Instituto Nacional de Matemática Pura e Aplicada

Planar continuum percolation: heavy tails and scale invariance

Daniel Ungaretti

supervised by
Prof. Augusto Teixeira

Rio de Janeiro, 2017

Abstract

This thesis is concerned with percolation models that have long-range dependencies. We study two related heavy-tailed models of planar continuum percolation, *ellipses model* and *scale-homogeneous Poisson stick soup*.

Ellipses model is a Boolean model of heavy-tailed isotropic ellipses that is very close to the Boolean model with heavy-tailed sticks. We study connectivity in ellipses model, proving that it presents a double phase transition in a parameter α that governs the tail decay of major axis distribution. Moreover, we prove that for the critical parameter $\alpha = 2$ the model is approximately scale-invariant and has a non-degenerate interval of densities for which the probability of crossing boxes of a fixed proportion is bounded away from zero and one.

Scale-homogeneous Poisson stick soup (SHPSS) is a natural candidate to scaling limit of ellipses model when we contract space through a homothety while adjusting the density of ellipses to obtain a non-degenerate process. This model has sticks of all scales and the sticks are actually dense on the plane. We analyze how its large and small scale sticks behave as we vary parameter α. However, our main interest is the particular case of SHPSS with $\alpha = 2$, which turns out to be a scale-invariant Poisson stick soup (SIPSS). We prove new properties of this object, by defining and analyzing families of exploration paths on boxes of the plane that follow the boundary between covered and vacant regions. We prove polynomial decay for the probability of covered 1-arm events in subcritical SIPSS and use it for proving tightness of our family of exploration paths in this regime, along with bounds on the regularity of its limit curves.

Acknowledgments

Any thesis takes a considerable part of a researcher's life. It takes a lot of time, effort and patience to verify the smallest details and to review all the work done for the tenth time. Oddly enough, some people (me included) are still able to find joy in the process and believe that this small amount of knowledge created (or discovered?) will contribute to develop science as a whole. However, the best reward obtained by our hard work is not having our name in some publication that only a negligible percentage of humankind will be able to read, but the network of friends and fellow researchers that has been formed and organized to support this wearisome work.

Making a list including everyone that I know that has some relevance to this thesis is a task, so I just mention the essential minimum and hope that people omitted here will not feel forgotten. If you know me and even bothered to read a small part of it, be certain that your name should be on this list.

First of all I must thank Augusto, my advisor, that accompanied me closely during this process. Guided and inspired by his ideas, my work flowed more smoothly and became more palpable. More than some kind of oracle, Augusto has been an advisor in the true meaning of the word, preparing me not only for a thesis but for research and academic life as a whole. After years of getting used to find the answers in the previous pages of the book, it is difficult to adapt yourself to this essentially different type of work. Being organized, dealing with rejection, keeping always active, are all abilities that are important but hard to teach and he somehow helped me developing them.

I also mention some people that had a direct influence in the results of this thesis. I would like to thank Caio Alves and Serguei Popov for insights on how to find the major axis distribution for the ellipses model derived from Poisson cylinder model; Leandro Cruz for Figure 1.1; Lucca Avena and Daniel Valesin for their warm welcome in my week-long stay in the Netherlands, and for the opportunity of spreading this work; Emanuel Carneiro, Roberto Imbuzeiro, Milton Jara, Hubert Lacoin, Cláudio Landim and Rob Morris, all of them inspiring teachers; Caio, Conrado, Guilherme, Leandro, Luiz Paulo, Mateus, Otávio, Pedro, Rangel, Rodrigo and many other students here at IMPA that provided fruitful discussions and interesting questions.

Finally, I would like to thank everyone that helped me in more indirect (although not less important) ways. All my family has been somehow involved with this project and were there for me 24-7. Many friends also played this role. To all of you, my most sincere gratitude.

Contents

Chapter 1

Introduction

Percolation processes are natural candidates to model environments in which connectivity is assumed to be random and have thus found many applications in different areas of research such as forest fires, spread of infections and polymerization, among others. We refer the reader to the classical books [5, 11] for a background on independent percolation on graphs.

Continuum percolation models (see [19] for a general account) are one of the possible variations. One common way to define percolation on a continuum space is by introducing a Poisson point process and adding a random defect at every point of this process. When the random defects are balls we obtain the Boolean model (see [1, 10, 12, 19]), possibly the most established continuum percolation model. In the plane, another common choice of defects are sticks with some random orientation (see [26, 29]). However, models with more general defects have also been studied [12, 22, 40, 41].

Most of the papers about continuum percolation with random defects assume that the defects have uniformly bounded diameter. Understanding models with unbounded defects is considerably more difficult, since these models have infinite range dependencies which prevents the use of some common tools like Peierls arguments. Apart from technical complications, models with unbounded defects also may present unexpected behaviors. For example, in [12] it is proven that there are Boolean models with random radii for which the critical point for the existence of an infinite cluster and the critical point for the mean cluster size to be infinite are different.

In the last years there has been major developments in understanding Boolean models with unbounded defects. In [10], Gouéré analyzes the existence of phase transition for percolation in Boolean models with heavy-tailed radius on $\mathbb{R}^d$. He proves that the existence of a subcritical phase for percolation is equivalent to $E[R^d]$ being finite, where R has the distribution of the ball radius. Ahlberg, Tassion and Teixeira [1] give a much more precise description for Boolean model in the plane, proving RSW estimates and sharpness of the phase transition.

This thesis contributes to this line of work by studying two related heavy-tailed models of planar continuum percolation, *ellipses model* and *scale-homogeneous Poisson stick soup*. Each model is developed in a separate chapter and we briefly describe the models addressed and our main results regarding them, leaving a more formal and detailed exposition to their respective chapters.

1.1 Ellipses Percolation

In Chapter 2 we study phase transition for a Boolean model in the plane when the defects are heavy-tailed random ellipses. The ellipses are centered on a Poisson point process of intensity $u > 0$, they have uniform direction and the size of their minor axis is always equal to one. Moreover, their major axis has distribution ρ supported on $[1, \infty)$ and satisfying

$$c_0^{-1} r^{-\alpha} \leq \rho[r, \infty) \leq c_0 r^{-\alpha}, \text{ for every } r \geq 1$$

for some positive constant c_0 and $\alpha > 0$. We refer to this process as the (u, ρ)-ellipses model. Details of the construction are given in Section 2.1.

Figure 1.1: Small simulation of ellipses model with $\rho[r, \infty) = r^{-2}$.

Since we are fixing the minor axis as always equal to one, studying ellipses model is essentially equivalent to studying stick model in the plane with heavy-tailed radius. By stochastic domination, ellipses model can be seen as an slightly enlarged stick model. All results proved here have direct analogues for stick models. Our preference for ellipses over sticks is due to the relation between ellipses model and Poisson Cylinder model, which was our original motivation to study the model. See the comparison with other models below.

We say that some random set *percolates* if it has an unbounded connected component. We are mainly interested in the phase transition properties that the model presents. We prove that the vacant set $\mathcal{V}$, the set not covered by any ellipse, presents a double phase transition in α.

Theorem 1.1.1. *Consider (u, ρ)-ellipses percolation model, where ρ has tail decay α. Then, with probability one:*

1. *If $\alpha \in (0, 1]$ we have $\mathcal{V} = \varnothing$ for every $u > 0$.*
2. *If $\alpha \in (1, 2)$ we have $\mathcal{V} \neq \varnothing$, but for any $u > 0$ there is no percolation.*
3. *If $\alpha \in (2, \infty)$ there exists a critical value $\bar{u}_c(\rho) \in (0, \infty)$ such that if $u < \bar{u}_c$ then $\mathcal{V}$ percolates and if $u > \bar{u}_c$ then $\mathcal{V}$ does not percolate.*

We also prove a version of Theorem 1.1.1 for the covered set $\mathcal{E}$, providing an overall picture of an ellipses model.

Theorem 1.1.2. *Consider (u, ρ)-ellipses percolation model, where ρ has tail decay α. Then, with probability one:*

1. *If $\alpha \in (0, 1]$ we have $\mathcal{E} = \mathbb{R}^2$ for every $u > 0$.*
2. *If $\alpha \in (1, 2)$ we have that $\mathcal{E}$ percolates for any $u > 0$.*
3. *If $\alpha \in (2, \infty)$ there exists a critical value $u_c(\rho) \in (0, \infty)$ such that if $u < u_c$ then $\mathcal{E}$ does not percolate and if $u > u_c$ then $\mathcal{E}$ percolates.*

The case $\alpha = 2$ is described separately in Theorem 1.1.3 because this case deserves special attention. It presents a phase transition in u which is related to fractal percolation.

Theorem 1.1.3. *Let ρ be a distribution with $\alpha = 2$. Then, there exists $\bar{u} = \bar{u}(c_0) > 0$ such that for any fixed $k > 0$, $u \in (0, \bar{u})$ and $l > 0$*

$$\delta \leq P_{u,\rho}[\text{there is a vacant horizontal crossing of box of height } l \text{ and width } kl] \leq 1 - \delta, \tag{1.1}$$

where $\delta = \delta(c_0, u, k) > 0$. Moreover, for $u \in (0, \bar{u})$ we have:

$$P_{u,\rho}[\text{neither } \mathcal{V} \text{ nor } \mathcal{E} \text{ percolate}] = 1. \tag{1.2}$$

In other words, equation (1.1) states that the probability of having vacant crossing of boxes is bounded away from zero and one, independently of the scale of the box. This property holds for an interval $(0, \bar{u})$, not only a point. Together with equation (1.2), Theorem 1.1.3 shows some similarity between critical bond percolation on $\mathbb{Z}^2$ and ellipses models with $\alpha = 2$ and u on $(0, \bar{u})$.

The existence of a non-trivial interval $(0, \bar{u})$ in which the model features non-degenerate crossing probabilities is very interesting, but not a novelty since it has already appeared in fractal percolation models like Mandelbrot percolation [7].

This result has the same flavor of some other phenomena already in the literature. In [32] it is proven for a random fragmentation model with long-range correlations that there is an entire off-critical region in which power-law scaling is observed. Another example can be found on Coordinate Percolation on $\mathbb{Z}^3$ [13]; in this model, each column that is parallel to one of the coordinate axis of $\mathbb{Z}^3$ is removed or not with a probability parameter depending only on its direction and columns are removed or not independently. This model has infinite range dependencies. In [13] it is shown that the tail distribution for the radius of the open cluster containing the origin decays exponentially fast when at least two of the parameters are fixed to be high, but if two of the parameters are taken relatively small, then the truncated version for this tail decays, at most, polynomially fast. Quoting reference [32], "these findings suggest that long-range directional correlations lead to a rich spectrum of critical phenomena which need to be understood".

Theorems 1.1.1 and 1.1.2 have statements that are quite similar. Looking closely at them, one can identify that when $\alpha \in (0, 2)$ the model is somewhat trivial and when $\alpha > 2$ there is a phase transition in u. Notice that the critical points $\bar{u}_c(\rho)$ and $u_c(\rho)$ mentioned on Theorems 1.1.1 and 1.1.2 are not known to be equal. However, we believe this holds

Conjecture 1.1.4. *When $\alpha > 2$ we have $\bar{u}(\rho) = u_c(\rho)$.*

This would imply that when $\alpha > 2$ the phase transition in u is rather classical, despite the long-range dependencies. This phenomenon is observed in planar Boolean models in reference [1]. In the beginning of Section 2.5 we define both $\bar{u}_c$ and u_c and discuss their relation more deeply (see remark 2.5.2).

1.1.1 Comparison with other models

The ellipses model can be compared with many other continuum percolation models. Since we fixed the minor axis of all ellipses as one, ellipses model trivially dominates Boolean model with radius one. However, it cannot be dominated by any Boolean model with fixed radius (see Remark 2.3.4 after Proposition 2.3.3). On the other hand, any (u, ρ)-ellipses model is dominated by a Boolean model with radius distribution ρ and this can be used to derive some of the results stated in Theorems 1.1.1 and 1.1.2. We notice that this domination is not useful for $\alpha \leq 2$ since in this case the plane is completely covered by balls.

Another straightforward comparison is that a (u, ρ)-ellipses model dominates stick model with same density and radius distribution ρ. As mentioned before, the two models are very similar and our results for ellipses percolation can be translated into results for heavy-tailed stick model. To our knowledge, heavy-tailed sticks have not been considered in the mathematics literature yet.

To justify our preference for ellipses over sticks, we have two main reasons. Firstly, the double phase transition in α described in Theorems 1.1.1 and 1.1.2 is more appealing than the phase transition a stick model presents. Secondly, ellipses models are a generalization of a construction that appears implicitly in a paper of Tykesson and Windisch [36], in which they have defined the Poisson cylinder model on $\mathbb{R}^d$. Many of their results follow by looking at the intersection of the collection of cylinders with a plane $\mathbb{R}^2 \times \{0\}^{d-2}$; by performing this intersection one obtains ellipses. This implicit construction was an inspiration for defining the ellipses model. We make explicit this relation in Section 2.2.

Another related continuum percolation model is the multi-scale version of Boolean percolation [19–21]. This model presents some interesting scale relations, resembling ellipses percolation for $\alpha = 2$. Let us recall however that this scale invariance is only possible for the multi-scale version of the model. Indeed, the standard Boolean percolation cannot present scale invariance as this would require α to be chosen as two and for this choice of tails decay the model covers the whole plane almost surely.

1.1.2 Main tools and ideas

To overcome the dependencies of the model, our study of ellipses percolation uses similar techniques as other models with long range dependencies such as the Poisson cylinder model of [36] and the random interlacements model [35].

Let us discuss the ideas of the proofs and the main tools used to study ellipses percolation. Our model can be defined as a Poisson point process on a larger space (see Definition 2.1.1). Thus, we are able to estimate the probability of many useful events by making an appropriate thinning of it, as described in Proposition 2.3.1 below. These estimates are at the core of many proofs.

We provide two proofs that the plane is completely covered by ellipses iff $\alpha \leq 1$. The first is a consequence of the estimates of Section 2.3 and Borel-Cantelli Lemma. The second makes use of an argument in Hall [12] to relate total covering to the expected area of an ellipse being infinite.

The proof that $\mathcal{E}$ percolates for every $u > 0$ when $\alpha \in (1, 2)$ follows from bounds on the probability of having a left-right covered crossing of a box done by exactly one ellipse. We build a sequence of nested boxes with the property that if we have covered crossings for all but finitely many of them we guarantee $\mathcal{E}$ percolates. In order to prove that for $\alpha \in (1, 2)$ the set $\mathcal{V}$ does not percolate for any $u > 0$, we adapt the proof of Proposition 5.6 from [36]; we prove that with probability one there is an infinite number of circuits made of exactly three ellipses surrounding the origin.

We finish Theorems 1.1.1 and 1.1.2 by proving that when $\alpha > 2$ is fixed there is a phase transition in u for the percolation of $\mathcal{V}$ and also of $\mathcal{E}$. This is done by dominating the ellipses model with a Boolean model with radius distribution ρ (see [10]). The phase transition for $\mathcal{E}$ follows directly from this domination, but for $\mathcal{V}$ we need to develop some additional arguments since [10] does not study the vacant set.

The proof of property (1.1) in Theorem 1.1.3 uses the estimates derived in the previous sections together with a coupling with fractal percolation, also known as Mandelbrot percolation [7]. One important characteristic of the regime $\alpha = 2$ that allows this coupling is the existence of some kind of scale invariance in the model. Our coupling uses the results of Ligget, Schonmann and Stacey [18] and is similar to some couplings already in the literature (see Theorem 8.1 from [19], or references [20, 21]; see also the discussion on Section 2.7). Finally, the fact that $\mathcal{E}$ does not percolate for small u follows from (1.1) and ergodicity. We also present an alternative proof using our bounds on decay of correlations and a generalization of Borel-Cantelli lemma from [24].

There are still some interesting unanswered questions regarding ellipses model. One of them is already stated as Conjecture 1.1.4. Another question is to understand better what actually happens when $\alpha = 2$. We only showed the existence of a phase transition in u related to the probability of crossing boxes of fixed ratio, but it is possible that there are other phase transitions. Let $\{0 \overset{\mathcal{V}}{\leftrightarrow} \partial B(n)\}$

denote the event in which the origin is in a connected component of $\mathcal{V}$ that intersects $\partial B(n)$. For instance, one could define

$$u_{\text{cross}}(\rho) := \sup\{\bar{u};\ (1.1) \text{ holds}\} \text{ and } u_{\text{exp}}(\rho) := \inf\{u;\ P_{u,\rho}[0 \overset{\mathcal{V}}{\leftrightarrow} \partial B(n)] \text{ decays exponentially}\}$$

and check if any of them coincide with $u_c(\rho)$. Finally, it would be interesting to say anything about what happens at the critical point when $\alpha = 2$.

1.2 Poisson Stick Soup

In Chapter 3 we define the scale-homogeneous Poisson stick soup (SHPSS) as the following Poisson point process (PPP) on $S := \mathbb{R}^2 \times [0, \infty) \times (-\pi/2, \pi/2]$.

Definition 1.2.1 (SHPSS and SIPSS)**.** Given u and $\alpha > 0$, consider the PPP on S with intensity measure $u\mu_\alpha := u\lambda \otimes \phi_\alpha \otimes \nu$ where λ is Lebesgue measure on $\mathbb{R}^2$, $\phi_\alpha[a, \infty) = a^{-\alpha}$ for every $a > 0$ and ν is the uniform probability on $(-\pi/2, \pi/2]$. This PPP is called a scale-homogeneous Poisson stick soup (SHPSS) of intensity u and decay α. The particular case in which $\alpha = 2$ is referred to as a scale-invariant Poisson stick soup (SIPSS) of intensity u.

A PPP ξ on S induces a collection of sticks on the plane by considering the union for all $s = (z, R, V) \in \text{supp}\ \xi$ of the stick centered at z with radius R and direction V, denoted by $E_0(s)$; we denote this stick soup by $\mathcal{E}_0 = \cup_{s \in \text{supp}\ \xi} E_0(s)$ and also refer to it as the covered set. In our nomenclature, the term 'scale-homogeneous' stresses that after applying a homothety of ratio $c > 0$ to the process we obtain a process with law given by $c^{2-\alpha}$ times the original law, as we prove in Proposition 3.1.3.

In Section 3.1 we see that SHPSS model is a natural candidate to scaling limit of ellipses model when we contract space through a homothety while adjusting the density of ellipses to obtain a non-degenerate process. This model has sticks of all scales and the sticks are actually dense on the plane.

Intuitively, the scaling limit of ellipses model should be a soup of segments, but we should specify in which sense we are taking a limit. Here, we consider weak convergence for the PPP's on S that represents the major axis of the ellipses. It might be interesting to study convergence in Smirnov Schramm topology [31], a topology defined on the space of quads (topological quadrilaterals), denoted by $(\mathcal{Q}, \mathcal{T})$, for studying scaling limits of planar percolation models. The idea behind this topology is that a percolation model can be associated to a probability distribution on $\mathcal{Q}$ that codifies all quads that have been crossed by open paths. Under some uniform Russo Seymour Welsh estimates on the sequence of percolation models, it holds that their respective probability distributions on $\mathcal{Q}$ are precompact with respect to weak convergence. If we consider a sequence of ellipses models with $\alpha = 2$, our Proposition 1.1.3 asserts that the probability of crossing boxes of fixed proportion with a left-right vacant path is bounded away from 0 and 1, implying that a subsequential limiting distribution might be non-trivial. We have not followed this line of work.

Section 3.2 provides a rough image of a general SHPSS through estimates for the probability of a SHPSS to intersect some sets. This analysis is done considering separately sticks with radius smaller and larger than a fixed $r > 0$. Our task is greatly simplified by a paper of Parker and Cowan [25], but is essentially analogous to the computations for ellipses model in Chapter 2. The analysis of small sticks, in turn, has some particularities of its own.

After a quick glance on general SHPSS models we focus on SIPSS, the particular case of SHPSS that is scale-invariant. Scale-invariant models generated through Poisson point processes are continuous versions of random fractals such as Mandelbrot percolation [7] and have been studied by Nacu and Werner in [23]. Reference [23] determines the Hausdorff dimension of what they call *carpet*, a random set of points that can be connected to the boundary of a domain without 'crossing' curves of the soup. Some of their results apply directly to SIPSS, see the discussion below for a summary.

Our study of SIPSS is based on crossings of boxes of $\mathbb{R}^2$. In Section 3.3 we define, for any fixed box, a natural family of exploration paths $\{\gamma_r^0\}_{r>0}$ on SIPPS related to the existence of left-right paths that do not 'cross' sticks of the soup on the underlying box. Exploration processes are quite common in the literature and have great importance in percolation theory. They appear in Russo Seymour Welsh theory [30, 33], in the study of noise sensitivity [9] and also on the proof that the interface of critical percolation on the triangular lattice converges to SLE_6 [6].

The exploration paths $\{\gamma_r^0\}_{r>0}$ follow the boundary between covered and vacant regions of approximations of SIPSS that get increasingly better as $r \to 0$. Our results are concerned with properties of this family of random curves. Applying the techniques in Aizenman and Burchard [2], we prove results on tightness and regularity for this family of exploration paths.

1.2.1 Historical remarks and main results

In order to motivate and contextualize our results, we take some time to analyze what is already known for SIPSS, based on reference [23].

In [23], Nacu and Werner study a class of Poissonian translation and scale-invariant models in the plane that include SIPSS. Instead of considering only sticks, they build scale-invariant models from measures on the space of compact planar curves. A special interest in [23] is studying Brownian loop soups and SIPSS appears only marginally. We focus only on reporting results that are specific for SIPSS.

Reference [23] proves that SIPSS is a *thin soup*, meaning that $\mu_2(s; R \geq r, E_0(s) \cap B(1) \neq \varnothing)$ is finite for any $r > 0$. Also, they prove that

$$\lim_{r \to \infty} \mu_2(s; R \geq r, E_0(s) \cap B(1) \neq \varnothing) = 0$$

and that for any fixed annulus in $\mathbb{R}^2$ only a finite number of sticks in ξ_0 intersect both its internal and external boundaries. These results for SIPSS can be seen as particular cases of our Proposition 3.2.3 and Lemma 3.2.7, that consider SHPSS models in general.

About percolation of $\mathcal{E}_0$, they prove that $\mathcal{E}_0$ has either zero or exactly one unbounded cluster. Actually, their argument is phrased for random curves with positive inner area instead of SIPSS, but the same argument works for SIPSS after adapting our Lemma 2.5.7 to deduce for any $u > 0$ there are infinitely many circuits of sticks surrounding the origin. Moreover, Nacu and Werner notice that the distance of the origin to the unbounded cluster is a finite and scale-invariant random variable and therefore must be a.s. zero. Thus, either SIPSS is composed of bounded covered clusters or it has a unique and dense unbounded covered cluster.

For most of the paper, Nacu and Werner work under what they call

Subcriticality Assumption 2. With positive probability, there exists a (random) closed loop l in the plane that surrounds the origin and does not 'cross' any sticks of the full soup (see Definition 3.3.5 for a formal description of 'crossing' sticks).

We refer to this assumption by SA2. By a 0-1 argument, under SA2 every point is a.s. surrounded by arbitrarily small and large such circuits. In principle, this assumption is stronger than asking for almost all clusters to be bounded.

They also prove results about the Hausdorff dimension of some important sets related to SIPSS in this subcritical regime (i.e., under SA2). For any simply connected domain $D \subset \mathbb{R}^2$ they define the random soup in D, denoted by Γ_D, as the random sticks of ξ_0 that are contained in D. For $D \neq \mathbb{R}^2$ they define

Definition 1.2.2. The *carpet* G is the set of points $z \in D$ such that for any neighborhood of z there is a path connecting it to ∂D that does not cross any stick of Γ_D.

Under SA2 the carpet is almost surely non-empty since any fixed point $z \in \partial D$ has arbitrarily small circuits surrounding it that do not 'cross' any sticks of the soup. Points in the interior of D that belong to some of these circuits are clearly on G. Nacu and Werner prove that for small densities u SIPSS is indeed subcritical, by claiming that the same coupling with fractal percolation present in [39] works. For completeness, we prove this result in Section 3.3

Proposition 1.2.3. *Let $\xi = \xi^u$ be a SIPSS of intensity u. There is $\bar{u} > 0$ such that if $u \in (0, \bar{u})$ then for any fixed $k > 0$ and $l > 0$*

$$\delta \leq P\,[\textit{there is a left-right path in } B_\infty(l;k) \textit{ that do not cross any stick}] \leq 1 - \delta, \tag{1.3}$$

where $\delta = \delta(u, k) > 0$. Moreover, for $u \in (0, \bar{u})$ we have:

$$P[\textit{neither } \mathcal{V}_0 \textit{ nor } \mathcal{E}_0 \textit{ percolate}] = 1. \tag{1.4}$$

Thus, by FKG inequality we can conclude SA2 holds for $u \in (0, \bar{u})$ and redefine

$$\bar{u} := \sup\{u;\ \xi^u \text{ satisfies the box-crossing property in (1.3)}\}. \tag{1.5}$$

Also under SA2, Nacu and Werner prove that the Hausdorff dimension of the carpet is a.s. a constant associated to the probability of existing certain paths in an annulus, roughly speaking a vacant 1-arm event. More precisely, let $D(l_1, l_2) := \{z \in \mathbb{R}^2; l_1 < |z| < l_2\}$ and consider event A_ϵ in which there is a path joining the inner and outer boundaries of $D(\epsilon, 1)$ that do not 'cross' any sticks of $\Gamma_{D(\epsilon,1)}$. Their Corollary 8 proves there are constants $\bar{\eta}(u)$ and $k'(u)$ such that $\epsilon^{\bar{\eta}} \leq P[A_\epsilon] \leq k'\epsilon^{\bar{\eta}}$, a power law for $P[A_\epsilon]$. Afterwards, in Proposition 11 they prove that if D is a bounded non-empty open domain then its carpet G has Hausdorff dimension $2-\bar{\eta}$ almost surely. Finally, they study how $\dim_{\mathcal{H}}(G)$ behaves as u decreases to zero, obtaining that $\dim_{\mathcal{H}}(G) = 2 - o(u)$.

In Section 3.4 we prove some results that have a similar flavor to the ones just described. Denote by C_ϵ the event in which there is a finite sequence of sticks of the full soup connecting the internal and external boundaries of annulus $D(\epsilon, 1)$, or in other words, in which we have a covered 1-arm. We prove in Proposition 3.4.6 a polynomial decay for $P[C_\epsilon]$, when $u \in (0, \bar{u})$. Although this result seems to be very similar to Corollary 8 of [23], new ideas are needed. We are not even able to fully recover a power law for $P[C_\epsilon]$ with matching exponents, as it holds for $P[A_\epsilon]$.

Finally, our Proposition 3.4.6 is used as a tool to prove that the family of exploration curves (see Figure 1.2) satisfies *Hypothesis H1* of Aizenman and Burchard [2], described in Section 3.3. This leads to

Theorem 1.2.4. *Let $u \in (0, \bar{u})$ and consider a SIPSS $\xi = \xi^u$ and a box $B \subset \mathbb{R}^2$. The family of exploration curves $\{\gamma_r^0\}_{r>0}$ in box B is tight and there is $c(u) > 0$ such that any subsequential limit law in the space of curves is supported on curves with upper-box dimension and Hausdorff dimension smaller than $2 - c(u)$.*

For a more precise statement, see Theorem 3.3.4. All properties in Theorem 1.2.4 are consequences of the theory developed in [2] after we establish that Hypothesis H1 indeed holds for our family of exploration paths. The main difficulty of our result is proving the validity of this hypothesis, which is done in Theorem 3.4.20. Our proof has two main steps. The first is to prove a polynomial decay for covered 1-arm events in SIPSS, i.e., for $P[C_\epsilon]$. As we mentioned above, this is done in Proposition 3.4.6. The second step is to prove a similar result for the probability of having k-arms in exploration

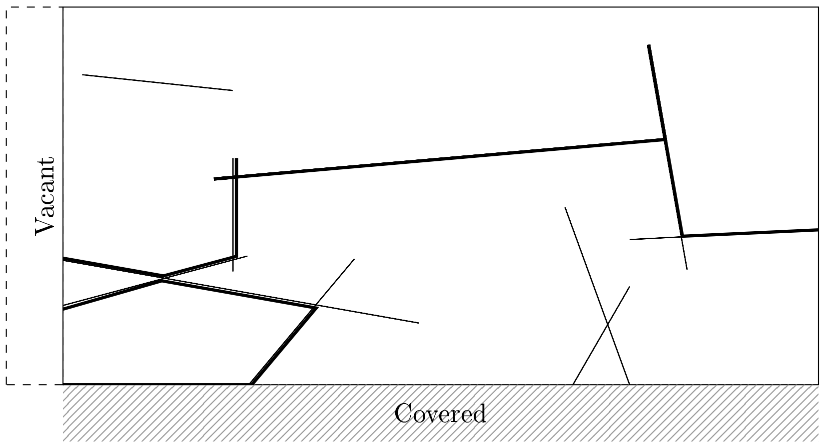


Figure 1.2: Depiction of a random curve from family $\{\gamma_r^0\}_{r>0}$. As $r \to 0$ more sticks from $\mathcal{E}_0$ are added to the picture and the exploration path is updated. A precise definition is in Section 3.3.

paths $\{\gamma_r^0\}$. Since our exploration paths 'follow' sticks, it is natural to expect that a similar result should hold. The key problem here is to find paths that use disjoint subsets of sticks, in order to apply BK inequality, a standard inequality for bounding the disjoint occurrence of events (see Section3.4.2). Section 3.4 is entirely devoted to Hypothesis H1.

Chapter 2

Ellipses Percolation

2.1 Description of the Model

We begin defining a model that provides a random collection of ellipses on $\mathbb{R}^2$. This process will be referred to as the *ellipses model*. To build it, we need three ingredients:

1. Denote by λ the Lebesgue measure on $\mathbb{R}^2$. Given $u \in (0,\infty)$, we consider a Poisson Point Process (PPP) on $\mathbb{R}^2$ with intensity $u\lambda$ which we denote by $\omega = \sum_i \delta_{x_i}$, where $\{x_i\} \subset \mathbb{R}^2$ is countable and locally finite. A concise notation for this definition is $\omega \overset{d}{=} \text{PPP}(u\lambda)$. A reference for PPP's can be found on [28].

2. Given $\alpha > 0$, let ρ be a distribution on $[1,\infty)$ such that $P[R \geq r] \asymp r^{-\alpha}$ for $r \geq 1$, i.e., there is a constant $c_0 = c_0(\rho) > 0$ such that $c_0^{-1} r^{-\alpha} \leq \rho[r,\infty) \leq c_0 r^{-\alpha}$, for all $r \geq 1$.

3. A random variable V with distribution $V \overset{d}{=} \text{U}(-\pi/2, \pi/2]$. The law of V will be denoted by ν.

Definition 2.1.1. Let $S = \mathbb{R}^2 \times [1,\infty) \times (-\pi/2, \pi/2] \subset \mathbb{R}^4$. The *ellipses model* is a PPP on S with intensity measure given by the product $(u\lambda) \otimes \rho \otimes \nu$. We denote it by $\xi = \sum_i \delta_{s_i}$.

Notice that ξ can be seen as an actual collection of ellipses on $\mathbb{R}^2$. Whenever we say ellipse, we mean the curve described by the ellipse together with its interior. For an element $(z,R,V) \in S$, we define $E(z,R,V)$ as the ellipse with center z and major axis of size R, minor axis of size 1 and direction V. We denote by $\mathcal{E} := \mathcal{E}(\xi)$ the random subset of $\mathbb{R}^2$ formed by the union of ellipses $E(s)$ with $s \in \text{supp}\ \xi$. We also write $\mathcal{V} := \mathbb{R}^2 \backslash \mathcal{E}$. The sets $\mathcal{E}$ and $\mathcal{V}$ will be called the *covered* and *vacant* sets, respectively. Our greatest concern with the ellipses model is to understand the connectivity behavior of these sets, as we change the parameters u and α of the model. Parameter u represents the density of ellipses and increasing u also increases the covered area. On the other hand, increasing α makes the major axis distribution have lighter tail decay.

Remark 2.1.2. Ellipses model has translational and rotational invariance. Indeed, this follows from two facts. The first is that the Lebesgue measure on $\mathbb{R}^2$ has rotational and translational invariance. The second one is our choice of uniform distribution for the directions of the ellipses.

Notation: We constantly use the following notation. For denoting boxes in $\mathbb{R}^2$, let

$$B_\infty(l;k) = [-lk/2, lk/2] \times [-l/2, l/2].$$

Denote by $L^-(l;k)$ and $L^+(l;k)$ the left and right sides of box $B_\infty(l;k)$; that is, the sets $\{-lk/2\} \times [-l/2, l/2]$ and $\{+lk/2\} \times [-l/2, l/2]$, respectively. We denote the euclidean ball on $\mathbb{R}^d$ with center on a point w and radius r by $B(w,r)$.

We also add a short note on our notation for constants. Constants that appear during calculations are generally denoted by c or C and can change from line to line. However, for more important constants we add a subscript number referring to their first appearance in the text.

2.2 Relation with Poisson Cylinder Model

Tykesson and Windisch in [36] define a continuum percolation model in $\mathbb{R}^d$, with $d \geq 3$, that provides a collection of random cylinders $\mathcal{L}$, the Poisson Cylinder model. Their model is built as a Poisson Point Process in the space of lines $\mathbb{L}$ of $\mathbb{R}^d$ and every line is the axis of a cylinder of radius 1. There is a very natural one-parameter family of intensity measures for this PPP, namely the ones that are rotation- and translation-invariant. Their definition is based on the Haar measure of SO_d, the group of rotations of $\mathbb{R}^d$ about the origin, and provides a family of intensity measures $u\mu$, where $u > 0$ and μ is a fixed measure. We refer to the original paper for the details.

The main objective of [36] is to study percolation properties of the vacant set $\mathcal{V} = \mathbb{R}^d \backslash \mathcal{L}$, the set of points not covered by any cylinder, when we vary u. Notice that when we restrict ourselves to the intersection of the random cylinders with the plane $e_1^\perp = \mathbb{R}^2 \times \{0\}^{d-2}$ we obtain a collection of random ellipses with minor axis equal to one and this simplification is important when they study percolation of $\mathcal{V}$ for low intensities u. Using this restriction, they prove that for $d \geq 4$, there is $u_*(d) > 0$ such that $\mathcal{V} \cap e_1^\perp$ percolates for $u \in (0, u_*(d))$. However, for $d = 3$ their argument breaks down: in Proposition 5.6 they prove that for any $u > 0$ the set $\mathcal{V} \cap e_1^\perp$ does not percolate.

