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MORITA EQUIVALENCE OF FORMAL POISSON STRUCTURES

Inocencio Ortiz

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Advisor: Henrique Bursztyn

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Abstract

In this work we introduce a notion of Morita equivalence for formal Poisson structures adapting to the formal setting Xu's notion of Morita equivalence of Poisson manifolds. Our main result is a classification of Morita equivalent formal Poisson structures deforming the zero structure via the action of B-field transformations. In order to obtain this result, we analyze the problem of deforming Poisson morphisms between ordinary Poisson structures into Poisson morphisms between their formal Poisson deformations, and use it to construct formal dual pairs. Combined with previous results on the classification of Morita equivalent star products, our main theorem yields a concrete way to link the notions of Morita equivalence in algebra and geometry through deformation quantization.

Contents

CHAPTER 1

Introduction

The notion of Morita equivalence first arose in the theory of rings and algebras [54], and it relies on comparing rings through their categories of representations. More precisely, Morita equivalence identifies rings possessing equivalent categories of modules. This notion, although reasonably weaker than the usual notion of ring isomorphism, turns out to preserve many relevant ring-theoretic properties (see e.g. [2] for an exposition of Morita theory). A key feature of Morita equivalence is that it is realized by means of certain bimodules, see Section 2.1.1 for a brief overview.

The notion of Morita equivalence has been transported from ring theory to many other contexts, including C^*-algebras [61], (topological, Lie, or symplectic) groupoids ([56], [72]), and Poisson manifolds [71]. In all these settings one has suitable notions of "module" (or "representation") and "bimodule" over the objects at hand, and these allow the concept of Morita equivalence to be adapted to each context (see [47] for a unified approach). Morita equivalence is the main equivalence relation between algebras that arise in noncommutative geometry ([19], [61]), playing also a key role in physical applications ([18], [63]). In this work, we will be particularly concerned with the notion of Morita equivalence in the realm of Poisson manifolds [71], where the role of bimodules is played by a refinement of the notion of *symplectic dual pairs*, due to Weinstein [68].

There are several natural constructions relating categories where Morita equivalence has been defined; for example, one can pass from groupoids to (C^*-)algebras by considering their convolution algebras (see e.g. [60], [16], [19]), or from (integrable) Poisson manifolds to symplectic groupoids through the integration of Lie algebroids (see e.g. [51], [22]). In doing so, an important question is whether Morita equivalence is preserved, i.e., whether these constructions are *functorial* with respect to Morita equivalence; several results are known in this direction, see e.g. [56], [55], [46] and [72]; for more recent papers, see [17] and [42].

This thesis concerns a similar issue of preservation of Morita equivalence, but in the context of passing from Poisson structures to noncommutative algebras by means of "quantization". More precisely, we will consider *deformation quantization* [3] as a way to associate noncommutative algebras, known as *star products*, to Poisson structures. We can summarize the guiding goal of this work as follows:

GOAL: *Find a concrete link between Morita equivalence in Poisson geometry and Morita equivalence in algebra through deformation quantization.*

To be more precise, let us recall the main result in deformation quantization, due to Kontsevich [44] (see [65], [27], [39] and [67] for surveys and introductions to deformation quantization). This result not only implies that any Poisson structure π on a manifold P admits a corresponding star-product algebra, but it gives a parametrization of all star products associated with π in terms of formal deformations of π. To have a clear statement, recall that a *formal Poisson structure* on P is a formal power series $\pi = \sum_{j=0}^{\infty} \lambda^j \pi_j$ of bivector fields π_j on P, such that $[\pi, \pi]_S = 0$ for the Schouten bracket $[\cdot, \cdot]_S$ of multivector fields extended bilinearly to formal power series. Note that, in such a series, the first nonzero term is an ordinary Poisson structure on P, so the series can be seen as a formal Poisson deformation of it. Kontsevich's theorem establishes a bijective correspondence

$$\mathcal{K}_*\colon \ \mathrm{FPois}_0(P) \to \mathrm{Def}(P), \tag{1.1}$$

where $\mathrm{FPois}_0(P)$ is the space of formal Poisson structures on P, deforming the zero Poisson tensor, modulo an equivalence relation given by the action of formal diffeomorphisms, and $\mathrm{Def}(P)$ is the moduli space of star-products on P.

To find a sense in which Morita equivalence is preserved under deformation quantization, a natural route is to extend the geometric notion of Morita equivalence from ordinary Poisson structures to formal Poisson structures (i.e., as an equivalence relation in $\mathrm{FPois}_0(P)$), and then analyze its behavior under Kontsevich's quantization map above. These issues are our main focus in this work.

Some background: Morita equivalence of star products and B-fields

The problem of describing Morita equivalent star products has been studied in [5], [13] and [6]. To formulate the results, we need to recall yet another notion of equivalence in the realm of Poisson manifolds, namely, the *gauge equivalence* of Poisson manifolds of Ševera and Weinstein [64], also known as *B-field transformations* in the context of generalized geometry [36]. Classically, given a Poisson structure π on a manifold P and a closed 2-form B on P, one can try to obtain a new Poisson structure $\tau_B\pi$ by adding the pullback of B to each leafwise symplectic form of π; assuming that the resulting closed 2-form on each leaf is nondegenerate, this operation yields a new Poisson structure. There is a natural adaptation of this "gauge action" by closed 2-forms to *formal* Poisson structures. In this context, it leads to a canonical action of $\mathbb{H}^2_{dR}(P)_\lambda$, the space of formal de Rham classes in degree 2, on $\mathrm{FPois}_0(P)$ [6], which we also denote by $([B],[\pi]) \mapsto [\tau_B\pi]$.

Meanwhile, on the algebraic side of Kontsevich's map above (star products), it is proven in [5] that Morita equivalence is described by the orbits of a natural action of the Picard group $\mathrm{Pic}(P) \simeq \mathbb{H}^2(P,\mathbb{Z})$ (i.e., the group of isomorphism classes of complex line bundles $L \to P$ with tensor product as group operation) on $\mathrm{Def}(P)$. A key result, proven by Bursztyn, Dolgushev and Waldmann in [6], is that Kontsevich's map (1.1) is equivariant with respect to these actions, as long as the B-field is an integral form. This result is summarized in the following diagram:

$$\{B\text{-field action}\} \xleftrightarrow{\text{deformation quantization (integral } B\text{-fields)}} \{\text{M.E. star products}\} \tag{1.2}$$

Building on this result, in order to close the circle of ideas relating Morita equivalences of Poisson structures and star products, one should:

(i) extend geometric Morita equivalence of Poisson structures to formal Poisson structures;

(ii) compare this notion with the gauge action by B-fields.

We will give a more detailed description of our results concerning (i) and (ii) in the next subsection.

The connection between Morita equivalence and gauge transformations for *ordinary* Poisson structures was studied by Bursztyn and Radko in [7], where they proved that integrable Poisson manifolds which are gauge equivalent up to a Poisson diffeomorphism are Morita equivalent; the converse is proven not to hold. This led Weinstein and Bursztyn to wonder how Xu's geometric notion of Morita equivalence might be related with the algebraic one on the quantum side ([9], [14]), and ultimately motivates the guiding goal stated above.

Contributions of this thesis

In order to motivate the definition of Morita equivalence for *formal* Poisson manifolds, let us recall here Xu's definition of Morita equivalence for Poisson manifolds.

Definition 1.0.1. *(**Xu's Morita equivalence** [71]) Two Poisson manifolds (P_1,π_0) and (P_2,σ_0) are Morita equivalent if there exists a symplectic manifold (S,ω_0) and complete surjective Poisson submersions $J_1\colon (S,\omega_0) \to (P_1,-\pi_0)$ and $J_2\colon (S,\omega_0) \to (P_2,\sigma_0)$, whose fibers are connected, simply connected and symplectically orthogonal to each other. Such a symplectic manifold (S,ω_0) is called an **equivalence symplectic bimodule**.*

In the above definition, the maps being *complete* means that, if X^i_f is a complete hamiltonian vector field in P_i, then $X_{J_i^*(f)}$ is a complete hamiltonian vector field in S. Despite the fact that Xu's Morita equivalence is defined for ordinary Poisson structures in a purely geometric way, it is built upon the notion of symplectic dual pairs, which admits (under some mild topological conditions) a nice algebraic characterization that can be naturally adapted to the formal setting. Concretely, a *dual pair* is a diagram of the form $P_1 \xleftarrow{J_1} S \xrightarrow{J_2} P_2$, where S is a symplectic manifold, P_1 and P_2 are Poisson manifolds, and the maps are Poisson submersions whose fibers are symplectically orthogonal to each other. The dual pair is called *full* if the maps are surjective, and *complete* if the maps are complete. In a full dual pair, the geometric property of symplectic orthogonality is equivalent to the algebraic condition of $J_1^*(C^\infty(P_1))$ and $J_2^*(C^\infty(P_2))$ being Poisson commutant of one another inside $C^\infty(S)$, as long as the J_i's fibers are connected (see [16]).

Hence, if $(P_1, -\pi_0) \xleftarrow{J_1} (S, \omega_0) \xrightarrow{J_2} (P_2, \sigma_0)$ is a Morita equivalence, then it satisfies, by definition, the topological condition needed to describe it algebraically, namely, by a diagram of the form

$$(C^\infty(P_1), \pi_0) \xrightarrow{J_1^*} (C^\infty(S), \omega_0) \xleftarrow{J_2^*} (C^\infty(P_2), \sigma_0),$$

were J_1^* is anti-Poisson, J_2^* is Poisson, and $J_1^*(C^\infty(P_1))$ and $J_2^*(C^\infty(P_2))$ are mutually Poisson commutant inside $(C^\infty(S), \omega_0)$.

This algebraic condition admits a natural version in the formal setting, which leads us to the following:

Definition 1.0.2. *(**Morita equivalence of formal Poisson manifolds**) Two formal Poisson manifolds, (P_i, π^i), for $i = 1, 2$, are **Morita equivalent** if the following two conditions hold:*

1. *There exists a symplectic manifold (S, ω_0) and a diagram of the form*

$$(C^\infty(P_1)_\lambda, \pi^1) \xrightarrow{\Phi^1} (C^\infty(S)_\lambda, \omega) \xleftarrow{\Phi^2} (C^\infty(P_2)_\lambda, \pi^2),$$

   for some $\omega = \omega_0 + \sum_{j=1}^\infty \lambda^j \omega_j \in \Omega^2_{cl}(S)_\lambda$, such that Φ^1 is anti-Poisson, Φ^2 is Poisson, and $\Phi^1(C^\infty(P_1)_\lambda)$ and $\Phi^2(C^\infty(P_2)_\lambda)$ are mutually centralizers inside $(C^\infty(S)_\lambda, \omega)$.

2. *In the classical limit ($\lambda \to 0$), (S, ω_0) is an equivalence symplectic bimodule between (P_1, π_0^1) and (P_2, π_0^2).*

With this definition, the main result of this work is a classification of Morita equivalent elements in $\mathrm{FPois}_0(P)$ via B-fields. More concretely, we have the following:

MAIN RESULT: *Two formal Poisson structures $[\pi], [\pi'] \in \mathrm{FPois}_0(P)$ are Morita equivalent if and only if $[\tau_B \pi'] = [\psi_* \pi]$, for some $\psi \in \mathrm{Diff}(P)$ and $[B] \in \mathbb{H}^2_{dR}(P)_\lambda$.*

This will follow from Theorem 4.2.3 and Theorem 5.1.19. Thus, we may extend diagram (1.2) in the following way

$$\begin{tikzcd}
\{B\text{-field action}\} \arrow[rr, leftrightarrow, "\text{deformation quantization (integral } B\text{-fields)}"] \arrow[dr, leftrightarrow] & & \{\text{ME star products}\} \\
& \{\text{ME formal Poisson structures}\} \arrow[ur, dashed] &
\end{tikzcd}$$

where the left diagonal arrow is due to our main result, and the dashed arrow is just given by composing the other two.

Next we elaborate a little on the tools and techniques behind this result, whose proof consists of two main steps: first, we study the construction of dual pairs for formal Poisson structures, which entails the problem of deforming Poisson morphisms to morphisms of formal Poisson structures; second, we note that formal deformations of dual pairs define a map that classifies Morita equivalent formal Poisson structures, which we then prove to agree with the action by B-fields. In more details:

Deformation of Poisson morphisms - Cohomology: First of all, we consider the problem of formal deformation of Poisson morphisms with respect to given deformations of the involved Poisson structures, identifying the cohomology controlling this deformation and finding the obstructions for iterative deformation and uniqueness up to a natural degree of freedom. Next we consider the symplectic realization $(T^*P, \omega_0) \xrightarrow{\rho} (P, 0)$, where ρ is the cotangent bundle projection and $\omega_0 := \omega_{can} + \rho^* B_0$, with an arbitrary $B_0 \in \Omega^2_{cl}(P)$. Applying the cohomological results to the Poisson morphism $(C^\infty(P), 0) \xrightarrow{\rho^*} (C^\infty(T^*P), \omega_0)$, we find that the obstructions are absent, and by analyzing commutants we obtain a map

$$\mathbb{H}^2_{dR}(P)_\lambda \times \mathrm{FPois}_0(P) \to \mathrm{FPois}_0(P); ([B], [\pi]) \mapsto [\pi^B], \tag{1.3}$$

which classifies Morita equivalent elements in $\mathrm{FPois}_0(P)$ up to the action of $\mathrm{Diff}(P)$ on $\mathrm{FPois}_0(P)$ via push-forward. This is the content of Theorem 4.2.3.

Relation with B-fields: Here we apply notions of Dirac geometry and Courant algebroids adapted to the formal setting. The main idea behind the identification of the previous map with B-field action is related to the problem of constructing symplectic realizations, which was locally solved in ([68]), and then globally in [20] and [23]. Generalizing the problem and the ideas of [23] to the context of Dirac geometry, in [32] Frejlich and Mărcuţ arrived to a dual-pair relation in the context of Dirac structures, which is suitable to be adapted into the formal setting of interest for us. To make sense of the result of Frejlich and Mărcuţ in the formal context, and to be able to proof our main result, we needed to adapt notions from Courant algebroids, its symmetries and derivations into the formal setting. In particular, for the formal version of the standard Courant algebroid in $TP \oplus T^*P$ we prove a formal version of Proposition

2.3 in Gualtieri's work [36], which describes the form of the flow generated by a derivation of the Courant algebroid. This allows us to prove that that the classifying map (1.3), and the gauge action are related by $[\pi^B] = [\tau_{-B}\pi]$, as stated in Theorem 5.1.19.

Organization of the thesis

In Chapter 2 we review some definitions and fundamental results about formal structures, deformation theory and its link to deformation quantization. We also recall the notions of dual pairs along with those of Morita equivalence which are relevant for us and review how the concept of B-fields come into play. We finish this chapter setting the definition of Morita equivalence for formal Poisson manifolds.

Chapter 3 begins by defining the deformation of Poisson morphisms and introduces a deformation problem, which will ultimately lead us to the classification map (1.3). Then we introduce the cohomology which controls the deformation problem and identify conditions for iterative construction of deformations as well as for uniqueness of such deformations. These results are summarized in Proposition 3.2.12 and Proposition 3.2.13. Meanwhile, in Proposition 3.2.18 we show how the deformation procedure yields a new formal Poisson structure, out of a given one.

In Chapter 4 we apply the previous algebraic result to the deformation of the Poisson morphism induced by the symplectic realization $(T^*P, \omega_0) \xrightarrow{\rho} (P, 0)$, where ρ is the cotangent bundle projection and $\omega_0 := \omega_{can} + \rho^* B_0$, for some $B_0 \in \Omega^2_{cl}(P)$. We find that, in this case, all the conditions identified in the previous chapter are fulfilled, thus, the deformation procedure yields the classifying map (1.3). That is the content of Theorem 4.2.1 (along with its corollary) and Theorem 4.2.3.

In Chapter 5 we establish the relation between Morita equivalence of formal Poisson structures and B-fields action. To do so, we start by adapting to the formal setting several facts concerning Courant algebroids, its symmetries and derivations, as well as some notions from Dirac geometry. A key ingredient to prove the main result, Theorem 5.1.19, is Proposition 5.1.18, for which the ideas from the work of Frejlich and Mărcuţ [32] were fundamental. We finish this chapter with some final remarks.

The last chapter gathers a couple of appendices. In Appendix A we recall some basic concepts of differential geometry, like semi-basic differential forms, vertical de Rham cohomology and cotangent lifting. In Appendix B we review some aspects of Courant algebroids and Dirac structures, in particular, the notions of symmetries and derivations of Courant algebroid, which we extend into the formal context in Chapter 5.

CHAPTER 2

Background on Morita equivalence and deformation theory

In this chapter we review some notions which are relevant for our purposes and introduce notations and definitions that we are going to use. In the last section we introduce our definition of Morita equivalence for formal Poisson manifolds.

2.1 Morita equivalence and gauge equivalence

In this section we recall the algebraic notion of Morita equivalence [54] as well as Xu's Morita equivalence for Poisson manifolds [71], lying this way the foundation upon which we will introduce the notion of Morita equivalence for formal Poisson manifolds. We also review the concept of gauge equivalence of Poisson manifolds [64] and its relation with Xu's Morita equivalence. Beside the original sources, further information can be found in [2], [14], [9] and [7].

2.1.1 Algebraic Morita equivalence

For a fixed unital ring R let ${}_{\mathsf{R}}\mathcal{M}$ be the category of left R-modules.

Definition 2.1.1. *Two unital rings,* R *and* S*, are said to be* ***Morita equivalent*** *if* ${}_{\mathsf{R}}\mathcal{M}$ *and* ${}_{\mathsf{S}}\mathcal{M}$ *are equivalent categories.*

Recall that two categories $\mathcal{C}$ and $\mathcal{D}$ are said to be equivalent if there exist functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ such that $F \circ G \approx 1_{\mathcal{D}}$ and $G \circ F \approx 1_{\mathcal{C}}$, where $\approx$ stands for natural isomorphism. Such functors are called **equivalence functors** (see, for instance, [52] and [2]).

Example 2.1.2. *The archetypical example of Morita equivalent rings is the following. Let* R *be a unital ring and let* $M_n(\mathsf{R})$ *be the ring of* $n \times n$ *matrices with entries in* R*, for some positive integer* n*. Given a left* R*-module* $V \in {}_{\mathsf{R}}\mathcal{M}$*, we associate to it the* $M_n(\mathsf{R})$*-module* $V^n \in {}_{M_n(\mathsf{R})}\mathcal{M}$

with the action given by the usual matrix action on column vectors. This defines a functor $F\colon {}_{\mathsf{R}}\mathcal{M} \to {}_{M_n(\mathsf{R})}\mathcal{M}$, which can be shown is an equivalence functor (see [45, Th. 17.20]). Thus, R and $M_n(\mathsf{R})$ are Morita equivalent for any natural number n. This result also follows immediately from the characterization theorem of Morita equivalent rings, which will be stated below.

Let R and S be unital rings, an **(R, S)-bimodule** is an abelian group M together with ring homomorphisms $\lambda\colon \mathsf{R} \to \mathrm{End}^l(M)$ and $\rho\colon \mathsf{S} \to \mathrm{End}^r(M)$, where $\mathrm{End}^l(M)$ (respectively $\mathrm{End}^r(M)$) is the set of group homomorphisms acting from the left (respectively, from the right), making M into a left R-module and a right S-module, respectively, and satisfying any (hence both) of the following equivalent conditions:

1. $\lambda\colon \mathsf{R} \to \mathrm{End}(M_{\mathsf{S}})$ is a ring homomorphism,

2. $\rho\colon \mathsf{S} \to \mathrm{End}({}_{\mathsf{R}}M)$ is a ring homomorphism.

Observe that $\mathrm{End}(M_{\mathsf{S}})$ is the set of S-module homomorphism $f\colon M_{\mathsf{S}} \to M_{\mathsf{S}}$, acting from the left, which is a sub-ring of $\mathrm{End}^l(M)$, and a similar consideration for the second condition.

A bimodule ${}_{\mathsf{R}}M_{\mathsf{S}}$ is called **balanced** if both the homomorphisms λ and ρ above are surjective. If they are isomorphisms, then the bimodule is called **faithfully balanced**. A right S-module P_{S} is called a **progenerator** if it is finitely generated, projective and a generator, i.e., there exist natural numbers m and n, and a module M_{S} such that $M_{\mathsf{S}} \oplus P_{\mathsf{S}} \simeq \mathsf{S}^m$ and $P_{\mathsf{S}}^n = M_{\mathsf{S}} \oplus \mathsf{S}$. A Similar definition holds for left modules.

Given a bimodule ${}_{\mathsf{R}}M_{\mathsf{S}}$, we can construct functors

$$F := {}_{\mathsf{R}}M_{\mathsf{S}} \otimes \cdot\colon {}_{\mathsf{S}}\mathcal{M} \to {}_{\mathsf{R}}\mathcal{M} \text{ and } G := \mathrm{Hom}({}_{\mathsf{R}}M_{\mathsf{S}}, \cdot)\colon {}_{\mathsf{R}}\mathcal{M} \to {}_{\mathsf{S}}\mathcal{M},$$

in the following way: For objects ${}_{\mathsf{S}}N \in {}_{\mathsf{S}}\mathcal{M}$, we define $F({}_{\mathsf{S}}N) := {}_{\mathsf{R}}M_{\mathsf{S}} \otimes {}_{\mathsf{S}}N \in {}_{\mathsf{R}}\mathcal{M}$ with module structure given by $r(x \otimes y) := rx \otimes y$, for $r \in \mathsf{R}$; for morphisms $f\colon {}_{\mathsf{S}}N \to {}_{\mathsf{S}}P$, we define $F(f)\colon {}_{\mathsf{R}}M_{\mathsf{S}} \otimes {}_{\mathsf{S}}N \to {}_{\mathsf{R}}M_{\mathsf{S}} \otimes {}_{\mathsf{S}}P$ by $F(f)(x \otimes y) := x \otimes f(y)$. On the other hand, for objects ${}_{\mathsf{R}}N \in {}_{\mathsf{R}}\mathcal{M}$ we put $G({}_{\mathsf{R}}N) := \mathrm{Hom}({}_{\mathsf{R}}M_{\mathsf{S}}, {}_{\mathsf{R}}N) \in {}_{\mathsf{S}}\mathcal{M}$, with module structure given by $(sf)(x) := f(xs)$, for $s \in \mathsf{S}$; for morphisms $g\colon {}_{\mathsf{R}}N \to {}_{\mathsf{R}}P$, we define $G(g)\colon \mathrm{Hom}({}_{\mathsf{R}}M_{\mathsf{S}}, {}_{\mathsf{R}}N) \to \mathrm{Hom}({}_{\mathsf{R}}M_{\mathsf{S}}, {}_{\mathsf{R}}P)$ by $G(g)(h)(x) := g(h(x))$, for $h \in \mathrm{Hom}({}_{\mathsf{R}}M_{\mathsf{S}}, {}_{\mathsf{R}}N)$ and $x \in M$.

Remark 2.1.3. *(**Notation**): For the sake of simplicity, the sub-indices emphasizing the side and the ring involved in the module structures are going to be omitted when there is no risk of confusion, for example, $\mathrm{Hom}_{\mathsf{R}}(M, N)$, when the bimodule ${}_{\mathsf{R}}M_{\mathsf{S}}$ is fixed, will be interpreted as $\mathrm{Hom}({}_{\mathsf{R}}M_{\mathsf{S}}, {}_{\mathsf{R}}N)$, and $M \otimes_{\mathsf{S}} P$ should be read ${}_{\mathsf{R}}M_{\mathsf{S}} \otimes {}_{\mathsf{S}}P$, etc.*

Now we can state the fundamental result of Morita theory for unital rings.

Theorem 2.1.4. *(Morita) Two unital rings R and S are Morita equivalent if and only if there exist a balanced bimodule ${}_{\mathsf{S}}P_{\mathsf{R}}$ such that ${}_{\mathsf{S}}P$ and P_{R} are progenerators. In such a case, $F := P \otimes_{\mathsf{R}} \cdot\colon {}_{\mathsf{R}}\mathcal{M} \to {}_{\mathsf{S}}\mathcal{M}$ and $G := \mathrm{Hom}_{\mathsf{S}}(P, \cdot)\colon {}_{\mathsf{S}}\mathcal{M} \to {}_{\mathsf{R}}\mathcal{M}$ are inverse equivalences. Moreover,* *if such a bimodule* ${}_{\mathsf{S}}P_{\mathsf{R}}$ *exists, then* $Q := \mathrm{Hom}_{\mathsf{R}}(P, \mathsf{R})$ *has a bimodule structure* ${}_{\mathsf{R}}Q_{\mathsf{S}}$ *with* ${}_{\mathsf{R}}Q$ *and* Q_{S} *being progenerators,* $F \approx \mathrm{Hom}_{\mathsf{R}}(Q, \cdot)$ *and* $G \approx Q \otimes_{\mathsf{S}} \cdot$.

The following corollary is a useful test for equivalence.

Corollary 2.1.5. *Given unital rings* R *and* S*, the following are equivalent,*

1. R *and* S *are Morita equivalent,*
2. *There is a progenerator* P_{R} *such that* $\mathsf{S} \simeq \mathrm{End}(P_{\mathsf{R}})$*,*
3. *There is a progenerator* ${}_{\mathsf{R}}Q$ *such that* $\mathsf{S} \simeq \mathrm{End}({}_{\mathsf{R}}Q)$*.*

For a thorough exposition and proofs of these fundamental concepts see, for example, [2].

Remark 2.1.6. *Note that, if* P_{R} *is a right* R*-module, say via a ring homomorphism* $\rho\colon \mathsf{R} \to \mathrm{End}^r(P)$*, then the set* $\mathrm{End}(P_{\mathsf{R}})$ *of* R*-module homomorphism is exactly the commutant of* R *inside* $\mathrm{End}^r(P)$*, i.e., in the shorthand notation* $f \circ \rho(a) = fa$ *for* $f \in \mathrm{End}^r(P)$ *and* $a \in \mathsf{R}$ *we have*

$$\mathrm{End}(P_{\mathsf{R}}) = \{f \in \mathrm{End}^r(P);\ fa = af\ \forall a \in \mathsf{R}\}.$$

Similar consideration holds for the left R*-module* ${}_{\mathsf{R}}Q$ *and the set* $\mathrm{End}({}_{\mathsf{R}}Q)$*.*

Picard groups

Definition 2.1.7. ***Algebraic Picard group***
Let $(\mathcal{A}, \mu_0)$ *be an associative commutative* R*-algebra. The Picard group* $\mathrm{Pic}(\mathcal{A})$ *is the set of self-equivalence functors* $F\colon {}_{\mathcal{A}}\mathcal{M} \to {}_{\mathcal{A}}\mathcal{M}$*, with composition of functors as group multiplication. By the Morita's characterization theorem, we can also view* $\mathrm{Pic}(\mathcal{A})$ *as the set of* $(\mathcal{A}, \mathcal{A})$*-equivalence bimodules* ${}_{\mathcal{A}}\mathcal{E}_{\mathcal{A}}$*, with the group multiplication given by tensor product of bimodules.*

If $\mathcal{A}$ is commutative, we denote by $\mathrm{Pic}_{\mathcal{A}}(\mathcal{A})$ the group of isomorphism classes of $(\mathcal{A}, \mathcal{A})$-equivalence bimodules ${}_{\mathcal{A}}\mathcal{E}_{\mathcal{A}}$ satisfying $ax = xa$, for all $x \in \mathcal{E}$ and $a \in \mathcal{A}$.

Definition 2.1.8. ***Geometric Picard group***
Given a smooth manifold P*, the Picard group* $\mathrm{Pic}(P)$ *is the set of isomorphism classes of (complex) line bundles* $L \to P$ *with fiber-wise tensor product as group multiplication.*

Example 2.1.9. *Let* $\mathcal{A} = C^\infty(P)$*, then relying on the Serre-Swan's theorem, the (algebraic) Picard group* $\mathrm{Pic}_{C^\infty(P)}(C^\infty(P))$ *can be identified with the (geometric) Picard group* $\mathrm{Pic}(P)$*. Moreover, the Chern map* $c_1\colon \mathrm{Pic}(P) \to \mathbb{H}^2(P, \mathbb{Z})$ *is a group isomorphism [40], where* $\mathbb{H}^2(P, \mathbb{Z})$ *is the second integral de Rham cohomology of* P*. Thus* $\mathrm{Pic}_{C^\infty(P)}(C^\infty(P)) \cong \mathbb{H}^2(P, \mathbb{Z})$*. Moreover, it can be shown (see Example 2.1 of [9]) that* $\mathbb{H}^2(P, \mathbb{Z})$*,* $\mathrm{Pic}(C^\infty(P))$ *and* $\mathrm{Diff}(P)$ *fit in an exact sequence*

$$1 \to \mathbb{H}^2(P, \mathbb{Z}) \to \mathrm{Pic}(P) \to \mathrm{Diff}(P),$$

thus, we obtain the following geometric description of $\mathrm{Pic}(P)$*:*

$$\mathrm{Pic}(P) = \mathrm{Diff}(P) \ltimes \mathbb{H}^2(P, \mathbb{Z}),$$

where the semidirect structure is given by the action of $\mathrm{Diff}(P)$ *on* $\mathbb{H}^2(P, \mathbb{Z})$ *via pull-back.*

2.1.2 Dual pairs and Morita equivalence of Poisson manifolds

A **dual pair** $P_1 \leftarrow S \rightarrow P_2$ consists of a symplectic manifold (S, ω), Poisson manifolds (P_1, π_1), (P_2, π_2) and submersive Poisson maps $J_1 \colon S \to P_1$, $J_2 \colon S \to \overline{P_2}$, where $\overline{P_2} = (P_2, -\pi_2)$, such that the fibres of J_1 and J_2 are mutually symplectically orthogonal. A dual pair is called **full** if the maps J_1 and J_2 are surjective, and **complete** if the maps J_1 and J_2 are complete, i.e., for any $f \in C^\infty(P_i)$ with complete hamiltonian X^i_f, the hamiltonian $X_{J_i^* f}$ is complete as well.

It follows that the algebras $J_1^*(C^\infty(P_1))$ and $J_2^*(C^\infty(P_2))$ are Poisson commutant in $C^\infty(S)$, i.e.,

$$\{J_1^*(C^\infty(P_1)), J_2^*\, C^\infty(P_2)\}_\omega = 0.$$

Example 2.1.10. *Let* (M, ω) *be a symplectic manifold, and let* G *be a Lie group acting on* M *in a regular way (*M/G *is a smooth manifold). Suppose that the action* $\psi \colon G \times M \to M$ *is hamiltonian with a constant rank momentum map* $J \colon M \to \mathfrak{g}^*$*, where* $\mathfrak{g}$ *is the Lie algebra of* G*. Then the fibration given by the level sets of* J *and the one given by orbits of* ψ *are symplectically orthogonal to each other, and the quotient* M/G *inherits from* M *a natural Poisson structure such that the projection map* $p \colon M \to M/G$ *is Poisson. Thus, the following diagram of Poisson maps,*

$$\mathfrak{g}^* \xleftarrow{J} M \xrightarrow{p} M/G,$$

where in $\mathfrak{g}^*$ *we consider the* KKS*-Poisson structure, constitutes a dual pair. It can be seen that, under some regularity conditions, there is a one to one correspondence between the symplectic leaves of* $\mathfrak{g}^*$*, which are known as coadjoint orbits, and those of* M/G*, which are given by symplectic reduction. Moreover, the transverse Poisson structures on corresponding leaves are anti-isomorphic. It follows then that* $\mathfrak{g}^*$ *and* M/G *have closely related Poisson geometry. (See [68] and [16] for details on this example.)*

Example 2.1.11. *Let* $G \rightrightarrows G_0$ *be a symplectic groupoid ([69], [43], [20]), i.e., it is a Lie groupoid [53] endowed with a symplectic structure* ω *on* G *such that graph*(m)*, the graph of the multiplication map* m*, is a lagrangian submanifold of* $G \times \overline{G} \times \overline{G}$*, with the natural simplectic structure on the product manifold, being* $\overline{G} = (G, -\omega)$*. It can be seen (see, for instance, [20]) that there exists a unique Poisson structure* π_0 *on* G_0 *such that the source map* s *is Poisson, and the target map* t *is anti-Poisson. Moreover, they are complete surjective submersion, hence* $(G_0, \pi_0) \xleftarrow{s} (G, \omega) \xrightarrow{t} (G_0, -\pi_0)$ *constitutes a complete full dual pair.*

Definition 2.1.12. *([71, Xu]) Two Poisson manifolds* P_1 *and* P_2 *are said to be* ***Morita equiv-***

alent *if they fit in a complete full dual pair*

$$P_1 \overset{J_1}{\longleftarrow} S \overset{J_2}{\longrightarrow} P_2,$$

with connected and simply connected fibres. Such an S is then called an ***equivalence (symplectic) bimodule****.*

Remark 2.1.13. *In a full dual pair $P_1 \overset{J_1}{\longleftarrow} (S, \omega) \overset{J_2}{\longrightarrow} P_2$, with connected J_1 and J_2 fibers, the condition of symplectic orthogonality of the J_i fibers is equivalent to the condition that the algebras $J_1^*(C^\infty(P_1))$ and $J_2^*(C^\infty(P_2))$ are mutually centralizers of each other, i.e., $J_1^*(C^\infty(P_1))$ is the Poisson commutator of $J_2^*(C^\infty(P_1))$ inside $(C^\infty(S), \{\cdot,\cdot\}_\omega)$, and vice versa [16]. This algebraic characterization is going to be essential for us.*

Example 2.1.14. *[71, Xu] Let S be a connected and simply connected symplectic manifold, and let M be a connected manifold endowed with the zero Poisson structure; then the direct product $S \times M$ is Morita equivalent to M via the symplectic bimodule*

$$S \times M \overset{\rho}{\leftarrow} (X := S \times T^*M) \overset{\sigma}{\rightarrow} M,$$

where $\sigma := pr\colon X \to M$ and $\rho := (id, pr)\colon X \to S \times M$.

The notion of algebraic Picard group (see Definition 2.1.7), as the group of isomorphism classes of self Morita equivalences can be directly translated into the Poisson setting, thus in [9], the authors proposed the following definition (see also [12]):

Definition 2.1.15. *The* ***Picard group*** *of a Poisson manifold (P, π), denoted by $\mathrm{Pic}(P, \pi)$, is the group of isomorphism classes of self Morita equivalence symplectic bimodules.*

In [9], the authors studied the Picard group of some particular examples of Poisson manifolds. Among them, the Picard group of the zero Poisson structure is particularly interesting for us, and the conclusion (see Proposition 6.6 in [9], see also [12]) is the following result:

Theorem 2.1.16. *Any self Morita equivalence symplectic bimodule over a zero Poisson manifold $(P, 0)$ is of the form*

$$(P, 0) \overset{\rho}{\leftarrow} (T^*P, \omega_B) \overset{\psi\circ\rho}{\longrightarrow} (P, 0),$$

where $T^*P \overset{\rho}{\rightarrow} P$ is the cotangent bundle, B is a closed two form on P, $\omega_B := \omega_{can} + \rho^*B$, and $\psi \in \mathrm{Diff}(P)$.

