in the next chapter.

Example 1. Let $f : [-1, 1] \to [-1, 1]$ be the tend map, that is, f is defined by $f(x) = 1 + 2x$ if $x \leq 0$ and $f(x) = 1 - 2x$ if $x \geq 0$. Let $k : [0, 1] \to [0, 1]$ be the Cantor map defined by $k(y) = 3y$ if $y \in [0, 1/3]$ and $k(y) = 3 - 3y$ if $y \in [2/3, 1]$. Take the toy model $F(x, y) := (f(x), K(x, y))$. Note that F satisfies both hypotheses of Theorem A and Theorem C.

Example 2. Let $f : [-1, 1] \to [-1, 1]$ be a unimodal map containing a wandering interval I_0 and 0 as turning point. Let $k : [0, 1] \to [0, 1]$ be the Cantor map defined by $k(y) = 3y$ if $y \in [0, 1/3]$ and $k(y) = 3 - 3y$ if $y \in [2/3, 1]$. Take the toy model $F(x, y) := (f(x), K(x, y))$. Note that F has a wandering interval, namely $I_0 \times \{y\}$.

Remark 6. It is clear that the construction of the toy model using unimodal maps has nothing special. So, in the Appendix A we build more general toy models.

CHAPTER 4

Proofs of the main results

In this chapter, we present the proofs of our main results.

4.1 Proof of Theorem 2

As before

$$\mathscr{C}(F) = \left\{(x,y) \in \mathrm{Dom}(F);\ \exists\, l \geq 0 \text{ such that } F^l(x,y) \in \mathscr{L}_c(F)\right\}$$

and

$$\mathscr{C}_n^F(y) := \left\{(x,y) \in \mathrm{Dom}(F) \cap I(y);\ F^l(x,y) \in \mathscr{L}_c(F) \text{ for some } 0 \leq l \leq n-1\right\},$$

for each $y \in [0,1]$ and $n \geq 1$.

Set

$$\mathscr{C}^F(y) := \bigcup_{n \geq 1} \mathscr{C}_n^F(y).$$

Hence

$$\mathscr{C}(F) = \coprod_{y \in [0,1]} \mathscr{C}^F(y).$$

Interchanging F and G we get $\mathscr{C}(G)$, $\mathscr{C}_n^G(y)$ and $\mathscr{C}^G(y)$ for each $y \in [0,1]$ and $n \geq 1$. Suppose that for all $y \in [0,1]$ and $n \geq 1$ there exists an strictly increasing bijection map $H_n(y) : \mathscr{C}_n^F(y) \to \mathscr{C}_n^G(\psi(y))$ such that $H_n(y) \circ F = G \circ H_n(y)$ on $\mathscr{C}_n^F(y) \backslash \{(0^\pm, y)\}$ and $H_n(y)_{|_{\mathscr{C}_{n-1}^F(y)}} = H_{n-1}(y)$. As $\psi : [0,1] \to [0,1]$ is a bijective map, it follows that

$$\mathscr{C}(G) = \coprod_{y \in [0,1]} \mathscr{C}^G(\psi(y)).$$

Therefore, we can define

$$\begin{array}{rcl} H : \mathscr{C}(F) & \longrightarrow & \mathscr{C}(G) \\ (x,y) & \longmapsto & H_n(y)(x,y), \end{array}$$

where n is such that $(x,y) \in \mathscr{C}_n^F(y)$. Then $H \circ F = G \circ H$ on $\mathscr{C}(F) \backslash \mathscr{L}_c(F)$. Hence to prove theorem 2 it is enough construct the maps $H_n(y)$. The proof follows by the same method as in the one-dimensional case.

There exists a partition of $I(y)$ induced by $\mathscr{C}_n^F(y)$, denoted by $\mathscr{P}_n^F(y)$. More specifically,

$$\mathscr{P}_n^F(y) = \left\{ I_i^F(y) \subset I(y);\ \partial I_i^F(y) \subset \mathscr{C}_n^F(y) \cup \{(-1,y),(1,y)\}, 1 \leq i \leq k_n^F(y) \right\},$$

where the increasing order of the index of $I_i^F(y)$ is the same order as the intervals are placed in $I(y)$ and $k_n^F(y) := \#\mathscr{P}_n^F(y)$. Note that the partition $\mathscr{P}_n^F(y)$ includes two intervals, one with left endpoint 0^+ and one with right endpoint 0^-.

Given a sequence $j = j_1 j_2 \dots j_m \in \{-,+\}^m$ we will use the following notation

$$f_j(y) := f_{j_m}(y_{j_{m-1}\cdots j_1}) \circ \cdots \circ f_{j_3}(y_{j_2 j_1}) \circ f_{j_2}(y_{j_1}) \circ f_{j_1}(y),$$

where $K_{j_l} \circ \cdots \circ K_{j_1}(y) := y_{j_l \cdots j_1}$, for all $1 \leq l \leq m$.

Let $(x,y) \in \mathscr{C}_n^F(y) \backslash \{(0^\pm, y)\}$ and we will identify the pair (x,y) with $x(y)_{j(x,y)}^F$. Let us remark that we will abuse the notation of $x(y)_{j(x,y)}^F$ to denote when we either apply F, seeing (x,y) as a vector, or applying the unimodal maps to it, seeing (x,y) as a real number. Here, $j(x,y) = j_1 j_2 \dots j_k \in \{-,+\}^k$ is the minimal sequence such that

$$F^k\left(x(y)_{j(x,y)}^F\right) = \left(f_{j(x,y)}(y)(x(y)_{j(x,y)}^F), y_{j_k \cdots j_1}\right) = \left(0^{j_k}, y_{j_k \cdots j_1}\right),$$

for some $1 \leq k \leq n-1$.

More precisely, for each $I_i^F(y) \in \mathscr{P}_n^F(y)$ there is a unique sequence $j(I_i^F(y)) = j_1^i j_2^i \dots j_n^i \in \{-,+\}^n$ such that the function

$$(z,y) \in \operatorname{int}\left(I_i^F(y)\right) \mapsto \left(f_{j(I_i^F(y))}(y)(z), y_{j_n^i \cdots j_1^i}\right)$$

is strictly monotone. Thus we get

$$\partial\left(I_i^F(y)\right) \subset \left\{(-1,y),(1,y),(0^{\pm},y),x(y)^F_{j_1^i j_2^i \cdots j_k^i}\right\},$$

for some $1 \le k \le n-1$. So for each $n \ge 1$ set

$$A_n^F(y) = \{j_1 j_2 \dots j_k \in \{-,+\}^k; \exists\, (x,y) \in \mathscr{C}_n^F(y) \backslash \{(0^{\pm},y)\} \text{ such that } (x,y) = x(y)^F_{j_1 j_2 \dots j_k} \text{ for some } 1 \le k \le n-1\}.$$

It is clear that we can define $\mathscr{P}_n^G(\psi(y))$ and $A_n^G(\psi(y))$ similarly. Now, we are able to finish the proof of the theorem. Our hypotheses imply that we can define $H_1(y) : \mathscr{C}_1^F(y) \to \mathscr{C}_1^G(\psi(y))$ as $(0^{\pm},y) \mapsto (0^{\pm},\psi(y))$.

Suppose by induction that there exists $H_n(y) : \mathscr{C}_n^F(y) \to \mathscr{C}_n^G(\psi(y))$ a strictly increasing map such that $H_n(y) \circ F = G \circ H_n(y)$ on $\mathscr{C}_n^F(y) \backslash \{(0^{\pm},y)\}$, for every $y \in [0,1]$. This implies that $\#\mathscr{P}_n^F(y) = \#\mathscr{P}_n^G(\psi(y))$ and $A_n^F(y) = A_n^G(\psi(y))$.

Let $I_i^F(y) \in \mathscr{P}_n^F(y)$. There is a unique sequence $j = j_1 j_2 \cdots j_n \in \{-,+\}^n$ such that

$$\partial\left(I_i^F(y)\right) \subset \left\{(\pm 1,y),(0^{\pm},y),x(y)^F_{j_1 \cdots j_k}\right\},$$

for some $1 \le k \le n-1$. First suppose that

$$I_i^F(y) = \left[\left[x(y)^F_{j_1 \cdots j_k}, x(y)^F_{j_1 \cdots j_l}\right], y\right],$$

with $1 \le k \ne l \le n-1$. By the induction hypothesis we have

$$I_i^G(\psi(y)) = \left[\left[x(\psi(y))^G_{j_1 \cdots j_k}, x(\psi(y))^G_{j_1 \cdots j_l}\right], y\right].$$

Moreover

$$H_n(y)(x(y)^F_{j_1 \cdots j_k}) = x(\psi(y))^G_{j_1 \cdots j_k} \text{ and } H_n(y)(x(y)^F_{j_1 \cdots j_l}) = x(\psi(y))^G_{j_1 \cdots j_l}.$$