Our ellipses model is a generalization of the random ellipses in $\mathcal{L} \cap e_1^\perp$. However, their construction does not give us the explicit distribution of the major axis of the random ellipses. In this section we prove

Proposition 2.2.1. *The Poisson cylinder model for $d = 3$ restricted to a plane is equivalent to an ellipses model, when we take $\rho(r, \infty) = r^{-2}$ for $r \geq 1$.*

This result was already known, although not exactly in this formulation (see [16], pages 93-100). We provide a complete proof, based on discussions with Caio Alves and Serguei Popov. Our proof derives from an alternative construction of the PPP on $\mathbb{L}$. Let $e_3 = (0,0,1) \in \mathbb{R}^3$ and denote the usual inner product on $\mathbb{R}^3$ by $\langle \cdot , \cdot \rangle$. Also, denote by D the upper half of S^2, that is

$$D = \{w \in \mathbb{R}^3 : \ ||w|| = 1 \text{ and } \langle w \,, e_3 \rangle > 0\}.$$

We would like to have a construction in the following line:

C.1) Start with a PPP γ on the plane $\mathbb{R}^2 \times \{0\}$ with intensity measure $u\lambda$, where λ is the Lebesgue measure. The points on supp γ will be the points where the axes of the cylinders intersect the plane.

C.2) Independently for every point $r \in \text{supp } \gamma$, choose a random direction on D. The distribution of this random direction will be denoted by β, and each $r \in \text{supp } \gamma$ has its own β_r, independent of everything else. The lines passing through r with direction β_r, for $r \in \text{supp } \gamma$ will be axes of the cylinders.

We would like to know what should be the distribution of β in order for this alternative construction to be equivalent to the construction in [36]. A first guess on what this distribution is could be the one given by uniform distribution on D. However, this guess is wrong.

Our strategy to find the desired distribution is the following. Denote by $\mathbb{L}$ the space of lines in $\mathbb{R}^3$. At first, we suppose there exists a Poisson Point Process on $\mathbb{L}$ which is invariant with respect to rotations and translations on $\mathbb{R}^3$ and conclude what should be the measure associated with it. Having a candidate for measure, it is simple to verify that it indeed is invariant.

Let μ be a measure on $\mathbb{L}$ which is invariant with respect to rotations and translations. We begin by identifying the space $\mathbb{L}$ with the $D \times \mathbb{R}^2$. More precisely, we do not identify all space $\mathbb{L}$, but instead we work with $\mathbb{L}^*$, the space of all lines in $\mathbb{R}^3$ which are not parallel to the plane $\mathbb{R}^2 \times \{0\}$. In Proposition 2.2.6 below, we characterize μ restricted to $\mathbb{L}^*$.

Notice that any line $l \in \mathbb{L}^*$ intersects the plane $\mathbb{R}^2 \times \{0\}$ at exactly one point; denote its (x, y) coordinates by $p(l)$. Also, every l has a direction in D, denoted by $V(l)$. Consider the function:

$$\Phi : \mathbb{L}^* \to D \times \mathbb{R}^2 \text{ such that } l \mapsto (V(l), p(l))$$

It is clear that Φ is a bijection. So, in order to know the measure μ we just need to understand what is the induced measure on the space $D \times \mathbb{R}^2$, and we denote this measure by $\tilde{\mu}$.

Lemma 2.2.2. *$\tilde{\mu}$ is a product measure on $D \times \mathbb{R}^2$: $\tilde{\mu} = \phi \otimes \lambda$, where λ is Lebesgue measure on $\mathbb{R}^2$.*

Proof. Fix A a borelian set in D and let $C \in \mathcal{B}(\mathbb{R}^2)$. Let τ be any translation along the xy plane of $\mathbb{R}^3$ and $\tilde{\tau}$ be translation τ when restricted to $\mathbb{R}^2$, the xy coordinates. Notice that the effect of applying τ is

$$\tau\, \Phi^{-1}(A \times C) = \Phi^{-1}(A \times \tilde{\tau} C).$$

If we define a measure $m_A : \mathcal{B}(\mathbb{R}^2) \to \mathbb{R}^+$ such that $C \mapsto \tilde{\mu}(A \times C)$, then translation invariance of μ implies:

$$m_A(\tilde{\tau}(C)) = \tilde{\mu}(A \times \tilde{\tau}(C)) = \mu(\tau\, \Phi^{-1}(A \times C)) = \mu(\Phi^{-1}(A \times C)) = \tilde{\mu}(A \times C) = m_A(C).$$

Since this is true for any translation $\tilde{\tau}$ on $\mathbb{R}^2$, we conclude that m_A must be a multiple of Lebesgue measure on $\mathbb{R}^2$. In other words, there exists a constant $\phi(A)$ such that

$$\tilde{\mu}(A \times C) = \phi(A)\lambda(C) \quad \forall A \in \mathcal{B}(D), C \in \mathcal{B}(\mathbb{R}^2).$$

Now, if we fix C it is simple to verify that the function $A \mapsto \phi(A)$ is indeed a measure on D, using that $\tilde{\mu}$ is a measure. $\square$

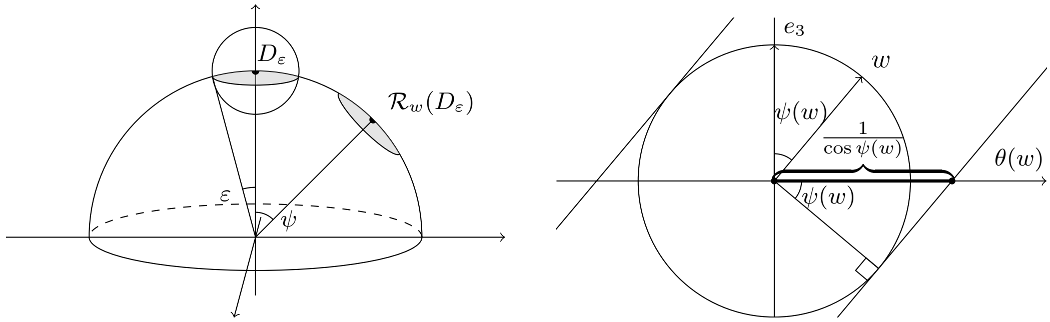
In order to find ϕ, we take advantage of rotational invariance. For any $w \in D$ define $\psi(w)$ as the angle between e_3 and w; this can be written as $\psi(w) = \arccos\langle e_3, w\rangle$. Also, define $\theta(w)$ as the angle between vector $e_1 = (1, 0, 0)$ and the projection of w onto the plane $\mathbb{R}^2 \times \{0\}$. Define the region $D_\varepsilon = \{w \in D : \psi(w) < \varepsilon\}$. Geometrically, D_ε is the region obtained by intersecting D with a ball in $\mathbb{R}^3$ centered in e_3 and whose radius is chosen so that the angle between e_3 and the boundary of the intersection is ε (see Figure 2.1a).

Consider the family of rotations $\{\mathcal{R}_w : w \in D\}$ where $\mathcal{R}_w : \mathbb{R}^3 \to \mathbb{R}^3$ is the only rotation whose invariant line is contained in the plane $\mathbb{R}^2 \times \{0\}$ and such that $\mathcal{R}_w(e_3) = w$. We want to control $\mathcal{R}_w[D_\varepsilon]$.

Lemma 2.2.3. *Fix a radius $r > 0$ and let $B(0, r) \subset \mathbb{R}^3$ be the euclidean ball. Let $w \in D$. Then,*

$$\{l \in \mathbb{L} : \ V(l) = w \text{ and } l \cap B(0, r) \neq \varnothing\} = \Phi^{-1}[\{w\} \times (r \cdot E_w)] \tag{2.1}$$

where E_w is the ellipsis centered at the origin whose major axis has size $\frac{1}{\cos\psi(w)}$ and direction $\theta(w)$ and whose minor axis has size 1. The product $r \cdot E_w$ represents the dilation of E_w by r.



(a) Region D_ε and its rotations. (b) Size of major axis when $V(l) = w$.

Figure 2.1: Auxiliary drawings for Lemmas 2.2.3 and 2.2.4

Proof. By a dilation argument, we may suppose that $r = 1$ without loss of generality. The result is obvious when we consider $w = e_3$, since $E_{e_3} = B_2(0,1) \subset \mathbb{R}^2$. For $w \neq e_3$, we can observe that applying rotation $\mathcal{R}_w$ takes e_3 to w and preserves the intersection with $B(0,1)$. Since the lines

$$\{l \in \mathbb{L} : V(l) = e_3 \text{ and } l \cap B(0,1)\}$$

form a cylinder and rotations preserve shape, we obtain that the subset of $\mathbb{R}^2$ we are looking for is the intersection of a cylinder and a plane, which is known to be an ellipsis. Since $\mathcal{R}_w$ preserves the vectors in $\mathbb{R}^2 \times \{0\}$ whose direction is orthogonal to $\theta(w)$, the minor axis of the ellipsis has size 1. Finally, the size of the major axis can be found by looking at the lines that are tangent to $B(0,1)$ (see Figure 2.1b). $\square$

Using Lemma 2.2.3 and the rotational invariance of μ we are able to relate $\phi[V_\varepsilon]$ and $\phi[\mathcal{R}_w(V_\varepsilon)]$.

Lemma 2.2.4. *Take $w \in D\backslash\{e_3\}$ and let $\psi = \psi(w)$. For all ε such that $0 < \varepsilon < \psi \wedge (\pi/2 - \psi)$ we have:*

$$\phi[\mathcal{R}_w(D_\varepsilon)]\frac{\cos^2\varepsilon}{\cos(\psi-\varepsilon)} \leq \phi[D_\varepsilon] \leq \phi[\mathcal{R}_w(D_\varepsilon)]\frac{1}{\cos(\psi+\varepsilon)}. \tag{2.2}$$

Proof. By Lemma 2.2.3 we know that

$$\mathcal{R}_v\Phi^{-1}[\{e_3\} \times B_2(0,1)] = \Phi^{-1}[\{v\} \times E_v], \ \forall v \in D. \tag{2.3}$$

Fix $w \in D$ and assume $0 < \varepsilon < \psi \wedge (\pi/2 - \psi)$. We divide the proof into the upper and lower bound. Both parts use the same reasoning: we play with equation (2.3) for $v \in D_\varepsilon$ to derive a set inclusion relation, and then use the rotational invariance of μ together with Fubini's theorem for the measure $\phi \otimes \lambda$.

Upper Bound: Notice that

$$\Phi^{-1}[\{v\} \times B_2(0,1)] \subset \Phi^{-1}[\{v\} \times E_v] = \mathcal{R}_v\Phi^{-1}[\{e_3\} \times B_2(0,1)], \tag{2.4}$$

by equation (2.3). Then, we have

$$\Phi^{-1}[D_\varepsilon \times B_2(0,1)] = \bigcup_{v\in D_\varepsilon} \Phi^{-1}[\{v\} \times B_2(0,1)] \subset \bigcup_{v\in D_\varepsilon} \mathcal{R}_v\ \Phi^{-1}[\{e_3\} \times B_2(0,1)]$$

and if we apply rotation $\mathcal{R}_w$, we get:

$$\mathcal{R}_w\ \Phi^{-1}[D_\varepsilon \times B_2(0,1)] \subset \bigcup_{v\in D_\varepsilon} \mathcal{R}_w\mathcal{R}_v\ \Phi^{-1}[\{e_3\} \times B_2(0,1)].$$

Now, we notice that $\mathcal{R}_w\mathcal{R}_v(e_3) = \mathcal{R}_w(v)$ and then by Lemma 2.2.3 we have:

$$\mathcal{R}_w\mathcal{R}_v\ \Phi^{-1}[\{e_3\} \times B_2(0,1)] = \Phi^{-1}[\{\mathcal{R}_w(v)\} \times E_{\mathcal{R}_w(v)}].$$

Making the union for $v \in D_\varepsilon$ we obtain the following set inclusion:

$$\mathcal{R}_w\Phi^{-1}[D_\varepsilon \times B_2(0,1)] \subset \Phi^{-1}[\cup_{v\in\mathcal{R}_w(D_\varepsilon)}\{v\} \times E_v]. \tag{2.5}$$

By rotational invariance of μ, we can write:

$$\mu[\Phi^{-1}(D_\varepsilon \times B_2(0,1))] = \mu[\,\mathcal{R}_w\ \Phi^{-1}[D_\varepsilon \times B_2(0,1)]\,] \le \mu[\,\Phi^{-1}[\cup_{v\in\mathcal{R}_w(D_\varepsilon)}\{v\} \times E_v]\,].$$

Thus, applying Fubini's theorem to measure $\tilde{\mu} = \phi \otimes \lambda$ we have:

$$\begin{aligned}\tilde{\mu}[D_\varepsilon \times B_2(0,1)] &\le \int_{\mathcal{R}_w(D_\varepsilon)} \lambda(E_v)\,\phi(\mathrm{d}v) = \int_{\mathcal{R}_w(D_\varepsilon)} \frac{\pi}{\cos\psi(v)}\,\phi(\mathrm{d}v)\\ &\le \int_{\mathcal{R}_w(D_\varepsilon)} \frac{\pi}{\cos[\psi(w)+\varepsilon]}\,\phi(\mathrm{d}v) = \frac{\pi}{\cos[\psi(w)+\varepsilon]}\phi(\mathcal{R}_w(D_\varepsilon)).\end{aligned}$$

Since $\tilde{\mu}[D_\varepsilon \times B_2(0,1)] = \phi(D_\varepsilon)\pi$, we have proved the upper bound on equation (2.2).

Lower Bound: For the lower bound, we begin by noticing

$$\Phi^{-1}[\{v\} \times B_2(0,1)] \supset \Phi^{-1}[\{v\} \times (\cos\psi(v)\cdot E_v)] = \mathcal{R}_v\ \Phi^{-1}[\{e_3\} \times B_2(0,\cos\psi(v))],$$

by Lemma 2.2.3. Taking the union for $v \in D_\varepsilon$:

$$\Phi^{-1}[D_\varepsilon \times B_2(0,1)] \supset \bigcup_{v\in D_\varepsilon} \mathcal{R}_v\ \Phi^{-1}[\{e_3\} \times B_2(0,\cos\psi(v))] \supset \bigcup_{v\in D_\varepsilon} \mathcal{R}_v\ \Phi^{-1}[\{e_3\} \times B_2(0,\cos\varepsilon)].$$

Apply rotation $\mathcal{R}_w$ on both sides to get

$$\mathcal{R}_w\ \Phi^{-1}[D_\varepsilon \times B_2(0,1) \supset \bigcup_{v\in D_\varepsilon} \mathcal{R}_w\mathcal{R}_v\ \Phi^{-1}[\{e_3\} \times B_2(0,\cos\varepsilon)] \supset \Phi^{-1}[\cup_{v\in\mathcal{R}_w(D_\varepsilon)}\,\{v\} \times (\cos\varepsilon\cdot E_v)]$$

and then use the rotational invariance of μ and Fubini:

$$\begin{aligned}\tilde{\mu}[D_\varepsilon \times B_2(0,1)] &= \mu[\Phi^{-1}[D_\varepsilon \times B_2(0,1)]] = \mu[\mathcal{R}_w\ \Phi^{-1}[D_\varepsilon \times B_2(0,1)]]\\ &\ge \tilde{\mu}[\cup_{v\in\mathcal{R}_w(D_\varepsilon)}\,\{v\} \times (\cos\varepsilon\cdot E_v)] = \int_{v\in\mathcal{R}_w(D_\varepsilon)} \lambda(\cos\varepsilon\cdot E_v)\,\phi(\mathrm{d}v)\\ &= \int_{v\in\mathcal{R}_w(D_\varepsilon)} \frac{\pi\cos^2\varepsilon}{\cos\psi(v)}\,\phi(\mathrm{d}v) \ge \int_{v\in\mathcal{R}_w(D_\varepsilon)} \frac{\pi\cos^2\varepsilon}{\cos[\psi(w)-\varepsilon]}\,\phi(\mathrm{d}v)\\ &= \frac{\pi\cos^2\varepsilon}{\cos[\psi(w)-\varepsilon]}\phi(\mathcal{R}_w(D_\varepsilon)).\end{aligned}$$

The lower bound follows when we notice that $\tilde{\mu}[D_\varepsilon \times B_2(0,1)] = \phi(D_\varepsilon)\pi$. □

Notice that Lemma 2.2.4 provides a natural candidate for the measure ϕ. If we assume that ϕ is absolutely continuous with respect to the Lebesgue measure on D, denoted by σ, then equation (2.2) shows which function is its Radon-Nikodym derivative:

Lemma 2.2.5. *If ϕ is a measure on D such that $\phi \ll \sigma$ and equation* (2.2) *is satisfied, then there exists $c \geq 0$ such that:*

$$\phi(A) = c \int_A \cos\psi(w)\ \sigma(dw). \tag{2.6}$$

Proof. Indeed, if we denote by f the Radon-Nikodym derivative, then by Lebesgue differentiation theorem we have:

$$\lim_{\varepsilon\to 0} \frac{\phi[\mathcal{R}_w(D_\varepsilon)]}{\sigma[\mathcal{R}_w(D_\varepsilon)]} = f(w), \qquad \sigma\text{-a.s.} \tag{2.7}$$

and the limit should exist for at least one $w_0 \in D$. On the other hand, Equation (2.2) implies that for any $w \in D$

$$\lim_{\varepsilon\to 0} \frac{\phi[\mathcal{R}_w(D_\varepsilon)]}{\sigma[\mathcal{R}_w(D_\varepsilon)]} = \cos\psi(w) \lim_{\varepsilon\to 0} \frac{\phi[D_\varepsilon]}{\sigma[D_\varepsilon]}. \tag{2.8}$$

We conclude the limit must exist for any $w \in D$. Moreover, if we define $c := \lim_{\varepsilon\to 0} \frac{\phi[D_\varepsilon]}{\sigma[D_\varepsilon]}$ then $f(w) = c\cos\psi(w)$, σ-a.s. □

We have proved that any measure ϕ that comes from a μ which is invariant with respect to rigid motions satisfies equation (2.2) in Lemma 2.2.4. Also, by Lemma 2.2.5 we know that the only measures ϕ satisfying (2.2) that are absolutely continuous with respect to σ are the ones given by equation (2.6). In order to conclude that the only possible measures ϕ must be given by equation (2.6), we need to guarantee $\phi \ll \sigma$.

Proposition 2.2.6. *Let μ be a non-trivial measure on $\mathbb{L}$ which is translational and rotational invariant and let $\tilde{\mu}$ be its induced measure on the space $D \times \mathbb{R}^2$ through the function Φ. Then $\tilde{\mu} = \phi \otimes \lambda$ where λ is Lebesgue measure on $\mathbb{R}^2$ and*

$$\phi(A) = c \int_A \cos\psi(w)\,\sigma(dw) \text{ for } A \in \mathcal{B}(D),$$

with σ being the Lebesgue measure on D and $c > 0$.

Remark 2.2.7. Proposition 2.2.6 explicits the intensity measure for the PPP restricted to $\mathbb{L}^*$. Using this representation and rotational invariance we can conclude that $\mu(\mathbb{L}\backslash\mathbb{L}^*) = 0$. Indeed, if $\mathcal{R}$ is any rotation that does not leave the xy plane invariant then $\{V(l); l \in \mathcal{R}(\mathbb{L}\backslash\mathbb{L}^*)\}$ is the intersection of D with a plane and must have σ measure zero.

Proof. To prove absolute continuity is equivalent to proving that

$$\forall\eta > 0,\ \exists\delta > 0 \text{ such that } \sigma(A) < \delta \implies \phi(A) < \eta.$$

We prove the following result, from which the absolute continuity is a trivial consequence:

Claim: There exists $K \in \mathbb{R}^+$ such that $\frac{\phi(A)}{\sigma(A)} \leq K$, $\forall A \in \mathcal{B}(D)$ with $\sigma(A) > 0$.

First, let us prove the claim for sets A of the form D_ε. We have mentioned that $D_\varepsilon = D \cap B_\varepsilon$, where B_ε is an euclidean ball in $\mathbb{R}^3$. By applying the law of cosines, we can deduce that $B_\varepsilon = B(e_3, \sqrt{2 - 2\cos\varepsilon}\,)$. For a point $w \in D$, define $B_\varepsilon(w) = B(w, \sqrt{2 - 2\cos\varepsilon}\,)$. Notice that the sets $\mathcal{R}_w(D_\varepsilon) = D \cap B_\varepsilon(w)$ are open balls on D and thus form a basis for its topology.

Consider the set $D_{\pi/4} \subset D$. For $\varepsilon \in (0, \pi/4)$, we will build coverings of the region $D_{\pi/4}$. For a fixed ε, consider a finite subset $\mathcal{N}_\varepsilon \subset D_{\pi/4}$ built through the following inductive procedure.

Define $N_1 = \{e_3\}$. For $i \geq 1$, if $N_i = \{w_1, \dots, w_i\}$ is such that $B_{\varepsilon/3}(w_j)$ are disjoint for all j and there exists $w_{i+1} \in D_{\pi/4}$ such that $B_{\varepsilon/3}(w_{i+1})$ is disjoint from the previous balls, then set $N_{i+1} = N_i \cup \{w_{i+1}\}$. Else, stop the procedure and call $\mathcal{N}_\varepsilon$ the final set of points obtained.

This procedure must stop after a finite number of steps, since the sets $\{B_{\varepsilon/3}(w)\}_{w\in\mathcal{N}_\varepsilon}$ are disjoint and there is a constant $c > 0$ (independent of ε) such that for sufficiently small ε:

$$\sigma(B_{\varepsilon/3}(w) \cap D_{\pi/4}) \geq c\,\sigma(B_{\varepsilon/3}(w)) = c\,2\pi(1 - \cos(\varepsilon/3)) \geq c\,\varepsilon^2 \tag{2.9}$$

where the constant c has changed but is always positive. Equation (2.9) implies

$$\sigma(D_{\pi/4}) \geq \sum_{w\in\mathcal{N}_\varepsilon} \sigma(B_{\varepsilon/3}(w) \cap D_{\pi/4}) \geq c|\mathcal{N}_\varepsilon|\varepsilon^2,$$

and then $|\mathcal{N}_\varepsilon| \leq c\varepsilon^{-2}$. Notice also that the family of balls $\{B_\varepsilon(w)\}_{w\in\mathcal{N}_\varepsilon}$ must cover $D_{\pi/4}$, since for any $v \in D_{\pi/4}$ there is some $w_j \in \mathcal{N}_\varepsilon$ with

$$\operatorname{dist}(v, w_j) < 2\sqrt{2 - 2\cos(\varepsilon/3)}$$

and for sufficiently small ε we have $2\sqrt{2 - 2\cos(\varepsilon/3)} < \sqrt{2 - 2\cos\varepsilon}$, which implies $v \in B_\varepsilon(w_j)$. From this we can derive a lower bound on $|\mathcal{N}_\varepsilon|$:

$$\sigma(D_{\pi/4}) \leq \sum_{w\in\mathcal{N}_\varepsilon} \sigma(B_\varepsilon(w)) \leq |\mathcal{N}_\varepsilon| 2\pi(1 - \cos\varepsilon) \leq C\,|\mathcal{N}_\varepsilon|\varepsilon^2 \implies |\mathcal{N}_\varepsilon| \leq C\varepsilon^{-2}$$

for another constant $C > 0$. Now, using the bounds we obtained for $|\mathcal{N}_\varepsilon|$, we can use this covering argument with the measure ϕ instead of σ:

$$\phi(D_{\pi/4}) \leq \sum_{w\in\mathcal{N}_\varepsilon} \phi(\mathcal{R}_w(D_\varepsilon)) \leq \sum_{w\in\mathcal{N}_\varepsilon} \phi(D_\varepsilon)\frac{\cos(\psi(w) - \varepsilon)}{\cos^2\varepsilon} \leq \phi(D_\varepsilon)\frac{|\mathcal{N}_\varepsilon|}{\cos^2\varepsilon} < \infty$$

since equation (2.2) implies $\phi(D_\varepsilon) \to 0$ when $\varepsilon \to 0$. Indeed, if this were not true the measure ϕ should be trivial: $\phi(A) = \infty$ for any $A \in \mathcal{B}(D)\backslash\{\varnothing\}$. On the other hand, we must also have:

$$\phi(D_{\pi/4}) \geq \sum_{w\in\mathcal{N}_\varepsilon} \phi(\mathcal{R}_w(D_{\varepsilon/3}) \cap D_{\pi/4}).$$

Remove from $\mathcal{N}_\varepsilon$ any w such that $\psi(w) \in (\frac{\pi}{4} - \varepsilon, \frac{\pi}{4})$, obtaining the set $\mathcal{N}'_\varepsilon$. Notice that we still have $|\mathcal{N}'_\varepsilon| \geq c\varepsilon^{-2}$ for some c and now $\mathcal{R}_w(D_{\varepsilon/3}) \subset D_{\pi/4}$, $\forall w \in \mathcal{N}'_\varepsilon$. Then:

$$\begin{aligned}
\phi(D_{\pi/4}) &\geq \sum_{w\in\mathcal{N}'_\varepsilon} \phi(\mathcal{R}_w(D_{\varepsilon/3})) \geq \phi(D_\varepsilon) \sum_{w\in\mathcal{N}'_\varepsilon} \cos(\psi(w) + \varepsilon) \\
&\geq \cos(\pi/4)\frac{\phi(D_\varepsilon)}{\sigma(D_\varepsilon)}|\mathcal{N}'_\varepsilon|\sigma(D_\varepsilon) \geq \frac{\phi(D_\varepsilon)}{\sigma(D_\varepsilon)} \cdot c\,\frac{1 - \cos\varepsilon}{\varepsilon^2}.
\end{aligned}$$

Since $\varepsilon^{-2}(1 - \cos\varepsilon) \to \frac{1}{2}$ when $\varepsilon \to 0$ and we have seen $\phi(D_{\pi/4}) < \infty$, we conclude that $\frac{\phi(D_\varepsilon)}{\sigma(D_\varepsilon)} \leq K$, $\forall\varepsilon < \tilde{\varepsilon}$ where $\tilde{\varepsilon}$ is sufficiently small. For sets of the form $\mathcal{R}_w(D_\varepsilon)$ that are contained in D and have $\varepsilon < \tilde{\varepsilon}$, we can use Lemma 2.2.4 to deduce:

$$\frac{\phi(\mathcal{R}_w(D_\varepsilon))}{\sigma(\mathcal{R}_w(D_\varepsilon))} \leq \frac{\cos(\psi(w) - \varepsilon)}{\cos^2\varepsilon}\,\frac{\phi(D_\varepsilon)}{\sigma(D_\varepsilon)} \leq \frac{1}{\cos^2\tilde{\varepsilon}}K,$$

a finite upper bound that is independent of w. Finally, for a general set $A \in \mathcal{B}(D)$ with $\sigma(A) > 0$ we can argue as follows. Since measure σ is regular, we can find a sequence $\{\mathcal{R}_{w_j}(D_{\varepsilon_j})\}_{j\in\mathbb{N}}$ of balls on D with:

$$\varepsilon_j < \tilde{\varepsilon}\ \forall j,\ A \subset \bigcup_j \mathcal{R}_{w_j}(D_{\varepsilon_j}) \text{ and } \sigma(A) \leq \sum_j \sigma(\mathcal{R}_{w_j}(D_{\varepsilon_j})) \leq 2\sigma(A).$$

From this we conclude

$$\phi(A) \leq \sum_j \phi(\mathcal{R}_{w_j}(D_{\varepsilon_j})) \leq K \sum_j \sigma(\mathcal{R}_{w_j}(D_{\varepsilon_j})) \leq 2K\sigma(A),$$

proving the claim and also Proposition 2.2.6. $\square$

Proposition 2.2.1 is a straightforward consequence of Proposition 2.2.6.

Proof of Proposition 2.2.1. The proof is immediate from Proposition 2.2.6, once we notice that

$$\rho(r,\infty) = P[R \geq r] = \phi(\{w \in D; \tfrac{1}{\cos\psi(w)} \geq r\}) \cdot \phi(D)^{-1} \tag{2.10}$$

and calculate $\phi(\{w \in D; \cos\psi(w) \leq \frac{1}{r}\})$. Parametrize D as $X(u,v) := (\sin v \cos u, \sin v \sin u, \cos v)$ for $u \in (0, 2\pi)$ and $v \in (0, \pi/2)$. Then, we can compute their partial derivatives

$$X_u = (-\sin v \sin u, \sin v \cos u, 0) \text{ and } X_v = (\cos v \cos u, \cos v \sin u, -\sin v),$$
$$\text{implying } ||X_u \times X_v|| = ||(\sin^2 v \cos u, -\sin^2 v \sin u, -\sin v \cos v)|| = \sin v.$$

Computing the integral in D, we have:

$$\begin{aligned}\phi(\{w \in D; \cos\psi(w) \leq r^{-1}\}) &= \int_0^{2\pi}\int_{\arccos\frac{1}{r}}^{\frac{\pi}{2}} \cos v \cdot ||X_u \times X_v||\, \mathrm{d}v\, \mathrm{d}u = \int_0^{2\pi}\int_{\arccos\frac{1}{r}}^{\frac{\pi}{2}} \cos v \cdot \sin v\, \mathrm{d}v\, \mathrm{d}u \\ &= 2\pi \int_{\sqrt{1-r^{-2}}}^{1} s\, \mathrm{d}s = 2\pi\left[\frac{1}{2} - \frac{1}{2}\left(1 - r^{-2}\right)\right] = \frac{\pi}{r^2},\end{aligned}$$

where in the last line we used the substitution $s = \cos v$. Notice also that taking $r = 1$ we get $\phi(D) = \pi$, hence we have from equation (2.10) that $\rho(r,\infty) = r^{-2}$. $\square$

2.3 Probability of Simple Events

It is useful to have in hands estimates on the probability of some basic events. These events will be used later to build more complex ones. This section is devoted to collecting these estimates. We want to estimate the probability of the intersection and covering events

$$\{\mathcal{E} \cap A \neq \varnothing\} \text{ and } \{A \subset \mathcal{E}\} \tag{2.11}$$

where $A \subset \mathbb{R}^2$ is some fixed set. Such estimates are done in two steps. First, we fix a point $z \in \mathbb{R}^2$ and try to bound the probability of a random ellipse centered at z to intersect (or cover) A. Of course, this probability will depend on z and A, and we would like that this dependence is not too complicated for calculations. For that reason, we will only be concerned with sets A which are reasonably simple, such as points, segments and balls.

The second step consists in taking into consideration the positions of all the centers of ellipses on the support of ω, the PPP on $\mathbb{R}^2$. Selecting only points of PPP ω that intersect A can be seen as a *thinning* and is essentially an application of Fubini's theorem. We use a proposition that can be found, for instance, in Meester and Roy [19], Proposition 1.3. We adapted their version to our notational conventions:

Proposition 2.3.1. *Let ω be a PPP on $\mathbb{R}^d$ with intensity measure $u\lambda$ and g be a measurable function $g: \mathbb{R}^d \to [0,1]$. Define ω_g the thinning of ω that keeps every point $z \in \operatorname{supp}\omega$ independently with probability $g(z)$. Then ω_g is a non-homogeneous PPP on $\mathbb{R}^d$ with intensity measure μ_g given by*

$$\mu_g(U) = u \int_U g(z)\, dz.$$

Notice that the invariances of the process (see Remark 2.1.2) ensure we can apply any rigid motion to A without altering the probability of the events on (2.11).

2.3.1 Estimates for covering a small ball

We investigate the probability of covering an euclidean ball $B(w,\varepsilon) := \{z \in \mathbb{R}^2; |z-w| \le \varepsilon\}$ for $0 \le \varepsilon \le 1/2$. Notice that we allow $\varepsilon = 0$ i.e., $B(w,0) = \{w\}$.

Lemma 2.3.2. *Let $w, z \in \mathbb{R}^2$ and $0 \le \varepsilon \le \frac{1}{2}$. Let E_z be a random ellipse centered on z with minor axis size equal to one and whose major axis is distributed as $\rho \otimes \nu$. Then, there is a constant $c_1 = c_1(c_0, \alpha) > 0$ such that for $|z-w| \ge 2$ we have*

$$c_1^{-1}|z-w|^{-(\alpha+1)} \le P(B(w,\varepsilon) \subset E_z) \le c_1|z-w|^{-(\alpha+1)}.$$

Proof. Without loss of generality, we may assume w is the origin and $z = (0,|z|)$. Thus, we have to prove $c_1^{-1}|z|^{-(\alpha+1)} \le P(B(\varepsilon) \subset E_z) \le c_1|z|^{-(\alpha+1)}$.

By symmetry, we can state that $P[0 \in E_z] = P[z \in E_0]$, where E_0 is a random ellipse centered in the origin with major axis size and direction given by R and V. Rotate the ellipse E_0 by the angle V clockwise and denote this rotation by $\mathcal{R}_V$. This rotation sends the ellipse E_0 to the ellipse $\mathcal{R}_V(E_0)$,whose equation in the plane is $\{(x,y); (x/R)^2 + y^2 \le 1\}$. Then:

$$P(z \in E_0) = P\left(\left(\frac{|z|\cos V}{R}\right)^2 + (|z|\sin V)^2 \le 1\right) = P\left(\left(\frac{1}{R}\right)^2 + \sin^2 V\left[1 - \frac{1}{R^2}\right] \le \frac{1}{|z|^2}\right). \tag{2.12}$$

Upper Bound: For the upper bound we notice that $P(B(\varepsilon) \subset E_z) \le P(0 \in E_z)$. We can assume $|z| > 2$, since we will choose $c(\varepsilon) \ge 2$. On the event in (2.12) we have $R \ge |z|$ and also that

$$\sin^2 V \le \frac{1}{|z|^2}\left[1 - \frac{1}{R^2}\right]^{-1} \le \frac{1}{|z|^2}\left[1 - \frac{1}{|z|^2}\right]^{-1} = \frac{1}{|z|^2 - 1}. \tag{2.13}$$

By equation (2.13), the event whose probability we want to estimate is contained into the rectangular event $\{R \ge |z|,\ |\sin V| \le (|z|^2-1)^{-1/2}\}$. This means

$$P(0 \in E_z) \le c_0|z|^{-\alpha}\left[\frac{2}{\pi}\arcsin\left(\frac{1}{(|z|^2-1)^{\frac{1}{2}}}\right)\right] \le \frac{4c_0}{\pi}|z|^{-(\alpha+1)}\frac{|z|}{\sqrt{|z|^2-1}} \le c_1|z|^{-(\alpha+1)}$$

where we have used that $\arcsin x \le 2x$ for $x \in [0,1]$ and $|z|[\,|z|^2 - 1\,]^{-1/2} \le 2$ for $|z| > 2$ and defined the constant $c_1(c_0) = \frac{8c_0}{\pi}$.

Lower Bound: For the lower bound, we notice that $P(B(\varepsilon) \subset E_z) = P(B(z,\varepsilon) \subset E_0)$. Once again, we apply a clockwise rotation $\mathcal{R}_V$ to get

$$P(B(z,\varepsilon) \subset E_0) = P(\mathcal{R}_V(z) + B(\varepsilon) \subset \mathcal{R}_V(E_0)).$$

Now, notice that $\mathcal{R}_V(z) = (|z|\cos V, -|z|\sin V)$ and for any $r = (r_1, r_2) \in B(\varepsilon)$ we have

$$\{\mathcal{R}_V(z) + r \in \mathcal{R}_V(E_0)\} = \left\{ \left(\tfrac{|z|\cos V + r_1}{R}\right)^2 + (|z|\sin V - r_2)^2 \le 1 \right\}.$$

Using the inequality $(a+b)^2 \le 2(a^2+b^2)$ twice and that $R \ge 1$, we can write

$$\begin{aligned}\left[\frac{|z|\cos V + r_1}{R}\right]^2 + (|z|\sin V - r_2)^2 &\le 2\left(\frac{|z|^2\cos^2 V + r_1^2}{R^2}\right) + 2(|z|^2\sin^2 V + r_2^2)\\ &\le 2|z|^2\left[\frac{1}{R^2} + \left(1 - \frac{1}{R^2}\right)\sin^2 V\right] + 2\varepsilon^2.\end{aligned}$$

Finally, notice that if we make both terms of the last sum smaller than $1/2$ we guarantee the event $\{\mathcal{R}_V(z) + B(\varepsilon) \subset \mathcal{R}_V(E_0)\}$ happens. It is easily checked that when $\varepsilon \le 1/2$ this event contains $\{R \ge \sqrt{2}|z|,\ |V| \le \arcsin\left(\frac{1}{\sqrt{2}|z|}\right)\}$ whose probability is greater than $c_1^{-1}|z|^{-(\alpha+1)}$ for some constant $c_1 = c_1(c_0, \alpha) > 0$ and $|z| \ge 2$. □

Having Lemma 2.3.2, we proceed in our two step strategy.