2.1.3 Gauge equivalence

The notion of *gauge equivalence*, introduced in [64], is easier to be understood in the context of Dirac structures. For the details of the following discussion, the reader is referred to Appendix B.

Let P be a smooth manifold. It can be seen that there is a bijection between Poisson tensors π on P and Dirac structures L on P satisfying the condition $L \cap TP = \{0\}$ (see Lemma B.1.5). On the other hand, as stated in Theorem B.2.4, the symmetry group of the standard Courant algebroid $TP \oplus T^*P$ is $\mathrm{Diff}(P) \ltimes \Omega^2_{cl}(P)$, where the sub-group $\Omega^2_{cl}(P)$ acts on $TP \oplus T^*P$ by $(X \oplus \alpha) \xmapsto{\tau_B} (X \oplus \alpha + \iota_X B)$, which is known as **gauge transformation** (also known as *B-field transformation* in the context of generalized complex geometry [36]). In particular, this bundle map τ_B sends Dirac structures to Dirac structures. Moreover, as discussed in Remark B.2.5, the condition $L \cap TP = \{0\}$ is preserved by this map if, and only if, the bundle map $I + B^\sharp \pi^\sharp \colon T^*P \to T^*P$ is invertible. In this situation, the map τ_B yields a transformation in the set of Poisson tensors on P, given by

$$\pi \xmapsto{\tau_B} \tau_B \pi; \quad (\tau_B \pi)^\sharp := \pi^\sharp (I + B^\sharp \pi^\sharp)^{-1},$$

and two Poisson tensors π and π' on P are called **gauge equivalent** if there exists $B \in \Omega^2_{cl}(P)$ such that $\pi' = \tau_B \pi$.

Remark 2.1.17. *The idea behind the gauge transformation is quite simply. Consider adding up the pullback of the closed 2-form B to the leafwise symplectic form induced by π. This yields a new leafwise closed 2-form that, in general, may fail to be nondegenerate. In the case it end up with a leafwise symplectic structure, it is associated to a new global Poisson structure. This new Poisson structure is the gauge transformation of the original one. In particular, suppose π is a symplectic Poisson structure on a manifold P, let us put $\pi^\sharp = (\omega^\sharp)^{-1}$. Then we have*

$$((\tau_B \pi)^\sharp)^{-1} = (I + B^\sharp (\omega^\sharp)^{-1})\omega^\sharp = \omega^\sharp + B^\sharp.$$

Hence, any two symplectic structures on P are gauge equivalent.

The relationship between Morita equivalence and Gauge equivalence of Poisson structures was explored in [7], where the following result was proved (see also [9]).

Theorem 2.1.18. *([9, Th. 7.1]) If two integrable Poisson manifolds (P_1, π_1) and (P_2, π_2) are gauge equivalent, up to a diffeomorphism, then they are Morita equivalent. The converse does not hold in general.*

2.2 Formal deformation theory

Here we recall some definitions and basic concepts related to deformation theory, focusing on formal deformation of associative algebras and of Poisson algebras. We also introduce here some notations and technicalities we are going to need. The exposition here is based on [48].

2.2.1 Formal power series

Let R be a ring (in this work, we will be mainly interested in the case $\mathsf{R} = \mathbb{C}$). A **formal power series** in the formal parameter (indeterminate) λ with coefficients in R is an object of the form $a = \sum_{n=0}^{\infty} \lambda^n a_n$, where $a_n \in \mathsf{R}$, for all n. The set of all such objects will be denoted by R_λ. Two formal power series a and b in R_λ are equal if, by definition, $a_n = b_n$ for all n.

The set R_λ can be given a ring structure by defining, for $a, b \in \mathsf{R}_\lambda$:

$$\begin{aligned} a + b &:= \sum_{n=0}^{\infty} \lambda^n (a_n + b_n), \\ ab &:= \sum_{n=0}^{\infty} \lambda^n \sum_{k=0}^{n} a_k b_{n-k}. \end{aligned} \tag{2.1}$$

If R is unital, with unit 1, then R_λ is unital with unit given by the series with all coefficients equal to $0 \in \mathsf{R}$ except the one at order zero, which is $1 \in \mathsf{R}$. Moreover, if R is commutative then so is R_λ. The following result concerning existence of inverses will be useful for us.

Proposition 2.2.1. *Given $a \in \mathsf{R}_\lambda$, there is an inverse $a^{-1} \in \mathsf{R}_\lambda$ (i.e., $aa^{-1} = a^{-1}a = 1 \in \mathsf{R}_\lambda$), if and only if $a_0 \in \mathsf{R}$ is invertible.*

Proof. Given $a = \sum_{n=0}^{\infty} \lambda^n a_n$, let $a^{-1} = \sum_{n=0}^{\infty} \lambda^n x_n$. Then, the condition $aa^{-1} = 1 \in \mathsf{R}_\lambda$ amounts to the following infinite system of equations:

$$a_0 x_0 = 1; \ \sum_{k=0}^{n} a_k x_{n-k} = 0, \quad n = 1, 2, ...,$$

which can be solved recursively for x_j if and only if $a_0 \in \mathsf{R}$ is invertible. $\square$

The space of coefficients of a formal power series need not be a ring, but can have other algebraic structures; for example, if $\mathcal{M}$ is an R-module, then the space of formal power series $\mathcal{M}_\lambda$ can be given an R_λ-module structure by defining, for $a \in \mathsf{R}_\lambda$ and $x \in \mathcal{M}_\lambda$:

$$ax := \sum_{n=0}^{\infty} \lambda^n \sum_{k=0}^{n} a_k x_{n-k}. \tag{2.2}$$

If $\mathcal{A}$ is an R-algebra, then the operations defined by equations (2.1) and (2.2), for $r \in \mathsf{R}_\lambda$ and $a, b \in \mathcal{A}_\lambda$, turn $\mathcal{A}_\lambda$ into an R_λ-algebra, which is associative and commutative if the algebra $\mathcal{A}$ is so. In the following, we will refer to the structures induced by equations (2.1) and (2.2) as the **undeformed structures**.

Example 2.2.2. *Let M be a manifold, and let $C^\infty(M)$ denote its algebra of complex-valued smooth functions, while $\mathcal{X}^\bullet(M)$ and $\Omega^\bullet(M)$ stand for the $C^\infty(M)$-modules of multivector fields and differential forms, respectively. Then $C^\infty(M)_\lambda$ is a $\mathbb{C}_\lambda$-algebra, while $\mathcal{X}^\bullet(M)_\lambda$ and $\Omega^\bullet(M)_\lambda$ are $C^\infty(M)_\lambda$-modules, all of them with the undeformed structures.*

Consider now two R-modules $\mathcal{M}$ and $\mathcal{N}$, with R being commutative. Then we have the R-module $\operatorname{Hom}_{\mathsf{R}}(\mathcal{M},\mathcal{N})$, while $\operatorname{Hom}_{\mathsf{R}}(\mathcal{M},\mathcal{N})_\lambda$ and $\operatorname{Hom}_{\mathsf{R}_\lambda}(\mathcal{M}_\lambda,\mathcal{N}_\lambda)$ are R_λ-modules. Concerning these structures we have the following basic observation.

Proposition 2.2.3. $\operatorname{Hom}_{\mathsf{R}_\lambda}(\mathcal{M}_\lambda,\mathcal{N}_\lambda) \cong \operatorname{Hom}_{\mathsf{R}}(\mathcal{M},\mathcal{N})_\lambda$.

Proof. Denote by $pr_i \colon \mathcal{N}_\lambda \to \mathcal{N}$ the projection $y \mapsto y_i$, for $y = \sum_{j=0}^\infty \lambda^j y_j \in \mathcal{N}_\lambda$. Now, given an R_λ-linear map $\Phi \colon \mathcal{M}_\lambda \to \mathcal{N}_\lambda$, let $\Phi_0 := \Phi|_{\mathcal{M}}$ and define $\Psi_\Phi := \sum_{j=0}^\infty \lambda^j pr_j \circ \Phi_0$. Since Φ_0 and pr_j are R-linear, we have $\Psi \in \operatorname{Hom}_{\mathsf{R}}(\mathcal{M},\mathcal{N})_\lambda$. Moreover, one can check that, for $\alpha \in \mathsf{R}_\lambda$ and Φ as before, $\Psi_{\alpha\Phi} = \alpha\Psi_\Phi$, thus we get an R_λ-morphism. To conclude it is an isomorphism, consider $\Psi = \sum_{j=0}^\infty \lambda^j \psi_j \in \operatorname{Hom}_{\mathsf{R}}(\mathcal{M},\mathcal{N})_\lambda$ and define $\Phi^\Psi \colon \mathcal{M}_\lambda \to \mathcal{N}_\lambda$ by $\Phi^\Psi(x) = \sum_{j=0}^\infty \lambda^j \sum_{m+n=j} \psi_m(x_n)$. Then one check that Φ^Ψ is R_λ-linear and actually we have $\Phi^{\Psi_\Phi} = \Phi$, thus these constructions are inverse of one another. □

By the previous proposition, we always will consider elements $\Phi \in \operatorname{Hom}_{\mathsf{R}_\lambda}(\mathcal{M}_\lambda,\mathcal{N}_\lambda)$ to be of the form $\Phi = \sum_{j=0}^\infty \lambda^j \phi_j$, with $\phi_j \in \operatorname{Hom}_{\mathsf{R}}(\mathcal{M},\mathcal{N})$. In particular, given $\Phi, \Psi \in \operatorname{End}_{\mathsf{R}_\lambda}(\mathcal{M}_\lambda) \cong \operatorname{End}_{\mathsf{R}}(\mathcal{M})_\lambda$, we have

$$\Phi\Psi = \sum_{j=0}^\infty \lambda^j \sum_{k=0}^j \phi_k \psi_{j-k}.$$

Let $\mathcal{A}$ be an R-algebra, and let $\operatorname{Der}(\mathcal{A})$ denote its space of derivations. Then we can check that the isomorphism $\operatorname{End}_{\mathsf{R}_\lambda}(\mathcal{A}_\lambda) \cong \operatorname{End}_{\mathsf{R}}(\mathcal{A})_\lambda$ restricts to derivations, thus $\operatorname{Der}(\mathcal{A}_\lambda) \cong \operatorname{Der}(\mathcal{A})_\lambda$. Indeed; given $D \in \operatorname{Der}(\mathcal{A}_\lambda)$, we know we can write it in the form $D = \sum_{j=0}^\infty \lambda^j D_j$, with D_j being R-linear. Then the derivation condition, $D(ab) = D(a)b + aD(b)$ for any $a, b \in \mathcal{A}$, forces each D_j to satisfy the same condition, thus $D_j \in \operatorname{Der}(\mathcal{A})$ for each j.

Given an R-algebra $\mathcal{A}$, define the following spaces:

$$\lambda \operatorname{Der}(\mathcal{A}_\lambda) := \{D \in \operatorname{Der}(\mathcal{A}_\lambda); D = \sum_{j=1}^\infty \lambda^j D_j\},$$

$$\operatorname{Aut}_0(\mathcal{A}_\lambda) := \{\phi \in \operatorname{Aut}(\mathcal{A}_\lambda); \phi = I + \sum_{j=1}^\infty \lambda^j \phi_j\}.$$

The next observation will be useful for us.

Proposition 2.2.4. *The map* $\exp \colon \lambda \operatorname{Der}(\mathcal{A}_\lambda) \to \operatorname{Aut}_0(\mathcal{A}_\lambda)$ *given by*

$$\exp(D) := I + \sum_{j=1}^\infty \frac{D^j}{j!},$$

is an isomorphism.

Proof. First of all let us observe that the map is well defined since D starts at order λ, thus, at each order of λ, there are finitely many terms summing in the above formula. Now, one can check that, for a and $b \in \mathcal{A}$, one gets $\exp(D)(ab) = \exp(D)(a)\exp(D)(b)$ by expanding both sides in terms of λ and comparing them order by order. Then, by R_λ-linearity, we conclude that the same holds for elements in $\mathcal{A}_\lambda$. Thus, exp actually maps into $\mathrm{Aut}_0(\mathcal{A}_\lambda)$. Injectivity of this map is clear. To show surjectivity, let $\phi = I + \sum_{j=0}^{\infty} \lambda^j \phi_j$ be an automorphism of $\mathcal{A}_\lambda$. We then want to solve the equation

$$\exp(D) = \phi,$$

where $D = \sum_{j=1}^{\infty} \lambda^j D_j$. Expanding the above equation in terms of λ, we get

$$I + \lambda D_1 + \lambda^2 \left(D_2 + \frac{D_1^2}{2!} \right) + \cdots = I + \lambda\phi_1 + \lambda^2\phi_2 + \cdots,$$

which can be solved recursively, starting with $D_1 := \phi_1$, then $D_2 := \phi_2 - \frac{D_1^2}{2!} = \phi_2 - \frac{\phi_1^2}{2!}$, and so on. The fact that each D_j defined this way is actually a derivation of $\mathcal{A}$ follows from the fact that ϕ is an automorphism of $\mathcal{A}_\lambda$, thus, in particular, for any a and $b \in \mathcal{A}$ we have $\phi(ab) = \phi(a)\phi(b)$, which forces the ${\phi_j}'s$ to satisfy certain equations. For instance, $\phi_1(ab) = \phi_1(a)b + a\phi_1(b)$, thus $\phi_1 \in \mathrm{Der}(\mathcal{A})$, and therefore, $D_1 := \phi_1$ gives us $D_1 \in \mathrm{Der}(\mathcal{A})$. Similarly, one can check that each D_j defined as above gives rise to a derivation of $\mathcal{A}$. $\square$

2.2.2 Deformation of algebras

From now on, we fix $\mathsf{R} = \mathbb{C}$. We want to review here the notion of formal deformation of associative algebras, in the sense of Gerstenhaber [33], and a similar notion regarding Poisson algebras, due initially to Richardson and Nijenhuis [59], introducing this way the concept of formal Poisson structures, which is a fundamental ingredient in this work.

Deformation of associative algebras

Let $(\mathcal{A}, \mu_0)$ be a commutative associative $\mathbb{C}$-algebra with unit. A **formal deformation** of $(\mathcal{A}, \mu_0)$, in the sense of Gerstenhaber [33], is an $\mathbb{C}_\lambda$-bilinear associative product $\mu \colon \mathcal{A}_\lambda \times \mathcal{A}_\lambda \to \mathcal{A}_\lambda$ of the form

$$\mu = \mu_0 + \sum_{j=1}^{\infty} \lambda^j \mu_j.$$

Given two deformed $\mathbb{C}_\lambda$-algebras $(\mathcal{A}_\lambda, \mu)$ and $(\mathcal{B}_\lambda, \eta)$, a morphism between them is an $\mathbb{C}_\lambda$-linear map $\Phi \colon \mathcal{A}_\lambda \to \mathcal{B}_\lambda$ such that $\Phi^*\eta = \mu$, i.e., for all $x, y \in \mathcal{A}_\lambda$:

$$\eta(\Phi(x), \Phi(y)) = \Phi(\mu(x, y)).$$

Definition 2.2.5. *Two formal deformations μ and μ' of μ_0 are* ***equivalent*** *if there exists a morphism $\Phi \colon (\mathcal{A}_\lambda, \mu) \to (\mathcal{A}_\lambda, \mu')$ of the form $\Phi = I + \sum_{j=1}^{\infty} \lambda^j \phi_j$. Such a map Φ is called an* ***equivalence map****.*

We denote by $\mathrm{Def}(\mathcal{A},\mu_0)$ the set of equivalence classes of formal deformations of $(\mathcal{A},\mu_0)$, or simply by $\mathrm{Def}(\mathcal{A})$ if the original product is understood from the context. The set $\mathrm{Aut}(\mathcal{A})$, of automorphisms of $\mathcal{A}$, naturally acts on $\mathrm{Def}(\mathcal{A})$ by

$$\mathrm{Aut}(\mathcal{A}) \times \mathrm{Def}(\mathcal{A}) \to \mathrm{Def}(\mathcal{A});\ (\psi,[\mu]) \mapsto [\psi^*\mu].$$

Star products: Let P be a smooth manifold and let $C^\infty(P)$ be its algebra of complex-valued smooth functions with the usual commutative product $\mu_0(f,g) = fg$, given by pointwise multiplication of functions. A **star-product** $\star$ on P is a formal deformation μ of μ_0 in the previous algebraic sense, i.e.,

$$f \star g = \mu(f,g) = fg + \sum_{i=1}^{\infty} \lambda^i \mu_i(f,g),$$

such that the maps $\mu_i\colon C^\infty(P) \times C^\infty(P) \to C^\infty(P)$ are bidifferential operators. We also require that the constant function $1 \in C^\infty(P)$ is still the unit for $\star$. It can be shown (see [34, Sec. 14]) that any star product is equivalent to one satisfying this last condition. Here, the notion of equivalence is the same as in the algebraic context; indeed, it can be seen that the operators ϕ_i realizing a purely algebraic equivalence as in Definition 2.2.5 must be, in this case, differential operators (see [38] and the references therein).

We denote by $\mathrm{Def}(P)$ the set of equivalence classes of star-products on $C^\infty(P)$, where the product being deformed is the usual commutative pointwise multiplication of functions on P. Consider now an automorphism ψ of $C^\infty(P)$. It can be shown ([1, Th. 4.2.36]) that such an automorphism can be realized by a unique diffeomorphism $\sigma \in \mathrm{Diff}(P)$ via pullback, i.e., $\psi(f) = \sigma^*(f) := f \circ \sigma$. Thus, the group of diffeomorphism $\mathrm{Diff}(P)$ acts on $\mathrm{Def}(P)$ via

$$\mathrm{Diff}(P) \times \mathrm{Def}(P) \to \mathrm{Def}(P);\ (\sigma,[\star]) \mapsto [\star_\sigma], \tag{2.3}$$

where, for all $f,g \in C^\infty(P)$,

$$\sigma^*(f \star_\sigma g) = \sigma^*(f) \star \sigma^*(g).$$

Deformation of Poisson algebras

Let $(\mathcal{A},\cdot,\pi_0)$ be a Poisson algebra, i.e., $(\mathcal{A},\cdot)$ is an associative commutative $\mathbb{C}$-algebra and $\pi_0\colon \mathcal{A}\times\mathcal{A}\to\mathcal{A}$ is a skew-symmetric biderivation satisfying the Jacobi identity, namely, for all $a,b,c \in \mathcal{A}$:

$$\pi_0(a,\pi_0(b,c)) + \pi_0(b,\pi_0(c,a)) + \pi_0(c,\pi_0(a,b)) = 0.$$

Equivalently, a Poisson structure on $(\mathcal{A},\cdot)$ can be defined in term of the Schouten bracket, whose definition we recall here for future references. To do so, define, for $k \geq 1$:

$$\mathcal{X}^k(\mathcal{A}) = \{D\colon \mathcal{A}^{\times k} \to \mathcal{A}; D \text{ is a skew-symmetric multiderivation}\}.$$

Let $\overline{k} := k-1$ and $\overline{\mathcal{X}}^{k-1} := \overline{\mathcal{X}}(\mathcal{A})$, thus $\overline{\mathcal{X}}^{\overline{k}} = \mathcal{X}(\mathcal{A})$.

Definition 2.2.6. *The* ***Schouten bracket*** $[\cdot,\cdot]_S$ *is a product on the space of skew-symmetric multi-derivations*

$$[\cdot,\cdot]_S\colon \overline{\mathcal{X}}^{\bar{k}}(\mathcal{A})\times\overline{\mathcal{X}}^{\bar{l}}(\mathcal{A})\to\mathcal{X}^{\bar{k}+\bar{l}}(\mathcal{A}),$$

given, for $D\in\overline{\mathcal{X}}^{\bar{k}}(\mathcal{A})$*,* $E\in\overline{\mathcal{X}}^{\bar{l}}(\mathcal{A})$ *and* $a_1,...,a_{\bar{k}+\bar{l}+1}\in\mathcal{A}$*, by*

$$\begin{aligned}[D,E]_S(a_1,...,a_{\bar{k}+\bar{l}+1}) := &\sum_{\sigma\in S_{l,\bar{k}}} sgn(\sigma)D(E(a_{\sigma(1)},...,a_{\sigma(l)}),a_{\sigma(l+1)},...,a_{\sigma(l+\bar{k})})\\ &-(-1)^{\bar{k}\ \bar{l}}\sum_{\sigma\in S_{k,\bar{l}}} sgn(\sigma)E(D(a_{\sigma(1)},...,a_{\sigma(k)}),a_{\sigma(k+1)},...,a_{\sigma(k+\bar{l})}),\end{aligned}$$

In terms of the Schouten bracket, a Poisson structure on $(\mathcal{A},\cdot)$ is an element $\pi_0\in\mathcal{X}^2(\mathcal{A})$ such that $[\pi_0,\pi_0]_S=0$ (see [28, Chapter 1] or [48, Chapter 3]).

Formal deformations of the Poisson structure π_0 are defined in a similar way as for the commutative associative product μ_0. More precisely, consider the $\mathbb{C}_\lambda$-algebra $(\mathcal{A}_\lambda,\cdot)$, endowed with the undeformed commutative product. A **formal Poisson deformation** π of π_0 is a skew-symmetric biderivation of $(\mathcal{A}_\lambda,\cdot)$, of the form

$$\pi=\pi_0+\sum_{j=1}^{\infty}\lambda^j\pi_j$$

which satisfies the Jacobi identity. Thus, $(\mathcal{A}_\lambda,\cdot,\pi)$ is a Poisson algebra.

Remark 2.2.7. *Some authors ([44, 29]) define a* ***formal Poisson structure*** *on* $(\mathcal{A},\cdot)$ *as the formal power series* π *given above and satisfying* $[\pi,\pi]_S=0$*, where* $[\cdot,\cdot]_S$ *is the Schouten bracket of multiderivations extended bilinearly to formal power series. It follows then that* π *satisfies the Jacobi identity, and at order zero,* $[\pi_0,\pi_0]_S=0$*; thus,* π_0 *is an ordinary Poisson structure on* $(\mathcal{A},\cdot)$*, and* π *can be seen as a formal deformation of* π_0*, according to the above definition. In the following, we will use both definitions interchangeably.*

Given $D\in\lambda\,\mathrm{Der}(\mathcal{A}_\lambda)$, and π a formal Poisson structure on $\mathcal{A}$, define $\pi':=\exp([D,\cdot]_S)\pi$. It can be shown (see [48, Sec. 13.3.4]) that $[\pi',\pi']_S=0$, thus π' is again a formal Poisson structure on $(\mathcal{A},\cdot)$.

Definition 2.2.8. *Two formal Poisson structures* π *and* π' *on* $(\mathcal{A},\cdot)$ *are said to be* ***equivalent*** *if there exists* $D\in\lambda\,\mathrm{Der}(\mathcal{A}_\lambda)$ *such that* $\pi'=\exp([D,\cdot]_S)\pi$*.*

A straightforward computation shows that two formal Poisson structures π and π' are related by $\pi'=\exp([D,\cdot]_S)\pi$ if and only if

$$\exp(D)\colon(\mathcal{A}_\lambda,\pi)\to(\mathcal{A}_\lambda,\pi')$$

is a Poisson morphism. This, together with Proposition 2.2.4 allow us to give an alternative definition of equivalence for formal Poisson structures, which is more in the spirit of the definition given in the context of deformation of associative products.

Definition 2.2.9. *Two formal Poisson structures π and π' on $(\mathcal{A}, \cdot)$ are said to be equivalent if there exists a Poisson morphism $\Phi\colon (\mathcal{A}_\lambda, \pi) \to (\mathcal{A}_\lambda, \pi')$ of the form $\Phi = I + \sum_{j=1}^{\infty} \lambda^j \phi_j$.*

We denote by $[\pi]$ the equivalence class of π, and by $\mathrm{FPois}(\mathcal{A}, \cdot)$ the set of equivalence classes of formal Poisson structures on $(\mathcal{A}, \cdot)$, or simply by $\mathrm{FPois}(\mathcal{A})$ if the associative product is understood from the context. Moreover, when a Poisson structure is given on the algebra $C^\infty(P)$, where P is a smooth manifold, we set $\mathrm{FPois}(P) := \mathrm{FPois}(C^\infty(P))$.

It follows immediately that two equivalent Poisson structures π and π', viewed as deformations, deform the same Poisson structure $\pi_0 = \pi'_0$. Thus, if a Poisson structure π_0 is fixed on $\mathcal{A}$ we set

$$\mathrm{FPois}(\mathcal{A}, \pi_0) := \{[\pi] \in \mathrm{FPois}(\mathcal{A}); \pi = \pi_0 \pmod{\lambda}\}.$$

Symplectic case: Let (M, ω_0) be a symplectic manifold. It can be shown (see, for instance, [37]) that any Poisson deformation π_ω of the Poisson tensor π_{ω_0} is of the form

$$\pi_\omega(df, dg) = -\omega(X_f, X_g),$$

with $\omega = \omega_0 + \sum_{j=1}^{\infty} \lambda^j \omega_j \in \Omega^2_{cl}(M)_\lambda$, being $X_f \in \mathcal{X}(M)_\lambda$ the hamiltonian formal vector field of $f \in C^\infty(M)_\lambda$, given by $\iota_{X_f}\omega = df$. The following classification result appeared in [49] (see also [37]):

Proposition 2.2.10. *The equivalence classes of Poisson deformations of the Poisson bracket on a symplectic manifold (M, ω_0) are parametrized by $\mathbb{H}^2_{dR}(M)_\lambda$.*

Thus, two Poisson deformations, $\pi = \pi_\omega$ and $\pi' = \pi_{\omega'}$, of the Poisson tensor π_{ω_0} are equivalent if and only if $[\omega] = [\omega'] \in \mathbb{H}^2_{dR}(M)_\lambda$.

2.2.3 Deformation quantization of Poisson manifolds

Consider again an associative $\mathbb{C}$-algebra $(\mathcal{A}, \mu_0)$ and let $\mu = \sum_{i=0}^{\infty} \lambda^i \mu_i$ be a formal deformation of μ_0. Let μ_1^- be the skew-symmetric part of μ_1, i.e., $\mu_1^-(a,b) := \frac{1}{2}(\mu_1(a,b) - \mu_1(b,a))$. The associativity of μ implies that μ_1^- is a Poisson structure on the algebra $(\mathcal{A}, \mu_0)$.

After this observation, let us go back to the situation of a star-product (Section 2.2.2). Suppose we have a Poisson manifold (P, π_0). then it is natural to ask whether or not it is possible to find a star product $\mu = \sum_{i=0}^{\infty} \lambda^i \mu_i$ deforming the natural commutative product μ_0 of $C^\infty(P)$ such that μ_1^- recovers the original Poisson structure. In the affirmative case, one say that such a μ **quantizes** π_0. More than being just mathematically natural, this question is deeply rooted in the physical problem of quantization, which can be traced back to Dirac [26].

Suppose now that $\mu = \sum_{i=0}^{\infty} \lambda^i \mu_i$ and $\mu' = \sum_{i=0}^{\infty} \lambda^i \mu'_i$ are two equivalent star-products on P, i.e., there exists a $\mathbb{C}_\lambda$-linear map $T : C^\infty(P)_\lambda \to C^\infty(P)_\lambda$ of the form $T = I + \sum_{k=1}^{\infty} T_i$ such that

$$T(\mu(f,g)) = \mu'(T(f), T(g)),\ \forall f, g \in C^\infty(P).$$

Expanding both sides of the equation above we find that the coefficients in λ must satisfy

$$\mu_1(f,g) = \mu_1'(f,g) + T_1(f)g + fT_1(g) - T_1(fg). \tag{2.4}$$

Interchanging the role of f and g, we get

$$\mu_1(g,f) = \mu_1'(g,f) + T_1(g)f + gT_1(f) - T_1(gf). \tag{2.5}$$

Thus, subtracting (2.5) from (2.4) we get

$$\mu_1(f,g) - \mu_1(g,f) = \mu_1'(f,g) - \mu_1'(g,f),$$

hence, according to the observation at the beginning of this section, two equivalent star-products on P quantizes the same Poisson structure. Thus, given a Poisson manifold (P,π_0), we define

$$\mathrm{Def}(P,\pi_0) := \{[\star] \in \mathrm{Def}(P); \star \text{ quantizes } \pi_0\}.$$

Let $\mathrm{Diff}_{\pi_0}(P)$ be the set of Poisson diffeomorphism of (P,π_0), then the action (2.3) restricts to an action of $\mathrm{Diff}_{\pi_0}(P)$ on $\mathrm{Def}(P,\pi_0)$.

The problem of deformation quantization of Poisson manifolds addresses the following natural questions:

1. Existence: Given a Poisson manifold (P,π_0), does there exist a star-product $\star$ on P quantizing π_0? i.e., Is $\mathrm{Def}(P,\pi_0)$ non-empty?

2. Classification: How can we parametrize $\mathrm{Def}(P,\pi_0)$?

The first known example of existence was the Moyal product on $(\mathbb{R}^{2n},\omega_{can})$, arising from canonical quantization ([26], [70]). See [67], [37] and the references therein for further information on Moyal Product. The problem of existence was solved for symplectic manifolds by De Wilde-Lecomte [24]. Soon after, also for symplectic manifolds, a more geometrical proof of existence was provided by Fedosov [31], which can be generalized for regular Poisson manifolds as well [30]. The classification problem for symplectic manifolds was solved by Nest-Tsygan [57], Deligne [25] and Bertelson-Cahen-Gutt [4], and the conclusion is that the space of equivalence classes of star-products on a symplectic manifold (M,ω) is parametrized by $\mathbb{H}^2_{dR}(M)_\lambda$. A proof of this result by Čech cohomological methods was given by Gutt-Rawnsley in [38]. For general Poisson manifolds, the problem (of existence and classification) was solved by Kontsevich [44] as a consequence of his Formality theorem. Kontsevich succeeded in constructing a bijection between the set $\mathrm{FPois}_0(P)$ of equivalence classes of formal deformations of the trivial Poisson structure on a manifold P and the set $\mathrm{Def}(P)$ of equivalence classes of star-products on P;

$$\mathcal{K}_*\colon\ \mathrm{FPois}_0(P) \to \mathrm{Def}(P),$$

such that, if $[\pi]$ and $[\mu_\star]$ correspond each other under this bijection, then $2\mu_1^- = \pi_1$. Thus, for a given Poisson manifold (P, π_0), the desired star product on P is any $\mu_\star$ such that $\mathcal{K}_*[\lambda\pi_0] = [\mu_\star]$.

2.3 Morita equivalence and deformation quantization

In this section we will review the principal results achieved, in a series of papers ([8], [5], [6]), by Bursztyn, Dolgushev and Waldmann, which establishes the link alluded to in diagram (1.2) in the introduction.

Classification of Morita equivalent star products: Given a Poisson manifold (P, π_0), let $\mathrm{Pic}(P)$ be its geometric Picard group, i.e.: complex line bundles $L \to P$ with group operation given by tensor product (see Definition 2.1.8). Let $\mathrm{Def}(P)$ be the moduli space of star products on P. In [5], the author showed that a deformation quantization procedure performed on complex line bundles $L \to P$ yields a canonical action

$$\Phi\colon\ \mathrm{Pic}(P) \times \mathrm{Def}(P, \pi_0) \to \mathrm{Def}(P, \pi_0),$$

such that two star-product $\star$ and $\star'$ on (P, π_0) are Morita equivalent if and only if there exists a Poisson diffeomorphism $\psi\colon P \to P$ such that $[\star]$ and $[\star'_\psi]$ lie in the same orbit of Φ. We denote by Φ_L the action map for a given element $L \in \mathrm{Pic}(P)$.

Formal gauge action: Given a manifold P, recall the action of B-fields on Poisson structures on P given by

$$(B, \pi) \mapsto \tau_B\pi; \quad (\tau_B\pi)^\sharp = \pi^\sharp(I + B^\sharp\pi^\sharp)^{-1},$$

provided the bundle map $I + B^\sharp\pi^\sharp\colon T^*P \to T^*P$ is invertible. Next we will see how this idea can be translated into the formal setting.

Let $\pi \in \lambda\mathcal{X}^2(P)_\lambda$ be a formal Poisson structure, and let $B \in \Omega^2_{cl}(P)_\lambda$ be a formal closed 2-form. We have the following $C^\infty(P)_\lambda$-linear maps:

$$\pi^\sharp := \sum_{j=1}^{\infty} \lambda^j \pi_j^\sharp\colon \Omega^1(P)_\lambda \to \lambda\mathcal{X}^1(P)_\lambda,$$

and

$$B^\sharp := \sum_{j=0}^{\infty} \lambda^j B_j^\sharp\colon \mathcal{X}^1(P)_\lambda \to \Omega^1(P)_\lambda.$$

Notice that $B^\sharp\pi^\sharp = \lambda(B_0^\sharp\pi_1^\sharp) + \mathcal{O}(\lambda^2)$, hence the map $I + B^\sharp\pi^\sharp$ is always invertible (see Proposition 2.2.1). Thus, we get a well defined map

$$(B, \pi) \overset{\tau}{\mapsto} \tau_B\pi;\ (\tau_B\pi)^\sharp := \pi^\sharp(I + B^\sharp\pi^\sharp)^{-1}.$$

It was shown in [6] that this map yields an action

$$H^2_{dR}(M,\mathbb{C})_\lambda \times \mathrm{FPois}_0(P) \to \mathrm{FPois}_0(P);\ ([B],[\pi]) \mapsto [\tau_B\pi].$$

The main result in [6] is the following:

Theorem 2.3.1. *Given a complex line bundle $L \to P$ and a formal Poisson structure $[\pi] \in \mathrm{FPois}_0(P)$, we have*

$$\Phi_L(\mathcal{K}_*[\pi]) = \mathcal{K}_*([\tau_B\pi]),$$

where B is a representative of the cohomology class $2\pi i c_1(L)$, being $c_1(L)$ the Chern class of $L \to P$.

2.4 Morita equivalence of formal Poisson manifolds

In this section we propose a definition of Morita equivalence in the realm of formal Poisson manifolds, building upon the algebraic characterization of equivalence symplectic bimodules.

As we saw in Remark 2.1.13, a key feature of an equivalence dual pair, $P \xleftarrow{J_P} (S,\omega_0) \xrightarrow{J_Q} Q$, is the Poisson commutativity condition:

$$\{J_P^*(C^\infty(P)), f\}_{\omega_0} = 0 \Rightarrow f \in J_Q^*(C^\infty(Q) \text{ and } \{g, J_Q^*(C^\infty(Q))\}_{\omega_0} = 0 \Rightarrow g \in J_P^*(C^\infty(P),$$

which says that, at the level of algebras (of functions), if P is Morita equivalent to Q then $C^\infty(P)$ and $C^\infty(Q)$ are mutually centralizers of one another inside $(C^\infty(S), \{\cdot,\cdot\}_{\omega_0})$, and this resembles the similar result concerning Morita equivalence of associative algebras (see Remark 2.1.6).

