Decoupling and noise sensitivity for models with conservative dependencies

Rangel Baldasso
Advisor: Augusto Teixeira
Co-advisor: Daniel Ahlberg

ABSTRACT

In this thesis we work on two different problems. First, we prove decoupling estimates for one-dimensional conservative particle systems. The second class of problems is related to noise sensitivity and sharp thresholds for percolation models.

In our setting, a decoupling is a type of correlation estimate for monotone functions of the space-time configurations with far enough supports. We prove these estimates for two models: The exclusion process and the zero range process. These estimates are used to study processes evolving on top of these particle systems.

For the exclusion process, we consider a detection model: At time zero, place nodes on each site independently with probability $\rho \in [0, 1)$ and let they evolve as a simple symmetric exclusion process. Also at time zero, a target is placed at the origin. The target moves only at integer times, and can move to any site that is within distance R from its current position. Assume also that the target can predict the future movement of all nodes. We prove that, for R large enough it is possible for the target to avoid detection forever with positive probability.

As for the decoupling of the zero range process, we use it to study the spread of an infection on top of this particle system. At time zero, the set of infected particles is composed by those which are in the negative axis, while particles at the right of the origin are considered healthy. A healthy particle immediately becomes infected if it shares a site with an infected particle. We prove that the front of the infection wave travels to the right with positive and finite velocities.

In the context of Boolean functions, noise sensitivity measures whether the outcome of such function can be predicted when one is given its value on a perturbation of the original configuration while threshold phenomena describes abrupt changes in the behavior of these functions.

We consider Poisson Voronoi percolation on $\mathbb{R}^2$ and prove that box-crossing events in this model are noise sensitive and present a threshold phenomena with polynomial window.

ACKNOWLEDGEMENTS

Every thesis demands a great deal of time, dedication and work. The journey is full of periods of joy and excitement, but it also teaches how to deal with frustration and disappointments. And these feelings are not unique to whoever is writing the thesis, but are to some degree shared with every friend.

I thank my advisor Augusto that has guided me during this process. I know he sparred no efforts during our work. His kindness and easygoing mindset helped me to overcome many difficulties, and his encouragement kept me motivated the whole way.

I also thank my co-advisor, Daniel Ahlberg, for all his guidance and availability. Working with him has been a truly wonderful experience.

I thank my teachers for their inspiring classes and for sharing their knowledge with me. In special, the probability group at IMPA, Cláudio Landim, Hubert Lacoin, Milton Jara, and Roberto Imbuzeiro, for all the help and support.

I am sure all the friends I made during these years will remain for the rest of my life. I thank Bruna, Conrado, Daniel, Felippe, Francisco, Franco, Giulia, Guilherme, Letícia, Lucas, Luiz Paulo, Majela, Maria Clara, Mariela, Mateus, Otávio, Pedro, Ricardo, Susana, and many others that I do not list here but kindly ask to not fell forgotten.

I also thank all the staff at IMPA for always being available and providing a good work environment. I thank Capes and FAPERJ for the financial support.

Last, but not least, I thank my family that has accompanied me during all this process. When I doubt myself, their support gives me strength to carry on.

CONTENTS

1

INTRODUCTION

Games of chance have always been part of the human nature and classical probability theory from before the second half of the 19th century revolved around these games. The toss of a die or a coin are examples of objects of study. The main concerns were in understanding trials from a finite number of equally likely outcomes. *Ars Conjectandi*, a book from Jacob Bernoulli published in 1713, contains a partial version of what is now known as weak law of large numbers, a good illustration of classical probability theory.

It is no secret that, many times, the development of mathematics is given by its necessity in other areas of science. And this is the case with modern probability theory. In physics, the idea that the atomic structure should help to understand the macroscopic properties of a material started gaining ground around 1850. However, assuming that the atomic structure could be completely understood by the laws of classical mechanics was not satisfactory, since this would end up contradicting the second law of thermodynamics. New tools where necessary to surpass these problems and this is where probability theory enters the scene. Maxwell, around 1860, was the first to assume molecules move randomly and derived his velocity distribution function. Later on, Boltzmann concluded that the second law of thermodynamics could be explained if one assumes random collisions between the molecules. From a mathematical point of view, this led to the development of Markov chains and diffusion processes.

The idea of indeterminism was seen by many scientists as ignorance, that the randomness actually came from the lack of complete information. Meanwhile, in the first half of the 20th century, quantum mechanics received great attention in the physics literature. The development of quantum mechanics explained the behavior of systems at atomic scales. Schrödinger's equation and Heisenberg's uncertainty principle once more contested absolute determinism. This gave even more importance to probability theory, since positions and velocities were now viewed as distributions and not exact values.

However, a complete axiomatization of probability theory was not yet complete. In fact, this was part of Hilbert's sixth problem, proposed in 1900. Kolmogorov, in 1933, founded what is now accepted as modern probability theory.

Nowadays, probability theory is a well established area of mathematics. It has developed many different branches and their applications go way beyond physics, touching biology, economy, political and social sciences.

In this thesis, we choose to work on two different aspects of the theory. First, we adapt a technique of decoupling to conservative interacting particle systems. These decouplings are also used to study processes on top of the particle systems considered. In the second part, a extensive study of noise sensitivity and sharp thresholds for planar Voronoi percolation is conducted. This thesis is based on the papers [11], [10] and [1]. In the following, we describe more precisely each of these two parts.

1.1 DECOUPLING OF INTERACTING PARTICLE SYSTEMS

Percolation theory, whose first appearance dates back to 1956 in Broadbent and Hammersley [16], is a branch of probability theory motivated by physics. It was originally a model for the spread of a fluid in a random medium. Unlike diffusion process, that assume the randomness comes from the behavior of the fluid, percolation models work with the premise that the randomness is intrinsic to the medium and not the fluid.

Since its introduction, many modifications of the model have been considered and the techniques used to understand such models improved greatly. Today, not only many robust techniques are available and can be used to understand other kinds of problems, but also the possible applications of these models are very wide, ranging from the study of epidemics to the understanding of social networks.

Arguably, the bond percolation model is the simplest and most studied among all percolation models. In it, the medium has an underlying graph structure and each edge of this graph receives an associated independent random variable that controls whether the passage of fluid through the edge is allowed or not. Usually, the underlying structures are infinite regular graphs and, in this case, the behavior of this model is very well understood.

The assumption of independence is a facilitating factor in the study of the bond percolation model and many of the classical proofs for central theorems rely on it. Decoupling techniques appear when one tries to drop this assumption. A decoupling is nothing more than a correlation estimate. It bounds the correlation of functions that depend on far enough subsets of the underlying structure. These estimates are central tools in the understanding of dependent percolation. Their strength is highlighted when combined with multiscale renormalisation using cascade events.

Models that lack independence are numerous in the literature and decouplings techniques have been used in many of them. The dependency in each model can present many different characteristics and the complexity of the decoupling depends on the model: For finite range percolation, functions that depend on far enough subsets have independent outcomes and Liggett, Schonmann and Stacey [34] studied this case. Some models with infinite range were also already considered in the literature: Voronoi percolation in Bollobás and Riordan [14], and Boolean percolation with random radii in the book [35], and in Ahlberg, Tassion and Teixeira [5].

In some models, different aspects of the dependency make the construction of a decoupling harder. This is the case of random interlacements, introduced by Sznitman [47]. This model consists of a Poissonian soup of random walks and decouplings were studied in this context, see Popov and Teixeira [39]. This is one of the first models where a sprinkling is used to blur the dependencies and obtain good decay bounds.

Interacting particle systems are another example of stochastic processes with non-trivial dependencies. Once again, the dependencies can have many different effects in the evolution of the process. Understanding conservative models is a challenging endeavor due to the nature of its dependency. Here, we focus on decoupling for conservative interacting particle systems on the integer lattice. We prove decouplings for two models introduced in Spitzer [45]: The exclusion process and for the zero range process.

THE EXCLUSION PROCESS. The exclusion process is defined as follows. Fix $\rho \in [0, 1]$ and place a particle in each integer site independently with probability ρ. Each particle moves as a continuous time random walk obeying the exclusion rule, that says two particles cannot occupy the same site at the same time. This means that, when a particle tries to jump to an occupied site, this jump is suppressed. Our initial distribution for the particles is, for every value of ρ, stationary to this evolution and the parameter ρ is the density of the process.

THE ZERO RANGE PROCESS. The zero range process in $\mathbb{Z}$ is a particle system where particles interact only when they are at the same site. The dependency is on the rate with which particles leave the site and it is controlled by a function $g : \mathbb{N}_0 \to \mathbb{R}_+$ of the number of particles with $g(0) = 0$. Particles jump to a uniformly chosen nearest neighbor.

We assume that there exist positive constants $\Gamma_- \leqslant 1 \leqslant \Gamma_+$ such that

$$\Gamma_- \leqslant g(k) - g(k-1) \leqslant \Gamma_+, \qquad \text{for all } k \in \mathbb{N}. \tag{1.1.1}$$

There exist explicit formulas for some invariant measures of the zero range process, see [32]. In fact, Assumption (1.1.1) implies that, for every $\rho \in \mathbb{R}_+$, there exists an associated invariant product measure with density ρ. This collection does not contain all the invariant measures for the zero range process, as observed in [32].

We now start to discuss the decoupling inequalities we prove here and that compound half of this thesis. The state space of the exclusion process and the zero range process are, respectively, $\{0, 1\}^{\mathbb{Z}}$ and $\mathbb{N}_0^{\mathbb{Z}}$. In both process, the state space has a natural partial order that allows to define when a function of the trajectories is monotone. Besides, we say that a function of the trajectories f has support on the space-time box $B \subset \mathbb{Z} \times \mathbb{R}_+$ if

$$\eta_t(x) = \xi_t(x), \text{ for all } (x, t) \in B, \text{ implies } f(\eta) = f(\xi). \tag{1.1.2}$$

We will prove that, if f_1 and f_2 are positive bounded non-decreasing functions of the space-time with respective supports in B_1 and B_2, and the boxes are sufficiently far away, then

$$\mathbb{E}_\rho(f_1 f_2) \leqslant \mathbb{E}_{\rho+\epsilon}(f_1)\mathbb{E}_{\rho+\epsilon}(f_2) + H(d, \epsilon, \rho), \tag{1.1.3}$$

where $\mathbb{E}_\rho$ denotes the expectation with respect to an exclusion process or a zero range process with density ρ, H is an error function, and d is used for the distance between the two boxes.

It is important to notice that the estimate above is not a correlation estimate. This is due to the increase of the density from ρ to $\rho + \epsilon$. This sprinkling blurs the dependency caused by conservation and helps to ensure that the decay rate of the error function H is stretched exponential. In fact, we will see that, if $\epsilon = 0$, then H decays at most polynomially with d.

The proofs of both results follow the same lines. The main step is to construct a coupling between two processes with densities ρ and $\rho + \epsilon$ in a way that the process with larger density dominates the other process in a fixed interval and some large time with high probability.

To illustrate the power of these decouplings, we provide two applications. We first work on a detection problem for the exclusion process. As for the zero range process decoupling, it is used to study an infection process on top of the particle system. We prove bounds for the velocity with which the infection process spreads.

1.1.1 *Detection*

In a detection model, we are given a random set of moving points, called nodes. We think of these nodes as detectors. Suppose we also have a target that can be mobile or not. We are interested in knowing whether any of the nodes will detect the target in finite time, and if so, what are the properties of the detection time. Of course the answer to these problems depend on the specific model in question.

There exists a rich literature concerning this class of problems. The mobile geometric graph model is an example of structure where this has been studied. In this model, the starting positions of the nodes is given by a Poisson point process in the plane with intensity $\lambda > 0$ and they evolve as independent Brownian Motions. A node detects everything that is within distance at most one from it. This model has been studied under different aspects. When the target is non-mobile, detection occurs in finite time almost surely. In this case, Kesidis, Konstantopoulos and Phoha [29] derive bounds on the tail distribution of the detection time (the first time a node detects the target). One can also consider a target that moves independently from the nodes, as in Peres, Sinclair, Sousi and Stauffer [38]. They also study the tail of the detection time, and which distributions for the movement of the target allows it to avoid detection for the longest time. They prove that, in dimension two, there are two equally good possibilities for it: Stay put or move as an independent Brownian Motion.

In the literature presented above, a central assumption is that the target moves independently from the nodes. It is also interesting to treat the case when we drop this assumption. We can suppose that the target is able to predict the future trajectories of all the nodes and moves cleverly trying to avoid detection. In Stauffer [46], this possibility is considered for the mobile geometric graph for dimensions $d \geqslant 2$. In this case, they prove a phase transition on the probability of detection as the value of λ changes. For dimension one, detection always occurs in finite time for this model.

DESCRIPTION OF RESULTS. In the model we consider here, the nodes follow an exclusion process with density $\rho \in (0, 1)$. As for the target, it starts at the origin and, unlike the nodes, moves only at integer times. On the other hand, we allow the target to jump to any site within distance at most $R > 0$ from its current position. The target is detected if it stays on top of some node. We assume that the target knows the future movement of all nodes, and we ask if it can escape detection with positive probability.

We will prove that, for fixed $\rho \in (0, 1)$, there exists a phase transition in the probability of detection in finite time, as we vary the value of R.

Theorem 1.1.1. *Suppose* $\rho \in [0, 1)$. *There exists* $R_0 = R_0(\rho)$ *such that, if* $R \geqslant R_0$, *then the probability that the target is never detected by some node is positive.*

Remark 1.1.2. It is not true that, for all values of R, the target can escape with positive probability. If we take any $\rho > 0$ and $R = 1$ it is possible to find two nodes at time zero, one at each side of the origin. Using a suitable construction of the exclusion process, we conclude that these nodes (as well as the empty sites) move as random walks. This implies that the two nodes we found will eventually meet and strangle the target, who is discovered.