Proposition 2.3.3. *Let $w \in \mathbb{R}^2$ and $0 \le \varepsilon \le 1/2$. Then, $P[B(w,\varepsilon) \subset \mathcal{E}] = 1$ if and only if $\alpha \le 1$.*

Notice that the result in Proposition 2.3.3 does not depend on u.

Proof. We begin assuming that $\varepsilon = 0$. By translation invariance, we may assume that w is the origin. Define the function $g(z) = P[0 \in E_z]$. Then, Lemma 2.3.2 provides the asymptotic behavior of $g(z)$ as $z \to \infty$: $g(z) \asymp |z|^{-(\alpha+1)}$. We use Proposition 2.3.1 with this function. Notice that

$$P[0 \notin \mathcal{E}] = P[0 \in \mathcal{V}] = P[\omega_g(\mathbb{R}^2) = 0] = \exp\left[-u \int_{\mathbb{R}^2} g(z)\,\mathrm{d}z\right] = 0 \quad \text{iff} \quad \int_{\mathbb{R}^2} g(z)\,\mathrm{d}z = \infty. \tag{2.14}$$

Since $g(z) \in [0,1]$ for all z, the integral in (2.14) is infinite iff $g(z)$ decays to zero sufficiently slow.

$$\int_{\mathbb{R}^2} g(z)\,\mathrm{d}z = \infty \text{ if and only if } \int_c^\infty r^{-\alpha}\,\mathrm{d}r = \infty, \text{ if and only if } \alpha \le 1. \tag{2.15}$$

Now, we handle the case $0 < \varepsilon \le 1/2$. If $\alpha \le 1$, then the same argument above with the function $g(z) := P[B(\varepsilon) \subset E_z]$ shows that $P[B(\varepsilon) \subset \mathcal{E}] = 1$. On the other hand, if $\alpha > 1$ then $P[B(\varepsilon) \subset \mathcal{E}] \le P[0 \in \mathcal{E}] < 1$. □

Remark 2.3.4. Let $x, y \in \mathbb{R}^2$ and $l(x,y)$ be the segment with endpoints in x and y. Notice that $P[l(x,y) \subset \mathcal{E}] \ge P[\exists z \in \operatorname{supp} \omega \cap B(x, 1/4); y \in E_z]$. Using Lemma 2.3.2 and the same thinning argument above we can see that $P[l(x,y) \subset \mathcal{E}] \ge 1 - \exp[-uc|x-y|^{-\alpha}]$. Then, the ellipses model cannot be dominated by any Boolean model of fixed radius since for the Boolean model the probability of covering $l(x,y)$ decays exponentialy on $|x-y|$ (see Remark 3.2 of [36]).

2.3.2 Estimates for intersecting a ball

In a completely analogous way we have just done, we can look into the case in which $\mathcal{E}$ intersects a ball $B(w,a)$.

Lemma 2.3.5. *Let C be any fixed constant with $C > 1$. Let $a > 0$ and $w, z \in \mathbb{R}^2$ be points with $|z-w| \geq \max\{a+1, Ca\}$. Also, let E_z be a random ellipse centered at z and whose size and direction of the major axis have distribution $\rho \otimes \nu$, and let α be the decay parameter of ρ. Then, there exists $c_2 = c_2(c_0, \alpha, C) > 0$ such that*

$$c_2^{-1} a|z-w|^{-(\alpha+1)} \leq P[E_z \cap B(w,a) \neq \varnothing] \leq c_2(a+1)|z-w|^{-(\alpha+1)}.$$

Proof. By applying a translation and a rotation we can suppose without loss of generality that w is the origin and $z = \big(|z|, 0\big)$. Thus, it is sufficient to prove there exists $c_2(c_0, \alpha, C) > 0$ such that

$$c_2^{-1} a|z|^{-(\alpha+1)} \leq P[E_z \cap B(a) \neq \varnothing] \leq c_2(a+1)|z|^{-(\alpha+1)}.$$

Upper Bound: In order to be possible for the ellipse E_z to intersect $B(a)$, it is necessary that $|V| \leq \arcsin(\frac{a+1}{|z|}) =: V_{a,z}$. Besides that, we also need $R \geq |z| - a$ independently of the value of V. Then, using independence

$$\begin{aligned} P[E_z \cap B(a) \neq \varnothing] &\leq P\big[\ R \geq |z| - a\ ,\ |V| \leq V_{a,z}\ \big] \leq c_0\big(|z|-a\big)^{-\alpha} \cdot \left(\frac{2}{\pi} V_{a,z}\right) \\ &= \frac{2}{\pi} c_0 \left(1 - \frac{a}{|z|}\right)^{-\alpha} |z|^{-\alpha} V_{a,z} \leq \frac{2}{\pi} c_0 \left(1 - \frac{1}{C}\right)^{-\alpha} |z|^{-\alpha} V_{a,z}. \end{aligned}$$

Finally, we use that $\arcsin x \leq 2x$ for $x \in [0,1]$ to obtain $V_{a,z} = \arcsin(\frac{a+1}{|z|}) \leq 2(a+1)|z|^{-1}$ and define a constant $c_2(c_0, \alpha, C)$.

Lower Bound: Define $\tilde{V}_{a,z} := \arcsin(a/|z|)$. We claim that the event $\{\,|V| < \tilde{V}_{a,z},\ R \geq \sqrt{|z|^2 - a^2}\,\}$ is contained in the event $\{E_z \cap B(a) \neq \varnothing\}$. Indeed, if $|V| < \tilde{V}_{a,z}$ then the direction of the major axis of E_z must intersect $B(a)$. Requiring also that $R \geq \sqrt{|z|^2 - a^2}$ ensures the intersection. Thus,

$$\begin{aligned} P[E_z \cap B(a) \neq \varnothing] &\geq P\Big[\ |V| < \tilde{V}_{a,z},\ R \geq \sqrt{|z|^2 - a^2}\ \Big] \geq \frac{2}{\pi} \tilde{V}_{a,z} \cdot c_0^{-1} \big(|z|^2 - a^2\big)^{-\frac{\alpha}{2}} \\ &\geq \frac{2}{\pi} \tilde{V}_{a,z} c_0^{-1} |z|^{-\alpha} = c_2(c_0, \alpha)^{-1} \arcsin\left(\frac{a}{|z|}\right) |z|^{-\alpha} \end{aligned}$$

for a constant c_2. The lemma is proved, since $\arcsin(a/|z|) \geq a/|z|$. □

From Lemma 2.3.5 we can deduce the asymptotic behavior of $P[B(w,a) \cap \mathcal{E} \neq \varnothing]$. Obviously, Proposition 2.3.3 shows that when $\alpha \leq 1$ this probability must be one, independently of a; so we can restrict ourselves to case where $\alpha > 1$.

Proposition 2.3.6. *Let ρ have decay $\alpha > 1$ and fix some $a \geq 1$. It holds:*

(i) *Let $w \in \mathbb{R}^2$ and $C \geq 2$ and define $g = g_{a,w,C}$ by $g(z) := P[E_z \cap B(w,a) \neq \varnothing]\mathbb{1}_{B(w,Ca)^c}(z)$, which means that the thinning will keep only ellipses centered outside $B(w, Ca)$ that intersect $B(w,a)$. Then there is a positive constant $c_3 = c_3(c_0, \alpha, C)$ such that*

$$\exp[-uc_3^{-1} a^{2-\alpha}] \leq P[\omega_g(\mathbb{R}^2) = 0] \leq \exp[-uc_3 a^{2-\alpha}].$$

(ii) *There is a positive constant $c_4 = c_4(c_0, \alpha)$ such that*

$$1 - \exp[-uc_4^{-1} a^2] \leq P[B(w,a) \cap \mathcal{E} \neq \varnothing] \leq 1 - \exp[-uc_4 a^2].$$

Remark 2.3.7. Proposition 2.3.6.*(i)* is indeed quite useful. It allows us to disregard the influence of ellipses too far away from the region we are interested in. For instance, when $\alpha > 2$ and a is sufficiently large, the probability of $\{\omega_g(\mathbb{R}^2) = 0\}$ is close to 1. This means that we can pay a small price for assuming that ellipses far away (centered on $B(Ca)^c$) do not interfere in what happens on the ball $B(a)$. The case in which $\alpha = 2$ is of special interest, as we will see.

Proof. Once again, we can assume without loss of generality that $w = 0$. Part *(i)* is a straightforward application of Proposition 2.3.1 and Lemma 2.3.5. The conditions $a \geq 1$ and $C \geq 2$ are just simple requirements to force $Ca \geq a + 1$, so that $g(z) \asymp |z|^{-(\alpha+1)}$ for $|z| > Ca$.

For part *(ii)*, notice that if we take g as in part *(i)* with $C = 2$ then

$$P[\omega(B(2a)) = 0, \omega_g(B(2a)^c) = 0] \leq P[B(a) \subset \mathcal{V}] \leq P[\omega(B(a)) = 0].$$

Since the random variables $\omega(B(2a))$ and $\omega_g(B(2a)^c)$ are independent and $P[\omega(B(2a)) = 0] = \exp[-u\pi 4a^2]$, we conclude from part *(i)* that

$$\exp[-u4\pi a^2 - uc_3^{-1}a^{2-\alpha}] \leq P[B(a) \subset \mathcal{V}] \leq \exp[-u\pi a^2]. \tag{2.16}$$

The result follows after we notice that $a^2 \geq a^{2-\alpha}$ for $a \geq 1$ and define $c_4(c_0, \alpha)$ appropriately. □

2.4 Phase Transition for Total Covering

The proof of Theorems 1.1.1 and 1.1.2 is split into two sections. In this section, we answer the question: are there values of α and u for which the vacant set $\mathcal{V}$ is empty almost surely? In other words, such that the plane is completely covered by ellipses, with probability one? Observe that as the value of α decreases, ellipses with very large axes will become more frequent. As a consequence, the region covered by the ellipses tends to be greater.

Proposition 2.3.3 is sufficient to provide an answer of how total covering depends on α.

Proposition 2.4.1. *We have* $P[\mathcal{E} = \mathbb{R}^2] = 1$ *if* $\alpha \leq 1$ *and* $P[\mathcal{E} = \mathbb{R}^2] = 0$ *otherwise.*

Proof of Proposition 2.4.1. Suppose $\alpha \leq 1$. It follows from Proposition 2.3.3 that $P(B(w, 1/2) \subset \mathcal{E},\ \forall w \in \mathbb{Q}^2) = 1$. Since this event is the same as $\{\mathcal{E} = \mathbb{R}^2\}$ we have proven one of the implications. To see the other, it suffices to see that when $\alpha > 1$ Proposition 2.3.3 says that $P(0 \in \mathcal{E}) < 1$. The stronger statement that $P[\mathcal{E} = \mathbb{R}^2] = 0$ follows from ergodicity with respect to translations, see Lemma 2.6.3. □

2.4.1 Infinite Area Argument

Recall that λ is the Lebesgue measure on $\mathbb{R}^2$. Noticing that $\lambda(E_0) = \pi R$ has infinite expected value if and only if $\alpha \leq 1$, Proposition 2.4.1 can be restated in the suggestive form:

Corollary 2.4.2. *We have* $P[\mathcal{E} = \mathbb{R}^2] = 1$ *if and only if* $E[\lambda(E_0)] = \infty$.

Corollary 2.4.2 says the probability of total covering is related to the expected value of the area of the random defects. This fact is not a coincidence. Consider a model of random defects made by taking ω a $\mathrm{PPP}(u\lambda)$ on $\mathbb{R}^2$ and associating to every $z \in \operatorname{supp} \omega$ a random closed subset $E_z \subset \mathbb{R}^2$ independently of everything else (see [12]). Let $\mathcal{E}$ be the union of these random defects.

In [12], it is proven that for any bounded measurable set $A \subset \mathbb{R}^2$ we have $P[\lambda(A \backslash \mathcal{E}) = 0] = 1$ if and only if $E[\lambda(E_0)] = \infty$, which implies that $\lambda(\mathcal{V}) = 0$ almost surely. In general, this does not mean that $\mathcal{V} = \varnothing$ a.s.. However, we can use this fact to prove total covering for any ellipses model.

Proof of Corollary 2.4.2 for ellipses model. Fix any ellipses model and choose $\varepsilon < 1/2$. Denote by $l(E_0)$ the perimeter of ellipse E_0 and notice that if an ellipse has axes of size a and b then its perimeter p satisfies $p \leq \pi\sqrt{2(a^2+b^2)}$ (see eg. [17]). In our case,

$$l(E_0) \leq \pi\sqrt{2(1+R^2)} \leq 2\pi R \leq 2\lambda(E_0). \tag{2.17}$$

For a set $K \subset \mathbb{R}^2$, denote by $K^{-\varepsilon}$ the ε-interior of the set K, that is $K^{-\varepsilon} = \{x; B(x,\varepsilon) \subset K\}$. Consider the model where the random subsets are given by the family $(E_z^{-\varepsilon})_{z\in\text{supp }\omega}$ and denote its vacant set by $\tilde{\mathcal{V}}$. Notice that the models we are considering are supported on bounded convex subsets. If K is a bounded convex subset, as a particular case of Steiner-Minkowski formula (see eg. [3]) we have $\lambda(K + B(r)) = \lambda(K) + \lambda(B(r)) + r\,l(K)$. Applying to $K = E_0^{-\varepsilon}$ and $r = \varepsilon$, we obtain:

$$\begin{aligned}\lambda(E_0) &= \lambda(E_0^{-\varepsilon} + B(\varepsilon)) = \lambda(E_0^{-\varepsilon}) + \lambda(B(\varepsilon)) + \varepsilon\, l(E_0^{-\varepsilon}) \\ &\leq \lambda(E_0^{-\varepsilon}) + \lambda(B(\varepsilon)) + \varepsilon\, l(E_0) \leq \lambda(E_0^{-\varepsilon}) + \lambda(B(\varepsilon)) + \varepsilon 2\,\lambda(E_0)\end{aligned} \tag{2.18}$$

where the first inequality comes from the fact that if $K_1 \subset K_2$ are two bounded convex subsets of $\mathbb{R}^2$ then $l(K_1) \leq l(K_2)$ and the second comes from (2.17). Then, it follows $(1-2\varepsilon)\lambda(E_0) - \lambda(B(\varepsilon)) \leq \lambda(E_0^{-\varepsilon}) \leq \lambda(E_0)$ and we conclude $E[\lambda(E_0)] = \infty$ if and only if $E[\lambda(E_0^{-\varepsilon})] = \infty$. By this, we have $\lambda(\tilde{\mathcal{V}}) = 0$ a.s. and thus $\mathcal{V} = \varnothing$. $\square$

Remark 2.4.3. The proof above can be immediately generalized for any model supported on bounded convex sets of $\mathbb{R}^2$ satisfying for some universal constant $C > 0$ the relation $l(E_0) \leq C\lambda(E_0)$. Although there are many papers on the total covering of sets, especially in relation to Dvoretsky's covering problem, we did not find any result that would apply to ellipses model directly (see eg. Kahane [15]).

2.4.2 Quantitative Estimates

Section 2.4.1 has a proof of the phase transition for total covering in the ellipses model that does not need any of the estimates of Section 2.3. This fact might put into question whether those estimates are useful at all. Actually, such estimates will have greater importance in the subsequent sections.

In this section we emphasize that the bounds from Section 2.3 provide a more precise description of the ellipses model. To exemplify that, we will prove a stronger result about the covering of a small ball by ellipses, generalizing Proposition 2.3.3.

For $\varepsilon > 0$, consider the random variables

$$N_n^{(\varepsilon)} := \#\{s \in \text{supp }\xi;\ B(\varepsilon) \subset E(s),\ z \in B(n)\},$$

where we recall our notation $s = (z, R, V)$ for points in S. In words, $N_n^{(\varepsilon)}$ counts the number of ellipses centered on the euclidean ball $B(n)$ that cover the ball $B(\varepsilon)$. For a fixed $\varepsilon < 1/2$, we prove a law of large numbers for $N_n^{(\varepsilon)}$. For positive functions $f(n)$ and $g(n)$ we denote $f(n) \asymp g(n)$ if there is a positive constant $C > 0$ such that $C^{-1}g(n) \leq f(n) \leq Cg(n)$ for every large n.

Proposition 2.4.4. *Let $\varepsilon \leq 1/2$. We have that:*

1) *For $0 < \alpha < 1$, it holds $E[N_n^{(\varepsilon)}] \asymp n^{1-\alpha}$ and $\frac{N_n^{(\varepsilon)} - E[N_n^{(\varepsilon)}]}{\sqrt{n^{1-\alpha}(\log n)^{1+\delta}}} \to 0$ as. when $n \to \infty$, for fixed $\delta > 0$.*

2) *For $\alpha = 1$, it holds $E[N_n^{(\varepsilon)}] \asymp \log n$ and $\frac{N_n^{(\varepsilon)} - E[N_n^{(\varepsilon)}]}{n^{1/2}(\log n)^{1+\delta}} \to 0$ as. when $n \to \infty$, for fixed $\delta > 0$.*

Proof. We omit the details. Notice that the random variables $N_n^{(\varepsilon)}$ have distribution

$$N_n^{(\varepsilon)} \stackrel{d}{=} \mathrm{Poi}\left(u\int_{B(n)} P[B(\varepsilon)\subset E_z]\,\mathrm{d}z\right).$$

Then, the asymptotic estimates for $E[N_n^{(\varepsilon)}]$ follow from Lemma 2.3.2. Define the random variables $X_n^{(\varepsilon)} := N_n^{(\varepsilon)} - N_{n-1}^{(\varepsilon)}$, which are independent Poisson random variables with

$$X_n^{(\varepsilon)} \stackrel{d}{=} \mathrm{Poi}\left(u\int_{B(n)\setminus B(n-1)} P[B(\varepsilon)\subset E_z]\,\mathrm{d}z\right).$$

In order to prove a strong law of large numbers, we resort to a theorem of Kolmogorov [34, Theorem 2 on p. 389]. Applied to $(X_n^{(\varepsilon)})$, it states that for any sequence of numbers $(b_n)\subset\mathbb{R}^+$ with $b_n\uparrow\infty$ and $\sum\frac{\mathrm{Var}X_n^{(\varepsilon)}}{b_n^2}<\infty$ we have

$$\frac{N_n^{(\varepsilon)}-E[N_n^{(\varepsilon)}]}{b_n} = \frac{\sum_{j=1}^n X_j^{(\varepsilon)} - \sum_{j=1}^n E[X_j^{(\varepsilon)}]}{b_n} \to 0 \text{ as. when } n\to\infty.$$

To finish the proof, we notice that for Poisson random variables the expectation and the variance coincide. The sequences b_n were chosen to use the fact that $\sum_n \frac{1}{n(\log n)^q}<\infty$ if and only if $q>1$. □

2.5 Phase Transition for Existence of Critical Point

Let us define two critical values for u in the (u,ρ)-ellipses model:

Definition 2.5.1. Define the critical values

$$\bar{u}_c(\rho) := \inf\{u\geq 0;\ P_{u,\rho}[\mathcal{V}\text{ percolates}]=0\} \text{ and } u_c(\rho) := \inf\{u\geq 0;\ P_{u,\rho}[\mathcal{E}\text{ percolates}]=1\}.$$

Remark 2.5.2. We make some comments about how $u_c(\rho)$ and $\bar{u}_c(\rho)$ are related:

1. Recall that we assumed ρ is supported on $[1,\infty)$ and then our model trivially dominates Boolean model with balls of radius 1 for any ρ. This fact ensures percolation for the covered set $\mathcal{E}$ when u is sufficiently large; there exists a finite constant C such that $u_c(\rho)\leq C$ and $\bar{u}_c(\rho)\leq C$ for all ρ we are considering. Moreover, notice that $u_c(\rho)$ may assume different values even for ρ's with the same decay α; the same goes for $\bar{u}_c(\rho)$.

2. One could try to adapt the classical proof of uniqueness of the infinite cluster in the supercritical phase to the covered set, together with Zhang's argument to conclude that in any ellipses model infinite vacant and covered clusters cannot coexist. We believe this holds, but did not carry out the computations. If true, this would imply $\bar{u}_c(\rho)\leq u_c(\rho)$. As a side note on uniqueness of the infinite cluster, notice that in Lemma 2.5.7 we prove that for $\alpha\leq 2$ and any $u>0$ there are infinitely many circuits of ellipses surrounding the origin; thus, if there is an infinite covered cluster then it must be unique.

3. Notice that the critical values do not need to be equal, since we prove with Theorem1.1.3 that $\bar{u}_c(\rho)=0<u_c(\rho)$ when $\alpha=2$. However, as we stated in Conjecture 1.1.4, when $\alpha>2$ we believe equality actually holds.

In the previous section we already proved that $\mathcal{V}\neq\varnothing$ for $\alpha>1$. Now we deal with the second phase transition. In this section we finish the proof of Theorems 1.1.1 and 1.1.2.

2.5.1 Crossing a box with one ellipse

Let us estimate the probability of the event that a single ellipse manages to connect opposite sides of a fixed box. Recall our notation for boxes $B_\infty(l;k)$ and its sides $L^-(l;k)$ and $L^+(l;k)$.

Definition 2.5.3. Define the events

$$LR(l;k) := \left\{ \exists\gamma : [0,1] \to \mathbb{R}^2; \begin{array}{l} \gamma \text{ is continuous, } \gamma([0,1]) \subset \mathcal{E} \cap B_\infty(l;k), \\ \gamma(0) \in L^-(l;k) \text{ and } \gamma(1) \in L^+(l;k) \end{array} \right\}$$

$$LR_1(l;k) := \{\exists s \in \text{supp } \xi; E(s) \cap L^-(l;k) \neq \varnothing \text{ and } E(s) \cap L^+(l;k) \neq \varnothing\}$$

In words, $LR(l;k)$ denotes the event in which there is a left-right crossing of $B_\infty(l;k)$ contained on $\mathcal{E}$ and $LR_1(l;k)$ is the event in which such a crossing is obtained by one ellipse alone. The subscript 1 in the above notation is to emphasize this. Obviously, $LR_1(l;k) \subset LR(l;k)$.

Let us prove bounds for $P[LR_1(l;k)]$. Firstly, we handle the easiest case. In Section 2.4 we proved that when $\alpha \leq 1$ any fixed point $x \in \mathbb{R}^2$ is covered by infinitely many ellipses. Therefore $P[LR_1(l;k)] = 1$ for these values of α and we omit the proof of this result. We only have to be concerned with the case where $\alpha > 1$. In this case, it holds

Proposition 2.5.4. *If $\alpha > 1$ and $k, l > 0$ satisfy $lk > 2$, then there is a constant $c_5 = c_5(\alpha, c_0) > 0$ such that:*

$$1 - \exp[-c_5^{-1} u(k \wedge k^{-\alpha}) l^{2-\alpha}] \leq P(LR_1(l;k)) \leq 1 - \exp[-c_5 u(k^{2-\alpha} \vee k^{-\alpha}) l^{2-\alpha}]. \tag{2.19}$$

Remark 2.5.5. The restriction $lk > 2$ avoids that the horizontal length of $B_\infty(l;k)$ were too small. Since we are mainly interested in cases in which $kl \to \infty$, this restriction is harmless.

Remark 2.5.6. When $\alpha = 2$ and k is fixed then $P[LR_1(l;k)]$ is bounded away from 0 and 1 uniformly on l. This scale invariance plays an important role in Section 2.7.

Proof. We begin proving the lower bound.

Lower Bound: Instead of searching all $\mathbb{R}^2$ for some ellipse that makes the crossing, we can restrict our search to a simpler region. We may force the center of the ellipse to be in the interior of $B_\infty(l;k)$. Using the notation above and recalling $s = (z, R, V)$, define the event

$$LR_1^-(l;k) := \{\exists s \in \text{supp } \xi; E(s) \cap L^-(l;k) \neq \varnothing, E(s) \cap L^+(l;k) \neq \varnothing \text{ and } z \in B_\infty(l/2;k)\}.$$

Independently of where z is, if we know that $z \in B_\infty(l/2;k)$ then

$$\{\, |V| \leq \arctan(\tfrac{1}{3k}), R \geq \tfrac{l}{4}\sqrt{1+9k^2} \,\} \text{ implies } \left\{E(s) \cap L^-(l;k) \neq \varnothing, E(s) \cap L^+(l;k) \neq \varnothing\right\}, \tag{2.20}$$

as is represented in Figure 2.2. Denote $V_{\max} := \arctan\left(\frac{1}{3k}\right)$ and $R_{\min} := \frac{l}{4}\sqrt{1+9k^2}$. For z in $B_\infty(l/2;k)$ and an ellipse E_z centered in z with major axis distributed like $\rho \otimes \nu$ we have

$$P\left[R \geq R_{\min}, |V| \leq V_{\max}\right] = P(|V| \leq V_{\max})\, P(R \geq R_{\min}) \geq 2V_{\max} \cdot c_0^{-1} R_{\min}^{-\alpha}. \tag{2.21}$$

In the last computation we needed to verify that $R_{\min} \geq 1$, otherwise $P(R \geq R_{\min}) = 1$. Hypothesis $kl > 2$ ensures $R_{\min} \geq 1$. Now, turn to Proposition 2.3.1 with the function

$$g(z) = P[E_z \cap L^-(l;k) \neq \varnothing,\ E_z \cap L^+(l;k) \neq \varnothing]\, \mathbb{1}_{B_\infty(l/2;k)}(z),$$

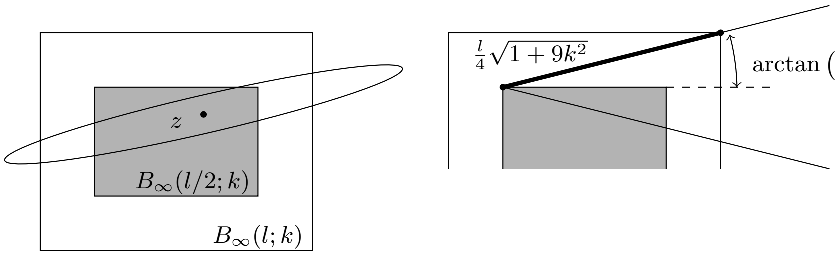


Figure 2.2: Condition in (2.20) implies the event $LR_1^-(l;k)$ happens.

which by (2.21) satisfies $g(z) \geq 2V_{\max} \cdot c_0^{-1} R_{\min}^{-\alpha} \, \mathbb{1}_{B_\infty(l/2;k)}(z)$. We have

$$\begin{aligned} P(LR_1(l;k)) &\geq P[\omega_g(\mathbb{R}^2) \geq 1] = 1 - \exp\left[-u \int_{\mathbb{R}^2} g(z) \, \mathrm{d}z\right] \geq 1 - \exp\left[-u \int_{B_\infty(l/2;k)} 2V_{\max} \cdot c_0^{-1} R_{\min}^{-\alpha} \, \mathrm{d}z\right] \\ &\geq 1 - \exp\left[-u \, c_5^{-1} \cdot \arctan\left(\tfrac{1}{3k}\right) k(k+1)^{-\alpha} \cdot l^{2-\alpha}\right] \end{aligned} \tag{2.22}$$

for some constant $c_5 = c_5(c_0, \alpha)$. To simplify the function $f(k) = \arctan\left(\frac{1}{3k}\right) k(k+1)^{-\alpha}$ on equation (2.22) we notice that $f(k) \sim \frac{\pi}{2} k$ when $k \to 0$, $f(k) \sim \frac{1}{3} k^{-\alpha}$ when $k \to \infty$ and f is a continuous, positive function on $(0, \infty)$. By this, changing the constant c_5 if needed we can assure that:

$$P(LR_1(l;k)) \geq P(LR_1^-(l;k)) \geq 1 - \exp[-u c_5^{-1} (k \wedge k^{-\alpha}) l^{2-\alpha}]. \tag{2.23}$$

Upper Bound: To prove the upper bound, we decompose $LR_1(l;k)$ into two independent events. We denote $a = (k \vee 1)l$. With this, notice that $B_\infty(l;k) \subset B(a) \subset B(2a)$. Define

$$\begin{aligned} LR_1^1(l;k) &:= \left\{\exists s \in \operatorname{supp} \xi; E(s) \cap L^-(l;k) \neq \varnothing, E(s) \cap L^+(l;k) \neq \varnothing \text{ and } z \in B(2a)\right\}, \\ LR_1^2(l;k) &:= \left\{\exists s \in \operatorname{supp} \xi; E(s) \cap L^-(l;k) \neq \varnothing, E(s) \cap L^+(l;k) \neq \varnothing \text{ and } z \notin B(2a)\right\}. \end{aligned}$$

Omitting the dependence on l and k, it holds that

$$P[LR_1] = P[LR_1^1 \cup LR_1^2] = 1 - P[(LR_1^1)^c \cap (LR_1^2)^c] \overset{\text{ind.}}{=} 1 - P[(LR_1^1)^c] P[(LR_1^2)^c]. \tag{2.24}$$

Bound for LR_1^2: Notice that on LR_1^2 the ellipse that makes the crossing must intersect $B(a)$ too. Apply Proposition 2.3.6.*(i)* with $C = 2$ and $g(z) = P[E_z \cap B(a) \neq \varnothing] \mathbb{1}_{B(2a)^c}(z)$ to deduce

$$P[LR_1^2(l;k)^c] \geq P[\omega_g(\mathbb{R}^2) = 0] \geq \exp[-u c_3^{-1} a^{2-\alpha}] \tag{2.25}$$

and then define a constant $\hat{c_5} = \hat{c_5}(c_0, \alpha)$ with the same value as c_3^{-1}.

Bound for LR_1^1: For an ellipse $E(s)$ with $s \in \operatorname{supp} \xi$ to be able to connect both sides of $B_\infty(l;k)$, it is necessary that $R \geq lk/2$. In this way, for any center of ellipse z in $B(2a)$ we have

$$P[E_z \cap L^-(l;k) \neq \varnothing, \; E_z \cap L^+(l;k) \neq \varnothing] \leq P[R \geq lk/2] \leq c_0 (lk/2)^{-\alpha}. \tag{2.26}$$

Apply Proposition 2.3.1 to the function $g(z) = P[E_z \cap L^-(l;k) \neq \varnothing, \; E_z \cap L^+(l;k) \neq \varnothing] \, \mathbb{1}_{B(2a)}(z)$. We have $g(z) \leq c_0 2^\alpha (lk)^{-\alpha} \, \mathbb{1}_{B(2a)}(z)$ by equation (2.26) and hence

$$P[LR_1^1(l;k)^c] = P[\omega_g(\mathbb{R}^2) = 0] \geq \exp\left[-u \int_{B(2a)} c_0 2^\alpha (lk)^{-\alpha} \, \mathrm{d}z\right] = \exp\left[-u \tilde{c_5} k^{-\alpha} (k \vee 1)^2 l^{2-\alpha}\right], \tag{2.27}$$

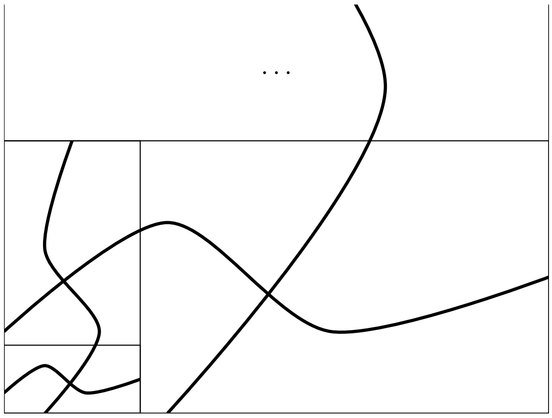


Figure 2.3: Contruction for the proof of Theorem 1.1.2.2

where we defined a constant $\tilde{c_5} = \tilde{c_5}(c_0, \alpha)$. Using the estimates from (2.25) and (2.27) on (2.24),

$$\begin{aligned} P[LR_1(l;k)] &\leq 1 - \exp\left[-u\tilde{c_5}k^{-\alpha}(k \vee 1)^2 l^{2-\alpha}\right] \exp[-u\hat{c_5}(k \vee 1)^{2-\alpha} l^{2-\alpha}] \\ &\leq 1 - \exp[-uc_5 k^{-\alpha}(k \vee 1)^2 l^{2-\alpha}] = 1 - \exp[-uc_5(k^{2-\alpha} \vee k^{-\alpha})l^{2-\alpha}]. \end{aligned}$$

Here, we used that $k^{-\alpha}(k \vee 1)^2 \geq (k \vee 1)^{2-\alpha}$ and took $c_5(c_0, \alpha) = \hat{c_5} + \tilde{c_5}$. □

2.5.2 Triviality of Critical Points when $\alpha \in (1, 2)$

We use Proposition 2.5.4 and the Borel-Cantelli Lemma. For the covered set our proof is straightforward.

Proof of Theorem 1.1.2.2. Consider the boxes $B_n = [0, 2^{n+1}] \times [0, 2^n]$ for n odd and $B_n = [0, 2^n] \times [0, 2^{n+1}]$ for n even. If for all sufficiently large n we have horizontal crossings of B_n for n odd and vertical crossings of B_n for n even, then it is clear that $\mathcal{E}$ percolates (see Figure 2.3). Proposition 2.5.4 proves that

$$P[\{\text{crossing of } B_n \text{ by one ellipse}\}^c] \leq \exp[-uc(2^{2-\alpha})^n],$$

which is summable, so Borel-Cantelli implies $P[\liminf_n \{\text{crossing of } B_n \text{ by one ellipse}\}] = 1$. □

Now, we prove that almost surely $\mathcal{V}$ does not percolate for any $u > 0$ when $\alpha \in (1, 2)$. One way to prove percolation does not happen is to use an argument of duality. We would like to prove that, almost surely, there is an infinite collection of circuits of covered areas around the origin.

It can be tricky to analyze general circuits of ellipses. In our proof, we replicate the idea in [36, Proposition 5.6]. We show that it is enough to look at a collection of circuits in which every circuit is made of three carefully positioned ellipses. We use the same notation of paper [36], Proposition 5.6, with minor modifications. For convenience, we replicate it here.