Now let P_1 and P_2 be two smooth manifolds with corresponding formal Poisson structures $\pi^1 = \sum_{j=0}^\infty \lambda^j\pi^1_j$ and $\pi^2 = \sum_{j=0}^\infty \lambda^j\pi^2_j$. We have seen that, in such a case, π^1_0 and π^2_0 define Poisson structures on P_1 and P_2, respectively, and from now on whenever we refer to the P_i's as Poisson manifolds, or they need to be Poisson manifolds from the context, we will assume that their Poisson structures are these zeroth order terms of their formal structures.

Definition 2.4.1. *A* ***formal dual pair*** *is a diagram of the form*

$$(C^\infty(P_1)_\lambda, \pi^1) \xrightarrow{\Phi^1} (C^\infty(S)_\lambda, \omega) \xleftarrow{\Phi^2} (C^\infty(P_2)_\lambda, \pi^2),$$

where; (P_i,π^i), for $i = 1,2$, are formal Poisson manifolds; $\omega = \sum_{j=0}^\infty \lambda^j\omega_j \in \Omega^2_{cl}(S)_\lambda$, with ω_0 symplectic; Φ^1 and Φ^2 are Poisson and anti-Poisson morphisms, respectively, such that $\Phi^1(C^\infty(P_1)_\lambda)$ and $\Phi^2(C^\infty(P_2)_\lambda)$ are mutually centralizers inside $(C^\infty(S)_\lambda, \{\cdot,\cdot\}_\omega)$.

We now propose the following definition of Morita equivalence for formal Poisson manifolds.

Definition 2.4.2. *Two formal Poisson manifolds, (P_i,π^i), for $i = 1,2$, are* ***Morita equivalent*** *if:*

1. *there exist a symplectic manifold* (S, ω_0) *such that*

$$(C^\infty(P_1)_\lambda, \pi^1) \xrightarrow{\Phi^1} (C^\infty(S)_\lambda, \omega) \xleftarrow{\Phi^2} (C^\infty(P_2)_\lambda, \pi^2),$$

is a formal dual pair, for some $\omega = \omega_0 + \sum_{j=1}^{\infty} \lambda^j \omega_j \in \Omega^2_{cl}(S)_\lambda$, *and*

2. *in the classical limit,* (S, ω_0) *is an equivalence symplectic bimodule between* (P_1, π^1_0) *and* (P_2, π^2_0).

It is not easy to provide examples of Morita equivalent formal Poisson manifolds. However, in this work we will give a general way of constructing them in the case of deformation of the zero structure. To properly state this result we need to go through some technicalities, yet we may give here an intuitive explanation. To do so, let us begin by recalling an example in the undeformed case.

Let $(P_1, \pi_1) \xleftarrow{J_1} (S, \omega) \xrightarrow{J_2} (P_2, \pi_2)$ be a Morita equivalence symplectic bimodule, then, it follows from the work of Frejlich and Mărcuţ in [32] that the single relation

$$\tau_\omega(J_1^* L_{\pi_1}) = J_2^* L_{\pi_2}, \tag{2.6}$$

between the backward Dirac images $J_i^* L_{\pi_i}$, $i = 1, 2$, codifies all the algebraic properties characterizing an equivalence symplectic bimodule. If $B \in \Omega^2_{cl}(P_2)$, essentially by equivariance of backward Dirac images with respect to B-field actions, we get $\tau_{(\omega + J_2^* B)}(J_1^* L_{\pi_1}) = J_2^* L_{\tau_B \pi_2}$. Thus, if $\tau_B \pi_2$ happens to be Poisson, we get the new Morita equivalence symplectic bimodule $(P_1, \pi_1) \xleftarrow{J_1} (S, \omega + J_2^* B) \xrightarrow{J_2} (P_2, \tau_B \pi_2)$.

Let us now consider an element $[\pi] \in \mathrm{FPois}_0(P)$, and suppose we have a self-equivalence formal dual pair

$$(C^\infty(P)_\lambda, \pi) \xrightarrow{\Phi^1} (C^\infty(S)_\lambda, \omega) \xleftarrow{\Phi^2} (C^\infty(P)_\lambda, \pi). \tag{2.7}$$

It follows from Theorem 2.1.16 and the condition on the classical limit Definition 2.4.2 that the manifold S must be diffeomorphic to T^*P, and ω must be a deformation of $\omega_0 = \omega_{can} + \rho^* B_0$, for some $B_0 \in \Omega^2_{cl}(P)$ and $\rho \colon T^*P \to P$ being the bundle projection. We will adapt into the formal context all the ingredients of the previous example, namely; the structures L_{π_i} and their backward transformation $\rho^* L_{\pi_i}$; the equivariance of the backward transformation with respect to the action of formal B-fields, and a formal version of relation (2.6), codifying the dual pair condition in Definition 2.4.2. This allows us to get, from the equivalence dual pair (2.7), another formal dual pair as follows

$$(C^\infty(P)_\lambda, \pi) \xrightarrow{\Psi^1} (C^\infty(S)_\lambda, \omega + \rho^* B) \xleftarrow{\Psi^2} (C^\infty(P)_\lambda, \tau_B \pi),$$

with $B \in \Omega^2_{cl}(P)_\lambda$. Hence, $\tau_B \pi$ is Morita equivalent to π, for any $B \in \Omega^2_{cl}(P)_\lambda$. Moreover, and this is the main result of this work, we will show that in this way (modulo a natural action of $\mathrm{Diff}(P)$ on $\mathrm{FPois}_0(P)$) we exhaust all formal Poisson structures $[\pi'] \in \mathrm{FPois}_0(P)$ which are Morita equivalent to a given one.

CHAPTER 3

Deformation of Poisson morphisms - Cohomological results

3.1 Deformation of Poisson morphisms

Let $\mathcal{A}$ and $\mathcal{B}$ be Poisson algebras over $\mathbb{C}$, and let

$$\phi_0 \colon (\mathcal{A}, \{\cdot,\cdot\}_{\mathcal{A}}) \to (\mathcal{B}, \{\cdot,\cdot\}_{\mathcal{B}}),$$

be a Poisson morphism.

Definition 3.1.1. *Given formal Poisson deformations $\pi^{\mathcal{A}} = \sum_{j=0}^{\infty} \lambda^j \pi_j^{\mathcal{A}}$, of $\pi_0^{\mathcal{A}} = \{\cdot,\cdot\}_{\mathcal{A}}$, and $\pi^{\mathcal{B}} = \sum_{j=0}^{\infty} \lambda^j \pi_j^{\mathcal{B}}$, of $\pi_0^{\mathcal{B}} = \{\cdot,\cdot\}_{\mathcal{B}}$, a formal Poisson deformation of the Poisson morphism $\phi_0 \colon \mathcal{A} \to \mathcal{B}$, with respect to $\pi^{\mathcal{A}}$ and $\pi^{\mathcal{B}}$, is a Poisson morphism of the form*

$$\Phi = \sum_{j=0}^{\infty} \lambda^j \phi_j \colon (\mathcal{A}_\lambda, \pi^{\mathcal{A}}) \to (\mathcal{B}_\lambda, \pi^{\mathcal{B}}).$$

Let $(S, \omega_0) \xrightarrow{J} (P, \pi_0)$ be a symplectic realization. Then let π be a formal Poisson deformation of π_0 and σ be a formal Poisson deformation of π_{ω_0}.

Definition 3.1.2. *A formal deformation of the symplectic realization $(S, \omega_0) \xrightarrow{J} (P, \pi_0)$, with respect to π and σ, is a formal Poisson deformation of the Poisson morphism*

$$J^* \colon (C^\infty(P), \pi_0) \to (C^\infty(S), \pi_{\omega_0}),$$

according to Definition 3.1.1.

We are interested in studying the following deformation problem:

Problem 1: *Given a Poisson morphism $\phi_0\colon (\mathcal{A}, \{\cdot,\cdot\}_{\mathcal{A}}) \to (\mathcal{B}, \{\cdot,\cdot\}_{\mathcal{B}})$ and formal Poisson deformations $\pi^{\mathcal{A}}$ of $\pi_0^{\mathcal{A}} := \{\cdot,\cdot\}_{\mathcal{A}}$, and $\pi^{\mathcal{B}}$ of $\pi_0^{\mathcal{B}} := \{\cdot,\cdot\}_{\mathcal{B}}$, find a Poisson deformation Φ of ϕ_0 with respect to $\pi^{\mathcal{A}}$ and $\pi^{\mathcal{B}}$, according to Definition 3.1.1.*

In the following section we will identify a cohomology controlling this deformation problem, and the main result in this chapter may be summarized as follows:

Main Result: *Triviality of the relevant cohomology allows us to solve Problem 1 and to establish a formal dual pair relation*

$$(\mathcal{A}'_\lambda, \pi') \xrightarrow{\Psi} (\mathcal{B}_\lambda, \pi^{\mathcal{B}}) \xleftarrow{\Phi} (\mathcal{A}_\lambda, \pi),$$

where $\mathcal{A}'$ is the Poisson commutator of $\phi_0(\mathcal{A})$ inside $\mathcal{B}$.

This is the outcome of propositions 3.2.12, 3.2.13 and 3.2.18.

3.2 Cohomological results

In this section we develop some cohomological tools and then we apply it to study Problem 1. Here we follow closely some private communications of Stefan Waldmann with Henrique Bursztyn [66].

3.2.1 A Chevalley-Eilenberg cohomology

Let $\phi_0\colon (\mathcal{A}, \{\cdot,\cdot\}_{\mathcal{A}}) \to (\mathcal{B}, \{\cdot,\cdot\}_{\mathcal{B}})$ be a Poisson morphism, as above. Recall that $(\mathrm{End}_{\mathbb{C}}(\mathcal{B}), [\cdot,\cdot])$, where $\mathrm{End}_{\mathbb{C}}(\mathcal{B})$ is the endomorphism group of $\mathcal{B}$ as a $\mathbb{C}$-vector space and $[\cdot,\cdot]$ is the commutator, is a Lie algebra. Then the map

$$(\mathcal{A}, \{\cdot,\cdot\}_{\mathcal{A}}) \to (\mathrm{End}_{\mathbb{C}}(\mathcal{B}), [\cdot,\cdot]);\ a \mapsto \{\phi_0(a), \cdot\}_{\mathcal{B}},$$

is a Lie algebra homomorphism, and thus we can see $\mathcal{B}$ as a left Lie algebra module over $(\mathcal{A}, \{\cdot,\cdot\}_{\mathcal{A}})$. Hence we consider the following Chevalley-Eilenberg complex: for $k = 0$ set $C^0_{CE}(\mathcal{A}, \mathcal{B}) := \mathcal{B}$, and for $k \in \mathbb{N}$, the k-cochains are

$$C^k_{CE}(\mathcal{A}, \mathcal{B}) := \{D : \mathcal{A}^{\times k} \to \mathcal{B};\ D \text{ is } \mathbb{C}\text{-multilinear and antisymmetric}\},$$

with differential δ given, for $D \in C^k_{CE}(\mathcal{A}, \mathcal{B})$ and $a_1, ..., a_{k+1} \in \mathcal{A}$, by

$$\begin{aligned}\delta D(a_1, ..., a_{k+1}) := &\sum_{j=1}^{k+1} (-1)^{j+1} \{\phi_0(a_j), D(a_1, ..., \overset{j}{\wedge}, ..., a_{k+1})\}_{\mathcal{B}} \\ &+ \sum_{i<j} (-1)^{i+j} D(\{a_i, a_j\}_{\mathcal{A}}, a_1, ..., \overset{i}{\wedge}, ..., \overset{j}{\wedge}, ..., a_{k+1}).\end{aligned}$$

The corresponding cohomology will be denoted by $H^\bullet_{CE}(\mathcal{A},\mathcal{B})$.

Remark 3.2.1. *The spaces $C^k_{CE}(\mathcal{A},\mathcal{B})$ can be given the structure of a $\mathcal{B}$-module, say from the left, via*

$$(bD)(a_1,...,a_k) := bD(a_1,...,a_k),$$

for $b\in\mathcal{B}$, $D\in C^k_{CE}(\mathcal{A},\mathcal{B})$ and $a_i\in\mathcal{A}$. Using ϕ_0, we can also give it a left $\mathcal{A}$-module structure, via

$$(aD)(a_1,...,a_k) := \phi_0(a)D(a_1,...,a_k),$$

where $a\in\mathcal{A}$. Notice that, in general, the differential is not compatible with these module structures. However, as we will see later, in a geometric case of interest for us, the differential will be $\mathcal{A}$-linear.

Remark 3.2.2. *This cohomology was considered by Richardson and Nijenhuis in [59] for the study of deformation of Lie structures, and then applied in [58], by the same authors, to study the deformation of Lie algebra and Lie group morphisms. In [59], given a vector space V they showed that $\mathrm{Alt}(V) := \oplus_n \mathrm{Alt}^n(V)$, where $\mathrm{Alt}^n(V)$ is the space of all alternating $(n+1)$-linear maps of V into itself, can be endowed with the structure of a differential graded Lie algebra and how this structure is related to the deformation problem they were working with. There, they mainly deal with finite dimensional Lie algebras. In the following, we will introduce a subcomplex of the above complex, in order to take care of the derivation property of the Poisson brackets.*

From now on we will focus on a special class of cochains, namely, those which are multiderivations of the associative product along ϕ_0, i.e., for $D\in C^k_{CE}(\mathcal{A},\mathcal{B})$, we ask for the following condition to hold:

$$D(a_1,...,a_lb_l,....,a_k) = \phi_0(a_l)D(a_1,...,b_l,...,a_k) + D(a_1,...,a_l,...,a_k)\phi_0(b_l), \tag{3.1}$$

for $a_1,...,a_k,b_l\in\mathcal{A}$ and $l=1,...,k$. Let $C^k_{CE,der}(\mathcal{A},\mathcal{B})$ denote the subset of $C^k_{CE}(\mathcal{A},\mathcal{B})$ satisfying (3.1).

Remark 3.2.3. *For the Poisson morphism induced by a symplectic realization $(S,\omega_0)\xrightarrow{J}(P,\pi_0)$, these multiderivation cochains correspond to multivector fields on P along J, which are sections of the pullback bundle $J^*\bigwedge^\bullet(TP)$ (see Proposition 4.1.3 in Section 4.1).*

Lemma 3.2.4. *$C^\bullet_{CE,der}(\mathcal{A},\mathcal{B})$ is a subcomplex of $C^\bullet_{CE}(\mathcal{A},\mathcal{B})$ with respect to δ.*

Proof. We need to show that for $D\in C^k_{CE,der}(\mathcal{A},\mathcal{B})$, the cochain δD is still a multiderivation. Since δD is already antisymmetric, we only need to check that it satisfies the Leibniz rule, say for, the first argument. Thus we compute:

$$
\begin{aligned}
\delta D(ab, a_2, ..., a_{k+1}) = &\{\phi_0(ab), D(a_2, ..., a_{k+1})\}_{\mathcal{B}} + \sum_{j=2}^{k+1}(-1)^{j+1}\{\phi_0(a_j), D(ab, a_2, ..., \overset{j}{\wedge}, ..., a_{k+1})\}_{\mathcal{B}} \\
&+ \sum_{j=2}^{k+1}(-1)^{j+1}D(\{ab, a_j\}_{\mathcal{A}}, a_2, ..., \overset{j}{\wedge}, ..., a_{k+1}) \\
&+ \sum_{i<j}(-1)^{i+j}D(\{a_i, a_j\}_{\mathcal{A}}, ab, ..., \overset{i}{\wedge}, ..., \overset{j}{\wedge}, ..., a_{k+1}).
\end{aligned}
$$

Then we use the fact that ϕ_0 is a morphism of the associative product, and the Leibniz rule for the bracket and for D to expand over the product ab and get

$$
\begin{aligned}
\delta D(ab, a_2, ..., a_{k+1}) = &\phi_0(a)\{\phi_0(b), D(a_2, ..., a_{k+1})\}_{\mathcal{B}} + \phi_0(b)\{\phi_0(a), D(a_2, ..., a_{k+1})\}_{\mathcal{B}} \\
&+ \sum_{j=2}^{k+1}(-1)^{j+1}\{\phi_0(a_j), \phi_0(a)D(b, a_2, ..., \overset{j}{\wedge}, ..., a_{k+1}) \\
&\qquad\qquad + \phi_0(b)D(a, a_2, ..., \overset{j}{\wedge}, ..., a_{k+1})\}_{\mathcal{B}} \\
&+ \sum_{j=2}^{k+1}(-1)^{j+1}D(a\{b, a_j\}_{\mathcal{A}} + b\{a, a_j\}_{\mathcal{A}}, a_2, ..., \overset{j}{\wedge}, ..., a_{k+1}) \\
&+ \sum_{i<j}(-1)^{i+j}\phi_0(a)D(\{a_i, a_j\}_{\mathcal{A}}, b, ..., \overset{i}{\wedge}, ..., \overset{j}{\wedge}, ..., a_{k+1}) \\
&+ \sum_{i<j}(-1)^{i+j}\phi_0(b)D(\{a_i, a_j\}_{\mathcal{A}}, a, ..., \overset{i}{\wedge}, ..., \overset{j}{\wedge}, ..., a_{k+1}).
\end{aligned}
$$

Applying the Leibniz rule and multilinearity of the brackets and of the cochain D, and the fact that ϕ_0 is a Poisson morphism we get

$$
\begin{aligned}
\delta D(ab, a_2, ..., a_{k+1}) = &\phi_0(a)\{\phi_0(b), D(a_2, ..., a_{k+1})\}_{\mathcal{B}} \\
&+ \phi_0(a)\sum_{j=2}^{k+1}(-1)^{j+1}\{\phi_0(a_j), D(b, a_2, ..., \overset{j}{\wedge}, ..., a_{k+1})\}_{\mathcal{B}} \\
&+ \sum_{j=2}^{k+1}(-1)^{j+1}\phi_0(\{a_j, a\}_{\mathcal{A}})D(b, a_2, ..., \overset{j}{\wedge}, ..., a_{k+1}) \\
&+ \phi_0(a)\sum_{j=2}^{k+1}(-1)^{j+1}D(\{b, a_j\}_{\mathcal{A}}, a_2, ..., \overset{j}{\wedge}, ..., a_{k+1}) \\
&+ \sum_{j=2}^{k+1}(-1)^{j+1}\phi_0(\{a, a_j\}_{\mathcal{A}})D(b, a_2, ..., \overset{j}{\wedge}, ..., a_{k+1}) \\
&+ \phi_0(a)\sum_{i<j}(-1)^{i+j}D(\{a_i, a_j\}_{\mathcal{A}}, b, ..., \overset{i}{\wedge}, ..., \overset{j}{\wedge}, ..., a_{k+1}) + (a \leftrightarrow b),
\end{aligned}
$$

where $(a \leftrightarrow b)$ means that we have again the entire expression with a and b interchanged. Observe now that the third and fifth lines cancel out, due to antisymmetry of the Poisson bracket, hence

$$\begin{aligned}
\delta D(ab, a_2, ..., a_{k+1}) = \phi_0(a)\Big(&\{\phi_0(b), D(a_2, ..., a_{k+1})\}_{\mathcal{B}} \\
&+ \sum_{j=2}^{k+1}(-1)^{j+1}\{\phi_0(a_j), D(b, a_2, ..., \overset{j}{\wedge}, ..., a_{k+1})\}_{\mathcal{B}} \\
&+ \sum_{j=2}^{k+1}(-1)^{j+1}D(\{b, a_j\}_{\mathcal{A}}, a_2, ..., \overset{j}{\wedge}, ..., a_{k+1}) \\
&+ \sum_{i<j}(-1)^{i+j}D(\{a_i, a_j\}_{\mathcal{A}}, b, ..., \overset{i}{\wedge}, ..., \overset{j}{\wedge}, ..., a_{k+1})\Big) \\
&+ (a \leftrightarrow b). \\
&= \phi_0(a)(\delta D)(b, a_2, ..., a_k) + \phi_0(b)(\delta D)(a, a_2, ..., a_k).
\end{aligned}$$

Hence, δD is a multiderivation cochain, as desired. $\square$

The corresponding cohomology will be denoted by $H^\bullet_{CE,der}(\mathcal{A}, \mathcal{B})$. Note that the inclusion at the level of cochains induces an inclusion $H^\bullet_{CE,der}(\mathcal{A}, \mathcal{B}) \to H^\bullet_{CE}(\mathcal{A}, \mathcal{B})$.

Consider now a second Poisson morphism $\psi\colon \mathcal{B} \to \mathcal{C}$. Thus $\psi \circ \phi_0\colon \mathcal{A} \to \mathcal{C}$ is also a Poisson morphism, so we may consider the complex $C^\bullet_{CE}(\mathcal{A}, \mathcal{C})$. Define, for each $k = 0, 1, ...$, the following map:

$$\phi_0^*\colon C^k_{CE}(\mathcal{B}, \mathcal{C}) \to C^k_{CE}(\mathcal{A}, \mathcal{C}); \quad \phi_0^* D(a_1, ..., a_k) := D(\phi_0(a_1), ..., \phi_0(a_k)),$$

where $D \in C^k_{CE}(\mathcal{B}, \mathcal{C})$ and $a_1, ..., a_k \in \mathcal{A}$.

Lemma 3.2.5. *The map $\phi_0^*\colon C^\bullet_{CE}(\mathcal{B}, \mathcal{C}) \to C^\bullet_{CE}(\mathcal{A}, \mathcal{C})$ is a chain map. It restricts to a chain map*

$$\phi_0^*\colon C^\bullet_{CE,der}(\mathcal{B}, \mathcal{C}) \to C^\bullet_{CE,der}(\mathcal{A}, \mathcal{C}).$$

Proof. Take $D \in C^k_{CE}(\mathcal{B}, \mathcal{C})$ and $a_1, ..., a_{k+1} \in \mathcal{A}$, then compute:

$$\begin{aligned}
(\phi_0^*\delta D)(a_1, ..., a_{k+1}) &= (\delta D)(\phi_0(a_1), ..., \phi_0(a_{k+1})) \\
&= \sum_{j=1}^{k+1}(-1)^{j+1}\{\psi(\phi_0(a_j)), D(\phi_0(a_1), ..., \overset{i}{\wedge}, ..., \phi_0(a_{k+1}))\}_{\mathcal{C}} \\
&+ \sum_{i<j}(-1)^{i+j}D(\{\phi_0(a_i), \phi_0(a_j)\}_{\mathcal{B}}, \phi_0(a_1), ..., \overset{i}{\wedge}, ..., \overset{j}{\wedge}, ..., \phi_0(a_{k+1})).
\end{aligned}$$

On the other hand, we have

$$
\begin{aligned}
(\delta\phi_0^* D)(a_1,...,a_{k+1}) &= \sum_{j=1}^{k+1}(-1)^{j+1}\{\psi\circ\phi_0(a_j), \phi_0^* D(a_1,...,\overset{i}{\wedge},...,a_{k+1})\}_{\mathcal{C}} \\
&+ \sum_{i<j}(-1)^{i+j}\phi_0^* D(\{a_i,a_j\}_{\mathcal{A}}, a_1,...,\overset{i}{\wedge},...,\overset{j}{\wedge},...,a_{k+1}) \\
&= \sum_{j=1}^{k+1}(-1)^{j+1}\{\psi\circ\phi_0(a_j), D(\phi_0(a_1),...,\overset{i}{\wedge},...,\phi_0(a_{k+1}))\}_{\mathcal{C}} \\
&+ \sum_{i<j}(-1)^{i+j} D(\{\phi_0(a_i),\phi_0(a_j)\}_{\mathcal{B}}, \phi_0(a_1),...,\overset{i}{\wedge},...,\overset{j}{\wedge},...,\phi_0(a_{k+1})).
\end{aligned}
$$

Thus, $\delta\phi_0^* = \phi_0^*\delta$, so ϕ_0^* is a chain map. The second statement follows from similar computations, taking into account that ϕ_0 is a morphism of the associative product. $\square$

Now we restrict our attention to pulling-back $\mathcal{B}$-valued multiderivations of $\mathcal{B}$, via ϕ_0. Thus, we consider $\mathcal{C} = \mathcal{B}$ and $\psi = I$, hence we get a chain map

$$\phi_0^* \colon C^\bullet_{CE,der}(\mathcal{B},\mathcal{B}) \to C^\bullet_{CE,der}(\mathcal{A},\mathcal{B}).$$

Definition 3.2.6. *(**Horizontal lift**) A horizontal lift along $\phi_0 \colon \mathcal{A} \to \mathcal{B}$ is a $\mathbb{C}$-linear map*

$$
\begin{aligned}
h \colon C^\bullet_{CE}(\mathcal{A},\mathcal{B}) &\to C^\bullet_{CE}(\mathcal{B},\mathcal{B}) \\
D &\mapsto D^h,
\end{aligned}
$$

such that, for all $a \in \mathcal{A}$ and $D \in C^\bullet_{CE}(\mathcal{A},\mathcal{B})$,

$$\phi_0^* D^h = D \text{ and } \quad \phi_0(a)D^h = (\phi_0(a)D)^h.$$

Remark 3.2.7. *Note that the existence of a horizontal lift implies, in particular, that ϕ_0^* is a surjective map. For $k = 0$, we have $C^0_{CE}(\mathcal{B},\mathcal{B}) = \mathcal{B} = C^0_{CE}(\mathcal{A},\mathcal{B})$, and $\phi_0^* = I \colon \mathcal{B} \to \mathcal{B}$. It follows from the first property in* (3.2.6) *that $b^h = b$, for all $b \in \mathcal{B}$.*

It follows from the definition that two horizontal lifts h_1 and h_2 differ by an element in the kernel of ϕ_0^*. We introduce the following:

Definition 3.2.8. *(**Vertical cochains**) A cochain $D \in C^\bullet_{CE,der}(\mathcal{B},\mathcal{B})$ is called vertical with respect to $\phi_0 \colon \mathcal{A} \to \mathcal{B}$ if*

$$\phi_0^* D = 0.$$

3.2.2 A deformation problem

For the reader convenience, let us recall here Problem 1.

<u>Problem 1:</u> *Given a Poisson morphism* $\phi_0\colon (\mathcal{A}, \{\cdot,\cdot\}_\mathcal{A}) \to (\mathcal{B}, \{\cdot,\cdot\}_\mathcal{B})$ *and formal Poisson deformations* $\pi^\mathcal{A}$ *of* $\pi_0^\mathcal{A} := \{\cdot,\cdot\}_\mathcal{A}$*, and* $\pi^\mathcal{B}$ *of* $\pi_0^\mathcal{B} := \{\cdot,\cdot\}_\mathcal{B}$*, find a Poisson deformation* Φ *of* ϕ_0 *with respect to* $\pi^\mathcal{A}$ *and* $\pi^\mathcal{B}$*, according to Definition 3.1.1.*

Remark 3.2.9. *For the geometrical context we are interested in, the deformations of* ϕ_0 *which are of geometric interest are given by formal diffeomorphisms, i.e., they are of the form*

$$\Phi = \exp(X)\phi_0,$$

where $X \in \lambda \operatorname{Der}(\mathcal{B})_\lambda$*. To simplify notation, we put* $\pi := \pi^\mathcal{A}$ *and* $\sigma := \pi^\mathcal{B}$*. Thus, we want to find* $X = \sum_{j=1}^\infty \lambda^j X_j$ *with* $X_j \in \operatorname{Der}(\mathcal{B})$ *such that, for all* $a, b \in \mathcal{A}$*, the map* $\Phi := \exp(X)\phi_0$ *satisfies*

$$\Phi(\pi(a,b)) = \sigma(\Phi(a), \Phi(b)). \tag{3.2}$$

Notice that, for Φ *to be a Poisson morphism, the condition* $\Phi(ab) = \Phi(a)\Phi(b)$ *must also be fulfilled, but this will be the case, since* ϕ_0 *is already a morphism of the associative product, and* $\exp(X) \in \operatorname{Aut}_0(\mathcal{B})$ *(see Proposition 2.2.4).*

Obstruction for the iterative construction

Suppose we have found $X_{(k)}$ such that $\Phi_{(k)} := \exp(X_{(k)})$ satisfies equation (3.2) up to order k, i.e., we have:

$$\Phi_{(k)}^{-1}\sigma\big(\Phi_{(k)}\phi_0(a), \Phi_{(k)}\phi_0(b)\big) = \phi_0(\pi(a,b)) + \lambda^{k+1}R_{k+1}(a,b) + \cdots \tag{3.3}$$

for all $a, b \in \mathcal{A}$, and some $\mathbb{C}$-bilinear map $R_{k+1}\colon \mathcal{A} \times \mathcal{A} \to \mathcal{B}$.

Note that for $k = 0$ we do have a solution: $X_{(0)} = 0$, thus $\Phi_{(0)} = I_\mathcal{B}$ and we have

$$R_1(a,b) = \sigma_1\big(\phi_0(a), \phi_0(b)\big) - \phi_0(\pi_1(a,b)).$$

Lemma 3.2.10. *Assume we have found* $X_{(k)}$ *such that* (3.3) *holds. Then the map* R_{k+1} *satisfies:*

1. $R_{k+1} \in C^2_{CE,der}(\mathcal{A}, \mathcal{B})$,
2. $\delta R_{k+1} = 0$.

Proof. For the first item, we see that since π and σ are both antisymmetric in each order of λ, this also holds for R_{k+1}. Thus, it left to check the Leibniz rule for R_{k+1}. For this recall that σ and π satisfy the Leibniz rule and $\Phi_{(k)}$ is an automorphism of $\mathcal{B}_\lambda$, thus the left hand side in (3.3) also satisfies the Leibniz rule. Therefore, the same holds for R_{k+1}. It remains to check the closedness of R_{k+1}. Let $a, b, c \in \mathcal{A}$, and compute:

$$
\begin{aligned}
\Phi_{(k)}^{-1}\sigma\Big(\Phi_{(k)}\phi_0(a),\sigma\big(\Phi_{(k)}\phi_0(b),\Phi_{(k)}\phi_0(c)\big)\Big) &= \Phi_{(k)}^{-1}\sigma\Big(\Phi_{(k)}\phi_0(a),\Phi_{(k)}\big(\phi_0(\pi(b,c)) \\
&\qquad\qquad + \lambda^{k+1}R_{k+1}(b,c)+\cdots\big)\Big) \\
&= \Phi_{(k)}^{-1}\sigma\big(\Phi_{(k)}\phi_0(a),\Phi_{(k)}\phi_0(\pi(b,c))\big) \\
&+ \Phi_{(k)}^{-1}\sigma\Big(\Phi_{(k)}\phi_0(a),\Phi_{(k)}\big(\lambda^{k+1}R_{k+1}(b,c)+\cdots\big)\Big) \\
&= \phi_0\big(\pi(a,\pi(b,c))\big)+\lambda^{k+1}R_{k+1}(a,\pi(b,c))+\cdots \\
&+ \Phi_{(k)}^{-1}\sigma\Big(\Phi_{(k)}\phi_0(a),\Phi_{(k)}(\lambda^{k+1}R_{k+1}(b,c)+\cdots)\Big),
\end{aligned}
$$

where we used the defining equation of R_{k+1}, (3.3), for the first and last equality. Now taking the cyclic sum over a, b, c, using the Jacobi identity for π and σ (recall that $\Phi_{(k)}$ is an automorphism, thus, it turns σ into a formal Poisson tensor again), and the fact that $\Phi_{(k)}$ is the identity at zeroth order, we get

$$
\begin{aligned}
0 &= \lambda^{k+1}\Big(R_{k+1}\big(a,\pi(b,c)\big)+R_{k+1}\big(b,\pi(c,a)\big)+R_{k+1}\big(c,\pi(a,b)\big)\Big) \\
&+ \lambda^{k+1}\Big(\sigma\big(\phi_0(a),R_{k+1}(b,c)\big)+\sigma\big(\phi_0(b),R_{k+1}(c,a)\big)+\sigma\big(\phi_0(c),R_{k+1}(a,b)\big)\Big)+\cdots.
\end{aligned}
$$

To obtain the order λ^{k+1} from the equation above, we have to expand the deformed Poisson structures π and σ and take their zeroth order, which are the original brackets on $\mathcal{A}$ and $\mathcal{B}$, respectively, thus we get

$$
\begin{aligned}
0 &= R_{k+1}(a,\{b,c\}_{\mathcal{A}})+R_{k+1}(b,\{c,a\}_{\mathcal{A}})+R_{k+1}(c,\{a,b\}_{\mathcal{A}}) \\
&+ \{\phi_0(a),R_{k+1}(b,c)\}_{\mathcal{B}}+\{\phi_0(b),R_{k+1}(c,a)\}_{\mathcal{B}}+\{\phi_0(c),R_{k+1}(a,b)\}_{\mathcal{B}},
\end{aligned}
$$

which is precisely the condition $\delta R_{k+1}=0$. □

Now we look for $X_{k+1}\in \mathrm{Der}(\mathcal{B})$ such that for $\Phi_{(k+1)} := \Phi_{(k)}\exp(\lambda^{k+1}X_{k+1})$ we have an analogous equation to (3.3) up to one order higher.