Sidoravicius and Stauffer [44] consider a model in $\mathbb{Z}^d$, $d \geqslant 2$, where nodes are placed according to a Poisson point process with intensity λ and move as independent random walks. The target also moves in continuous time, but with bounded speed. They compare this process with oriented percolation to prove a phase transition in the probability of detection as the value of λ changes. One may wonder if the techniques from [44] can be used to prove survival in our setting. There, the authors use a well-chosen subspace of $\mathbb{Z}^{d+1}$ and prove that each node only influences a small area of this subspace. Hence, they can remove the forbidden sites and disregard the trajectory of the node. However, when $d = 1$, this fails because, since each node intersects a fixed subspace infinitely many times, the area of influence of each node extends infinitely and can not be disregarded so easily. This makes the proof in dimension one more intricate, and requires some different machinery. We expect, however, that the proof presented in [44] can be adapted to our case for larger dimensions.

An additional difficulty comes from the choice of the exclusion process as an underlying dynamic due to its lack of good mixing properties. For this reason, the usual techniques do not apply in a straightforward way and this is where our decoupling comes in play. The existence of dependence among the movement of the particles is also a complicating factor.

PROOF OVERVIEW. We use multiscale renormalisation to prove the existence of oriented percolation in dependent models. This allows us to perform some comparison, in a similar flavor of [44]. The main advantage is that our renormalisation does not rely on the specifics of the model and can be used in other contexts.

Stauffer [46] uses a multiscale renormalisation to prove the existence of a phase where detection always occurs. Our theorem goes in the opposite direction. It gives sufficient conditions to the existence of percolation, and we use it to prove that, in our model, detection may not happen. To prove that detection may fail, it is necessary to compare the process with oriented percolation, instead of looking into non-oriented models. This adds a new complicating factor, since oriented paths are harder to exhibit.

The renormalisation scheme we develop is a general statement about percolation in oriented models. It proves percolation using a fixed set of oriented paths. The main advantage of this technique is that it does not require independence. Instead, we only need to take care of the decay of correlations on the environment. We will focus here in site percolation in dimension $1+1$, but the proof techniques can be adapted to more general models.

We present a particular case of the percolation models we are interested in. Let $\mathcal{I} \subset \mathbb{Z}^2$ be a random subset of the integer lattice with distribution $\mathbb{P}$ that is translation invariant. Fix also S as the set of paths $f : \mathbb{N}_0 \to \mathbb{Z}^2$ that satisfy

$$f(n+1) - f(n) \in \{(0,1),(1,0)\}, \text{ for all } n \in \mathbb{N}_0. \tag{1.1.4}$$

We look for conditions over $\mathbb{P}$ that ensure the existence of an infinite open path, i.e., a infinite path of S contained in $\mathcal{I}$.

When $\mathbb{P}$ is obtained by independently declaring each vertex open with probability p, one can easily prove that percolation occurs for large values of p, see [25]. Our objective here is to drop the independence assumption. We will assume instead a good decay on the correlations of the environment.

Our theorem states that, under the right correlation decay on the environment, if the probability that a site is open is big enough, then

$$\text{"the probability that there exists a path in } S \text{ that is open is positive."} \tag{1.1.5}$$

Combining this result with the exclusion process decoupling will easily imply the existence of a phase where detection may not occur.

1.1.2 *Infection*

Infection processes model the evolution of a population divided into two groups: The healthy individuals and the infected ones. The interest lies in understanding what happens with the infected set in the long run.

DESCRIPTION OF RESULTS. The model we consider here is defined as follows. Given an initial configuration η_0 for the zero range process, we declare all particles to the left of zero, ξ_0, infected. Define also $\zeta_0 = \eta_0 - \xi_0$ as the configuration of healthy particles.

We assume that the process $\xi + \zeta$ evolves as a zero range process with rate function g. Besides, a healthy particle becomes immediately infected when it shares a site with some already infected particle. In particular, in any non-empty site, either all particles are healthy or all particles are infected.

We define the front of the infection wave as

$$r_t = \sup\{x : \xi_t(x) > 0\}. \tag{1.1.6}$$

If we choose η_0 according to the invariant product measure with density $\rho > 0$, then $r_t \in \mathbb{Z}$ for all $t \geqslant 0$ almost surely. We prove two theorems regarding the behavior of r_t. We obtain uniform upper and lower bounds for the quantities r_t/t.

The first result states that r_t goes to the right with finite speed.

Theorem 1.1.3. *For any $\rho > 0$, there exist $v_+ > 0$ such that, for all $L > 0$,*

$$\mathbb{P}_\rho \left[\begin{array}{c} r_t \geqslant v_+ t + L, \\ \textit{for some } t \geqslant 0 \end{array} \right] \leqslant c_1 e^{-c_1^{-1} \log^{5/4} L}, \tag{1.1.7}$$

for some positive constant c_1 that depends only on the density ρ and the rate function g.

Our second result states that the velocity is also positive.

Theorem 1.1.4. *For any $\rho > 0$, there exist $v_- > 0$ such that, for all $L > 0$,*

$$\mathbb{P}_\rho \left[\begin{array}{c} r_t \leqslant v_- t - L, \\ \textit{for some } t \geqslant 0 \end{array} \right] \leqslant c_2 e^{-c_2^{-1} \log^{5/4} L}, \tag{1.1.8}$$

for some positive constant c_2 that depends only on the density ρ and the rate function g.

The process r_t increases by one whenever an infected particle at position r_t jumps to $r_t + 1$. However, in order for r_t to decrease, it is necessary that all infected particles at r_t jump to $r_t - 1$. This suggests that the process r_t should have a tendency to go to the right. Turning this heuristics into a proof may seem easy at first sight. An indicative that this is not the case is the collection of works Ramírez and Sidoravicius [41], Comets, Quastel and Ramírez [18], and Bérard and Ramírez [13] where a similar model is considered. There, healthy particles remain still until they become infected. Besides, infected particles move independently from each other. These works establish a law of large numbers, central limit theorem and large deviations for the model.

Our theorem is a first step in understanding how influences spread in the zero range process: As a corollary, we obtain a correlation estimate for functions that depend only on sets that are far enough in space.

PROOF OVERVIEW. First, we prove that r_t travels to the right with finite velocity. We use multiscale renormalisation to bound the probability of events where r_t travels fast to the right at some fixed times. When we have a good bound for this fixed sequence of times, all the work remaining is to do an interpolation argument to conclude that the statement holds uniformly in time.

The proof of the second theorem is also based in multiscale renormalisation. However, we cannot apply the same argument using events where the front does not travel with some small but positive speed. We use an alternative strategy considering a broad class of paths and prove that, for each of these paths, there is a positive fraction of time where at least two particles are close to it. We observe that the front wave is one such path and, when two particles are close to it, there is a positive chance that these particles will meet in the front and produce a drift to the right. A central step in both proofs is the decoupling for the zero range process.

RELATED WORKS. There exists a rich literature concerning infection processes. Giacomelli [24] proves that, for our model, in the independent case, i.e., when the rate function g equals the identity, the velocity of the infection wave is greater than one.

Jara, Moreno and Ramírez [27] consider an infection process evolving on top of the exclusion process. Based on a regeneration argument, they prove a law of large numbers and central limit theorem for this model.

Higher dimensional modes have also been considered. Popov [40] contains a detailed review of the so-called frog model. An extensive estudy of this model is conducted in Alves, Machado and Popov [8, 6], and Alves, Machado, Popov and Ravishankar [7].

In Kesten and Sidoravicius [31], the authors consider a model that is similar to ours, but for any dimension: Particles evolve as independent random walks and only the origin begins infected. In this case, they prove a shape theorem.

1.2 NOISE SENSITIVITY FOR VORONOI PERCOLATION

The concept of a Boolean function, $f : \{0,1\}^n \to \{0,1\}$, is of fundamental importance in theoretical computer science. Moreover, many of the most well-studied problems in the intersection between combinatorics and probability theory may be phrased in terms of (often monotone) Boolean functions. One is, in this context, interested in the typical behavior of a Boolean function for an element in $\{0,1\}^n$ chosen according to product measure with marginal density p, henceforth denoted by $\mathbb{P}_p$. The study of Boolean functions has led to a vast literature on a range of fascinating phenomena, such as the existence of thresholds and the effect of small perturbations, see e.g. [23, 36]. The second kind of problems we consider in this thesis consists in understanding these concepts in the context of Voronoi percolation.

Threshold phenomena of monotone Boolean functions was first discovered by Erdős and Rényi [20] in their pioneering study of random graphs. The existence of a sharp threshold is the essence of Kesten's celebrated 1980 proof that the critical probability for the existence of an infinite connected component in bond percolation on $\mathbb{Z}^2$ equals 1/2 [30].

A sequence $(f_n)_{n\geqslant 1}$ of monotone[1] Boolean functions $f_n : \{0,1\}^n \to \{0,1\}$ is said to have a *threshold* at $p \in (0,1)$ if, for every $\epsilon > 0$, we have

$$\lim_{n\to\infty} \mathbb{P}_{p-\epsilon}[f_n = 1] = 0 \quad \text{and} \quad \lim_{n\to\infty} \mathbb{P}_{p+\epsilon}[f_n = 1] = 1.$$

The understanding of thresholds has increased with works by Russo [42], Kahn, Kalai and Linial [28], Friedgut and Kalai [21], and Talagrand [48].

The notion of noise sensitivity was introduced in a seminal paper by Benjamini, Kalai and Schramm [12]. Given $\omega \in \{0,1\}^n$, we obtain an ϵ-perturbation ω^ϵ of ω by resampling each bit of ω independently with probability ϵ. A sequence $(f_n)_{n\geqslant 1}$ of functions $f_n : \{0,1\}^n \to \{0,1\}$ is said to

[1] A Boolean function is monotone if $f_n(\omega') \geqslant f_n(\omega)$ whenever $\omega' \geqslant \omega$ coordinate-wise.

be *noise sensitive* at level p (NS_p for short) if $f_n(\omega)$ and $f_n(\omega^\epsilon)$ are asymptotically uncorrelated, i.e., if

$$\mathbb{E}_p[f_n(\omega)f_n(\omega^\epsilon)] - \mathbb{E}_p[f_n(\omega)]^2 \to 0, \quad \text{as } n \to \infty. \tag{1.2.1}$$

The study of noise sensitivity has led to a detailed understanding of certain planar percolation models, both in the discrete setting: Benjamini, Kalai and Schramm [12], Schramm and Steif [43], Garban, Pete and Schramm [22], and in the continuum: Ahlberg, Broman, Griffiths and Morris [2], and Ahlberg, Griffiths, Morris and Tassion [3].

We study threshold phenomena and the effect of small perturbations in the context of Poisson Voronoi percolation on $\mathbb{R}^2$. Our contributions in this direction are two-fold. First, we describe the discretization method developed in [2], by which we reduce the continuum problem to its discrete counterpart, and emphasize the close relation between threshold phenomena and noise sensitivity of Boolean functions via the study of randomized algorithms. Combining the two techniques we derive quantitative estimates on the width of the threshold window and the rate of decorrelation in (1.2.1). Second, we discuss a range of different but related notions of perturbations in the context of Voronoi percolation.

We remark that the application of the discretization approach is here somewhat simpler than as originally developed in [2]. Moreover, the techniques we use apply to a range of continuum percolation models such as Poisson Boolean percolation and confetti percolation, as opposed to the approach in [3] that exploits color-switching tricks. For self-dual models, such as Voronoi and confetti percolation, our approach offers an alternative proof that the critical probability for percolation equals 1/2, as originally proved by Bollobás and Riordan [14].

DESCRIPTION OF VORONOI PERCOLATION. Poisson Voronoi percolation is a model for the study of long-range connections in a two-coloring of $\mathbb{R}^2$ based on a tessellation. The large-scale behavior in models of this kind is well-known to be governed by its behavior in finite regions, and we shall for this reason work with the restriction of the model to the unit square. Let, hence, $S := [0,1]^2$ and let Ω denote the space of finite subsets of $S \times \{0,1\}$, equipped with the Borel σ-algebra. Formally we construct a Voronoi configuration on S based on a Poisson point process η on Ω with intensity measure $n\lambda_S \otimes [p\delta_1 + (1-p)\delta_0]$, where λ_S denotes the Lebesgue measure on S.

Given $\eta \in \Omega$, we define the Voronoi cell associated to $(x,u) \in \eta$ as

$$V(x) := \{y \in S : d(y,x) \leqslant d(y,x') \text{ for all } (x',u') \in \eta\},$$

where d denotes the Euclidean distance. Based on the tessellation we declare a point in S *red* or *blue* depending on whether it is contained in the cell corresponding to a point in η with u-coordinate 0 or 1, respectively.[2] To rule out degenerate cases, we color all points in S red in the case that $\eta = \emptyset$. We shall denote the associated measure by $\mathbb{P}_{n,p}$, and we will occasionally suppress the subscript to ease the notation.

Given a rectangle $R \subseteq S$, let H_R denote the event defined by the existence of a continuous blue path crossing R horizontally, and let $f_R : \Omega \to \{0,1\}$ denote the indicator of the event H_R. Conditioned on $\eta \neq \emptyset$, at $p = 1/2$ the model is self-dual, meaning that the red and blue components are equidistributed. Since any rectangle $R \subseteq S$ is either crossed horizontally by a blue path or vertically by a red path, it follows by symmetry that[3]

$$\mathbb{P}_{n,1/2}[f_S = 1] \to 1/2.$$

Besides, the function f_R is non-degenerate at $p = 1/2$ for any rectangle $R \subseteq S$: There exists a constant $c_3 > 0$, depending only on the aspect ratio of R, such that

$$c_3 \leqslant \mathbb{P}_{n,1/2}[f_R = 1] \leqslant 1 - c_3, \tag{1.2.2}$$

uniformly in n. This was first proved by Tassion [49] for Voronoi percolation on $\mathbb{R}^2$, and later extended in [3] to subsets of $\mathbb{R}^2$ with boundary. The box-crossing property in (1.2.2) is a typical critical phenomenon and a suggestive indication that the critical threshold for the existence of an unbounded connected blue component in Poisson Voronoi percolation on $\mathbb{R}^2$ equals $1/2$.

[2] It is not hard to see that, with probability one, every Voronoi cell is a closed bounded convex set. A point on the boundary of some set may belong to more than one cell, but no point of S can belong to more than three cells. Besides, if two cells share a vertex, they share an entire edge. We can therefore ignore the fact that points on the boundary of two cells may be declared both red and blue.