Let $S_1^{\pm}(a) = \{\pm\frac{\sqrt{3}}{2}a\} \times \left[-\frac{a}{2}, -\frac{a}{4}\right]$. Both $S_1^+(a)$ and $S_1^-(a)$ are segments on $\mathbb{R}^2$. Denote by $\mathcal{R}_{2\pi/3}$ the counter-clockwise rotation of angle $2\pi/3$ around the origin on $\mathbb{R}^2$ and define $S_j^{\pm}(a) = \mathcal{R}_{2\pi/3}^j S_1^{\pm}(a)$, for $j = 2, 3$. If for each j we have an ellipse connecting $S_j^+(a)$ and $S_j^-(a)$, then we have formed a circuit of ellipses around the origin (see Figure 2.4).

Our objective is to check for which values of α and u we can guarantee that these circuits will appear infinitely often. Fortunately, the proof holds even in the case $\alpha = 2$ and thus Lemma 2.5.7 below will be used also in the proof of Theorem 1.1.3.

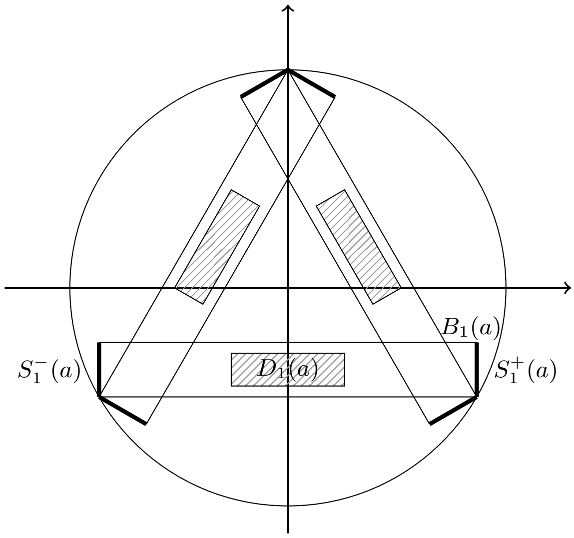


Figure 2.4: Boxes $B_j(a)$ and $D_j(a)$ for $j \neq 1$ are obtained by rotations.

Lemma 2.5.7. *Fix $\alpha \leq 2$. Then, for any ρ with decay α we have $\bar{u}_c(\rho) = 0$.*

Proof. We use Proposition 2.5.4 for a fixed proportion of the box we would like to cross. Consider the box $B_1(a) := \left[-\frac{\sqrt{3}}{2}a, \frac{\sqrt{3}}{2}a\right] \times \left[-\frac{a}{2}, -\frac{a}{4}\right]$, which is a translation of the box $B_\infty(\frac{a}{4}; 4\sqrt{3})$. Also, define $B_j(a)$ for $j = 2, 3$ by rotating the already defined box $B_1(a)$. Notice that the events

$$C_j(a) := \left\{\exists s \in \text{supp } \xi;\ E(s) \cap S_j^+(a) \neq \varnothing \text{ and } E(s) \cap S_j^-(a) \neq \varnothing\right\}$$

are not independent for different j. To get independence, we restrict ourselves to ellipses centered on smaller boxes contained on $B_j(a)$, exactly like we did on the proof of the lower bound of Proposition 2.5.4. Recall that in the proof of the lower bound we considered the event $LR_1^-(l;k)$ in which our ellipses had to be centered on $B_\infty(l/2;k)$. However, the choice of $1/2$ was arbitrary and if we consider only ellipses centered on $B_\infty(cl;k)$ for some fixed $c \in (0,1)$, we obtain the same lower bound with a different constant c_5.

Choose constants $c \in (0,1)$, h_1 and h_2 so that the box $D_1(a) := [-ca, ca] \times [h_1 a, h_2 a]$ satisfies $D_1(a) \subset B_1(a)$ and $D_1(a) \cap B_i(a) = \varnothing$, for all $i \neq 1$.

As we did before, we can use rotations to define the analogous regions $D_j(a)$ for $j = 2, 3$. Using the regions D_j, we can define events that are similar to $C_j(a)$ and are independent indeed. Define

$$\tilde{C}_j(a) := \left\{\exists s \in \text{supp } \xi;\ E(s) \cap S_j^+(a) \neq \varnothing, E(s) \cap S_j^-(a) \neq \varnothing \text{ and } z \in D_j(a)\ \right\}.$$

By the proof of the lower bound in Proposition 2.5.4, we get

$$P[\tilde{C}_j(a)] \geq 1 - \exp[-c_5^{-1} u a^{2-\alpha}] \text{ for any } j.$$

Define $\Delta_a := \cap_{j=1}^3 \tilde{C}_j(a)$. Using this lower bound, we have

$$P[\Delta_a] = P[\tilde{C}_1(a) \cap \tilde{C}_2(a) \cap \tilde{C}_3(a)] \geq (1 - \exp[-c_5^{-1} u a^{2-\alpha}])^3.$$

Finally, notice that taking the sequence $a_n = 3^n$ makes the events Δ_{a_n} independent, since they depend on what the PPP ξ looks like on disjoint regions of $\mathbb{R}^2$. Since

$$\sum_{n\geq 1} P[\Delta_{a_n}] \geq \sum_{n \geq 1} (1 - \exp[-c_5^{-1} u 3^{n(2-\alpha)}])^3 = \infty \text{ for } \alpha \leq 2,$$

we conclude by Borel-Cantelli that $P[\Delta_{a_n},\ \text{i.o.}] = 1$ and then $P[\mathcal{V} \text{ percolates}] = 0$. □

2.5.3 Proving Phase Transition in u for $\alpha > 2$

The last ingredient to finish the proof of Theorems 1.1.1 and 1.1.2 is to prove the behavior of ellipses model when $\alpha > 2$. As we mentioned in the introduction, this can be done by dominating (u,ρ)-ellipses model by Boolean model of radius distribution ρ and intensity $u\lambda$, because of the results of Gouéré [10].

Reference [10] proves for Boolean model with balls of radius distribution ρ that the connected component of $\mathcal{E}$ that contains the origin, which we denote by $\mathcal{C}$, satisfies

$$\mathcal{C} \text{ is a.s. finite if and only if } E_\rho[R^d] < \infty.$$

Their result is based on a multiscale renormalization construction; it essentially proves that $P[\text{diam } \mathcal{C} \geq l]$ is smaller than $P[\text{diam } \mathcal{C} \geq l/10]^2$ up to some error term related to the existence of large balls, see equation (2.29) below. The techniques in [10] are enough to prove the existence of a phase transition in u for this values of α for both $\mathcal{E}$ and $\mathcal{V}$. However, studying the vacant set was not a priority in [10]. For convenience of the reader we provide a full argument for this case, through Lemmas 2.5.8 and 2.5.9.

Denote by $P^\circ_{u,\rho}$ the probability measure associated to the Boolean model above defined. Notice that since ρ has tail decay α and support on $[1,\infty)$, we have

$$E[R^t] = \int_1^\infty R^t\,\rho(\mathrm{d}R) = \int_1^\infty \int_0^R t y^{t-1}\,\mathrm{d}y\,\rho(\mathrm{d}R) = \int_0^\infty t y^{t-1}\,\rho(R \geq y)\,\mathrm{d}y \tag{2.28}$$

which implies $E[R^t] < \infty$ for $t \in (0,\alpha)$. Since $\alpha > 2$, we have that $E[R^2] < \infty$ and thus by Theorem 2.1 of [10] there is a positive constant c such that

$$P_{u,\rho}[\mathcal{E} \text{ percolates}] \leq P^\circ_{u,\rho}[\mathcal{E} \text{ percolates}] = 0,\ \forall u \in (0,\, cE[R^2]^{-1}).$$

Moreover, by Theorem 2.2 of [10] if we define $\mathcal{C}$ as the connected (covered) component of the origin and $D := \text{diam } \mathcal{C}$ then for any fixed $t \in (0, \alpha - 2)$ we have

$$E[R^{2+t}] < \infty \text{ implies } E^\circ_{u,\rho}[D^t] < \infty \text{ for } u < cE[R^2]^{-1}$$

and by Markov's inequality we conclude $P_{u,\rho}[D \geq l] \leq P^\circ_{u,\rho}[D \geq l] \leq E^\circ_{u,\rho}[D^t] \cdot l^{-t}$. This means that the probability of the origin being connected to $\partial B(l)$ decays at least polynomially in l. This provides the correct decay of $P_{u,\rho}[0 \overset{\mathcal{E}}{\longleftrightarrow} \partial B(l)]$, since by Proposition 2.3.6 we have

$$P_{u,\rho}[0 \overset{\mathcal{E}}{\longleftrightarrow} \partial B(l)] \geq P_{u,\rho}[\exists s \in \text{supp } \xi; z \in B(2l)^c, E(s) \cap B(l) \neq \varnothing] \geq 1 - \exp[-uc_3 l^{2-\alpha}] \geq uc_3 l^{2-\alpha}$$

when $l \to \infty$. The only statement we still have not proved in Theorems 1.1.1 and 1.1.2 is that for small u the vacant set percolates. We try to keep the same notation of [10]. Define

$$G(l) := \{\partial B(l) \overset{\mathcal{E}}{\longleftrightarrow} \partial B(8l) \text{ using only balls centered on } B(10l)\}$$

and let $\pi(l) := P^\circ_{u,\rho}[G(l)]$. Proposition 3.1 of [10] proves there is a constant $C > 0$ such that

$$\pi(10l) \leq C\pi(l)^2 + uC\int_l^\infty R^2\,\rho(\mathrm{d}R) \leq C\pi(l)^2 + uc_6 l^{2-\alpha}, \forall l \geq 1, \tag{2.29}$$

in which the last inequality follows from a straightforward computation and $\rho[l,\infty) \le c_0 l^{-\alpha}$ and $c_6 = c_6(c_0,\alpha)$ is a constant. Also, if $\partial B(l)$ is connected to $\partial B(8l)$ then either $G(l)$ happened or there is a ball centered on $B(10l)^c$ intersecting $B(8l)$. This leads to the bound

$$P^{\circ}_{u,\rho}[\partial B(l) \overset{\mathcal{E}}{\longleftrightarrow} \partial B(8l)] \le \pi(l) + 1 - \exp\left[-u \int_{B(10l)^c} \rho\big[|z| - 8l, \infty\big)\, \mathrm{d}z \right] \le \pi(l) + uc_6 l^{2-\alpha} \tag{2.30}$$

by a computation similar to the one in equation (2.29). Define $q_k(u,\rho) = P^{\circ}_{u,\rho}[\partial B(10^k) \overset{\mathcal{E}}{\longleftrightarrow} \partial B(8 \cdot 10^k)]$ for $k \ge 0$. We have:

$$q_{k+1} \le \pi(10^{k+1}) + uc_6(10^{2-\alpha})^{k+1} \le C\pi(10^k)^2 + uc_6(10^{2-\alpha})^k \le c_6 q_k^2 + uc_6(10^{2-\alpha})^k. \tag{2.31}$$

Using the recurrence relation in (2.31) we can prove that for small values of u the sequence q_k tends to zero very fast.

Lemma 2.5.8. *Fix $\alpha > 2$. There exists $u_0 = u_0(\alpha, c_0) > 0$ such that $q_k(u,\rho) \le \exp[-2(\alpha-2)k]$, for all $k \ge 1$ and for all $u < u_0$.*

Proof. Fix $\varepsilon = 2(\alpha - 2)$ and notice that $0 < \varepsilon < (\log 10)(\alpha - 2)$. After that, take $k_0 = k_0(\alpha, c_0)$ sufficiently large so that

$$c_6 \exp[\varepsilon - \varepsilon k_0] < \frac{1}{2} \quad \text{and} \quad c_6 \exp\left[(\varepsilon - (\log 10)(\alpha-2))\, k_0 + \varepsilon\right] < \frac{1}{2}. \tag{2.32}$$

The choices above are possible only because our previous choice of ε and the fact that $\alpha > 2$ together imply the left hand sides on equation (2.32) tend to zero when $k_0 \to \infty$. Now that we fixed k_0, let us choose u_0. Notice that q_k must be increasing in u and besides,

$$\lim_{u \to 0+} q_k(u) = 0, \text{ for any fixed } k.$$

One way to see this is combining (2.30) and Lemma 3.6 of [10]:

$$q_k(u) \le \pi(10^k) + uc_6(10^{2-\alpha})^k \le uC100^k + uc_6(10^{2-\alpha})^k \tag{2.33}$$

Thus, take $u_0 = u_0(\alpha, c_0)$ sufficiently small such that $u_0 \le 1$ and $q_{k_0}(u_0) \le \exp[-\varepsilon k_0]$. Proceeding by induction, we will extend this inequality for all $k \ge k_0$. Suppose $q_k(u_0) \le \exp[-\varepsilon k]$. Using equation (2.31), we have that

$$\begin{aligned} \frac{q_{k+1}(u_0)}{\exp[-\varepsilon(k+1)]} &\le c_6 q_k^2 \exp[\varepsilon(k+1)] + u_0 c_6 10^{k(2-\alpha)} \exp[\varepsilon(k+1)] \\ &\le c_6 \exp[-2\varepsilon k + \varepsilon k + \varepsilon] + u_0 c_6 10^{k(2-\alpha)} \exp[\varepsilon(k+1)] \\ &= c_6 \exp[-\varepsilon k + \varepsilon] + u_0 c_6 \exp[-(\log 10)(\alpha-2)k + \varepsilon(k+1)] \\ &= c_6 \exp[-\varepsilon k + \varepsilon] + u_0 c_6 \exp[(\varepsilon - (\log 10)(\alpha-2))k + \varepsilon]. \end{aligned} \tag{2.34}$$

Since the right-hand side of the last equation is decreasing in k, we can use k_0 in the place of k. But then, by our choice of ε, k_0 and u_0 we can conclude $q_{k+1}(u_0) \le \exp[-\varepsilon(k+1)]$, completing the induction step. To extend the bound to values of k smaller than k_0 we can simply decrease u_0 even more using the crude bound on (2.33). Finally, since $q_k(u)$ is increasing in u the bound is valid for all $u < u_0$. □

Using Lemma 2.5.8 we can show that $P_{u,\rho}(\mathcal{V} \text{ percolates}) = 1$ for $u < u_0(\alpha, c_0)$.

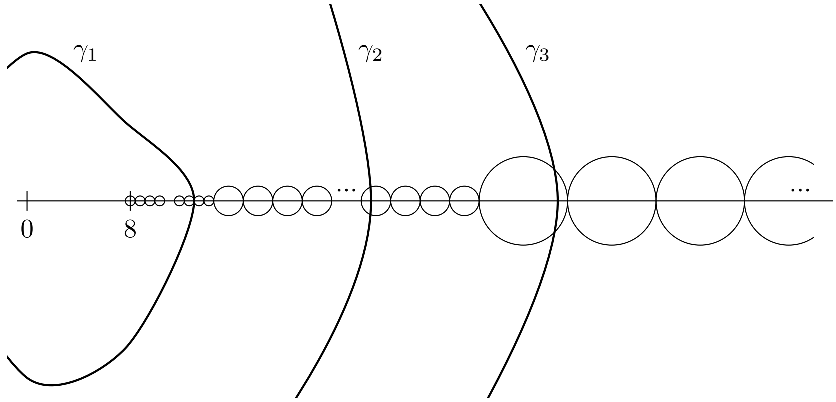


Figure 2.5: If $\mathcal{V}$ does not percolate, circuits γ_n must intersect balls B_j^i with arbitrarily large j.

Lemma 2.5.9. *Fix $\alpha > 2$. Then, for any ρ with tail decay α we have $\bar{u}_c(\rho) > 0$.*

Proof. Take $u_0(\alpha, c_0)$ from Lemma 2.5.8. If under measure $P^\circ_{u_0,\rho}$ the set $\mathcal{V}$ a.s. does not percolate then there must exist a sequence γ_n of disjoint circuits around the origin with $\gamma_n \subset \mathcal{E}$ and such that

$$\operatorname{dist}(0, \gamma_n \cap (\mathbb{R}^+ \times \{0\})) \to \infty.$$

Focusing on this observation, consider the sequence of balls $(B_j^i)_{1 \le i \le 32, j \ge 0}$ where (see Figure 2.5):

- All B_j^i have their centers on the horizontal axis and radius 10^j.
- The ball B_0^1 has its center at point $(8, 0)$.
- The balls B_j^i and B_j^{i+1} are adjacent with B_j^{i+1} on the right $\forall j, \forall 1 \le i \le 31$.
- The balls B_j^{32} and B_{j+1}^1 are adjacent with B_{j+1}^1 on the right $\forall j$.

The choice of this construction of balls aims to ensure that whenever a circuit around the origin passes through B_j^i a translation of the event $\{\partial B(10^j) \overset{\mathcal{E}}{\longleftrightarrow} \partial B(8 \cdot 10^j)\}$ happens. Indeed, if we define $\tilde{B}_j^i := \{z \in \mathbb{R}^2;\ \operatorname{dist}(z, B_j^i) \le 7 \cdot 10^j\}$ then the event

$$A(B_j^i) = \{\partial B_j^i \overset{\mathcal{E}}{\longleftrightarrow} \partial \tilde{B}_j^i\}$$

is just a translation of $\{\partial B(10^j) \overset{\mathcal{E}}{\longleftrightarrow} \partial B(8 \cdot 10^j)\}$ and has also probability q_j. Notice that if there is a closed circuit around the origin γ such that $\gamma \cap B_j^i \neq \varnothing$ then, since $\tilde{B}_j^i \subset \mathbb{R}^+ \times \mathbb{R}$ by construction, we can deduce that $A(B_j^i)$ happened.

It follows from the definition of B_j^i and the observation made above that if $\mathcal{V}$ does not percolate then the circuits γ_n must pass through balls B_j^i with arbitrarily large j. Let B_n be the ordering of balls B_j^i sorted by their distance to the origin. Thus, using Lemma 2.5.8 and Borel-Cantelli's Lemma, since $\sum_n 32 \cdot q_n < \infty$, we have

$$\begin{aligned} P_{u_0,\rho}(\{\mathcal{V} \text{ perc.}\}^c) &\le P^\circ_{u_0,\rho}(\{\mathcal{V} \text{ perc.}\}^c) \le P^\circ_{u_0,\rho}\begin{pmatrix} \exists(\gamma_n) \text{ circuits around origin with } \gamma_n \subset \mathcal{E} \\ \text{and } \operatorname{dist}(0, \gamma_n \cap (\mathbb{R}^+ \times \{0\})) \to \infty \end{pmatrix} \\ &\le P^\circ_{u_0,\rho}[A(B_n),\ \text{i.o.}] = 0. \end{aligned}$$

□

2.6 Decay of Correlations

In this section we derive a lemma that is useful to handle the dependence of some events in ellipses model. It provides bounds that prove that some events are almost independent from one another if the distance between their dependence regions is large. Essentially, Lemma 2.6.2 gives a quantitative description of the rate in which the system mixes, making Lemma 2.6.2 important by its own.

One possible application of Lemma 2.6.2 is to provide an alternative derivation of Lemma 2.5.8, without using reference [10]. Lemma 2.6.2 can also be used to prove that when $\alpha = 2$ and u is small the set $\mathcal{E}$ does not percolate, almost surely.

Let K be a measurable subset of $\mathbb{R}^2$. Recall $\xi = \sum_i \delta_{s_i}$, where s_i are points in S for all i, is the PPP on S that can be identified with the random collection of ellipses. Define

$$\xi_K := \sum_{i; E(s_i)\cap K \neq \varnothing} \delta_{s_i}$$

that is, the PPP obtained from ξ by taking only the points of supp ξ whose ellipses intersect K.

Definition 2.6.1. We say a function f from the point processes on S to $\mathbb{R}$ depends only on the ellipses touching $K \subset \mathbb{R}^2$ if $f(\xi) = f(\xi_K)$.

We are now ready to state the decoupling we want to prove. The proof is similar to an argument of Sznitman [35, Theorem 2.1]:

Lemma 2.6.2. *Take $\alpha > 1$. Let $K_1 = B(l_1)$ and $K_2 = B(l_2)^c$, with $l_2 - l_1 \geq 2$ and $l_1 \geq 1$ and define $a := l_2/l_1$. Let f_1 and f_2 be real functions of ξ such that $|f_j| \leq 1$, f_1 depends only on ellipses touching K_1 and f_2, on K_2. Then, there is a constant $c_7 = c_7(c_0, \alpha) > 0$ such that*

$$\big|E[f_1 f_2] - E[f_1]E[f_2]\big| \leq u c_7 l_1^{2-\alpha}(a-1)^{1-\alpha}. \tag{2.35}$$

Proof. Take two independent copies of ξ and denote them by ξ and ξ'. Fixed one of these copies, we decompose it into four independent PPP's on $\mathbb{R}^4$. Consider the following partition of S:

$$\Gamma_1 = \{s; E(s)\cap K_1 \neq \varnothing \text{ and } E(s)\cap K_2 = \varnothing\},\quad \Gamma_2 = \{s; E(s)\cap K_1 = \varnothing \text{ and } E(s)\cap K_2 \neq \varnothing\},$$
$$\Gamma_{12} = \{s; E(s)\cap K_1 \neq \varnothing \text{ and } E(s)\cap K_2 \neq \varnothing\},\quad \Gamma_0 = \{s; E(s)\cap K_1 = \varnothing \text{ and } E(s)\cap K_2 = \varnothing\}.$$

These restrictions give birth to independent PPP's [28]. We decompose $\xi = \xi_1 + \xi_2 + \xi_{12} + \xi_0$, where $\xi_\square$ denotes the restriction of ξ to the region $\Gamma_\square$. Analogously, we have $\xi' = \xi_1' + \xi_2' + \xi_{12}' + \xi_0'$. Define $\gamma_1 := \xi_1 + \xi_2' + \xi_{12} + \xi_0'$ and $\gamma_2 := \xi_1' + \xi_2 + \xi_{12}' + \xi_0$. Naturally, this construction makes γ_1 and γ_2 independent and with the same distribution of ξ. Besides, we have

$$(\gamma_1)_{K_1} = \xi_1 + \xi_{12} = \xi_{K_1}, \quad \xi_{K_2} = \xi_2 + \xi_{12} \quad \text{and} \quad (\gamma_2)_{K_2} = \xi_2 + \xi_{12}' \tag{2.36}$$

by construction. Using the equations on (2.36) we can relate the left hand side of equation (2.35) with the coupling we have just defined. Notice that

$$\begin{aligned} E[f_1(\xi)f_2(\xi)] &= E[f_1(\xi_{K_1})f_2(\xi_{K_2})] \\ &= E\Big[f_1(\gamma_1)\ [f_2(\gamma_2)\mathbb{1}_{\{\xi_{K_2}=(\gamma_2)_{K_2}\}} + f_2(\xi_{K_2})\mathbb{1}_{\{\xi_{K_2}\neq(\gamma_2)_{K_2}\}}]\Big] \end{aligned} \tag{2.37}$$

and also $$\begin{aligned} E[f_1(\xi)]E[f_2(\xi)] &= E[f_1(\gamma_1)]E[f_2(\gamma_2)] \overset{\text{Ind.}}{=} E[f_1(\gamma_1)f_2(\gamma_2)] \\ &= E\Big[f_1(\gamma_1)\ [f_2(\gamma_2)\mathbb{1}_{\{\xi_{K_2}=(\gamma_2)_{K_2}\}} + f_2(\gamma_2)\mathbb{1}_{\{\xi_{K_2}\neq(\gamma_2)_{K_2}\}}]\Big]. \end{aligned} \tag{2.38}$$

Take the absolute value of the difference between the left hand sides in equations (2.37) and (2.38). Using that $|f_j| \le 1$ and the triangular inequality, we get

$$\big|E[f_1 f_2] - E[f_1]E[f_2]\big| = \Big|E\left[f_1(\gamma_1)(f_2(\xi_{K_2}) - f_2(\gamma_2))\mathbb{1}_{\{\xi_{K_2}\neq(\gamma_2)_{K_2}\}}\right]\Big| \le 2P[\xi_{K_2} \neq (\gamma_2)_{K_2}]$$

Moreover, using (2.36) once again we can write

$$\begin{aligned} P(\xi_{K_2} \neq (\gamma_2)_{K_2}) &= P(\xi_{12} \neq \xi'_{12}) \le P\left(\{\xi(\Gamma_{12}) \neq 0\} \cup \{\xi'(\Gamma_{12}) \neq 0\}\right) \le 2P(\xi(\Gamma_{12}) \ge 1) \\ &= 2\{1 - \exp[-(u\lambda \times \rho \times \nu)(\Gamma_{12})]\} \le 2(u\lambda \times \rho \times \nu)(\Gamma_{12}), \end{aligned}$$

implying that

$$\big|E[f_1 f_2] - E[f_1]E[f_2]\big| \le 4(u\lambda \times \rho \times \nu)(\Gamma_{12}). \tag{2.39}$$

Estimating the measure of Γ_{12}: Defining the notation $\mu := \lambda \otimes \rho \otimes \nu$, we want to estimate $\mu(\Gamma_{12})$. Notice that the distance between K_1 and K_2 is given by $(l_2 - l_1) = l_1(a-1)$. Thus, if $z \in \mathbb{R}^2$ is the center of an ellipse intersecting both sets, we must have $R \ge l_1(a-1)/2$. To estimate the measure of Γ_{12} we decompose $\mathbb{R}^2$ into three different regions, according to the position of z. For $z \in B_1 := B(l_1+1)$, we have the trivial bound

$$\mu(z \in B_1, s \in \Gamma_{12}) \le \mu(z \in B_1, R \ge l_1(a-1)/2) \le c_7 l_1^{2-\alpha}(a-1)^{-\alpha}.$$

If $B_2 := \{z;\ l_1 + 1 < |z| \le \frac{l_1+l_2}{2}\}$ then for $z \in B_2$ we already have some restrictions on the possible values of V. Analogously to Lemma 2.3.5, V must be in an interval $V_{l_1,z}$ of total length $2\arcsin(\frac{l_1+1}{|z|})$ and thus

$$\begin{aligned} \mu(z \in B_2, s \in \Gamma_{12}) &\le c_7 \int_{B_2}\int_{\frac{l_1(a-1)}{2}}^{\infty} \frac{l_1}{|z|}\,\rho(\mathrm{d}R)\,\mathrm{d}z \le c_7 l_1 (l_1(a-1))^{-\alpha}\int_{l_1+1}^{\frac{l_1+l_2}{2}} \frac{1}{r}\, r\,\mathrm{d}r \\ &\le c_7 l_1^{1-\alpha}(a-1)^{-\alpha}\cdot \tfrac{1}{2}(l_2 - l_1) = c_7 l_1^{2-\alpha}(a-1)^{1-\alpha}. \end{aligned}$$

Finally, if $z \in B_3 := \{z; |z| > \frac{l_1+l_2}{2}\}$ then the restriction on V still holds and now we use also $R \ge |z| - l_1$. We have

$$\begin{aligned} \mu(z \in B_3, s \in \Gamma_{12}) &\le c_7 \int_{B_3}\int_{|z|-l_1}^{\infty} \frac{l_1}{|z|}\,\rho(\mathrm{d}R)\,\mathrm{d}z \le c_7 l_1 \int_{B_3}\left(1 - \frac{l_1}{|z|}\right)^{-\alpha} |z|^{-(\alpha+1)}\,\mathrm{d}z \\ &\le c_7 l_1 \left(1 - \frac{2l_1}{l_1+l_2}\right)^{-\alpha}\int_{B_3} |z|^{-(\alpha+1)}\,\mathrm{d}z = c_7 l_1 \left(\frac{a+1}{a-1}\right)^{\alpha}\int_{\frac{l_1+l_2}{2}}^{\infty} r^{-\alpha}\,\mathrm{d}r \\ &\le c_7 l_1 \left(\frac{a+1}{a-1}\right)^{\alpha}\left[\frac{l_1+l_2}{2}\right]^{1-\alpha} = c_7 l_1^{2-\alpha}(a+1)(a-1)^{-\alpha} \le c_7 l_1^{2-\alpha}(a-1)^{1-\alpha}. \end{aligned}$$

Taking the worst of the three bounds gives $u\mu(\Gamma_{12}) \le uc_7 l_1^{2-\alpha}(a-1)^{1-\alpha}$. After substituting this bound on equation (2.39), the proof is over. □

In order to give an explicit application of Lemma 2.6.2, we use it to prove ergodicity of ellipses model with respect to translations in $\mathbb{R}^2$. Ergodicity should not come as a surprise, since Lemma 2.6.2 proves ellipses models is mixing. The proof of Lemma 2.6.3 below is presented for didactic purposes only.

Consider the family of translations $(\tau_x)_{x\in\mathbb{R}^2}$, where $\tau_x : \mathbb{R}^2 \ni v \mapsto v + x$. We already know that $P_{u,\rho}$ is invariant with respect to any τ_x. With Lemma 2.6.2 we can prove more:

Lemma 2.6.3. *Let A be an event such that $\tau_x(A) = A$, $P_{u,\rho}$-a.s., $\forall x \in \mathbb{R}^2$. Then, $P_{u,\rho}[A] \in \{0,1\}$.*

Proof. We omit u and ρ from $P_{u,\rho}$. Fix some function f on the point processes of S that depends only on $B(l_1)$. Using Lemma 2.6.2 with $f_1 = f$ and $f_2 = f \circ \tau_x$ we obtain for $x > 2l_1$ that

$$|E[f \cdot f \circ \tau_x] - E[f]E[f \circ \tau_x]| \leq uc(l_1, \alpha)(x - l_1 - 1)^{1-\alpha} \to 0$$

when $x \to \infty$. The proof follows if we are able to prove that $\mathbb{1}_A$ can be approximated in $L^1(P)$ by functions depending only on a finite ball. In fact, the indicator function of any measurable event can be approximated in such a way. The canonical σ-algebra for a point process in S, denoted by $\mathcal{S}$, is the smallest one that makes the evaluation maps $\xi \mapsto \xi(I)$ measurable for $I \in \mathcal{B}(S)$. Instead of checking measurability for every I it is sufficient to check for $I \in \mathcal{I}$, a family of relatively compact sets with $\sigma(\mathcal{I}) = \mathcal{B}(S)$ that is closed to intersections and contains a partition of $S = \cup I_n$ with $I_n \in \mathcal{I}$ [28, Proposition 3.2].

To obtain a suitable collection, let us discretize S by defining a sequence of nested partitions of S. Define the cell

$$C(n, k, r, v) := \prod_{i=1,2} \left(\frac{k_i - 1}{2^n}, \frac{k_i}{2^n}\right] \times \left(\frac{r-1}{2^n}, \frac{r}{2^n}\right] \times \left(\frac{\pi(v-1)}{2^{n+1}}, \frac{\pi v}{2^{n+1}}\right]$$

for each $n \in \mathbb{N}$, $k = (k_1, k_2) \in \mathbb{Z}^2$, $r \in \mathbb{N}$ and $v \in (-2^n, 2^n] \cap \mathbb{Z}$. Then, for any fixed n we have that

$$\mathcal{P}_n := \{\, C(n, k, r, v);\ k \in \mathbb{Z}^2, r \in \mathbb{N}, v \in (-2^n, 2^n] \cap \mathbb{Z} \,\}$$

forms a partition of S and also that $\mathcal{P}_{n+1}$ is finer than $\mathcal{P}_n$ for every n. Notice that family $\cup \mathcal{P}_n$ is large enough to ensure measurability on $\mathcal{S}$. To obtain the approximation by events on finite balls, we modify the construction above. Define for each n the family

$$\mathcal{Q}_n := \{\, C \in \mathcal{P}_n;\ C \cap \big(B(2^n) \times \mathbb{R}^+ \times (-\pi/2, \pi/2]\big) \neq \varnothing \,\}.$$

and let $\mathcal{F}_n := \sigma\big(\xi(C); C \in \mathcal{Q}_n\big)$. By construction $(\mathcal{F}_n)$ is a increasing sequence of σ-algebras, since any $C \in \mathcal{Q}_n$ is the union of finitely many C_j from $\mathcal{Q}_{n+1}$. It is also clear that $\sigma(\cup \mathcal{F}_n) = \mathcal{S}$. Fix any event $A \in \mathcal{S}$. We use family $(\mathcal{F}_n)$ to approximate $\mathbb{1}_A$ by defining

$$f_n := E\big[\mathbb{1}_A \mid \mathcal{F}_n\big] \text{ for } n \in \mathbb{N}.$$

Thus, $(f_n, \mathcal{F}_n)_{n \in \mathbb{N}}$ is a martingale that converges to $\mathbb{1}_A$ a.s. and in $L^1(P)$, by [8, Theorem 5.5.7]. Notice that f_n depends only on ellipses centered near $B(2^n)$. Take any $\varepsilon > 0$ and choose n such that

$$\big|\, E[f_n \cdot f_n \circ \tau_x] - E[\mathbb{1}_A] \,\big| \leq \varepsilon \quad \text{and} \quad \big|\, E[f_n] - E[\mathbb{1}_A] \,\big| \leq \varepsilon,$$

where we used above that $\mathbb{1}_A \circ \tau_x = \mathbb{1}_A$. Applying the triangular inequality, we can write

$$|P[A] - P[A]^2| \leq \varepsilon + \big|\, E[f_n \cdot f_n \circ \tau_x] - E[f_n]^2 \,\big| + 2\varepsilon \leq 3\varepsilon + uc(n, \alpha)x^{1-\alpha}$$

for x sufficiently large. Making $x \to \infty$, we get $|P[A] - P[A]^2| \leq 3\varepsilon$ for arbitrary $\varepsilon > 0$. □

2.7 Vacant Crossing of Boxes for $\alpha = 2$

Theorems 1.1.1 and 1.1.2 prove a phase transition in α for the percolative behavior of $\mathcal{V}$ and $\mathcal{E}$. During their proofs we discovered with Proposition 2.5.4 that when $\alpha = 2$ the probability of one ellipse crossing $B_\infty(l; k)$ is bounded away from 0 and 1, which came as a surprise.

By this reason, we focus on $\alpha = 2$ and study the vacant crossing of boxes. We fix α as 2 in this whole section. We begin defining the event in which we have a vacant crossing of a box $B_\infty(l; k)$ and restating Theorem 1.1.3.

Definition 2.7.1. Define the event:

$$\overline{LR}(l;k) := \left\{ \exists \gamma : [0,1] \to \mathbb{R}^2; \begin{array}{l} \gamma \text{ is continuous, } \gamma([0,1]) \subset \mathcal{V} \cap B_\infty(l;k), \\ \gamma(0) \in L^-(l;k) \text{ and } \gamma(1) \in L^+(l;k) \end{array} \right\}$$

Remark 2.7.2. Event $\overline{LR}(l;k)$ is measurable in any ellipses model. This follows from the fact that a.s. either $\mathcal{E} = \mathbb{R}^2$ or any finite box is intersected by a finite number of ellipses, implying $\mathcal{V}$ is open. We omit a formal proof, that is essentially a rephrasing of Lemma 2.3 of [36].