Remark 3.2.11. *Note that $\Phi_{(k+1)}$ is an automorphism of $\mathcal{B}_\lambda$ starting with identity, hence (see Proposition 2.2.4), it must be of the form $\exp(X_{(k+1)})$ for some $X_{(k+1)}\in\lambda\,\mathrm{Der}(\mathcal{B})_\lambda$, thus, it has the desired form.*

We compute the analogous to (3.3) up to the order λ^{k+1}:

$$
\begin{aligned}
\Phi_{(k+1)}^{-1}\sigma\big(\Phi_{(k+1)}\phi_0(a),\Phi_{(k+1)}\phi_0(b)\big) &= \Phi_{k+1}^{-1}\Phi_{(k)}^{-1}\sigma\big(\Phi_{(k)}\Phi_{k+1}\phi_0(a),\Phi_{(k)}\Phi_{k+1}\phi_0(b)\big) \\
= (I-\lambda^{k+1}X_{k+1}+\cdots)\Phi_{(k)}^{-1}\sigma\big(\Phi_{(k)}((I&+\lambda^{k+1}X_{k+1}+\cdots)\phi_0(a)),a\leftarrow b\big),
\end{aligned}
$$

where $a \leftarrow b$, in the second argument of σ, means that we repeat the first argument with a replaced by b. Expanding the right hand side and using the defining equation of R_{k+1}, the fact that $\Phi_{(k)}$ and $\Phi_{(k)}^{-1}$ are identity at zeroth order, and that ϕ_0 is Poisson morphism between $\{\cdot,\cdot\}_{\mathcal{A}}$ and $\{\cdot,\cdot\}_{\mathcal{B}}$, we get

$$\begin{aligned}
\Phi_{(k+1)}^{-1}\sigma\big(\Phi_{(k+1)}\phi_0(a),\Phi_{(k+1)}\phi_0(b)\big) &= \Phi_{(k)}^{-1}\sigma\big(\Phi_{(k)}\phi_0(a),\Phi_{(k)}\phi_0(b)\big) - \lambda^{k+1}X_{k+1}\sigma\big(\phi_0(a),\phi_0(b)\big) \\
&+ \lambda^{k+1}\sigma\big(\phi_0(a),X_{k+1}\phi_0(b)\big) + \lambda^{k+1}\sigma\big(X_{k+1}\phi_0(a),\phi_0(b)\big) + \cdots \\
&= \phi_0(\pi(a,b)) + \lambda^{k+1}R_{k+1}(a,b) - \lambda^{k+1}X_{k+1}\phi_0\{a,b\}_{\mathcal{A}} \\
&+ \lambda^{k+1}\{\phi_0(a),X_{k+1}\phi_0(b)\}_{\mathcal{B}} + \lambda^{k+1}\{X_{k+1}\phi_0(a),\phi_0(b)\}_{\mathcal{B}} + \cdots .
\end{aligned}$$

Thus, to have (3.2) satisfied up to order λ^{k+1} we need to fulfill

$$R_{k+1}(a,b) = X_{k+1}\phi_0\{a,b\}_{\mathcal{A}} - \{\phi_0(a),X_{k+1}\phi_0(b)\}_{\mathcal{B}} - \{X_{k+1}\phi_0(a),\phi_0(b)\}_{\mathcal{B}},$$

for all $a,b\in\mathcal{A}$. Note that the right hand side in the equation above is just $-\delta\phi_0^*X_{k+1}(a,b)$, hence we get the condition

$$-R_{k+1} = \delta\phi_0^*X_{k+1}.$$

This equation, together with Lemma 3.2.10, tell us that $[R_{k+1}]\in H^2_{CE,der}(\mathcal{A},\mathcal{B})$ is an obstruction to finding a solution X_{k+1}. Moreover, even if $[R_{k+1}]=0$, we still have to realize it as a coboundary $-\delta\xi_{k+1}$, with ξ_{k+1} in the image of ϕ_0^*. Thus we can state the following result.

Proposition 3.2.12. *Suppose that we have a k^{th}-order Poisson deformation $\Phi_{(k)}=\exp(X_{(k)})$ of ϕ_0, thus yielding $R_{k+1}\in C^2_{CE,der}(\mathcal{A},\mathcal{B})$ according to Lemma 3.2.10. If ϕ_0^* is surjective, then the obstruction to extend $\Phi_{(k)}$ into a $(k+1)^{th}$-order Poisson deformation of ϕ_0 is given by $[R_{k+1}]\in H^2_{CE,der}(\mathcal{A},\mathcal{B})$.*

Uniqueness

Suppose we have found a solution to (3.2) of the form $\Phi=\exp(X)\phi_0$. We are now interested in studying the degree of freedom of such a solution. First notice that, if $Y\in\lambda\,\mathrm{Der}(\mathcal{B})_\lambda$ is a derivation of σ, then $\exp(Y)$ is an automorphism of σ, then defining $\overline{\Phi}:=\exp(Y)\Phi$, we still have a solution of (3.2). Consider now an element $V\in\lambda\,\mathrm{Der}(\mathcal{B})_\lambda$ which is vertical with respect to ϕ_0, i.e., $\phi_0^*V_j=V_j\circ\phi_0=0$. It follows that $\exp(V)\phi_0=\phi_0$, therefore $\overline{\Phi}:=\exp(X)\exp(V)\phi_0=\Phi$ is again solution of (3.2).

Thus, so far we have that any solution of (3.2) of the form $\Phi=\exp(X)\phi_0$, can be modified by transformations of the form

$$\exp(X)\rightsquigarrow\exp(Y)\exp(X)\exp(V),$$

with V vertical and Y being a derivation of σ. Now we want to see whether this is the only way to modify a given solution $\exp(X)\phi_0$ of (3.2).

Thus let $\Phi = \exp(X)\phi_0$ and $\overline{\Phi} = \exp(\overline{X})\phi_0$ be two solutions of (3.2), where $X, \overline{X} \in \lambda \operatorname{Der}(\mathcal{B})_\lambda$. Suppose also that we have found $Y_{(k)}, V_{(k)} \in \lambda \operatorname{Der}(\mathcal{B})_\lambda$ such that, $V_{(k)}$ is vertical, $Y_{(k)}$ is a derivation of σ and $\exp(Y_{(k)})\exp(X)\exp(V_{(k)})$ agree with $\exp(\overline{X})$ up to order k. Notice that, $Y_{(0)} = V_{(0)} = 0$ does the job for $k = 0$.

The fact that $\exp(Y_{(k)})\exp(X)\exp(V_{(k)})$ agree with $\exp(\overline{X})$ up to order k means that there is a $Z_{(k+1)} \in \lambda^{k+1} \operatorname{Der}(\mathcal{B})_\lambda$ such that

$$\exp(Y_{(k)})\exp(X)\exp(V_{(k)})\exp(-\overline{X}) = \exp(Z_{(k+1)}) = Id + \lambda^{k+1} Z_{k+1} + \cdots. \tag{3.4}$$

Now we look for some vertical derivation $V_{k+1} \in \operatorname{Der}(\mathcal{B})_\lambda$ and a σ derivation $Y_{k+1} \in \operatorname{Der}(\mathcal{B})_\lambda$ such that the maps $\exp(\lambda^{k+1}Y_{k+1})\exp(Y_{(k)})\exp(X)\exp(V_{(k)})\exp(\lambda^{k+1}V_{k+1})$ and $\exp(\overline{X})$ agree up to order $k+1$.

To do so, we compute the analogue to equation (3.4) up to on order higher.

$$\begin{aligned}
&\exp(\lambda^{k+1}Y_{k+1})\exp(Y_{(k)})\exp(X)\exp(V_{(k)})\exp(\lambda^{k+1}V_{k+1})\exp(-\overline{X}) \\
&\quad = (I + \lambda^{k+1}Y^0_{k+1} + \cdots)\exp(Y_{(k)})\exp(X)\exp(V_{(k)})(I + \lambda^{k+1}V^0_{k+1} + \cdots)\exp(-\overline{X}) \\
&\quad = \exp(Y_{(k)})\exp(X)\exp(V_{(k)})\exp(-\overline{X}) + \lambda^{k+1}(Y^0_{k+1} + V^0_{k+1}) + \cdots \\
&\quad = I + \lambda^{k+1}Z_{k+1} + \lambda^{k+1}(Y^0_{k+1} + V^0_{k+1}) + \cdots.
\end{aligned}$$

Hence, the zeroth order terms Y^0_{k+1} and V^0_{k+1} need to satisfy

$$Y^0_{k+1} + V^0_{k+1} = -Z_{k+1}. \tag{3.5}$$

Now, since $\exp(Y_{(k)})$ is an automorphism of σ we have that $\exp(Y_{(k)})\exp(X)\exp(V_{(k)})\phi_0$ solves (3.2), thus we compute

$$\begin{aligned}
&\exp(Z_{(k+1)})\sigma\big(\exp(-Z_{(k+1)})\exp(Y_{(k)})\exp(X)\exp(V_{(k)})\phi_0(a), a \leftarrow b\big) \\
&\quad = \exp(Y_{(k)})\exp(X)\exp(V_{(k)})\exp(-\overline{X})\sigma\big(\exp(\overline{X})\phi_0(a), \exp(\overline{X})\phi_0(b)\big) \\
&\quad = \exp(Y_{(k)})\exp(X)\exp(V_{(k)})\phi_0(\pi(a,b)) \\
&\quad = \sigma\big(\exp(Y_{(k)})\exp(X)\exp(V_{(k)})\phi_0(a), a \leftarrow b\big).
\end{aligned} \tag{3.6}$$

Now, expanding the left hand side of this equation we get

$$\begin{aligned}
&\exp(Z_{(k+1)})\sigma\big(\exp(-Z_{(k+1)})\exp(Y_{(k)})\exp(X)\exp(V_{(k)})\phi_0(a), a \leftarrow b\big) \\
&\quad = (I + \lambda^{k+1}Z_{k+1} + \cdots)\sigma\big((I - \lambda^{k+1}Z_{k+1} + \cdots)\exp(Y_{(k)})\exp(X)\exp(V_{(k)})\phi_0(a), a \leftarrow b\big) \\
&\quad = \sigma\big(\exp(Y_{(k)})\exp(X)\exp(V_{(k)})\phi_0(a), \exp(Y_{(k)})\exp(X)\exp(V_{(k)})\phi_0(b)\big) \\
&\quad + \lambda^{k+1}\Big(Z_{k+1}\phi_0\{a,b\}_{\mathcal{A}} - \{Z_{k+1}\phi_0(a), \phi_0(b)\}_{\mathcal{B}} - \{\phi_0(a), Z_{k+1}\phi_0(b)\}_{\mathcal{B}}\Big) + \cdots \\
&\quad = \sigma\big(\exp(Y_{(k)})\exp(X)\exp(V_{(k)})\phi_0(a), a \leftarrow b\big) - \lambda^{k+1}(\delta\phi_0^* Z_{k+1})(a,b) + \cdots.
\end{aligned}$$

Hence, comparing with the right hand side of (3.6), we get

$$\delta\phi_0^* Z_{k+1} = 0.$$

Suppose then that $H^1_{CE,der}(\mathcal{A}, \mathcal{B}) = \{0\}$. Hence, since $\phi_0^* Z_{k+1} \in C^1_{CE,der}(\mathcal{A}, \mathcal{B})$ is closed, we can find $H_{k+1} \in \mathcal{B}$ such that $\delta H_{k+1} = \phi_0^* Z_{k+1}$, i.e.:

$$\{H_{k+1}, \phi_0(a)\}_{\mathcal{B}} = -\phi_0^* Z_{k+1}(a) = -Z_{k+1}(\phi_0(a)).$$

Observe that $H_{k+1} \in C^0_{CE,der}(\mathcal{A}, \mathcal{B}) = \mathcal{B} = C^0_{CE,der}(\mathcal{B}, \mathcal{B})$, hence, it follows from the previous equation that $V^0_{k+1} := \delta H_{k+1} - Z_{k+1} \in C^1_{CE,der}(\mathcal{B}, \mathcal{B})$ vanishes along ϕ_0. Hence, it is a vertical cochain. Then we have

$$\{H_{k+1}, \cdot\}_{\mathcal{B}} + V^0_{k+1} = -Z_{k+1},$$

and we fulfill (3.5) by taking $Y^0_{k+1} := \{H_{k+1}, \cdot\}_{\mathcal{B}}$. Now, we may take $V_{k+1} := V^0_{k+1}$ and $Y_{k+1} := \sigma(H_{k+1}, \cdot)$.

In the following proposition we summarize the previous discussion.

Proposition 3.2.13. *Assume that $H^1_{CE,der}(\mathcal{A}, \mathcal{B}) = \{0\}$. If $\Phi = \exp(X)\phi_0$, with $X \in \lambda \operatorname{Der}(\mathcal{B})_\lambda$, is a solution of (3.2), then $\exp(X)$ is unique up to transformations of the form*

$$\exp(X) \rightsquigarrow \exp(X_H)\exp(X)\exp(V),$$

where $V \in \lambda \operatorname{Der}(\mathcal{B})_\lambda$ is vertical and $X_H = \sigma(H, \cdot)$ with $H \in \lambda\mathcal{B}_\lambda$.

The commutant

Suppose we have found a deformation $\Phi = \exp(X)\phi_0$ of ϕ_0 corresponding to some given deformations π of $\{\cdot,\cdot\}_{\mathcal{A}}$ and σ of $\{\cdot,\cdot\}_{\mathcal{B}}$. Let $\mathcal{A}'$ be the Poisson commutant of $\mathcal{A}$ inside $\mathcal{B}$, i.e.:

$$\mathcal{A}' := (\phi_0(\mathcal{A}))^c := \{b \in \mathcal{B}; \{b, \phi_0(a)\}_{\mathcal{B}} = 0 \ \forall a \in \mathcal{A}\}.$$

Now let C stand for the Poisson commutator of $\Phi(\mathcal{A}_\lambda)$ inside $(\mathcal{B}_\lambda, \sigma)$. We are then interested in finding a transformation of the form $\exp(X')\colon \mathcal{B}_\lambda \to \mathcal{B}_\lambda$ such that $\exp(X')(\mathcal{A}'_\lambda) = C$, i.e, such that the restriction

$$\exp(X')\colon \mathcal{A}'_\lambda \to C,$$

is an isomorphism, and studying how it is linked to the deformations π and σ.

Remark 3.2.14. *Let us first observe that, in order to have the above isomorphism, it is enough to have*

$$\sigma\big(\exp(X')c, \Phi(a)\big) = 0, \tag{3.7}$$

for all $c \in \mathcal{A}'$ and $a \in \mathcal{A}$. Indeed, this equation implies that $\exp(X')$ maps $\mathcal{A}'_\lambda$ injectively inside C. Conversely, if $b = \sum_{j=0}^{\infty} \lambda^j b_j \in \mathcal{B}_\lambda$ commutes with $\Phi(a)$ for all $a \in \mathcal{A}$, then the zeroth *order of equation (3.7) reads $\{b_0, \phi_0(a)\}_{\mathcal{B}} = 0$ for all $a \in \mathcal{A}$, which implies $b_0 \in \mathcal{A}'$. Then $b - \exp(X')b_0 = \lambda c_1 + \cdots$ belongs to C, hence we get $\{c_1, \phi_0(a)\}_{\mathcal{B}} = 0$ for all $a \in \mathcal{A}$, and therefore $c_1 \in \mathcal{A}'$. Iterating this procedure yields an element $c = b_0 + \lambda c_1 + \cdots \in \mathcal{A}'_\lambda$ such that $b = \exp(X')c$.*

Thus, we look for $X' \in \lambda \operatorname{Der}(\mathcal{B})_\lambda$ such that equation (3.7) holds, for any $c \in \mathcal{A}'$ and $a \in \mathcal{A}$. As in the previous cases, we want to find such an X' in an inductive way. Thus, suppose $X'_{(k)}$ satisfies (3.7) up to order k, for all $c \in \mathcal{A}'$ and $a \in \mathcal{A}$, then the error term has the form

$$\sigma\big(\exp(X'_{(k)})c, \Phi(a)\big) = \lambda^{k+1} F_{k+1}(c, a) + \cdots, \tag{3.8}$$

for some $\mathbb{C}$-bilinear map $F_{k+1}\colon \mathcal{A}' \times \mathcal{A} \to \mathcal{B}$. In particular, $X'_{(0)} = 0$ will do the job for $k = 0$. Then we want to find the obstruction in the next order for $X'_{k+1} \in \operatorname{Der}(\mathcal{B})$. We compute

$$\begin{aligned}
\sigma\big(\exp(\lambda^{k+1} X'_{k+1}) \exp(X'_{(k)})c, \Phi(a)\big) &= \sigma\big((I + \lambda^{k+1} X'_{k+1} + \cdots) \exp(X'_{(k)})c, \Phi(a)\big) \\
&= \sigma\big(\exp(X'_{(k)})c, \Phi(a)\big) + \lambda^{k+1} \{X'_{k+1}c, \phi_0(a)\}_{\mathcal{B}} + \cdots \\
&= \lambda^{k+1} F_{k+1}(c, a) + \lambda^{k+1} \{X'_{k+1}c, \phi_0(a)\}_{\mathcal{B}} + \cdots.
\end{aligned}$$

Hence, the equation we need to solve is

$$\{X'_{k+1}c, \phi_0(a)\}_{\mathcal{B}} = -F_{k+1}(c, a), \tag{3.9}$$

for all $c \in \mathcal{A}'$ and $a \in \mathcal{A}$.

Lemma 3.2.15. *For each $c \in \mathcal{A}'$, the map $D := F_{k+1}(c, \cdot)$ satisfies $\delta D = 0$.*

Proof. Consider $a, b \in \mathcal{A}$ and $c \in \mathcal{A}'$. We use the Jacoby identity for σ, the defining equation for F_{k+1} (3.8), and the fact that Φ is a Poisson morphism to compute

$$\begin{aligned}
\sigma\Big(\Phi(a), \sigma\big(\Phi(b), \exp(X'_{(k)})c\big)\Big) &= \sigma\Big(\sigma\big(\Phi(a), \Phi(b)\big), \exp(X'_{(k)})c\Big) + \sigma\Big(\Phi(b), \sigma\big(\Phi(a), \exp(X'_{(k)})c\big)\Big) \\
&= \sigma\big(\Phi(\pi(a, b)), \exp(X'_{(k)})c\big) + \sigma\big(\Phi(b) - \lambda^{k+1} F_{k+1}(c, a) + \cdots\big) \\
&= -\lambda^{k+1} F_{k+1}(c, \pi(a, b)) + \cdots - \lambda^{k+1} \{\phi_0(b), F_{k+1}(c, a)\}_{\mathcal{B}} + \cdots \\
&= -\lambda^{k+1} F_{k+1}(c, \{a, b\}_{\mathcal{A}}) - \lambda^{k+1} \{\phi_0(b), F_{k+1}(c, a)\}_{\mathcal{B}} + \cdots.
\end{aligned}$$

On the other hand, expanding the left hand side, we have

$$\begin{aligned}
\sigma\Big(\Phi(a), \sigma\big(\Phi(b), \exp(X'_{(k)})c\big)\Big) &= \sigma\big(\Phi(a), -\lambda^{k+1} F_{k+1}(c, b) + \cdots\big) \\
&= -\lambda^{k+1} \{\phi_0(a), F_{k+1}(c, b)\}_{\mathcal{B}} + \cdots.
\end{aligned}$$

Hence, the map $D := F_{k+1}(c, \cdot)$ satisfies

$$\begin{aligned}\delta D(a,b) &= \{\phi_0(a), D(b)\}_{\mathcal{B}} - \{\phi_0(b), D(a)\}_{\mathcal{B}} - D(\{a,b\}_{\mathcal{A}}) \\ &= \{\phi_0(a), F_{k+1}(c,b)\}_{\mathcal{B}} - \{\phi_0(b), F_{k+1}(c,a)\}_{\mathcal{B}} - F_{k+1}(c, \{a,b\}_{\mathcal{A}}) = 0,\end{aligned}$$

thus, $\delta D = 0$, as claimed. $\square$

Under the assumption that the first cohomology $H^1_{CE,der}(\mathcal{A}, \mathcal{B}) = \{0\}$ is trivial, we have that there is an element $X'_{k+1}(c) \in \mathcal{B}$ such that

$$\delta X'_{k+1}(c) = F_{k+1}(c, \cdot),$$

which means that $\{\phi_0(a), X'_{k+1}(c)\}_{\mathcal{B}} = (\delta X'_{k+1}(c))(a) = F_{k+1}(c, a)$, thus, equation (3.9) would be solved. Since F_{k+1} depends linearly on c, we could construct X'_{k+1} by mapping c into $X'_{k+1}(c)$ in a linear way as well, but such a linear map X'_{k+1} need not be a derivation.

Now, suppose we can prove the vanishing of the cohomology by means of a homotopy, i.e., an $\mathbb{C}$-linear map $h_\bullet \colon C^\bullet_{CE,der}(\mathcal{A}, \mathcal{B}) \to C^{\bullet-1}_{CE,der}(\mathcal{A}, \mathcal{B})$ such that, for $l \geq 1$

$$h_{l+1}\delta_l + \delta_{l-1}h_l = I_{C^l_{CE,der}(\mathcal{A},\mathcal{B})}.$$

In this case, we have for F_{k+1} the equation

$$F_{k+1}(c, \cdot) = \delta_0 h_1 F_{k+1}(c, \cdot),$$

for every $c \in \mathcal{A}'$. Hence, we could take

$$X'_{k+1}(c) := h_1(F_{k+1}(c, \cdot)),$$

which clearly satisfies (3.2.2) and is $\mathbb{C}$-linear in c, since both F_{k+1} and h_1 are $\mathbb{C}$-linear. For the Leibniz rule we get

$$X'_{k+1}(cc') = h_1(F_{k+1}(cc', \cdot)) = h_1(cF_{k+1}(c', \cdot) + c'F_{k+1}(c, \cdot)).$$

Thus, it depends on whether the homotopy $h_\bullet$ is $\mathcal{A}'$-linear or not.

So far, assuming the existence of such $\mathcal{A}'$-linear homotopy, we have X' defined only on $\mathcal{A}'_\lambda$. In order to guarantee that we can extend X' to a derivation of $\mathcal{B}_\lambda$, we need a second horizontal lift, this time along the embedding $\phi'_0 \colon \mathcal{A}' \to \mathcal{B}$. Let us summarize the previous discussion as follows.

Lemma 3.2.16. *Suppose Φ is a solution of (3.2). If $H^1_{CE,der}(\mathcal{A}, \mathcal{B}) = \{0\}$ by means of an $\mathcal{A}'$-linear homotopy, then there exists a formal derivation $X' \colon \mathcal{A}'_\lambda \to \lambda\mathcal{B}_\lambda$ such that (3.7) holds. If in addition there exists a horizontal lift to a derivation $X' \in \lambda\,\mathrm{Der}(\mathcal{B})_\lambda$ then we have a formal* *automorphism* $\exp(X')$ *of* $\mathcal{B}_\lambda$ *satisfying (3.7). Moreover, this establishes an isomorphism*

$$\exp(X')\phi'_0\colon \mathcal{A}'_\lambda \to C,$$

where C *stands for the commutator of* $\Phi(\mathcal{A}_\lambda)$ *inside* $(\mathcal{B}_\lambda, \sigma)$.

Degree of freedom for X': Suppose we have a horizontal lift with respect to $\phi'_0\colon \mathcal{A}' \to \mathcal{B}$, i.e., a map $h'\colon \operatorname{Der}(\mathcal{A}') \to \operatorname{Der}(\mathcal{B}); \xi \mapsto \xi^{h'}$ such that $\phi'_0(\xi(g)) = \xi^{h'}(\phi'_0(g))$. In such a situation, if $\exp(X')$ is a solution of (3.7), according to the previous lemma, for any $\xi \in \operatorname{Der}(\mathcal{A}'_\lambda)$, we have that $\exp(X')\exp(\xi^{h'})$ is also a solution, indeed:

$$\sigma\big(\exp(X')\exp(\xi^{h'})\phi'_0(c), \Phi(a)\big) = \sigma\big(\exp(X')\phi'_0\exp(\xi)(c), \Phi(a)\big) = 0.$$

Now we observe that this is the only freedom we have in choosing X'.

Lemma 3.2.17. *Suppose we already have* $\exp(X')$ *according to Lemma 3.2.16. If* $\exp(X'')$ *also satisfies*

$$\sigma\big(\exp(X'')\phi'_0(a'), \Phi(a)\big) = 0,$$

for all $a' \in \mathcal{A}'$ *and* $a \in \mathcal{A}$*, then, we must have*

$$\exp(X'')\phi'_0 = \exp(X')\exp(\xi^{h'})\phi'_0,$$

with $\xi^{h'}$ *being the horizontal lift of some* $\xi \in \lambda\operatorname{Der}(\mathcal{A}'_\lambda)$.

Proof. It follows from Lemma 3.2.16 that we must have

$$\exp(X'')\phi'_0(a') = \exp(X')\phi'_0(b'),$$

for some $b' \in \mathcal{A}'_\lambda$. We have actually, $b' = \phi_0'^{-1}\exp(-X')\exp(X'')\phi'_0(a')$, hence the map $a' \mapsto b'$ is an automorphism of $\mathcal{A}'_\lambda$ starting at identity. Thus, $b' = \exp(\xi)(a')$ for some $\xi \in \lambda\operatorname{Der}(\mathcal{A}'_\lambda)$ and

$$\exp(X'')\phi'_0(a') = \exp(X')\phi'_0\exp(\xi)(a') = \exp(X')\exp(\xi^{h'})\phi'_0(a'),$$

for all $a' \in \mathcal{A}'$. Therefore,

$$\exp(X'')\phi'_0 = \exp(X')\exp(\xi^{h'})\phi'_0,$$

with $\xi \in \lambda\operatorname{Der}(\mathcal{A}'_\lambda)$. □

Proposition 3.2.18. *Suppose that we can deform* $\phi_0\colon (\mathcal{A}, \{\cdot,\cdot\}_\mathcal{A}) \to (\mathcal{B}, \{\cdot,\cdot\}_\mathcal{B})$*, with respect to deformations* π *of* $\{\cdot,\cdot\}_\mathcal{A}$ *and* σ *of* $\{\cdot,\cdot\}_\mathcal{B}$*, to a map* $\Phi = \exp(X)\phi_0$ *uniquely up to transformations of the form*

$$\exp(X) \rightsquigarrow \exp(Y)\exp(X)\exp(V),$$

where V *is vertical and* $\exp(Y)$ *is a morphism of* σ*. Suppose further that* $H^1_{CE,der}(\mathcal{A}, \mathcal{B}) = \{0\}$ *by means of an* $\mathcal{A}'$*-linear homotopy, and that there exists a horizontal lift along* $\phi'_0\colon \mathcal{A}' \to \mathcal{B}$*.*

Then, the equivalence class $[\pi]$, of a formal deformation π of π_0, induces a formal Poisson structure π' in $\mathcal{A}'_\lambda$, which is unique up to equivalence.

Proof. For a fixed deformation π of π_0, and a corresponding deformation Φ of ϕ_0, the isomorphism

$$\Phi' := \exp(X')\phi'_0 \colon \mathcal{A}'_\lambda \to C,$$

of Lemma 3.2.16 defines a formal Poisson structure π' on $\mathcal{A}'_\lambda$ by the formula

$$\pi'(a,b) = \Phi'^{-1}(\sigma(\Phi' a, \Phi' b)).$$

Now we check that the class $[\pi']$ is independent of any choices we made in the process.

First consider the degree of freedom of Φ', so let $\Phi'' \colon \mathcal{A}'_\lambda \to C$ be another isomorphism, inducing a corresponding formal Poisson structure π''. By Lemma 3.2.17 we must have $\Phi'' = \exp(X')\exp(\xi^{h'})\phi'_0$, for some $\xi \in \lambda\,\mathrm{Der}(\mathcal{A}'_\lambda)$. But then, π' and π'' are equivalent via the map

$$\exp(\xi) \colon (\mathcal{A}'_\lambda, \pi'') \to (\mathcal{A}'_\lambda, \pi').$$

Next consider the degree of freedom of $\Phi = \exp(X)\phi_0$. Let $\overline{\Phi} = \exp(\overline{X})\phi_0$ be another deformation of ϕ_0. By the uniqueness assumption, we must have

$$\overline{\Phi} = \exp(Y)\exp(X)\exp(V)\phi_0 = \exp(Y)\Phi$$

for some Poisson morphism $\exp(Y)$ of σ and some vertical derivation V (notice that V vertical implies $\exp(V)\phi_0 = \phi_0$). Let $\overline{C}$ stands for the corresponding commutator, and let $\overline{\Phi}' \colon \mathcal{A}'_\lambda \to \overline{C}$ be the corresponding isomorphism given by Lemma 3.2.16, inducing the formal Poisson structure $\overline{\pi}'$. Notice that $\exp(Y) \colon C \to \overline{C}$ is an isomorphism, and therefore the map

$$\overline{\Psi} := \overline{\Phi}'^{-1}\exp(Y)\Phi' \colon (\mathcal{A}'_\lambda, \pi') \to (\mathcal{A}'_\lambda, \overline{\pi}')$$

is an isomorphism starting at identity, thus, an equivalence map. Hence, π' and $\overline{\pi}'$ are equivalent.

Finally, let $\hat{\pi} \sim \pi$, so $\hat{\pi} = \exp(\mathcal{L}_\xi)\pi$ for some $\xi \in \lambda\,\mathrm{Der}(\mathcal{B}_\lambda)$. Let $\hat{\Phi}$ be a corresponding deformation of ϕ_0, and let $\hat{\Phi}' \colon \mathcal{A}'_\lambda \to \hat{C}$ be the corresponding isomorphism of Lemma 3.2.16, with associated formal Poisson structure $\hat{\pi}'$. Then the map $\overline{\Phi} := \hat{\Phi}\exp(\xi)$ must be, by the uniqueness assumption, of the form $\hat{\Phi} = \exp(Y)\Phi$, for some Poisson morphism $\exp(Y)$ of σ. Notice that $\overline{C} = \hat{C}$, and that $\exp(Y) \colon C \to \overline{C}$ is an isomorphism. Thus, the map

$$\hat{\Phi}'^{-1}\exp(Y)\Phi' \colon (\mathcal{A}'_\lambda, \pi') \to (\mathcal{A}'_\lambda, \hat{\pi}')$$

is an equivalence. Hence, π' and $\hat{\pi}'$ are equivalent after all. $\square$

CHAPTER 4

Classification of Morita equivalent formal Poisson structures

The main result in this chapter, which comes from Theorem 4.2.1, its corollary, and Theorem 4.2.3 is the following:

Classifying map: *Given a manifold P, there exists a map*

$$\mathbb{H}^2_{dR}(P)_\lambda \times \mathrm{FPois}_0(P) \to \mathrm{FPois}_0(P); \ \ ([B],[\pi]) \mapsto [\pi^B],$$

such that two elements $[\pi],[\pi'] \in \mathrm{FPois}_0(P)$ are Morita equivalent if and only if $[\pi'] = [\psi_*\pi^B]$ for some $B \in \Omega^2_{cl}(P)_\lambda$ and $\psi \in \mathrm{Diff}(P)$.

4.1 The zero Poisson structure

Here we apply the previous considerations to the symplectic realization

$$(T^*P, \omega_0) \xrightarrow{\rho} (P, \pi_0 = 0),$$

where ρ is the cotangent bundle projection and $\omega_0 := \omega_{can} + \rho^* B_0$ with $B_0 \in \Omega^2_{cl}(P)$.

Given a deformation of ω_0 of the form $\omega_B = \omega_0 + \rho^* B$, with $B = B_0 + \lambda B_1 + \cdots \in \Omega^2_{cl}(P)_\lambda$ and a deformation π of $\pi_0 = 0$, we will see that the Poisson deformation procedure described in the previous chapter yields a new formal Poisson structure π^B on P, defined uniquely up to equivalence by the classes $[\pi]$ and $[B]$, and which is Morita equivalent to π according to Definition 2.4.2, this is the content of Theorem 4.2.1 and its corollary. Moreover, in Theorem 4.2.3 we show that any pair of Morita equivalent elements $[\pi],[\pi'] \in \mathrm{FPois}_0(P)$ must satisfy $[\pi'] = [\psi_*\pi^B]$, with $\psi \in \mathrm{Diff}(P)$, $B \in \Omega^2_{cl}(P)_\lambda$ and π^B obtained via the Poisson deformation of ρ^* with respect to π and ω_B.

We begin by studying the following problem: Let $\pi = \sum_{j=1}^{\infty} \lambda^j \pi_j$ be a formal Poisson structure deforming the trivial Poisson structure on P, and let $B \in \Omega^2_{cl}(P)_\lambda$. Putting $\omega_0 := \omega_{can} + \rho^* B_0$, we want to find a deformation Φ of

$$\phi_0 := \rho^* \colon (C^\infty(P), 0) \to (C^\infty(T^*P), \{\cdot,\cdot\}_{\omega_0})$$

with respect to π and $\sigma = \{\cdot,\cdot\}_B$, according to Definition 3.1.1, where $\{\cdot,\cdot\}_B$ is the Poisson bracket induced by $\omega_B := \omega_{can} + \rho^* B$.

To conclude existence and uniqueness, according to Proposition 3.2.12 and Proposition 3.2.13 we need to compute the cohomology $H^\bullet_{CE,der}(P, T^*P) := H^\bullet_{CE,der}(C^\infty(P), C^\infty(T^*P))$ (at least at degree 1 and 2), and to find a horizontal lift

$$h \colon C^\bullet_{CE,der}(P, T^*P) \to C^\bullet_{CE,der}(T^*P, T^*P).$$

Before going to these computations, let us observe some properties of the symplectic realization we are working with.

Lemma 4.1.1. *Consider $(T^*P, \omega_0) \xrightarrow{\rho} P$, with $\omega_0 := \omega_{can} + \rho^* B_0$. Then we have*

1. *$T^*P \xrightarrow{\rho} P$ is a lagrangian fibration, i.e., each fiber is a lagrangian submanifold of (T^*P, ω_0).*
2. *For any $g \in C^\infty(P)$, the hamiltonian $X_{\rho^* g}$ is tangent to the fibers.*
3. *For any $g \in C^\infty(P)$ and $f \in C^\infty(T^*P)$, we have $\{\rho^* g, f\}_{\omega_0} = \{\rho^* g, f\}_{can}$. In particular;*

$$\{\rho^* g, f\}_{\omega_0} = 0,\ \forall g \in C^\infty(P) \Rightarrow f = \rho^* h \text{ for some } h \in C^\infty(P).$$

Proof. Let $x \in P$ be any point, and let (q^i) be a coordinate chart centered at x. Let $L_x := \rho^{-1}(x)$ be the fiber at x. Then, on the induced chart (q^i, p_i) L_x is characterized by $(q^1 = \cdots = q^n = 0)$ and at any point $z \in L_x$ we have $T_z L_x = span(\partial_{p_1}, ..., \partial_{p_n})$. Hence, we get

$$\omega_0|_z(\partial_{p_i}|_z, \partial_{p_j}|_z) = \omega_{can}|_z(\partial_{p_i}|_z, \partial_{p_j}|_z) + B_0|_{\rho(z)}(\rho_*|_z \partial_{p_i}|_z, \rho_*|_z \partial_{p_j}|_z) = 0,$$

thus, L_x is isotropic and by dimension it is lagrangian.

For the second item let $V \subset T(T^*P)$ be the vertical distribution, namely, at any $z \in T^*P$, $V_z = T_z L_{\rho(z)}$. Given $g \in C^\infty(P)$, pick any section Y of V and consider

$$\omega_0(X^{\omega_0}_{\rho^* g}, Y) = \iota_{X^{\omega_0}_{\rho^* g}} \omega_0(Y) = d(\rho^* g)(Y) = Y(\rho^* g) = 0.$$

Thus, $X^{\omega_0}_{\rho^* g} \in V^{\omega_0}$, but $V^{\omega_0} = V$ by item 1, hence $X^{\omega_0}_{\rho^* g} \in V$, which means it is tangent to the fibers.

