Partial Actions, Partial Group Rings and Crossed Products

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Abstract

Partial actions of groups appeared in the theory of operator algebras as a powerful tool of their study. In the most general setting of partial actions on abstract sets the definition is as follows:

DEFINITION 1. Let G be a group with identity element e and $\mathcal{X}$ be a set. A partial action α of G on $\mathcal{X}$ is a collection of subsets $\Delta_g \subseteq \mathcal{X}$ $(g \in G)$ and bijections $\alpha_g : \Delta_{g^{-1}} \mapsto \Delta_g$ such that

(i) $\Delta_e = \mathcal{X}$ and α_e is the identity map of $\mathcal{X}$;

(ii) $\Delta_{(gh)^{-1}} \supseteq \alpha_h^{-1}(\Delta_h \cap \Delta_{g^{-1}})$;

(iii) $(\alpha_g \circ \alpha_h)(x) = \alpha_{gh}(x)$ for each $x \in {\alpha_h}^{-1}(\Delta_h \cap \Delta_{g^{-1}})$.

If instead of an abstract set $\mathcal{X}$ we take a vector space V over a field K, then in the definition of partial action of G on V the subsets Δ_g are supposed to be subspaces and the maps $\alpha_g : \Delta_{g^{-1}} \mapsto \Delta_g$ are linear isomorphisms. Thus we come to the concept of partial representation of G on a vector space V. Similarly to the case of (usual) representations of groups, there exists an algebra $K_{\mathrm{par}}G$, called the partial group algebra of G over K, which governs the partial representations of G. It turns out that $K_{\mathrm{par}}G$ keeps much more information about the structure of G than does KG.

In order to define a partial action α of a group G on a unital K-algebra $\mathcal{A}$ we suppose in Definition 1 that each Δ_g $(g \in G)$ is an ideal of $\mathcal{A}$ and that every map $\alpha_g : \Delta_{g^{-1}} \mapsto \Delta_g$ is an isomorphism of algebras. Using partial actions one can generalize the concept of crossed product. For simplicity of notation we assume that the twisting is trivial, so we give the definition in the context of corresponding skew group rings.

DEFINITION 2. Given a partial action α of G on $\mathcal{A}$, the skew group ring $\mathcal{A}_\alpha * G$ corresponding to α is the set of all formal sums $\{\sum_{g\in G} a_g\delta_g : a_g \in \Delta_g\}$, where δ_g are symbols. The addition is defined by the obvious way and the multiplication is determined by $(a_g\delta_g)\cdot(b_h\delta_h) = \alpha_g(\alpha_{g^{-1}}(a_g)b_h)\delta_{gh}$.

We give a survey of results on (abstract) partial representations of groups, partial group rings, their isomorphism problem and the associativity question of crossed products defined by partial actions of groups.