From the definition of F and G it follows that

$$\begin{aligned} F^n(I_i^F(y)) \;\; = \;\; & \big[\big(f_{j_n}(y_{j_{n-1}\cdots j_1}) \circ \cdots \circ f_{j_{k+1}}(y_{j_k \cdots j_1})(0^{j_{k+1}}), \\ & f_{j_n}(y_{j_{n-1}\cdots j_1}) \circ \cdots \circ f_{j_{l+1}}(y_{j_l \cdots j_1})(0^{j_{l+1}})\big), y_{j_n \cdots j_1}\big] \cup \Lambda(I_i^F(y)) \end{aligned}$$

and

$$
\begin{aligned}
G^n(I_i^G(\psi(y))) &= \big[\big(g_{j_n}(\psi(y)_{j_{n-1}\cdots j_1}) \circ \cdots \circ g_{j_{k+1}}(\psi(y)_{j_k\cdots j_1})(0^{j_{k+1}}), \\
&\quad g_{j_n}(\psi(y)_{j_{n-1}\cdots j_1}) \circ \cdots \circ g_{j_{l+1}}(\psi(y)_{j_l\cdots j_1})(0^{j_{l+1}})\big), y_{j_n\cdots j_1}\big] \\
&\quad \cup \Lambda(I_i^G(\psi(y))),
\end{aligned}
$$

where $\Lambda(I_i^F(y)) = \{F^{n-k}(0^{j_k}, y_{j_k\cdots j_1}), F^{n-l}(0^{j_l}, y_{j_l\cdots j_1})\}$ and $\Lambda(I_i^G(\psi(y))) = \{G^{n-k}(0^{j_k}, \psi(y)_{j_k\cdots j_1}), G^{n-l}(0^{j_l}, \psi(y)_{j_l\cdots j_1})\}$.

Since $\mathcal{K}_F(\mathscr{L}_c) = \mathcal{K}_G(\mathscr{L}_c)$, we get

$$F^n(I_i^F(y)) \cap \mathscr{L}_c(F) \neq \emptyset$$

if, and only if

$$G^n(I_i^G(\psi(y))) \cap \mathscr{L}_c(G) \neq \emptyset.$$

If $F^n(I_i^F(y)) \cap \mathscr{L}_c(F) = \emptyset$, then $I_i^F(y) \in \mathscr{P}_{n+1}^F(y)$. On the other hand, if $\Lambda(I_i^F(y)) \cap \mathscr{L}_c(F) \neq \emptyset$ we get $I_i^F(y) \in \mathscr{P}_{n+1}^F(y)$, since F is a one-to-one map. In any case, we also have $I_i^G(\psi(y)) \in \mathscr{P}_{n+1}^G(\psi(y))$.

Now, if $\big(F^n(I_i^F(y)) \backslash \Lambda(I_i^F(y))\big) \cap \mathscr{L}_c(F) \neq \emptyset$ then there exists a unique

$$x(y)_{j_1 j_2\cdots j_n}^F := \big(f_{j_1}^{-1}(y) \circ \cdots \circ f_{j_n}^{-1}(y_{j_{n-1}\cdots j_1})(0^{j_n}), y\big) \in \operatorname{int}\big(I_i^F(y)\big).$$

Hence, $\big(G^n(I_i^G(\psi(y))) \backslash \Lambda(I_i^G(\psi(y)))\big) \cap \mathscr{L}_c(G) \neq \emptyset$ and also there exists a unique

$$x(\psi(y))_{j_1 j_2\cdots j_n}^G := \big(g_{j_1}^{-1}(\psi(y)) \circ \cdots \circ g_{j_n}^{-1}(\psi(y)_{j_{n-1}\cdots j_1})(0^{j_n}), \psi(y)\big) \in \operatorname{int}\big(I_i^G(\psi(y))\big).$$

So we can define $H_{n+1}(y)(x(y)_{j_1 j_2\cdots j_n}^F) := x(\psi(y))_{j_1 j_2\cdots j_n}^G$. Moreover, if $(x, y) \in \mathscr{C}_n^F(y)$, then we put $H_{n+1}(x, y) = h_n(x, y)$. The cases where $I_i^F = [[-1, x(y)_{j_1 j_2\ldots j_k}^F], y]$, $I_i^F = [[x(y)_{j_1 j_2\ldots j_k}^F, 1], y]$, or $I_i^F = [[x(y)_{j_1 j_2\ldots j_k}^F, 0^{\pm}], y]$ are similar. Thus we have

$$H_{n+1}(y) : \mathscr{C}_{n+1}^F(y) \to \mathscr{C}_{n+1}^G(\psi(y))$$

a strictly increasing map such that

$$H_{n+1}(y) \circ F = G \circ H_{n+1}(y)$$

on $\mathscr{C}_{n+1}^F(y) \backslash \{(0^{\pm}, y)\}$. Moreover, $H_{n+1}(y)|_{\mathscr{C}_n^F(y)} = H_n(y)$. This concludes the proof of theorem 2.

4.2 Proof of Theorem A

We will first prove the following

Theorem 3. *Let F and G be two toy models. Assume that G has no wandering intervals, no interval of periodic points and no weakly attracting periodic points. If F and G have the same kneading sequences, then there is a continuous surjective map $H : \mathrm{Dom}(F) \to \mathrm{Dom}(G)$ such that $H \circ F = G \circ H$.*

We need the following lemma.

Lemma 1. *Let $y \in [0,1] \mapsto f(y) : [-1,1] \to [-1,1]$ be the family on the definition of a toy model. For all $y \in [0,1]$ and $\varepsilon > 0$ there is $\delta > 0$ satisfying: If $y' \in [0,1]$, $x, x' \in [-1,1]$ are such that $d(y,y') < \delta$, $d(x,x') < \delta$ and there exist $x(y) = f_j^{-1}(y)(x)$ and $x'(y') = f_j^{-1}(y')(x')$, where $j = -$ or $+$, then $d(x(y), x'(y')) < \varepsilon$.*

Proof. Suppose that there is $\varepsilon > 0$ such that for all $n \geq 1$ there are $y_n \in [0,1]$ and $x_n^y, x_n^{y_n} \in [-1,1]$ with $d(y,y_n) < 1/n$ and $d(x_n^y, x_n^{y_n}) < 1/n$ such that $x_n(y) = f_j^{-1}(y)(x_n^y)$ and $x_n(y_n) = f_j^{-1}(y_n)(x_n^{y_n})$ exist and $d(x_n(y), x_n(y_n)) \geq \varepsilon$. By compactness, we may suppose that $\lim x_n^y = \lim x_n^{y_n} = x_0$, $\lim x_n(y) = x_1$ and $\lim x_n(y_n) = x_2$. Note that $x_1 \neq x_2$ and $x_1, x_2 \in [-1,0]$ or $x_1, x_2 \in [0,1]$. So $f(y)(x_1) \neq f(y)(x_2)$. On the other hand, as $f(y_n)$ converges uniformly to $f(y)$ we get $f(y)(x_1) = \lim f(y)(x_n(y)) = \lim x_n^y = \lim x_n^{y_n} = \lim f(y_n)(x_n(y_n)) = f(y)(x_2)$. This contradiction finishes the proof of the lemma. ■

Let $\xi_- : ([-1,0^-] \cup (0^+,1]) \times [0,1] \to [-1,0^-]$ and $\xi_+ : ([-1,0^-) \cup [0^+,1]) \times [0,1] \to [0^+,1]$ be two maps defined by

$$\xi_j(x,y) = \begin{cases} f_j^{-1}(y)(x) & , \text{ if } x \in \mathrm{Im}(f_j(y)) \\ 0^j & , \text{ otherwise.} \end{cases}$$

As a consequence of Lemma 1 we get the following.

Corollary 1. *The maps ξ_j defined above are continuous.*

Proof. Let $(x,y) \in \mathrm{Dom}(\xi_j)$ and $\varepsilon > 0$. Assume that $x \in [-1, f_j(y)(0^j))$. Since the family $\{f(y)\}_{y \in [0,1]}$ depends continuously on y, there exists $\delta > 0$ such that for every x' and y' satisfying $d(x,x') < \delta$ and $d(y,y') < \delta$, we have $x' \in [-1, f_j(y')(0^j))$. From Lemma 1 it follows that

$$d(\xi_j(x,y), \xi_j(x',y')) = d(f_j^{-1}(y)(x), f_j^{-1}(y')(x')) < \varepsilon$$

if $\delta > 0$ is small enough.

Suppose now that $x > f_j(y)(0^j)$. Again by continuity, there exists $\delta > 0$ such that for all x' and y' satisfying $d(x, x') < \delta$ and $d(y, y') < \delta$ we have $x' > f_j(y')(0^j)$. Therefore, $\xi_j(x, y) = 0^j = \xi_j(x', y')$ and so

$$d(\xi_j(x, y), \xi_j(x', y')) = 0.$$

The case where $x = f_j(y)(0^j)$ is analogous.

∎

The following lemma is the fundamental step in the proof of Theorem 3.