For the last item, observe that $X^{\omega_0}_{\rho^* g} \in V$ implies $\iota_{X^{\omega_0}_{\rho^* f}} \rho^* B_0 = 0$, hence $X^{\omega_0}_{\rho^* f} = X^{can}_{\rho^* f}$, and we get

$$\{\rho^* g, f\}_{\omega_0} = df(X^{\omega_0}_{\rho^* g}) = df(X^{can}_{\rho^* g}) = \omega_{can}(X^{can}_f, X^{can}_{\rho^* g}) = \{\rho^* g, f\}_{can}.$$

In particular, $0 = \{\rho^* g, f\}_{\omega_0} = \{\rho^* g, f\}_{can}$ for all $g \in C^\infty(P)$ implies $f = \rho^* h$ for some $h \in C^\infty(P)$. $\square$

Now observe that, in this geometric case, since the Poisson bracket on $C^\infty(P)$ is zero, and by item 3 in the previous lemma, the differential is just

$$\delta D(f_1, ..., f_{k+1}) = \sum_{j=1}^{k+1} (-1)^{j+1} \{\rho^*(f_j), D(f_1, ..., \overset{j}{\wedge}, ..., f_{k+1})\}_{can},$$

for $D \in C^k_{CE,der}(P, T^*P)$ and $f_1, ..., f_{k+1} \in C^\infty(P)$.

Recall also the $C^\infty(P)$-module structure on $C^\bullet_{CE,der}(P, T^*P)$, given according to Remark 3.2.1 by

$$(fD)(f_1, ..., f_k) := \rho^* f D(f_1, ..., f_k),$$

for $D \in C^k_{CE,der}(P, T^*P)$ and $f_1, ..., f_k \in C^\infty(P)$.

Lemma 4.1.2. *The differential δ is $C^\infty(P)$-linear with respect to the $C^\infty(P)$-module structure of the complex $C^\bullet_{CE,der}(P, T^*P)$.*

Proof. We need to check that $\delta(gD) = g\delta D$ for any $g \in C^\infty(P)$ and $D \in C^k_{CE,der}(P, T^*P)$. To do so, consider $g_1, ..., g_{k+1} \in C^\infty(P)$ and compute

$$\begin{aligned}
\delta(gD)(g_1, ..., g_{k+1}) &= \sum_{j=1}^{k+1} (-1)^{j+1} \{\rho^* g_j, (gD)(g_1, ..., \overset{j}{\wedge}, ..., g_{k+1})\}_{can} \\
&= \sum_{j=1}^{k+1} (-1)^{j+1} \{\rho^* g_j, \rho^* g D(g_1, ..., \overset{j}{\wedge}, ..., g_{k+1})\}_{can} \\
&= \sum_{j=1}^{k+1} (-1)^{j+1} \{\rho^* g_j, D(g_1, ..., \overset{j}{\wedge}, ..., g_{k+1})\}_{can} \rho^* g \\
&+ \sum_{j=1}^{k+1} (-1)^{j+1} \underbrace{\{\rho^* g_j, \rho^* g\}_{can}}_{=0} D(g_1, ..., \overset{j}{\wedge}, ..., g_{k+1}) \\
&= g(\delta D)(g_1, ..., g_{k+1}),
\end{aligned}$$

hence, $\delta(gD) = g\delta D$, as desired. $\square$

4.1.1 Horizontal lift

To understand the horizontal lifting on this geometric case, let us first understand the geometric interpretation of the elements in the complex we are working with. It is well-known that derivations of the algebra of smooth functions $C^\infty(P)$ of a smooth manifold P are in one to one correspondence with smooth vector fields on P, i.e., with smooth sections of the bundle $TP \to P$. Similarly we can identify multiderivations of $C^\infty(P)$ with multivector fields, i.e., with sections of the bundle $\bigwedge^k(TP) \to P$. Now, consider an element D in $C^k_{CE,der}(P, T^*P)$, which we defined to be a multiderivation along ρ^*, i.e., a multilinear map $D\colon C^\infty(P)^{\times k} \to C^\infty(T^*P)$ satisfying

$$D(f_1, ..., f_l g_l, ..., f_k) = \rho^*(f_l)D(f_1, ..., g_l, ..., f_k) + \rho^*(g_l)D(f_1, ..., f_l, ..., f_k),$$

for $f_1, ..., f_k, g_l \in C^\infty(P)$ and $l = 1, ..., k$. Then we would like to have a geometric counterpart of this. What we need to consider geometrically is the concept of multivector field along a map.

Consider a smooth map $\phi\colon N \to P$, then a vector field on P along ϕ is, by definition, a smooth section of the pull-back bundle $\phi^*TP \to N$. Any such section σ can be identified with a smooth map $\sigma\colon N \to TP$ such that $\tau_P \circ \sigma = \phi$, where τ_P is the tangent bundle projection.

On the algebraic side, we consider linear maps $D : C^\infty(P) \to C^\infty(N)$ which are derivations along ϕ^*, i.e.,

$$D(gh) = \phi^*(g)D(h) + \phi^*(h)D(g),$$

whose collection we denote by $\mathrm{Der}(\phi)$. Then we have the following result.

Proposition 4.1.3. *Let $\phi\colon N \to P$ be a smooth map. Then $\Gamma(\phi^*TP)$ and $\mathrm{Der}(\phi)$ are isomorphic.*

The proof (which can be found in [62, Prop. 3.2.7]) is a simple adaptation of the proof that derivations of $C^\infty(P)$ and vector fields on P are in one to one correspondence. Also, it can be easily generalized for the context of multiderivations and multivector fields along the given map, thus we have an isomorphism $\Gamma(\phi^* \bigwedge^k(TP)) \cong \mathrm{Der}^k(\phi)$, where $\mathrm{Der}^k(\phi)$ is the set of multiderivations along ϕ^*.

In our case, $N = T^*P$ and the map is the cotangent bundle projection ρ. A cochain, $D \in C^k_{CE,der}(P, T^*P)$, thus corresponds to some section of the bundle $\rho^* \bigwedge^k(TP) \to T^*P$, hence in a chart (U, q^i) on P, with induced chart (T^*U, q^i, p_i) on T^*P, we have the local expression

$$D(g_1, ..., g_k) = \sum_{i_1,...,i_k} D^{i_1,...,i_k} \partial_{q^{i_1}} g_1 \cdots \partial_{q^{i_k}} g_k, \tag{4.1}$$

where $\{i_1, ..., i_k\} \subset \{1, ..., n\}$ and $D^{i_1,...,i_k} \in C^\infty(T^*U)$ are totally skew-symmetric in the indices $i_1, ..., i_k$.

Now, by a horizontal lifting we mean a map sending a multiderivation $D\colon C^\infty(P)^{\times k} \to C^\infty(T^*P)$ along ρ^* to a multiderivation $D^{hor}\colon C^\infty(T^*P)^{\times k} \to C^\infty(T^*P)$. Here it is important to notice that, for the geometric case we are working with, we are mainly interested on degree 1, thus, on derivations $D\colon C^\infty(P) \to C^\infty(T^*P)$ along ρ^*. Since the geometric counterpart of these are vector fields on P along $T^*P \xrightarrow{\rho} P$, it follows from the local expression (4.1) that it suffices for us to lift vector fields from P to its cotangent bundle T^*P. This can be accomplished, in this geometric case, in at least two ways. One is by horizontal lifting of vector fields from a manifold to its cotangent bundle, which is a particular case of a lifting procedure on vector bundles. Since this procedure, though well established, involves the introduction of a connection on the base manifold, we choose to use here a more canonical lifting procedure available in the cotangent case. The interested reader on the more general lifting procedure may consult [73], [62] and the references therein.

Given a cotangent bundle $T^*P \to P$, the cotangent lifting is a canonical way to lift some geometric objects from P to T^*P. In the case of a vector field X on P, it yields a vector field $\overline{X}$ on T^*P (see Appendix A.3). If (U, q^i) is a chart on P and $X = X^i\partial_{q^i}$ on this chart, then on the induced cotangent chart (T^*U, q^i, p_i) we have $\overline{X} = \rho^*(X^i)\partial_{q^i} - p_k\rho^*(\partial_{q^i}X^k)\partial_{p_i}$ (see Proposition A.3.2).

Now, for a derivation $D\colon\ C^\infty(P) \to C^\infty(T^*P)$, with local expression

$$D = D^i\partial_{q^i},$$

we define its horizontal lift to be

$$D^{hor} = D^i\overline{\partial_{q^i}},$$

where $\overline{\partial_{q^i}}$ is the cotangent lift of the vector field ∂_{q^i}. For a fixed $1 \le i_0 \le n$ we have $\overline{\partial_{q^{i_0}}} = \partial_{q^{i_0}}$, hence, for $D^{hor} = D^i\overline{\partial_{q^i}}$ we have $D^{hor} = D^i\partial_{q^i}$.

Then we check, for any $f \in C^\infty(P)$:

$$(\phi_0^*D^{hor})(f) := D^{hor}(\phi_0(f)) = D^{hor}(\rho^*(f)) = D^i\partial_{q^i}(\rho^*(f)) = D(f),$$

and

$$(\phi_0(f)D)^{hor} = (\rho^*(f)D^i\partial_{q^i})^{hor} = ((\rho^*(f)D^i)\partial_{q^i})^{hor} = (\rho^*(f)D^i)\overline{\partial_{q^i}} = \rho^*(f)D^i\overline{\partial_{q^i}} = \phi_0(f)D^{hor}.$$

Hence, this ^{hor} map satisfies the conditions asked for in Definition 3.2.6.

4.1.2 The cohomology

To compute $H^0_{CE,der}(P, T^*P)$ we consider the beginning of the complex:

$$0 \xrightarrow{\delta_0} C^0_{CE,der}(P, T^*P) \cong C^\infty(T^*P) \xrightarrow{\delta_1} C^1_{CE,der}(P, T^*P) \to \cdots,$$

hence,

$$H^0_{CE,der}(P, T^*P) = \ker(\delta_1) := \{f \in C^\infty(T^*P);\ \delta_1 f = 0\}.$$

Now, $\delta_1 f = 0$ if and only if $\delta_1 f(g) = 0$ for any $g \in C^\infty(P)$, i.e.:

$$0 = \{\rho^*g, f\}_{can},\ \forall g \in C^\infty(P) \Leftrightarrow f = \rho^*h;\ h \in C^\infty(P).$$

Hence, $H^0_{CE,der}(C^\infty(P), C^\infty(T^*P)) \cong \rho^*(C^\infty(P))$.

For $k \geq 1$ we will break the problem into three steps. First, we localize the problem by considering $C^{\bullet}_{CE,der}(U, T^*U)$, where $U \subset P$ is a chart. In the second step, we construct a complex isomorphism between the local model $C^{\bullet}_{CE,der}(U, T^*U)$ and the vertical de Rham complex of T^*U, denoted by $\Omega^{\bullet}_V(T^*U)$ (see Appendix A.2). Now, the complex $\Omega^{\bullet}_V(T^*U)$ admits a $C^{\infty}(U)$-linear homotopy (see Proposition A.2.3), and then, by means of the previous isomorphism we get a $C^{\infty}(U)$-linear homotopy in the complex $C^{\bullet}_{CE,der}(U, T^*U)$. Finally, we use Lemma 4.1.2 to construct a global $C^{\infty}(P)$-linear homotopy for $C^{\bullet}_{CE,der}(P, T^*P)$ by means of a partition of unity on P.

First of all, let us fix some notations: given a chart $(U, q^i) \subset P$, with induced chart $(T^*U, q^i, p_i) \subset T^*P$, let $I_k = (i_1 < \cdots < i_k) \subset \{1, ..., n\}$ denote an strictly increasing multi-index of length k. Then, we put $dq^{I_k} := dq^{i_1} \wedge \cdots \wedge dq^{i_k}$, and likewise for dp_{I_k}, $\frac{\partial}{\partial q^{I_k}}$ and $\frac{\partial}{\partial p_{I_k}}$.

Notice that $D \in C^k_{CE,der}(P, T^*P)$ induces by restriction a cochain $D_U \in C^k_{CE,der}(U, T^*U)$. The meaning of such a restriction is clear by the identification of D as a section of the bundle $\rho^* \bigwedge^k(TP) \to T^*P$. Hence, D_U is uniquely written as

$$D_U = \sum_{I_k} D_U^{I_k} \frac{\partial}{\partial q^{I_k}},$$

for some $D_U^{I_k} \in C^{\infty}(T^*U)$, totally skew-symmetric in the indices $\{i_1, ..., i_k\}$. Notice that we have $D(f_1, ..., f_k)|_{T^*U} = D_U(f_1|_U, ..., f_k|_U)$.

Let δ_U be the differential in the complex $C^{\bullet}_{CE,der}(U, T^*U)$, given, for $F \in C^k_{CE,der}(U, T^*U)$ and $g_1, ..., g_{k+1} \in C^{\infty}(T^*U)$, by

$$\delta_U F(g_1, ..., g_{k+1}) = \sum_{l=1}^{k+1} (-1)^{l+1} \{\rho^*(g_l), F(g_1, ..., \overset{l}{\wedge}, ..., g_{k+1})\}_{can}.$$

Notice then that $(\delta D)_U = \delta_U D_U$.

Now we work with the local model, thus let $U \subset \mathbb{R}^n$ be an open set with global coordinates (q^i). For each $k \geq 1$ we consider the following map

$$\Psi \colon C^k_{CE,der}(U, T^*U) \to \Omega^k_V(T^*U);\ D \mapsto \Psi(D) := \sum_{I_k} D^{I_k} dp_{I_k},$$

where D^{I_k} are the coefficients of D in the global frame. Recall that both complexes, $\Omega^{\bullet}_V(T^*U)$ and $C^{\bullet}_{CE,der}(U, T^*U)$ are $C^{\infty}(P)$-modules (see Remark A.2.1 and Remark 3.2.1). The map Ψ is a $C^{\infty}(P)$-module isomorphism. Indeed, it is clearly a bijection, and for $D = \sum D^{I_k} \frac{\partial}{\partial q^{I_k}}$ and $f \in C^{\infty}(U)$ we get

$$\Psi(fD) = \Psi\left(f \sum D^{I_k} \frac{\partial}{\partial q^{I_k}}\right) = \Psi\left(\sum \rho^* f D^{I_k} \frac{\partial}{\partial q^{I_k}}\right) = \sum \rho^* f D^{I_k} dp_{I_k} = f \sum D^{I_k} dp_{I_k} = f\Psi(D).$$

Now we prove that Ψ is a complex map. (See Appendix A.2 for the complex structure in $\Omega^{\bullet}_V(T^*U)$.)

Lemma 4.1.4. *In the above setting, we have $d_{ver} \circ \Psi = \Psi \circ \delta_U$.*

Proof. Given $D \in C^k_{CE,der}(U, T^*U)$ we have uniquely $D = \sum_{I_k} D^{I_k} \frac{\partial}{\partial q^{I_k}}$. Then we get

$$\delta_U D = \tilde{D} = \sum_{I_{k+1}} \tilde{D}^{I_{k+1}} \frac{\partial}{\partial q^{I_{k+1}}},$$

where

$$\tilde{D}^{I_{k+1}} = \sum_{l=1}^{k+1} (-1)^{l+1} \frac{\partial D^{i_1 \ldots \overset{l}{\wedge} \ldots i_{k+1}}}{\partial p_{i_l}}.$$

Hence, we get

$$\Psi(\delta_U D) = \sum_{I_{k+1}} \left(\sum_{l=1}^{k+1} (-1)^{l+1} \frac{\partial D^{i_1 \ldots \overset{l}{\wedge} \ldots i_{k+1}}}{\partial p_{i_l}} \right) dp_{I_{k+1}}. \tag{4.2}$$

Now we compute $d_{ver}(\Psi(D))$ (see Example A.2.2).

$$d_{ver}(\Psi(D)) = \left(d\Big(\sum_{I_k} D^{I_k} dp_{I_k} \Big) \right)^v = \sum_{I_k} \left(\sum_{r \notin I_k} \frac{D^{I_k}}{\partial p_r} dp_r \right) dp_{I_k}.$$

Note that for each multi-index I_{k+1} we have $(k+1)$ multi-indices I_k:

$$I_{k+1} = (i_1 < \ldots < i_{k+1}) \rightsquigarrow I_k^l = (i_1 < \ldots \overset{l}{\wedge} \ldots < i_{k+1}); \ 1 \leq l \leq k+1.$$

For each of these (k+1) multi-indices I_k^l we have the expression

$$\sum_{r \notin I_k^j} \frac{\partial D^{I_k^j}}{\partial p_r} dp_r. \tag{4.3}$$

To get the coefficient of $d_{ver}(\Psi(D))$ at a fixed $dp_{I_{k+1}}$, we look at the $(k+1)$ multi-indices I_k^l with coefficients given by (4.3) and choose the ones to complete the multi-index I_{k+1}, thus we will get an expression like

$$\sum_{l=1}^{k+1} \frac{\partial D^{i_1 \ldots \overset{l}{\wedge} \ldots i_{k+1}}}{\partial p_l} dp_j \wedge dp_{I_k},$$

up to a sign depending on l. Then we observe that to get $dp_{I_{k+1}}$ we have to move dp_l across $dp_{I_k} = dp_{i_1} \wedge \cdots \wedge dp_{i_k}$ up to the l^{th} position, which gives us the sign $(-1)^{l+1}$. Thus, the coefficient at $dp_{I_{k+1}}$ is given by

$$\sum_{l=1}^{k+1} (-1)^{l+1} \frac{\partial D^{i_1 \ldots \overset{l}{\wedge} \ldots i_{k+1}}}{\partial p_l},$$

which is exactly the coefficient of $dp_{I_{k+1}}$ in the R.H.S. of (4.2). Thus, Ψ is a complex map. $\square$

Now, let $\phi_U \colon \Omega^\bullet_V(T^*U) \to \Omega^{\bullet-1}_V(T^*U)$ be a $C^\infty(U)$-linear homotopy, i.e., $d_{ver}\phi_U + \phi_U d_{ver} = I$ (see Proposition A.2.3), then $\varphi_U := \Psi^{-1}\phi_U\Psi$ is a $C^\infty(U)$-linear homotopy in $C^\bullet_{CE,der}(U, T^*U)$. Indeed, since φ_U is a composition of $C^\infty(U)$-linear maps, it is clearly $C^\infty(U)$-linear as well. For the homotopy property, we compute

$$\begin{aligned}
\delta_U\varphi_U + \varphi_U\delta_U &= \delta_U\Psi^{-1}\phi_U\Psi + \Psi^{-1}\phi_U\Psi\delta_U \\
&= \Psi^{-1}d_{ver}\phi_U\Psi + \Psi^{-1}\phi_U d_{ver}\Psi \\
&= \Psi^{-1}(d_{ver}\phi_U + \phi_U d_{ver})\Psi = \Psi^{-1}\Psi = I.
\end{aligned}$$

Now we can prove the following result.

Proposition 4.1.5. *For $k \geq 1$, we have $H^k_{CE,der}(P, T^*P) = \{0\}$, by means of a $C^\infty(P)$-linear homotopy.*

Proof. We will construct a $C^\infty(P)$-linear homotopy φ on the complex $C^\bullet_{CE,der}(P, T^*P)$. It will follow then that the cohomology is trivial.

Let (U_α, q^i_α) be an atlas on P, and let (η_α) be a locally finite partition of unity subordinated to the covering (U_α), i.e., $\eta_\alpha \in C^\infty(P)$, $supp(\eta_\alpha) \subset U_\alpha$, and $\sum_\alpha \eta_\alpha = 1$.

For each α we have the isomorphisms Ψ_α given by Lemma 4.1.4, and thus we get the $C^\infty(U_\alpha)$-linear homotopies φ_α on $C^\bullet_{CE,der}(U_\alpha, T^*U_\alpha)$. Notice then that we can multiply φ_α by functions on U_α as follows: given $g \in C^\infty(U_\alpha)$ we define, for any $D \in C^k_{CE,der}(U_\alpha, T^*U_\alpha)$:

$$(g\varphi_\alpha)(D) = g(\varphi_\alpha(D)).$$

We define global maps $\tilde{\varphi}_\alpha \colon C^\bullet_{CE,der}(P, T^*P) \to C^{\bullet-1}_{CE,der}(P, T^*P)$ in the following way: for $D \in C^k_{CE,der}(P, T^*P)$ we put

$$\tilde{\varphi}_\alpha(D)|_{U_\alpha} := (\eta_\alpha|_{U_\alpha})\varphi_\alpha(D_{U_\alpha}),$$

and

$$\tilde{\varphi}_\alpha(D)|_{P\setminus supp(\eta_\alpha)} \equiv 0.$$

Nitice that, for each α, these are $C^\infty(P)$-linear maps. Indeed: given $f \in C^\infty(P)$ and $D \in C^k_{CE,der}(P, T^*P)$ we have

$$\tilde{\varphi}_\alpha(fD)|_{U_\alpha} = \eta_\alpha|_{U_\alpha}\varphi_\alpha(fD_{U_\alpha}) = f|_{U_\alpha}\eta_\alpha|_{U_\alpha}\varphi_\alpha(D_{U_\alpha}) = f|_{U_\alpha}\tilde{\varphi}_\alpha(D)|_{U_\alpha},$$

while $\tilde{\varphi}_\alpha(fD)|_{P\setminus supp(\eta_\alpha)} = 0 = f|_{P\setminus supp(\eta_\alpha)}\tilde{\varphi}_\alpha(D)|_{P\setminus supp(\eta_\alpha)}$.

We claim that $\delta\tilde{\varphi}_\alpha + \tilde{\varphi}_\alpha\delta = \eta_\alpha I$, where I is the identity operator in $C^\bullet_{CE,der}(P, T^*P)$. Indeed, let $D \in C^k_{CE,der}(P, T^*P)$ and $g_1, ..., g_k \in C^\infty(P)$. Then, first observe that:

$$(\eta_\alpha I)(D)|_{P\setminus supp(\eta_\alpha)} = 0 = (\delta\tilde{\varphi}_\alpha + \tilde{\varphi}_\alpha\delta)(D)|_{P\setminus supp(\eta_\alpha)}.$$

On the other hand:

$$(\eta_\alpha I)(D)|_{U_\alpha} := \eta_\alpha|_{U_\alpha} D_{U_\alpha}$$

and

$$\begin{aligned}
(\delta\tilde{\varphi}_\alpha + \tilde{\varphi}_\alpha\delta)(D)|_{U_\alpha} &= \delta(\tilde{\varphi}_\alpha(D))|_{U_\alpha} + \tilde{\varphi}_\alpha(\delta D)|_{U_\alpha} \\
&= \delta_\alpha(\tilde{\varphi}_\alpha(D)|_{U_\alpha}) + \eta_\alpha|_{U_\alpha}\varphi_\alpha(\delta D|_{U_\alpha}) \\
&= \delta_\alpha(\eta_\alpha|_{U_\alpha}\varphi_\alpha(D_{U_\alpha}) + \eta_\alpha|_{U_\alpha}\varphi_\alpha(\delta_\alpha D_{U_\alpha}) \\
&= \eta_\alpha|_{U_\alpha}\delta_\alpha(\varphi_\alpha(D_{U_\alpha})) + \eta_\alpha|_{U_\alpha}\varphi_\alpha(\delta_\alpha D_{U_\alpha}) \\
&= \eta_\alpha|_{U_\alpha}(\delta_\alpha\varphi_\alpha + \varphi_\alpha\delta_\alpha)D_{U_\alpha} \\
&= \eta_\alpha|_{U_\alpha} D_{U_\alpha}.
\end{aligned}$$

Thus $\delta\tilde{\varphi}_\alpha + \tilde{\varphi}_\alpha\delta = \eta_\alpha I$, as claimed.

Now we define $\varphi := \sum_\alpha \tilde{\varphi}_\alpha$, which is well defined due to the finiteness property of the partition of unity. Finally, we have, for $f \in C^\infty(P)$ and $D \in C^k_{CE,der}(P, T^*P)$:

$$\varphi(fD) = \Big(\sum_\alpha \tilde{\varphi}_\alpha\Big)(fD) = \sum_\alpha \tilde{\varphi}_\alpha(fD) = \sum_\alpha f\tilde{\varphi}_\alpha(D) = f\sum_\alpha \tilde{\varphi}_\alpha(D) = f\varphi(D),$$

and

$$\delta\varphi + \varphi\delta = \delta\Big(\sum_\alpha \tilde{\varphi}_\alpha\Big) + \Big(\sum_\alpha \tilde{\varphi}_\alpha\Big)\delta = \sum_\alpha(\delta\tilde{\varphi}_\alpha + \tilde{\varphi}_\alpha\delta) = \sum_\alpha \eta_\alpha I = \Big(\sum_\alpha \eta_\alpha\Big) I = I.$$

Hence, the map φ is a $C^\infty(P)$-linear homotopy. $\square$

4.2 The classifying map

We can now state the following result.

Theorem 4.2.1. *Given $B \in \Omega^2_{cl}(P)_\lambda$ and π on P deforming $\pi_0 = 0$, put $\omega_B := \omega_{can} + \rho^*B$. Then there exist $X \in \lambda\mathcal{X}(T^*P)_\lambda$ such that we have a Poisson morphism*

$$\Phi := \exp(X)\rho^* \colon (C^\infty(P)_\lambda, \pi) \to (C^\infty(T^*P)_\lambda, \omega_B),$$

unique up to transformations of the form $\exp(X) \rightsquigarrow \exp(X_H)\exp(X)\exp(V)$, with $X_H = \{H, \cdot\}_{\omega_B}$ and $V \circ \rho^* = 0$. Moreover, if C is the Poisson commutator of $\Phi(C^\infty(P)_\lambda)$ inside $(C^\infty(T^*P)_\lambda, \{\cdot,\cdot\}_{\omega_B})$, there exists $Y \in \mathcal{X}(T^*P)_\lambda$ such that

$$\Psi = \exp(Y)\rho^* \colon \ C^\infty(P)_\lambda \to C$$

is an isomorphism, unique up to formal diffeomorphisms of P.

Proof. The existence and uniqueness of Φ follows from Propositions 3.2.12 and 3.2.13, in view of the triviality of the cohomology $H^k_{CE,der}(P, T^*P)$, for $k \geq 1$, and the existence of horizontal lift as discussed previously. Notice now that from the last item in Lemma 4.1.1 it follows that

$$C^\infty(P)' := \{f \in C^\infty(T^*P), \{f, \rho^* g\}_{\omega_0} = 0 \ \forall g \in C^\infty(P)\} = \rho^*(C^\infty(P)),$$

hence, the existence and uniqueness of Ψ follows from Lemma 3.2.16 and Lemma 3.2.17, in view of the existence of a $C^\infty(P)$-linear homotopy φ given by Proposition 4.1.5. □

Corollary 4.2.2. *There is a map $(B, \pi) \mapsto [\pi^B]$, such that π^B and π are Morita equivalent, according to Definition 2.4.2.*

Proof. The existence of such a map follows from the previous theorem and Proposition 3.2.18. Then observe that π^B and π satisfies the conditions in Definition 2.4.2, hence, they are Morita equivalent. □

Next we show that the map $(B, \pi) \mapsto [\pi^B]$, given by the corollary above descends to the equivalence classes and characterizes Morita equivalent formal Poisson structures. Precisely, we have:

Theorem 4.2.3. *The map $(B, \pi) \mapsto [\pi^B]$ given by Corollary 4.2.2, for $B \in \Omega^2_{cl}(P)_\lambda$ and π formal Poisson structure on P deforming $\pi_0 = 0$, descends to a well defined map*

$$\mathbb{H}^2_{dR}(P)_\lambda \times \mathrm{FPois}_0(P) \to \mathrm{FPois}_0(P); ([B], [\pi]) \mapsto [\pi^B].$$

Two elements $[\pi], [\pi'] \in \mathrm{FPois}_0(P)$ are Morita equivalent, according to Definition 2.4.2, if and only if, $[\pi'] = [\varphi_* \pi^B]$ for some $[B] \in \mathbb{H}^2_{dR}(P)_\lambda$ and $\varphi \in \mathrm{Diff}(P)$.

Proof. The fact that $[\pi^B]$ is independent of the representative of $[\pi]$ was already proved in Proposition 3.2.18.

Now let $B' = B + dA$, for some $A \in \Omega(P)_\lambda$. Then ω_B and $\omega_{B'}$ are cohomologous, thus from Proposition 2.2.10 there exists a Poisson morphism

$$\exp(Z) \colon (C^\infty(T^*P)_\lambda, \omega_{B'}) \to (C^\infty(T^*P)_\lambda, \omega_B).$$

Consider the formal dual pair

$$(C^\infty(P)_\lambda, \pi^{B'}) \xrightarrow{\Psi'} (C^\infty(T^*P)_\lambda, \omega_{B'}) \xleftarrow{\Phi'} (C^\infty(P)_\lambda, \pi),$$

given by the pair (B', π) according to Theorem 4.2.1. Then we get the formal dual pair

$$(C^\infty(P), \pi^{B'}) \xrightarrow{\tilde{\Psi}} (C^\infty(T^*P)_\lambda, \omega_B) \xleftarrow{\tilde{\Phi}} (C^\infty(P)_\lambda, \pi),$$

where $\tilde{\Phi} := \exp(Z)\Phi'$ and $\tilde{\Psi} := \exp(Z)\Psi'$. Thus, by Corollary 4.2.2 we have that π^B and $\pi^{B'}$ are equivalent.

Now consider $[\pi'] = [\psi_* \pi^B]$. Then, from the dual pair

$$(C^\infty(P)_\lambda, \pi^B) \xrightarrow{\Psi} (C^\infty(T^*P)_\lambda, \omega_B) \xleftarrow{\Phi} (C^\infty(P)_\lambda, \pi)$$

given by the pair (B, π) according to Theorem 4.2.1, we get the following formal dual pair

$$(C^\infty(P)_\lambda, \pi') \xrightarrow{\hat{\Psi}} (C^\infty(T^*P)_\lambda, \omega_B) \xleftarrow{\Phi} (C^\infty(P)_\lambda, \pi),$$

with classical limit $(P, 0) \xleftarrow{\varphi\rho} (T^*P, \omega_0) \xrightarrow{\rho} (P, 0)$, where $\omega_0 = \omega_{can} + \rho^* B_0$, which is an equivalence symplectic bimodule. Hence $[\pi']$ is Morita equivalent to $[\pi]$.

For the converse, suppose $[\pi]$ and $[\pi']$ are Morita equivalent by means of a formal dual pair

$$(C^\infty(P)_\lambda, \pi') \xrightarrow{\Psi} (C^\infty(S)_\lambda, \omega) \xleftarrow{\Phi} (C^\infty(P)_\lambda, \pi),$$

with classical limit $(P, 0) \xleftarrow{J_1} (S, \omega_0) \xrightarrow{J_2} (P, 0)$. Then, it follows from Theorem 2.1.16 that $S = T^*P$, $\omega_0 = \omega_{can} + \rho^* B_0$, for some $B_0 \in \Omega^2_{cl}(P)$, $J_2 = \rho$ and $J_1 = \varphi\rho$, for some $\varphi \in \mathrm{Diff}(P)$. Then we observe that, if $s\colon P \to T^*P$ is the zero section, each ω_i in the formal power series ω is cohomologous to $\rho^* B_i$, with $B_i := s^* \omega_i \in \Omega^2_{cl}(P)$. Hence, ω is cohomologous to one of the form $\omega_B = \omega_{can} + \rho^* B$, with $B \in \Omega^2_{cl}(P)_\lambda$. It follows that there exists a Poisson isomorphism

$$\exp(Z)\colon (C^\infty(T^*P)_\lambda, \omega) \to (C^\infty(T^*P)_\lambda, \omega_B).$$

Hence, if

$$(C^\infty(P)_\lambda, \pi^B) \xrightarrow{\hat{\Psi}} (C^\infty(T^*P)_\lambda, \omega_B) \xleftarrow{\hat{\Phi}} (C^\infty(P)_\lambda, \pi)$$

is the formal dual pair given by the pair (B, π), then

$$\Psi^{-1} \exp(-Z) \hat{\Psi} \varphi^*\colon (C^\infty(P)_\lambda, \varphi_* \pi^B) \to (C^\infty(P)_\lambda, \pi')$$

is an equivalence. Hence $[\pi'] = [\varphi_* \pi^B]$, as desired. $\square$

In the next chapter we will find the relation between this classifying map and the B-fields action, arriving to the last ingredient to conclude the main result stated in the introduction.

CHAPTER 5

Morita equivalence of formal Poisson structures and B-fields

In this chapter we establish the relation between Morita equivalence of formal Poisson structures and B-fields action. The main result is:

Classification of Morita equivalence formal Poisson structures via B-fields: *Given a manifold P, two elements $[\pi], [\pi'] \in \mathrm{FPois}_0(P)$ are Morita equivalent if and only if $[\tau_B \pi'] = [\psi_* \pi]$ for some $B \in \Omega^2_{cl}(P)_\lambda$ and $\psi \in \mathrm{Diff}(P)$, where τ is the B-field action map*

5.1 Morita equivalence vs. B-field transformation

The main result in this section is Theorem 5.1.19, which essentially says that the map given by Theorem 4.2.3 is actually an action map, whose orbits coincide with the orbits of B-field actions. It follows that Definition 2.4.2 defines an equivalence relation in $\mathrm{FPois}_0(P)$, and relying on Theorem 2.3.1 we conclude that Morita equivalent elements in $\mathrm{FPois}_0(P)$ related by an integral B-field quantize to Morita equivalent star-produts on P under Kontsevich's deformation quantization.

5.1.1 Formal Courant algebroids

In order to prove Theorem 5.1.19 we need to introduce some technical tools. Essentially, we will extend the notions of Courant algebroids, along with its symmetries and derivations, as presented in Appendix B, to the realm of formal structures.

Courant algebroids on $\Gamma(\mathbb{T}M)_\lambda$

Given a smooth manifold M, let $\mathbb{T}M := TM \oplus T^*M$. A Courant algebroid structure on $\mathbb{T}M$ is given by some structural maps defined on the $C^\infty(M)$-module $\Gamma(\mathbb{T}M)$ (see Appendix B.2). Here we want to extend those structural maps into maps defined on the $C^\infty(M)_\lambda$-module $\Gamma(\mathbb{T}M)_\lambda$, in such a way that they are linked together by the same axioms defining a Courant algebroid structure on $\Gamma(\mathbb{T}M)$, and extend into this formal context some results about symmetries and derivations of Courant algebroids. First of all, notice that there is a natural isomorphism

$$\Gamma(\mathbb{T}M)_\lambda \cong \mathcal{X}(M)_\lambda \oplus \Omega(M)_\lambda,$$

thus, any element $\sigma \in \Gamma(\mathbb{T}M)_\lambda$ can be uniquely written as $\sigma = X \oplus \alpha$, with $X \in \mathcal{X}(M)_\lambda$ and $\alpha \in \Omega(M)_\lambda$.

We then start by extending to the formal realm some elements of the Cartan calculus.