[3] Would it not be for the possibility that η may be empty, equality would hold here.

DESCRIPTION OF RESULTS. In the continuum setting, a natural notion of perturbation of a Voronoi configuration is obtained as follows. For $\epsilon \in (0,1)$ let $\eta(\epsilon)$ be obtained from η by first thinning η by a factor $1-\epsilon$ and then sprinkling an independent density of ϵn points to regain the initial density n. The proportion p of blue points are in each step kept constant. We shall say that the function $f_R : \Omega \to \{0,1\}$, encoding the existence of a blue crossing of the rectangle R, is *noise sensitive* at level p if, for every $\epsilon > 0$, we have

$$\mathbb{E}_{n,p}\big[f_R(\eta)f_R(\eta(\epsilon))\big] - \mathbb{E}_{n,p}\big[f_R(\eta)\big]^2 \to 0, \quad \text{as } n \to \infty. \tag{1.2.3}$$

Moreover, we say that f_R has positive noise sensitivity exponent if (1.2.3) holds with ϵ replaced by $\epsilon_n = n^{-\alpha}$ for some $\alpha > 0$.

Notice that in (1.2.3) we have defined what it means for a single function to be noise sensitive, contrary to the discrete setting, where a sequence of functions was considered. The two definitions are the natural analogues of one another, and the reason for the difference lies in how dimensionality is expressed differently in the two settings.

Our first theorem shows that box crossings in Poisson Voronoi percolation are noise sensitive at the critical parameter $p = 1/2$, and that the probability of a horizontal blue crossing of a rectangle R tends to either 0 or 1 outside of a polynomial-sized window around $1/2$.

Theorem 1.2.1. *For every rectangle $R \subseteq S$, the function f_R is noise sensitive at level $p = 1/2$ with a positive noise sensitivity exponent. Moreover, there exists a constant $\gamma > 0$ such that*

$$\lim_{n\to\infty} \mathbb{P}_{n,1/2-n^{-\gamma}}[f_R = 1] = 0 \quad \textit{and} \quad \lim_{n\to\infty} \mathbb{P}_{n,1/2+n^{-\gamma}}[f_R = 1] = 1.$$

We remark the fact that $\mathbb{P}_p[f_R = 1]$ converges to either zero or one for $p \neq 1/2$, together with Cauchy-Schwarz inequality, implies that Voronoi percolation is trivially noise sensitive for $p \neq 1/2$. In addition, the above provides an alternative proof of Bollobás and Riordan's theorem that the critical probability for Poisson Voronoi percolation on $\mathbb{R}^2$ equals $1/2$.

One way to think of the perturbation in (1.2.3) is as the following dynamical process evolving in time: Let points appear in $S \times \{0,1\}$ at rate n, where they remain for an exponentially distributed time before disappearing. The measure $\mathbb{P}_{n,1/2}$ is stationary for this process, and for $\epsilon = 1 - e^{-t}$

the pair $(\eta, \eta(\epsilon))$ corresponds to the dynamical process observed at times 0 and t.

In greater generality we may think of a perturbation as a reversible time-homogeneous Markov process $(\eta(t))_{t\geqslant 0}$ on Ω evolving in equilibrium. For each such process, the Markov property and reversibility together give that

$$\begin{aligned}
&\mathbb{E}\big[f_R(\eta(0))f_R(\eta(t))\big] - \mathbb{E}\big[f_R(\eta(0))\big]^2 \\
&\quad = \mathbb{E}\Big[\mathbb{E}\big[f_R(\eta(0))\big|\eta(t/2)\big]\mathbb{E}\big[f_R(\eta(t))\big|\eta(t/2)\big]\Big] - \mathbb{E}\big[f_R(\eta(0))\big]^2 \\
&\quad = \mathrm{Var}\Big(\mathbb{E}\big[f_R(\eta(t/2))\big|\eta(0)\big]\Big).
\end{aligned}$$

Hence, for each dynamical process of this kind, the correlation between two points in time measures the amount of information in some σ-algebra $\mathcal{F}$ – the σ-algebra generated by the glimpse of the process in one of the time points – and being sensitive with respect to this information is equivalent to

$$\mathrm{Var}_{n,1/2}\left(\mathbb{E}[f_R(\eta)|\mathcal{F}]\right) \to 0, \quad \text{as } n \to \infty.$$

Clearly, the more information contained in $\mathcal{F}$ the larger the variance. This indicates, in particular, that more conservative dynamics tend to affect a system to a lesser extent. Two natural notions of perturbations that conserve the number of points are

- re-randomize colors of a small proportion of points;
- re-randomize locations of a small proportion of points.

The former of these two notions was studied in [3], where the authors showed that the existence of crossings in Voronoi percolation are sensitive with respect to resampling a small proportion of the colors. The latter we study here, and show that Voronoi crossings are sensitive also with respect to relocation of points within S.

Theorem 1.2.2. *Let η^* be obtained from η by re-randomizing the location of each point in η independently and uniformly within S with probability $\epsilon > 0$. For every $\epsilon > 0$ and rectangle $R \subseteq S$, we have*

$$\mathbb{E}_{n,1/2}\big[f_R(\eta)f_R(\eta^*)\big] - \mathbb{E}_{n,1/2}\big[f_R(\eta)\big]^2 \to 0, \quad \text{as } n \to \infty.$$

PROOF OVERVIEW. We will follow the approach developed in [2], and revisited in [5], by which the continuum problem is reduced to its discrete counterpart via a two-stage construction. The central idea is to consider a Poisson point process η_k on Ω chosen according to $\mathbb{P}_{kn,p}$ for some $k \geqslant 1$, and obtain a configuration η from η_k via thinning. Conditional on η_k, we may think of η as an element in, and f_R as a function on, $\{0,1\}^{\eta_k}$. This will allow us to study the behavior of f_R via techniques developed for the analysis of Boolean functions.

Russo's approximate 0-1 law [42] says that any sequence of monotone Boolean functions for which the influence of each bit tends to zero exhibits a threshold behavior. A more modern approach to threshold phenomena comes from randomized algorithms via the OSSS inequality [37]. That randomized algorithms can be used to study threshold phenomena has previously been observed by Gady Kozma (see the Appendix of [4]) and in a recent paper by Duminil-Copin, Raoufi and Tassion [19]. The latter also gives an alternative proof of the result due to Bollobás and Riordan that $p_c = 1/2$ for Voronoi percolation on $\mathbb{R}^2$. Randomized algorithms are also connected to noise sensitivity via the Schramm-Steif revealment Theorem [43]. In order to prove Theorem 1.2.1 we shall thus devise an algorithm that, conditional on η_k, queries points in η_k sequentially until the outcome of $f_R(\eta)$ is determined. If, with high probability, the algorithm has low revealment, that is, it is unlikely to query any specific point in η_k, then the result will follow.

The proof of Theorem 1.2.2 will also rely on the reduction to a discrete setting. The dynamical process studied there is conservative, and in that sense related to the concept of exclusion sensitivity studied by Broman, Garban and Steif [17]. We shall follow their approach, and instead of a direct study of the conservative dynamics, we shall show that there is a coupling between $(\eta,\eta(\epsilon))$ and (η,η^*) such that $(f_R(\eta), f_R(\eta(\epsilon)))$ and $(f_R(\eta), f_R(\eta^*))$ agree with high probability. This will be possible due to a result in [17] which says that any noise sensitive sequence of Boolean functions $(f_n)_{n\geqslant 1}$ is unlikely to change when resampling up to order $\sqrt{n}$ of the variables. The result then follows by Theorem 1.2.1 and the observation that

$$\Big|\mathbb{E}_{n,p}\big[f_R(\eta)f_R(\eta(\epsilon))\big] - \mathbb{E}_{n,p}\big[f_R(\eta)f_R(\eta^*)\big]\Big| \leqslant \mathbb{P}_{n,p}\big[f_R(\eta(\epsilon)) \neq f_R(\eta^*)\big].$$

2

DECOUPLING FOR THE EXCLUSION PROCESS AND DETECTION

Apart from independent random walks, the exclusion process is among the simplest conservative interacting particle systems. Therefore, to obtain a general decoupling for conservative systems, it is natural to begin the study by proving such estimates for this model. In this chapter, we develop a decoupling for the exclusion process. As an application, we study a detection model evolving on top of this particle system.

We split our discussion in two sections. First, we prove the decoupling for the exclusion process. The second section is devoted to the study of the detection model and the proof of Theorem 1.1.1.

2.1 THE DECOUPLING

In this section, we prove a decoupling for the exclusion process. This particle system is conservative and particles move as random walks. In this case, the major difficulty is to overcome the slow mixing of the process and this is done using a sprinkling.

In the next subsection, we present some preliminary results about the process. The remaining of the section is devoted to the proof of the decoupling.

2.1.1 *A brief review of the exclusion process*

The exclusion process $(\eta_t)_{t\geqslant 0}$ on $\mathbb{Z}$ is a Markov process with state space $\{0,1\}^{\mathbb{Z}}$ and generator given by

$$Lf(\eta) = \frac{1}{2}\sum_{x\in\mathbb{Z}}\sum_{h=\pm 1}\eta(x)(1-\eta(x+h))\left[f(\eta^{x,x+h})-f(\eta)\right], \tag{2.1.1}$$

where $f:\{0,1\}^{\mathbb{Z}}\to\mathbb{R}$ is any local function and $\eta^{x,y}$ is the configuration given by

$$\eta^{x,y}(z) = \begin{cases} \eta(y), & \text{if } z = x, \\ \eta(x), & \text{if } z = y, \\ \eta(z), & \text{otherwise.} \end{cases}$$

For each $\rho \in [0,1]$, define μ_ρ as the product measure on $\{0,1\}^{\mathbb{Z}}$ with marginals given by

$$\mu_\rho\{\eta : \eta(k) = 1\} = 1 - \mu_\rho\{\eta : \eta(k) = 0\} = \rho, \quad \text{for all } k \in \mathbb{Z}. \tag{2.1.2}$$

It is a well known fact that the process $(\eta_t)_{t\geqslant 0}$ is reversible with respect to the measure μ_ρ. We call the parameter ρ the density of the process if its starting configuration is distributed as μ_ρ. Denote by $\mathbb{P}_\rho$ the distribution of the exclusion process $(\eta_t)_{t\geqslant 0}$ with density ρ.

We also recall a classical graphical construction of the exclusion process that will be useful. This construction is made with the help of the interchange process, that we denote by γ. First consider an independent Poisson process of rate $1/2$ for each edge $(x, x+1)$ of $\mathbb{Z}$. We will represent the Poisson processes in the edges by arrows (as in Figure 2.1). Observe that, for each site $x \in \mathbb{Z}$ and $t \geqslant 0$, there exists an almost sure unique path that starts at (x,t), ends in $\mathbb{Z}\times\{0\}$, goes downwards and is forced to cross all arrows it encounters. We denote the end position of this path by $\gamma_t(x) \in \mathbb{Z}$, the label of the interchange process in site x at time t (see Figure 2.1).

Given an initial configuration η_0 for the exclusion process, we obtain the configuration at time t by setting

$$\eta_t(x) = \eta_0(\gamma_t(x)), \quad \text{for all } x \in \mathbb{Z}. \tag{2.1.3}$$

This construction results in the Markov process with generator given by (2.1.1). Observe that each particle as well as each hole in η_0 performs a continuous time random walk.

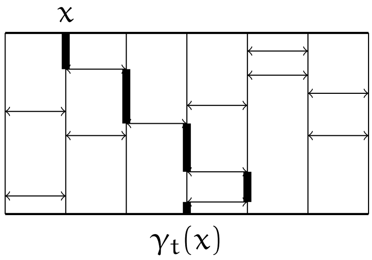

Figure 2.1: The points x and $\gamma_t(x)$.

The space of configurations $\{0,1\}^{\mathbb{Z}}$ has a partial order given by

$$\eta \preceq \xi \text{ if, and only if, } \eta(x) \leqslant \xi(x), \quad \text{for all } x \in \mathbb{Z}. \tag{2.1.4}$$

If we use the same Poisson clocks in the graphical construction presented above for two different initial conditions, this partial order is preserved.

When the starting configuration of the exclusion process is distributed as μ_ρ, by reversibility it is possible to use the graphic construction presented above to construct the exclusion process for negative times: Simply consider an independent copy of the graphical construction, that gives rise to a process $\tilde{\eta}$, and define $\eta_{-t}(x) = \tilde{\eta}_t(x)$, for all $t \leqslant 0$ and $x \in \mathbb{Z}$.

Using a coupling of the measures $(\mu_\rho)_{\rho \in [0,1]}$ that is increasing in the partial order of $\{0,1\}^{\mathbb{Z}}$, all the processes $(\eta_t^\rho)_{t \in \mathbb{R}, \rho \in [0,1]}$ can be constructed in the same probability space in a way that

1. $(\eta_t^\rho)_{t \in \mathbb{R}}$ is an exclusion process with density ρ;
2. if $\rho \leqslant \rho'$, then $\eta_t^\rho \preceq \eta_t^{\rho'}$, for all $t \in \mathbb{R}$.

2.1.2 *Decoupling*

In this subsection, the exclusion process decoupling is proved. In the proof we will assume the existence of the coupling stated in Proposition 2.1.5.

We will work with functions defined in the space of càdlàg trajectories $\mathbf{S} = D_{\mathbb{R}}\{0,1\}^{\mathbb{Z}}$ of the exclusion process. It will be useful to think of the domain of the functions as the set $\{0,1\}^{\mathbb{Z}\times\mathbb{R}}$. If $B \subset \mathbb{R}^2$ is a rectangular box, we denote by $\mathrm{per}(B)$ its perimeter. We are ready to state our decoupling.