Lemma 2. *For every $(x, y) \in \mathscr{C}(F)$ there exists a continuous curve $\gamma^F : [0, 1] \to \mathscr{C}(F)$ of the form $\gamma^F(w) = (\widetilde{\gamma}^F(w), w)$ such that it satisfies $\gamma^F(y) = (x, y)$.*

Proof. Given $(x, y) \in \mathscr{C}(F)$ there exist $n \geq 1$ and a sequence $(j_1 \cdots j_n) \in \{-, +\}^n$ such that $(x, y) = (f_{j_1}^{-1}(y) \circ \cdots \circ f_{j_n}^{-1}(y_{j_{n-1} \cdots j_1})(0^{j_n}), y)$. Consider the following sequence of curves defined inductively:

- $\gamma_{j_1}^F : [0, 1] \to \mathrm{Dom}(F)$ is defined by $\gamma_{j_1}^F(w) = (\xi_{j_1}(0^{j_1}, w), w)$;
- $\gamma_{j_1 j_2}^F : [0, 1] \to \mathrm{Dom}(F)$ is defined by $\gamma_{j_1 j_2}^F(w) = (\xi_{j_1}(\pi_1(\gamma_{j_2}^F(w_{j_1})), w), w)$;

$$\vdots$$

- $\gamma_{j_1 \cdots j_n}^F : [0, 1] \to \mathrm{Dom}(F)$ is defined by $\gamma_{j_1 \cdots j_n}^F(w) = (\widetilde{\gamma}_{j_1 \cdots j_n}^F(w), w)$, where $\widetilde{\gamma}_{j_1 \cdots j_n}^F(w) := \xi_{j_1}(\pi_1(\gamma_{j_2 \cdots j_n}^F(w_{j_1})), w)$.

It follows from Corollary 1 that $\gamma_{j_1 \cdots j_n}^F : [0, 1] \to \mathrm{Dom}(F)$ is a continuous curve. Furthermore, by construction we have that $\gamma_{j_1 \cdots j_n}^F(y) = (x, y)$. Now it is quite easy to see that $\gamma_{j_1 \cdots j_n}^F([0, 1]) \subset \mathscr{C}(F)$. For each $w \in [0, 1]$ put

$$l_{j_1 \cdots j_n}(w) := \max\{1 \leq k \leq n;\ f_{j_1}^{-1}(w) \circ \cdots \circ f_{j_k}^{-1}(w_{j_{k-1} \cdots j_1})(0^{j_k}) \text{ exists}\}.$$

Hence, if $l_{j_1 \cdots j_n}(w)$ exists, then

$$\gamma_{j_1 \cdots j_n}^F(w) = \left(f_{j_1}^{-1}(w) \circ \cdots \circ f_{j_{l_{j_1 \cdots j_n}(w)}}^{-1}(w_{j_{l_{j_1 \cdots j_n}(w)-1} \cdots j_1})(0^{j_{l_{j_1 \cdots j_n}(w)}}), w\right).$$

On the other hand, if $l_{j_1 \cdots j_n}(w)$ does not exist, then

$$\gamma_{j_1 \cdots j_n}^F(w) = (0^{j_1}, w).$$

So in any case we get $\gamma_{j_1 \cdots j_n}^F([0, 1]) \subset \mathscr{C}(F)$.

∎

Remark 7. Suppose that $(x, y), (z, w) \in \mathscr{C}(F)$ are such that there exist $n \geq 1$ and a sequence $(j_1 \cdots j_n) \in \{-, +\}^n$ so that $(x, y) = (f_{j_1}^{-1}(y) \circ \cdots \circ f_{j_n}^{-1}(y_{j_{n-1} \cdots j_1})(0^{j_n}), y)$ and $(z, w) = (f_{j_1}^{-1}(w) \circ \cdots \circ f_{j_n}^{-1}(w_{j_{n-1} \cdots j_1})(0^{j_n}), w)$. The Lemma 2 implies that there exists a continuous curve $\gamma^F : [0, 1] \to \mathscr{C}(F)$ satisfying $\gamma^F(y) = (x, y)$ and $\gamma^F(w) = (z, w)$.

Now we are able to prove Theorem 3.

Proof of Theorem 3. From the proof of Theorem 2, for every $y \in [0, 1]$, let $H(y) : \mathscr{C}^F(y) \to \mathscr{C}^G(\psi(y))$ be the map defined by $H(y)(x, y) := H_n(y)(x, y)$, where n is such that $(x, y) \in \mathscr{C}_n^F(y)$. Like in the one-dimensional case, we will extend $H(y)$ to $I(y)$. Take $(z, y) \in \overline{\mathscr{C}^F(y)} \backslash \mathscr{C}^F(y)$ and suppose that there exist $(z_n, y)^j \in \mathscr{C}^F(y)$, $j = 1, 2$, such that $(z_n, y)^1 \uparrow (z, y)$ and $(z_n, y)^2 \downarrow (z, y)$. The cases where we have only $(z_n, y) \uparrow (z, y)$ or $(z_n, y) \downarrow (z, y)$, with $(z_n, y) \in \mathscr{C}^F(y)$ are similar. Since G has no wandering intervals, no intervals of periodic points and no weakly attracting periodic point, the limits $\lim H((z_n, y)^1) := H^1(z, y)$ and $\lim H((z_n, y)^2) := H^2(z, y)$ exist and $H^1(z, y) = H^2(z, y)$. Hence, if $(z, y) \in \overline{\mathscr{C}^F(y)} \backslash \mathscr{C}^F(y)$ we define

$$H(z, y) := \lim H(z_n, y),$$

where $(z_n, y) \in \mathscr{C}^F(y)$ is any sequence such that $(z_n, y) \to (z, y)$. As $\mathscr{C}^G(\psi(y))$ is dense on $I(\psi(y))$, for all $W = [(\alpha, \beta), y]$ connected component of $I(y) \backslash \overline{\mathscr{C}^F(y)}$ and $(z, y) \in W$ we can extend $H(y)$ as $H(y)(z, y) := H(y)(\alpha, y)$, once $H(y)(\alpha, y) = H(y)(\beta, y)$. Let $H : \mathrm{Dom}(F) \to \mathrm{Dom}(G)$ be the map defined by $H(x, y) := H(y)(x, y)$. Note that H is non-decreasing on each fiber $I(y)$.

Claim: $H : \mathrm{Dom}(F) \to \mathrm{Dom}(G)$ is continuous.

Take $(x, y) \in \mathrm{Dom}(F) \backslash \{(\pm 1, y), (0^{\pm}, y); y \in [0, 1]\}$. If we get $(x, y) \in \{(\pm 1, y), (0^{\pm}, y); y \in [0, 1]\}$ the argument is similar. Let $B_\varepsilon(H(x, y))$ be a neighborhood of $H(x, y)$ for some $\varepsilon > 0$. Since $\overline{\mathscr{C}^G(\psi(y))} = I(\psi(y))$, there are $z_1 < x < z_2$ such that $(z_1, y), (z_2, y) \in \mathscr{C}(F)$ and $H(z_i, y) \in B_\varepsilon(H(x, y)) \cap I(\psi(y))$. Note that $\pi_1(H(z_1, y)) < \pi_1(H(x, y)) < \pi_1(H(z_2, y))$. From Lemma 2, it follows that there exist continuous curves $\gamma^G_{j_1 \cdots j_l}$ and $\gamma^G_{e_1 \cdots e_k}$ such that $\gamma^G_{j_1 \cdots j_l}(\psi(y)) = H(z_1, y)$ and $\gamma^G_{e_1 \cdots e_k}(\psi(y)) = H(z_2, y)$. By continuity, there are $y_1 < y < y_2$ so that the domain $D(H(x, y))$ limited by curves $\gamma^G_{j_1 \cdots j_l}([\psi(y_1), \psi(y_2)])$, $\gamma^G_{e_1 \cdots e_k}([\psi(y_1), \psi(y_2)])$ and the intervals

$[(\gamma^G_{j_1\cdots j_l}(\psi(y_1)), \gamma^G_{e_1\cdots e_k}(\psi(y_1))), \psi(y_1)]$, $[(\gamma^G_{j_1\cdots j_l}(\psi(y_2)), \gamma^G_{e_1\cdots e_k}(\psi(y_2))), \psi(y_2)]$ contains $H(x,y)$ and satisfies $D(H(x,y)) \subset B_\varepsilon(H(x,y))$. Again by Lemma 2, for every sequence $t_1\cdots t_m \in \{-,+\}^m$ and $w \in [0,1]$ we get $H(\gamma^F_{t_1\cdots t_m}(w)) = \gamma^G_{t_1\cdots t_m}(\psi(w))$. Let $D(x,y)$ be the subset limited by curves $\gamma^F_{j_1\cdots j_l}([y_1,y_2])$, $\gamma^F_{e_1\cdots e_k}([y_1,y_2])$ and the intervals $[(\gamma^F_{j_1\cdots j_l}(y_1), \gamma^F_{e_1\cdots e_k}(y_1)), y_1]$, $[(\gamma^F_{j_1\cdots j_l}(y_2), \gamma^F_{e_1\cdots e_k}(y_2)), y_2]$. Note that $D(x,y)$ contains a neighborhood of (x,y). As H is monotone on the fibers we have $H(D(x,y)) = D(H(x,y))$. See Figure 4.1. This proves the claim.

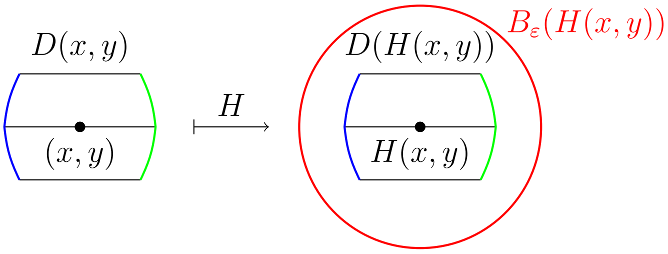

Figure 4.1: Action of H on $D(x,y)$.