Theorem 1.1.3. *Let ρ be a distribution with $\alpha = 2$. Then, there exists $\bar{u} = \bar{u}(c_0) > 0$ such that for any fixed $k > 0$, $u \in (0, \bar{u})$ and $l > 0$*

$$\delta \le P_{u,\rho}[\overline{LR}(l;k)] \le 1 - \delta, \tag{2.40}$$

where $\delta = \delta(c_0, u, k) > 0$. Moreover, for $u \in (0, \bar{u})$ we have:

$$P_{u,\rho}[\textit{neither } \mathcal{V} \textit{ nor } \mathcal{E} \textit{ percolate}] = 1. \tag{2.41}$$

Proof. We begin proving the upper bound on (2.40). Notice that by duality the event $\overline{LR}(l;k)$ is the complementary event of the one in which there is a vertical covered crossing of box $B_\infty(l;k)$ and if we apply a rotation by $\frac{\pi}{2}$ we get a left-right covered crossing of the box $B_\infty(kl; \frac{1}{k})$. Thus, using rotational invariance of the model and Proposition 2.5.4 we can deduce

$$P[\overline{LR}(l;k)] = P[LR(kl; \tfrac{1}{k})^c] \le P[LR_1(kl; \tfrac{1}{k})^c] \le \exp[-c_5^{-1} u (k^{-1} \wedge k^2)] \tag{2.42}$$

for a constant $c_5 = c_5(c_0)$ whenever $l > 2$. Choosing $\delta(c_0, u, k)$ accordingly we get $P[\overline{LR}(l;k)] \le 1 - \delta$ for $l > 0$. However, for the lower bound we will need to deal with all ellipses.

Idea for the lower bound: Let us discuss how to prove the lower bound. A short argument (Lemma 2.7.3 below) implies that we only need to study the case $k = 2$ and $l > 1$. After that, all we have to do is to build an event that implies the vacant crossing of $B_\infty(l;2)$ and whose probability we can bound more easily. For that reason, we start to decompose the PPP ξ into a sum of independent PPPs and analyze their contributions to the event we want to study.

To control ellipses with large major axis we use Proposition 2.3.6.*(i)*. As we stressed before on Remark 2.3.7, it proves that we can pay a small price to prevent interference from ellipses centered too far away from our region of interest. Moreover, since we can now worry only about ellipses centered in a finite ball, we can pay a reasonable price to ensure there are no ellipses with very large major axis inside it.

The second step is to control the ellipses with major axis not too large. This is the most delicate part of the proof; since we are dealing with arbitrarily large scales, it is too much to expect that there will be no ellipses inside that region. We need to have some control on the connectivity even when there are many ellipses that could potentially block the vacant crossing. The idea behind this step is to make a comparison with fractal percolation (see [7], Theorem 1). This comparison is done by dominating small ellipses by a multi-scale Boolean model (see remark 2.7.9; a reference for multi-scale Boolean model is Section 8.1 of [19]).

We begin simplifying our problem by restricting the values of k and l we need to analyze.

Lemma 2.7.3. *If the lower bound holds for $k = 2$ then it holds for any $k > 0$.*

Proof. Suppose that we can find $\bar{u}(c_0)$ and $\delta(c_0, u, 2)$ satisfying $P[\overline{LR}(l;2)] \ge \delta(c_0, u, 2)$ for any $u \in (0, \bar{u})$. By event inclusion, notice that if $k \le k'$ then

$$\overline{LR}(l;k') \subset \overline{LR}(l;k). \tag{2.43}$$

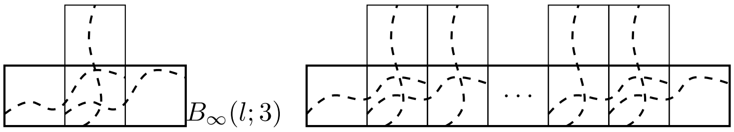


Figure 2.6: Building vacant crossings of $B_\infty(l;3)$ and $B_\infty(l;m)$.

In particular, if $k \leq 2$ we have $P[\overline{LR}(l;k)] \geq P[\overline{LR}(l;2)]$ for all $l > 0$. Thus, in the case $k \leq 2$ we can take $\delta(c_0, u, k) = \delta(c_0, u, 2)$.

If $k > 2$, we apply FKG inequality for Poisson point processes, a correlation inequality for increasing functions. In the context of PPPs, we say that a function f from the point processes on S to $\mathbb{R}$ is *increasing* if $f(\xi) \geq f(\xi')$ whenever $\xi \supset \xi'$ and say f is *decreasing* if $-f$ is increasing. We also say that an event is increasing if its indicator function is increasing. It holds:

Lemma 2.7.4 (FKG inequality)**.** *Any two bounded increasing functions f, g on the point processes of S are positively correlated, that is*

$$E[fg] \geq E[f] \cdot E[g]. \tag{2.44}$$

For a proof of FKG inequality for Poisson processes, see [14]. Notice that $\overline{LR}(l;2)$ and any translated event in which we have a vacant left-right crossing of a box $z + B_\infty(l;2)$, with $z \in \mathbb{R}^2$, are all decreasing events. By this, we can take advantage of planarity to build a crossing of a larger box by making the intersection of vacant crossings of translates of $B_\infty(l;2)$.

Taking $m = \lceil k \rceil$, we can write from equation (2.43) that

$$P[\overline{LR}(l;k)] \geq P[\overline{LR}(l;m)], \text{for all } l > 0.$$

Notice that using three vacant crossings of boxes congruent to $B_\infty(l;2)$ we can build a crossing of the box $B_\infty(l;3)$. A completely analogous construction shows that we can build a vacant crossing of $B_\infty(l;m)$ using $2m-3$ vacant crossings of boxes congruent to $B_\infty(l;2)$ (see Figure 2.6). Denote these boxes by $(B_j)_{j=1}^{2m-3}$. Then, by FKG inequality:

$$P[\overline{LR}(l;m)] \geq P\left(\cap_{j=1}^{2m-3}\{\text{vacant crossing of } B_j\}\right) \geq P[\overline{LR}(l;2)]^{2m-3} \geq P[\overline{LR}(l;2)]^{2k-1} \tag{2.45}$$

and thus for $k > 2$ we can take $\delta(c_0, u, k) = \delta(c_0, u, 2)^{2k-1}$. □

Because of Lemma 2.7.3 we know it is enough to prove the lower bound when $k = 2$, which we assume from now on. Notice that we can also assume l is large. Indeed, we have the trivial bound

$$P[\,\overline{LR}(l;2)\,] \geq P[B(2l) \subset \mathcal{V}] = P[B(2l) \cap \mathcal{E} = \varnothing], \ \forall l > 0. \tag{2.46}$$

The probability on the right hand side of equation (2.46) is decreasing on l, so for $l \leq 1$ we have by Proposition 2.3.6.*(ii)* with $a = 2$ that $P[\,\overline{LR}(l;2)\,] \geq P[B(2) \cap \mathcal{E} = \varnothing] \geq \exp[-uc_4 2^2]$.

Then, assume $k = 2$ and $l > 1$. The next step is to start the decomposition of ξ into more treatable PPPs. For that, we introduce some new notation. We will partition the set S into many parts, according to the ellipses positioning and size and also the sets they intersect. We recall that restricting a PPP to disjoint subsets gives birth to independent PPPs.

Definition 2.7.5. Let $B, C \subset \mathbb{R}^2$ and $D \subset \mathbb{R}$. We define:

$$\Gamma[B||C,D] := \{s \in S;\ E(s) \cap B \neq \varnothing,\ z \in C \text{ and } R \in D\}.$$

Definition 2.7.6. We define $\xi_{B||C,D}$ as the restriction of PPP ξ to the set $\Gamma[B||C,D]$. Also, we define the shorter versions $\xi_B := \xi_{B||\mathbb{R}^2,\mathbb{R}}$ and $\xi_{||C,D} := \xi_{\mathbb{R}^2||C,D}$.

Remark 2.7.7. Notice that the notation on Definition 2.7.6 is consistent with Section 2.6.

Our final objective is to use the decomposition of ξ to build an event $H = H(l)$ with probability bounded away from zero for all l, and such that $H \subset \overline{LR}(l;2)$. To begin our partitioning, we restate Proposition 2.3.6.*(i)* with the notation above.

Lemma 2.7.8. *Let $w \in \mathbb{R}^2$, $a \geq 1$ and $M \geq 2$. Then, we have*

$$\exp[-uc_3^{-1}a^{2-\alpha}] \leq P[\xi_{B(w,a)||B(w,Ma)^c,\mathbb{R}}(S) = 0] \leq \exp[-uc_3a^{2-\alpha}] \tag{2.47}$$

for some constant $c_3 = c_3(c_0, \alpha, M) > 0$.

Notice that for any value of l we have $B_\infty(l;2) \subset B(2l) \subset B(4l)$. Taking $\alpha = 2$, w as the origin, $M = 2$ and $a = 2l$ in Lemma 2.7.8, we get

$$P[\xi_{B(2l)||B(4l)^c,\mathbb{R}}(S) = 0] \geq \exp[-uc_3^{-1}] \tag{2.48}$$

for some constant $c_3 = c_3(c_0) > 0$. Notice that on this event we do not have to worry about ellipses too far away (centered on $B(4l)^c$) interfering with the vacant left-right crossing of $B_\infty(l;2)$, since none of them intersect the ball $B(2l)$.

So we can restrict ourselves to the ellipses centered on $B(4l)$. Now, being confined to a finite region, we can pay a price to avoid ellipses with too large major axis. More precisely, notice that

$$\begin{aligned} P[\xi_{||B(4l),[l/2,\infty)}(S) = 0] &= \exp[-u\,\lambda(B(4l))\,P[R \geq l/2]] \\ &\geq \exp[-u \cdot 16\pi l^2 \cdot c_0 2^2 l^{-2}] \geq \exp[-u\,c(c_0)]. \end{aligned}$$

Define $H_1 := \{\xi_{B(2l)||B(4l)^c,\mathbb{R}}(S) = 0,\ \xi_{||B(4l),[l/2,\infty)}(S) = 0\}$. By independence, we have that

$$P[H_1] \geq \exp[-u\,c(c_0)] \tag{2.49}$$

where the constants c and c_3 have been combined into a new one. On H_1, the only ellipses that could prevent $\overline{LR}(l;2)$ from happening must be centered on $B(4l)$ with major axis size in $[1, l/2)$.

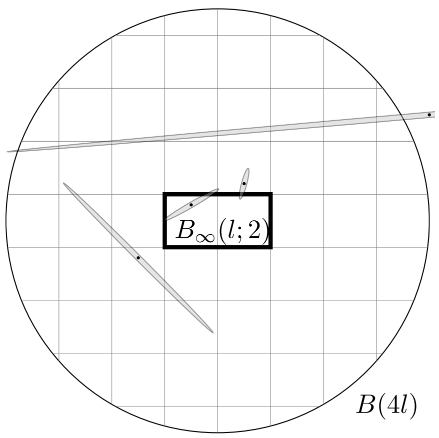


Figure 2.7: On H_1, ellipses that might influence $B_\infty(l;2)$ are centered in $B(4l)$ and have $R \in [1, l/2)$.

2.7.1 Introducing Fractal Percolation

We want to compare the model we are currently studying with fractal percolation, as defined in [7]. Let us give some definitions and results from [7]. Consider a square box $B = [0, l]^2$ (in the original paper the boxes had side length 1, but this makes no difference in what follows).

Given a parameter $p \in [0, 1]$ and $N \in \mathbb{N}$, $N \geq 2$, we do the following inductive procedure: we divide all boxes into N^2 equal boxes and let any of them remain in the process with probability p, independently. Define the remaining set after the n-th step by A_n. More formally, we can define the boxes

$$B_{ij}^n := \left[\frac{(i-1)l}{N^n}, \frac{il}{N^n}\right] \times \left[\frac{(j-1)l}{N^n}, \frac{jl}{N^n}\right] \text{ for } 0 \leq i, j \leq N^n$$

and take iid. random variables $(\epsilon_{ij}^n)_{0\leq i,j\leq N^n}^{n\in\mathbb{N}}$ with $\epsilon_{ij}^n \overset{d}{=} \mathrm{Ber}(p)$. Setting $A_0 = B$, we can define

$$A_n = A_{n-1} \cap \Big(\bigcup_{i,j;\, \epsilon_{ij}^n = 1} B_{ij}^n \Big).$$

In an analogous way, we can study fractal percolation in sets different from B. In order to do that, we can consider unions of disjoint boxes of side l and then make the same procedure in each one of them.

In [7], Theorem 1, the authors prove that for p sufficiently close to 1 the set $A_\infty = \cap A_n$ connects the opposing sides of box B with high probability. Let us define a similar process in the model we are working with.

2.7.2 The Removal Process

We can relate our model with fractal percolation through the following procedure. We fix $N = 2$. Initially, we divide the interval $[1, l/2)$, into $n_0 = n_0(l)$ disjoint intervals

$$I_n = \begin{cases} \left[\frac{l}{2^{n+1}}, \frac{l}{2^n}\right) & \text{if } 1 \leq n < n_0 \\ \left[1, \frac{l}{2^n}\right) & \text{if } n = n_0 \end{cases} \tag{2.50}$$

where n_0 is the only integer such that $l/2^{n_0+1} \leq 1 < l/2^{n_0}$, or equivalently $n_0 = \lceil \log_2 l \rceil - 1$. The box $B_\infty(l; 2)$ is composed of 2 square boxes of side l. For each n we can partition each of them into 4^n boxes of side $l/2^n$. We denote by $(B_z^n)_{z\in\Lambda_n}$ the collection of boxes of this partition. Since we want to find a vacant path connecting $L^-(l; 2)$ and $L^+(l; 2)$, we will successively test, for n ranging from 1 to n_0, which are the sub boxes that were not intersected by ellipses with major axis in I_n. We define

$$X_z^n := \mathbb{1}\{\xi_{B_z^n || \mathbb{R}^2, I_n}(S) = 0\}. \tag{2.51}$$

For every n, the family $(X_z^n)_{z\in\Lambda_n}$ is a random field. Notice that the collection of random fields $(X_\cdot^n)_{n=1}^{n_0}$ is independent. However, for a fixed n the values of X_z^n on this random field are not independent; if $X_z^n = 0$, we have that B_z^n has been intersected by an ellipse and then it is more probable that one of its neighboring boxes has also been intersected. Now, the choice of intervals I_n becomes clearer. Since the random field X_z^n is only concerned with ellipses whose major axis is in I_n, we have that one single ellipse cannot intersect two boxes at distance greater than $2l/2^n$. This means the random field X_z^n is 2-dependent. Moreover, if we denote by $\tilde{B}_z^n$ the region B_z^n enlarged by $l/2^n$, we have

$$P[X_z^n = 1] \geq P[\xi(\tilde{B}_z^n \times I_n \times (-\tfrac{\pi}{2}, \tfrac{\pi}{2}]) = 0] \geq \exp[-uc(c_0)] =: p(u).$$

By this, we can apply the results of Liggett, Schonmann and Stacey [18]; in their notation, each random field (X_z^n) is in the class $\mathcal{C}(d,k,p)$ of random fields on $\mathbb{Z}^d$ such that for every $z \in \mathbb{Z}^d$ one has

$$P[X_z = 1 \mid (X_w)_{\|w-z\|_\infty > 2}] \geq p,$$

with $d = 2$, $k = 2$ and $p = p(u)$. We conclude that if $p(u)$ is sufficiently close to one there is a $\beta = \beta(p(u))$ such that for each n we can find an independent random field $(Y_z^n)_{z\in\Lambda_n}$ that is dominated by $(X_z^n)_{z\in\Lambda_n}$ and has a product law with $P[Y_z^n = 1] = \beta$. Moreover, we can take $\beta \to 1$ when $p \to 1$. This domination is enough to complete our proof, since if we take $\bar{u} = \bar{u}(c_0)$ sufficiently small then $\beta(u)$ will be close to one for $u \in (0, \bar{u})$. Hence, the n_0-th step of fractal percolation A_{n_0} obtained through

$$A_0 = B_\infty(l; 2) \text{ and } A_n = A_{n-1} \cap \Big(\bigcup_{\substack{z\in\Lambda_n;\\ Y_z^n = 1}} B_z^n \Big) \tag{2.52}$$

will contain a crossing of $B_\infty(l;2)$ with probability close to one, by Theorem 1 of [7]. Define the random subset $\tilde{A}_{n_0}$ in the same way as A_{n_0}, substituting Y_z^n by X_z^n. Stochastic domination implies we also have a left-right crossing of $B_\infty(l;2)$ in the set $\tilde{A}_{n_0}$. Translating to ellipses intersection, this means we found a random path on $B_\infty(l;2)$ that is not intersected by any ellipse with major axis in $[1, l/2)$. Denoting this event by H_2 and noticing that H_2 is independent from H_1, we conclude

$$P[\overline{LR}(l;2)] \geq P[H_1 \cap H_2] \geq P[H_1]P[H_2] \geq \delta(u) > 0, \ \forall u \in (0, \bar{u}(c_0)),$$

finishing the proof of the first claim on Theorem 1.1.3.

Remark 2.7.9. The construction above is very close to the one in Werner [39], for studying CLE. The stochastic domination can also be seen as a comparison with multi-scale Boolean model with fixed radius ([19], Section 8.1). If we replace every ellipse with major axis in I_n by a ball with radius $l \cdot 2^{-n}$ we obtain a union of $n_0(l)$ Boolean models, where the n-th model has density

$$uP[R \in I_n] = u\rho[l\cdot 2^{-(n+1)}, \infty) - u\rho[l \cdot 2^{-n}, \infty) \leq u(l\cdot 2^{-n})^{-2}[4c_0 - c_0^{-1}].$$

Subcriticality of multi-scale Boolean model for small densities is already proved in [19]. References [20, 21] prove stronger results in this direction.

2.7.3 Covered set does not percolate for small intensities

Using (2.40), let us prove that for $u < \bar{u}(c_0)$ neither $\mathcal{V}$ nor $\mathcal{E}$ percolate, almost surely. The proof for $\mathcal{V}$ is already done in Lemma 2.5.7; we already know that $\mathcal{V}$ does not percolate for any $u > 0$. To prove $\mathcal{E}$ does not percolate, we combine equation (2.40) and FKG inequality. We present 2 proofs:

First proof. Notice that with probability at least $\delta(u, c_0) > 0$ we have a vacant circuit around the origin on $B(3l)\backslash B(l)$. Define the event $\tilde{A}_n$ in which there is such a circuit on $B(3^n)\backslash B(3^{n-1})$. By Fatou's lemma, we have $P[\limsup_n \tilde{A}_n] \geq \limsup_n P[\tilde{A}_n] \geq \delta$ and thus $P[\mathcal{E} \text{ percolates}] \leq 1 - \delta < 1$. Since Lemma 2.6.3 shows that ellipses model is ergodic with respect to translations, we conclude $P[\mathcal{E} \text{ percolates}] = 0$. □

Second proof. Alternatively, we can pick an increasing sequence $(L_n) \subset \mathbb{R}^+$, define the event A_n in which there is such a circuit on $B(3L_n)\backslash B(L_n)$ and try to apply Borel-Cantelli's lemma, since $\sum P(A_n) = \infty$.

However, the events A_n are not independent. The idea is then to choose L_n increasing sufficiently fast so that the events A_n get almost independent quickly. We apply Lemma 2.6.2 for the functions

$\mathbb{1}_{A_n}$ when $\alpha = 2$. Let us denote $a_n := \frac{L_n}{L_{n-1}}$. If we choose $L_1 \geq 1$ and $a_n \geq 4$ the hypotheses of Lemma 2.6.2 are satisfied, giving

$$|P[A_i \cap A_j] - P[A_i]P[A_j]| \leq \frac{uc_7}{\frac{L_j}{3L_i} - 1} = \frac{3uc_7}{a_{i+1} \dots a_j - 3} \leq \frac{3uc_7}{a_{i+1} - 3}.$$

Then, we can use a generalization of Borel-Cantelli due to Ortega and Wschebor [24]. The result states that a sufficient condition for $P[A_n, \text{ i.o.}] = 1$ is that $\sum_n P(A_n) = \infty$ and

$$\liminf_n \frac{\sum_{1 \leq i < j \leq n} [P(A_i \cap A_j) - P(A_i)P(A_j)]}{\left[\sum_{i=1}^n P(A_i)\right]^2} \leq 0. \tag{2.53}$$

Using that $P[A_i] \geq \delta(u, c_0)$, we obtain

$$\begin{aligned}\left| \frac{\sum_{1 \leq i < j \leq n} [P(A_i \cap A_j) - P(A_i)P(A_j)]}{\left[\sum_{i=1}^n P(A_i)\right]^2} \right| &\leq [n\delta(u, c_0)]^{-2} \sum_{1 \leq i < j \leq n} \frac{3uc_7}{a_{i+1} - 3} \leq \frac{c(u, c_0)}{n^2} \sum_{i=1}^{n-1} \frac{n}{a_{i+1} - 3} \\ &\leq \frac{c(u, c_0)}{n} \sum_{i=1}^{\infty} \frac{1}{a_{i+1} - 3}.\end{aligned}$$

where $c(u, c_0)$ is a positive constant. If we additionally require that sequence a_i satisfies $\sum_i \frac{1}{a_i} < \infty$ then condition (2.53) is satisfied. □

This finishes the proof of Theorem 1.1.3. □

Chapter 3

Poisson Stick Soup

3.1 SHPSS as a limit distribution

The model we study in this chapter is a PPP on $S := \mathbb{R}^2 \times [0, \infty) \times (-\pi/2, \pi/2]$, given by Definition 1.2.1. Let us recall it here:

Definition 1.2.1 (SHPSS and SIPSS)**.** Given u and $\alpha > 0$, the PPP on S with intensity measure $u\mu_\alpha := u\lambda \otimes \phi_\alpha \otimes \nu$ where $\phi_\alpha[a, \infty) = a^{-\alpha}$ is called a scale-homogeneous Poisson stick soup (SHPSS) of intensity u and decay α. The particular case in which $\alpha = 2$ is referred to as a scale-invariant Poisson stick soup (SIPSS) of intensity u.

Recall also that any PPP ξ on S can be seen as a random collection of sticks in the plane. This can be accomplished by considering the union for all $s = (z, R, V) \in \text{supp } \xi$ of the stick centered at z with radius R and direction V, denoted by $E_0(s)$; we denote this stick soup by $\mathcal{E}_0 = \cup_{s \in \text{supp } \xi} E_0(s)$.

In this section we argue that SHPSS is a natural candidate for scaling limit of ellipses model. By scaling limit we mean we will apply a homothety of ratio $l > 0$ to the random collection of ellipses $\mathcal{E}$ given by ellipses model and take $l \to 0$, while choosing densities of ellipses $u(l)$ in order to get a non-degenerate limit. To better describe models obtained from the original ellipses model through a homothety, we generalize our notation.

Definition 3.1.1. Let $S = \mathbb{R}^2 \times \mathbb{R}^+ \times (-\pi/2, \pi/2]$ and $l \geq 0$. Given a point $s = (z, R, V) \in S$ with $R \geq l$, define $E_l(s)$ as the ellipse with center in z, major axis size and direction given by R and V, and minor axis of size l. When $l = 0$ we obtain a stick. Thus, from a point process ξ on $S \cap \{R \geq l\}$ we are able to build ellipses models with minor axis fixed as l, by considering $\cup_{s \in \xi} E_l(s)$.

Let us first study what happens when we apply an homothety of ratio l to the random subset $\mathcal{E} = \cup_{s \in \xi} E_1(s)$. The homothety of ratio $l > 0$ takes ellipses with minor axis size one to ellipses with minor axis size l, preserving directions. The major axis will have a distribution ρ_l satisfying

$$\rho_l[r, \infty) = P[lR \geq r] = \rho[\tfrac{r}{l}, \infty).$$

Thus, we have that ρ_l is a measure on $[l, \infty)$ satisfying

$$c_0^{-1} r^{-\alpha} l^\alpha \leq \rho_l[r, \infty) \leq c_0 r^{-\alpha} l^\alpha, \ \forall r \geq l$$

and for any fixed $l > 0$, ρ_l has the same tail decay as ρ. Besides that, we are also changing the density of the Poisson point process on $\mathbb{R}^2$, from u to ul^{-2}.

Summing it up, we end up with a new PPP. The homothety of ratio $l > 0$ induces a transformation on S defined as $\mathcal{H}_l : S \to S$ such that $(z, R, V) \mapsto (lz, lR, V)$. Denoting $\mathcal{H}_l : \xi = \sum_i \delta_{s_i} \mapsto \sum_i \delta_{\mathcal{H}_l(s_i)}$, we have

Proposition 3.1.2. *Fix a distribution ρ and $u > 0$. If ξ is the PPP associated to the (u,ρ)-ellipses model of minor axis one and we apply a homothety of ratio $l > 0$ to the ellipses soup, then the resulting process is a ellipses soup of minor axis l built from PPP on S given by $\mathcal{H}_l\xi$. Its intensity measure is $(ul^{-2}\lambda) \otimes \rho_l \otimes \nu$.*

3.1.1 Convergence of PPP's

We want to obtain a non-degenerate object when we make a scaling limit of ellipses model by taking $l \to 0$. We focus on weak convergence of PPP's on space S. One possible reference is [28], where PPP's are studied through Laplace functional techniques. We can use for instance Proposition 3.22 of [28]. It states that if the PPP ξ is simple, in order to prove that ξ_n converges in distribution to ξ it is sufficient to check if

$$\lim_n E[\xi_n(I)] = E[\xi(I)] \text{ and } \lim_n P[\xi_n(I) = 0] = P[\xi(I) = 0],\ \forall I \in \mathcal{I}, \tag{3.1}$$

where $\mathcal{I}$ is a basis of open, relatively compact sets $I \subset S$ that is closed to finite unions and intersections and such that $P[\xi(\partial I) = 0]$ for every I.

We take $\mathcal{I}$ as the finite union of sets of the form $K \times (a,b) \times J$ in $\mathbb{R}^2 \times \mathbb{R}^+ \times (-\pi/2, \pi/2]$ with $a, b > 0$ and $K = (x_1, y_1) \times (x_2, y_2)$ and $J = (x_4, y_4)$. Using notation of Proposition 3.1.2, we have to take a function $u = u(l)$ and check whether $\mathcal{H}_l\xi^{u(l)}$ satisfies (3.1). Since

$$P[\mathcal{H}_l\xi(K \times (a,b) \times J) = 0] = \exp[-u(l)l^{-2}\lambda(K)\rho(\tfrac{a}{l}, \tfrac{b}{l})\nu(J)]$$

we should choose $u(l)$ in order that $u(l)l^{-2}\rho(\frac{a}{l}, \frac{b}{l})$ converges to a non-degenerate distribution. Now, we restrict our choice of allowed distributions ρ to those satisfying $\rho[r, \infty) = L(r)r^{-\alpha}$, where L is a slowly varying function; this will ensure the convergence needed. Indeed, notice that if we fix $0 < a < b$ we get

$$\rho(\tfrac{a}{l}, \tfrac{b}{l}) = \rho(\tfrac{a}{l}, \infty) - \rho(\tfrac{b}{l}, \infty) = l^\alpha L(1/l) \left[a^{-\alpha}\frac{L(a/l)}{L(1/l)} - b^{-\alpha}\frac{L(b/l)}{L(1/l)} \right] \sim l^\alpha L(1/l)[a^{-\alpha} - b^{-\alpha}].$$

This implies that we can take $u(l) = L(1/l)^{-1}l^{2-\alpha}$, to get

$$P[\mathcal{H}_l\xi(K \times (a,b) \times J) = 0] \xrightarrow{l \to 0} \exp[-\lambda(K)(a^{-\alpha} - b^{-\alpha})\nu(J)]$$

and thus $u(l)l^{-2}\rho(\frac{\cdot}{l})$ converges to the distribution $\phi_\alpha = \alpha x^{-(1+\alpha)}\mathrm{d}x$ on $\mathbb{R}^+$. Then, we have the convergence

$$\mathcal{H}_l\xi \Rightarrow \mathrm{PPP}(\lambda \otimes \phi_\alpha \otimes \nu) \text{ when } l \to \infty$$

and to simplify notation, define $\mu_\alpha := \lambda \otimes \phi_\alpha \otimes \nu$. Naturally, if we fix a parameter $u \in (0, \infty)$ and consider a PPP on S of intensity $u\mu_\alpha$, we also get a possible scaling limit, corresponding to taking $u(l) = uL(1/l)^{-1}l^{2-\alpha}$.

It is interesting to ask when the model obtained is translation and scale-invariant, in the spirit of Nacu and Werner [23]. The discussion above together with Proposition 3.1.3 justifies Definition 1.2.1.

Proposition 3.1.3. *For any $\alpha, c > 0$ we have the homogeneity relation $\mu_\alpha(\mathcal{H}_c(\cdot)) = c^{2-\alpha}\mu_\alpha$. In particular, measure μ_2 is scale-invariant.*

Proof. Notice that for any rectangular event of the form $K \times (a,b) \times J$ we have:

$$\begin{aligned}
\mu_\alpha(\mathcal{H}_c[K \times (a,b) \times J]) &= \mu_\alpha(cK \times (ca, cb) \times J) = c^2\lambda(K) \cdot \int_{ac}^{bc} \alpha x^{-(1+\alpha)}\,\mathrm{d}x \cdot \nu(J) \\
&= c^2\lambda(K) \cdot [(ac)^{-\alpha} - (bc)^{-\alpha}] \cdot \nu(J) = c^{2-\alpha}\lambda(K) \cdot [a^{-\alpha} - b^{-\alpha}] \cdot \nu(J) \\
&= c^{2-\alpha}\mu_\alpha(K \times (a,b) \times J).
\end{aligned}$$

Since this equality holds for a sufficiently large family of sets, the measures $\mu_\alpha(\mathcal{H}_c(\cdot))$ and $c^{2-\alpha}\mu_\alpha$ are equal. □

3.2 Analysis of large and small sticks

In this section we study the behavior of small sticks and large sticks on a SHPSS. Here, homogeneity is a very useful tool. A good example is applying homogeneity to study percolation for the covered and the vacant sets, as shown by

Proposition 3.2.1. *For $\alpha \neq 2$ and any intensity $u > 0$, a SHPSS satisfies $P[\mathcal{E}$ percolates$] = 1$ and $P[\mathcal{V}$ percolates$] = 0$.*

Proof. One important observation is that whenever we restrict the radius of the sticks of a SHPSS to a finite interval, we can easily compare the soup obtained with a Poisson stick soup (PSS) of sticks with fixed length, a model that is already quite well-understood (see [19]). Indeed, notice that for any $r > 0$

$$u\phi_\alpha((r/2, r])\nu((-\tfrac{\pi}{2}, \tfrac{\pi}{2}]) = u(2^\alpha - 1)r^{-\alpha}$$

is the intensity of the process obtained when we only consider sticks with radius in the interval $(r/2, r]$. Thus, the PPP of intensity $u\lambda \otimes \mathbb{1}_{(r/2,r]}(x)\phi_\alpha(\mathrm{d}x) \otimes \nu$ dominates a PSS with sticks of radius $r/2$ and intensity $u(2^\alpha - 1)r^{-\alpha}$ and is dominated by a PSS of radius r and same intensity. Moreover, if we denote by $u_c(l)$ the critical parameter for PSS of radius l, we must have by scaling that $u_c(l) = u_c(1)l^{-2}$. Notice that if $\alpha > 2$ then

$$\frac{u(2^\alpha - 1)r^{-\alpha}}{u_c(r/2)} = \frac{u(2^\alpha - 1)r^{-\alpha}}{4u_c(1)r^{-2}} = \frac{u(2^\alpha - 1)}{4u_c(1)}\frac{1}{r^{\alpha-2}} \gg 1 \tag{3.2}$$

if r is sufficiently small. This means that for $\alpha > 2$ small sticks dominate fixed length PSS's with arbitrarily high intensities, and thus $\mathcal{E}$ percolates but $\mathcal{V}$ does not. Analogously, if we fix $\alpha < 2$ we have $u(2^\alpha - 1)r^{-\alpha} > u_c(r/2)$ for sufficiently large r and the same conclusion applies, changing small sticks for large ones. We emphasize that this reasoning is valid for any fixed $u > 0$. □

Remark 3.2.2. When $\alpha = 2$ the density ratio on (3.2) does not depend on r, a consequence of the already mentioned scale invariance. The behavior for $\alpha = 2$ was detailed in Proposition 1.2.3 and its proof is at the end of Section 3.3.

3.2.1 Behavior of large sticks

One possible approach to understand better the behavior of a SHPSS is to break the original PPP into two independent PPP's by allowing only small sticks in the first and large sticks in the second. As we will see, the exact breaking point is not important.

Fix some $\alpha > 0$. We denote by ξ_0 the PPP of intensity measure $u\lambda \otimes \phi_\alpha \otimes \nu$ from which we build a SHPSS of intensity u and decay α. The restriction of ξ_0 to sticks with radius greater than r, is a PPP of intenstity measure

$$u\mu_{\alpha,r} := u\lambda \otimes \phi_\alpha \mathbb{1}\{x \geq r\}\,\mathrm{d}x \otimes \nu,$$

which we denote by ξ_r. The restriction of ξ_0 to sticks with radius strictly smaller than r is denoted by $\xi_{<r}$. Also, let $\mathcal{E}_r$ and $\mathcal{E}_{<r}$ be the sets obtained as the union of sticks $E_0(s)$ for $s \in \operatorname{supp} \xi_r$ and $s \in \operatorname{supp} \xi_{<r}$, respectively. We study these two random sets separately.

To study large sticks, one could use the same methods that were used on Chapter 2. In fact, the ellipses on Chapter 2 can be seen as 'slightly enlarged sticks' and many computations remain valid. However, there are already some useful results in the literature that simplify this task. Cowan and

Parker [25] compute the expectation of some useful quantities for general random processes of line segments on the plane. Our Poisson stick soup ξ_r is contained in the class of models they study.

Notice that our expression for $u\mu_{\alpha,r}$ can be rewritten as $ur^{-\alpha}\lambda \otimes r^{\alpha}\phi_{\alpha}\mathbb{1}\{x \geq r\}\,\mathrm{d}x \otimes \nu$, with the advantage that now the distribution on $\mathbb{R}^+$ composing the product is a probability measure, allowing a more straightforward application of the results in [25]. Fix $r > 0$ and consider the soup ξ_r. For any convex Borel set A, their equation (7) prove that the expected number of segments intersecting A, denoted by $N(A)$, is given by

$$E[N(A)] = ur^{-\alpha}\lambda(A) + ur^{-\alpha}\pi^{-1}E[R]\mathrm{per}(A)$$

where $E[R]$ is the expected length of a segment and $\mathrm{per}(A)$ is the perimeter of A. Since we are working with a PPP, we know that $E[N(A)] = \mu_{\alpha,r}(s; E_0(s) \cap A \neq \varnothing)$. Computing $E[R]$, we get

$$E[R] = \infty \text{ for } \alpha \leq 1 \text{ and } E[R] = \alpha r^{\alpha}\int_r^{\infty} x^{-\alpha}\,\mathrm{d}x = \tfrac{\alpha}{\alpha-1}r \text{ for } \alpha > 1,$$

implying

$$E[N(A)] = ur^{-\alpha}\lambda(A) + \frac{u}{\pi}\cdot\frac{\alpha}{1-\alpha}\cdot r^{1-\alpha}\mathrm{per}(A). \tag{3.3}$$

We begin estimating the probability of some stick with radius greater than r intersecting a ball of radius a.

Proposition 3.2.3. *For any $a, r > 0$, we have that:*

(i) If $\alpha \leq 1$, then $\mu_{\alpha}(E_0(s) \cap B(a) \neq \varnothing, R \geq r) = \infty$.