Theorem 2.1.1 (Exclusion process decoupling)**.** *There exist positive constants* c_4 *and* C_1 *such that, if* $f_1, f_2 : \{0,1\}^{\mathbb{Z}\times\mathbb{R}} \to [0,1]$ *are two non-decreasing functions with respective supports on space-time boxes* B_1 *and* B_2 *satisfying*

$$d = d(B_1, B_2) \geqslant 6(\mathrm{per}(B_1) + \mathrm{per}(B_2)) + C_1, \tag{2.1.5}$$

then, for any densities $\rho < \rho' \in [0,1]$*,*

$$\mathbb{E}_\rho(f_1 f_2) \leqslant \mathbb{E}_{\rho'}(f_1)\mathbb{E}_{\rho'}(f_2) + c_4 d^2 \exp\left\{-c_4^{-1}(\rho'-\rho)^2 d^{1/4}\right\}. \tag{2.1.6}$$

Remark 2.1.2. We can also take f_1 and f_2 to be two non-increasing functions and assume that $\rho' < \rho \in [0,1]$. The proof carries out in the same way in this case.

Remark 2.1.3. Observe that (2.1.6) is not a correlation estimate, since we need to add the sprinkling in order to have this bound on the error function.

Remark 2.1.4. Recall the construction of the exclusion process from the interchange process in (2.1.3). Using the independence between the configuration η_0 and the interchange process γ, we compute

$$\begin{aligned}
\mathrm{Cov}_\rho(\eta_t(0), \eta_0(0)) &= \mathbb{E}_\rho(\eta_t(0)\eta_0(0)) - \rho^2 \\
&= \mathbb{E}_\rho(\eta_t(0)\eta_0(0)(\mathbf{1}_{\{\gamma_t(0)=0\}} + \mathbf{1}_{\{\gamma_t(0)\neq 0\}})) - \rho^2 \\
&= \rho^2\mathbb{P}[\gamma_t(0) \neq 0] + \rho\mathbb{P}[\gamma_t(0) = 0] - \rho^2 \\
&= (\rho - \rho^2)\mathbb{P}[\gamma_t(0) = 0] \\
&\geqslant \frac{c_5}{\sqrt{t}}.
\end{aligned}$$

The last inequality is a consequence of the fact that $\gamma_t(0)$ has the distribution of a continuous time random walk. In particular, this example implies that the sprinkling is necessary to obtain a rapidly decaying error function in (2.1.6).

For the proof of Theorem 2.1.1, there are two different cases to take care of: Either the horizontal distance between the boxes is large or the vertical distance is. In the first case, we only need to use some moderate deviation estimates to get the bounds we need. In the second case, we use a coupling between two exclusion processes with densities $\rho < \rho'$. This coupling ensures us that the process with density ρ is dominated by the process with higher density in an interval I if the time is large enough. The existence of this coupling is the content of the following proposition.

Proposition 2.1.5. *There exist positive constants c_6 and C_2 such that the following holds. For $\rho < \rho' \in [0, 1]$, any given interval $I = [c, d] \subset \mathbb{R}$ with $c, d \in \mathbb{Z}$ and time $t \geqslant C_2$, there exists a coupling $\mathbb{P}$ of two exclusion processes with independent initial conditions $\eta_0 \sim \mu_\rho$ and $\bar{\eta}_0 \sim \mu_{\rho'}$ in a way that $(\bar{\eta}_s)_{s \geqslant 0}$ is independent of η_0 and*

$$\mathbb{P}\Big[\exists\, x \in I \cap \mathbb{Z} : \eta_t(x) > \bar{\eta}_t(x)\Big] \leqslant c_6 t(t + |I|) \exp\Big\{-c_6^{-1}(\rho' - \rho)^2 t^{1/4}\Big\}.$$

The proof of this Proposition is contained in the next subsection. We now use it to conclude Theorem 2.1.1.

Proof of Theorem 2.1.1. Let d_H and d_V denote the horizontal and vertical distances between the boxes B_1 and B_2:

$$d_H = \inf\{|x - y| : (x, t) \in B_1 \text{ and } (y, s) \in B_2\}, \tag{2.1.7}$$

and

$$d_V = \inf\{|t - s| : (x, t) \in B_1 \text{ and } (y, s) \in B_2\}. \tag{2.1.8}$$

Assume first that

$$d_H \geqslant 3(\operatorname{per}(B_1) + \operatorname{per}(B_2) + d_V). \tag{2.1.9}$$

In this case, observe that, if a particle (or a hole) of the exclusion process touched both boxes, it jumped at least d_H times in at most $\operatorname{per}(B_1) + \operatorname{per}(B_2) + d_V$ units of time. Since these particles (as well as the empty sites) move as random walks, the number of jumps in a given period of time has Poisson distribution. This implies that

$$\begin{aligned}\mathbb{P}&\left[\begin{array}{c}\text{a fixed particle (or hole)}\\ \text{touches both boxes}\end{array}\right]\\ &\leqslant \mathbb{P}\left[\text{Poisson}(\operatorname{per}(B_1) + \operatorname{per}(B_2) + d_V) \geqslant d_H\right] \leqslant e^{-c_{30}(d_V + d_H)},\end{aligned} \tag{2.1.10}$$

with $c_{30} > 0$ given by Lemma A.0.5 in the Appendix.

Now we only need to count how many particles can touch both boxes. Fix an arbitrary site between the two boxes. Each particle that touches both boxes must cross this fixed site. This implies that we can bound the number of particles by the number of clocks that ring in the neighboring edges of this site. It turns out that this also has Poisson distribution with parameter at most $\text{per}(B_1) + \text{per}(B_2) + d_V$. Hence, with probability at least $1 - e^{-c_{30}(d_V + d_H)}$, there are at most d_H particles that can cross this site. Using a union bound, we get

$$\begin{aligned}
\mathbb{P}\left[\begin{array}{c}\text{some particle (or hole)}\\ \text{touches both boxes}\end{array}\right] &\leqslant \mathbb{P}\left[\begin{array}{c}\text{more than } d_H \text{ clocks ring in the}\\ \text{neighboring edges of the fixed site}\end{array}\right] \\
&\quad + d_H \mathbb{P}\left[\begin{array}{c}\text{a fixed particle (or hole)}\\ \text{touches both boxes}\end{array}\right] \\
&\leqslant (1 + d_H) e^{-c_{30}(d_V + d_H)}.
\end{aligned}$$

Now, if we condition on the trajectories inside B_1, we can split the expectation below according to the existence of particles that touch both boxes and conclude that

$$\mathbb{E}_\rho(f_1 f_2) \leqslant \mathbb{E}_\rho(f_1)\mathbb{E}_\rho(f_2) + (1 + d_H) e^{-c_{30}(d_V + d_H)},$$

a stronger estimate than (2.1.6).

Assume now that (2.1.9) does not hold, i.e., assume that $d_H \leqslant 3(\text{per}(B_1) + \text{per}(B_2) + d_V)$. This, combined with Equation (2.1.5), implies that

$$d_V \geqslant \frac{C_4}{4}. \tag{2.1.11}$$

If we take $C_4 \geqslant 4C_2$, the equation above allows us to use Proposition 2.1.5. In this case we use a different approach.

We will assume that the boxes have the form

$$\begin{aligned}
B_1 &= [\tilde{a}, \tilde{b}] \times [-\tilde{t}, 0], \\
B_2 &= [a, b] \times [t, t + s].
\end{aligned}$$

Figure 2.2 can be used as a reference in this part of the proof.

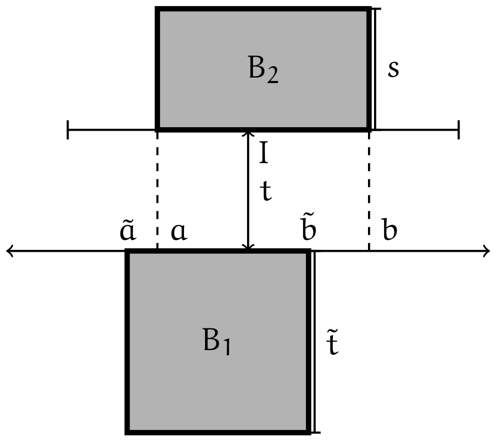

Figure 2.2: The boxes B_1, B_2 and the interval I (used in the proof of Theorem 2.1.1).

Let $\mathscr{F} = \sigma(\eta_u; u \leqslant 0)$ and use the Markov property to get

$$\begin{aligned} \mathbb{E}_\rho(f_1 f_2) &= \mathbb{E}_\rho(\mathbb{E}_\rho(f_1 f_2 | \mathcal{F})) \\ &= \mathbb{E}_\rho(f_1 \mathbb{E}_\rho(f_2 | \mathcal{F})) \\ &= \mathbb{E}_\rho(f_1 \mathbb{E}_\rho(f_2 | \eta_0)). \end{aligned} \tag{2.1.12}$$

To estimate the conditional expectation above, we apply Proposition 2.1.5 with $I = [a - 2s - t, b + 2s + t]$ (see Figure 2.2) and define the event

$$A = \left\{ \eta : \begin{array}{c} \text{all particles of } \eta \text{ that pass through} \\ \text{the box } B_2 \text{ are inside I at time } t \end{array} \right\}.$$

We now split the conditional expectation in Estimate (2.1.12), use Proposition 2.1.5, and the fact that both functions f_1 and f_2 are positive, bounded by one and non-decreasing to get

$$\begin{aligned} \mathbb{E}_\rho(f_1 f_2) &= \mathbb{E}_\rho\Big(f_1(\eta)\mathbb{E}_\rho\Big(f_2(\eta)\Big|\eta_0\Big)\Big) \\ &\leqslant \mathbb{E}\Big(f_1(\eta)\mathbb{E}\Big(f_2(\bar{\eta})\mathbf{1}_{\{\forall x \in I\cap\mathbb{Z}: \eta_t(x) \leqslant \bar{\eta}_t(x)\}}\mathbf{1}_A\Big|\eta_0\Big)\Big) \\ &\quad + \mathbb{P}\left[A^c\right] + \mathbb{P}\left[\exists x \in I \cap \mathbb{Z} : \eta_t(x) > \bar{\eta}_t(x)\right] \\ &\leqslant \mathbb{E}_\rho(f_1)\mathbb{E}_{\rho'}(f_2) + \mathbb{P}\left[A^c\right] + \mathbb{P}\left[\exists x \in I \cap \mathbb{Z} : \eta_t(x) > \bar{\eta}_t(x)\right]. \end{aligned}$$

To bound the last probability above we use Proposition 2.1.5. Now all we need to do is estimate the probability of A^c. We use a similar argument to the one used in the first part of the proof. Begin by observing that, in order for a particle that is at the k-th site at the right of I at time t to enter the box B_2, it is necessary that it jumps more than $2s+t+k$ times before time $t+s$. We also know that the number of jumps of a particle during the time interval $[t, t+s]$ has Poisson distribution with parameter s. This allows us to estimate

$$\mathbb{P}[A^c] \leqslant 2\sum_{k=0}^{+\infty} \mathbb{P}[\text{Poisson}(s) \geqslant 2s+t+k] \leqslant 2\sum_{k=0}^{+\infty} e^{-c_{28}(t+k)} = ce^{-c_{28}t},$$

where in the second inequality we used a simple large deviation estimate given by Lemma A.0.3 in the Appendix.

If we combine all of this, we get

$$\begin{aligned}\mathbb{E}_\rho(f_1 f_2) &\leqslant \mathbb{E}_\rho(f_1)\mathbb{E}_{\rho'}(f_2) + ce^{-c_{28}t} \\ &\quad + c_6 t(t+|I|)\exp\left\{-c_6^{-1}(\rho'-\rho)^2 t^{1/4}\right\} \\ &\leqslant \mathbb{E}_{\rho'}(f_1)\mathbb{E}_{\rho'}(f_2) + c_6 t(t+|I|)\exp\left\{-c_6^{-1}(\rho'-\rho)^2 t^{1/4}\right\},\end{aligned} \tag{2.1.13}$$

by possibly changing the constants in the last estimate.

Now, since $t = d_V$ and $d_H \leqslant 3(\text{per}(B_1) + \text{per}(B_2) + d_V)$, we have

$$d \leqslant \sqrt{2}(d_H + d_V) \leqslant \sqrt{2}\,(4d_V + 3(\text{per}(B_1) + \text{per}(B_2)) \leqslant \sqrt{2}\left(\frac{d}{2} + 4d_V\right),$$

and hence $d \leqslant 4\left(1 - \frac{\sqrt{2}}{2}\right)^{-1} d_V$. Substituting this on estimate (2.1.13) concludes the proof. □

2.1.3 *Coupling*

To conclude the proof of the exclusion process decoupling, we only need to prove Proposition 2.1.5, the content of this subsection. We begin by giving an informal description of the coupling and then we make all the estimates needed to get the domination.

Consider two independent initial configurations $\eta_0 \sim \mu_\rho$ and $\bar{\eta}_0 \sim \mu_{\rho'}$ with $\rho < \rho'$. In our coupling, we want to obtain domination in an interval I for a large time t. Due to the bounded velocity that particles have, we only need to look at particles that at time zero are inside a sufficiently large interval H that contains I. Once we have a bound on the probability that some particle spends time outside H and is inside I at time t, we can restrict ourselves to particles that stay inside the interval H for all times before time t.

We know that each particle, in both processes, performs a random walk. We want our coupling to behave in a way that, if two particles of different process are in the same site of H at some time $s \leqslant t$, then they move together from this time on.

With this greedy strategy it is not possible to get good bounds on the probabilities we need. To get around this problem, at time zero we will match the particles in pairs that will stay together if they meet.

We would like that these particles do not take a long time to do so and to control this we need to ensure that they are close at time zero. Therefore, we introduce a partition of the interval H with intervals $(I_j)_{j=1}^N$ of controlled length. Due to the difference of densities, we expect that, with high probability, each of the intervals I_j has more particles of the configuration $\bar{\eta}_0$ than particles of η_0. When this happens, we can match all the particles of η_0 to some particle of $\bar{\eta}_0$ in a way that they belong to the same interval of the covering at time zero.

Once we have the couples at time zero we need to set the evolution. We will make use of two independent copies of the graphical construction of the exclusion process presented in Subsection 2.1.1. We make the process $\bar{\eta}$ follow one of them and the evolution of the process η will alternate between the two graphical constructions in order to obtain the property that coupled particles stay together.