The Claim implies that H is surjective. Now, take $(z,y) \in \overline{\mathscr{C}^F(y)}\backslash\mathscr{C}^F(y)$ and let $(z_n,y) \in \mathscr{C}^F(y)\backslash\{0^\pm, y\}$ such that $\lim(z_n,y) = (z,y)$. As F is continuous on $\mathrm{Dom}(F)\backslash\mathscr{C}(F)$ and $H(z,y) \in \overline{\mathscr{C}^G(\psi(y))}\backslash\mathscr{C}^G(\psi(y))$ it follows that

$$\begin{aligned}(H\circ F)(z,y) &= \lim(H\circ F)(z_n,y_n)\\ &= \lim(G\circ H)(z_n,y_n)\\ &= (G\circ H)(z,y).\end{aligned}$$

On the other hand, for all connected component $W \subset I(y)\backslash\overline{\mathscr{C}^F(y)}$ and $n \geq 0$ there is a connected component $W'_n \subset I(\pi_2(F^n(y)))\backslash\overline{\mathscr{C}^F(\pi_2(F^n(y)))}$ such that $F^n(W) \subset W'_n$. As F is injective, we get $F^n(W) = W'_n$. Let $[(\alpha,\beta),y]$ be a connected component of $I(y)\backslash\overline{\mathscr{C}^F(y)}$. Since $(\alpha,y) \in \overline{\mathscr{C}^F(y)}$ we have $H(F(\alpha,y)) = G(H(\alpha,y))$. Thus

$$H(F(z,y)) = H(F(\alpha,y)) = G(H(\alpha,y)) = G(H(z,y)),$$

for all $(z,y) \in [(\alpha,\beta),y]$. This finishes the proof of Theorem. ∎

Theorem A is now an immediate consequence of Theorem 3.

Proof of Theorem A. As, for all $y \in [0,1]$ we have $\overline{\mathscr{C}^F(y)} = I(y)$ and $\overline{\mathscr{C}^G(y)} = I(y)$, Theorem A follows from Theorem 3. ∎

4.3 Proof of Theorem C

Let's start with a lemma which proof is similar to the proof of Proposition 1.

Lemma 3. *Let F be a toy model. Suppose that F has no wandering domain. Then*

$$\overline{\mathscr{C}(F)} = \mathrm{Dom}(F).$$

Proof. Take $V \subset \mathrm{Dom}(F)$ open set such that $F^m(V) \cap \mathscr{L}_c(F) = \emptyset$ for all $m \geq 0$. We will assume that V is of the form $I \times J$, where I and J are open intervals. Since F has no wandering domain, there exist $1 \leq m_1 < m_2$ such that $F^{m_1}(V) \cap F^{m_2}(V) \neq \emptyset$. By the choice of V, we may suppose, without loss of generality, that $m_1 = 0$, $m_2 = m$ for some $m \geq 1$ and m is the smallest integer with such property $F^m(V) \cap V \neq \emptyset$. We now construct, as in the proof of Proposition 1, a connected set L containing V such that $F^l(L) \subset L$ for some $0 < l \leq m$. In addition, $F^i(L) \cap \mathscr{L}_c(F) = \emptyset$ and $F^i(L) \cap F^j(L) = \emptyset$ for all $i, j \in \{0, \dots l-1\}$ with $i \neq j$. Therefore, there exists a sequence $j_0 j_1 \cdots j_{l-1} \in \{-,+\}^l$ such that, for any $(a,b) \in L$ and $n \geq 0$ we have $F^{nl}(a,b) = (a_{nl}, b_{(j_{l-1}\cdots j_1 j_0)^n})$, where $b_{(j_{l-1}\cdots j_1 j_0)^n} := \left(K_{j_{l-1}} \circ \cdots \circ K_{j_0}\right)^n(b)$ and $a_{nl} = f_{j_{l-1}}(b_{j_{l-2}\cdots j_0 (j_{l-1}\cdots j_0)^{n-1}}) \circ \cdots \circ f_{j_0}(b_{(j_{l-1}\cdots j_0)^{n-1}}) \circ \cdots \circ f_{j_{l-1}}(b_{j_{l-2}\cdots j_0}) \circ \cdots \circ f_{j_1}(b_{j_0}) \circ f_{j_0}(b)(a)$. Since $K_{j_{l-1}} \circ \cdots \circ K_{j_0}$ is a contraction, there is $w \in \pi_2(\overline{L})$ such that $K_{j_{l-1}} \circ \cdots \circ K_{j_0}(w) = w$. For each $(x,y) \in V$, we can prove that $y = w$. Therefore $V \subset I(w)$, which contradicts the fact that V is an open set. ■

For all $n \geq 1$ consider the sets

$$\mathscr{C}_n^F = \left\{(x,y) \in \mathrm{Dom}(F);\ \exists\, 0 \leq l \leq n-1 \text{ such that } F^l(x,y) \in \mathscr{L}_c(F)\right\}.$$

For each $y \in [0,1]$, let $H_n(y) : \mathscr{C}_n^F(y) \to \mathscr{C}_n^G(\psi(y))$ be the map constructed in the proof of Theorem 2. Let $\widetilde{H_n}(y) : I(y) \to I(\psi(y))$ be the piecewise linear homeomorphism such that

$$H_n(y)(\pm 1, y) = (\pm 1, \psi(y)),\ \widetilde{H_n}(y)_{|_{\mathscr{C}_n^F(y)}} = H_n(y)$$

and $\widetilde{H_n}(y)$ is linear in each connected component of $I(y) \backslash \mathscr{C}_n^F(y)$.

Now we define $\widetilde{H_n} : \mathrm{Dom}(F) \to \mathrm{Dom}(G)$ as $\widetilde{H_n}(x,y) := \widetilde{H_n}(y)(x,y)$. See Figure 4.2.

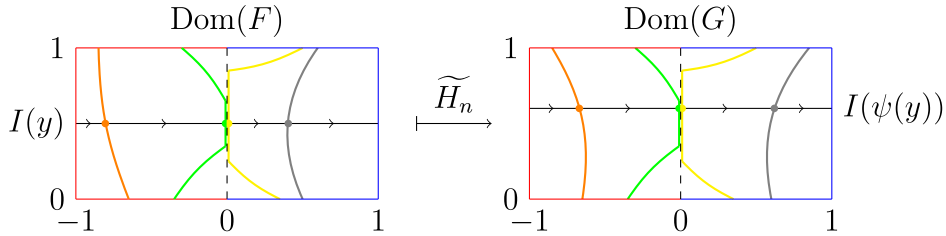

Figure 4.2: Dynamics of $\widetilde{H}_n$ and some curves of $\mathscr{C}_n^F$ and $\mathscr{C}_n^G$.

Lemma 4. *For each* $n \geq 1$, $\widetilde{H}_n : \mathrm{Dom}(F) \to \mathrm{Dom}(G)$ *is a continuous function.*

Proof. Let us start by invoking the curves defined in the proof of Lemma 2. For each sequence $j_1 j_2 \cdots j_l \in \{-,+\}^l$, with $0 \leq l \leq n-1$, let

$$\gamma^F_{j_1 j_2 \cdots j_l} : [0,1] \to \mathscr{C}_n^F$$

be the continuous curve that appears in the construction of $\mathscr{C}_n^F$. Note that $\mathscr{C}_n^F$ is a finite union of curves of this kind, that is,

$$\mathscr{C}_n^F = \bigcup_{\substack{j_1 j_2 \cdots j_l \in \{-,+\}^l \\ 0 \leq l \leq n-1}} \gamma^F_{j_1 j_2 \cdots j_l}([0,1]).$$

Take $(x,y) \in \mathrm{Dom}(F) \backslash \mathscr{C}_n^F$. Since we have finite curves in $\mathscr{C}_n^F$ there exists $\delta > 0$ such that $B_\delta(x,y) \cap \mathscr{C}_n^F = \emptyset$. Let $x_1 < x < x_2$ such that $(x_1,y),(x_2,y) \in \mathscr{C}_n^F$ and does not exist point of $\mathscr{C}_n^F(y)$ on the interval $[(x_1,x_2),y]$. [1] In addition, there are curves $\gamma^F_{j_1 j_2 \cdots j_l}$ and $\gamma^F_{e_1 e_2 \cdots e_k}$ such that $\gamma^F_{j_1 j_2 \cdots j_l}(y) = (x_1,y)$ and $\gamma^F_{e_1 e_2 \cdots e_k}(y) = (x_2,y)$ for some $1 \leq l,k \leq n-1$. For any other curve $\gamma^F_{t_1 t_2 \cdots t_i}$, with $1 \leq i \leq n-1$, we have $\gamma^F_{t_1 t_2 \cdots t_i}(y) \cap [(x_1,x_2),y] = \emptyset$. See Figure 4.3.

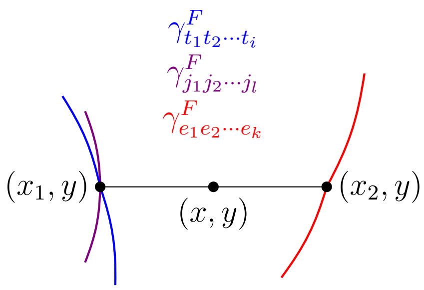

Figure 4.3: Construction of the domain $D(x,y)$.