Definition 5.1.1. ***(Lie derivative, contraction, and differential)*** *Given $X \in \mathcal{X}(M)_\lambda$, we define the Lie derivative along X,*

$$\mathcal{L}_X : \mathcal{X}(M)_\lambda \oplus \Omega(M)_\lambda \to \mathcal{X}(M)_\lambda \oplus \Omega(M)_\lambda,$$

by

$$\mathcal{L}_X(Y \oplus \alpha) := \sum_{j=0}^{\infty} \lambda^j \sum_{k+l=j} \mathcal{L}_{X_k}(Y_l \oplus \alpha_l),$$

the contraction by X,

$$\iota_X : \Omega^\bullet(M)_\lambda \to \Omega^{\bullet-1}(M)_\lambda,$$

by

$$\iota_X \eta := \sum_{j=0}^{\infty} \lambda^j \sum_{k+l=j} \iota_{X_k} \eta_l,$$

and the differential,

$$d : \Omega^\bullet(M)_\lambda \to \Omega^{\bullet+1}(M)_\lambda,$$

by

$$d\eta := \sum_{j=0}^{\infty} \lambda^j d\eta_j.$$

Everywhere in the R.H.S. we use the usual Lie derivative, contraction and differential.

Notice that in the definition above of Lie derivative along formal vector fields, there is already included the definition of Lie bracket of formal vector fields. With these definitions, we recover usual properties of the Cartan calculus, like the following ones:

1. $\mathcal{L}_X = \iota_X d + d\iota_X$. "Cartan's magic formula",

2. $\mathcal{L}_{[X,Y]} = [\mathcal{L}_X, \mathcal{L}_Y]$,

3. $\iota_{[X,Y]} = [\mathcal{L}_X, \iota_Y]$,

with the brackets on the R.H.S. being just commutators. These can be checked by simply expanding by λ-linearity both sides of the equalities and comparing terms at each order.

In this same spirit, we now extend the structural maps of the standard Courant algebroid on $\Gamma(\mathbb{T}M)$ to maps defined on $\Gamma(\mathbb{T}M)_\lambda$. Thus, for $\sigma = \sum_{j=0}^\infty \lambda^j \sigma_j$ and $\eta = \sum_{j=0}^\infty \lambda^j \eta_j$, elements of $\Gamma(\mathbb{T}M)_\lambda$, we consider the pairing,

$$\langle \sigma, \eta \rangle := \sum_{j=0}^\infty \lambda^j \sum_{k+l=j} \langle \sigma_k, \eta_l \rangle,$$

the Courant bracket,

$$[\sigma, \eta] := \sum_{j=0}^\infty \lambda^j \sum_{k+l=j} [\sigma_k, \eta_l],$$

the anchor map,

$$\rho : \Gamma(E)_\lambda \to \mathcal{X}(M)_\lambda;\ \sigma \mapsto \sum_{j=0}^\infty \lambda^j \rho(\sigma_j),$$

and the differential,

$$d : C^\infty(M)_\lambda \to \Omega(M)_\lambda;\ f \mapsto \sum_{j=0}^\infty \lambda^j df_j,$$

using in the R.H.S. the structural maps defined on sections of $\mathbb{T}M$ (see Appendix B.2).

Writing $\sigma = X \oplus \alpha$ and $\eta = Y \oplus \beta$ via the isomorphism $\Gamma(\mathbb{T}M)_\lambda \cong \mathcal{X}(M)_\lambda \oplus \Omega(M)_\lambda$, these definitions recover the formulas for the pairing, the bracket, the anchor and the differential given for sections of $\mathbb{T}M$ in Example B.2.1; for instance, we have:

$$\langle \sigma, \eta \rangle = \langle X \oplus \alpha, Y \oplus \beta \rangle = \iota_X \beta + \iota_Y \alpha.$$

In particular, these extended structural maps also satisfy the Courant algebroid axioms listed in Appendix B.2. We will refer to $\Gamma(\mathbb{T}M)_\lambda$ with these structural maps as a *formal Courant algebroid on* $\Gamma(\mathbb{T}M)_\lambda$, or simply as the *Courant algebroid* $\Gamma(\mathbb{T}M)_\lambda$ if there is no risk of confusion.

Symmetries and derivations of $\Gamma(\mathbb{T}M)_\lambda$

Definition 5.1.2. ***(Formal Courant Symmetries)*** *Symmetries of the formal Courant algebroid on* $\Gamma(\mathbb{T}M)_\lambda$ *are pairs* (F, ϕ)*, where* ϕ *is a formal diffeomorphism of* M*, i.e., it is a map of the form*

$$\phi = \exp(X)\colon\ C^\infty(M)_\lambda \to C^\infty(M)_\lambda,$$

for some $X \in \lambda\mathcal{X}(M)_\lambda$*, and*

$$F\colon \Gamma(\mathbb{T}M)_\lambda \to \Gamma(\mathbb{T}M)_\lambda,$$

is a $\mathbb{C}_\lambda$-linear map, satisfying, for any $\sigma, \eta \in \Gamma(\mathbb{T}M)_\lambda$, and $g \in C^\infty(M)_\lambda$:

1. $F(g\sigma) = \phi^{-1}(g)F(\sigma)$,
2. $\phi\langle F(\sigma), F(\eta)\rangle = \langle \sigma, \eta\rangle$,
3. $[F(\sigma), F(\eta)] = F([\sigma, \eta])$.

Example 5.1.3. ***(Symmetries induced by formal diffeomorphisms)*** *Let $\phi = \exp(X)$ be a given formal diffeomorphism. Define $F := \exp(-\mathcal{L}_X)$. Then (F, ϕ) is a formal Courant symmetry. The proof of this assertion is a kind of straightforward but tedious computation to check the three defining properties above. Let's give some indication for the first one. First note that, by linearity, it suffices to check for $g = g_0 \in C^\infty(M)$ and $\sigma = \sigma_0 \in \Gamma(\mathbb{T}M)$. Then observe that $X = \lambda X_1 + \lambda^2 X_2 + \cdots = \lambda\hat{X}$, with $\hat{X} = X_1 + \lambda X_2 + \cdots$, thus we have $\exp(-X) = \exp(-\lambda\hat{X})$ and $\exp(-\mathcal{L}_X) = \exp(-\lambda\mathcal{L}_{\hat{X}})$. Then expand both $\exp(-\lambda\hat{X})(g_0)\exp(-\lambda\mathcal{L}_{\hat{X}})(\sigma_0)$ and $\exp(-\lambda\mathcal{L}_{\hat{X}})(g_0\sigma_0)$ in terms of λ and check the equality order by order using induction. The other properties can be proved in the same way. Notice that the inverse of this symmetry is just the pair $(\exp(\mathcal{L}_X), \exp(-X))$.*

Example 5.1.4. ***(Symmetries induced by B-fields)*** *Consider now an element $B \in \Omega^2_{cl}(M)_\lambda$. It induces on $\Gamma(\mathbb{T}M)_\lambda$ the formal version of a B-field transformation, via*

$$e^B(X \oplus \alpha) := X \oplus \alpha + \iota_X B.$$

Then we have that the pair (e^B, I) is a formal symmetry of $\Gamma(\mathbb{T}M)_\lambda$. This is proved in the same way as in the geometric case, by using the Cartan calculus properties. For example:

$$\begin{aligned}
[e^B(X \oplus \alpha), e^B(X \oplus \alpha)] &= [X \oplus \alpha + \iota_X B, Y \oplus \beta + \iota_Y B] \\
&= [X, Y] \oplus \mathcal{L}_X(\beta + \iota_Y B) - \iota_Y d(\alpha + \iota_X B) \\
&= [X, Y] \oplus \mathcal{L}_X\beta - \iota_Y d\alpha + \mathcal{L}_X\iota_Y B - \iota_Y\mathcal{L}_X B \\
&= e^B[X \oplus \alpha, Y \oplus \beta],
\end{aligned}$$

where; in the third equality we use the Cartan's magic formula and the fact that $dB = 0$; for the fourth equality, we use $\iota_{[X,Y]} = [\mathcal{L}_X, \iota_Y]$ and the definition of e^B.

The set $\mathrm{Diff}_f(M)$ of formal diffeomorphisms on M has a group structure given by

$$\exp(X)\exp(Y) = \exp(BCH(X, Y)),$$

where

$$BCH(X, Y) = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}([X, [X, Y]] + [Y, [Y, X]]) + \cdots,$$

is the Baker-Campbell-Hausdorff formula (see [41, Chap. V]).

This group acts on the set $\Omega^\bullet(M)_\lambda$ of formal differential forms on M, via

$$\exp(X) \mapsto \exp(\mathcal{L}_X),$$

moreover, since $d\mathcal{L}_X = \mathcal{L}_X d$, the action restricts to closed formal differential forms.

In the following lemma, we establish some technical result we are going to need.

Lemma 5.1.5. *For any $X, Y \in \mathcal{X}(M)_\lambda$, $B \in \Omega^2(M)_\lambda$ and $n \geq 1$ we have*

1. $\iota_{\mathcal{L}_Y^n X} = \mathcal{L}_Y^n \iota_X - \binom{n}{1}\mathcal{L}_Y^{n-1}\iota_X\mathcal{L}_Y + \cdots (-1)^k\binom{n}{k}\mathcal{L}_Y^{n-k}\iota_X\mathcal{L}_Y^k + \cdots (-1)^n \iota_X \mathcal{L}_Y^n$.
2. $\exp(\lambda\mathcal{L}_Y)\iota_X = \iota_{\exp(\lambda\mathcal{L}_Y)X}\exp(\lambda\mathcal{L}_Y)$.
3. $e^B\exp(-\lambda\mathcal{L}_Y) = \exp(-\lambda\mathcal{L}_Y)e^{\exp(\lambda\mathcal{L}_Y)B}$,

   where $\exp(\lambda\mathcal{L}_Y)B$ *is the action of* $\mathrm{Diff}_f(M)$ *on* $\Omega^2_{cl}(M)_\lambda$*, mentioned above.*

Proof. 1. For $n = 1$ this is just the property $\iota_{\mathcal{L}_Y X} = [\mathcal{L}_Y, \iota_X]$. We now proceed by induction on n, thus:

$$\begin{aligned}
\iota_{\mathcal{L}_Y^n X} &= \iota_{\mathcal{L}_Y(\mathcal{L}_Y^{n-1}X)} = \mathcal{L}_Y\iota_{\mathcal{L}_Y^{n-1}X} - \iota_{\mathcal{L}_Y^{n-1}X}\mathcal{L}_Y \\
&= \mathcal{L}_Y\left(\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}\mathcal{L}_Y^{n-1-k}\iota_X\mathcal{L}_Y^k\right) - \left(\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}\mathcal{L}_Y^{n-1-k}\iota_X\mathcal{L}_Y^k\right)\mathcal{L}_Y \\
&= \sum_{k=0}^{n}(-1)^k\binom{n}{k}\mathcal{L}_Y^{n-k}\iota_X\mathcal{L}_Y^k.
\end{aligned}$$

2. First notice that the proposed identity is equivalent to

$$\exp(\lambda\mathcal{L}_Y)\iota_X\exp(-\lambda\mathcal{L}_Y) = \iota_{\exp(\lambda\mathcal{L}_Y)X}. \tag{5.1}$$

Thus we expand both sides of the above expression as follows:

$$\iota_{\exp(\lambda\mathcal{L}_Y)X} = \iota_X + \lambda\iota_{\mathcal{L}_Y X} + \cdots + \frac{\lambda^k}{k!}\iota_{\mathcal{L}_Y^k X} + \cdots, \tag{5.2}$$

and

$$\iota_X\exp(-\lambda\mathcal{L}_Y) = \iota_X - \lambda\iota_X\mathcal{L}_Y + \cdots + (-1)^k\frac{\lambda^k}{k!}\iota_X\mathcal{L}_Y^k + \cdots,$$

thus

$$\begin{aligned}
\exp(\lambda\mathcal{L}_Y)\iota_X\exp(-\lambda\mathcal{L}_Y) = \iota_X + \lambda(\mathcal{L}_Y\iota_X - \iota_X\mathcal{L}_Y) + \cdots + \lambda^k\bigg(\frac{\mathcal{L}_Y^k}{k!}\iota_X - \frac{\mathcal{L}_Y^{k-1}}{(k-1)!}\iota_X\mathcal{L}_Y \\
+ \frac{\mathcal{L}_Y^{k-2}}{(k-2)!}\iota_X\frac{\mathcal{L}_Y^2}{2!} - \cdots + (-1)^k\iota_X\frac{\mathcal{L}_Y^k}{k!}\bigg) + \cdots.
\end{aligned} \tag{5.3}$$

Then, expanding the term in λ^k in (5.2) using the formula in item 2, we see that it is exactly the term in λ^k in (5.3). Thus, equality (5.1) is proved, and the result follows.

3. Evaluating both sides of the proposed equality in an arbitrary element $X \oplus \alpha \in \Gamma(\mathbb{T}M)_\lambda$, we have:

$$e^B \exp(-\lambda \mathcal{L}_Y)(X \oplus \alpha) = \exp(-\lambda \mathcal{L}_Y)X \oplus \exp(-\lambda \mathcal{L}_Y)\alpha + \iota_{\exp(-\lambda \mathcal{L}_Y)X} B,$$

and

$$\exp(-\lambda \mathcal{L}_Y)e^{\exp(\lambda \mathcal{L}_Y)B}(X \oplus \alpha) = \exp(-\lambda \mathcal{L}_Y)X \oplus \exp(-\mathcal{L}_Y)\alpha + \exp(-\lambda \mathcal{L}_Y)\iota_X \exp(\lambda \mathcal{L}_Y)B.$$

Thus, it suffices to show that for any $X \in \mathcal{X}(M)_\lambda$:

$$\iota_{\exp(-\lambda \mathcal{L}_Y)X} B = \exp(-\lambda \mathcal{L}_Y)\iota_X \exp(\lambda \mathcal{L}_Y)B, \ \forall B \in \Omega^2(M)_\lambda,$$

which is equivalent to:

$$\iota_{\exp(-\lambda \mathcal{L}_Y)X} = \exp(-\lambda \mathcal{L}_Y)\iota_X \exp(\lambda \mathcal{L}_Y).$$

Hence, the result follows from item 2, by replacing Y with $-Y$.

□

The same argument applied by Gualtieri in [36] to prove Theorem B.2.4 can be applied here, in the formal context, to obtain the following result.

Proposition 5.1.6. *Let (F, ϕ) be a formal symmetry of $\Gamma(\mathbb{T}M)$, with $\phi = \exp(X)$. Then*

$$F = \exp(-\mathcal{L}_X)e^B,$$

for some $B \in \Omega^2_{cl}(M)_\lambda$. Thus we have $\mathrm{Aut}(\Gamma(\mathbb{T}M)_\lambda) \simeq \mathrm{Diff}_f(M) \ltimes \Omega^2_{cl}(M)_\lambda$, where the semi-direct product structure is given by the action $\exp(X) \mapsto \exp(\mathcal{L}_X)$.

Proof. Consider $G := \exp(\mathcal{L}_X)F$. Then we get that (G, I) is a formal symmetry. Let $\sigma, \eta \in \Gamma(\mathbb{T}M)_\lambda$ and $g \in C^\infty(M)_\lambda$ be arbitrary, then by developing $G[\sigma, g\eta] = [G(\sigma), G(g\eta)]$ using the properties of the bracket and the $C^\infty(M)_\lambda$-linearity of G, we get that $\rho = \rho G$. Then writing

$$G = \begin{pmatrix} G_1 & G_2 \\ G_3 & G_4 \end{pmatrix},$$

according to the splitting $\Gamma(\mathbb{T}M)_\lambda = \mathcal{X}(M)_\lambda \oplus \Omega(M)_\lambda$, we conclude that $G_2 \equiv 0$ and $G_1 \equiv I$. Using orthogonality, we conclude that $2\iota_X G_3(X) + 2\iota_X G_4(\alpha) = 2\iota_X \alpha$, for any $X \oplus \alpha$. Thus, taking $\alpha = 0$, we get that $B := G_3 \colon \mathcal{X}(M)_\lambda \to \Omega(M)_\lambda$ is a $C^\infty(M)_\lambda$-linear skew-symmetric map, hence it can be seen as an element in $\Omega^2(M)_\lambda$. Finally, we have $2\iota_X G_4(\alpha) = 2\iota_X \alpha$ for any $X \oplus \alpha$, thus $G_4 = I$. Therefore, $G = e^B$, which being a symmetry, forces B to be actually closed. Then we conclude $F = \exp(-\mathcal{L}_X)e^B$, with $B \in \Omega^2_{cl}(M)_\lambda$, as claimed.

The map $\Psi\colon \mathrm{Diff}_f(M) \times \Omega^2_{cl}(M)_\lambda \to \mathrm{Aut}(\Gamma(\mathbb{T}M)_\lambda); (\exp(X), B) \mapsto \exp(-\mathcal{L}_X)e^B$ is a bijection, and the semi-direct product indicated in the statement is, due to item 3 in the previous lemma, precisely the group product induced by Ψ. Indeed,

$$\begin{aligned}(\exp(X), e^B)(\exp(Y), e^C) &:= \Psi^{-1}(\exp(-\mathcal{L}_X)e^B \exp(-\mathcal{L}_Y)e^C) \\ &= \Psi^{-1}(\exp(-\mathcal{L}_X) \exp(-\mathcal{L}_Y)e^{\exp(\mathcal{L}_Y)B}e^C) \\ &= (\exp(BCH(X, Y)), \exp(\mathcal{L}_Y)B + C).\end{aligned}$$

□

Definition 5.1.7. ***(Formal Courant Derivations)*** *Derivations of the formal Courant algebroid $\Gamma(\mathbb{T}M)_\lambda$ are $\mathbb{C}_\lambda$-linear maps*

$$D\colon \Gamma(\mathbb{T}M)_\lambda \to \Gamma(\mathbb{T}M)_\lambda,$$

for which there exists a formal vector field $X \in \mathcal{X}(M)_\lambda$ such that:

1. $D(g\sigma) = gD(\sigma) + X(g)\sigma$.
2. $D[\sigma, \eta] = [D(\sigma), \eta] + [\sigma, D(\eta)]$.
3. $X\langle\sigma, \eta\rangle = \langle D(\sigma), \eta\rangle + \langle\sigma, D(\eta)\rangle$.

The formal vector field X is usually called the symbol of D.

Any element $e \in \Gamma(\mathbb{T}M)_\lambda$ yields a derivation via $(D, X) = ([e, \cdot], \rho(e))$, as can be verified directly from the properties of the bracket and the anchor.

Let $F_t = \exp(-t\mathcal{L}_X)e^{B_t}$ be a one-parameter family of formal Courant symmetries, with $F_0 = I$. We may write it as

$$F_t = \begin{pmatrix} \exp(-t\mathcal{L}_X) & 0 \\ \exp(-t\mathcal{L}_X)B_t & \exp(-t\mathcal{L}_X) \end{pmatrix},$$

according to the splitting $\Gamma(\mathbb{T}M)_\lambda \cong \mathcal{X}(M)_\lambda \oplus \Omega(M)_\lambda$. Then we compute its derivative at $t = 0$, noticing that $-\frac{d}{dt}\big|_{t=0} \exp(-t\mathcal{L}_X) = \mathcal{L}_X$ and $B_0 = 0$, to get

$$-\frac{d}{dt}\bigg|_{t=0} F_t = \begin{pmatrix} \mathcal{L}_X & 0 \\ -b & \mathcal{L}_X \end{pmatrix},$$

where $b := \frac{d}{dt}\big|_{t=0}B_t \in \Omega^2_{cl}(M)_\lambda$. Thus, we can identify $-\frac{d}{dt}\big|_{t=0}F_t$ with a pair $(X, b) \in \mathcal{X}(M)_\lambda \oplus \Omega^2_{cl}(M)_\lambda$ acting on $\Gamma(\mathbb{T}M)_\lambda$ by

$$(X, b)Y \oplus \beta = [X, Y] \oplus \mathcal{L}_X\beta - \iota_Y b. \tag{5.4}$$

It can be seen, by using the Cartan calculus properties, that this yields a derivation of $\Gamma(\mathbb{T}M)_\lambda$ with symbol X.

If $e = X \oplus \alpha \in \Gamma(\mathbb{T}M)_\lambda$, we get a derivation as the pair $(X, d\alpha)$, and

$$(X, d\alpha)Y \oplus \beta = [X, Y] \oplus \mathcal{L}_X\beta - \iota_Y d\alpha = [X \oplus \alpha, Y \oplus \beta].$$

Again in the same spirit of [36], we can now prove an analogous of Theorem B.2.6 in the formal setting.

Proposition 5.1.8. *Let $(X, b) \in \mathcal{X}(M)_\lambda \oplus \Omega^2_{cl}(M)_\lambda$, with $X = \lambda X_1 + \cdots$, be a derivation of $\Gamma(\mathbb{T}M)_\lambda$ acting via (5.4). Then, it induces a one-parameter subgroup of symmetries of $\Gamma(\mathbb{T}M)_\lambda$,*

$$F_t = \exp(-t\mathcal{L}_X)e^{B_t},$$

where

$$B_t = \int_0^t \exp(s\mathcal{L}_X)bds.$$

Proof. To see that F_t is a symmetry of $\Gamma(\mathbb{T}M)_\lambda$ just notice that

$$dB_t = \int_0^t \exp(s\mathcal{L}_X)dbds = 0,$$

since b is closed. To see that it yields a one-parameter subgroup of symmetries, we compute

$$\begin{aligned}
B_{t+s} &= \int_0^{t+s} \exp(\epsilon\mathcal{L}_X)bd\epsilon = \int_0^{s} \exp(\epsilon\mathcal{L}_X)bd\epsilon + \int_s^{s+t} \exp(\epsilon\mathcal{L}_X)bd\epsilon \\
&= B_s + \int_0^t \exp((s+u)\mathcal{L}_X)bdu = B_s + \exp(s\mathcal{L}_X)\int_0^t \exp(u\mathcal{L}_X)bdu \\
&= B_s + \exp(s\mathcal{L}_X)B_t.
\end{aligned}$$

Hence, due to item 3 in Lemma 5.1.5, we get

$$\begin{aligned}
F_tF_s &= \exp(-t\mathcal{L}_X)e^{B_t}\exp(-s\mathcal{L}_X)e^{B_s} = \exp(-t\mathcal{L}_X)\exp(-s\mathcal{L}_X)e^{\exp(s\mathcal{L}_X)B_t + B_s} \\
&= \exp(-(t+s)\mathcal{L}_X)e^{B_{t+s}} = F_{t+s},
\end{aligned}$$

and we are done. $\square$

We call a $C^\infty(M)_\lambda$-submodule $L \subset \Gamma(\mathbb{T}M)_\lambda$ **involutive** if $[L, L] \subset L$, and **lagrangian** if $L^\perp = L$.

Remark 5.1.9. *The same argument as in Remark B.1.2 gives us here the same result, namely, the involutivity of a lagrangian L is equivalent to the vanishing of*

$$\Upsilon(a, b, c) := \langle [a, b], c\rangle,$$

for any $a, b, c \in L$.

Example 5.1.10. *Given a formal bivector field π on M, we may consider $L_\pi := graph(\pi^\sharp) \subset \Gamma(\mathbb{T}M)_\lambda$, which is a $C^\infty(M)_\lambda$-submodule. Then, the same computation as in Lemma (B.1.3) gives us here that L_π is lagrangian, and it is involutive if and only if π is a formal Poisson tensor.*

Remark 5.1.11. *If L is an involutive submodule of $\Gamma(\mathbb{T}M)_\lambda$ and $e = X \oplus \alpha \in L$, it follows that the one-parameter group of symmetries F_t, generated by $(X, d\alpha)$ according to Proposition 5.1.8, preserves L.*

5.1.2 Classification via B-fields

For the computations in this section we will consider point-wise evaluation as well as restriction to open sets of formal objects, in the following sense: let M be a smooth manifold and consider, for instance, a formal complex-valued smooth function $f = \sum_{j=0}^{\infty} \lambda^j f_j \in C^\infty(M)_\lambda$, then given $q \in M$ we get $f_q := \sum_{j=0}^{\infty} \lambda^j f_j(q) \in \mathbb{C}_\lambda$. On the other hand, given an open subset $U \subset M$, we may define $f_U := \sum_{j=0}^{\infty} \lambda^j f_j|_U \in C^\infty(U)_\lambda$. Likewise, given a formal vector field X, a formal 1-form η or a formal Poisson structure π on M, inducing the map $\pi^\sharp \colon \Omega(M)_\lambda \to \mathcal{X}(M)_\lambda$, we get a formal series of tangent vectors, $X_q \in (T_qM)_\lambda$, and a formal vector field $X_U \in \mathcal{X}(U)_\lambda$; a formal series of covectors, $\eta \in (T_q^*M)_\lambda$, and a formal 1-form $\eta_U \in \Omega(U)_\lambda$, or a map $\pi_q^\sharp \colon (T_q^*M)_\lambda \to (T_qM)_\lambda$ and a map $\pi^\sharp|_U \colon \Omega(U)_\lambda \to \mathcal{X}(U)_\lambda$.

Given a smooth map $\varphi \colon M \to N$ we have an induced map $\varphi^* \colon \Omega^\bullet(N)_\lambda \to \Omega^\bullet(M)_\lambda$, and for each $q \in M$:

$$d\varphi_q \colon (T_qM)_\lambda \to (T_{\varphi(q)}N)_\lambda,$$

and

$$(d\varphi_q)^* \colon (T^*_{\varphi(q)}N)_\lambda \to (T_q^*M)_\lambda.$$

Then, an element $a = X \oplus \alpha \in \mathcal{X}(M)_\lambda \oplus \Omega(M)$ and an element $b = Y \oplus \beta \in \mathcal{X}(N) \oplus \Omega(N)_\lambda$ are called φ-related, denoted by $a \sim_\varphi b$, if $\alpha = \varphi^*\beta$ and $d\varphi_q X_q = Y_{\varphi(q)}$, for all $q \in M$.

Given a smooth manifold P, let $\rho \colon T^*P \to P$ be its cotangent bundle, and consider the induced map

$$\rho^* \colon \Omega^\bullet(P)_\lambda \to \Omega^\bullet(T^*P)_\lambda.$$

Any formal vector field $\mathcal{V} \in \lambda\mathcal{X}(T^*P)_\lambda$ induces a deformation of ρ^* as follows:

$$\varrho^* := \exp(\mathcal{L}_\mathcal{V})\rho^* \colon \Omega^\bullet(P)_\lambda \to \Omega^\bullet(T^*P)_\lambda.$$

If π is a formal Poisson structure on P, we consider $L_\pi := graph(\pi^\sharp) \subset \mathcal{X}(P)_\lambda \oplus \Omega(P)_\lambda$, while for a given $x \in P$ we consider $(L_\pi)_x := graph(\pi_x^\sharp) \subset (T_xP)_\lambda \oplus (T_x^*P)_\lambda$, and for a given open subset $U \subset P$, we put $(L_\pi)|_U := graph(\pi^\sharp|_U)$. Notice then that we have $X \oplus \alpha \in (L_\pi)|_U$ if and only if $(X \oplus \alpha)_x \in (L_\pi)_x$ for all $x \in U$. We also define, for a given $q \in T^*P$:

$$(\rho^* L_\pi)_q := \{v \oplus (d\rho_q)^*\xi \in (T_qT^*P)_\lambda \oplus (T_q^*T^*P)_\lambda; d\rho_q v \oplus \xi \in (L_\pi)_{\rho(q)}\},$$

and then

$$\rho^* L_\pi := \{X \oplus \alpha \in \mathcal{X}(T^*P)_\lambda \oplus \Omega(T^*P)_\lambda; (X \oplus \alpha)_q \in (\rho^* L_\pi)_q, \forall q \in T^*P\}.$$

Example 5.1.12. *If $X \oplus \alpha$ satisfies $\alpha = \rho^*\beta$ and $X \sim_\rho \pi^\sharp(\beta)$, then $X \oplus \alpha \in \rho^* L_\pi$.*

Remark 5.1.13. *The elements $X \oplus \alpha$ of $\rho^* L_\pi$ can be more general than the ones captured in the previous example. If $X \oplus \alpha \in \rho^* L_\pi$, we have, first of all, $\alpha = \sum_{j=0}^\infty \lambda^j \alpha_j$ with $\alpha_j \in \Omega(T^*P)$. Then, by definition of $\rho^* L_\pi$, for any $q \in T^*P$ we have $\alpha_q = (d\rho_q)^* \xi_{\rho(q)}$, for some unique $\xi_{\rho(q)} \in (T^*_{\rho(q)}P)_\lambda$, hence, by Corollary A.1.2, $\alpha_j \in \Gamma(\rho^* T^*P)$, for each j, and therefore $\alpha \in \Gamma(\rho^* T^*P)_\lambda$. Now, by Example A.1.4, for any induced cotangent chart T^*U, with coordinates (x^i, ξ_i), over a chart U on P, with coordinates (x^i), we have $\alpha_j = \sum_{r=1}^n f_r^j \rho^*(dx^r)$, for some $f_r^j \in C^\infty(T^*U)$. Then, for any $q \in T^*U$, the formal vector field X must satisfies $d\rho_q X_q = \pi^\sharp_{\rho(q)}(\sum_{j=0}^\infty \lambda^j (\sum_{r=1}^n f_r^j(q) dx^r|_{\rho(q)}))$.*

Finally, we define

$$\varrho^* L_\pi := \exp(\mathcal{L}_{\mathcal{V}})(\rho^* L_\pi).$$

Regarding these definitions, we now prove some technical result we are going to need.

Lemma 5.1.14. *Given any $a_0 = X_0 \oplus \alpha_0 \in (\rho^* L_\pi)_q$, there exists, for some open sets $U \subset P$ and $T^*U \subset T^*P$ with $q \in T^*U$, local formal sections $a = X \oplus \alpha \in (\rho^* L_\pi)|_{T^*U}$ and $b = Y \oplus \beta \in (L_\pi)|_U$ with $a \sim_\rho b$ and $a_q = a_0$.*

Proof. Pick a trivializing chart U around $p = \rho(q)$, and consider $T^*U \xrightarrow{\rho} U$. Since $X_0 \oplus \alpha_0 \in (\rho^* L_\pi)_q$, we have $\alpha_0 = (d\rho_q)^* \beta_0$, for a unique $\beta_0 \in (T^*_p U)_\lambda$ and $d\rho_q X_0 = \pi^\sharp_p(\beta_0)$. Extend β_0 to an element $\beta \in \Omega(U)_\lambda$ and define $Y := \pi^\sharp(\beta) \in \mathcal{X}(U)_\lambda$. Then we have $b := Y \oplus \beta \in (L_\pi)|_U$. On the other hand, define $\alpha := \rho^*\beta \in \Omega(T^*U)_\lambda$ and pick any formal vector field $Z \in \mathcal{X}(T^*U)_\lambda$ which is ρ-related to Y (which exists since ρ is a surjective submersion). Then notice that $V_0 := Z_q - X_0$ satisfies

$$d\rho_q V_0 = d\rho_q Z_q - d\rho_q X_0 = Y_p - \pi^\sharp_p(\beta_0) = 0,$$

hence it is vertical. Extend V_0 into a vector field $V \in \Gamma(\mathrm{Ver}(T^*U))_\lambda$ and define $X := Z - V \in \mathcal{X}(T^*U)_\lambda$. Then $a := X \oplus \alpha \in (\rho^* L_\pi)|_{T^*U}$ is ρ-related to b and satisfies $a_q = a_0$. ☐

Lemma 5.1.15. *With the previous definition, $\rho^* L_\pi \subset \mathcal{X}(T^*P)_\lambda \oplus \Omega(T^*P)_\lambda$ is involutive.*

Proof. We first prove that $\rho^* L_\pi$ is lagrangian for the natural pairing on $\mathcal{X}(T^*P)_\lambda \oplus \Omega(T^*P)_\lambda$. To show $\rho^* L_\pi \subset (\rho^* L_\pi)^\perp$, we need to check that $\langle a_1, a_2 \rangle_q = 0$ for all $a_1, a_2 \in \rho^* L_\pi$ and $q \in T^*P$. Thus, let $X_1 \oplus \alpha_1$ and $X_2 \oplus \alpha_2$ be arbitrary elements in $\rho^* L_\pi$ and pick a trivializing open set $U \subset P$ around $\rho(q)$. By the previous lemma, for each $i = 1, 2$ we can find $Y_i \oplus \rho^*\beta_i \in (\rho^* L_\pi)|_{T*U}$ extending $(X_i \oplus \alpha_i)_q$ and $Z_i \oplus \beta_i \in (L_\pi)|_U$ with $Y_i \sim_\rho Z_i$. Then we compute

$$\langle Y_1 \oplus \rho^*\beta_1, Y_2 \oplus \rho^*\beta_2 \rangle_q = (\iota_{Y_1} \rho^*\beta_2 + \iota_{Y_2} \rho^*\beta_1)_q = (\rho^*(\iota_{Z_1}\beta_2 + \iota_{Z_2}\beta_1))_q = \langle Z_1 \oplus \beta_1, Z_2 \oplus \beta_2 \rangle_{\rho(q)} = 0.$$

But $\langle X_1 \oplus \alpha_1, X_2 \oplus \alpha_2\rangle_q = \langle Y_1 \oplus \rho^*\beta_1, Y_2 \oplus \rho^*\beta_2\rangle_q$, hence, $\rho^* L_\pi \subset (\rho^* L_\pi)^\perp$.

On the other hand, given $Z \oplus \eta \in (\rho^* L_\pi)^\perp$, we have, for any $X \oplus \alpha \in \rho^* L_\pi$:

$$\langle Z \oplus \eta, X \oplus \alpha\rangle = \iota_Z \alpha + \iota_X \eta = 0.$$

In particular, taking $\alpha = 0$ we get $X \oplus 0 \in \rho^* L_\pi$ if and only if $X \in \Gamma(\mathrm{Ver}(T^*P))_\lambda$, thus $\iota_X \eta = 0$ for any $X \in \Gamma(\mathrm{Ver}(T^*P))_\lambda$, in particular, $(\iota_{X_0}\eta)_q = 0$ for any $q \in T^*P$ and $X_0 \in \ker(d\rho_q)$. Hence, $\eta = \sum_{j=0}^\infty \lambda^j \eta_j$ satisfies that $\eta_j|_q \in \mathrm{Ann}(\ker(d\rho_q)) = (d\rho_q)^*(T_p^*P)$, where $p = \rho(q)$. Thus, $\eta_q = (d\rho_q)^*\tilde{\eta}_p$ for some $\tilde{\eta}_p \in (T_p^*P)_\lambda$. Now observe that, being ρ a surjective submersion, for any $\alpha \in \Omega(P)_\lambda$ there exists $X \in \mathcal{X}(T^*P)_\lambda$ such that $X \sim_\rho \pi^\sharp(\alpha)$, hence $X \oplus \rho^*\alpha \in \rho^* L_\pi$, and we get

$$0 = (\iota_Z \rho^*\alpha + \iota_X \eta)_q = \alpha_p(d\rho_q Z_q) + \tilde{\eta}_p(\pi_p^\sharp(\alpha_p)) = \alpha_p(d\rho_q Z_q - \pi_p^\sharp(\tilde{\eta}_p)).$$

Since α was arbitrary and ρ^* is injective, we conclude that $d\rho_q Z_q = \pi_p^\sharp(\tilde{\eta}_p)$, thus, $Z \oplus \eta \in \rho^* L_\pi$.

