ROBUSTLY EXPANSIVE CODIMENSION-ONE HOMOCLINIC CLASSES ARE HYPERBOLIC

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ABSTRACT. We shall prove that C^1-robustly expansive codimension-one homoclinic classes are hyperbolic.

1. INTRODUCTION

Let M be a d-dimensional manifold and $\mathrm{Diff}^1(M)$ be the set of C^1 diffeomorphisms f on M endowed with the C^1 topology. Let p be a hyperbolic periodic point of f and $H(p)$ be its homoclinic class. The diffeomorphism f is α-expansive in $H(p)$ if $\mathrm{dist}(f^n(x), f^n(y)) \leq \alpha$ for all $n \in \mathbb{Z}$, with $x, y \in H(p)$, implies $x = y$. It is well known that hyperbolicity implies α-expansiveness for some $\alpha > 0$. But expansiveness alone does not guarantee hyperbolicity, even when one is dealing with a codimension one expansive homoclinic class, as can be seen in [PPV, Section 2]. We note that there are even examples of expansive codimension one homoclinic classes such all of its periodic orbits are hyperbolic that are not hyperbolic.

To see this consider a Smale horse-shoe H in a plane and Λ a non trivial minimal subset of H. In a complementary direction multiply H by a non uniform contraction $\lambda(w)$, depending on the distance from w to Λ, and in such way that $\lambda(w) = 1$ for $w \in \Lambda$. The resulting homoclinic class is expansive, has all periodic points hyperbolic but it is not hyperbolic.

Then we assume that expansiveness holds in a C^1-neighborhood of the homoclinic class, that is, for all diffeomorphism g C^1 near f the homoclinic class $H(p_g)$ of the continuation p_g of p is α-expansive. In [PPV] it was proved that robustly expansive homoclinic classes of a three dimensional manifold are generically hyperbolic. We generalize this result in two ways. First we drop the assumption $\dim(M) = 3$ and get that robustly expansive codimension-one homoclinic classes have a codimension-one dominated splitting. Second we prove that robustly expansive codimension-one homoclinic classes with a dominated splitting are hyperbolic.

The main step to obtain this result is to prove that center unstable manifolds for all point x in the homoclinic class $H(p)$ are true unstable manifolds, that is, center unstable manifolds are dynamically defined. To achieve this it is enough to get this property for periodic points homoclinically related to p. For this we use a result of Sambarino that controls the behavior of periodic points homoclinically related to p. Once this is settled, under the hypothesis that the dominated splitting $E \oplus F$ is not hyperbolic, the fact that center unstable manifolds are dinamically defined allows us to obtain a hyperbolic periodic point q with arbitrarily large period as near $H(p)$ as we wish, and with arbitrarily small rate of contraction along the E-direction. Then we prove that

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such hyperbolic periodic points are in fact in $H(p)$, contradicting uniform E-contraction **in** the period for periodic points in $H(p)$. As in [Ma2] this implies that the sub-bundle F is uniformly expanding, and then $E \oplus F$ is hyperbolic.

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