[1] It is clear that (x_1,y) or (x_2,y) may not exist. In this case, either $x_1 = -1$ or $x_2 = 1$.

By finiteness, there are $y_1 < y < y_2$ and continuous curves $\alpha_i : [y_1, y_2] \to \mathscr{C}_n^F$, with $i = 1, 2$, such that $\alpha_i(y) = (x_i, y)$ and, if $D(x, y)$ is the domain limited by the curves $\alpha_1([y_1, y_2])$, $\alpha_2([y_1, y_2])$ and the intervals $[(\alpha_1(y_1), \alpha_2(y_1)), y_1]$, $[(\alpha_1(y_2), \alpha_2(y_2)), y_2]$, then $D(x, y) \cap \mathscr{C}_n^F = \alpha_1([y_1, y_2]) \cup \alpha_2([y_1, y_2])$. Besides that $\widetilde{H}_n(D(x, y))$ is a domain $D(\widetilde{H}_n(x, y))$ such that

$$D(\widetilde{H}_n(x, y)) \cap \mathscr{C}_n^G = \widetilde{\alpha_1}[\psi(y_1), \psi(y_2)] \cup \widetilde{\alpha_2}[\psi(y_1), \psi(y_2)],$$

where $\widetilde{\alpha_i} = \widetilde{H}_n(\alpha_i)$. See Figure 4.4.

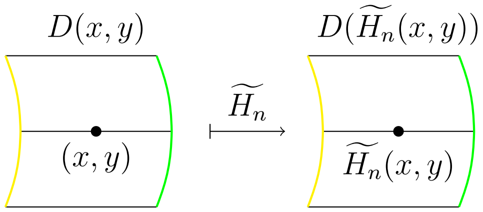

Figure 4.4: Action of $\widetilde{H}_n$ on neighborhood of (x, y).

Now, by definition of $\widetilde{H}_n$, for all $(z, w) \in D(x, y)$ we get $\widetilde{H}_n(z, w) = (\xi_{x,y,w}(z), \psi(w))$ where

$$\xi_{x,y,w}(z) = \pi_1(\widetilde{\alpha_1}(\psi(w))) + \frac{\pi_1(\widetilde{\alpha_2}(\psi(w))) - \pi_1(\widetilde{\alpha_1}(\psi(w)))}{\pi_1(\alpha_2(w)) - \pi_1(\alpha_1(w))} \left[z - \pi_1(\alpha_1(w))\right].$$

This implies that $\widetilde{H}_n$ is continuous in (x, y). In fact, $\widetilde{H}_n$ is continuous in $D(x, y) \cup \partial D(x, y)$.

Suppose now that $(x, y) \in \mathscr{C}_n^F$. Let $s \in \{-, +\}^l$ such that $(x, y) = (\gamma_s^F(y), y)$ for some $0 \leq l \leq n - 1$. Consider $(x_j, y_j) \to (x, y)$. Note that there exist curves $\gamma_{a_j}^F$ and $\gamma_{b_j}^F$ such that (x_j, y_j) belongs to the interval $[[\gamma_{a_j}^F(y_j), \gamma_{b_j}^F(y_j)], y_j]$ and does not exist point of $\mathscr{C}_n^F(y)$ on the intervals $[(\gamma_{a_j}^F(y_j), \gamma_{b_j}^F(y_j)), y_j]$, for each $j \geq 1$. Therefore, in order to prove that $\widetilde{H}_n(x_j, y_j) \to \widetilde{H}_n(x, y)$ it is enough to prove that any subsequence (x_{j_k}, y_{j_k}) contains a subsequence $(x_{j_{k_l}}, y_{j_{k_l}})$ so that $\widetilde{H}_n(x_{j_{k_l}}, y_{j_{k_l}}) \to \widetilde{H}_n(x, y)$. Let (x_{j_k}, y_{j_k}) be any subsequence of (x_j, y_j). There is a subsequence such that $(x_{j_{k_l}}, y_{j_{k_l}}) \in [[\gamma_a^F(y_{j_{k_l}}), \gamma_b^F(y_{j_{k_l}})], y_{j_{k_l}}]$ for some sequences a and b fix. Since $\gamma_a^F(y_{j_{k_l}}) \leq x_{j_{k_l}} \leq \gamma_b^F(y_{j_{k_l}})$, we get that $\gamma_a^F(y) \leq \gamma_s^F(y) \leq \gamma_b^F(y)$ by taking the limit when l goes to infinity. If $\gamma_a^F(y) = \gamma_b^F(y)$, then $\widetilde{H}_n(x_{j_{k_l}}, y_{j_{k_l}}) \to \widetilde{H}_n(x, y)$, since

$$\widetilde{H}_n(\gamma_a^F(y_{j_{k_l}}), y_{j_{k_l}}) \leq \widetilde{H}_n(x_{j_{k_l}}, y_{j_{k_l}}) \leq \widetilde{H}_n(\gamma_b^F(y_{j_{k_l}}), y_{j_{k_l}})$$

and

$$\begin{aligned}\lim_{l\to\infty}\widetilde{H}_n(\gamma_a^F(y_{j_{k_l}}),y_{j_{k_l}}) &= \widetilde{H}_n(\gamma_a^F(y),y)\\ &= \widetilde{H}_n(\gamma_s^F(y),y)\\ &= \widetilde{H}_n(\gamma_b^F(y),y)\\ &= \lim_{l\to\infty}\widetilde{H}_n(\gamma_b^F(y_{j_{k_l}}),y_{j_{k_l}}).\end{aligned}$$

Assume now that $\gamma_a^F(y)\neq\gamma_b^F(y)$. Then either $\gamma_s^F(y)=\gamma_a^F(y)$ or $\gamma_s^F(y)=\gamma_b^F(y)$. Thus, we can build a domain $D(x,y)$ such that $(x_{j_{k_l}},y_{j_{k_l}})\in D(x,y)$ for all $l\geq 1$. Hence $\widetilde{H}_n(x_{j_{k_l}},y_{j_{k_l}})\to\widetilde{H}_n(x,y)$. It follows that $\widetilde{H}_n$ is continuous.

■

The goal now is to prove that the sequence $\{\widetilde{H}_n\}_{n\geq 1}$ is a Cauchy sequence.

Lemma 5. *The sequence $\{\widetilde{H}_n\}_{n\geq 1}$ form a Cauchy sequence.*

Proof. Let $\varepsilon>0$. By equicontinuity of the family $\{\phi_n^G\}$, there is $\delta>0$ such that $|y_1-y_2|<\delta$ implies $d_{\mathcal{H}}(\mathscr{C}_n^G(y_1),\mathscr{C}_n^G(y_2))<\varepsilon$ for all $n\geq 1$. From Lemma 3 it follows that $\mathscr{C}(G)$ is dense in $\mathrm{Dom}(G)$, so there exists a number $N>0$ such that $\{B_\delta(x,y);\ (x,y)\in\mathscr{C}_N^G\}$ covers $\mathrm{Dom}(G)$. Let $(x,y)\in\mathrm{Dom}(F)$ and $n,m\geq N$. As $\widetilde{H}_n$ is a surjective map, we can take $x_1<x<x_2$ such that $|\widetilde{H}_n(x_1,y)-\widetilde{H}_n(x,y)|=|\widetilde{H}_n(x_2,y)-\widetilde{H}_n(x,y)|=2\varepsilon$. Thus there is $(x_1',y'),(x_2'',y'')\in\mathscr{C}_N^F$ such that $\widetilde{H}_n(x_1,y)\in B_\delta(\widetilde{H_N}(x_1',y'))$ and $\widetilde{H}_n(x_2,y)\in B_\delta(\widetilde{H_N}(x_1'',y''))$. Again by equicontinuity there exist $(\widetilde{x_1},y),(\widetilde{x_2},y)\in\mathscr{C}_N^F$ such that $\widetilde{H}_n(x,y)$ belongs to the interval $[[\widetilde{H_N}(\widetilde{x_1},y),\widetilde{H_N}(\widetilde{x_2},y)],\psi(y)]$ and $|\widetilde{H_N}(\widetilde{x_1},y)-\widetilde{H_N}(\widetilde{x_2},y)|<6\varepsilon$. Since $\widetilde{H}_l|_{\mathscr{C}_N^F}=\widetilde{H_N}|_{\mathscr{C}_N^F}$ for all $l\geq N$ we get $\widetilde{H}_n(x,y),\widetilde{H_m}(x,y)\in[[\widetilde{H_N}(\widetilde{x_1},y),\widetilde{H_N}(\widetilde{x_2},y)],\psi(y)]$. Therefore, $|\widetilde{H}_n(x,y)-\widetilde{H_m}(x,y)|<6\varepsilon$. Hence,

$$\|H_n-H_m\|=\sup_{(x,y)\in\mathrm{Dom}(F)}|\widetilde{H}_n(x,y)-\widetilde{H_m}(x,y)|<6\varepsilon.$$

Thus, it follows that $\{\widetilde{H}_n\}_{n\geq 1}$ is a Cauchy sequence.