(ii) If $\alpha > 1$, we have $\mu_{\alpha}(E_0(s) \cap B(a) \neq \varnothing, R \geq r) = \pi a^2 r^{-\alpha} + \frac{2\alpha}{\alpha-1}ar^{1-\alpha}$.

(iii) It holds that $P[\mathcal{E}_r \cap B(a) \neq \varnothing] = 1$ if and only if $\alpha \leq 1$.

Proof. Items *(i)* and *(ii)* follow directly from equation (3.3) with $\mu_{\alpha,r}$ and the convex set $A = B(a)$. For *(iii)*, just notice that the random variable $N(B(a))$ that counts the number of sticks intersecting $B(a)$ has distribution $\mathrm{Poi}(\mu_{\alpha,r}(s;\ E_0(s) \cap B(a) \neq \varnothing))$. □

We also state an analogous proposition for intersecting line segments. Its proof is identical to the proof of Proposition 3.2.3.

Proposition 3.2.4. *Let $L(a) \subset \mathbb{R}^2$ be any segment with length a. For any $a, r > 0$, we have that:*

(i) If $\alpha \leq 1$, then $\mu_{\alpha}(E_0(s) \cap L(a) \neq \varnothing, R \geq r) = \infty$.

(ii) If $\alpha > 1$, we have $\mu_{\alpha}(E_0(s) \cap L(a) \neq \varnothing, R \geq r) = \pi^{-1} \cdot \frac{2\alpha}{\alpha-1}ar^{1-\alpha}$.

(iii) It holds that $P[\mathcal{E}_r \cap L(a) \neq \varnothing] = 1$ if and only if $\alpha \leq 1$.

Proposition 3.2.3 has some easy consequences which are versions of results we proved for ellipses percolation. An interesting use of Proposition 3.2.3 is to control long-range interactions. A simple example is applying it with $r = a$ to obtain

Corollary 3.2.5. *For any $\alpha > 1$, there is a constant $c_{\alpha} > 0$ such that for all $a > 0$ the SHPSS with decay α and intensity u satisfies*

$$P[\nexists s = (z, R, V) \in \mathrm{supp}\ \xi;\ z \notin B(2a),\ E_0(s) \cap B(a) \neq \varnothing] \geq \exp[-uc_{\alpha}a^{2-\alpha}].$$

In Chapter 2 we proved some estimates on the probability that a box is crossed by one ellipse. We end this subsection with a stick version of this result. Its proof is a simple adaptation of the ellipses version, together with Corollary 3.2.5. Define the event

$$LR_1(l;k) := \{\exists s \in \text{supp } \xi; E_0(s) \cap L^-(l;k) \neq \varnothing \text{ and } E_0(s) \cap L^+(l;k) \neq \varnothing\}.$$

We can prove the following estimates on the probability of $LR_1(l;k)$.

Proposition 3.2.6. *If $\alpha > 1$ and $k, l > 0$, then there is a constant $c_\alpha > 0$ such that:*

$$1 - \exp[-c_\alpha^{-1} u(k \wedge k^{-\alpha}) l^{2-\alpha}] \leq P(LR_1(l;k)) \leq 1 - \exp[-c_\alpha u(k^{2-\alpha} \vee k^{-\alpha}) l^{2-\alpha}]. \tag{3.4}$$

3.2.2 Decay of correlations

Proposition 3.2.3 is also useful to derive bounds for decay of correlations. We briefly discuss this application. We begin with an auxiliary result that will be very useful throughout the text.

Lemma 3.2.7. *Take $\alpha > 1$ and let $\Gamma_{12} := \{s;\ E_0(s) \cap \partial B(l_j) \neq \varnothing, j = 1, 2\}$ with $l_2 > l_1 > 0$ and $a := l_2/l_1$. There is a constant $c_\alpha > 0$ such that*

$$c_\alpha^{-1} l_1^{2-\alpha}[a^{-\alpha} + a^{1-\alpha}] \leq \mu_\alpha(\Gamma_{12}) \leq c_\alpha l_1^{2-\alpha}[(a-1)^{-\alpha} + (a-1)^{1-\alpha}]. \tag{3.5}$$

In particular, if $a \geq 2$ then

$$\mu_\alpha(s; E_0(s) \cap \partial B(l_j) \neq \varnothing, j = 1, 2) \leq c_\alpha l_1^{2-\alpha} a^{1-\alpha}. \tag{3.6}$$

Proof. By Proposition 3.2.3 we have

$$\begin{aligned}\mu_\alpha(\Gamma_{12}) &\leq \mu_\alpha[E_0(s) \cap B(l_1) \neq \varnothing,\ 2R \geq (l_2 - l_1)] \leq c_\alpha[l_1^2(l_2 - l_1)^{-\alpha} + l_1(l_2 - l_1)^{1-\alpha}] \\ &= c_\alpha l_1^{2-\alpha}[(a-1)^{-\alpha} + (a-1)^{1-\alpha}],\end{aligned} \tag{3.7}$$

as we claimed for the upper bound. The lower bound follows the same reasoning, beginning with the bound

$$\mu_\alpha(\Gamma_{12}) \geq \mu_\alpha[E_0(s) \cap B(l_1) \neq \varnothing,\ R \geq l_2].$$

□

Lemma 3.2.8 (Decay of Correlations)**.** *Take $\alpha > 1$. Let $K_1 = B(l_1)$ and $K_2 = B(l_2)^c$, with $l_2 > l_1$. Let f_1 and f_2 be real functions of ξ such that $|f_j| \leq 1$ and f_i depends only on sticks touching K_i. Then, we have that*

$$\big|E[f_1 f_2] - E[f_1]E[f_2]\big| \leq 4u\mu_\alpha(s; E_0(s) \cap \partial B(l_j) \neq \varnothing, j = 1, 2) \tag{3.8}$$

In particular, if we take $l_2 = al_1$ with $a \geq 2$ then

$$\big|E[f_1 f_2] - E[f_1]E[f_2]\big| \leq uc_\alpha l_1^{2-\alpha} a^{1-\alpha}. \tag{3.9}$$

Proof. Inequality (3.8) was proved in equation (2.39) of Lemma 2.6.2. The bound (3.9) follows from equation (3.8) and Lemma 3.2.7. □

Remark 3.2.9. Lemma 3.2.8 gives a quantitative measure of how fast a SHPSS mixes. Like ellipses model, any SHPSS is ergodic with respect to translations on $\mathbb{R}^2$.

3.2.3 Behavior of small sticks

It is clear that for any fixed Borel set B with $\lambda(B) > 0$, there are infinitely many small sticks centered on B, since $\phi_\alpha(0,r) = \infty$ for any $r, \alpha > 0$. Thus, $P[\mathcal{E}_{<r} \cap B \neq \varnothing] = 1$. Let us estimate the probability of a SHPSS intersecting $L(a)$, a segment with length a, when we only consider small sticks. To sum it up in one statement, the probability of intersecting a fixed segment with small and large sticks is intimately related to integrability of $r^{-\alpha}$ near 0 and near infinity, respectively.

Proposition 3.2.10. *For any $a, r > 0$, we have $P[\mathcal{E}_{<r} \cap L(a) \neq \varnothing] = 1$ if and only if $\alpha \geq 1$.*

Proof. By homogeneity, we can suppose $r = 1$. Using the upper bound

$$\begin{aligned} u\mu_\alpha(R \in [l/2, l), E_0(s) \cap L(a) \neq \varnothing) &\leq u\mu_\alpha(R \in [l/2, l), \mathrm{dist}(z, L(a)) \leq l) \\ &\leq uc_\alpha l^{-\alpha} \cdot (al + \pi l^2) = uc_\alpha(al^{1-\alpha} + l^{2-\alpha}) \end{aligned}$$

we can conclude that

$$\begin{aligned} u\mu_\alpha(R \in (0,1), E_0(s) \cap L(a) \neq \varnothing) &= \sum_{n=0}^{\infty} u\mu_\alpha(R \in [2^{-(n+1)}, 2^{-n}), E_0(s) \cap L(a) \neq \varnothing) \\ &\leq uc_\alpha \sum_{n=0}^{\infty} [a(2^{\alpha-1})^n + (2^{\alpha-2})^n] \end{aligned}$$

which is clearly finite for $\alpha < 1$.

For $\alpha \geq 1$, we can prove that $u\mu_\alpha(E_0(s) \cap L(a) \neq \varnothing) = \infty$ by estimating the measure of sticks crossing $L(a)$ when we know its center z is in the isosceles triangle of base $L(a)$ and height $a/2$, which we denote by $\triangle$. Let us consider that the center of segment $L(a)$ is in the origin of $\mathbb{R}^2$ and the segment is parallel to the x axis. Notice that for a random stick centered on $z = (x, y)$ in the triangle to intersect $L(a)$ we can ask that V is in an interval of length $\frac{\pi}{2}$ and $R \geq \sqrt{2}y$. Thus,

$$\begin{aligned} u\mu_\alpha(E_0(s) \cap L(a) \neq \varnothing) &\geq u\frac{\pi}{2}\ \lambda \otimes \phi_\alpha(z \in \triangle, R \geq \sqrt{2}y) \\ &= u\frac{\pi}{2} \cdot 2 \int_0^{a/2} \int_0^{1-x} 2^{-\alpha/2} y^{-\alpha}\ \mathrm{d}y\, \mathrm{d}x = \infty. \end{aligned}$$

□

We also prove an analogue of Proposition 3.2.10 for a circle. It should be expected that any fixed circle is intersected by infinitely many sticks if and only if $\alpha \geq 1$ and this result is indeed true (see Remark 3.2.12). However, in order to provide a cleaner argument we only prove here a partial result. Proposition 3.2.11 below covers the case $\alpha = 2$, that is the only one we actually need in what follows.

Proposition 3.2.11. *For any $M, l > 0$ it holds $P[\mathcal{E}_{<M} \cap \partial B(l) \neq \varnothing] = 1$ if $\alpha \geq 2$ and is strictly less than 1 if $\alpha < 1$.*

Proof. We can suppose by homogeneity that $l = 1$. We want to know for which α we have that $\mu_\alpha(E_0(s) \cap \partial B(1) \neq \varnothing, R < M)$ is infinite. Denote by A_M the event above. We have for any $\varepsilon \in (0, M)$ that

$$\mu_\alpha(E_0(s) \cap \partial B(1) \neq \varnothing, R \in [\varepsilon, M)) \leq \lambda(B(1+M)) \cdot (\varepsilon^{-\alpha} - M^{-\alpha}) < \infty. \tag{3.10}$$

Thus, either $\mu_\alpha(A_M)$ is finite for every $M > 0$ or it is infinite for every $M > 0$. Notice that any stick with $R < \varepsilon$ that intersects $\partial B(1)$ must have also $\big| |z| - 1 \big| < \varepsilon$. This implies

$$\mu_\alpha(A_\varepsilon) \leq \mu\big(\big| |z| - 1 \big| \leq \varepsilon, R \in \big[\big| |z| - 1 \big|, \varepsilon\big)\big) \leq \int_{1-\varepsilon<|z|<1+\varepsilon} \big| |z| - 1 \big|^{-\alpha}\ \mathrm{d}z = c \int_{1-\varepsilon}^{1+\varepsilon} \big| r - 1 \big|^{-\alpha} r\ \mathrm{d}r$$

which is finite for $\alpha < 1$. For the lower bound, it suffices to show that for any $\alpha \geq 2$ we have $\mu_\alpha(A_M) = \infty$. Recall the power of a point z with respect to circle $\partial B(1)$ is given by $|z|^2 - 1$. It is known that for any line that intersects the circle at points p_1 and p_2 it holds that $|p_1 - z| \cdot |p_2 - z| = \big| |z|^2 - 1 \big|$. Thus, for any line intersecting the circle we must have

$$|p_1 - z| \wedge |p_2 - z| \leq \big| |z|^2 - 1 \big|^{1/2} \tag{3.11}$$

For our lower bound on $\mu_\alpha(A_M)$ we only consider points z inside $\partial B(1)$. By inequality (3.11), if $R \geq \big| |z|^2 - 1 \big|^{1/2}$ the stick is long enough to intersect the circle, for any V. Fix $\varepsilon = \varepsilon_M > 0$ to ensure $\big| 1 - |z|^2 \big|^{1/2} < M$ when $|z| \in (1 - \varepsilon, 1)$. Then

$$\begin{aligned}\mu_\alpha(A_M) &\geq \mu(|z| \in (1-\varepsilon, 1),\, R \in [\, \big| |z|^2 - 1 \big|^{1/2}, M)) = \int_{1-\varepsilon<|z|<1} c_\alpha \cdot [(1-|z|^2)^{-\alpha/2} - M^{-\alpha}]\, \mathrm{d}z \\ &= c_\alpha \int_{1-\varepsilon}^{1} (1-r^2)^{-\alpha/2} r\, \mathrm{d}r - c_{\alpha,M}.\end{aligned}$$

The integral above is infinite for $\alpha \geq 2$, since making the change of variables $y = 1 - r$ leads to

$$\int_{1-\varepsilon}^{1} (1-r^2)^{-\alpha/2} r\, \mathrm{d}r = \int_0^{\varepsilon} (2-y)^{-\alpha/2}(1-y) \cdot y^{-\alpha/2}\, \mathrm{d}y. \qquad \square$$

Remark 3.2.12. Our lower bound in Proposition 3.2.11 only works for $\alpha \geq 2$ because in order to simplify calculations we take $R \geq (1 - |z|^2)^{1/2} \geq c(1 - |z|)^{1/2}$ independently of the direction. A more precise calculation would be to take points with $|z| \in (1 - \varepsilon, 1)$ and restrict directions V according to $|z|$, to ensure a stick (z, R, V) with $R \geq c(1 - |z|)$ crosses $\partial B(1)$. Since this result is not very important in what follows, we only present the simpler version above. The same argument just described for $\partial B(1)$ should in principle work for any sufficiently smooth curve in $\mathbb{R}^2$.

3.3 Exploration paths in SIPSS and ellipses model

The SHPSS obtained when we fix $\alpha = 2$ is the most interesting SHPSS, since it presents scale invariance. As stated earlier, we refer to this model as the scale-invariant Poisson stick soup, or SIPSS. In this section we introduce exploration paths and the machinery we will employ in our results about SIPSS.

3.3.1 The space of curves

We use the same definition of curves as in Aizenman and Burchard [2]. We refer to [2] for a more complete description.

Definition 3.3.1 (Space of Curves)**.** We denote by $\mathcal{S}_\Lambda$ the set of continuous functions from $[0,1]$ to a closed subset $\Lambda \subset \mathbb{R}^2$ modulo reparametrizations. We endow $\mathcal{S}$ with the metric

$$d(\mathcal{C}_1, \mathcal{C}_2) = \inf_{\psi_1, \psi_2} \sup_{t \in [0,1]} |f_1(\psi_1(t)) - f_2(\psi_2(t))|$$

where the infimum is over all continuous monotone bijections ψ_1 and ψ_2 on $[0,1]$. In this case, $(\mathcal{S}_\Lambda, d)$ is a complete separable metric space, see [2].

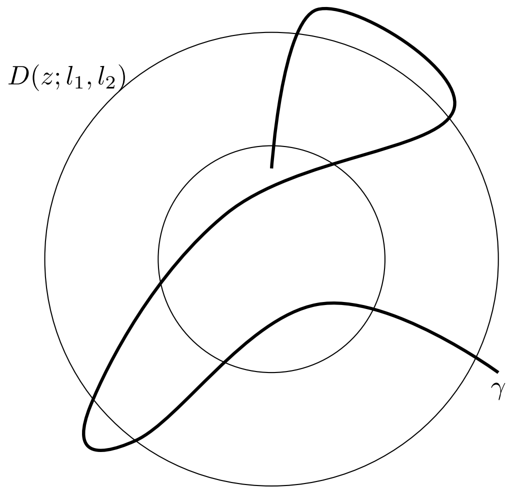


Figure 3.1: Traversing annulus $D(z;l_1,l_2)$ with 5 separate segments of γ.

There are some useful ways to quantify the regularity of a curve $\mathcal{C}$. In [2] there are four different quantities used for this purpose: Hausdorff dimension, upper box dimension, tortuosity and Hölder exponent, denoted by $\dim_{\mathcal{H}}(\mathcal{C}), \dim_B(\mathcal{C}), \tau(\mathcal{C})$ and $\alpha(\mathcal{C})$, respectively. They satisfy the relation

$$\dim_{\mathcal{H}}(\mathcal{C}) \leq \dim_B(\mathcal{C}) \leq \tau(\mathcal{C}) = \alpha(\mathcal{C})^{-1}$$

and a sufficient condition for $\dim_B(\mathcal{C}) = \tau(\mathcal{C})$ to hold is that the curve $\mathcal{C}$ has what Aizenman and Burchard call the *tempered crossing property*, a property that restricts how wiggly a curve can be at small scales.

The main purpose of paper [2] is to develop tools to study regularity and tightness of a family of random curves. Let us define for $z \in \mathbb{R}^2$ and $0 < l_1 < l_2$ the annulus $D(z;l_1,l_2) := \{w \in \mathbb{R}^2; l_1 < |z-w| \leq l_2\}$. When z is the origin, we simply write $D(l_1,l_2)$. We refer to $\partial B(z,l_1)$ and $\partial B(z,l_2)$ as the internal and external boundaries of $D(z;l_1,l_2)$.

Definition 3.3.2. Given an annulus $D(z;l_1,l_2)$ and a random continuous curve γ, let $f:[0,1] \to \mathbb{R}^2$ be a parametrization of γ. We say that $D(z;l_1,l_2)$ is *traversed by k separate segments of* γ if there are k disjoint intervals $[t_1^i, t_2^i] \subset [0,1]$ for $i = 1,2,\ldots,k$ such that for every i it holds $f(t_1^i)$ belongs one of the boundaries $\partial B(z,l_j)$, $f(t_2^i)$ belongs to the other boundary of $D(z;l_1,l_2)$ and $|f(t)-z| \in (l_1,l_2)$ for $t \in (t_1^i, t_2^i)$. (see Figure 3.1)

If we consider a family $\{\gamma_r\}_{r\in(0,1]}$ of random curves their *Hypothesis H1* is

Hypothesis H1. For all $k < \infty$ and for all annulus $D(z;l_1,l_2)$ with $r \leq l_1 \leq l_2 \leq 1$ we have uniformly in r that

$$P\left[\begin{array}{l} D(z;l_1,l_2) \text{ is traversed by } k \text{ separate} \\ \text{segments of curve } \gamma_r \end{array}\right] \leq K_k \left(\frac{l_1}{l_2}\right)^{\eta(k)} \tag{3.12}$$

for some $K_k < \infty$ and $\eta(k) \to \infty$ as $k \to \infty$.

A family of random curves that satisfies Hypothesis H1 has many useful properties, which we list in Theorem 3.3.3 below. Before that, we need more definitions. We say that a sequence of random variables $\{X_r\}_{0<r\leq 1}$ is stochastically bounded as $r \to 0$ if for every $0 < r \leq 1$ and $\varepsilon > 0$ there is $a < \infty$ with $P[|X_r| \geq a] \leq \varepsilon$. Also, recall that a family of probability measures on a complete separable metric space is relatively compact with respect to convergence in distribution if and only if it is tight (see [4]). With this, we can state a summary of Theorems 1.1 and 1.2 from [2]:

Theorem 3.3.3 (Aizenman and Burchard [2]). *Suppose $\{\gamma_r\}_{0<r\leq 1}$ is a family of curves satisfying Hypothesis H1* (3.12). *For any $\varepsilon > 0$ the curves γ_r have parametrizations $f_r : [0,1] \to \mathbb{R}^2$ satisfying for $0 \leq t_1 \leq t_2 \leq 1$ that*

$$|f_r(t_2) - f_r(t_1)| \leq \kappa_{\varepsilon,r}(\xi) g(\text{diam } \gamma_r)^{1+\varepsilon} |t_1 - t_2|^{1/(2-\eta(1)+\varepsilon)}$$

with random variable $\kappa_{\varepsilon,r}$ stochastically bounded as $r \to 0$ and $g(t) = t^{-\eta(1)/(2-\eta(1))}$. If family $\{\gamma_r\}$ is supported on $\mathcal{S}_\Lambda$ for Λ compact then their probability laws on $(\mathcal{S}_\Lambda, d)$ are tight. Moreover, in this case any limiting probability distribution $\mathcal{C}$ on $\mathcal{S}_\Lambda$ is supported on paths with $\dim_B \mathcal{C} = \alpha(\mathcal{C})^{-1} \leq 2 - \eta(1)$ *and* $\dim_{\mathcal{H}} \mathcal{C} \leq 2 - \eta(2)$.

In our Theorem 3.4.20 we prove Hypothesis H1 (3.12) for a certain family of curves $\{\gamma_r^0\}$, which are exploration paths on SIPSS. As a consequence of Theorem 3.3.3, this implies that the family $\{\gamma_r^0\}$ is tight and has some regularity bounds. This is the main result of this chapter and was already sketched in Theorem 1.2.4. We end this section stating a more precise version of Theorem 1.2.4. Its proof is immediate from Theorem 3.4.20 and Theorem 3.3.3.

Theorem 3.3.4. *Let $u \in (0, \bar{u})$ and consider a SIPSS $\xi = \xi^u$ and a box $B \subset \mathbb{R}^2$. The family of exploration curves $\{\gamma_r^0\}_{0<r\leq 1}$ in box B a.s. has, for any $\varepsilon > 0$, parametrizations $f_r : [0,1] \to \mathbb{R}^2$ satisfying for $0 \leq t_1 \leq t_2 \leq 1$ that*

$$|f_r(t_2) - f_r(t_1)| \leq \kappa_{\varepsilon,r,B}(\xi) |t_1 - t_2|^{1/(2-c(u)+\varepsilon)}$$

with random variable $\kappa_{\varepsilon,r,B}$ stochastically bounded as $r \to 0$ and $c(u) > 0$, implying that the family $\{\gamma_r^0\}$ is tight. Moreover, any subsequential limit $\mathcal{C}$ is supported on curves of $\mathcal{S}_B$ with $\alpha(\mathcal{C})^{-1} =$ $\dim_B \mathcal{C} \leq 2 - c(u)$ *and* $\dim_{\mathcal{H}} \mathcal{C} \leq 2 - 2c(u)$.

3.3.2 Defining box-crossing in SIPSS

Let us discuss how one can define the event 'there is a vacant crossing of a box' in a SHPSS. Recall that Proposition 3.2.1 proves SHPSS processes with $\alpha \neq 2$ are too unbalanced; either there are too many large sticks (case $\alpha < 2$) or too many small sticks (case $\alpha > 2$), preventing any possibility of a vacant crossing. So, we study vacant box-crossing on SIPSS.

A first attempt might be to find a curve contained in $\mathcal{V}_0$ that makes the crossing. However, this is not convenient (see Remark 3.3.9).

Definition 3.3.5 (Crossing of a segment by a curve). Given a segment $[z, w] \subset \mathbb{R}^2$, notice that the infinite line supporting $[z, w]$ divides the plane into two open half-spaces. We say a curve $\mathcal{C}$ crosses $[z, w]$ if for some parametrization f there are $0 < t_1 \leq t_2 < 1$ satisfying $f([t_1, t_2]) \subset (z, w)$ and for some $\delta > 0$ we have $f((t_1 - \delta, t_1))$ and $f((t_2, t_2 + \delta))$ are contained in different open half-spaces.

Remark 3.3.6. Clearly, the property of crossing a segment is independent of parametrization.

Definition 3.3.7 (Vacant crossing for SIPSS). Fix a rectangular box $B \subset \mathbb{R}^2$ parallel to the coordinate axes. For a SIPSS of intensity u, we define $\overline{LR}_r(B)$ as the event that there is a curve $\mathcal{C}$ contained in B with one endpoint on the left side and the other on the right side of B that do not cross any stick from ξ_r.

Notice that for any $r > 0$ the event $\overline{LR}_r$ is measurable, by the same reasoning of Remark 2.7.2. It is not clear yet that $\overline{LR}_0$ is measurable. We would like to say $\overline{LR}_0 = \cap_{r>0} \overline{LR}_r$, noticing that events $\overline{LR}_{r_1} \subset \overline{LR}_{r_2}$ for $r_1 \leq r_2$. A step in this direction is

Proposition 3.3.8. *Let $\mathcal{C}_n$ be a sequence of curves that do not cross $[z, w]$ and suppose that $\mathcal{C}_n$ converge to $\mathcal{C}$ in the metric d. Then, the limiting curve does not cross $[z, w]$.*

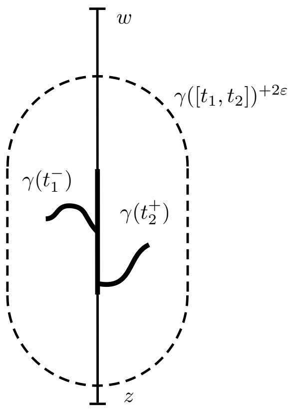


Figure 3.2: Construction on Proposition 3.3.8

Proof. Since $\mathcal{C}_n$ converges to $\mathcal{C}$, we have a sequence of parametrizations γ_n of the curves $\mathcal{C}_n$ such that $\gamma_n \to \gamma$ uniformly, where γ is a parametrization of $\mathcal{C}$. Suppose by contradiction that $\mathcal{C}$ crosses $[z, w]$. Then, there are $0 < t_1 \leq t_2 < 1$ and $\delta > 0$ such that $\gamma([t_1, t_2]) \subset (z, w)$ and the sets $\gamma((t_1 - \delta, t_1))$ and $\gamma((t_2, t_2 + \delta))$ are contained in different open half-spaces created by $[z, w]$.

Define $\varepsilon = \frac{1}{4} \operatorname{dist}(\gamma([t_1, t_2]), \{z, w\})$. Decreasing δ if needed, we can assume that

$$\gamma([t_1 - \delta, t_2 + \delta]) \subset \gamma([t_1, t_2])^{+\varepsilon} =: \{y \in \mathbb{R}^2;\ \operatorname{dist}(y, \gamma([t_1, t_2])) \leq \varepsilon\}.$$

Let $t_1^- = t_1 - \delta$ and $t_2^+ = t_2 + \delta$. By uniform convergence we have for $t \in [t_1^-, t_2^+]$ and n large enough that

$$\operatorname{dist}(\gamma([t_1, t_2]), \gamma_n(t)) \leq \operatorname{dist}(\gamma([t_1, t_2]), \gamma(t)) + \operatorname{dist}(\gamma(t), \gamma_n(t)) \leq 2\varepsilon,$$

which implies $\gamma_n([t_1^-, t_2^+]) \subset \gamma([t_1, t_2])^{+2\varepsilon}$ for all large n. Notice that $[z, w]$ divides $\gamma([t_1, t_2])^{+2\varepsilon}$ into two connected components. Taking n sufficiently large, we can ensure that $\operatorname{dist}(\gamma_n(t_1^-), \gamma(t_1^-))$ is so small they are in the same half-space, and the same reasoning applies for $\operatorname{dist}(\gamma_n(t_2^+), \gamma(t_2^+))$. Since $\mathcal{C}_n$ does not cross $[z, w]$ by hypothesis, we have a contradiction. $\square$

Remark 3.3.9. Take any decreasing sequence $\varepsilon_n \downarrow 0$. Notice that if $\mathcal{C}_n$ is a sequence of curves contained in $\mathcal{V}_{\varepsilon_n}$ that converges to some curve $\mathcal{C}$, then it is possible that $\mathcal{C} \cap \mathcal{E}_0 \neq \varnothing$. For that reason, when looking for vacant crossings we allow $\mathcal{C}$ to intersect but not cross $\mathcal{E}_0$.

An easy consequence of Proposition 3.3.8 is

Corollary 3.3.10. *Let $\mathcal{C}_n$ be a sequence of curves that do not cross any stick of a collection $\mathcal{E}^{(n)} = \{[z_i, w_i]\}$. Also, suppose that $\mathcal{C}_n$ converge to $\mathcal{C}$ in the metric d and that $\mathcal{E}^{(n)}$ is increasing. Then, the limiting curve does not cross any stick of $\mathcal{E} = \cup_n \mathcal{E}^{(n)}$.*

By definition, on event $\overline{LR}_r(B)$ there is certainly some curve $\mathcal{C}_r$ that makes the crossing of B without crossing sticks in ξ_r. However, it is possible that we are on event $\cap_{r>0} \overline{LR}_r(B)$ but not on event $\overline{LR}_0(B)$. There could be a configuration with curves $\mathcal{C}_r$ that do not converge to a curve when $r \to 0$. This phenomenon is related to the fact that the intersection of non-empty compact path-connected sets in the plane is a continuum, i.e., compact and connected, but is not necessarily path-connected. For a concrete example, consider the topologist's sine curve

$$K := \{(x, \sin(1/x));\ 0 < x \leq 1\} \cup (\{0\} \times [-1, 1])$$

and notice $K = \cap_{r>0} K^{+r}$, where $K^{+r} := \{y \in \mathbb{R}^2;\ \mathrm{dist}(y, K) \le r\}$. For box $B = [0,1] \times [-1,1]$ we can easily build a path connecting its left and right sides entirely contained in $B \cap K^{+r}$ for a fixed $r > 0$. However, there is no path connecting $\{0\} \times [-1,1]$ to $(1, \sin 1)$ contained in K.

If we are able to prove that almost surely there are among these curves $\mathcal{C}_r$ a convergent subsequence when $r \to 0$ then stating $\overline{LR}_0 := \cap_{r>0} \overline{LR}_r$ will be a good definition. In the next subsections we introduce exploration paths, a very natural family of curves to look for this property. After we prove Theorem 1.2.4 to this family of curves, we can conclude that a.s. every configuration in $\cap_{r>0} \overline{LR}_r$ is in $\overline{LR}_0$, implying $\overline{LR}_0$ is measurable in the completion of the original probability measure.

3.3.3 Exploration paths

Let us introduce the exploration path, a curve that follows the interface between covered and vacant sets. Our aim is to verify whether there is a vacant left-right crossing of a box $B_\infty(l;k)$ or not in a fixed configuration. We discuss exploration processes for both the ellipses model and the SIPSS. It is useful to build both of them on a common framework. Recall our notation ξ_0 for the PPP of intensity measure $u\lambda \otimes \phi_2 \otimes \nu$ from which we build a SIPSS of intensity u. Also, recall ξ_r is the restriction of ξ_0 to sticks with radius greater than r; it is a PPP of intenstity $u\mu_{2,r} = u\lambda \otimes \phi_2 \mathbb{1}\{x \ge r\}\, \mathrm{d}x \otimes \nu$. In this way, we have a coupling for all ξ_r with $r \ge 0$.

Fix some $r > 0$. Let $\mathcal{E}_r$ be the union of sticks $E_0(s)$ for $s \in \operatorname{supp} \xi_r$. If we take $0 < l \le r$ and make the union with ellipses $E_l(s)$ for $s \in \operatorname{supp} \xi_r$, we obtain a set $\mathcal{E}_r^l$ that is some homothety of an ellipses model. This construction, which we call the *standard coupling*, is a very natural way of comparing a SIPSS and some homothety of ellipses model. We mainly want to compare $\mathcal{E}_r$ and $\mathcal{E}_r^r$.

On this framework, let us define exploration paths for $\mathcal{E}_r^l$ where $l \in (0, r]$. Fix a box B (with sides parallel to xy axes, without loss of generality) and consider that the bottom side of B is covered, while its left side is vacant. The exploration path γ_r^l is a random path defined for each configuration of the ellipses model. It begins at the lower left corner of B and follows the interface between the covered set $\mathcal{E}_r^l$ and its complement, keeping the covered region always at its right. The path is allowed to walk over the left and bottom sides of the box and also over the arcs of ellipses intersecting the box until it meets either the top side or the right side of B (see Figure 3.3). Since we know that for any $l \in (0, r]$ almost surely there is only a finite number of ellipses in $\mathcal{E}_r^l$ intersecting B, the exploration paths γ_r^l are well-defined. Also, we can parametrize curve γ_r^l to ensure it has no self-intersections.

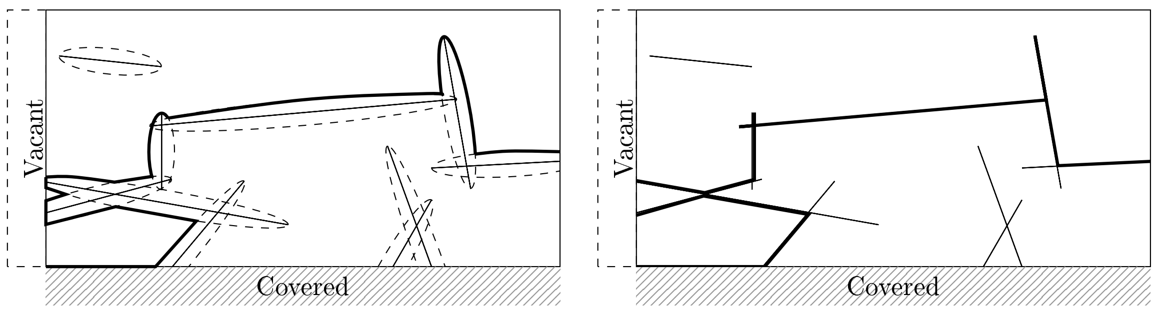


Figure 3.3: Exploration paths for an ellipses model and its stick soup. The latter exploration path can be seen as an exploration path for ellipses of minor axis size equal to zero and can have self intersections.

Following the exploration path γ_r^l, we have only two possible outcomes. If γ_r^l ends in the right side of B, then there is a left-right vacant crossing of B. On the other hand, if γ_r^l ends in the top side there is a top-bottom covered crossing of the box. Intuitively, we are looking for the lowest crossing of the box. Notice also that curves γ_r^l almost never touch the interior of any ellipse of the soup, with the only exceptions being when it walks over the left side of B. If we consider the subpath of γ_r^l that starts at the last time γ_r^l touches the left side of B, then we obtain the lowest crossing.

3.3.4 Exploration paths for SIPSS

It is not clear how we can define a notion of exploration path directly on SIPSS. However, if we consider instead of the full soup $\mathcal{E}_0$ its restriction for sticks having radius greater than some $r > 0$, ie. $\mathcal{E}_r$, we can apply essentially the same definition.

We can see the exploration path γ_r^0 as a degenerate case of the ellipses model exploration path γ_r^l, in which $l = 0$. Indeed, it can be shown that exploration paths γ_r^l converge to a curve when $l \downarrow 0$. This claim is quite easy to believe and we do not provide a formal proof, just discuss why the result is true. If you fix a finite configuration of sticks (s_i) in B and take

$$l < \min\{\operatorname{dist}(E_0(s_i), E_0(s_j));\ E_0(s_i) \cap E_0(s_j) = \varnothing\ \forall i \neq j\},$$

then the covered connected components of $\cup_i E_0(s_i)$ and $\cup_i E_l(s_i)$ are close to one another (check Figure 3.3).

Notice that curve γ_r^0 obtained in the limit can touch the left sides of B and can walk over its bottom side and also over the sticks intersecting the box. Moreover, similarly to paths γ_r^l with $l > 0$, curve γ_r^0 cannot cross any stick of the soup in general, except sticks touching the left side of B (recall Definition 3.3.5). Even when γ_r^0 crosses some stick of the soup, if we consider the part of γ_r^0 starting from the last visit to the left side of B, we obtain a legitimate path that do not cross any stick of the soup.

