Bounding Betti numbers of patchworked real hypersurfaces by Hodge numbers

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The Smith-Thom inequality bounds the sum of the Betti numbers of a real algebraic variety by the sum of the Betti numbers of its complexification. In this talk I will explain our proof of a conjecture of Itenberg which, for a particular class of real algebraic projective hypersurfaces, bounds the individual Betti numbers in terms of the Hodge numbers of the complexification. The real hypersurfaces we consider arise from Viro's patchworking construction, which is a powerful combinatorial method for constructing topological types of real algebraic varieties.

To prove the bounds conjectured by Itenberg we develop a real analogue of tropical homology and use spectral sequences to compare it to the usual tropical homology of Itenberg, Katzarkov, Mikhalkin, Zharkov with coefficients in $\mathbb{Z}/2\mathbb{Z}$. Their homology theory with rational coefficients gives the Hodge numbers of a complex projective variety from its tropicalization. We prove the analogous statement for tropical homology with $\mathbb{Z}/2\mathbb{Z}$ coefficients and establish that the integral tropical homology groups of a hypersurface are torision free.

Lurking in the spectral sequence of the proof are the keys to having combinatorial control of the topology of the real hypersurface produced from a patchwork.