■

From Lemma 5 there exist a continuous map $\widetilde{H}:\mathrm{Dom}(F)\to\mathrm{Dom}(G)$ such that $\lim\widetilde{H}_n=\widetilde{H}$. The proof is completed by showing that $\widetilde{H}$ is a semiconjugacy between F and G.

Note that the mapping $\widetilde{H}$ agrees with $\widetilde{H_n}$ on $\mathscr{C}_n^F$ whence $\widetilde{H} \circ F = G \circ \widetilde{H}$ on $\mathscr{C}(F)$. Take $(x, y) \in \overline{\mathscr{C}(F)} \backslash \mathscr{C}(F)$ and $(x_n, y_n) \in \mathscr{C}(F)$ such that $\lim(x_n, y_n) = (x, y)$. As F is continuous on $\mathrm{Dom}(F) \backslash \mathscr{C}(F)$ and $\widetilde{H}(x, y) \in \overline{\mathscr{C}(G)} \backslash \mathscr{C}(G)$ it follows that

$$\begin{aligned}(\widetilde{H} \circ F)(x, y) &= \lim(\widetilde{H} \circ F)(x_n, y_n) \\ &= \lim(G \circ \widetilde{H})(x_n, y_n) \\ &= (G \circ \widetilde{H})(x, y).\end{aligned}$$

So $\widetilde{H} \circ F$ and $G \circ \widetilde{H}$ agree on $\overline{\mathscr{C}(F)} = \mathrm{Dom}(F)$. Thus $\widetilde{H}$ is the required semiconjugacy and Theorem C is proved.

Remark 8. Notice that if the family $\{\phi_n^F\}$ is equicontinuous, then $\widetilde{H}$ is a conjugacy.

CHAPTER 5

On Singer's Theorem for toy models

The goal of this short chapter is to prove a version of Singer's theorem for toy models. We will recall the definition of the Schwarzian derivative.

Let $f : I \to I$ be a C^3 interval map and let $x \in I$ such that $Df(x) \neq 0$. The Schwarzian derivative $Sf(x)$ of f at x is defined by

$$Sf(x) := \frac{D^3 f(x)}{Df(x)} - \frac{3}{2}\left(\frac{D^2 f(x)}{Df(x)}\right)^2.$$

We say that f has *negative Schwarzian derivative* if $Sf(x) < 0$ for all points $x \in I$ except, possibly, the turning points. In these points we define $Sf(x) = -\infty$.

From the definition, if $f, g : I \to I$ are C^3 interval maps and $x \in I$ is such that $Df(x) \neq 0$ and $Dg(f(x)) \neq 0$ then

$$S(g \circ f)(x) = Sg(f(x)) \cdot (Df(x))^2 + Sf(x).$$

This implies that if f and g have negative Schwarzian derivative, then $g \circ f$ also has negative Schwarzian derivative.

The lemma below gives an intuition about the types of graphs that can not occur for functions f with negative Schwarzian derivative.

Lemma 5.1. (Minimum Principle) *Let I be a closed interval with endpoints a, b and $f : I \to I$ a map with negative Schwarzian derivative. If $Df(x) \neq 0$ for all $x \in I$ then*

$$|Df(x)| > \min\{|Df(a)|, |Df(b)|\}, \quad \text{for all} \quad x \in (a, b).$$

We say that c is a critical point of a C^1 map f if $Df(c) = 0$. As consequence of the Minimum Principle we have

Theorem 5.1. (Singer) *If $f : I \to I$ is a C^3 map with negative Schwarzian derivative then the immediate basin of any attracting periodic orbit contains either a critical point of f or a boundary point of the interval I.*

Now we can state the main result of this chapter.

Theorem B. *Let $F(x,y) = (f(y)(x), K^F_{sign(x)}(y))$ be a toy model. Suppose that $f(y) : [-1,1] \to [-1,1]$ has negative Schwarzian derivative for all $y \in [0,1]$. Then the closure of the immediate basin of any strong attracting periodic orbit contains either a point of the critical line or a point of $\Lambda := \{-1,1\} \times [0,1]$.*

The proof of this Theorem is similar to one dimensional case. Before proving Theorem B, let us establish some notation. We write the derivative of the toy model $F(x,y) = (f(y)(x), K^F_{sign(x)}(y))$ as

$$DF = \begin{pmatrix} f_x & f_y \\ 0 & K_y \end{pmatrix}.$$

We know that given $(x,y) \in \mathrm{Dom}(F)$, there is a sequence $j_{(x,y)} = (j_1 \cdots j_m \cdots) \in \{-,+\}^{\mathbb{N}}$ such that

$$F^m(x,y) = \left(f_{j_m}(y_{j_{m-1}\cdots j_1}) \circ \cdots \circ f_{j_2}(y_{j_1}) \circ f_{j_1}(y)(x), y_{j_m \cdots j_1}\right),$$

for all $m \geq 0$. So the derivative of F^m at (x,y) is given by

$$DF^m(x,y) = \begin{pmatrix} A^m(x,y) & B^m(x,y) \\ 0 & D^m(x,y) \end{pmatrix},$$

where

$$A^m(x,y) = \prod_{l=0}^{m-1} f_x(F^l(x,y)), \quad D^m(x,y) = \prod_{l=0}^{m-1} K_y(F^l(x,y)),$$

$$B^m(x,y) = \sum_{j=0}^{m-1} \left(f_y(F^j(x,y)) \prod_{l=0}^{j-1} K_y(F^l(x,y)) \prod_{l=j+1}^{m-1} f_x(F^l(x,y)) \right).$$

Proof of Theorem B. Let $(p,q) \in \text{Dom}(F)$ be a strong attracting periodic point of period m and B be the closure of its immediate basin. We assume that B does not contain points of Λ neither points of the critical line. Let B_0 be the interior of the connected component containing (p,q). Then $F^m(B_0) \subset B_0$. Let $[(a,b),q]$ be the connected component of $B_0 \cap I(q)$ containing (p,q). See Figure 5.1.

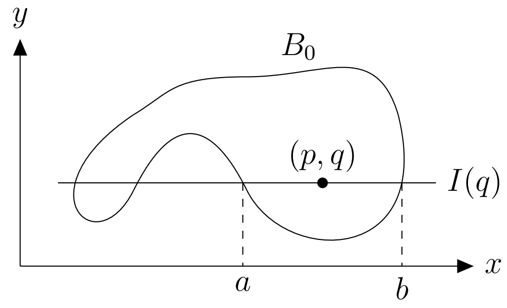

Figure 5.1: Construction of the interval $[(a,b),q]$.

Since $F^m(p,q) = (p,q)$ this implies

$$F^m([(a,b),q]) \subset ([(a,b),q]) \, .$$

By our assumptions, there exists a sequence $j = (j_1 \cdots j_m) \in \{-,+\}^m$ such that

$$F^m(x,q) = (f_j(q)(x), q)$$

and

$$A^m(x,q) \neq 0$$

for all $x \in (a,b)$, where $f_j(q) = f_{j_m}(q_{j_{m-1}\cdots j_1}) \circ \cdots \circ f_{j_2}(q_{j_1}) \circ f_{j_1}(q)$.

Claim: $f_j(q)(\{a,b\}) \subset \{a,b\}$.

Suppose that $f_j(q)(a) \in (a,b)$ (the case where $f_j(q)(b) \in (a,b)$ is similar). As (a,q) is a continuity point of F^m there is a neighborhood V of (a,q) such that $F^m(V) \subset B_0$. This is a contradiction since B_0 is the connected component of the immediate basin containing (p,q). Therefore the claim is proved.

We can assume that $A^m(x,q) > 0$ for all $x \in (a,b)$ and $f_j(q)(\alpha) = \alpha$ for $\alpha \in \{a,b\}$ (otherwise, consider $A^{2m}(x,q)$ instead of $A^m(x,q)$). As (a,q) and (b,q) can not be attractor periodic points we get $A^m(\alpha,q) \geq 1$ for $\alpha \in \{a,b\}$.

Note that $f_j(q)$ has negative Schwarzian derivative. From the Minimum Principle, it follows that $A^m(x,q) > 1$ for all $(x,q) \in [(a,b),q]$, contradicting that $F^m([(a,b),q]) = [(a,b),q]$. ■

CHAPTER 6

Open questions and future work

The present work has at least two possible directions of research. One is related to increase the knowledge on the toy model and, the other is to try to extend the results of the toy model to the two dimensional dissipative diffeomorphisms of the disc.

6.1 Toy model and further results

It is known that in the context of one-dimensional dynamics the set of periodic points is dense in the set of recurrent points for all $f : I \to I$ continuous. Inspired by this result we would like to prove the following:

Question 1: Let μ a F-invariant ergodic measure. Is $\overline{\mathrm{supp}(\mu)}$ contained in $\overline{\mathrm{Per}(F)}$?

Once Question 1 is answered, it is natural to wonder if the periodic points of F are dense in its recurrent set (or, more generally, non-wandering set). For this it is enough to answer the next question:

Question 2: $\overline{\mathscr{C}(F|_{\mathscr{K}^F})} \subset \overline{\mathrm{Per}(F)}$?