As exploration paths γ_r^l with $l > 0$, if γ_r^0 ends on the top side, we can conclude that there was no left-right vacant crossing of the box, even in the non-truncated SIPSS. On the other hand, if γ_r^0 ends on the right side, we can only conclude there was a crossing when ignoring sticks with radius smaller than r, i.e., that event $\overline{LR}_r(B)$ happened.

Our goal now is to ensure that the families of curves $\{\gamma_r^0\}_{0<r\leq 1}$ satisfy Hypothesis H1 (3.12) of Aizenman and Burchard, which implies this family of curves is tight. Notice that any subsequential limit is a random curve γ that does not cross any sticks of the soup with the only possible exception of sticks intersecting the left side of B, by Proposition 3.3.10. Hence, curve γ must have a subcurve that crosses B from left to right without crossing any sticks of the soup. This would justify our claim that $\overline{LR}_0(B)$ is the same event as $\cap_{r>0}\overline{LR}_r(B)$ apart from a zero measure set. This goal is accomplished in Theorems 3.4.20 and 3.3.4.

Remark 3.3.11. It is also interesting to ensure that family $\{\gamma_r^r\}_{0<r\leq 1}$ also satisfies Hypothesis H1 (3.12). Possibly the same argument used for $\{\gamma_r^0\}_{0<r\leq 1}$ could be adapted to this other family of curves. However, we are still not able to fill in the details. In a more ambitious line of work, one could reasonably expect that in the limit curves γ_r^r and γ_r^0 get arbitrarily close almost surely.

3.3.5 Properties of exploration paths

In this subsection we collect some useful facts about exploration paths γ_r^l. In order to prove some properties of exploration paths, we introduce notation for paths and concatenation. If $p, q \in \gamma$, we denote by $p\gamma q$ the curve from p to q following γ. For points p, q in the boundary of a convex $C \subset \mathbb{R}^2$ with non-empty interior, we denote by pCq the clockwise arc of ∂C from p to q and by pC^-q the

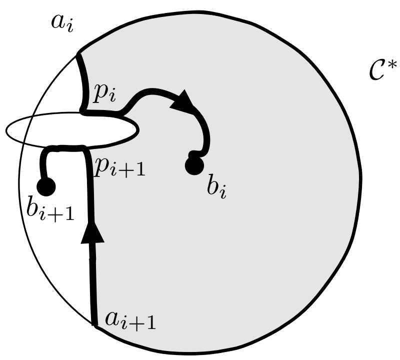


Figure 3.4: Proving Property 2; if $p_iEp_{i+1} \cap \partial_1 D = \varnothing$, then b_i and b_{i+1} are separated by $\mathcal{C}^*$.

anti-clockwise arc from p to q. Concatenation of paths is represented by combining these notations, eg. $a\gamma bEc$.

Property 1: Any fixed ellipse is touched in clockwise order by the exploration path. Indeed, fix $0 < l \leq r$, let E be some ellipse that intersects B and $f : [0,1] \to \mathbb{R}^2$ be an injective parametrization of γ_r^l. Suppose that for $0 \leq t_1 < t_2 \leq 1$ we have $f(t_1), f(t_2) \in \partial E$ and $f((t_1,t_2)) \subset \mathbb{R}^2 \backslash E$. If we consider the Jordan curve $\mathcal{C} = f(t_1)\gamma f(t_2)E^- f(t_1)$, then $f((t_2,1])$ cannot intersect $\mathcal{C}$ or its interior. This implies that if f returns to ∂E, it must be on a point in $f(t_2)Ef(t_1)$.

On the course of proving Hypothesis H1, we will be interested in checking if exploration path γ_r^l traverses a fixed annulus $D = D(z; l_1, l_2)$ many times or not, as in Definition 3.3.2. Define $\partial_i D := z + \partial B(l_i)$ for $i = 1, 2$.

Definition 3.3.12. Let $f : [0,1] \to \mathbb{R}^2$ be an injective parametrization of $\gamma = \gamma_r^l$ with $f(0)$ being the lower left corner of B. We say a segment of the curve $f([t_1,t_2])$ with $0 \leq t_1 < t_2 \leq 1$ is an *entering arm* for D if $f(t_1) \in \partial_2 D$, $f((t_1,t_2))$ is in the interior of D and $f(t_2) \in \partial_1 D$. Interchanging the roles of $\partial_1 D$ and $\partial_2 D$ we have the definition of a *leaving arm* for D. An ellipse E is used by an entering (or leaving) arm $f([t_1,t_2])$ if their intersection is non-empty.

Fix $D = D(z; l_1, l_2)$ and some ellipse E of the soup with $D \cap E \neq \varnothing$. Almost surely, we have that E is not tangent to $\partial_1 D$. We denote the i-th entering arm for D that use E by $\mathcal{P}_i$. Entering arms $\mathcal{P}_i$ intersect $\partial_2 D$ and $\partial_1 D$ in a_i and b_i, respectively, and let p_i be the first visit of $\mathcal{P}_i$ to E (see Figure 3.5).

Property 2: If $\mathcal{P}_i$ and $\mathcal{P}_{i+1}$ exist then $\partial_1 D$ must intersect p_iEp_{i+1} and $p_{i+1}Ep_i$.

Proof. Suppose $\mathcal{Q}_i = p_iEp_{i+1}$ does not intersect $\partial_1 D$. If $\mathcal{Q}_i$ intersects $\partial_2 D$, let q be the first point of $\partial_2 D$ that intersects $\mathcal{Q}_i$. Then, $\mathcal{C} = a_i\gamma p_iEq\partial_2 D^- a_i$ is a Jordan curve that does not intersect $\partial_1 D$. Notice that b_i is a point of $\partial_1 D$ inside $\mathcal{C}$ and b_{i+1} is a point of $\partial_1 D$ outside $\mathcal{C}$. Since $\partial_1 D$ is path-connected it must cross $\mathcal{C}$, contradiction. The case in which $\mathcal{Q}_i \cap \partial_2 D$ is empty is similar. Notice that curve $\mathcal{C}^* = a_i\gamma p_iEp_{i+1}\gamma a_{i+1}\partial_2 D^- a_i$ cannot intersect $\partial_1 D$ and separates $b_i \subset \mathrm{int}\mathcal{C}^*$ from b_{i+1}, contradiction (see Figure 3.4). A similar reasoning implies $\partial_1 D$ intersects $p_{i+1}Ep_i$. $\square$

Lemma 3.3.13. *Fix $0 \leq l \leq r$ and consider the exploration path $\gamma = \gamma_r^l$. If an ellipse (or stick if $l = 0$) E is used in an exploration arm of γ entering $D(z; l_1, l_2)$ then it can only be used again in a new exploration arm entering the same annulus if $\partial_1 D \backslash E$ has two connected components. Moreover, E cannot be used by three entering exploration arms.*

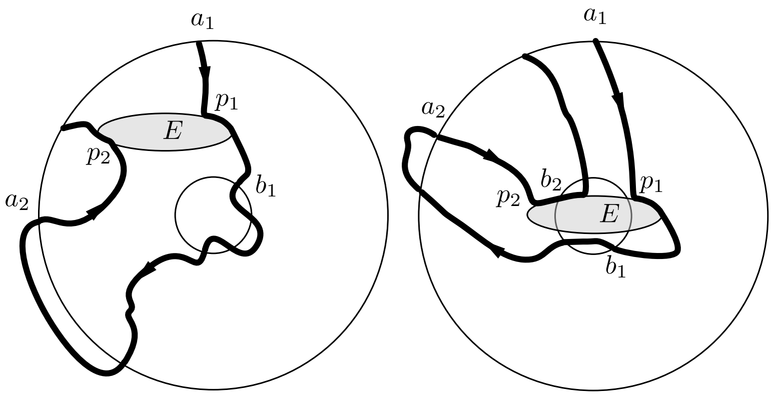


Figure 3.5: An ellipse E can be used at most twice by entering exploration paths.

Proof. Fix $l \in (0, r]$. Notice that p_i cannot be inside $\partial_1 D$ since then $a_i \gamma p_i$ would intersect $\partial_1 D$. Then, the lemma follows from Property 2 and the fact that $B(l_1) \backslash E$ can have at most 2 connected components. The case $l = 0$ can be deduced by making $l \downarrow 0$, since $\gamma_r^l \to \gamma_r^0$. □

3.3.6 Box-crossing and percolation on SIPSS

We now focus on the question of whether $\mathcal{V}_0$ or $\mathcal{E}_0$ percolate in a SIPSS of intensity u. By paper [23], we already know that for small densities u the carpet is non empty. For completeness, we provide two proofs of this result. The first is a straightforward adaptation of Theorem 1.1.3 and the second is based on a theorem of Popov and Vachkovskaia for multi-scale Poisson stick model [26]. Let us recall Proposition 1.2.3:

Proposition 1.2.3. *Let $\xi = \xi^u$ be a SIPSS of intensity u. There is $\bar{u} > 0$ such that if $u \in (0, \bar{u})$ then for any fixed $k > 0$ and $l > 0$*

$$\delta \leq P\left[\cap_{r>0} \overline{LR}_r(B_\infty(l; k))\right] \leq 1 - \delta, \tag{3.13}$$

where $\delta = \delta(u, k) > 0$. Moreover, for $u \in (0, \bar{u})$ we have:

$$P[\textit{neither } \mathcal{V}_0 \textit{ nor } \mathcal{E}_0 \textit{ percolate}] = 1. \tag{3.14}$$

Proof. We simply follow the proof of Theorem 1.1.3. Since our model is already scale-invariant, we can fix $l = 1$. The argument in Lemma 2.7.3 that reduces the problem to studying $k = 2$ is solely based on FKG inequality and hence also holds. Proposition 3.2.6 provides the same bounds for the probability of crossing a box with one stick that Proposition 2.5.4 does for ellipses, implying $P\left[\cap_{r>0} \overline{LR}_r(B_\infty(l; k))\right] \leq 1 - P[LR_1(kl; 1/k)] \leq 1 - \delta(u, k)$.

For the lower bound, we make the same coupling with fractal percolation that was described in the removal process. For each $n \in \mathbb{N}$ we partition our original box $B = B_\infty(1; 2)$ into square boxes of side 2^{-n} and denote this family of boxes by $(B_z^n)_{z \in \Lambda_n}$. Let also $I_n := (2^{-(n+1)}, 2^{-n}]$. For each n we define the random field

$$X_z^n := \mathbb{1}\{\xi_0(s; R \in I_n,\ E_0(s) \cap B_z^n \neq \varnothing) = 0\}$$

and notice that once again $P[X_z^n = 1] \geq \exp[-cu]$ for some universal positive constant. Thus, applying the results of Liggett, Schonmann and Stacey we obtain for each n a product random field

$(Y^n_z)_{z\in\Lambda_n}$ that is dominated by $(X^n_z)_{z\in\Lambda_n}$ and $P[Y^n_z = 1] = \beta(u) \to 1$ as $u \to 0$. Recall the sets

$$A_0 := B_\infty(l;2) \text{ and } A_n := A_{n-1} \cap \Big(\bigcup_{\substack{z\in\Lambda_n;\\ Y^n_z=1}} B^n_z \Big)$$

represent the n-th step of construction of fractal percolation model with parameter $\beta(u)$. Event $\{\,\xi_0(s; E_0(s) \cap B \neq \varnothing, R > 1) = 0\,\} \cap \{A_n \text{ crosses } B \text{ from left to right}\}$ is contained on event $\underline{LR}_{2^{-n}}(B)$ for every n. Applying Theorem 1 of Chayes, Chayes and Durrett [7], we know that if we take $\bar{u}$ sufficiently small and $u \in (0, \bar{u})$ then $\cap_{n\in\mathbb{N}}\{A_n \text{ crosses } B \text{ from left to right}\}$ happens with positive probability, implying $P[\cap_{r>0}\underline{LR}_r(B)] \geq \delta(u)$.

Finally, the proof of (3.14) is the same as its ellipses counterpart, since the bounds for decay of correlations in Lemma 3.2.8 and crossing a box with one stick in Proposition 3.2.6 are the same. □

Second proof of Proposition 1.2.3. We describe a particular case of Theorem 1 of [26]. Consider a PSS $U^{(0)}$ with uniform direction and sticks of radius 1 as level-0 of the process; take the intensity of this PPP to be η. Fix $M > 1$ and let level-i of the process be a PSS with uniform direction and radius M^{-i} and intensity ηM^{2i}; that means applying an homothety of ratio M^{-i} to an independent copy of $U^{(0)}$. If $U^{(i)}$ denotes the union of all level-i sticks, we are interested in $U = \cup_{i=0}^{\infty} U^{(i)}$. Their Theorem 1 states that if intensity η is subcritical for $U^{(0)}$ and $M = M(\eta)$ is chosen sufficiently large, the covered set of U does not percolate.

Let us compare their model with our PPP on S with intensity $u\lambda \otimes \phi_2(\mathrm{d}x)\mathbb{1}\{x < 1\} \otimes \nu$. Notice that we have for any $K > 1$:

$$u\phi_2 \otimes \nu(R \in (K^{-(i+1)}, K^{-i}]) = u(K^2 - 1)K^{2i}.$$

Take level-i to be a PSS with uniform direction, sticks of size M^{-i} and intensity ηM^{2i}. In order for U to dominate $u\lambda \otimes \phi_2(\mathrm{d}x)\mathbb{1}\{x < 1\} \otimes \nu$ we only need to ask:

$$\eta M^{2i} \geq u(K^2 - 1)K^{2i} \text{ and } M^{-i} \geq K^{-i}, \ \forall i \in \mathbb{N}. \tag{3.15}$$

Notice that in order to satisfy both relations on (3.15) for all i, we need to have $K = M$. First choose $\eta < u_c(1)$ and find $M = M(\eta)$ such that U is also subcritical. After that, choose $u \leq \eta[M^2 - 1]^{-1}$. Then, the domination is valid and the SIPSS of intensity u is subcritical, when we restrict it to sticks smaller than one. Since ξ_1 is independent of $\xi_{<1}$ and

$$P[\xi_1(s; E_0(s) \cap B_\infty(1;2) \neq \varnothing) = 0] \geq \exp[-uc] > 0$$

for some positive c, we are done. □

Remark 3.3.14. Proposition 1.2.3 ensures that for $u \in (0, \bar{u})$ we have Subcriticality Assumption 2 and that it implies almost all clusters are bounded. If $u_c := \sup\{u;\ \mathcal{E}_0 \text{ does not percolate}\}$, then we have $\bar{u} \leq u_c$. Nacu and Werner argue that if $u < u_c$ then Subcriticality Assumption 2 should hold, since we could follow the boundary of a bounded covered cluster surrounding the origin, which we know to exist almost surely. In this case, we would conclude that $\bar{u} = u_c$.

3.4 Hölder regularity of exploration paths

In this section we develop our proof that exploration paths γ^0_r satisfy Hypothesis H1 (3.12), implying this sequence of curves is tight and useful information on their regularity. The proof is divided into two steps. First, we prove bounds on covered 1-arm events for SIPSS. The second step is to relate the existence of k-arms in a path γ^0_r with the existence of a positive proportion (independent of r) of disjoint covered arms using sticks in ξ_r; this allows to use BK inequality to conclude the result.

3.4.1 1-Arm events for SIPSS

In this subsection, we estimate the probability of covered 1-arm events for SIPSS. This bound is new, to the best of our knowledge. As we mentioned in Section 3.3, Nacu and Werner in [23] prove a power law (Corollary 8) for 'vacant' 1-arms in a general scale-invariant soup of random curves. Let A_ϵ be the event in which there is a path connecting the inner and outer boundaries of $D(\epsilon, 1)$ that do not cross curves in $\Gamma_{D(\epsilon,1)}$ for SIPSS. The reason behind the power law for $P[A_\epsilon]$ is in

Lemma 3.4.1 (Lemma 7 of [23]). *For any $u > 0$ there exists $k(u) > 0$ and $R(u) > 2$ such that for any $\epsilon, \epsilon' \in (0, 1/R)$,*

$$kP(A_\epsilon)P(A_{\epsilon'/R}) \leq P(A_{\epsilon\epsilon'}) \leq P(A_\epsilon)P(A_{\epsilon'}). \tag{3.16}$$

One could try to adapt its proof for covered 1-arms. In fact, the same argument provided for the lower bound in Equation (3.16) works, as we detail in Lemma 3.4.3 below. However, the upper bound in Lemma 3.4.1 is not directly adaptable. Since Nacu and Werner disregard boundary effects when defining Γ_D, their upper bound follows from independence of finding paths that do not cross sticks in $D(\epsilon\epsilon', \epsilon)$ and in $D(\epsilon, 1)$. Notice we cannot use the same argument for covered 1-arms, since paths of sticks in the two annuli above can share sticks.

If we compare SIPSS with other percolation models in general, it is quite plausible that the covered and vacant sets of SIPSS have different behaviors since the model does not present a clear symmetry between covered and vacant sets. As a close example of this lack of symmetry in continuum percolation, we can cite reference [1]. Their Corollary 4.4 presents bounds for the probability of covered and vacant arm events in Boolean models on the plane that are unavoidably different. For some radius distributions, the probability of having a vacant arm decays polynomially but the decay for covered arms is slower. In Proposition 3.4.6 we show that for $u \in (0, \bar{u})$ the probability of having a covered 1-arm in SIPPS decays polynomially. However, the exponents in our upper and lower bounds do not coincide.

Definition 3.4.2. For $0 < l_1 < l_2$, define $\mathrm{Arm}_0(l_1, l_2)$ as the event in which there is a finite sequence of sticks of ξ_0 crossing $D(l_1, l_2)$ in SIPSS and let $C_\epsilon := \mathrm{Arm}_0(\epsilon, 1)$. Also, define $\mathrm{Circ}_0(l_1, l_2)$ as the event in which there is a finite sequence $(s_i)_{i=1}^n \in \mathrm{supp}\ \xi_0$ such that $E_0(s_i) \cap E_0(s_{i+1}) = p_i \in D(l_1, l_2)$ for all $i = 1, \ldots, n$ (with $s_{n+1} = s_1$) and the polygon $p_1, p_2, \ldots, p_n, p_1$ is a simple curve that contains $B(l_1)$ in its interior.

Notice that scale-invariance implies $P[\mathrm{Arm}_0(l_1, l_2)] = P[C_{l_1/l_2}]$. The lower bound for $P(C_{\epsilon\epsilon'})$ follows from FKG inequality and is based on the proof of Lemma 3.4.1.

Lemma 3.4.3. *Fix $u > 0$ and $R > 1$. For any SIPSS of intensity u there is $k = k_R(u) > 0$ such that for any $\epsilon, \epsilon' \in (0, 1/R)$*

$$P[C_{\epsilon\epsilon'}] \geq kP[C_\epsilon]P[C_{\epsilon'/R}]. \tag{3.17}$$

As a consequence, the limit $\beta(u) := \lim_{\epsilon \to 0} \frac{\log P[C_\epsilon]}{\log \epsilon}$ exists and belongs to $(0, 2]$. Also, it holds $P[C_\epsilon] \geq k^{-1}R^\beta \epsilon^\beta$.

Remark 3.4.4. Lemma 3.4.3 is trivial for $u > u_c$. In this case $P[C_\epsilon] = 1$ for any ϵ since the unbounded cluster is dense, almost surely.

Proof. Notice that for $\epsilon, \epsilon' \in (0, 1/R)$ one has $\epsilon\epsilon' < \epsilon < R\epsilon < 1$. Moreover, $k(u) := P[\mathrm{Circ}_0(1, R)] > 0$ and by scale-invariance we have $k(u) = P[\mathrm{Circ}_0(\epsilon, R\epsilon)]$ for any $\epsilon > 0$. Using FKG inequality we can write

$$P[C_{\epsilon\epsilon'}] \geq P[C_\epsilon, \mathrm{Circ}_0(\epsilon, R\epsilon), \mathrm{Arm}_0(\epsilon\epsilon', R\epsilon)] \geq kP[C_\epsilon]P[C_{\epsilon'/R}], \tag{3.18}$$

since all events above are increasing. Thus, if we define the function $g(\epsilon) := kP[C_{\epsilon/R}]$, it follows from equation (3.18) that

$$g(\epsilon\epsilon') = kP[C_{\epsilon\epsilon'/R}] \geq k^2 P[C_{\epsilon/R}]P[C_{\epsilon'/R}] = g(\epsilon)g(\epsilon').$$

Supermultiplicativity implies the existence of the limit

$$\tilde{\beta}(u) := \lim_{\epsilon\to 0} \frac{\log g(\epsilon)}{\log \epsilon} = \sup_{\epsilon<1/R} \frac{\log g(\epsilon)}{\log \epsilon} > 0.$$

Existence of the limit characterizing β follows from noticing

$$\beta(u) = \lim_{\epsilon\to 0} \frac{\log P[C_{\epsilon/R}]}{\log(\epsilon/R)} = \lim_{\epsilon\to 0} \left[\frac{\log g(\epsilon)}{\log \epsilon}\frac{\log \epsilon}{\log(\epsilon/R)} - \frac{\log k}{\log \epsilon}\frac{\log \epsilon}{\log(\epsilon/R)}\right] = \lim_{\epsilon\to 0} \frac{\log g(\epsilon)}{\log \epsilon} = \tilde{\beta}(u).$$

To see that $\beta(u) \leq 2$, we apply Proposition 3.2.6. Notice that for $\epsilon < 1/2$ we have

$$P[C_\epsilon] \geq P[LR_1(2\epsilon; 1/(2\epsilon))] \geq 1 - \exp[-uc(2\epsilon)^2],$$

implying that

$$\beta(u) = \lim_{\epsilon\to 0} \frac{\log P[C_\epsilon]}{\log \epsilon} \leq \lim_{\epsilon\to 0} \frac{\log(1 - \exp[-uc\epsilon^2])}{\log \epsilon} = 2.$$

Finally, by the supremum characterization of $\tilde{\beta}(u)$ we get for $\epsilon \in (0, 1/R)$ that $\log g(\epsilon) \geq \beta \log \epsilon$ and hence $f(\epsilon) = k^{-1}g(R\epsilon) \geq k^{-1}R^\beta \epsilon^\beta$. □

Definition 3.4.5. For any $r > 0$ and $0 < l_1 < l_2$ let $\mathrm{Arm}_r(l_1, l_2) := \{\mathcal{E}_r \text{ crosses } D(l_1, l_2)\}$. We also define its complement $\overline{\mathrm{Circ}}_r(l_1, l_2)$, the event in which there is a vacant circuit in $D(l_1, l_2)$ with $B(l_1)$ in its interior.

A straightforward consequence of Definition 3.4.5 is that the function $r \mapsto P[\mathrm{Arm}_r(l_1, l_2)]$ is decreasing, since by our standard coupling we have $\mathcal{E}_{r'} \supset \mathcal{E}_r$ whenever $r' \leq r$. Also, notice that $\mathrm{Arm}_0(l_1, l_2) = \cup_{r>0}\mathrm{Arm}_r(l_1, l_2)$. By this, we have

$$P[\mathrm{Arm}_r(l_1, l_2)] \leq \lim_{r'\downarrow 0} P[\mathrm{Arm}_{r'}(l_1, l_2)] = P[\mathrm{Arm}_0(l_1, l_2)]. \tag{3.19}$$

Proposition 3.4.6. *Let $u \in (0, \bar{u})$. There exists $K(u) > 0$ and $\eta(u) \in (0, 1)$ such that for any $0 < l_1 < l_2$ we have*

$$P[\mathrm{Arm}_0(l_1, l_2)] \leq K\,(l_1/l_2)^\eta\,.$$

Remark 3.4.7. Proposition 3.4.6 implies $P[C_\epsilon] \leq K\epsilon^\eta$. Thus, we have also that the limit given by β in Lemma 3.4.3 satisfies $\beta \geq \eta$.

Remark 3.4.8. Using equation (3.19) we deduce $P[\mathrm{Arm}_r(l_1, l_2)] \leq K(l_1/l_2)^\eta$, an upper bound that is uniform in $r > 0$. This uniformity is important for obtaining Hypothesis H1 for the family $(\gamma_r^0)_{r>0}$.

Proof. Using scale-invariance, we can consider only $\mathrm{Arm}_0(1, l)$ for $l > 1$. It is sufficient to prove bounds for the values $l = 2^m$ where $m \in \mathbb{N}$. Indeed, if Proposition 3.4.6 holds for these values of l then for a generic $l > 1$ we can find $m \in \mathbb{N}$ with $2^{m-1} < l \leq 2^m$ and notice

$$P[\mathrm{Arm}_0(1, l)] \leq P[\mathrm{Arm}_0(1, 2^{m-1})] \leq K2^{-(m-1)\eta} \leq (2^\eta K)l^{-\eta}. \tag{3.20}$$

Thus, we want to prove there are $K(u), \eta(u) > 0$ such that $P[\mathrm{Arm}_0(1, 2^m)] \leq K2^{-m\eta}$ for every $m \in \mathbb{N}$. Let us partition $\mathbb{R}^2\backslash\{0\}$ into disjoint annuli with ratio 2 using $A_j := D(2^{j-1}, 2^j)$, where

$j \in \mathbb{Z}$. Abusing notation a little, we denote $\mathrm{Arm}_0(2^{j-1}, 2^j)$ by $\mathrm{Arm}_0(A_j)$ and use the analogous notation for $\overline{\mathrm{Circ}}_0$.

Notice that on event $\mathrm{Arm}_0(1, 2^m)$ none of the events $\overline{\mathrm{Circ}}_0(A_j)$ for $j = 1, \dots, m$ happened. By our choice of $u \in (0, \bar{u})$ we know that there is $\delta(u) > 0$ such that

$$\delta \le P(\overline{\mathrm{Circ}}_0(A_j)^c) \le 1 - \delta, \text{ for every } j \in \mathbb{Z}. \tag{3.21}$$

If events on (3.21) were independent, the proof would be already over. However, we know by Lemma 3.2.8 that these events have a correlation that decays slowly with distance. This decay is so slow that even the techniques employed in Lemma 4.7 of [1] are not enough. If we pick some fixed subset of indices $N \subset [m]$ and use the bound $P[\mathrm{Arm}_0(1, 2^m)] \le P[\overline{\mathrm{Circ}}_0(A_j)^c$, for $j \in N]$ trying to take advantage of the decay of correlations, the best bound we can obtain is $K2^{-\eta\sqrt{m}}$ which is weaker than the one in Proposition 3.4.6. To get stronger bounds, we consider random sets of indices.

For any set $B \subset \mathbb{R}^2$ we can consider the restriction of ξ_0 in which we only look for sticks intersecting the set B. In this case, we denote the covered set observed in this restriction by $\mathcal{E}_0(B)$. Define the random variables

$$D_j := \max\{n \in \mathbb{Z}; n < j \text{ and } \mathcal{E}_0(A_j) \cap A_n = \varnothing\}. \tag{3.22}$$

In words, D_j searches the index of the first annulus smaller than A_j that was not intersected by sticks touching A_j. Since $P[0 \in \mathcal{E}_0] = 0$, we have D_j are always finite. Scale-invariance implies that all D_j are identically distributed, but clearly they are not independent. As a first computation, notice that

$$P[D_m \le 0] \le P[\exists s \in \operatorname{supp} \xi_0; E_0(s) \cap B(1) \neq \varnothing, R \ge (2^m - 1)/2] \le uc2^{-m}$$

for some constant c independent of u. Notice that this event has the decay we want. On the other hand, on event $\{D_m \ge 1\}$ we have

$$P[\mathrm{Arm}_0(1, 2^m), D_m \ge 1] \le P[\overline{\mathrm{Circ}}_0(A_{D_m})^c, D_m \ge 1] \le 1 - \delta.$$

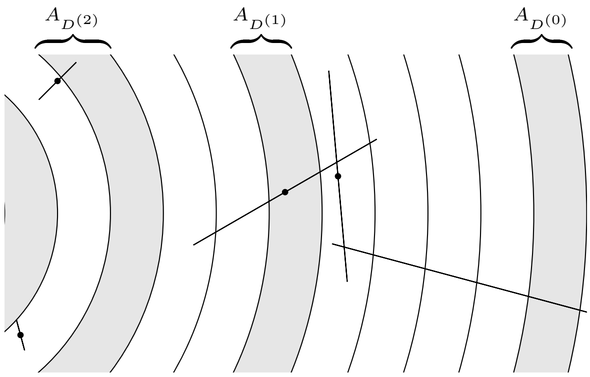


Figure 3.6: Successively discovering new tries for vacant circuits. Annuli that were used are colored in gray. Conditionally on $A_{D^{(j)}}$ for $j = 1, \dots, t-1$ and all sticks intersecting them, it is easier to have a vacant circuit in $A_{D^{(t)}}$ (see Lemma 3.4.9).

The idea is then to iterate this reasoning, discovering step by step how many annuli we should discard in order to try again finding a new vacant circuit. From now on we fix $m \in \mathbb{N}$ and consider the sequence of random variables $(D^{(j)})_{j \ge 0}$ with $D^{(0)} = m$ and $D^{(j)} = D_{D^{(j-1)}}$. Sequence $D^{(j)}$ is clearly decreasing. It holds

Lemma 3.4.9. *For every $t \in \mathbb{N}$ one has $P[\cap_{j=1}^{t} \overline{\mathrm{Circ}}_0(A_{D^{(j)}})^c] \leq (1-\delta)^t$.*

Proof. Let $\mathcal{F}_0$ be the smallest σ-algebra that makes $\xi_0(A_{D^{(0)}})$ measurable. Inductively, for every $t \in \mathbb{N}$ we can define $\mathcal{F}_t$, the smallest σ-algebra that makes $\xi_0(A_{D^{(j)}})$ for $j = 1, \ldots, t$ all measurable. Notice that $D^{(t)}$ is $\mathcal{F}_{t-1}$ measurable. Also, on $\mathcal{F}_{t-1}$ when $D^{(t)} = l$ and $D^{(t-1)} = n$ the only sticks intersecting A_l that were not explored yet are the ones in $\Gamma(l,n) := \{s; E_0(s) \cap A_l \neq \varnothing, E_0(s) \cap A_n = \varnothing\}$. We have for any t that

$$
\begin{aligned}
P[\overline{\mathrm{Circ}}_0(A_{D^{(t)}})^c \,|\mathcal{F}_{t-1}] &= P\left[\bigcup_{s \in \xi_0(\Gamma(D^{(t)},D^{(t-1)}))} E_0(s) \quad \text{prevents crossing of } A_{D^{(t)}} \middle| \mathcal{F}_{t-1} \right] \\
&= \sum_{l,n} \frac{\mathbb{1}\{D^{(t)} = l, D^{(t-1)} = n\}}{P[D^{(t)} = l, D^{(t-1)} = n]} P\left[\begin{array}{r} \cup_{s\in\xi_0(\Gamma(l,n))} E_0(s) \text{ prevents crossing of } A_l, \\ D^{(t)} = l, D^{(t-1)} = n \end{array} \right] \\
&\leq \sum_{l,n} \frac{\mathbb{1}\{D^{(t)} = l, D^{(t-1)} = n\}}{P[D^{(t)} = l, D^{(t-1)} = n]} P\left[\overline{\mathrm{Circ}}_0(A_l)^c\right].
\end{aligned}
$$

Since we know $P\left[\overline{\mathrm{Circ}}_1(A_l)^c\right] \leq 1 - \delta$ for any l, after conditioning on $\mathcal{F}_{t-1}$ we have

$$
\begin{aligned}
P[\cap_{j=1}^{t} \overline{\mathrm{Circ}}_0(A_{D^{(j)}})^c] &= P[\cap_{j=1}^{t-1} \overline{\mathrm{Circ}}_0(A_{D^{(j)}})^c]\; E[P[\overline{\mathrm{Circ}}_0(A_{D^{(t)}})^c \,|\mathcal{F}_{t-1}]] \\
&\leq P[\cap_{j=1}^{t-1} \overline{\mathrm{Circ}}_0(A_{D^{(j)}})^c](1-\delta).
\end{aligned}
$$

Iterating, t times, we are done. □

Now, we consider the possibilities for how many tries we have before $D^{(t)} < 1$. Define

$$L^{(j)} = D^{(j-1)} - D^{(j)} \text{ for every } j \geq 1,$$

the random variable that counts the amount of rings discarded moving from step $j-1$ to step j. Notice that knowing the family $(L^{(j)})$ is equivalent to knowing $(D^{(j)})$. The coordinates of these processes are not independent. Noticing that the $L^{(j)}$ are identically distributed, let us denote by Y the distribution of $L^{(j)}$, i.e.,

$$Y \stackrel{d}{=} j - \max\{n; n < j \text{ and } \mathcal{E}_0(A_j) \cap A_n = \varnothing\}.$$

Applying the same reasoning of Lemma 3.4.9 we can prove a very useful stochastic domination.