To conclude that $\rho^* L_\pi$ is involutive, we need to verify (see Remark 5.1.9) that

$$\langle [a_1, a_2], a_3\rangle = 0, \ \forall a_1, a_2, a_3 \in \rho^* L_\pi.$$

First notice that, for $a_i \in \mathcal{X}(T^*P)_\lambda \oplus \Omega(T^*P)_\lambda$ and $b_i \in \mathcal{X}(P)_\lambda \oplus \Omega(P)_\lambda$, for $i = 1, 2, 3$, and $a_i \sim_\rho b_i$, we have

$$\langle [a_1, a_2], a_3\rangle = \rho^*\langle [b_1, b_2], b_3\rangle.$$

Indeed, let $a_i = X_i \oplus \rho^*\alpha_i$ and $b_i = Y_i \oplus \alpha_i$ with $X_i \sim_\rho Y_i$, then we compute:

$$\begin{aligned}
\langle [X_1 \oplus \rho^*\alpha_1, X_2 \oplus \rho^*\alpha_2], X_3 \oplus \rho^*\alpha_3\rangle &= \langle [X_1, X_2] \oplus \mathcal{L}_{X_1}\rho^*\alpha_2 - \iota_{X_2} d\rho^*\alpha_1, X_3 \oplus \rho^*\alpha_3\rangle \\
&= \iota_{[X_1,X_2]}\rho^*\alpha_3 + \iota_{X_3}(\mathcal{L}_{X_1}\rho^*\alpha_2 - \iota_{X_2} d\rho^*\alpha_1) \\
&= \rho^*\iota_{[Y_1,Y_2]}\alpha_3 + \iota_{X_3}(\rho^*\mathcal{L}_{Y_1}\alpha_2 - \rho^*\iota_{Y_2} d\alpha_1) \\
&= \rho^*\iota_{[Y_1,Y_2]}\alpha_3 + \rho^*\iota_{Y_3}(\mathcal{L}_{Y_1}\alpha_2 - \iota_{Y_2} d\alpha_1) \\
&= \rho^*\langle [Y_1 \oplus \alpha_1, Y_2 \oplus \alpha_2], Y_3 \oplus \alpha_3\rangle.
\end{aligned}$$

Now given $a_i|_q \in (\rho^* L_\pi)_q$, for $i = 1, 2, 3$, by Lemma 5.1.14 we extend them locally to elements $a_i \in (\rho^* L_\pi)|_{T^*U}$ and find elements $b_i \in (L_\pi)|_U$ with $a_i \sim_\rho b_i$. Thus we get

$$\Upsilon_{\rho^* L_\pi}(a_1|_q, a_2|_q, a_3|_q) = \Upsilon_{\rho^* L_\pi}(a_1, a_2, a_3)_q = \Upsilon_{L_\pi}(b_1, b_2, b_3)_{\rho(q)} = 0,$$

hence, $\rho^* L_\pi$ is involutive. □

Lemma 5.1.16. *Given a formal Poisson structure $\pi = \sum_{j=1}^\infty \lambda^j \pi_j$ on P and $B \in \Omega_{cl}^2(P)_\lambda$ we have*

$$\tau_{\rho^* B}(\rho^* L_\pi) = \rho^*(\tau_B L_\pi),$$

Proof. Pick $X \oplus \alpha + \iota_X \rho^* B \in \tau_{\rho^* B}(\rho^* L_\pi)$ and put $\beta := \alpha + \iota_X \rho^* B$. We want to show that $X \oplus \beta \in \rho^*(\tau_B L_\pi)$, i.e., for any $q \in T^*P$, putting $p = \rho(q)$, we need to find $\tilde{\beta}_p \in (T_p^*P)_\lambda$ such that

$$\beta_q = (d\rho_q)^*\tilde{\beta}_p \quad \text{and} \quad d\rho_q X_q = (\tau_B\pi)^\sharp_p(\tilde{\beta}_p) \tag{5.5}$$

hold. We know that there exists $\tilde{\alpha}_p \in (T^*_pP)_\lambda$ such that

$$\alpha_q = (d\rho_q)^*\tilde{\alpha}_p \quad \text{and} \quad d\rho_q X_q = \pi^\sharp_p(\tilde{\alpha}_p), \tag{5.6}$$

hold. Then $\tilde{\beta}_p := (I + B^\sharp_p\pi^\sharp_p)\tilde{\alpha}_p$ satisfies both conditions in (5.5). Indeed:

$$\beta_q = \alpha_q + (\iota_X\rho^*B)_q = (d\rho_q)^*\tilde{\alpha}_p + (d\rho_q)^*(\iota_{d\rho_qX_q}B_p) = (d\rho_q)^*(\tilde{\alpha}_p + \iota_{\pi^\sharp_p(\tilde{\alpha}_p)}B_p) = (d\rho_q)^*(\tilde{\beta}_p),$$

and

$$d\rho_qX_q = \pi^\sharp_p(\tilde{\alpha}_p) = \pi^\sharp_p(I + B^\sharp_p\pi^\sharp_p)^{-1}\tilde{\beta}_p = (\tau_B\pi)^\sharp_p(\tilde{\beta}_p).$$

Thus we have $\tau_{\rho^*B}(\rho^*L_\pi) \subset \rho^*(\tau_BL_\pi)$.

Conversely, let $X \oplus \beta \in \rho^*(\tau_BL_\pi)$ and put $\alpha := \beta - \iota_X\rho^*B$. Then $X \oplus \beta = X \oplus \alpha + \iota_X\rho^*B$ and we want to show that $X \oplus \alpha \in \rho^*L_\pi$, i.e., for any $q \in T^*P$, putting $p = \rho(q)$, we need to find $\tilde{\alpha}_p \in (T^*_pP)_\lambda$ such that both conditions in 5.6 are satisfied. We know there exists $\tilde{\beta}_p \in (T^*_pP)_\lambda$ satisfying both conditions in (5.5). We claim that $\tilde{\alpha}_p := (I - B^\sharp_p(\tau_B\pi)^\sharp_p)\tilde{\beta}_p$ does the job, indeed, for the first condition in (5.6) we have

$$\alpha_q = \beta_q - (\iota_X\rho^*B)_q = (d\rho_q)^*(\tilde{\beta}_q - \iota_{d\rho_qX_q}B_p),$$

which together with $d\rho_qX_q = (\tau_B\pi)^\sharp_p(\tilde{\beta}_p)$ yields $\alpha_q = (d\rho_q)^*\tilde{\alpha}_p$, as desired. As for the second condition, we compute:

$$d\rho_qX_q = (\tau_B\pi)^\sharp_p(\tilde{\beta}_p) = (\tau_B\pi)^\sharp_p(I - B^\sharp_p(\tau_B\pi)^\sharp_p)^{-1}\tilde{\alpha}_p = (\tau_{-B}(\tau_B\pi))^\sharp_p(\tilde{\alpha}_p) = \pi^\sharp_p(\tilde{\alpha}_p).$$

Hence, $\tau_{\rho^*B}(\rho^*L_\pi) \supset \rho^*(\tau_BL_\pi)$. $\square$

Lemma 5.1.17. *Given $(T^*P, \omega_0) \xrightarrow{\rho} P$, where $\omega_0 = \omega_{can} + \rho^*B_0$ with $B_0 \in \Omega^2_{cl}(P)$, let $\mathcal{V} \in \lambda\mathcal{X}(T^*P)_\lambda$ and $\varrho^* := \exp(\mathcal{L}_\mathcal{V})\rho^*$. Consider a diagram of the form*

$$(C^\infty(P)_\lambda, \pi_1) \xrightarrow{\rho^*} (C^\infty(T^*P)_\lambda, \omega) \xleftarrow{\varrho^*} (C^\infty(P)_\lambda, \pi_2),$$

where π_1 and π_2 are formal Poisson structures on P, $\omega = \omega_0 + \sum_{j=1}^\infty \lambda^j\omega_j \in \Omega^2_{cl}(T^*P)_\lambda$ and suppose the relation

$$\rho^*L_{\pi_1} = e^\omega(\varrho^*L_{\pi_2})$$

holds. Then:

1. *Putting $\eta := \omega^\sharp|_{\Gamma(\mathrm{Ver}(T^*P))_\lambda}$, the map $\exp(-\mathcal{L}_\mathcal{V})\eta\colon \Gamma(\mathrm{Ver}(T^*P))_\lambda \to \Gamma(\rho^*(T^*P))_\lambda$ is an isomorphism.*

2. *For all $f, g \in C^\infty(P)_\lambda$, we have $\{\varrho^*f, \rho^*g\}_\omega = 0$, and the maps ρ^* and ϱ^* are Poisson and*

anti-Poisson morphism, respectively.

Proof. 1. First observe that $V \in \Gamma(\mathrm{Ver}(T^*P))_\lambda$ implies $V \oplus 0 \in \rho^* L_{\pi_1}$, hence

$$\exp(-\mathcal{L}_\mathcal{V})V \oplus -\exp(-\mathcal{L}_\mathcal{V})\iota_V\omega \in \rho^* L_{\pi_2}.$$

In particular, this means that $\exp(-\mathcal{L}_\mathcal{V})(\omega^\sharp(V)) \in \Gamma(\rho^* T^*P)_\lambda$, hence we get an injective map

$$\varphi := \exp(-\mathcal{L}_\mathcal{V})\eta \colon \Gamma(\mathrm{Ver}(T^*P))_\lambda \to \Gamma(\rho^*(T^*P))_\lambda.$$

Now observe that this map is $\mathbb{C}_\lambda$-linear, thus it is of the form $\varphi = \sum_{j=0}^\infty \lambda^j \varphi_j$, with

$$\varphi_0 = \omega_0^\sharp|_{\Gamma(\mathrm{Ver}(T^*P))} \colon \Gamma(\mathrm{Ver}(T^*P)) \to \Gamma(\rho^*(T^*P)).$$

Then observe that, for X vertical, $\iota_X \rho^* B_0 \equiv 0$, hence $\varphi_0 = \omega_0^\sharp|_{\Gamma(\mathrm{Ver}(T^*P))} = \omega_{can}^\sharp|_{\Gamma(\mathrm{Ver}(T^*P))}$, which, from Example A.1.5, is an isomorphism. It follows from Proposition 2.2.1 that φ is an isomorphism.

2. To conclude the statement we need to establish, for any $f, g \in C^\infty(P)_\lambda$, the following three equations:

$$\begin{aligned} \{\rho^* f, \varrho^* g\}_\omega &= 0, \\ \{\rho^* f, \rho^* g\}_\omega &= \rho^*\{f, g\}_{\pi_1}, \\ \{\varrho^* f, \varrho^* g\}_\omega &= -\varrho^*\{f, g\}_{\pi_2}. \end{aligned} \tag{5.7}$$

Let $f, g \in C^\infty(P)_\lambda$ be arbitrary formal smooth functions, and let $X_{\rho^* f}$ and $X_{\varrho^* g}$ be hamiltonians with respect to ω. We claim that the three conditions in (5.7) follow if the vector field

$$X := X_{\rho^* f} - X_{\varrho^* g} \tag{5.8}$$

satisfies

$$X \sim_\rho X_f^1 \quad \text{and} \quad \exp(-\mathcal{L}_\mathcal{V})X \sim_\rho X_g^2, \tag{5.9}$$

where X_f^1 and X_g^2 are hamiltonians with respect to π_1 and π_2, respectively.

Indeed, from $X \sim_\rho X_f^1$ we get $X(\rho^* f) = \rho^*(X_f^1(f)) = 0$, which together with (5.8) implies

$$\{\rho^* f, \varrho^* g\}_\omega = 0. \tag{5.10}$$

Likewise, we have $X(\rho^* g) = \rho^*(X_f^1(g)) = \rho^*\{f, g\}_{\pi_1}$, which, together with (5.8) and (5.10) yield

$$\{\rho^* f, \rho^* g\}_\omega = \rho^*\{f, g\}_{\pi_1}.$$

On the other hand, from $\exp(-\mathcal{L}_\mathcal{V})X \sim_\rho X_g^2$ we get $(\exp(-\mathcal{L}_\mathcal{V})X)(\rho^* f) = \rho^*(X_g^2(f))$. We may rewrite $(\exp(-\mathcal{L}_\mathcal{V})X)(\rho^* f)$ as $\iota_{\exp(-\mathcal{L}_\mathcal{V})X}\rho^*(df)$, and from Lemma (5.1.5) we have

$\iota_{\exp(-\mathcal{L}_{\mathcal{V}})X} = \exp(-\mathcal{L}_{\mathcal{V}})\iota_X \exp(\mathcal{L}_{\mathcal{V}})$, hence

$$\rho^*(X_g^2(f)) = \iota_{\exp(-\mathcal{L}_{\mathcal{V}})X}\rho^*(df) = \exp(-\mathcal{L}_{\mathcal{V}})\iota_X \exp(\mathcal{L}_{\mathcal{V}})(\rho^*(df)).$$

Therefore, $\varrho^*(X_g^2(f)) = \iota_X(d(\varrho^* f))$, which together with (5.8) and (5.10) implies

$$\{\varrho^* f, \varrho^* g\}_\omega = -\varrho^*\{f, g\}_{\pi_2}.$$

Now we proceed to show that the vector field X given by (5.8) actually satisfies both conditions in (5.9). But first notice that, since the conditions in (5.7) make sense point-wise, we may fix a point $p_0 \in P$ and work on a trivializing chart $U \subset P$ around p_0, with coordinates $(x^r)_{r=1}^n$, yet, to simplify notation, we will avoid writing the open sets to indicate any restrictions.

Since ρ is a surjective submersion, there exists $Y \in \mathcal{X}(T^*U)_\lambda$ with $Y \sim_\rho X_f^1$, hence, $Y \oplus \rho^*(df) \in \rho^* L_{\pi_1} = (\varrho^* L_{\pi_2})^\omega$. It follows that $\exp(-\mathcal{L}_{\mathcal{V}})Y \oplus \exp(-\mathcal{L}_{\mathcal{V}})(\rho^*(df) - \iota_Y\omega) \in \rho^* L_{\pi_2}$, i.e.:

$$\exp(-\mathcal{L}_{\mathcal{V}})(\rho^*(df) - \iota_Y\omega) = \alpha = \sum_{j=0}^{\infty} \lambda^j \alpha_j; \quad \alpha_j = \sum_{r=1}^{n} h_r^j \rho^*(dx^r),$$

and, for any $q \in T^*U$, with $p = \rho(q)$,

$$d\rho_q(\exp(-\mathcal{L}_{\mathcal{V}})Y)_q = \pi_2^\sharp|_p\left(\sum_r h_r^j(q) dx^r|_p\right).$$

Let $\eta := \exp(\mathcal{L}_{\mathcal{V}})\alpha = \rho^*(df) - \iota_Y\omega$. By item 1 there exists $\tilde{Y} \in \Gamma(\mathrm{Ver}(T^*U))_\lambda$ such that $\exp(-\mathcal{L}_{\mathcal{V}})(\iota_{\tilde{Y}}\omega) = \alpha - \rho^*(dg)$, thus $\iota_{\tilde{Y}}\omega = \eta - \varrho^*(dg)$.

Define $X := Y + \tilde{Y}$. Then we have $\iota_X\omega = \rho^*(df) - \varrho^*(dg)$, hence

$$X = X_{\rho^* f} - X_{\varrho^* g}.$$

Since $Y \sim_\rho X_1^f$ and $d\rho_q \tilde{Y}_q = 0$ for any $q \in T^*U$, we also get

$$X \sim_\rho X_f^1.$$

Finally, thought after a slightly more involved argumentation, we also get

$$\exp(-\mathcal{L}_{\mathcal{V}})X \sim_\rho = X_g^2. \tag{5.11}$$

Indeed, notice that $\tilde{Y} \in \Gamma(\mathrm{Ver}(T^*U))_\lambda$ implies $\tilde{Y} \oplus 0 \in \rho^* L_{\pi_1}$, hence, from $\rho^* L_{\pi_1} =$

$(\varrho^* L_{\pi_2})^\omega$, we get $\exp(-\mathcal{L}_\mathcal{V})\tilde{Y} \oplus - \exp(-\mathcal{L}_\mathcal{V})\iota_{\tilde{Y}}\omega \in \rho^* L_{\pi_2}$, which in turns implies that

$$-\exp(-\mathcal{L}_\mathcal{V})\iota_{\tilde{Y}}\omega = -\exp(-\mathcal{L}_\mathcal{V})(\eta - \varrho^*(dg)) = -\alpha + \rho^*(dg) = \sum_{j=0}^{\infty} \lambda^j \left(\rho^*(dg_j) - \sum_{r=1}^{n} h^j_r \rho^*(dx^r) \right),$$

and, for any $q \in T^*U$, with $p = \rho(q)$,

$$d\rho_q(\exp(-\mathcal{L}_\mathcal{V})\tilde{Y})_q = \pi_2^\sharp|_p \left(\sum_{j=0}^{\infty} \lambda^j \left(dg_j|_p - \sum_{r=1}^{n} h^j_r(q) dx^r|_p \right) \right) = \pi_2^\sharp|_p(dg_p) - d\rho_q(\exp(-\mathcal{L}_\mathcal{V})Y)_q.$$

Hence, we get

$$d\rho_q(\exp(-\mathcal{L}_\mathcal{V})X)_q = d\rho_q(\exp(-\mathcal{L}_\mathcal{V})Y)_q + \pi_2^\sharp|_p(dg_p) - d\rho_q(\exp(-\mathcal{L}_\mathcal{V})Y)_q = \pi_2^\sharp|_p(dg_p),$$

which implies (5.11), as claimed.

□

The next proposition, which is fundamental to prove the main result of this section, is a formal version of, and is inspired directly by, a result due to Frejlich and Mărcuţ in the context of realizations of Dirac structures by closed two-forms [32].

Proposition 5.1.18. *Given a formal Poisson manifold (P, π) with cotangent bundle $\rho\colon T^*P \to P$, define $\mathcal{V} \in \lambda\mathcal{X}(T^*P)_\lambda$ by $\mathcal{V}_\xi = hor(\pi^\sharp(\xi))$, where hor is the horizontal lifting associated to a linear connection ∇ on T^*P, and let $L_\pi := graph(\pi)$. Consider the map $\varrho^* := \exp(\mathcal{L}_\mathcal{V})\rho^*$, and let*

$$\omega := \int_0^1 \exp(s\mathcal{L}_\mathcal{V})\omega_{can} ds.$$

We then have

1. *$e^{-\omega}(\rho^* L_\pi) = \varrho^* L_\pi$.*

2. *There exists $Z \in \lambda\mathcal{X}(T^*P)_\lambda$ such that*

$$(C^\infty(P)_\lambda, \pi) \xrightarrow{\Phi} (C^\infty(T^*P)_\lambda, \{\cdot,\cdot\}_{can}) \xleftarrow{\hat{\Phi}} (C^\infty(P)_\lambda, \pi),$$

where $\Phi := \exp(Z)\rho^*$ and $\hat{\Phi} := \exp(Z)\varrho^*$, is a formal dual pair.

Proof. 1. Let $\lambda_{can} \in \Omega(T^*P)_\lambda$ be the tautological 1-form on T^*P and consider $\mathcal{V} \oplus \lambda$, which generates the flow

$$F_t = \exp(-t\mathcal{L}_\mathcal{V})e^{B_t},$$

with $B_t = -\int_0^t \exp(s\mathcal{L}_\mathcal{V})\omega_{can} ds$, according to Proposition 5.1.8. Since $\rho^* L_\pi$ is involutive and $\mathcal{V} \oplus \lambda_{can} \in \rho^* L_\pi$, it follows from Remark 5.1.11 that F_t preserves $\rho^* L_\pi$. Then we have

$$\exp(-t\mathcal{L}_\mathcal{V})e^{B_t}(\rho^* L_\pi) = \rho^* L_\pi \Rightarrow e^{B_t}(\rho^* L_\pi) = \exp(t\mathcal{L}_\mathcal{V})(\rho^* L_\pi).$$

Since $t=1$ implies $B_1=-\omega$, we get

$$e^{-\omega}(\rho^* L_\pi) = \exp(\mathcal{L}_\mathcal{V})(\rho^* L_\pi) = \varrho^* L_\pi.$$

2. It follows from item 1 and Lemma 5.1.17 that in the diagram

$$(C^\infty(P)_\lambda, \pi) \xrightarrow{\rho^*} (C^\infty(T^*P)_\lambda, \omega) \xleftarrow{\varrho^*} (C^\infty(P)_\lambda, \pi),$$

ρ^* is Poisson, ϱ^* is anti-Poisson, and $\{\rho^* f, \varrho^* g\}_\omega = 0$ for any $f, g \in C^\infty(P)_\lambda$. Now observe that

$$\omega = \int_0^1 \exp(s\mathcal{L}_\mathcal{V}) d\lambda_{can} ds = d\left(\int_0^1 \exp(s\mathcal{L}_\mathcal{V})\lambda_{can} ds\right),$$

hence, $[\omega] = [\omega_{can}] \in \mathbb{H}^2_{dR}(P)_\lambda$. It follows from Proposition 2.2.10, that the formal Poisson structures π_ω and π_{can} are equivalent, i.e., there exists $Z \in \lambda\mathcal{X}(T^*P)_\lambda$ such that

$$\exp(Z)\colon (C^\infty(T^*P)_\lambda, \omega) \to (C^\infty(T^*P)_\lambda, \omega_{can})$$

is a Poisson isomorphism. Thus, in the diagram

$$(C^\infty(P)_\lambda, \pi) \xrightarrow{\Phi} (C^\infty(T^*P)_\lambda, \{\cdot,\cdot\}_{can}) \xleftarrow{\hat{\Phi}} (C^\infty(P)_\lambda, \pi), \tag{5.12}$$

$\Phi := \exp(Z)\rho^*$ is Poisson, $\hat{\Phi} := \exp(Z)\varrho^*$ is anti-Poisson, and $\{\Phi f, \hat{\Phi} g\}_{can} = 0$ for all $f, g \in C^\infty(P)_\lambda$. It follows then by an argument like in Remark 3.2.14 that $\Phi(C^\infty(P)_\lambda)$ and $\hat{\Phi}(C^\infty(P)_\lambda)$ are mutually centralizers. Thus, diagram (5.12) constitutes a formal dual pair.

□

Now we are ready to prove the main result of this chapter

Theorem 5.1.19. *Given a formal Poisson structure $\pi = \lambda\pi_1 + \cdots$ on P, and $B \in \Omega^2_{cl}(P)_\lambda$ let π^B be the formal Poisson structure on P given by Corollary 4.2.2. Then we have $[\pi^B] = [\tau_{-B}\pi]$.*

Proof. Consider the formal Poisson structure $\tau_{-B}\pi$. Then we have

$$\rho^* L_{\tau_{-B}\pi} = e^\omega(\varrho^* L_{\tau_{-B}\pi}),$$

for ω and ϱ^* as in Proposition 5.1.18. Then we get, from Lemma 5.1.16:

$$e^{\omega+\rho^* B}(\varrho^* L_{\tau_{-B}\pi}) = e^{\rho^* B}(e^\omega(\varrho^* L_{\tau_{-B}\pi})) = e^{\rho^* B}(\rho^* L_{\tau_{-B}\pi}) = \rho^* L_\pi.$$

Then, by Lemma 5.1.17 we have that

$$(C^\infty(P)_\lambda, \pi) \xrightarrow{\rho^*} (C^\infty(T^*P)_\lambda, \omega + \rho^* B) \xleftarrow{\varrho^*} (C^\infty(P)_\lambda, \tau_{-B}\pi),$$

is a formal dual pair. Now recall from the proof of item 2 in Proposition 5.1.18 that ω is cohomologous to ω_{can}, thus we get that $\omega + \rho^*B$ is cohomologous to $\omega_{can} + \rho^*B$, hence, there exists a Poisson isomorphism

$$\exp(Z)\colon (C^\infty(T^*P)_\lambda, \omega + \rho^*B) \to (C^\infty(T^*P)_\lambda, \omega_{can} + \rho^*B),$$

and therefore we get a formal dual pair

$$(C^\infty(P)_\lambda, \pi) \xrightarrow{\Phi} (C^\infty(T^*P)_\lambda, \omega_{can} + \rho^*B) \xleftarrow{\Psi} (C^\infty(P)_\lambda, \tau_{-B}\pi),$$

with $\Phi := \exp(Z)\rho^*$ and $\Psi := \exp(Z)\varrho^*$. It follows then, by Corollary 4.2.2 that $\tau_{-B}\pi$ and π^B are equivalent. □

5.2 Final remarks

Here we address some remarks about the work presented so far, as well as some related open questions. A natural line of generalization concerns the zeroth order term of the formal Poisson structure on P. Letting this zeroth order π_0 to be an arbitrary (integrable) Poisson structure would lead us directly to the problem of characterizing the Picard group of (P, π_0), which is currently mostly an open question (see [9] and [12]). Also, in an attempt to show that Morita equivalence of formal Poisson structures is actually an equivalence relation, let us point out that for the general zeroth order, it is likely that reflexivity may fail, even with our more restrictive definition keeping the symplectic structure undeformed. To get a feeling of that, suppose π is a formal Poisson structure on P, deforming π_0. We would like to find a symplectic manifold (S, ω_0), formal maps Φ and Ψ such that

$$(C^\infty(P)_\lambda, \pi) \xrightarrow{\Psi} (C^\infty(S)_\lambda, \omega_0) \xleftarrow{\Phi} (C^\infty(P)_\lambda, \pi)$$

is a formal dual pair, and in the classical limit, (S, ω_0) to be an equivalence symplectic bimodule between (P, π_0) and $(P, -\pi_0)$. One way to find such structures would be by deforming an equivalence symplectic dual pair $(P, \pi_0) \xleftarrow{J} (S, \omega_0) \xrightarrow{\overline{J}} (P, \pi_0)$. Since we are assuming that π_0 is integrable, such an equivalence symplectic bimodule is given by the s-simply connected symplectic groupoid $(\Gamma(P), \omega_0)$ integrating (P, π_0). But, as we have seen, the deformation procedure is not unobstructed, hence, in general the simple iterative construction will not work. We leave these observations for future work.

Another direction of research is to investigate Morita equivalence of Poisson algebras, which could be seen as the algebraic structure underlying the geometric case we worked with. In this direction, there exists some initial ideas due to Stefan Waldmann and Henrique Bursztyn in private communications. The idea is to define a category by introducing a notion of Poisson bimodules and an appropriate tensor product between them. Then, objects of the category would be Poisson algebras and morphism between them would be the bimodules introduced previously (or better, isomorphism classes of bimodules). In such a category, Morita equivalent of Poisson algebras would be just isomorphic objects. This is in the same spirit of the unified approach to Morita equivalences as can be found in [47].

As a third line we could consider a connection with the work of Cabrera-Dherin [15]. There they consider, for an arbitrary Poisson structure π on $\mathbb{R}^n$, deformation of the symplectic realization $q_0\colon (\mathbb{R}^{2n}, \omega_{can}) \to (\mathbb{R}^n, \pi_0 = 0)$ of the form $q_\lambda = q_0 + \mathcal{O}(\lambda)\colon (\mathbb{R}^{2n}, \omega_{can}) \to (\mathbb{R}^n, \lambda\pi)$, finding an explicit formula for q_λ, related to Kontsevich's deformation quantization [44]. A continuation of this result could be to consider the "quantum" side (noncommutative algebras), finding a deformation $Q_\lambda = Q_0 + \mathcal{O}(\lambda)\colon\ C^\infty(\mathbb{R}^n)[[\lambda]] \to C^\infty(\mathbb{R}^{2n})[[\lambda]]$ of the natural inclusion $Q_0 := q_0^*\colon\ C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^{2n})$, in such a way that sends Kontsevich's star product on $(\mathbb{R}^n, \pi)$ to the star product on $\mathbb{R}^{2n}$ quantizing ω_{can} (Moyal product). It follows from general arguments that such a deformation always exists, the point here is to find an explicit formula for it. The deformation problem of Poisson morphism we considered in my thesis is a deformation of the map with respect to deformations of the Poisson structures, keeping the associative products undeformed. It could be interesting to find out how this deformation problem (and its solution) is related with the deformation with respect to the star products.

APPENDIX A

Differential geometry of vector bundles

Here we review some basic concepts of differential geometry of fibred manifolds. We are mostly interested in vector bundles and more precisely in the tangent and cotangent bundles of a given manifold. For details the reader may consult [35] and [62].

A.1 Semi-basic differential forms

Let $\rho\colon E \to M$ be a surjective submersion. Given $z \in E$, consider the vector space $\mathrm{Ver}(E)_z := \ker(d\rho|_z)$. The *vertical bundle with respect to* ρ is $\mathrm{Ver}(E) := \bigcup_z \mathrm{Ver}(E)_z$. This is a subbundle of the tangent bundle TE, whose sections are vector fields on E which are everywhere tangent to the fibers of ρ.

Let $\tau\colon F \to M$ be a vector bundle. Recall that the pull-back bundle of F by ρ, denoted ρ^*F, is a vector bundle over E with total space given by

$$\rho^*F := E \times_M F = \{(u, v) \in E \times F; \rho(u) = \tau(v)\},$$

thus, at any point $z \in E$, the fiver $\rho^*F|_z$ is given by $F|_{\rho(z)}$.

We may define a bundle morphism $\Phi\colon TE \to \rho^*TM$ by

$$\Phi(z, v) = (z, d\rho_z(v)).$$

Its transpose $\Psi\colon \rho^*T^*M \to T^*E$ is then given by

$$\Psi(z, \varphi) = (z, (d\rho_z)^*\varphi).$$

Since ρ is a surjective submersion, Φ is a surjective bundle morphism, and Ψ is an injective bundle morphism. Moreover, we have the following observation.

Proposition A.1.1. *The map Φ yields an isomorphism $TE/\operatorname{Ver}(E) \cong \rho^*TM$, and the map Ψ is an isomorphism between ρ^*T^*M and the annihilator $(\operatorname{Ver}(E))^0$ of $\operatorname{Ver}(E)$.*

Proof. For each $z \in E$, the restriction $\Phi_z \colon T_zE \to \rho^*TM_z$ of Φ is a surjective linear map whose kernel is $\ker(d\rho_z) = \operatorname{Ver}(E)_z$. Thus we have $T_zE/\operatorname{Ver}(E)_z \cong \rho^*T^*M_z$, and so $TE/\operatorname{Ver}(E) \cong \rho^*TM$. On the other hand, the restriction $\Psi_{\rho(z)}$ is the transpose of Φ_z, hence we have $\operatorname{Im}(\Psi_{\rho(z)}) = (\ker(\Phi_z))^0 = (\operatorname{Ver}(E)_z)^0$. Since $\Psi_{\rho(z)}$ is injective, we have an isomorphism $\Psi_{\rho(z)} \colon T^*_{\rho(z)}M \to (\operatorname{Ver}(E)_z)^0$, hence the bundle isomorphism $\Psi \colon \rho^*T^*M \to (\operatorname{Ver}(E))^0$. □

Corollary A.1.2. *Given a surjective submersion $\rho \colon E \to M$, let $\eta \colon E \to T^*E; z \mapsto (z, \eta_z)$ be a 1-form. The following are equivalent:*

1. *For any $X \in \Gamma(\operatorname{Ver}(E))$, $\iota_X \eta = 0$.*
2. *For each $z \in E$, there exists a unique $\varphi \in T^*_{\rho(z)}M$ such that $\eta_z = (d\rho_z)^*(\varphi)$.*
3. *The 1-form η can be regarded as a section of the bundle ρ^*T^*M, under the isomorphism Ψ.*

Proof.

1. 1) $\Rightarrow$ 2): notice that condition 1) implies that for any $z \in E$ and $v \in \operatorname{Ver}(E)_z$ we have $\eta_z(v) = 0$, thus $\eta_z \in (\operatorname{Ver}(E)_z)^0$, and then by the isomorphism $\Psi_{\rho(z)} \colon T^*_{\rho(z)}M \to (\operatorname{Ver}(E)_z)^0$, there exists a unique $\varphi \in T^*_{\rho(z)}M$ such that $(d\rho_z)^*\varphi = \eta_z$.
2. 2) $\Rightarrow$ 3): just notice that condition 2) means that the section η satisfies $z \mapsto (z, \eta_z)$ with $\eta_z \operatorname{Im}(\Psi_{\rho(z)})$. Hence, by the isomorphism Ψ, η can be regarded as a section of ρ^*T^*M.
3. 3) $\Rightarrow$ 1): If $\eta \in \Gamma(\rho^*T^*M)$, under the identification given by Ψ, then η is a section of $(\operatorname{Ver}(E)_z)^0$, hence, for any $X \in \Gamma(\operatorname{Ver}(E))$ we have $\iota_X\eta = 0$.

□

A 1-form on E that satisfies any of the three equivalent conditions above is called a *semi-basic* differential form

Example A.1.3. *Taking the surjective submersion $\rho \colon E \to M$ to be the cotangent bundle T^*M, the morphism Ψ identifies $\rho^*(T^*M)$ with $(\operatorname{Ver}(T^*M))^0$. Recall the coordinate free definition of the canonical 1-form on T^*M, $\lambda \colon T^*M \to T^*T^*M; z = (q, \xi) \mapsto (z, \lambda_z)$, where $\lambda_z(v) = \xi(d\rho_z v) = (d\rho_z)^*(\xi)(v)$. Thus, λ satisfies, by definition, condition 2 in the previous corollary. Hence, λ is a semi-basic differential form.*

Local expression

Continuing with the surjective submersion $\rho\colon E \to M$, by the local form of submersions, around any $z \in E$ there exists an adapted chart $U \subset E$, i.e., a local chart with coordinates $(x^1, ..., x^n, y^1, ..., y^k)$ in U and $(x'^1, ..., x'^n)$ in $\rho(U)$ such that, on U, the map ρ takes the form

$$(x^1, ..., x^n, y^1, ..., y^k) \mapsto (x'^1 = x^1, ..., x'^n = x^n).$$

It follows that a 1-form η is semi-basic if and only if its restriction to any adapted chart may be written as

$$\eta = \sum_{i=1}^{n} a_i dx^i,$$

where $a_i \in C^\infty(U)$, for each i.