On the other hand, it is well-known that for unimodal maps f with negative Schwarzian derivative we get absence of wandering intervals. More precisely,

Theorem 6.1. (Guckenheimer) *Let $f : I \to I$ be a C^3 unimodal map with negative Schwarzian derivative and such that $f''(c) \neq 0$ at the unique critical point c of f. Then f has no wandering intervals.*

As we have seen in the proof of Theorem A, the hypothesis of nonexistence of wandering intervals is fundamental to construct our conjugacy. So it is natural to ask:

Question 3: Let $F(x, y) = (f(y)(x), K^F_{sign(x)}(y))$ be a toy model. If $f(y) : [-1, 1] \to [-1, 1]$ has negative Schwarzian derivative for all $y \in [0, 1]$ does F have no wandering intervals?

6.2 Looking for combinatorial structure for two dimensional dissipative diffeomorphisms

As we saw on Chapter 2, the classical results in one-dimensional dynamics follow from the fact that the line is totally ordered: the orbit equivalence (provided by the preimages of the turning points) can be extended to the closure as a semi-conjugacy, since it is monotone.

In our two dimensional counterpart, the one-dimensional order structure is preserved along "unstable leaves" providing a partial order allowing us to extend the orbit equivalence to the closure.

We wonder if a similar type of approach can be used for two dimensional diffeomorphisms. In [CP17], it was introduced an open class of diffeomorphisms of the disc, namely strongly dissipative: In short that means that stable manifolds of regular points of a non-trivial ergodic measure separate the disc.

Definition 6. Let S be a boundaryless surface. A C^r-diffeomorphism $f : S \to f(S) \subset S$ is *strongly dissipative* if

- $f(S)$ is contained in a compact subset of S,
- f is dissipative, that is, $|\det(Df(x))| < 1$ for any $x \in S$,

- for any ergodic measure μ which is not supported on a hyperbolic sink, and for μ-almost every point x, each of the two connected components of $W^s_S(x)\backslash\{x\}$ meets $S\backslash f(S)$.

In [CP17], for each f strongly dissipative diffeomorphism of the disc a one-dimensional dynamics is associated: there exists a real tree (that is, a path connected metric space such that for any two points x, y there exists a unique subset homeomorphic to $[0,1]$ whose endpoints are x and y) and a continuous map defined on the tree which is measure equivalent to f. More precisely, they proved the following:

Theorem 6.2. *Let f be a C^r, $r > 1$, strongly dissipative diffeomorphism of the disc. Then there exists a semi-conjugacy $\pi : (\mathbb{D}, f) \to (X, h)$ to a continuous map h on a compact real tree which induces an injective map on the set of non-atomic ergodic measures μ of f. Moreover the entropies of μ and $\pi_*(\mu)$ are the same.*

Furthermore, conjugate dynamics on the disc have their induced dynamics on the tree conjugated. The tree, by definition, is partially ordered, so it is natural to associate to it a kneading sequence involving the turning points and regions of monotonicity. The goal is to answer the following question.

Question 4: Let f and g be two strongly dissipative diffeomorphisms of the disc such that the kneading sequences of their associated dynamics over their trees are equal. Does there exist a (semi)conjugacy between f and g?

Appendix A

Generalized toy models

In this appendix, we build toy models with families of l-modal maps.

Consider a one-parameter family

$$f(y) : [0,1] \to [0,1], \text{ with } y \in [0,1],$$

depending continuously on y, such that $f(y)$ is a l-modal map verifying that $0 < c_1 < \cdots < c_l$ are the turning points and $f(y)(\{0,1\}) \subset \{0,1\}$ for all $y \in [0,1]$. This means that $f(y)$ has local extrema at $0 < c_1 < \cdots < c_l$ and that $f(y)$ is strictly monotone in each of the $l+1$ intervals $I_1 = [c_0, c_1)$, $I_2 = (c_1, c_2)$, $\ldots$, $I_{l+1} = (c_l, c_{l+1}]$, where $c_0 = 0$ and $c_{l+1} = 1$.

For each $y \in [0,1]$ we use the notation $f_i(y)$ for $f(y)_{|_{I_i}}$. See Figure A.1.

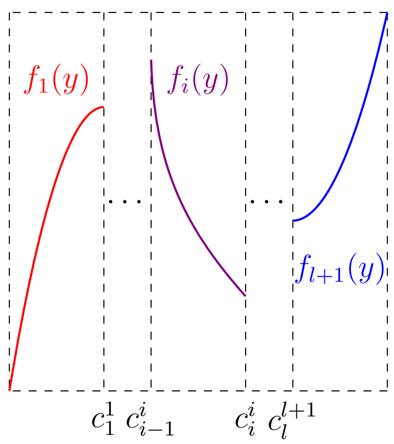

Figure A.1: Dynamics of $f(y)$.

Let $0 = a_1 < b_1 < a_2 < b_2 < \cdots < a_{l+1} < b_{l+1} = 1$ and

$$k : [a_1, b_1] \cup \cdots \cup [a_{l+1}, b_{l+1}] \to [0, 1]$$

be a differentiable function such that $k(\{a_i, b_i\}) \subset \{0, 1\}$ and $|k'| > \gamma > 1$. We put

$$K(x, y) = K_i(y), \text{ if } x \in I_i,$$

where $K_i = (k_{|_{[a_i,b_i]}})^{-1}$ for all $i = 1, \ldots, l + 1$.

We call *Generalized Toy Model* the map defined as

$$\begin{array}{rcccl} F & : & ([0,1]\backslash\{c_1,\ldots,c_l\}) \times [0,1] & \longrightarrow & [0,1]\times[0,1] \\ & & (x,y) & \longmapsto & (f(y)(x), K(x,y)). \end{array}$$

As before we will make the domain of F compact introducing the points c_i^i and c_i^{i+1} with $i = 1, \ldots, l$ and extend F to them via the formula $F(c_i^i, y) = (f_i(y)(c_i), K_i(y))$ and $F(c_i^{i+1}, y) = (f_{i+1}(y)(c_i), K_{i+1}(y))$. See Figure A.2.

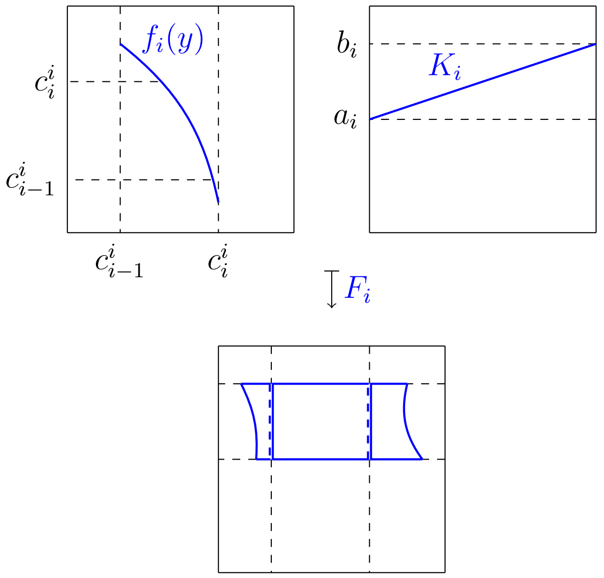

Figure A.2: Action of F_i on $[c_{i-1}^i, c_i^i] \times [0, 1]$.

Let I_i, $i = 1, \ldots, l + 1$, be an interval on the construction of the family $y \mapsto f(y)$. We make the following convention. If $f(y)$ is strictly increasing at I_i, then we put

$$\begin{array}{rcccl} F_i & : & [c_{i-1}^i, c_i^i] \times [0,1] & \longrightarrow & ([0,c_1^1] \cup (c_1^2, c_2^2] \cup \cdots \cup (c_l^{l+1}, 1]) \times [0,1] \\ & & (x,y) & \longmapsto & (f_i(y)(x), K_i(y)). \end{array}$$

If $f(y)$ is strictly decreasing at I_i, then we put

$$\begin{aligned} F_i \; : \; [c_{i-1}^i, c_i^i] \times [0,1] &\longrightarrow ([0, c_1^1) \cup [c_1^2, c_2^2) \cup \cdots \cup [c_l^{l+1}, 1]) \times [0,1] \\ (x,y) &\longmapsto (f_i(y)(x), K_i(y)). \end{aligned}$$

So, for each $(x,y) \in \mathrm{Dom}(F)$ there is a sequence $j(x,y) = (j_1 j_2 \cdots j_m \cdots) \in \{1, \ldots, l+1\}^{\mathbb{N}}$ such that

$$F^{m+1}(x,y) = F_{j_{m+1}}(x_m, y_m)$$

for all $m \geq 0$, where $x_m = f_{j_m}(y_{j_{m-1} \cdots j_1}) \circ \cdots \circ f_{j_2}(y_{j_1}) \circ f_{j_1}(y)(x)$ and $y_m = K_{j_m} \circ \cdots \circ K_{j_1}(y) := y_{j_m \cdots j_1}$.