We can get bounds on the probability that two particles do not meet up to time t, but the decay is not as good as the one in Proposition 2.1.5. To get the desired bound we have to repeat the same procedure more than one time. So we split the time interval $[0, t]$ into smaller intervals $[t_{i-1}, t_i)$, where $0 = t_0 < t_1 < \cdots < t_k = t$, and set the evolution on these intervals. When we reach the end point t_i, we take another matching (this is done in a way that couples that already met stay together) and let the system evolve once again. This will allow us to get stretched exponential bounds as claimed in Proposition 2.1.5.

We now present the rigorous construction of this coupling. First we introduce the intervals we will consider and the matching that we need. The second step is to set the evolution of the coupled process and the last step is to repeat this procedure.

Given the interval $I = [a, b]$, we define $H = [a - \lceil 3t \rceil, b + \lceil 3t \rceil]$ and cover it with disjoint subintervals with length $L = \lfloor t^{1/4} \rfloor$. Let us call $(I_j)_{j=1}^N$ these intervals. Observe that we have at most $|H|$ intervals in the covering. Figure 2.3 can be used to keep track of the notation. It may be necessary to increase the size of H to make sure that every interval of the covering $(I_j)_{j=1}^N$ has exactly L integer points. We need, however, to increase the size of H by at most L.

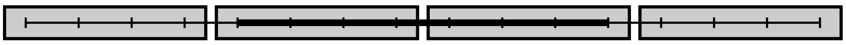

Figure 2.3: The grid is the interval H, the thicker line is the interval I and the covering $(I_j)_{j=1}^N$ is represented by the gray rectangles.

We want to match particles that are inside the same interval of the partition $(I_j)_{j=1}^N$. It is necessary to control the number of particles inside each one of these intervals for the given configurations. This leads us to define $\sigma_j(\eta) = \sum_{x \in I_j} \eta(x)$, the number of particles inside the interval I_j for the configuration η.

Claim 2.1.6. *If* $\eta_0 \sim \mu_\rho$, $\bar{\eta}_0 \sim \mu_{\rho'}$ *are initial configurations with* $\rho < \rho'$ *and* $\bar{\rho} = 1/2(\rho + \rho')$ *then*

$$\mathbb{P}\left[\min_{j \in [N]} \{\sigma_j(\bar{\eta}_0)\} \leqslant \bar{\rho} L\right] \leqslant |H| \exp\left\{-\frac{L(\rho' - \rho)^2}{8}\right\},$$

and

$$\mathbb{P}\left[\max_{j \in [N]} \{\sigma_j(\eta_0)\} \geqslant \bar{\rho} L\right] \leqslant |H| \exp\left\{-\frac{L(\rho' - \rho)^2}{8}\right\}.$$

Proof. Since the invariant measures are product measures, the number of particles in a given interval has the distribution of a sum of i.i.d. random variables that assume only value 0 and 1. This claim is a consequence of a simple large deviation bound, see Corollary A.0.2 in Appendix. □

Remark 2.1.7. Notice that the last claim implies

$$\begin{aligned}\mathbb{P}\left[\exists\, j \leqslant N : \sigma_j(\eta_0) \geqslant \sigma_j(\bar\eta_0)\right] &\leqslant \mathbb{P}\left[\min_{j\in[N]}\{\sigma_j(\bar\eta_0)\} \leqslant \bar\rho L\right] \\ &+ \mathbb{P}\left[\exists j \leqslant N : \sigma_j(\eta_0) \geqslant \sigma_j(\bar\eta_0) \geqslant \bar\rho L\right] \leqslant 2|H| \exp\left\{-\frac{L(\rho'-\rho)^2}{8}\right\}. \quad (2.1.14)\end{aligned}$$

It is really important to notice also that, in the estimate above, we do not need to assume independence between the configurations η_0 and $\bar\eta_0$.

When two configurations $(\eta, \bar\eta)$ are not in the event above, we call it a *good pair of configurations* and denote this by $\eta \preceq_I \bar\eta$. In a good pair of configurations, the matching is possible.

This matching must satisfy two important properties. The first condition is that, if two particles are in the same site, they are paired. The other property we need is that two matched particles are in the same interval of the partition $(I_j)_{j=1}^N$.

Suppose we are given a pair $(\eta, \bar\eta)$ of good configurations. It is easy to construct a deterministic pairing of the particles inside each of the intervals $(I_j)_{j=1}^N$ satisfying the properties listed above. We fix from now on any deterministic construction. Figure 2.4 shows an example of a matching between two configurations.

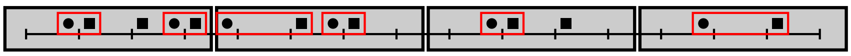

Figure 2.4: A matching of two configurations. Balls represent the process η and squares represent the configuration $\bar\eta$.

Now that we have the matching, it is possible to set the evolution in our coupling. Keep in mind that we start with two independent configurations $\eta_0 \sim \mu_\rho$ and $\bar\eta_0 \sim \mu_{\rho'}$ on $\mathbb{Z}$, with $\rho < \rho'$. We need auxiliary random variables for the evolution: Consider two families of independent Poisson processes $(N_t^{x,i})_{t\geqslant 0, x\in\mathbb{Z}, i=1,2}$ with rate $1/2$. Assume also that the Poisson processes are independent of the configurations η_0 and $\bar\eta_0$.

For the process $\bar\eta$, associate each edge $(x, x+1)$ with the Poisson process $(N_t^{x,2})_{t\geqslant 0}$ and use the graphical construction given in (2.1.3) of Subsection 2.1.1. The matching is used to evolve the process η. If $(\eta_0, \bar\eta_0)$ is not a good pair of configurations, we use $(N_t^{x,1})_{t\geqslant 0, x\in\mathbb{Z}}$ for η in the same way we did with the process $\bar{\eta}$. Suppose now that $\eta_0 \preceq_I \bar{\eta}_0$ and fix an edge $(x, x+1)$. The occupations for the process η in the sites x and $x+1$ are exchanged according to $N^{x,2}$ if one of these sites has a pair of matched particles. Otherwise, the occupation changes in these sites for the process η obey the exponential times given by $N^{x,1}$. Figure 2.5 presents some examples of evolutions.

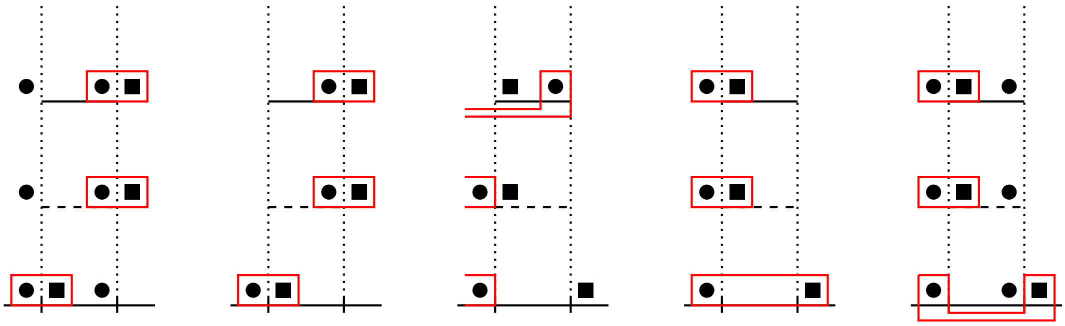

Figure 2.5: Some cases of the evolution in our coupling. We use the same conventions of Figure 2.4. The Poisson process N^1 is represented by the lines and N^2 is represented by the dashed lines.

By construction, $\bar{\eta}$ performs an exclusion process. We need to see that the same happens for the process η.

Claim 2.1.8. *The process η in this coupling is an exclusion process.*

Proof. We will prove that, up to time $T > 0$, the process η is an exclusion process. Notice that, by Borel-Cantelli Lemma, there exist infinitely many edges whose both clocks do not ring up to time T. This implies that we can split the integer lattice into intervals that do not exchange particles until time T. It is not hard to see that in each one of these intervals the process η behaves like an exclusion process: Simply wait until the first of the clocks we are using rings and then, if necessary, update the clocks to mark the next interchange time, according to the coupling. This is exactly an exclusion process in a finite set. This observation implies the claim. □

There are some features about this construction that are important to mention. Observe that, for positive times, it is possible to have two particles on the same site (one from each process) that are not matched. The second observation is that the process $\bar{\eta}$ is independent of η_0, since its evolution depends only on the Poisson processes and its own initial condition.

Finally, observe that the distance between a pair of matched particles (that we call $(Z_s)_{s\geqslant 0}$) follows the law of a continuous time symmetric random walk $(X_s)_{s\geqslant 0}$ sped up by a factor of 2 that dies when it reaches the origin. Besides, since the matched particles lie in the same interval of the partition $(I_j)_{j=1}^N$, the initial position of Z_s is at most L. Using the reflection principle for random walks and also the heat kernel estimates presented in Appendix B we obtain:

$$\begin{aligned}
&\mathbb{P}\left[\begin{array}{c}\text{a fixed pair matched of particles}\\ \text{do not meet before time } t\end{array}\right] \leqslant \max_{0\leqslant k\leqslant L} \mathbb{P}_k\left[\inf_{u\leqslant t} Z_u > 0\right]\\
&\leqslant \max_{0\leqslant k\leqslant L} \mathbb{P}_k\left[\inf_{u\leqslant 2t} X_u > 0\right] = \max_{0\leqslant k\leqslant L} \mathbb{P}_0\left[\sup_{u\leqslant 2t} X_u < k\right] \qquad (2.1.15)\\
&= \mathbb{P}_0\left[\sup_{u\leqslant 2t} X_u < L\right] = 1 - \mathbb{P}_0\left[\sup_{u\leqslant 2t} X_u \geqslant L\right]\\
&\leqslant 1 - 2\mathbb{P}_0\left[X_{2t} > L\right] = \mathbb{P}_0\left[|X_{2t}| \leqslant L\right]\\
&= \sum_{k=-L}^{L} \mathbb{P}_0\left[X_{2t} = k\right] \leqslant \frac{c_{32}(2L+1)}{\sqrt{2t}} \leqslant t^{-1/8},
\end{aligned}$$

if t is large enough, since $L = \lfloor t^{1/4} \rfloor$.

The decay obtained in the last estimate is not good enough to get the bounds we need in the error term. To improve this, we change the pairs at some fixed times, obeying the same matching rule. This implies that the particles that already met remain together and give a new chance for those that did not meet their pair yet.

Let the *coupling times* be the sequence $(kt^{3/4})_{k=1}^{\lfloor t^{1/4}\rfloor}$. At these times, we remake the pairing and continue the evolution as explained before. Notice that, if a particle has met its couple before some coupling time, then, in the new pairing, this particle receives the same partner, since they are at the same site.

Let us now list all the possible ways that domination might fail to hold. First, since all the pairings are made inside the interval H, we must consider the case where some particle of the process η spends time outside H and at time t is inside the interval I. To bound this probability, we can simply observe that the endpoints of the interval H are at linear distance from the interval I and use concentration on the number of particles that can make such journey.

Once we know all the particles remain inside H all the time up to time t, we look at the coupling times. At these times, the matching is remade. Hence, if the configurations are not good for any of them, our coupling fails. To bound this probability we will make use of Remark 2.1.7.

Now, if we ensure also that in all coupling times the configurations are good, the only possibility is that a particle of the process η does not find its couple in any of its allowed attempts. With the aid of (2.1.15) we can bound this last probability. Our task now is to estimate the probability of all events described above.

We begin by setting

$$A = \left\{ \begin{array}{c} \text{there are particles of the process } \eta \text{ that spend time} \\ \text{outside } H \text{ before time } t \text{ and are inside } I \text{ at time } t \end{array} \right\}. \qquad (2.1.16)$$

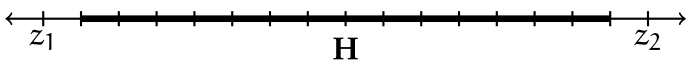

Figure 2.6: The interval H and the points z_1 and z_2.

Let z_1 be the rightmost site at the left of H and z_2 the leftmost site at the right of H (see Figure 2.6). If a particle makes an excursion outside H, it must necessarily pass through z_1 or z_2. Since the number of clocks that ring up to time t in the neighboring edges of each of these points is a random variable with Poisson distribution with parameter t, we have a good control on the number of particles that spend some time in z_1. Now, if a particle passes through z_1 (or z_2, by symmetry) and at time t it is in the interval I, it necessarily jumped at least $\lceil 3t \rceil$ steps in time at most t. Since we also know that all particles evolve as random walks, we conclude that in time t, the number of jumps that a given particle performs is distributed as Poisson with parameter t. Since the particle has at most time t to travel from outside H to I we can estimate:

$$
\begin{aligned}
\mathbb{P}[A] &\leqslant 2\mathbb{P}\left[\begin{array}{c}\text{the number of clocks in the neighbouring edges of } z_1\\ \text{that ring before time t is bigger than 3t}\end{array}\right]\\
&\quad + 2\mathbb{P}\left[\begin{array}{c}\text{there are at most 3t particles that passes through } z_1\\ \text{before time t and at least one of them is inside I at time t}\end{array}\right] \qquad (2.1.17)\\
&\leqslant 2\Big(\mathbb{P}[\mathrm{Po}(t) > 3t] + 3t\mathbb{P}[\mathrm{Po}(t) > 3t]\Big)\\
&\leqslant 2(e^{-t} + 3te^{-t}) \leqslant (6t+2)e^{-t}.
\end{aligned}
$$

Now we focus on the second probability we need to bound. Define

$$
B = \left\{\begin{array}{c}\text{there exists } k \leqslant \lfloor t^{1/4} \rfloor \text{ such that}\\ \text{the pair } (\eta_{kt^{3/4}}, \bar{\eta}_{kt^{3/4}}) \text{ is not good}\end{array}\right\}. \qquad (2.1.18)
$$

Notice that the initial law is invariant, as pointed out in (2.1.2), but the configurations $(\eta_{kt^{3/4}}, \bar{\eta}_{kt^{3/4}})$ are not independent. Combining union bounds with estimate (2.1.14), that also holds for non-independent configurations, gives us

$$
\mathbb{P}[B] \leqslant 2t|H| \exp\left\{-\frac{L(\rho' - \rho)^2}{8}\right\}. \qquad (2.1.19)
$$