Example A.1.4. *For the cotangent bundle T^*M, the natural chart induced on T^*M by a chart on M is an adapted chart. Thus, a 1-form η on T^*M is semi-basic if and only if, for any cotangent chart (T^*V, x^i, ξ_i) the local expression of η is of the form $\eta = \sum_{i=1}^n a_i dx^i$, with $a_i \in C^\infty(T^*V)$. Thus, for any $z \in T^*V$, $\eta_z = \sum_{i=1}^n a_i(z) dx^i|_z$, and the unique element $\varphi \in T^*_{\rho(z)}M$ given by condition 2 in Corollary A.1.2, is given by $\varphi = \sum_{i=1}^n a_i(z) dx^i|_{\rho(z)}$.*

Example A.1.5. *Let ω_{can} be the canonical 2-form on T^*M, and let $\omega^\sharp_{can}\colon T(T^*M) \to T^*(T^*M)$ be the induced bundle map: $(z, v) \mapsto (z, \omega_{can}|_z(v, \cdot))$. We claim that $\omega^\sharp_{can}$ maps vertical vector fields on T^*M isomorphically onto semi-basic 1-forms on T^*M. Indeed: for any $z \in T^*M$, using a cotangent chart (T^*V, x^i, ξ_i) around z we have $\omega_{can} = \sum_{j=1}^n dx^i \wedge d\xi_i$ and any vertical vector at z is of the form $v = \sum_{j=1}^n f_j \partial\xi_j \in \mathrm{Ver}(T^*M)_z$, for some $f_j \in C^\infty(T^*U)$. Then we have $\omega^\sharp_{can}(z, v) = (z, \sum_{j=1}^n f_j(z) dx^j|_z)$, hence, by the previous example: $\omega^\sharp_{can}|_z\colon \mathrm{Ver}(T^*M)_z \to \rho^*(T^*M)_z$. This map is injective, since ω_{can} is non-degenerate, and by counting dimensions, it is an isomorphism. It follows that the bundle map $\omega^\sharp_{can}\colon Ver(T^*M) \to \rho^*T^*M$ is an isomorphism, hence so is the induced map on sections.*

A.2 Vertical cohomology

Consider a surjective submersion $\rho\colon E \to M$. Let $\mathrm{Ver}(E)$ be the vertical subbundle of TE, and let $\mathrm{Ver}(E)^*$ be its dual bundle. Sections of these bundles are finitely generated projective $C^\infty(E)$-modules, as usual, due to Serre-Swan theorem. We want to consider also, for each integer $k \geq 0$ the bundle $\bigwedge^k \mathrm{Ver}(E)^*$, whose section we denote by $\Omega^k_V(E)$, with $\Omega^0_V(E)$ identified with $C^\infty(E)$.

Let us denote the sections of $\mathrm{Ver}(E)$ by $\mathcal{X}_V(E)$. Notice that an element $X \in \mathcal{X}(E)$ is in $\mathcal{X}_V(E)$ if and only if, at any $z \in E$, we have $d\rho_z X_z = 0$, thus, if and only if, X is ρ-related to zero. It follows that, if X and Y are elements of $\mathcal{X}_V(E)$, then so is their Lie bracket $[X, Y]$, hence $\mathcal{X}_V(E)$ is a Lie sub algebra of $\mathcal{X}(E)$. This allows us to define exterior derivative, contraction and Lie derivative with respect to elements in $\mathcal{X}_V(E)$ in the algebra $\Omega^\bullet_V(E)$.

For each $k \geq 1$, we may consider elements of $\Omega^k_V(E)$ as alternate $C^\infty(E)$-multilinear maps from $\mathcal{X}_V(E)^{\times k}$ to $C^\infty(E)$, thus, we may define the following operators in $\Omega^\bullet_V(E)$: The *vertical* de Rham differential $d_{ver}\colon \Omega^k_V(E) \to \Omega^{k+1}_V(E)$, the contraction by $X \in \mathcal{X}_V(E)$, $\iota_X\colon \Omega^k_V(E) \to \Omega^{k-1}_V(E)$, and the Lie derivative, with respect to $X \in \mathcal{X}_V(E)$, $\mathcal{L}_X\colon \Omega^k_V(E) \to \Omega^k_V(E)$. These maps are defined, for $\eta \in \Omega^k_V(E)$ and $X_1, ..., X_{k+1} \in \mathcal{X}_V(E)$, by

$$
\begin{aligned}
(d_{ver}\eta)(X_1, ..., X_{k+1}) = &\sum_{j=1}^{k+1} (-1)^{j+1} X_j\big(\eta(X_1, ..., \overset{j}{\wedge}, ..., X_{k+1})\big) \\
&+ \sum_{i<j} (-1)^{i+j} \eta\big([X_i, X_j], X_1, ..., \overset{i}{\wedge}, ..., \overset{j}{\wedge}, ..., X_{k+1}\big),
\end{aligned}
$$

$$(\iota_X\eta)(X_1, ..., X_{k-1}) := \eta(X, X_1, ..., X_{k-1}),$$

and

$$(\mathcal{L}_X\eta)(X_1, ..., X_k) = X(\eta(X_1, ..., X_k)) - \sum_{j=1}^{k} \eta(X_1, ..., [X, X_i], ..., X_k). \tag{A.1}$$

For $k = 0$, we extend these maps by $(d_{ver}f)(X) = X(f)$, $\iota_X f = 0$ and $\mathcal{L}_X f = X(f)$. For $\eta \in \Omega^k_V(E)$, $\xi \in \Omega^\bullet_V(E)$, and $X, Y \in \mathcal{X}_V(E)$, they satisfy, all together, the following properties.

1. $\iota_X(\eta \wedge \xi) = \iota_X\eta \wedge \xi + (-1)^k \eta \wedge \iota_X\xi$.

2. $\iota_{[X,Y]} = \mathcal{L}_X\iota_Y - \iota_Y\mathcal{L}_X$.

3. $\mathcal{L}_X(\eta \wedge \xi) = \mathcal{L}_X\eta \wedge \xi + \eta \wedge \mathcal{L}_X\xi$.

4. $\mathcal{L}_{[X,Y]} = [\mathcal{L}_X, \mathcal{L}_Y]$.

5. $\mathcal{L}_X = \iota_X d_{ver} + d_{ver}\iota_X$.

6. $d_{ver}(\eta \wedge \xi) = d_{ver}\eta \wedge \xi + (-1)^k \eta \wedge d_{ver}\xi$.

7. $d^2_{ver} = 0$.

These properties can be proved by the same steps used to prove the analogous results for ordinary calculus on manifolds as can be found, for instance, in [35, Chap. IV]. The complex $(\Omega^\bullet_V(E), d_{ver})$ is called the vertical cohomology associated to the submersion $\rho\colon E \to M$.

Remark A.2.1. *As elements in $\Omega^\bullet_V(E)$ takes values on $C^\infty(E)$, it acquires a $C^\infty(M)$-module via ρ^*: If $\eta \in \Omega^k V(E)$, we put $(f\eta)(X_1, ..., X_k) := \rho^* f\eta(X_1, ..., X_k)$.*

Our next task is to consider the cohomology of this complex in the particular case of vector bundles. More precisely, we will only need the result for a trivial vector bundle. From now on, we pick $\rho\colon E \to M$ to be a finite dimensional vector bundle.

A *horizontal* subbundle $\mathcal{H}_E$ of TE is any subbundle that complements the vertical subbundle, i.e.: $TE = \mathrm{Ver}(E) \oplus \mathcal{H}_E$. Existence of hizontal subbundles for any finite dimensional vector bundle is a well known fact (see, for instance, [35, Chap. II]).

Suppose then that a horizontal subbundle $\mathcal{H}_E$ has been fixed. A vector field X on E is called *horizontal* if, at any $z \in E$, it satisfies $X_z \in \mathcal{H}_E|_z$. As sections of a vector bundle, the set $\mathcal{X}_H(E)$ of horizontal vector fields on E forms a finitely generated projective $C^\infty(E)$-module, however, it is not in general a Lie subalgebra of $\mathcal{X}_V(E)$. Any vector field X on E can be uniquely decomposed as $X = X_v \oplus X_h$, according to the direct sum $TE = \mathrm{Ver}(E) \oplus \mathcal{H}_E$.

We may define a graded subalgebra $A_V(E) \subset \Omega^\bullet(E)$ by

$$A_V(E) := \{\eta \in \Omega^\bullet(E); \iota_X \eta = 0 \text{ for all } X \in \mathcal{X}_H(E)\}.$$

Following [35], we call $A_V(E)$ the *vertical subalgebra* of $\Omega^\bullet(E)$. Notice that it depends on the choice of $\mathcal{H}_E$. We are interested now in the following isomorphism of $C^\infty(E)$-modules

$$f_V \colon \Omega_V^\bullet(E) \to A_V(E),$$

given, for $\eta \in \Omega_V^k(E)$, and $X_1, ..., X_k \in \mathcal{X}(E)$, by

$$(f_V \eta)(X_1, ..., X_k) = \eta(X_{1v}, ..., X_{kv}),$$

according to the decomposition $X_i = X_{iv} \oplus X_{ih}$.

Example A.2.2. *Consider $M = \mathbb{R}^n$ with global coordinates (q^i) and let $E = T^*M$, with global coordinates (q^i, p_i). Then we have $\mathrm{Ver}(E) = span\{\partial_{p_i}\}$, and we have a natural horizontal subbundle given by $span\{\partial_{q^i}\}$. The vertical subalgebra $A_V(E)$ is given, at degree k, by elements of the form $\eta = \sum_{I_k} \eta^{I_k} dp_{I_k}$, where $I_k = \{i_1 < \cdots < i_k\} \subset \{1, ..., n\}$ is an strictly increasing multi-index of length k, $\eta^{I_k} \in C^\infty(E)$, and $dp_{I_k} := dp_{i_1} \wedge \cdots \wedge dp_{i_k}$. Under the isomorphism $f_V \colon \Omega_V^\bullet(E) \to A_V(E)$, the vertical de Rham differential d_{ver} is then computed in coordinates by $d_{ver}\eta = (d\eta)^v$, where d is the usual de Rham differential in $\Omega^\bullet(E)$ and $(\cdot)^v$ is the projection onto $A_V(E)$ given by discarding all components of the argument that contains d_{q^i}.*

Next we show, for the previous example, that there exists a $C^\infty(M)$-linear homotopy operator $h \colon \Omega_V^\bullet(T^*M) \to \Omega_V^{\bullet-1}(T^*M)$.

Let us denote a point $z \in T^*M$ by $z = (q, p)$. Let $\sigma \in \Omega_V^k(T^*M)$, hence it may be written as $\sigma = \frac{1}{k!}\sigma_{i_1 \dots i_k}(x, p) dp_{i_1} \wedge \cdots \wedge dp_{i_k}$. We define then, for $t \in \mathbb{R}$, $\overline{\sigma}_t$ by

$$\overline{\sigma}_t = \frac{1}{k!}\sigma_{i_1 \dots i_k}(x, tp) dp_{i_1} \wedge \cdots \wedge dp_{i_k}.$$

Observe that $\overline{\sigma}_0 = \sigma(x, 0)$ and $\overline{\sigma}_1 = \sigma$. Introducing the new independent variables $\overline{p}_i := tp_i$, we get $\overline{\sigma}_{i_1 \dots i_k}(q, \overline{p}) = \sigma_{i_1 \dots i_k}(x, tp)$ and

$$
\begin{aligned}
d_{ver}\overline{\sigma} &= \left(\frac{1}{k!}d\overline{\sigma}_{i_1\ldots i_k}(q,\overline{p}) \wedge dp_{i_1} \wedge \cdots dp_{i_k}\right)^v = \frac{1}{k!}\frac{\partial\overline{\sigma}_{i_1\ldots i_k}}{\partial\overline{p}_i}d\overline{p}_i \wedge dp_{i_1} \wedge dp_{i_k} \\
&= t\frac{1}{k!}\frac{\partial\overline{\sigma}_{i_1\ldots i_k}}{\partial\overline{p}_i}dp_i \wedge dp_{i_1} \wedge dp_{i_k} = t\overline{d_{ver}\sigma}.
\end{aligned}
$$

Now let $\mathcal{V}$ be the vertical vector field defined by $\mathcal{V}(q,p) := p_i\partial_{p_i}$. Then we define, for each integer $k \geq 1$, the operator $h\colon \Omega^k_V(T^*M) \to \Omega^{k-1}_V(T^*M)$, given by

$$
h\sigma = \int_0^1 (\iota_{\mathcal{V}}\overline{\sigma})t^{k-1}dt. \tag{A.2}
$$

Proposition A.2.3. *The operator h, given by (A.2), is $C^\infty(M)$-linear and satisfies $d_{ver}h + hd_{ver} = I$, where I is the identity operator in $\Omega^k_V(T^*M)$.*

Proof. The fact that, for each k, the operator h is $C^\infty(M)$-linear is clear since the integration is made over the fibres, and $\rho^* f$ is constant along the fibres for any $f \in C^\infty(M)$.

For the homotopy property, let $\sigma \in \Omega^k_V(T^*M)$, then

$$
\begin{aligned}
(d_{ver}h + hd_{ver})\sigma &= d_{ver}(h\sigma) + h(d_{ver}\sigma) = \int_0^1 d_{ver}(\iota_{\mathcal{V}}\overline{\sigma})t^{k-1}dt + \int_0^1 (\iota_{\mathcal{V}}\overline{d_{ver}\sigma})t^k dt \\
&= \int_0^1 t^{k-1}\big(d_{ver}(\iota_{\mathcal{V}}\overline{\sigma}) + \iota_{\mathcal{V}}(d_{ver}\overline{\sigma})\big) = \int_0^1 t^{k-1}\mathcal{L}_{\mathcal{V}}\overline{\sigma}dt.
\end{aligned} \tag{A.3}
$$

Writing down $\mathcal{L}_{\mathcal{V}}\overline{\sigma}$ in coordinates we have

$$
\mathcal{L}_{\mathcal{V}}\overline{\sigma} = \frac{1}{k}(\mathcal{L}_{\mathcal{V}}\overline{\sigma})_{i_1\ldots i_k}dp_{i_1} \wedge \cdots \wedge dp_{i_k},
$$

where, from the definition of Lie derivative (A.1) we get

$$
\begin{aligned}
(\mathcal{L}_{\mathcal{V}}\overline{\sigma})_{i_1\ldots i_k} &= p_i\frac{\overline{\sigma}_{i_1\ldots i_k}}{\partial p_i} + \sum_{r=1}^k \overline{\sigma}_{i_1\ldots i_{r-1}ii_{r+1}\ldots i_k}\frac{\partial p_i}{\partial p_{i_r}} = p_i\frac{\partial\overline{\sigma}_{i_1\ldots i_k}}{\partial p_i} + \sum_{r=1}^k \overline{\sigma}_{i_1\ldots i_{r-1}ii_{r+1}\ldots i_k}\delta^i_{i_r} \\
&= tp_i\frac{\overline{\sigma}_{i_1\ldots i_k}}{\partial\overline{p}_i} + k\overline{\sigma}_{i_1\ldots i_k} = t\frac{d}{dt}\overline{\sigma}_{i_1\ldots i_k} + k\overline{\sigma}_{i_1\ldots i_k}.
\end{aligned} \tag{A.4}
$$

Substituting in (A.3) we get

$$
\begin{aligned}
(d_{ver}h + hd_{ver})\sigma &= \frac{1}{k!}\int_0^1 \left(t^k\frac{d\overline{\sigma}_{i_1\ldots i_k}}{dt} + kt^{k-1}\overline{\sigma}_{i_1\ldots i_k}\right)dp_{i_1} \wedge \cdots \wedge dp_{i_k}dt \\
&= \frac{1}{k!}\int_0^1 \frac{d}{dt}(t^k\overline{\sigma}_{i_1\ldots i_k})dp_{i_1} \wedge \cdots \wedge dp_{i_k}dt = \frac{1}{k!}(t^k\overline{\sigma}_{i_1\ldots i_k})\big|_0^1 dp_{i_1} \wedge \cdots \wedge dp_{i_k} \\
&= \frac{1}{k!}\sigma_{i_1\ldots i_k}dp_{i_1} \wedge \cdots \wedge dp_{i_k} = \sigma
\end{aligned}
$$

□

A.3 Cotangent lifting

Let M and N be smooth manifolds and let $f\colon M \to N$ be a diffeomorphism. We may lift it to a diffeomorphism $\overline{f}\colon T^*M \to T^*N$ in the following way: given $(q,\xi) \in T^*M$, we define

$$\overline{f}(q,\xi) = (f(q), (df_q^*)^{-1}\xi),$$

where $df_q^*\colon T^*_{f(q)}N \to T^*_qM$ is the transpose of df_q. The map $f^\sharp$ turns out to be a diffeomorphism, and it is known as the *cotangent lift* of f. Let us summarize some of its properties, which proofs are straightforward manipulations of the definitions.

Proposition A.3.1. *The cotangent lift of a diffeomorphism has the following properties.*

1. *Given a diffeomorphism $f\colon M \to N$, the map $\overline{f}\colon T^*M \to T^*N$ is a bundle isomorphism over f.*

2. *If $g\colon N \to P$ is another diffeomorphism, then $\overline{g \circ f} = \overline{g} \circ \overline{f}$.*

3. *If α_N and α_M are the tautological 1-forms on T^*N and T^*M, respectively, then $\overline{f}^*\alpha_N = \alpha_M$.*

We may use now this cotangent lift to lift vector fields from a manifold M to its cotangent bundle T^*M. To do so, first recall that any vector field X on M generates a flow $\varphi_{(X,t)}$, i.e., a local one-parameter group of diffeomorphisms. If we apply the cotangent lifting procedure to $\varphi_{(X,t)}$ we get a one-parameter family of local diffeomorphisms $\overline{\varphi}_{(X,t)}$, which, by item 2) in the previous proposition, turns out to be a flow on T^*M, and thus, it gives rise to a vector field $\overline{X}$ on T^*M, which is known as the *cotangent lift* of X. Notice that, since $\overline{\varphi}_{(X,t)}$ covers $\varphi_{(X,t)}$, with respect to the cotangent bundle projection $\rho\colon T^*M \to M$, it follows that $d\rho_z\overline{X}_z = X_{\rho(z)}$, for any $z \in T^*M$. Thus, $\overline{X}$ and X are ρ-related. Next we find the local expression of the cotangent lift $\overline{X}$.

Proposition A.3.2. *Let X be a vector field on M, and let (U, q^i) be a chart on M. If the local expression of X on U is $X = X^i\partial_{q^i}$, then the local expression of $\overline{X}$ on the induced chart (T^*U, q^i, p_i) is*

$$\overline{X} = \rho^*X^i\partial_{q^i} - p_j\rho^*(\partial_{q^i}X^j)\partial_{p_i}.$$

Proof. Let us first observe that $\overline{X}$ is a hamiltonian vector field, with respect to the canonical symplectic structure of T^*M. In fact, if α is the tautological 1-form on T^*M, then $H := \iota_{\overline{X}}\alpha$ is a hamiltonian function for $\overline{X}$. Indeed, since the flow $\overline{\varphi}_{(X,t)}$ preserves α, it follows that $\mathcal{L}_{\overline{X}}\alpha = 0$. Then, by Cartan's magic formula we get $d\iota_{\overline{X}}\alpha = -\iota_{\overline{X}}d\alpha$, hence $\iota_{\overline{X}}\omega_{can} = dH$.

Now, if $\overline{X} = a_i\partial_{q^i} + b_i\partial_{p_i}$, it follows from $\iota_{\overline{X}}\omega_{can} = dH$ that $a_l = \partial_{p_l}H$ and $b_l = -\partial_{q^l}H$. In order to compute these partial derivatives we express H in terms of the local data as follows. Observe that, for $z = (q,p) \in T^*U$, $H = \iota_{X^\sharp}\alpha$ implies that

$$H(z) = (\iota_{X^\sharp}\alpha)(z) = \alpha|_z(X^\sharp_z) = p(d\rho_zX^\sharp_z) = p(X_p) = p(X^i_q\partial_{q^i}|_q) = (p_r\rho^*X^r)(z).$$

Hence we get $a_l = \rho^* X^l$, and $b_l = -p_r \partial_{q^l}(\rho^* X^r) = -p_r \rho^*(\partial_{q^l} X^r)$. Thus, we have

$$\overline{X} = \rho^* X^i \partial_{q^i} - p_j \rho^*(\partial_{q^i} X^j) \partial_{p_i},$$

as claimed. $\square$

APPENDIX B

Dirac structures and Courant algebroids

Here we recall some basic concepts about Dirac Geometry [21] and Courant algebroids [50]. We limit ourselves to the case of interest for our purposes. For more about these topics the reader is referred, beside the original sources, to [10], [11], [36] and the references therein.

B.1 Dirac structures

Given a smooth manifold M, consider the vector bundle $\mathbb{T}M := TM \oplus T^*M$. A typical section of this bundle will be denoted as $X \oplus \xi$ or (X, ξ), where $X \in \Gamma(TM) \cong \mathcal{X}(M)$ and $\xi \in \Gamma(T^*M) \cong \Omega(M)$. The $C^\infty(M)$-module $\Gamma(\mathbb{T}M)$ carries a natural non-degenerate symmetric bilinear pairing

$$\langle \cdot, \cdot \rangle \colon \Gamma(\mathbb{T}M) \times \Gamma(\mathbb{T}M) \to C^\infty(M);\ \langle X \oplus \xi, Y \oplus \eta \rangle := \eta(X) + \xi(Y), \tag{B.1}$$

as well as a bilinear bracket on the sections $\Gamma(\mathbb{T}M)$, known as **Dorfman bracket**, defined by

$$[\cdot, \cdot] \colon \Gamma(\mathbb{T}M) \times \Gamma(\mathbb{T}M) \to \Gamma(\mathbb{T}M);\ [X \oplus \xi, Y \oplus \eta] := ([X, Y], \mathcal{L}_X \eta - \iota_Y d\xi), \tag{B.2}$$

where the bracket in the right is the usual Lie bracket of vector fields, $\mathcal{L}_X$ is the Lie derivative with respect to X and ι_Y is the contraction by Y.

Definition B.1.1. *Given a smooth manifold M, a* ***Dirac structure*** *on M is a subbundle $L \subset \mathbb{T}M$ such that*

1. *$L = L^\perp$, i.e., L is self-orthogonal (lagrangian) with respect to the pairing* $(B.1)$.

2. $\Gamma(L)$ *is involutive with respect to the Courant bracket (B.2).*

Remark B.1.2. *Subbundles $L \subset \mathbb{T}M$ satisfying only condition 1 above are usually called **almost Dirac structures**, and condition 2 is referred to as the **integrability condition**. In the presence of self-orthogonality, the involutivity property is equivalent to the vanishing of*

$$\Upsilon(a, b, c) := \langle [a, b], c \rangle,$$

for any $a, b, c \in L$. Indeed, let L be involutive, and let $a, b, c \in L$, then $[a, b] \in L$, and from $L^\perp = L$, we get

$$\langle [a, b], c \rangle = 0.$$

Conversely, if Υ vanishes identically on L, it means that for any $a, b, c \in L$, $[a, b] \in L^\perp = L$, thus L is involutive.

Given a bivector field π on M, let $\pi^\sharp \colon T^*M \to TM$ be its associated bundle map. For $f, g \in C^\infty(M)$, define $\{f, g\}_\pi := \pi(df, dg)$, and consider also its Jacobiator;

$$Jac_\pi(f, g, h) := \{f, \{g, h\}_\pi\}_\pi \{g, \{h, f\}_\pi\}_\pi + \{h, \{f, g\}_\pi\}_\pi.$$

We then observe the following fact.

Lemma B.1.3. *Given a bivector field π on M, define*

$$L_\pi := graph(\pi^\sharp) = \{\pi^\sharp(\alpha) \oplus \alpha; \alpha \in T^*M\} \subset \mathbb{T}M.$$

Then $L_\pi^\perp = L_\pi$, and for $a_i := \pi^\sharp(df_i) \oplus df_i$, with $f_i \in C^\infty(M)$, $i = 1, 2, 3$, we have

$$\Upsilon(a_1, a_2, a_3) = Jac_\pi(f_1, f_2, f_3).$$

In particular, L_π is Dirac if and only if π is Poisson.

Proof. The fact that $L_\pi^\perp = L_\pi$, follows from the skewsymmetry of π. Indeed, for any $\alpha, \beta \in \Omega(M)$,

$$\langle \pi^\sharp(\alpha) \oplus \alpha, \pi^\sharp(\beta) \oplus \beta \rangle = \iota_{\pi^\sharp(\alpha)}\beta + \iota_{\pi^\sharp(\beta)}\alpha = \pi(\alpha, \beta) + \pi(\beta, \alpha) = 0,$$

thus $L_\pi \subset L_\pi^\perp$. Conversely, let $X \oplus \alpha \in L_\pi^\perp$, then

$$0 = \langle X \oplus \alpha, \pi^\sharp(\beta) \oplus \beta \rangle = \iota_X\beta + \iota_{\pi^\sharp(\alpha)}\beta = \iota_X\beta - \iota_{\pi^\sharp(\beta)}\alpha = \iota_{X - \pi^\sharp(\alpha)}\beta, \ \forall \beta \in \Omega(M),$$

hence $X = \pi^\sharp(\alpha)$. Therefore, $L_\pi^\perp \subset L_\pi$.

Now let $a_i = \pi^\sharp(df_i) \oplus df_i$, $f_i \in C^\infty(M)$, $i = 1, 2, 3$, then we compute:

$$\begin{aligned}
\langle [\pi^\sharp(df_1) \oplus df_1, \pi^\sharp(df_2) \oplus df_2], \pi^\sharp(df_3) \oplus df_3 \rangle &= \langle [\pi^\sharp(df_1), \pi^\sharp(df_2)] \oplus \mathcal{L}_{\pi^\sharp(df_1)} df_2, \pi^\sharp(df_3) \oplus df_3 \rangle \\
&= \mathcal{L}_{\pi^\sharp(df_1)} \iota_{\pi^\sharp(df_2)} df_3 - \iota_{\pi^\sharp(df_2)} \mathcal{L}_{\pi^\sharp(df_1)} df_3 \\
&+ \iota_{\pi^\sharp(df_3)} (\mathcal{L}_{\pi^\sharp(df_1)} df_2) \\
&= \iota_{\pi^\sharp(df_1)} d\iota_{\pi^\sharp(df_2)} df_3 - \iota_{\pi^\sharp(df_2)} d\iota_{\pi^\sharp(df_1)} df_3 \\
&+ \iota_{\pi^\sharp(df_3)} d\iota_{\pi^\sharp(df_1)} df_2 \\
&= \{f_1, \{f_2, f_3\}\} - \{f_2, \{f_1, f_3\}\} + \{f_3, \{f_1, f_2\}\} \\
&= Jac_\pi(f_1, f_2, f_3).
\end{aligned}$$

Thus, by Remark B.1.2 we conclude that L_π is a Dirac structure if and only if π is a Poisson tensor. □

Remark B.1.4. *Notice that if π is a Poisson tensor, then the Dirac structure L_π satisfies $L_\pi \cap TM = \{0\}$. Indeed, given $\pi^\sharp(\alpha) \oplus \alpha \in L_\pi \cap TM$, we get $\alpha = 0$, thus $\pi^\sharp(\alpha) = 0$.*

The converse of this fact is also true. More precisely, we have:

Lemma B.1.5. *A Dirac structure $L \subset E$ is the graph of a bivector field π if and only if $L \cap TM = \{0\}$.*

Proof. The "only if" part is the remark above. Let $\rho \colon TM \oplus T^*M \to T^*M$ be the natural projection. Notice that $L = L^\perp$ implies that at any $p \in M$ the dimension of the fiber L_p is the same as de dimension of T_p^*M, thus $\rho(L) = T^*M$. Now define a map $\pi^\sharp \colon T^*M \to TM$ by $\pi^\sharp(\alpha) := X$ such that $X \oplus \alpha \in L$. If $Y \in TM$ also satisfies $Y \oplus \alpha \in L$, then $X - Y \in L \cap TP = \{0\}$, thus $X = Y$ and so $\pi^\sharp$ is well defined. Now observe that $L = L^\perp$ implies

$$0 = \langle \pi^\sharp(\alpha) \oplus \alpha, \pi^\sharp(\beta) \oplus \beta \rangle = \alpha(\pi^\sharp(\beta)) + \beta(\pi^\sharp(\alpha)),$$

hence, $\pi^\sharp \colon T^*P \to TP$ is a skew-symmetric bundle map with $graph(\pi^\sharp) = L$. It follows from Lemma B.1.3 that the associated bivector π is Poisson. □

B.2 Standard Courant algebroid

Given a smooth manifold M, consider again a vector bundle $E \to M$. A **Courant algebroid** structure on E consists of

1. A bundle map $\rho \colon E \to TM$, called the **anchor**.
2. A non-degenerate, symmetric, bilinear pairing $\langle \cdot, \cdot \rangle$ on its module of section $\Gamma(E)$.
3. A bilinear bracket $[\cdot, \cdot]$ on its module of sections $\Gamma(E)$.

4. A differential map $d\colon C^\infty(M) \to \Gamma(E)$.

Such that for any $e_1, e_2, e_3 \in \Gamma(E)$ and $f \in C^\infty(M)$, they satisfy the following conditions:

1. $[e_1, [e_2, e_3]] = [[e_1, e_2], e_3] + [e_2, [e_1, e_3]]$,
2. $\rho(e_1)\langle e_2, e_3\rangle = \langle [e_1, e_2], e_3\rangle + \langle e_2, [e_1, e_3]\rangle$,
3. $\rho([e_1, e_2]) = [\rho(e_1), \rho(e_2)]$,
4. $[e_1, fe_2] = f[e_1, e_2] + \rho(e_1)(f)e_2$,
5. $[e_1, e_1] = \mathcal{D}\langle e_1, e_1\rangle$,

where $\mathcal{D} = \frac{1}{2}\rho^* d$, for $d\colon C^\infty(M) \to \Gamma(E)$.

Example B.2.1. *The bundle $E := TM \oplus T^*M$ endowed with the pairing (B.1) and the Courant bracket (B.2) is a Courant algebroid with anchor map given by $X \oplus \alpha \mapsto X$ and differential $d\colon C^\infty(M) \to \Gamma(E)$ given by $f \mapsto 0 \oplus df$, being df the de Rham differential of f.*

From now on, we will mainly be interested in the Courant algebroid of Example B.2.1, and will refer to it as the standard Courant algebroid.

A **symmetry** of a Courant algebroid E is a bundle map (F, f), with $f \in \mathrm{Diff}(M)$ such that the induced map on sections, $F\colon \Gamma(E) \to \Gamma(E); e \mapsto F \circ e \circ f^{-1}$ satisfies:

1. $f^*\langle F(e_1), F(e_2)\rangle = \langle e_1, e_2\rangle$,
2. $[F(e_1), F(e_2)] = F([e_1, e_2]$.

Remark B.2.2. *Notice that, implicit in the concept of a bundle map there is the fact that the induced map on sections and the automorphism of $C^\infty(M)$ given by f^* are related by*

$$F(g\sigma) = (f^*)^{-1}(g)F(\sigma).$$

This property will be important for us in the formal context.

Example B.2.3. *A diffeomorphism $f \in \mathrm{Diff}(M)$ induces a symmetry $(\hat{f}, f)$ via:*

$$\hat{f} := diag(f_*, f^{*-1}).$$

A closed 2-form B on M also induces a symmetry (e^B, I) via:

$$e^B(X \oplus \alpha) = X \oplus \alpha + \iota_X B.$$

*This is known as B-**field transformation** and denoted sometimes by τ_B.*

It can be seen that this example covers essentially all the symmetries of the standard Courant algebroid. More precisely, in [36] it was proved the following result.

Theorem B.2.4. *Let (F, f) be a symmetry of the standard Courant algebroid structure on $E = TM \oplus T^*M$. Then we must have*

$$F = \hat{f} e^B,$$

for some closed 2-form B on M. Thus, the group of automorphisms of the Courant algebroid E is isomorphic to $\mathrm{Diff}(M) \ltimes \Omega^2_{cl}(M)$, where the semi-direct product is given by the action of $\mathrm{Diff}(M)$ on $\Omega^2_{cl}(M)$ via pull-back.

Remark B.2.5. *Given a Poisson structure π and $B \in \Omega^2_{cl}(P)$, the Courant algebroid symmetry τ_B transforms the Dirac structure L_π as follows:*

$$\tau_B(L_\pi) = \{(\pi^\sharp(\alpha), \alpha + \iota_{\pi^\sharp(\alpha)} B);\ \alpha \in T^*P\},$$

which is again a Dirac structure. Thus, in order to define a new Poisson structure, it must satisfy $\tau_B(L_\pi) \cap TM = \{0\}$, which is to say that $\alpha + \iota_{\pi^\sharp(\alpha)} B$ is never zero whenever $\alpha \neq 0$, which in turn is equivalent to the condition that the bundle map

$$Id + B^\sharp \pi^\sharp : T^*M \to T^*M$$

is invertible. In the affirmative case, we denote by π_B the corresponding Poisson structure, which is characterized by

$$(\pi_B)^\sharp = \pi^\sharp \circ (Id + B^\sharp \pi^\sharp)^{-1}.$$

A **derivation** of the Courant algebroid structure on E is a $\mathbb{C}$-linear map

$$D : \Gamma(E) \to \Gamma(E),$$

for which there exists a derivation X of the algebra $C^\infty(M)$ such that, for any $e \in \Gamma(E)$ and $f \in C^\infty(M)$, we have

$$D(fe) = f D(e) + X(f) e.$$

It follows from the Courant algebroid axioms listed above that any section $e \in \Gamma(E)$ defines a derivation via $(D, X) := ([e, \cdot], \rho(e))$.

Differentiating a one-parameter family of Courant automorphisms $F_t = \hat{f}_t e^{B_t}$, with $F_0 = I$, and recalling that $\mathcal{L}_X = -\frac{d}{dt}\big|_{t=0} \hat{f}$, we see that derivations can be identified with pairs $(X, b) \in \mathcal{X}(M) \oplus \Omega^2_{cl}(M)$, acting by

$$(X, b)(Y \oplus \beta) = [X, Y] \oplus \mathcal{L}_X \beta - \iota_Y b. \tag{B.3}$$

Thus, this gives another way to get a derivation out of any section of E: given $e = X \oplus \alpha \in \Gamma(E)$, we consider the pair $(X, d\alpha)$, which can be seen yields a derivation via the action above.

Moreover, this derivation coincides with the one constructed using the bracket and the anchor map.

Regarding the derivations of the standard Courant algebroid $E = TM \oplus T^*M$, in [36] it was proved the following result.

Theorem B.2.6. *Let (X, b) be a derivation of the standard Courant algebroid $E = TM \oplus T^*M$, acting via (B.3). Then it induces a one-parameter subgroup of Courant automorphisms $F_t = \hat{\varphi}_t e^{B_t}$, where φ_t is the flow of X, and B_t is given by*

$$B_t = \int_0^t \varphi_t^* b dt.$$

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