Chromatic index of random multigraphs

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For a (multi)graph $G = (V, E)$, we denote by $\chi'(G)$ the minimum number of colors needed to color the edges of G properly. Clearly, $\Delta(G) \leq \chi'(G)$. Vizing proved that $\chi'(G) \leq \Delta(G) + \mu(G)$, where $\mu(G) = \max\{\mu(e) \mid e \in \binom{V}{2}\}$ is the maximum edge multiplicity of G.

Let $S \subseteq V$ and let $\rho(G) = \max\{\frac{e(G[S])}{\lfloor |S|/2 \rfloor} \mid S \subseteq V\}$. Since every color class forms a matching, we have that $\chi'(G) \geq \lceil \rho(G) \rceil$. In the '70s, Goldberg, and independently Seymour, conjectured that for any multigraph G, $\chi'(G) \in \{\Delta(G), \Delta(G) + 1, \lceil \rho(G) \rceil\}$. We show that their conjecture is true w.h.p. for random multigraphs.

The model $M(n, m)$ is the probability space consisting of all loopless multigraphs with n vertices and m edges, in which m pairs from $[n]$ are chosen independently at random with repetitions (that is, we permit edge repetitions in the standard random graph model $G(n, m)$). Our result states that, for a given $m := m(n)$, $M \sim M(n, m)$ typically satisfy $\chi'(G) = \max\{\Delta(G), \lceil \rho(G) \rceil\}$. In particular, we show that if n is even and $m := m(n)$, then $\chi'(M) = \Delta(M)$ for a typical $M \sim M(n, m)$. Furthermore, for a fixed $\varepsilon > 0$, if n is odd, then a typical $M \sim M(n, m)$ has $\chi'(M) = \Delta(M)$ for $m \leq (1 - \varepsilon)n^3 \log n$, and $\chi'(M) = \lceil \rho(M) \rceil$ for $m \geq (1 + \varepsilon)n^3 \log n$.