Assume now that we are on the event $A^c \cap B^c$ and let

$$
C = \{\text{there exists } x \in I \cap \mathbb{Z} \text{ such that } \eta_t(x) > \bar{\eta}_t(x)\}. \qquad (2.1.20)
$$

In order for $C \cap A^c \cap B^c$ to hold, it is necessary that in all attempts, a particle fails to meet its couple. Since each attempt takes time $t^{3/4}$, we can use the same computations of estimate (2.1.15) (notice that the value of L does not change) to get

$$
\mathbb{P}\left[\begin{array}{c}\text{a fixed pair of particles}\\ \text{do not meet before time } t^{3/4}\end{array}\right] \leqslant t^{-1/16}. \qquad (2.1.21)
$$

To obtain better bounds we use the fact that the matching is remade. We can use union bounds and the fact that our coupling is Markovian to obtain

$$
\begin{aligned}
\mathbb{P}\left[C \cap A^c \cap B^c\right] &\leqslant |H| \sup_{x \in H}\left\{\mathbb{P}\left[\begin{array}{c}\text{a fixed particle of the process } \eta\\ \text{that starts at } x \text{ does not find}\\ \text{any of its couples before time t, } A^c \cap B^c\end{array}\right]\right\}\\
&\leqslant |H| t^{\frac{-\lfloor t^{1/4} \rfloor}{16}} \leqslant |H| \exp\left\{-\frac{t^{1/4}}{32}\log t\right\}. \qquad (2.1.22)
\end{aligned}
$$

Recall the events (2.1.16), (2.1.18) and (2.1.20). We use estimates (2.1.17), (2.1.19) and (2.1.22) to get our final bound

$$\begin{aligned}
&\mathbb{P}\left[\exists x \in I \cap \mathbb{Z} : \eta_t(x) > \bar{\eta}_t(x)\right] \leqslant \mathbb{P}\left[C \cap A^c \cap B^c\right] + \mathbb{P}[A] + \mathbb{P}[B] \\
&\leqslant (6t+2)e^{-t} + 2t|H| \exp\left\{-\frac{L(\rho'-\rho)^2}{8}\right\} + |H| \exp\left\{-\frac{t^{1/4}}{32}\log t\right\}. \quad (2.1.23)
\end{aligned}$$

We can further simplify Equation (2.1.23) by increasing if necessary the value of C_2 and get

$$\mathbb{P}\left[\exists x \in I \cap \mathbb{Z} : \eta_t(x) > \bar{\eta}_t(x)\right] \leqslant c_6 t(t+|I|) \exp\left\{-c_6^{-1}(\rho'-\rho)^2 t^{1/4}\right\},$$

which concludes the proof of Proposition 2.1.5.

2.2 DETECTION

We now consider our detection problem. Recall we let the nodes evolve as an exclusion process with density $\rho \in (0,1)$. The target starts at the origin, moves only at integer times and it can jump to any site within distance R from its current position. Besides, a node detects a target if it shares the same site with it.

Our main theorem states that the detection probability suffers a phase transition as the value of R increases. The proof uses multiscale renormalisation and a comparison of the model with oriented percolation on $\mathbb{Z}^2$. We work with a probability distribution $\mathbb{P}$ of subsets $\mathcal{J} \in \mathbb{Z}^2$ and a set S of paths. Our objective is to find conditions on $\mathbb{P}$ and S that ensure that

$$\mathbb{P}\left[\begin{array}{c}\text{there exists a path in } S \\ \text{whose image is completely contained in } \mathcal{J}\end{array}\right] > 0$$

In the next two subsections, we describe all the necessary hypothesis we need for S and $\mathbb{P}$. The third and fourth subsections are devoted to the construction of a renormalisation scheme for oriented percolation. Finally, in Subsection 2.2.5, we prove Theorem 1.1.1.

2.2.1 *The set* S

In this subsection, we discuss the properties we need for the set of paths.

Fix a convex set $\mathcal{C} \subset \mathbb{R} \times [0,1]$ with $0 \in \partial\mathcal{C}$. We will assume that S is formed by all the functions $f : \mathbb{N}_0 \to \mathbb{Z}^2$ such that

$$f(n+1) - f(n) \in \mathcal{C} \setminus \{0\}. \tag{2.2.1}$$

We also need to ensure the set S is rich enough to allow us to construct crossings of boxes. Hence, we assume

H1. $(0,1) \in \mathcal{C}$;

H2. either $(1,0)$ or $(3,1)$ is in $\mathcal{C}$.

These hypothesis allow the construction of horizontal crossings in boxes of the form $[0,3L] \times [0,L]$ and vertical crossings in boxes of the form $[0,L] \times [0,3L]$, for $L \in \mathbb{N}$.

One of the main reasons why the set S is constructed in this way is a concatenating property we will make use of. For $f \in S$, define $\tilde{f} : \mathbb{R}_+ \to \mathbb{R}^2$ as the linear interpolation of f:

$$\tilde{f}(t) = (\lfloor t \rfloor + 1 - t) f(\lfloor t \rfloor) + (t - \lfloor t \rfloor) f(\lfloor t \rfloor + 1). \tag{2.2.2}$$

Suppose we are given $f, g \in S$ and that there exist $s, t \in \mathbb{R}_+$ such that $\tilde{f}(s) = \tilde{g}(t)$. Then the concatenation of f and g, given by $h : \mathbb{N}_0 \to \mathbb{Z}^2$ as

$$h(n) = \begin{cases} f(n) & \text{if } n \leqslant s, \\ g(\lfloor t \rfloor - \lfloor s \rfloor + n) & \text{if } n > s, \end{cases} \tag{2.2.3}$$

is also in S. This is easily verified by observing that

$$g(\lfloor t \rfloor + 1) - f(\lfloor s \rfloor) \in \mathcal{C}. \tag{2.2.4}$$

We end this subsection with examples of sets that can be considered as the possible paths in our oriented model.

Example 2.2.1. Notice that the set defined in (1.1.4) clearly satisfy all hypothesis above, if we consider the convex set $\mathcal{C}$ to be the convex hull of the points $(0,0)$, $(0,1)$ and $(1,0)$.

Example 2.2.2. The second example is important for the proof of Theorem 1.1.1. Fix $R \geqslant 3$ and define $\tilde{S}_R$ to be the set of paths obtained by using the set $\mathcal{C}_R$ given by the convex hull of $(-R, 1)$, $(0, 0)$ and $(R, 1)$. It is easy to see that these sets also satisfy all the hypothesis above.

2.2.2 *The probability measure* $\mathbb{P}$

In this subsection we state the necessary hypothesis on the measure $\mathbb{P}$.

It will be useful to think of $\mathbb{P}$ as a measure on $\{0, 1\}^{\mathbb{Z}^2}$ and write $\eta : \{0, 1\}^{\mathbb{Z}^2} \to \{0, 1\}$ for the (random) characteristic function given by the (also random) set $\mathcal{I}$.

We require the probability $\mathbb{P}$ to satisfy two conditions that will be discussed in the following.

First we assume that

$$\mathbb{P} \text{ is translation invariant.} \tag{2.2.5}$$

The second condition deals with the decay of correlations. To state this precisely we need some additional notation.

Observe that the set $\{0, 1\}^{\mathbb{Z}^2}$ has a partial order given by

$$\eta \preceq \xi \text{ if and only if } \eta(x) \leqslant \xi(x), \text{ for all } x \in \mathbb{Z}^2. \tag{2.2.6}$$

This allows us to say that a function $f : \{0, 1\}^{\mathbb{Z}^2} \to \mathbb{R}$ is non-increasing if

$$\eta \preceq \xi \text{ implies } f(\eta) \geqslant f(\xi). \tag{2.2.7}$$

We also say that $f : \{0, 1\}^{\mathbb{Z}^2} \to \mathbb{R}$ has support on the box $B = [a, b] \times [c, d] \subset \mathbb{R}^2$ if for every pair of configurations η and ξ

$$\eta|_{B \cap \mathbb{Z}^2} = \xi|_{B \cap \mathbb{Z}^2} \text{ implies } f(\eta) = f(\xi). \tag{2.2.8}$$

We set $\mathrm{per}(B) = 2(|b - a| + |d - c|)$.

We are now ready to state our second assumption on $\mathbb{P}$. It says that there exist constants $C_3, C_4 \geqslant 0$ such that for any non-increasing functions $f, g : \{0, 1\}^{\mathbb{Z}^2} \to [0, 1]$ with respective supports on boxes B_1 and B_2 that satisfy

$$d(B_1, B_2) \geqslant C_3(\mathrm{per}(B_1) + \mathrm{per}(B_2)) + C_4, \tag{2.2.9}$$

we have

$$\mathbb{E}(f(\eta)g(\eta)) \leqslant \mathbb{E}(f(\eta))\mathbb{E}(g(\eta)) + H(d(B_1, B_2)), \tag{2.2.10}$$

where the error term $H : \mathbb{R} \to [0, +\infty)$ is a non-increasing function satisfying

$$\limsup_{x \to +\infty} x^7 H(x) < \frac{1}{200}. \tag{2.2.11}$$

Remark 2.2.3. In Equation (2.2.9) above, we will assume that $C_3 \geqslant 1$. This does not weaken our hypothesis, it just simplifies some computations.

2.2.3 *The box notation*

Before stating precisely our theorem we need some notation. We begin by the scale notation that will be used in our renormalisation scheme.

First we define the sequence of scales as

$$l_0 = 10^{100}, \qquad l_{k+1} = \lfloor l_k^{1/2} \rfloor l_k \qquad \text{and} \qquad L_k = \left\lfloor \left(\frac{3}{2} + \frac{1}{k}\right) l_k \right\rfloor. \tag{2.2.12}$$

Observe that $\frac{l_k^{3/2}}{2} \leqslant l_{k+1} \leqslant l_k^{3/2}$ and that $l_k \leqslant L_k \leqslant 2l_k$ if k is large enough.

This allows us to define the sequence of sets (see Figure 2.7)

$$A_k = [0, l_k] \times [0, L_k] \cup [L_k, l_k + L_k] \times [L_k + l_k, l_k + L_k]. \tag{2.2.13}$$

We also set the box of A_k as

$$B_k = [0, l_k + L_k] \times [0, l_k + L_k]. \tag{2.2.14}$$

Recall the linear interpolation of a function $f \in S$, defined in (2.2.2). We say that $f \in S$ is a crossing of A_k (see Figure 2.7) if there exists $T_f \in \mathbb{R}_+$ such that

$\tilde{f}(0) = f(0) \in [0, l_k] \times \{0\}$;

$\tilde{f}(T_f) \in \{l_k + L_k\} \times [L_k, l_k + L_k]$;

$f(n) \in A_k$, for all $n \in [0, T_f] \cap \mathbb{N}_0$.

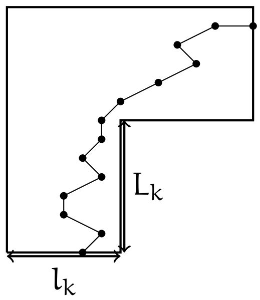

Figure 2.7: The set A_k and a crossing of it.

We say that a crossing f of A_k is open if $\eta(f(n)) = 1$, for all $n \in [0, T_f] \cap \mathbb{N}_0$. Define the events

$$D_k = \{\text{there exists no open crossing } f \text{ of } A_k\}. \tag{2.2.15}$$

These are the events whose probability we are interested in bounding. We also define

$$p_k(S) = \mathbb{P}[D_k]. \tag{2.2.16}$$

Although the probabilities $p_k(S)$ depend on the set S, we will usually omit this dependence and write only p_k.

An important observation is that the event D_k has support in the box B_k, in the sense of (2.2.8). Notice also that the characteristic function of D_k is a non-increasing function.

For $x \in \mathbb{Z}^2$, define the translated sets $A_k(x)$, $B_k(x)$ and write $D_k(x)$ for the event in (2.2.15), when replacing A_k by $A_k(x)$ in its definition.

2.2.4 *Planar oriented percolation*

We begin by stating the theorem we prove in this subsection

Theorem 2.2.4. *Suppose $\mathbb{P}$ satisfy all the hypothesis in Subsection 2.2.2. There exists a $\tilde{k} \in \mathbb{N}$ such that, for any set S satisfying the hypothesis in Subsection 2.2.1, if*

$$p_k \leqslant l_k^{-4}, \qquad \textit{for some } k \geqslant \tilde{k}, \tag{2.2.17}$$

then

$$p_n \leqslant l_n^{-4}, \qquad \textit{for all } n \geqslant k. \tag{2.2.18}$$

Besides,

$$\mathbb{P}[\text{there exists an infinite open path } f \in S] > 0. \tag{2.2.19}$$

Remark 2.2.5. The value of $\tilde{k}$ does not depend on the set S. In fact, its dependence on the probability measure $\mathbb{P}$ is only through the error function H in (2.2.10).

The proof of this theorem begins with a lemma that relates the events D_k and D_{k-1}. We will prove that if D_k holds, then there exists two events in the scale $k-1$ that hold and are far apart, in the sense of (2.2.9).

Lemma 2.2.6. *There exists $k_0 \in \mathbb{N}$ such that for each $k \geqslant k_0$, there exists $M_k \subset \mathbb{Z}^2$ satisfying*

1. $|M_k| \leqslant 10 l_{k-1}^{1/2}$*;*
2. *If D_k happens, there exists $x, y \in M_k$ such that $D_{k-1}(x)$ and $D_{k-1}(y)$ happen and*

$$d(B_{k-1}(x), B_{k-1}(y)) \geqslant C_3(\mathrm{per}(B_{k-1}(x)) + \mathrm{per}(B_{k-1}(y))) + C_4, \tag{2.2.20}$$

with the constants C_3 and C_4 as in (2.2.9).

Proof. We will look into the event D_k for fixed k. The idea of the proof is to construct two chains of events in the scale $k-1$ in a way that, if D_k holds, then one event in each chain necessarily holds.