Similarly, if we consider the alphabet $\mathcal{A} = \{I_1, c_1^1, c_1^2, I_2, \ldots, c_l^l, c_l^{l+1}, I_{l+1}\}$, the address $t_F(x,y)$ of a point $(x,y) \in \mathrm{Dom}(F)$ is defined by

$$t_F(x,y) = \begin{cases} I_i & \text{if} \quad \pi_1(x,y) \in I_i \\ c_i^i & \text{if} \quad \pi_1(x,y) = c_i^i \\ c_i^{i+1} & \text{if} \quad \pi_1(x,y) = c_i^{i+1} \end{cases}$$

Besides that, the itinerary of (x,y) is the sequence

$$T_F(x,y) = (t_F(x,y), t_F(F(x,y)), \ldots, t_F(F^n(x,y)), \ldots).$$

The *critical line* of F is the set

$$\mathscr{L}_c(F) = \{(c_i^i, y), (c_i^{i+1}, y); y \in [0,1] \text{ and } i = 1, \ldots, l\}$$

and the *Kneading Sequences* of F, denoted by $\mathcal{K}_F(\mathscr{L}_c)$, are the set of all itineraries of points of $\mathscr{L}_c(F)$.

Now we can state a Theorem analogue to Theorem A.

Theorem A'. *Let F and G be two generalized toy models. Assume that G has no wandering intervals, no interval of periodic points and no weakly attracting periodic points. If F and G have the same kneading sequences, then F and G are topologically semiconjugate.*

The proof of Theorem A' follows the same ideas of the proof of Theorem 3.

APPENDIX B

Combinatorial equivalence for nonautonomous discrete dynamical systems

A *nonautonomous discrete dynamical system* (short NDS) is a pair $(\mathcal{X}, \mathcal{F})$, where $\mathcal{X} = (X_n)_{n\geq 1}$ is a sequence of metric spaces and $\mathcal{F} = (f_n)_{n\geq 1}$ is a sequence of continuous maps $f_n : X_n \to X_{n+1}$. Orbits of the system are described by the maps $f_n^l : X_n \to X_{n+l}$, defined by

$$f_n^l(x) := (f_{n+(l-1)} \circ \cdots \circ f_n)(x) \text{ for each } n, l \in \mathbb{N} \text{ and } x \in X_n,$$

$$f_n^0 := \mathrm{id}_{X_n} \text{ for each } n \in \mathbb{N}.$$

The classical autonomous setting is obtained by letting $f_n = f$ and $X_n = X$, for every $n \geq 1$. Furthemore, we define $f_n^{-l} := (f_n^l)^{-1}$, which is only applied to sets. (We do not assume that the maps f_n are invertible.)

Nonautonomous discrete dynamical systems were introduced in [KS96]. In this paper, S. Kolyada and L. Snoha introduced and studied the notion of topological entropy for the nonautonomous discrete dynamical systems given by a sequence of endomorphisms $(f_n)_{n\geq 1}$ of a compact topological space X. They were motivated by the desire to understand better the topological entropy of skew products.

In the past twenty years, a large number of papers have been devoted to dynamical properties in nonautonomous discrete systems. Kolyada *et al* [KMS99] generalized the definitions of [KS96]. Huang *et al* [HWZ08] introduced and studied the notion of topological pressure for the nonautonomous discrete dynamical systems. Metric entropy of NDS has been studied in [Kaw14] and [KL16]. The notion of chaos was extended to NDS setting by many researchers (see, e.g., [ZSS16],[WZ13],[Shi12], [TC06]).

So, although recognizably distinct from that of classical autonomous dynamic systems, the theory of the nonautonomuos discrete dynamical systems has developed into a highly active field of research. In particular, on [KS96, Section 5] Kolyada and Snoha studied the invariance of the topological entropy by conjugacy between two NDS's.

Definition 7. Let $(\mathcal{X}, \mathcal{F})$ and $(\mathcal{Y}, \mathcal{G})$ be two nonautonomous discrete dynamical systems. We say that $(\mathcal{X}, \mathcal{F})$ and $(\mathcal{Y}, \mathcal{G})$ are topologically semiconjugate if there exist a sequence $h_n : X_n \to Y_n$ of continuous surjective maps such that $h_{n+1} \circ f_n = g_n \circ h_n$ for every $n \geq 1$. If h_n is a homeomorphism for each $n \geq 1$, then we say that $(\mathcal{X}, \mathcal{F})$ and $(\mathcal{Y}, \mathcal{G})$ are topologically conjugate.

So it is natural to ask:

Question 1: Let $(\mathcal{X}, \mathcal{F})$ and $(\mathcal{Y}, \mathcal{G})$ be two nonautonomous discrete dynamical systems. Under which conditions $(\mathcal{X}, \mathcal{F})$ and $(\mathcal{Y}, \mathcal{G})$ are topologically (semi)conjugate?

The main aim of this Appendix is to give a notion weaker than topological conjugacy to NDS's on the particular case where X_n are intervals and f_n are unimodal maps for every $n \geq 1$, namely *combinatorial equivalence.*

B.1 Combinatorial equivalence for NDS

Let $(\mathcal{X}, \mathcal{F})$ be the NDS defined as follows: For each $n \geq 1$ let $J_n = [a_n, b_n]$ be an interval and $f_n : J_n \to J_{n+1}$ be a continuous map satisfying $f_n(a_n) = f_n(b_n) = a_{n+1}$. Besides that, there is $c_n \in (a_n, b_n)$ such that $f_n|_{[a_n,c_n]}$ is strictly increasing and $f_n|_{[c_n,b_n]}$ is strictly decreasing. The points $\{c_n;\ n \geq 1\}$ are the *turning points* of the NDS. We call the NDS defined above *unimodal nonautonomous discrete dynamical system* (short UNDS)

For each $n \geq 1$ set

$$\mathscr{C}_n(\mathcal{F}) = \left\{ x \in J_n^{\mathcal{F}};\ \exists\, l \geq 0 \text{ such that } f_n^l(x) = c_{n+l} \right\}.$$

Definition 8. We say that two UNDS's $(\mathcal{X}, \mathcal{F})$ and $(\mathcal{Y}, \mathcal{G})$ with turning points $\{c_n^{\mathcal{F}};\ n \geq 1\}$ respectively $\{c_n^{\mathcal{G}};\ n \geq 1\}$ are **combinatorially equiva-**

lent if there exist a family of order preserving bijections $h_n : \mathscr{C}_n(\mathcal{F}) \to \mathscr{C}_n(\mathcal{G})$ such that $h_{n+1} \circ f_n = g_n \circ h_n$ on $\mathscr{C}_n(\mathcal{F})\backslash\{c_n\}$ for all $n \geq 1$.

Let's proceed, as in previous chapters, by constructing the symbolic space for UNDS. Consider the alphabet $\mathcal{A} = \{L, c_n, R; n \geq 1\}$. Let $x \in J_n$ for some $n \geq 1$. The address $i_{\mathcal{F}}(x)$ is defined by

$$i_{\mathcal{F}}(x) = \begin{cases} L & \text{if} \quad x \in [a_n, c_n) \\ c_n & \text{if} \quad x = c_n \\ R & \text{if} \quad x \in (c_n, b_n] \end{cases}$$

Applying this map to an orbit of a given point $x \in J_n$, we associate to that orbit one infinite symbolic sequence.

Definition 9. Consider the sequence of symbols in $\mathcal{A}$

$$I_{\mathcal{F}}(x) = (i_{\mathcal{F}}(x), i_{\mathcal{F}}(f_n^1(x)), \ldots, i_{\mathcal{F}}(f_n^l(x)), \ldots) \in \mathcal{A}^{\mathbb{N}}.$$

This infinite sequence is called itinerary of $x \in J_n$.

The *kneading invariant* of $(\mathcal{X}, \mathcal{F})$ is the sequence $\mathbb{V}(\mathcal{F}) = \{\mathbb{V}_n\}_{n\geq 1}$, where $\mathbb{V}_n := I_{\mathcal{F}}(c_n)$.

Now we can state the main result of this appendix.

Theorem D. Let $(\mathcal{X}, \mathcal{F})$ and $(\mathcal{Y}, \mathcal{G})$ be two UNDS's with kneading invariants $\mathbb{V}(\mathcal{F})$ and $\mathbb{V}(\mathcal{G})$. If $\mathbb{V}(\mathcal{F}) = \mathbb{V}(\mathcal{G})$, then $(\mathcal{X}, \mathcal{F})$ and $(\mathcal{Y}, \mathcal{G})$ are combinatorially equivalent. Furthermore, if $\mathscr{C}_n(\mathcal{G})$ is dense in $J_n^{\mathcal{G}}$ for all $n \geq 1$, then $(\mathcal{X}, \mathcal{F})$ and $(\mathcal{Y}, \mathcal{G})$ are topologically semiconjugate.

Again, the proof of the theorem above is similar to Theorem 2.1.

Note that many concepts used in classical dynamics such as periodicity, recurrence and wandering domains do not seem to make sense for nonautonomous systems, while other concepts such as entropy can be generalized. Thus, if we want analogous hypotheses to Theorem 2.1 to ensure that $\mathcal{G}$ be dense in $J_n^{\mathcal{G}}$, firstly we have define compatible hypotheses. So we can ask:

Question 2: Let $(\mathcal{X}, \mathcal{F})$ be a unimodal nonautonomous discrete dynamical systems. Under which conditions we get $\overline{\mathscr{C}_n(\mathcal{F})} = J_n^{\mathcal{F}}$ for all $n \geq 1$?

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