Lemma 3.4.10. *Let $(Y^{(j)})_{j \geq 1}$ be a family of iid. random variables distributed as Y. Then $(Y^{(j)})$ dominates stochastically $(L^{(j)})$.*

Proof. It is sufficient to prove that for any fixed $j \in \mathbb{N}$ and any vector $(l_1, \ldots, l_j)$ with integer $l_i \geq 3$ for all i we have

$$P[\cap_{i=1}^{j} \{L^{(i)} \geq l_i\}] \leq P[\cap_{i=1}^{j} \{Y^{(i)} \geq l_i\}] = \prod_{i=1}^{j} P[Y \geq l_i].$$

Indeed, we notice that Proposition 3.2.11 implies $P[Y \geq 2] = 1$. Conditioning on the σ-algebra generated by $(D^{(i)})_{i=1}^{j-1}$, which we denote by $\mathcal{F}_{j-1}$, we have

$$P[\cap_{i=1}^{j} \{L^{(i)} \geq l_i\}] = P[\cap_{i=1}^{j-1} \{L^{(i)} \geq l_i\}]\; E[\, P[L^{(j)} \geq l_j \,|\mathcal{F}_{j-1}]\,]. \tag{3.23}$$

Besides that, we can write

$$
\begin{aligned}
E[\,P[L^{(j)} \geq l_j \,|\mathcal{F}_{j-1}]\,] &= E[\,P[\mathcal{E}_0(A_{D^{(j-1)}}) \cap A_{D^{(j-1)}-l_j+1} \neq \varnothing \,|\mathcal{F}_{j-1}]\,] \\
&= E\left[\sum_l \mathbb{1}\{D^{(j-1)} = l\} \frac{P[\mathcal{E}_0(A_l) \cap A_{l-l_j+1} \neq \varnothing, D^{(j-1)} = l]}{P[D^{(j-1)} = l]}\right].
\end{aligned}
$$

Notice that for any value of l we have

$$
P[\mathcal{E}_0(A_l) \cap A_{l-l_j+1} \neq \varnothing, D^{(j-1)} = l] \leq P[\mathcal{E}_0(A_l) \cap A_{l-l_j+1} \neq \varnothing] = P[Y \geq l_j].
$$

Substituting into equation (3.23) and iterating the argument, the result follows. □

Moving on with the proof of Proposition 3.4.6, we split the event $\mathrm{Arm}_0(1, 2^m)$ into two events according to whether we were able to find a sufficient amount of tries for vacant circuits or not. Let

$$
T := \max\{j; D^{(j)} \geq 1\} = \max\{j; L^{(1)} + \ldots + L^{(j)} \leq m - 1\}. \tag{3.24}
$$

In the next lemma, we prove that with high probability we have a linear number of tries:

Lemma 3.4.11. *For any $u \in (0, \bar{u})$ there exists positive constants $\kappa = \kappa(u)$ and $c = c(u)$ such that*

$$
P\left[\sum_{j=1}^{\lfloor \kappa m \rfloor} L^{(j)} \geq m\right] \leq \exp[-cm].
$$

Proof. Using Lemma 3.4.10, we can write for any $t \in \mathbb{N}$ that $P\left[\sum_{j=1}^{t} L^{(j)} \geq m\right] \leq P\left[\sum_{j=1}^{t} Y^{(j)} \geq m\right]$. Fix some $\theta \in (0, \log 2)$. We have for $n \geq 3$ that

$$
\begin{aligned}
P[Y \geq n] &= P[\exists s \in \mathrm{supp}\ \xi_0; E_0(s) \cap A_1 \neq \varnothing,\ E_0(s) \cap A_{2-n} \neq \varnothing] \\
&= P[\exists s \in \mathrm{supp}\ \xi_0; E_0(s) \cap \partial B(1) \neq \varnothing,\ E_0(s) \cap \partial B(2^{2-n}) \neq \varnothing] \\
&= 1 - \exp[-u\mu_2(s; E_0(s) \cap \partial B(1) \neq \varnothing,\ E_0(s) \cap \partial B(2^{2-n}) \neq \varnothing)] \\
&\leq 1 - \exp[-uc2^{-n}] \leq uc2^{-n}.
\end{aligned}
$$

where the inequality derives from equation (3.6) in Lemma 3.2.7. This implies

$$
\begin{aligned}
E[e^{\theta Y}] &= \sum_{n \geq 2} e^{\theta n} P[Y = n] \leq e^{2\theta} + \sum_{n \geq 3} e^{\theta n} P[Y \geq n] \leq e^{2\theta} + \sum_{n \geq 3} uce^{-(\log 2 - \theta)n} \\
&\leq e^{2\theta} + uc_\theta < \infty,
\end{aligned} \tag{3.25}
$$

for some constant $c_\theta > 0$. Using the exponential moment in (3.25) and Markov inequality, we can write

$$
P\left[\sum_{j=1}^{t} Y^{(j)} \geq m\right] \leq \exp[-\theta m] E[e^{\theta \sum_{j=1}^{t} Y^{(j)}}] = \exp[\,-\theta m + t \log E[e^{\theta Y}]\,].
$$

Take $t = \lfloor \kappa m \rfloor$ where $0 < \kappa < \frac{\theta}{\log(e^{2\theta} + c_\theta u)}$. We obtain

$$
P\left[\sum_{j=1}^{t} Y^{(j)} \geq m\right] \leq \exp[-\theta m + \kappa m \log(e^{2\theta} + uc_\theta)] = \exp[-m(\theta - \kappa \log(e^{2\theta} + c_\theta u))]. \qquad \square
$$

Split event $\mathrm{Arm}_0(1,2^m)$ according to whether $T < \lfloor \kappa m \rfloor$ or not. Notice that $\{T < \lfloor \kappa m \rfloor\} = \{L^{(1)} + \ldots + L^{(\lfloor \kappa m \rfloor)} \geq m\}$. Hence, we have by Lemma 3.4.11

$$\begin{aligned} P[\mathrm{Arm}_0(1,2^m)] &\leq P[T < \lfloor \kappa m \rfloor] + P[\mathrm{Arm}_0(1,2^m), T \geq \lfloor \kappa m \rfloor] \\ &\leq \exp[-cm] + P\big[\cap_{j=1}^{\lfloor \kappa m \rfloor} \overline{\mathrm{Circ}}_0(A_{D^{(j)}})^c\big] \\ &\leq \exp[-cm] + (1-\delta)^{-1} \exp[-\log(1-\delta)^{-1} \kappa m] \\ &\leq 2(1-\delta)^{-1} \exp[-[c \wedge (\kappa \log(1-\delta)^{-1})] m\ ]. \end{aligned}$$

Lemma 3.4.9 finishes the proof of Proposition 3.4.6, showing we can take $K := 2(1-\delta)^{-1}$ and

$$\eta := \frac{c(u) \wedge (\kappa \log(1-\delta)^{-1})}{\log 2} \leq \frac{c(u)}{\log 2} < \frac{\theta}{\log 2} < 1$$

in equation (3.20). □

3.4.2 BK inequality

In this section we extend the covered 1-arm bound we obtained for SIPSS in Proposition 3.4.6 to a bound involving more than one arm. The canonical way to make this passage is to apply the van den Berg-Kesten inequality [38], also known as BK inequality. This inequality provides an upper bound on the probability of disjoint occurrence of two events A and B. To put it simply, we say that events A and B occur disjointly if there are two (random) disjoint regions U and W of your space such that we can ensure A (respectively, B) happens just by looking at region U (respectively, W). A formal definition is given below.

Reference [38] introduced BK inequality only for increasing events and a extension to any two events is due to Reimer [27]. Both results are on a different context and are not directly applicable for Poisson point processes. We present an extension of BK inequality from [37], which is suitable for marked PPPs (see Lemma 3.4.15).

Reference [37] formalizes the concept of disjoint occurrence for any marked PPP on $\mathbb{R}^d$ in the following way. Denote a realization of the process by ω and notice it can be seen as a collection of pairs

$$\{(x_i, s_i);\ x_i \in \mathbb{R}^d,\ s_i \in \mathbb{M},\ i \in \mathbb{N}\},$$

where $\mathbb{M}$ denotes the space of marks. For any region $U \in \mathcal{B}(\mathbb{R}^d)$ we denote by $\omega_{\|U}$ the restriction of ω to points in U, i.e., $\omega_{\|U} := \{(x_i, s_i) \in \omega;\ x_i \in U\}$. Moreover, we also define the cylinder event $[\omega_{\|U}] := \{\omega';\ \omega'_{\|U} = \omega_{\|U}\}$, the collection of all realizations that coincide with ω in U.

Definition 3.4.12. We say that an event A is *centered* on $U \subset \mathbb{R}^d$ if whenever $\omega \in A$ we have also $[\omega_{\|U}] \subset A$. In other words, we have A is centered on U if $\mathbb{1}_A(\omega) = \mathbb{1}_A(\omega_{\|U})$.

Remark 3.4.13. In [37], instead of 'A being centered on U' the authors use that 'A lives on U'. However, this name can be misleading when dealing with SIPSS, since ellipses centered outside region $U \subset \mathbb{R}^2$ may still intersect U (see discussion below).

Definition 3.4.14 (Disjoint occurrence)**.** Let A and B be events. We define the event that A and B *occur disjointly* by

$$A \circ B := \left\{ \omega;\ \begin{array}{l} \text{there are disjoint regions } U \text{ and } W \text{ of } \mathbb{R}^d \\ \text{such that } [\omega_{\|U}] \subset A \text{ and } [\omega_{\|W}] \subset B \end{array} \right\}. \tag{3.26}$$

For ensuring measurability, we restrict regions U and W above to the family of countable unions of rectangles with rational coordinates. We also recursively define disjoint occurrence for events $(A_i)_{i=1}^k$ with $k \geq 3$:

$$\bigcirc_{i=1}^k A_i := \big(\bigcirc_{i=1}^{k-1} A_i\big) \circ A_k.$$

As we mentioned earlier, van den Berg [37] has a version of BK inequality for marked Poisson point processes. His version is restricted to increasing events; recall that A is increasing if $\omega \in A$ and $\omega \subset \omega'$ then $\omega' \in A$.

Lemma 3.4.15 (Theorem from [37]). *Let $U \subset \mathbb{R}^2$ be a bounded region and consider two increasing events A and B that are centered on U. Then, we have*

$$P[A \circ B] \leq P[A]P[B]. \tag{3.27}$$

Focusing on SIPSS model, we would like to apply BK inequality to events $A = B = \mathrm{Arm}_r(l_1, l_2)$, but this event is not centered on any bounded region of $\mathbb{R}^2$. Similarly to Definition 2.6.1 that introduces functions depending only on ellipses touching a region $K \subset \mathbb{R}^2$, let

$$\xi_K := \{s_i \in \xi;\ E_0(s_i) \cap K \neq \varnothing\}.$$

Definition 3.4.16. An event A depends only on sticks touching $K \subset \mathbb{R}^2$ if $\mathbb{1}_A(\xi) = \mathbb{1}_A(\xi_K)$.

The straightforward relation $\xi_{\|K} \subset \xi_K \subset \xi$ shows that if A is centered on K then A depends only on sticks touching K, as one would hope. A slight improvement of Lemma 3.4.15 is

Lemma 3.4.17. *Let $K \subset \mathbb{R}^2$ be a bounded region. For a SIPSS model, consider two increasing events A and B that depend only on sticks touching K. Then, we have*

$$P[A \circ B] \leq P[A]P[B].$$

Proof. Since K is bounded we have $K \subset B(n)$ for any large euclidean ball, say $n \geq n_0$. Let us define

$$\xi_{K\|B(n)} := \{s_i = (z_i, R_i, V_i) \in \xi;\ E_0(s_i) \cap K \neq \varnothing,\ z_i \in B(n)\}.$$

and for $n \geq n_0$ let $A_n := \{\xi;\ \xi_{K\|B(n)} \in A\}$. Notice that each A_n is an increasing event that is centered on $B(n)$. Moreover, events A_n are nested and satisfy $A = \cup_n A_n$. One can define events B_n in an analogous way. Notice that

$$A \circ B \subset (A_n \circ B_n) \cup \{\exists s \in \xi;\ E_0(s) \cap K \neq \varnothing,\ z \notin B(n)\}. \tag{3.28}$$

Thus, by Lemma 3.2.8 and the BK inequality from Lemma 3.4.15, we get

$$P[A \circ B] \leq P[A_n]P[B_n] + uc(n/n_0)^{-1} \to P[A]P[B] \text{ when } n \to \infty.$$ □

Notice that $\mathrm{Arm}_r(l_1, l_2)$ is an event that depends only on sticks touching $B(l_2)$. Moreover, if events A_i are increasing then $\bigcirc_{i=1}^k A_i$ is also increasing. Iterating Lemma 3.4.17 we obtain

Corollary 3.4.18. *Let $r > 0$ and $l_2 > l_1 > 0$. For any $k \in \mathbb{N}$ we have*

$$P[\bigcirc_{i=1}^k \mathrm{Arm}_r(l_1, l_2)] \leq P[\mathrm{Arm}_r(l_1, l_2)]^k.$$

3.4.3 Proof of Hypothesis H1

Having uniform bounds for the probability of covered 1-arms in truncated SIPSS and the topological property given by Lemma 3.3.13, we need to work a little bit to conclude that the random family $\{\gamma_r^0\}$ satisfies Hypothesis H1 (3.12).

Definition 3.4.19. For any $r > 0$ and $l \geq 0$, define the k-arm events for exploration path γ_r^l as

$$k\text{-EArm}_{\gamma_r^l}(z; l_1, l_2) = \{D(z; l_1, l_2) \text{ is traversed by } k \text{ separate segments of } \gamma_r^l\}.$$

Restating Hypothesis H1 for our model, we want to prove

Theorem 3.4.20 (Hypothesis H1 for SIPSS)**.** *Let $u \in (0, \bar{u})$. For all $k \in \mathbb{N}$ and $0 < l_1 < l_2$ we have uniformly in $r > 0$ and $z \in \mathbb{R}^2$ that*

$$P[k\text{-EArm}_{\gamma_r^0}(z; l_1, l_2)] \leq K_k \left(\frac{l_1}{l_2}\right)^{c(u)k} \tag{3.29}$$

for positive constants $K_k(u) < \infty$ and $c(u) > 0$.

Firstly, notice that the existence of k exploration arms in $D(z; l_1, l_2)$ implies the existence of at least $\lfloor k/2 \rfloor$ entering exploration arms. If they did not share any sticks, the proof would be a trivial application of BK inequality. However, we know by Lemma 3.3.13 that some sticks can be shared. We need some control on the number of sticks shared to ensure this will not be a problem. Proposition 3.4.22 below is a first step in this direction. For any fixed $l > 0$ we are interested in calculating

$$m_2(l) := \mu_2(s; \#E_0(s) \cap \partial B(l) = 2),$$

that is, the measure of all sticks (in full SIPSS) that intersect the boundary of $B(l)$ twice. Notice that although $\mu_2(s; \#E_0(s) \cap \partial B(l) \geq 1) = \infty$ by Proposition 3.2.11, the quantity we are interested into is indeed finite. Another comment is that the quantity above does not depend on l by a simple application of scale-invariance. Thus, we take $l = 1$.

We divide the calculation into two cases, according to whether z is inside $B(1)$ or not.

Lemma 3.4.21. *Let $z \in \mathbb{R}^2 \backslash \{0\}$ and consider a straight line through z making an angle $\tilde{V}$ with z and $|\tilde{V}| \leq \pi/2$. If $0 < |z| < 1$, any line through z intersects $\partial B(1)$ at two points. If $|z| \geq 1$, any line with $|\tilde{V}| < \arcsin \frac{1}{|z|}$ intersects $\partial B(1)$ twice. In both cases, if we denote by y_+ the distance to the farthest point of intersection, we have $y_+ = |z| \cos \tilde{V} + \sqrt{1 - |z|^2 \sin^2 \tilde{V}}$.*

Proof. It is straightforward to check for which interval in $\tilde{V}$ the line intersects $\partial B(1)$ twice. To find out the distance to the farthest intersection we simply apply the law of cosines. For $|z| \in (0, 1)$ the distances to the intersections are given by the positive roots of

$$1 = y^2 + |z|^2 \pm 2y|z| \cos \tilde{V}$$

where we notice there is one positive root for each choice of sign. For $|z| \geq 1$, the distances are the roots of $1 = y^2 + |z|^2 - 2y|z| \cos \tilde{V}$. In both cases, the largest value is

$$y_+ = |z| \cos \tilde{V} + \sqrt{|z|^2 \cos^2 \tilde{V} + 1 - |z|^2} = |z| \cos \tilde{V} + \sqrt{1 - |z|^2 \sin^2 \tilde{V}}. \qquad \square$$

We use Lemma 3.4.21 to calculate explicitly the value of $m_2(l)$.

Proposition 3.4.22. *We have for any $l > 0$ that $m_2(l) = 2\pi$.*

Proof. For each $z \in \mathbb{R}^2$ we denote by I_z the interval for V in which a line through z with direction V intersects $\partial B(1)$ twice. By Fubini, we can write

$$m_2(1) = \int_{\mathbb{R}^2}\int_{V\in I_z}\int_{y+}^{\infty}\mu_2(\mathrm{d}s) = \left(\int_{B(1)}\int_{-\pi/2}^{\pi/2} + \int_{|z|\geq 1}\int_{V\in I_z}\right)\left[|z|\cos\tilde{V} + \sqrt{1-|z|^2\sin^2\tilde{V}}\right]^{-2}\nu(\mathrm{d}\tilde{V})\mathrm{d}^2z.$$

Notice that $\tilde{V}$ is just a translation of V by the angle between z and the x-axis modulo π. Applying a linear change of variables, we get $\mathrm{d}V = \mathrm{d}\tilde{V}$ and thus

$$m_2(1) = \left(\int_{B(1)}\int_{-\pi/2}^{\pi/2} + \int_{|z|\geq 1}\int_{|\tilde{V}|<\arcsin\frac{1}{|z|}}\right)\left[|z|\cos\tilde{V} + \sqrt{1-|z|^2\sin^2\tilde{V}}\right]^{-2}\frac{\mathrm{d}\tilde{V}}{\pi}\,\mathrm{d}^2z.$$

We calculate the two integrals separately. Notice that we get a better expression if we multiply and divide the integrand by $\sqrt{1-|z|^2\sin^2\tilde{V}} - |z|\cos\tilde{V}$. For the first integral, we have

$$\begin{aligned}
(\mathrm{I}) &= \int_{B(1)}\int_{-\pi/2}^{\pi/2}\left[|z|\cos\tilde{V} + \sqrt{1-|z|^2\sin^2\tilde{V}}\right]^{-2}\frac{\mathrm{d}\tilde{V}}{\pi}\,\mathrm{d}^2z\\
&= \int_{B(1)}\int_{-\pi/2}^{\pi/2}\frac{1-|z|^2\sin^2\tilde{V} + |z|^2\cos^2\tilde{V} - 2|z|\cos\tilde{V}\sqrt{1-|z|^2\sin^2\tilde{V}}}{(1-|z|^2)^2}\frac{\mathrm{d}\tilde{V}}{\pi}\,\mathrm{d}^2z\\
&= \int_{B(1)}(1-|z|^2)^{-2}\int_{-\pi/2}^{\pi/2}1 - 2|z|\cos\tilde{V}\sqrt{1-|z|^2\sin^2\tilde{V}}\,\frac{\mathrm{d}\tilde{V}}{\pi}\,\mathrm{d}^2z\\
&= \int_{B(1)}(1-|z|^2)^{-2}\left[1 - \int_{-|z|}^{|z|}2\sqrt{1-U^2}\,\frac{\mathrm{d}U}{\pi}\right]\mathrm{d}^2z = \int_{B(1)}\frac{1-\frac{2}{\pi}(|z|\sqrt{1-|z|^2} + \arcsin|z|)}{(1-|z|^2)^2}\mathrm{d}^2z\\
&= 2\pi\int_0^1\frac{t - \frac{2}{\pi}t^2\sqrt{1-t^2} - \frac{2}{\pi}t\arcsin t}{(1-t^2)^2}\,\mathrm{d}t = \pi.
\end{aligned}$$

where we used $\int_{-\pi/2}^{\pi/2}\cos^2 x\,\mathrm{d}x = \int_{-\pi/2}^{\pi/2}\sin^2 x\,\mathrm{d}x$ in the third line, we used the change of variables $U = |z|\sin\tilde{V}$ in the fourth line and the last equality follows from noticing $\frac{\pi+2\arcsin t - 4t^2\arcsin t - 2t\sqrt{1-t^2}}{2\pi(1-t^2)}$ is a primitive. A very similar reasoning can be done for the second integral.

$$\begin{aligned}
(\mathrm{II}) &= \int_{|z|>1}\int_{|\tilde{V}|<\arcsin\frac{1}{|z|}}\frac{1-|z|^2\sin^2\tilde{V} + |z|^2\cos^2\tilde{V} - 2|z|\cos\tilde{V}\sqrt{1-|z|^2\sin^2\tilde{V}}}{(1-|z|^2)^2}\frac{\mathrm{d}\tilde{V}}{\pi}\,\mathrm{d}^2z\\
&= \int_{|z|>1}2(1-|z|^2)^{-2}\int_0^{\arcsin\frac{1}{|z|}}1 + |z|^2\cos(2\tilde{V}) - 2|z|\cos\tilde{V}\sqrt{1-|z|^2\sin^2\tilde{V}}\,\frac{\mathrm{d}\tilde{V}}{\pi}\,\mathrm{d}^2z\\
&= \int_{|z|>1}\frac{2}{\pi(1-|z|^2)^2}\left[\arcsin\frac{1}{|z|} + \sqrt{|z|^2-1} - 2\int_0^1\sqrt{1-U^2}\,\mathrm{d}U\right]\mathrm{d}^2z\\
&= 2\pi\int_1^{\infty}\frac{2t}{\pi(1-t^2)^2}\left[\arcsin\frac{1}{t} + \sqrt{t^2-1} - \frac{\pi}{2}\right]\mathrm{d}t = \pi.
\end{aligned}$$

where we used that a primitive for the last integral is $\frac{\pi - 2\sqrt{t^2-1} - 2t^2\arcsin\frac{1}{t}}{2\pi(t^2-1)}$. □

Proof of Theorem 3.4.20. Since we deal mainly with family $\{\gamma_r^0\}$, we adopt the shorter notation $k\text{-EArm}_r(z; l_1, l_2) := k\text{-EArm}_{\gamma_r^0}(z; l_1, l_2)$. To prove inequality (3.29), we notice that by scale-invariance we can write

$$P[k\text{-EArm}_r(l_1, l_2)] = P\left[k\text{-EArm}_1\left(l_1 r^{-1}, l_2 r^{-1}\right)\right] = P[k\text{-EArm}_{\frac{r}{l_1}}(1, l_2/l_1)], \tag{3.30}$$

so it suffices to prove that for any $r > 0$ and $m \in \mathbb{N}$ we have for a constant $c(u) > 0$ that

$$P[k\text{-EArm}_r(1, 2^m)] \leq K_k e^{-cmk}. \tag{3.31}$$

Consider the circles $\partial B(2^j)$ for $j = 1, \dots, m$ and let $N_r(j)$ denote the number of sticks with radius greater than r that intersect $\partial B(2^j)$ twice. By our standard coupling, we have that $(N_r(1), \dots, N_r(m))$ is stochastically smaller than $(N_0(1), \dots, N_0(m))$. Moreover, by Proposition 3.4.22 we know $N_0(j)$ has distribution $\mathrm{Poi}(2\pi u)$ for every j.

Define $J = J_{r,k} := \min\{j \in \mathbb{N}; N_r(j) \leq \frac{k}{4}\}$, with $\min \varnothing = \infty$. On event $\{k\text{-EArm}_r(1, 2^m), J = j\}$ with $j < m$ we have also $\{k\text{-EArm}_r(2^j, 2^m), N_r(j) \leq \frac{k}{4}\}$. Notice that on this event we can deduce there are a lot of disjoint stick arms:

- By Lemma 3.3.13, each stick intersecting $\partial B(2^j)$ can participate in at most 2 entering exploration arms, so we can ensure at least $\lfloor \frac{k}{2} \rfloor - \frac{k}{4} > \frac{k}{4} - 1$ entering arms that do not share any sticks.
- An entering arm may walk over the bottom side of our box B and such an arm does not make a covered 1-arm of sticks connecting the internal and external boundaries of $D(z; 2^j, 2^m)$. However, the same reasoning in Property 1 of Section 3.3.5 shows us that at most one entering arm of $D(z; 2^j, 2^m)$ can use the boundary of B. Thus, we have at least $\frac{k}{4} - 2$ disjoint stick arms in $D(2^j, 2^m)$.

By this, we can conclude using BK inequality in Corollary3.4.18 that for $k \geq 6$ one has

$$\begin{aligned} P\left[\begin{array}{c} k\text{-EArm}_r(1, 2^m), \\ J \leq \lfloor m/2 \rfloor \end{array}\right] &= \sum_{j=1}^{\lfloor m/2 \rfloor} P[k\text{-EArm}_r(1, 2^m), J = j] \leq \sum_{j=1}^{\lfloor m/2 \rfloor} P\left[\bigcirc_{i=1}^{\lceil \frac{k}{4} - 2 \rceil} \mathrm{Arm}_r(2^j, 2^m)\right] \\ &\leq \lfloor \frac{m}{2} \rfloor P[\mathrm{Arm}_r(2^{\lfloor m/2 \rfloor}, 2^m)]^{\lceil \frac{k}{4} - 2 \rceil} \leq \lfloor \frac{m}{2} \rfloor \left[K e^{-cm}\right]^{\lceil \frac{k}{4} - 2 \rceil} \leq K_k \exp[-cmk], \end{aligned}$$

where the penultimate inequality comes from Proposition 3.4.6. On the other hand, we have

$$P[k\text{-EArm}_r(1, 2^m), J > \lfloor m/2 \rfloor] \leq P\left[N_r(j) \geq \frac{k}{4}, j \in [m/2]\right] \leq P\left[N_0(j) \geq \frac{k}{4}, j \in [m/2]\right]$$

by the stochastic domination we just described. To finish the proof it suffices to show

Lemma 3.4.23. *There is $c = c(u) > 0$ such that for every $k \in \mathbb{N}$ there is $K_k = K_k(u) > 0$ with*

$$P\left[N_0(j) \geq k, j \in [m]\right] \leq K_k \exp[-cmk], \quad \textit{for every } m \geq 1. \tag{3.32}$$

Proof. Let us define $\Lambda_m := \{s; \#E_0(s) \cap \partial B(2^j) = 2 \text{ for some } j \in [m]\}$ and $\mu_2(\Lambda_m) =: \lambda_m$. By union bound, we know that $\lambda_m \leq 2\pi m$. Also, define

$$\Lambda_m(i, j) := \{s; \#E_0(s) \cap \partial B(2^l) = 2 \text{ iff } l \in [i, j]\}.$$

and notice that the sets $\Lambda_m(i, j)$ with $1 \leq i \leq j \leq m$ form a partition of Λ_m.

The number of sticks in $\operatorname{supp} \xi_0 \cap \Lambda_m$ is a random variable M with distribution $\mathrm{Poi}(u\lambda_m)$. After checking the number of sticks in Λ_m, we want to know for each stick s_l the number circles $\partial B(2^j)$ with $j \in [m]$ it has crossed, which we denote by D_l. Since we are working with a PPP, the s_l are chosen uniformly in Λ_m and D_l are iid. random variables. Using the bound of Lemma 3.2.7 we get

$$\begin{aligned} P[D_l = n] &= \sum_{i=1}^{m-n+1} P[s_l \in \Lambda_m(i, i + n - 1)] = \sum_{i=1}^{m-n+1} \frac{\mu_2(\Lambda_m(i, i - n + 1))}{\mu_2(\Lambda_m)} \\ &\leq \sum_{i=1}^{m-n+1} \frac{\mu_2(s; E_0(s) \cap \partial B(2^i) \neq \varnothing, E_0(s) \cap \partial B(2^{i-n+1}) \neq \varnothing)}{\mu_2(\Lambda_m)} \leq \frac{mc}{\lambda_m} 2^{-n}. \end{aligned} \tag{3.33}$$

Fixing any $\theta \in (0, \log 2)$ we have that

$$P\left[N_0(j) \geq k, j \in [m]\right] \leq P\left[\sum_{l=1}^{M} D_l \geq mk\right] \leq \exp[-\theta mk] \cdot E\left[\exp\left[\theta \sum_{l=1}^{M} D_l\right]\right]$$
$$= \exp[-\theta mk + u\lambda_m(E[e^{\theta D_l}] - 1)],$$

by Markov inequality and the independence of M and random variables D_l. Estimating $E[e^{\theta D_l}]$ with equation (3.33), we get

$$E[e^{\theta D_l}] = \sum_{n=1}^{m} e^{\theta n} P[D_l = n] \leq \frac{cm}{\lambda_m} \sum_{n=1}^{m} \exp[(\theta - \log 2)n] \leq \frac{c_\theta m}{\lambda_m}.$$

and hence

$$P\left[N_0(j) \geq k, j \in [m]\right] \leq \exp[-\theta mk + uc_\theta m - u\lambda_m] \leq \exp\left[-(\theta - \frac{uc_\theta}{k})mk\right].$$

Taking $k_0(u) = \lceil 2\frac{uc_\theta}{\theta} \rceil$, we have for $k \geq k_0(u)$ that $\theta - \frac{uc_\theta}{k} \geq \theta - \frac{uc_\theta}{k_0(u)} \geq \frac{\theta}{2} > 0$.

To extend our bounds for $k < k_0(u)$, we prove it first for $k = 1$. Recall that starting from random variables D_j in (3.22) we defined $D^{(0)} := m$, $D^{(j)} := D_{D^{(j-1)}}$ and $T := \max\{j; D^{(j)} \geq 1\}$. Replicating the argument in Lemma 3.4.9 for events $\{N_0(D^{(j)}) \geq 1\}$ instead of $\overline{\mathrm{Circ}}_0(A_{D(j)})^c$ (notice that both are increasing), we get

$$P\left[N_0(j) \geq 1, j \in [m]\right] \leq P[T < \lfloor \kappa m \rfloor] + P[T \geq \lfloor \kappa m \rfloor, \cap_{j=1}^{\lfloor \kappa m \rfloor} N_0(D^{(j)}) \geq 1]$$
$$\leq \exp[-c(u)m] + (1 - \exp[-u2\pi])^{\lfloor \kappa m \rfloor} \leq 2\exp[-c(u)m].$$

Thus, for any $k < k_0(u)$ we have

$$P\left[N_0(j) \geq k, j \in [m]\right] \leq P\left[N_0(j) \geq 1, j \in [m]\right] \leq 2\exp[-c(u)m] \leq 2\exp\left[-\frac{c(u)}{k_0(u)} mk\right]$$

and we can pick constants $c(u), K_k(u) > 0$ that verify inequality (3.32) for all $k \in \mathbb{N}$. □

We conclude equation (3.29) holds and the proof of Theorem 3.4.20 for family $\{\gamma_r^0\}$ is over. □

Having Theorem 3.4.20, the proof of Theorem 3.3.4 is immediate from Theorem 3.3.3.

Remark 3.4.24. It seems possible to adapt the proof above for family $\{\gamma_r^r\}$. Notice that the same reasoning should also work for ellipses exploration paths $\{\gamma_r^r\}$ provided we can prove Hypothesis H1 for $k = 1$ and redo the BK inequality argument. For the second part we are in good shape since Lemma 3.3.13 ensures that for any ellipse $E_r(s)$ that shares 2 entering arms of γ_r^r, the set $B(l_1)\backslash E_r(s)$ has two connected components. Thus, its supporting stick $E_0(s)$ also divides $B(l_1)$ into two connected components and we have $\mu_2(s; \#E_r(s) \cap \partial B(l_1) = 2) \leq m_2(l_1) = 2\pi$. However, the argument of successively discovering new tries for vacant circuits becomes more involved and we were not able to fill in the details yet.

Bibliography

[1] D Ahlberg, V Tassion, and A Teixeira. Sharpness of the phase transition for continuum percolation in $\mathbb{R}^2$. *arXiv preprint arXiv:1605.05926*, 2016.

[2] M Aizenman and A Burchard. Hölder regularity and dimension bounds for random curves. *Duke mathematical journal*, 99(3):419–453, 1999.

[3] M Berger. *Geometry II*. Springer-Verlag, 2009.

[4] P Billingsley. *Convergence of probability measures*. John Wiley & Sons, 2013.

[5] B Bollobas and O Riordan. *Percolation*. Cambridge University Press, 2006.

[6] F Camia and CM Newman. Critical percolation exploration path and sle_6: a proof of convergence. *Probability theory and related fields*, 139(3):473–519, 2007.

[7] JT Chayes, L Chayes, and R Durrett. Connectivity properties of mandelbrot's percolation process. *Probability Theory and Related Fields*, 77(3):307–324, 1988.

[8] R Durrett. *Probability: theory and examples*. Cambridge university press, 2010.

[9] C Garban and JE Steif. *Noise sensitivity of Boolean functions and percolation*, volume 5. Cambridge University Press, 2014.

[10] JB Gouéré. Subcritical regimes in the poisson boolean model of continuum percolation. *The Annals of Probability*, pages 1209–1220, 2008.

[11] GR Grimmett. *Percolation*. Springer: Berlin, Germany, 2010.

[12] P Hall. On continuum percolation. *The Annals of Probability*, pages 1250–1266, 1985.

[13] MR Hilário. *Coordinate percolation on Z3*. PhD thesis, IMPA, 2011.

[14] S Janson. Bounds on the distributions of extremal values of a scanning process. *Stochastic processes and their applications*, 18(2):313–328, 1984.

[15] JP Kahane. Random coverings and multiplicative processes. In *Fractal Geometry and Stochastics II*, pages 125–146. Springer, 2000.

[16] MG Kendall and PAP Moran. *Geometrical probability*. Griffin London, 1963.

[17] MS Klamkin. Elementary approximations to the area of n-dimensional ellipsoids. *The American Mathematical Monthly*, 78(3):pp. 280–283, 1971.

[18] TM Liggett, RH Schonmann, and AM Stacey. Domination by product measures. *The Annals of Probability*, 25(1):71–95, 1997.

[19] R Meester and R Roy. *Continuum Percolation*. Cambridge Tracts in Mathematics. Cambridge Univ Press, 1996.

[20] MV Menshikov, S Yu Popov, and M Vachkovskaia. On the connectivity properties of the complementary set in fractal percolation models. *Probab. Theory Related Fields*, 119(2):176–186, 2001.

[21] MV Menshikov, S Yu Popov, and M Vachkovskaia. On a multiscale continuous percolation model with unbounded defects. *Bull. Braz. Math. Soc.*, 34(3):417–435, 2003.

[22] MV Menshikov and AF Sidorenko. Coincidence of critical points for poisson percolation models. *Theory Probab. Appl.*, 32:603–606, 1987.

[23] Ş Nacu and W Werner. Random soups, carpets and fractal dimensions. *Journal of the London Mathematical Society*, page jdq094, 2011.

[24] J Ortega and M Wschebor. On the sequence of partial maxima of some random sequences. *Stochastic processes and their applications*, 16(1):85–98, 1984.

[25] P Parker and R Cowan. Some properties of line segment processes. *Journal of Applied Probability*, 13(01):96–107, 1976.

[26] S Yu Popov and M Vachkovskaia. A note on percolation of poisson sticks. *Brazilian Journal of Probability and Statistics*, pages 59–67, 2002.

[27] D Reimer. Proof of the van den berg-kesten conjecture. *Combinatorics, Probability and Computing*, 9(1):27–32, 2000.

[28] SI Resnick. *Extreme values, regular variation and point processes*. Springer, 2013.

[29] R Roy. Percolation of poisson sticks on the plane. *Probab. Theory Related Fields*, 89(4):503–517, 1991.

[30] Lucio Russo. A note on percolation. *Probability Theory and Related Fields*, 43(1):39–48, 1978.

[31] O Schramm, S Smirnov, and C Garban. On the scaling limits of planar percolation. In *Selected Works of Oded Schramm*, pages 1193–1247. Springer, 2011.

[32] KJ Schrenk, MR Hilário, V Sidoravicius, NAM Araújo, HJ Herrmann, M Thielmann, and A Teixeira. Critical fragmentation properties of random drilling: How many holes need to be drilled to collapse a wooden cube? *Physical review letters*, 116(5):055701, 2016.

[33] PD Seymour and DJA Welsh. Percolation probabilities on the square lattice. *Annals of Discrete Mathematics*, 3:227–245, 1978.

[34] AN Shiryaev. *Probability*. Springer-Verlag, New York,, 1996.

[35] AS Sznitman. Vacant set of random interlacements and percolation. *Annals of mathematics*, pages 2039–2087, 2010.

[36] J Tykesson and D Windisch. Percolation in the vacant set of Poisson cylinders. Probability Theory and Related Fields, Volume 154, Issue 1 (2012), Page 165-191, 2010. arXiv:1010.5338v2.

[37] J van den Berg. A note on disjoint-occurrence inequalities for marked poisson point processes. *Journal of applied probability*, 33(02):420–426, 1996.

[38] J van den Berg and H Kesten. Inequalities with applications to percolation and reliability. *Journal of applied probability*, 22(03):556–569, 1985.

[39] Wendelin Werner. Course 2 some recent aspects of random conformally invariant systems. *Les Houches*, 83:57–99, 2006.

[40] SA Zuev and AF Sidorenko. Continuous models of percolation theory. i. *Theoretical and Mathematical Physics*, 62(1):51–58, 1985.

[41] SA Zuev and AF Sidorenko. Continuous models of percolation theory. ii. *Theoretical and Mathematical Physics*, 62(2):171–177, 1985.