We will construct a chain of sets of the form A_{k-1} and take the corresponding events D_{k-1}. First, define

$$x_j = j(l_{k-1}, L_{k-1}), \;\; 0 \leqslant j < \frac{L_k + l_k}{L_{k-1}}. \tag{2.2.21}$$

Observe that $(A_{k-1}(x_j))_{0 \leqslant j < \frac{L_k + l_k}{L_{k-1}}}$ crosses the set A_k from the bottom to the top, as in Figure 2.8. Notice that the sequence $(A_{k-1}(x_j))_{0 \leqslant j < \frac{L_k + l_k}{L_{k-1}}}$ does not touch the point (l_k, L_k). This is a simple consequence of

$$l_{k-1}\frac{L_k}{L_{k-1}} + L_{k-1} \leqslant L_k. \tag{2.2.22}$$

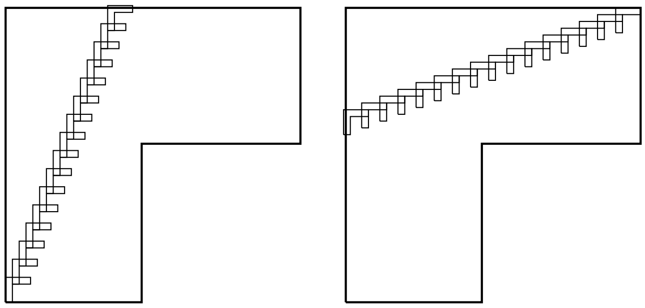

Figure 2.8: The first collection of sets and its reflection.

Reflecting this construction across the diagonal of the set A_k it is possible to find a sequence that connects the left boundary of A_k to its upper right boundary (see Figure 2.8).

We now take the first chain of events to be the corresponding events D_{k-1}, i.e., take $D_{k-1}(x)$ for the values of x in (2.2.21) or in its reflection. This concludes the construction of the first chain.

For the second chain we consider

$$y_j = (l_k, L_k) - j(l_{k-1}, L_{k-1}), \ \ 1 \leqslant j \leqslant \frac{L_k}{L_{k-1}}. \tag{2.2.23}$$

Again in this case we use a reflection argument and construct the events as in the first chain (see Figure 2.9).

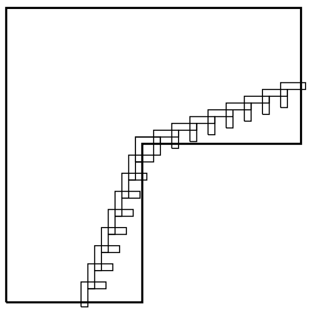

Figure 2.9: The second chain constructed.

Take M_k to be the set of all points $x \in \mathbb{Z}^2$ such that $A_{k-1}(x)$ is in some of the two chains described above. Observe that, by (2.2.21) and (2.2.23),

$$|M_k| \leqslant 2\left(\frac{L_k + l_k}{L_{k-1}} + \frac{L_k}{L_{k-1}}\right) \leqslant 10 l_{k-1}^{1/2},$$

that is exactly the first conclusion of the lemma.

Now, suppose that D_k holds. We will prove that one event in each of the two chains necessarily occurs. Suppose not, and assume, without loss of generality, that all events in the second chain do not happen. In this case, every set $A_{k-1}(x)$ with x as in (2.2.23) or in its reflection, has an open crossing by some function of S. If we concatenate these open paths (see Equation (2.2.3)), we obtain an open crossing of A_k, contradicting out assumption that D_k holds.

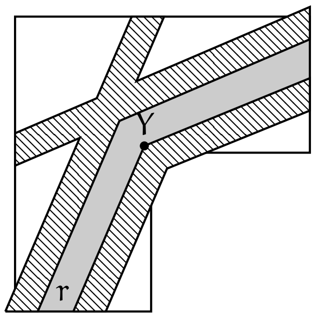

Figure 2.10: The strip, the point Y and the line r. The hashed areas correspond to the space used by both chains of events in the smaller scale and they contain the whole boxes B_{k-1}. The line r is defined by the bottom-right vertices of these boxes.

To get the distance estimate in Equation (2.2.20) observe that there is a strip (see Figure 2.10) that splits the two chains. As a consequence, we can bound the distance in (2.2.20) by the distance between the line r and the point Y of Figure 2.10. Since the line r has equation

$$y = \frac{L_{k-1}}{l_{k-1}} \left(x - L_{k-1}\right), \tag{2.2.24}$$

a simple computations yields

$$\begin{aligned} d(B_{k-1}(x), B_{k-1}(y)) \geqslant d(r, Y) & \\ &= \frac{\left|L_{k-1}l_k - L_{k-1}l_{k-1} - l_{k-1}L_k - l_k^2 - L_{k-1}^2\right|}{\sqrt{l_{k-1}^2 + L_{k-1}^2}} \\ &\geqslant l_{k-1}\frac{1}{\sqrt{5}l_{k-1}}\left[L_k - \lfloor l_{k-1}^{1/2} \rfloor L_{k-1} + L_{k-1}^2 + l_{k-1}^2 + \frac{L_{k-1}^2}{l_{k-1}}\right] \\ &\geqslant l_{k-1}\frac{\lfloor l_{k-1}^{1/2} \rfloor}{\sqrt{5}}\left(\frac{1}{k} - \frac{1}{k-1}\right), \end{aligned} \tag{2.2.25}$$

for k large enough.

Now, since l_k has super-exponential growth (see (2.2.12)), it is easy to conclude that

$$\lim_{k\to\infty} \lfloor l_{k-1}^{1/2} \rfloor \left(\frac{1}{k} - \frac{1}{k-1}\right) = +\infty \tag{2.2.26}$$

If we combine equations (2.2.25) and (2.2.26) and use that $\mathrm{per}(B_{k-1}) \leqslant 12l_{k-1}$, it is easy to conclude that for k large enough we have

$$d(B_{k-1}(x), B_{k-1}(y)) \geqslant C_3(\mathrm{per}(B_{k-1}(x)) + \mathrm{per}(B_{k-1}(y))) + C_4,$$

which is exactly Estimate (2.2.20). □

The lemma above provides us with a way to estimate the probability p_k in terms of p_{k-1}, if k is large. Since we know the realisation of D_k implies that two events of order $k-1$ with indices in M_k hold, and that they satisfy (2.2.20), we can use (2.2.10) with an union bound to obtain

$$p_{k+1} \leqslant |M_k|^2(p_k^2 + H(C_3 l_k)), \tag{2.2.27}$$

since the distance between the boxes is at least $C_3 l_k$ and the error function H is non-increasing. This will help us to conclude the proof of Theorem 2.2.4, our next goal.

Proof of Theorem 2.2.4. Take $\tilde{k} \geqslant k_0$ so that, for all $k \geqslant \tilde{k}$

$$100\left(l_k^{-1} + l_k^7 H(C_3 l_k)\right) \leqslant 1, \tag{2.2.28}$$

where H is the error function in (2.2.10) and C_3 is the constant in (2.2.9). Observe that this is possible by (2.2.11), since $l_k \to +\infty$ as $k \to \infty$.

Suppose now that (2.2.17) holds, i.e., for some $k \geqslant \tilde{k}$, $p_k \leqslant l_k^{-4}$. Inductively, using (2.2.27) we get

$$\begin{aligned} l_{n+1}^4 p_{n+1} &\leqslant 100 l_{n+1}^4 l_n \left(p_n^2 + H(C_3 l_n)\right) \\ &\leqslant 100 l_n^7 \left(l_n^{-8} + H(C_3 l_n)\right) \leqslant 1, \end{aligned}$$

which concludes the proof of (2.2.18).

Let us now verify that percolation occurs with positive probability. We will use an adaptation of the construction in the proof of Lemma 2.2.6. Begin by observing that

$$\sum_{k=1}^{\infty} 10 l_k^{1/2} p_k < \infty. \tag{2.2.29}$$

For each k, let $U_k \subset M_k$ to be the set of points $x \in \mathbb{Z}^2$ such that $D_{k-1}(x)$ is in the second chain constructed in the proof of Lemma 2.2.6. By Borel-Cantelli Lemma, (2.2.29) implies that only finitely many events in the collection $\{D_{k-1}(x), x \in U_k, k \in \mathbb{N}\}$ can hold. Thus, we may assume that $D_{k-1}(x)$ does not hold for all $x \in U_k$ and all sufficiently large k. This implies that for each of these points x it is possible to find an open crossing of $A_{k-1}(x)$ by some function of S. We use a concatenation of the crossings to find an infinite open path $f \in S$, which concludes the proof. $\square$

Remark 2.2.7. If one is interested in the vacant set, the verification of Equation (2.2.10) for non-decreasing functions allows to prove an analogous result from Theorem 2.2.4, but looking for closed paths in S.

It may be the case that the probability measure $\mathbb{P}$ allows us to construct a family $(\mathcal{I}_u)_{u\in\mathcal{U}}$, with either $\mathcal{U} = [0,1]$ or $\mathcal{U} = \mathbb{R}_+$, of increasing subsets of $\mathbb{Z}^2$. In this case we can replace the correlation decay (2.2.10) by the inequality

$$\mathbb{E}_u(fg) \leqslant \mathbb{E}_{u(1-\varepsilon)}(f)\mathbb{E}_{u(1-\varepsilon)}(g) + H(\varepsilon, d(B_1, B_2)), \tag{2.2.30}$$

with error function $H : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ that is non-increasing in each of the variables and satisfies that, for some $\delta > 0$,

$$\lim_{x\to+\infty} x^7 H(x^{-\delta}, x) = 0. \tag{2.2.31}$$

Theorem 2.2.4 can be extended for such values of $u \in \mathcal{U}$. Fix $u_\infty \in \mathcal{U}$, set

$$u_0 = u_\infty \prod_{k=0}^{\infty}(1 - l_k^{-\delta}), \quad u_{k+1} = \frac{u_k}{(1 - l_k^{-\delta})}, \tag{2.2.32}$$

and define

$$p_k = \mathbb{P}_{u_k}[D_k]. \tag{2.2.33}$$

With these definitions, the proof of Theorem 2.2.4 carries on in the same way. In (2.2.29), we can replace p_k by $\mathbb{P}_{u_\infty}(D_k)$ just by noticing that $\mathbb{P}_u(D_k)$ is non-increasing in $u \in \mathcal{U}$.

This generalization states that in order to conclude the existence of percolation for the density u_∞, one needs to understand the probability of the existence of crossings in smaller densities.

Remark 2.2.8. The error decay on (2.2.31) is not sharp and may be modified to fit in other cases. Sometimes, for example, it may be the case that the error depends also on u and not on ε as in (2.2.30). In these cases, we only need to find $\tilde{k}$ large enough so that (2.2.28) holds for all $k \geqslant \tilde{k}$. To do so, one can change the scales u_k, but not the ones in (2.2.12), since the proof of Lemma 2.2.6 strongly uses their growth rate.

2.2.5 *Proof of Theorem 1.1.1*

Here we use all the tools constructed so far to conclude the proof of Theorem 1.1.1. The first step is to modify the problem to fit the hypothesis in our percolation model. The decoupling in the exclusion process will be used to verify the decay correlation on our percolation model.

We begin by simply observing that, since the empty spaces of the exclusion process with density ρ also perform an exclusion process with density $1 - \rho$, we can prove that it is possible for our target to stay always on top of the exclusion process. This is what we will prove here.

Suppose we constructed in the same probability space the collection $(\eta_t^\rho)_{t \in \mathbb{R}, \rho \in [0,1]}$ of exclusion processes with all possible densities in a way that if $\rho \leqslant \rho'$, then $\eta_t^\rho \preceq \eta_t^{\rho'}$ for all real times t.

We will construct the family of sets $(\mathcal{I}_\rho)_{\rho \in [0,1]}$ as described above Equation (2.2.30). We say that a point $(x, t) \in \mathbb{Z}^2$ is closed for the density ρ if there exists some time $s \in [t, t+1)$ such that $\eta_s^\rho(x) = 0$. A point is open if it is not closed. Define the set $\mathcal{I}_\rho$ as the collection of open points for the density ρ. This set is exactly the places where our target is not detected by a hole of the exclusion process η^ρ for a period of time of size one. An important observation is that, if there exists $\tilde{g} \in \tilde{S}_R$ (see Example 2.2.2) such that

$$\mathrm{Range}(\tilde{g}) \subset \mathcal{I}_\rho, \tag{2.2.34}$$

then the projection g on the first coordinate axis of $\tilde{g}$ satisfies

$$\eta_t(g(\lfloor t \rfloor)) = 1, \text{ for all } t \in \mathbb{R}_+. \tag{2.2.35}$$

This implies that non-detection is equivalent to percolation of the set $\mathcal{I}_\rho$ using the set of paths given by Example 2.2.2.

Let us verify that the sets $(\mathcal{I}_\rho)_{\rho\in[0,1]}$ satisfy all the necessary hypothesis. First observe that these sets have a translation invariant distribution, since the same is true for the exclusion process. The decay correlation in (2.2.30) is a direct consequence of Theorem 2.1.1, the decoupling for the exclusion process. Fix $\rho_\infty > 0$ and define $(\rho_k)_{k\geqslant 0}$ as in (2.2.32). Observe that $\rho_0 > 0$

Hence, to conclude Theorem 1.1.1, it is suffice to verify (2.2.17) for some large value of k. We now take $R_k = l_k + L_k + 1$. This implies that D_k holds for the set S_{R_k} if and only if there is no open vertical crossing of the set $[0, l_k] \times [0, L_k]$. To estimate the probability of this event we define

$$J(x) = \left\{ \begin{array}{c} \eta_0(x) = 0 \text{ or there is a poisson clock in a} \\ \text{neighboring edge of } x \text{ that rings before time } 1 \end{array} \right\}. \tag{2.2.36}$$

There are two important observations about the events $(J(x))_{x\in\mathbb{Z}}$: First, observe that $\mathbb{P}_\rho(J(x)) = 1 - \rho e^{-1}$. The second fact is that if $|x-y| \geqslant 2$, then $J(x)$ and $J(y)$ are independent.

