On subgraphs of C_{2k}-free graphs and some generalised Turán problems

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Let C_l denote a cycle of length l. Kühn and Osthus showed that every bipartite C_{2k}-free graph G contains a C_4-free subgraph with at least $1/(k-1)$ fraction of the edges of G. We present a new and simple proof of this result.

Győri et. al. showed that if c denotes the largest constant such that every C_6-free graph G contains a bipartite C_4-free subgraph with c fraction of edges of G, then $\frac{3}{8} \leq c \leq \frac{2}{5}$. We show that $c = \frac{3}{8}$. Our proof uses the following statement, which we prove using probabilistic method, generalizing a theorem of Erdős: For any $\varepsilon > 0$, and any integers a, b, $k \geq 2$, there exists an a-uniform hypergraph H of girth greater than k which does not contain any b-colorable subhypergraph with more than $\left(1 - \frac{1}{b^{a-1}}\right)(1+\varepsilon)$ fraction of the hyperedges of H. This is based on joint work with Grósz and Tompkins.

We will also present results on generalized Turán problems for even cycles, extending the results of Alon and Shikhelman. Given graphs H and F, let $ex(n, H, F)$ denote the maximum possible number of copies of H in an F-free graph on n vertices. We determine the order of magnitude of $ex(n, C_{2l}, C_{2k})$ for any $l, k \geq 2$. Moreover, we determine $ex(n, C_4, C_{2k})$ asymptotically for all k.

Solymosi and Wong proved that if Erdős's Girth Conjecture holds, then for any $l \geq 3$, the maximum number of C_{2l}'s in a graph of girth $2l$ is $\Theta(n^{2l/(l-1)})$. We prove that their result is sharp in the sense that if an even cycle of any other length is also forbidden, then the order of magnitude is smaller. More precisely, we show that for any $k > l$, the maximum number of C_{2l}'s in a C_{2k}-free graph of girth $2l$ is $\Theta(n^2)$. This is based on joint work with Gerbner, Győri and Vizer.