The choice of $R_k = l_k + L_k + 1$ helps us to estimate

$$\begin{aligned} p_k &= \mathbb{P}_{\rho_k}\left[\begin{array}{c} [0, l_k] \times [0, L_k] \text{ does not} \\ \text{have a vertical open crossing} \end{array} \right] \\ &= \mathbb{P}_{\rho_k}\left[\begin{array}{c} \text{there exists } u \in [0, L_k) \text{ such that for all } x \in [0, l_k] \\ \text{there exists } t \in [u, u+1) \text{ such that } \eta_t(x) = 0 \end{array} \right] \\ &\leqslant L_k \mathbb{P}_{\rho_k}\left(\bigcap_{x\in[0,l_k]} J(x) \right) \leqslant L_k \mathbb{P}_{\rho_k}\left(\bigcap_{x\in[0,l_k]\cap 2\mathbb{N}} J(x) \right) \\ &\leqslant L_k (1 - \rho_k e^{-1})^{l_k/2} \leqslant l_k^{3/2} (1 - \rho_0 e^{-1})^{l_k/2}. \end{aligned}$$

Now, if we take k large enough, we conclude that $l_k^4 p_k \leqslant 1$ for $R_k = L_k + l_k + 1$. For such a choice of k, and fixing R_k from now on, we can apply Theorem 2.2.4 to conclude the proof of Theorem 1.1.1.

3

ZERO RANGE PROCESS DECOUPLING AND SPREAD OF INFECTIONS

Another conservative interacting particle system that is natural to consider is the zero range process. In this chapter, we prove a decoupling for this model and, as an application, study an infection process evolving on top of this particle system.

Our discussion is split in four section. The first section contains a brief review of the zero range process and the proof of the decoupling. Section 3.2 contains the precise definition of the infection process and some preliminary results about it. Sections 3.3 and 3.4 contain the proofs of Theorems 1.1.3 and 1.1.4, respectively.

3.1 THE DECOUPLING

The decoupling we prove in this section is similar to the exclusion process decoupling proved in the last chapter. However, we consider only the case when the boxes are far away in time. The case where the boxes are distant in the space coordinate will be obtained later on as a consequence of our study of the infection process.

In this section, we first review some basic facts about the zero range process and then focus on the proof of the decoupling.

3.1.1 *The zero range process*

In this subsection we define and recall some properties of the zero range process.

The zero range process in $\mathbb{Z}$ is a particle system where particles interact only when they are at the same site. This interaction alters the jump rate of a particle according to the number of particles that share its site.

Fix a non-negative function $g : \mathbb{N}_0 \to \mathbb{R}_+$ with $g(0) = 0$ and a translation invariant transition probability $p(\cdot,\cdot)$ on $\mathbb{Z}$. The zero range process with rate function g, transition probability p and initial state $\eta_0 \in \mathbb{Z}^{\mathbb{N}_0}$ is the particle system on $\mathbb{N}_0^{\mathbb{Z}}$ with infinitesimal generator given by

$$Lf(\eta) = \sum_{x\in\mathbb{Z}} g(\eta(x)) \sum_{y\in\mathbb{Z}} p(x,y)\Big[f(\eta^{x,y}) - f(\eta)\Big],$$

where $\eta^{x,y}$ is the configuration obtained from η by taking one particle from site x and placing it at site y and f is any bounded local function. We will provide soon classical conditions for the existence of the process.

In this process, particles interact only when they are at the same site. The interaction is given by the function g that controls the jump rate.

We are interested in the case where p is the nearest-neighbor symmetric transition probability, $p(0,1) = p(0,-1) = 1/2$, and g satisfies (1.1.1).

For $\phi \in \mathbb{R}_+$, consider the product measure with marginals ν_ϕ given by

$$\nu_\phi(k) = \frac{1}{Z(\phi)}\frac{\phi^k}{g(k)!}, \text{ for all } k \in \mathbb{N}_0, \tag{3.1.1}$$

where $g(k)! = g(k)\cdot g(k-1)\cdots g(1)$, $g(0)! = 1$ and $Z(\phi)$ is a normalizing constant:

$$Z(\phi) = \sum_{k=0}^{\infty} \frac{\phi^k}{g(k)!}. \tag{3.1.2}$$

Observe that the lower bound in Assumption (1.1.1) implies that, for all $\phi \in \mathbb{R}_+$, $Z(\phi) < \infty$ and hence ν_ϕ is well-defined for all values of $\phi \in \mathbb{R}_+$. These probabilities measures are invariant and compose the collection of invariant measures for the zero range process that we consider. We remark however that these are not a complete set of invariant measures for the zero range process, as proved in [9].

Remark 3.1.1. We will use a slight abuse of notation, by denoting the product measure and its marginals by the same symbols.

In general, the parameter ϕ is not the density of the process. In fact, for the measure ν_ϕ, the expected number of particles in each site is given by

$$R(\phi) = \frac{1}{Z(\phi)} \sum_{k=0}^{\infty} \frac{k\phi^k}{g(k)!} = \phi \frac{Z'(\phi)}{Z(\phi)}. \tag{3.1.3}$$

The function $R : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing bijection. This implies that we can parametrize the measures in (3.1.1) by density:

$$\mu_\rho = \nu_{R^{-1}(\rho)}. \tag{3.1.4}$$

We refer to Section 2.3 of [32] for further information about these measures.

Theorem 1.4 from [9] implies that the process starting from any measure μ_ρ exists with probability one.

In order to prove our decoupling, an important ingredient is concentration of the invariant measures. This is the content of the next proposition.

Proposition 3.1.2. *Assume $X_k \sim \mu_\rho$, for $k \leqslant n$, are independent and fix $\epsilon \in (0, 1]$. Then*

$$\mathbb{P}_\rho \left[\sum_{k=1}^{n} X_k \geqslant (\rho + \epsilon)n \right] \leqslant e^{-c(\rho)\epsilon^2 n}, \tag{3.1.5}$$

and

$$\mathbb{P}_\rho \left[\sum_{k=1}^{n} X_k \leqslant (\rho - \epsilon)n \right] \leqslant e^{-c(\rho)\epsilon^2 n}, \tag{3.1.6}$$

where $c(\rho)$ is a constant that depends on ρ and is uniformly bounded on compact intervals of $[0, \infty)$.

We defer the proof of this proposition to the Appendix.

For future reference, we introduce a constant $c_7 > 0$ defined by the fact that

$$\frac{Z(eR^{-1}(\rho))}{Z(R^{-1}(\rho))} e^{-c_7\rho} \leqslant 1, \quad \text{for all } \rho \in [0, \rho_+]. \tag{3.1.7}$$

This constant satisfies that, for all $\rho \in [0, \rho_+]$,

$$\mathbb{P}_\rho \left[\sum_{k=1}^{n} X_k \geqslant c_7 \rho n + t \right] \leqslant e^{-t}. \tag{3.1.8}$$

3.1.2 *A graphical construction for the zero range process*

This subsection is devoted to a graphical construction for the zero range process. This construction will be used in the coupling presented in Subsection 3.1.4.

In this construction of the process, every site $x \in \mathbb{Z}$ has an associated Poisson point process $\mathcal{P}(x)$ that will control the jumps on its corresponding site. The points of the process have the form (t, n, u, h), where t describes the time of a jump, n describes the height of the particle that is moved, u is an uniformly distributed auxiliary random variable that will help in controlling the jump rate, and h is the direction of the jump. Each Poisson point process takes values in $\mathbb{R}_+ \times \mathbb{N} \times [0,1] \times \{-1,+1\}$ and has intensity measure $\Gamma_+\lambda \otimes \mu \otimes \lambda \otimes {}^1/_2(\delta_{-1} + \delta_{+1})$, where Γ_+ is the constant defined in (1.1.1), μ is the counting measure and λ is the usual Lebesgue measure.

The evolution is set in the following way. Suppose that, at some site x, we have a point from the Poisson point process of the form (t, n, u, h) and that the configuration, at this time, has at least n particles at x. The particle at height n will perform a jump directed according to h if

$$u \leqslant \frac{g(n) - g(n-1)}{\Gamma_+}. \tag{3.1.9}$$

If the jump is allowed, all particles that are above the selected particle at site x go down one position and the particle that jumps lands at the top of its next pile.

Whenever (3.1.9) does not hold or the pile contains less than n particles, the jump is simply suppressed.

This construction allows us to bound the probability that a site has many particles at some time in $[0, t]$, as stated in the next lemma.

Lemma 3.1.3. *Let* $A(u,t) = (2u + 4\Gamma_+ t + 1)(\rho + 1) + 1$. *There exists* $c_8 > 0$ *such that, for all* $\rho \in [0, \rho_+]$,

$$\mathbb{P}_\rho \left[\begin{array}{c} \eta_s(0) \geqslant A(u,t), \\ \textit{for some } s \in [0,t] \end{array} \right] \leqslant c_8 (t+1) e^{-c_8^{-1} u}. \tag{3.1.10}$$

This lemma is not sharp and can be regarded as a rough estimate that will be used to obtain some bounds later in the text. The quantity $A(u,t)$ is chosen so that we can use concentration of the invariant measure in a large interval around the origin. The strategy of the proof is to observe that if the event in the lemma holds, either some large interval has many particles or some particle reaches the origin from very far away.

Proof. Let B be the event described in the lemma. Observe that, if B holds, either the interval $J_t = [-\lfloor 2\Gamma_+ t + u \rfloor, \lfloor 2\Gamma_+ t + u \rfloor]$ contains more that $A(u,t) - 1$ particles at time zero or some particle that started outside J_t reaches zero before time t. Since each particle jumps at most Poisson$(\Gamma_+ t)$ times, Proposition 3.1.2, and Equation (3.1.8) can be used to bound

$$\begin{aligned}\mathbb{P}[B] &\leqslant \mathbb{P}[\sum_{k \in J_t} \eta_0(k) \geqslant A(u,t) - 1] \\ &\quad + 2 \sum_{y \geqslant 2\Gamma_+ t + u} \mathbb{P}[\eta_0(y) \geqslant c_7 \rho + y] + (c_7 \rho + y)\mathbb{P}[\text{Poisson}(\Gamma_+ t) \geqslant y] \\ &\leqslant e^{-c_8^{-1} u} + 2 \sum_{y \geqslant 2\Gamma_+ t + u} e^{-y} + (c_7 \rho + y) e^{-\frac{y}{3}} \leqslant c_8 (t+1) e^{-c_8^{-1} u},\end{aligned} \tag{3.1.11}$$

concluding the proof. □

3.1.3 *Vertical decoupling*

This subsection contains the proof of the vertical decoupling for the zero range process. The techniques are similar to the ones used in the proof of the decoupling for the exclusion process.

Given two space-time boxes $B_1, B_2 \subset \mathbb{Z} \times \mathbb{R}_+$, recall that the definition of vertical distance between them given in (2.1.8) is

$$d_V = \inf\{|t - s| : (x,t) \in B_1 \text{ and } (y,s) \in B_2\}. \tag{3.1.12}$$

Our decoupling considers functions whose vertical distance between the supports is large.

Theorem 3.1.4. *Fix $\rho_+ > 0$. There exist positive constants C_5 and c_9 such that, for any two square boxes B_1 and B_2 of side-length s satisfying*

$$d_V = d_V(B_1, B_2) \geqslant C_5, \tag{3.1.13}$$

and any two non-decreasing functions of the space-time configurations $f_1, f_2 : \mathbb{N}_0^{\mathbb{Z}\times\mathbb{R}_+} \to [0,1]$ *with respective supports in* B_1 *and* B_2*, we have, for any* $\rho \in [0, \rho_+]$ *and* $\epsilon \in (0,1]$*,*

$$\mathbb{E}_\rho[f_1 f_2] \leqslant \mathbb{E}_{\rho+\epsilon}[f_1]\mathbb{E}_{\rho+\epsilon}[f_2] + c_9\, d_V(d_V + s + 1)e^{-c_9^{-1}\epsilon^2 d_V^{1/4}}. \tag{3.1.14}$$

Remark 3.1.5. One can also take f_1 and f_2 non-increasing and assume that $\epsilon \in [-1, 0)$. The proof carries out in the same way in this case.

Remark 3.1.6. As in the case of the exclusion process, (3.1.14) is not a correlation estimate, since we need to add a sprinkling in order to have this bound on the error function. A question that rises naturally from the theorem above is if it is possible to take $\epsilon = 0$, and do not use the sprinkling. In [26], the authors consider a particle system composed by independent random walks evolving in discrete time. The continuous version of their model corresponds to a zero range process with rate funciton $g(n) = n$. They prove that the correlations do not decay as fast as the bound given in our theorem. In fact, Equation (2.11) from [26] provides an example where the correlations decay polynomially.

The next proposition is a central tool used in the proof of the vertical decoupling. It provides a coupling between two zero range processes with densities ρ and $\rho + \epsilon$ in a way that the process with larger density dominates the other in a fixed interval for some large time t.

Proposition 3.1.7. *Given* $\rho_+ > 0$*, there exist positive constants* c_{10} *and* C_6 *such that, for any* $t \geqslant C_{10}$*, interval* $I \subset \mathbb{R}$*, density* $\rho \in [0, \rho_+]$ *and* $\epsilon \in (0,1]$*, there exists a coupling between two zero range processes* $(\eta_s)_{s\geqslant 0}$ *and* $(\bar{\eta}_s)_{s\geqslant 0}$ *such that*

1. $(\eta_s)_{s\geqslant 0}$ *has density* ρ *and* $(\bar{\eta}_s)_{s\geqslant 0}$ *has density* $\rho + \epsilon$*;*
2. $(\bar{\eta}_s)_{s\geqslant 0}$ *is independent from* η_0*;*
3. $$\mathbb{P}\left[\begin{array}{c}\textit{there exists } x \in I \\ \textit{such that } \eta_t(x) > \bar{\eta}_t(x)\end{array}\right] \leqslant c_6 t(|I| + t)e^{-c_6^{-1}\epsilon^2 t^{1/4}}. \tag{3.1.15}$$

We postpone the proof of this proposition to the next subsection. Assuming it, we are in position to prove the decoupling for the zero range process.

Proof of Theorem 3.1.4. In the proof, it is useful to keep Figure 3.1 in mind. Without loss of generality, we assume that the boxes have the form

$$B_1 = [-s/2, s/2] \times [-s, 0],$$
$$B_2 = [a, a+s] \times [d_V, d_V + s],$$

where $s/2$ and a are positive integer numbers.

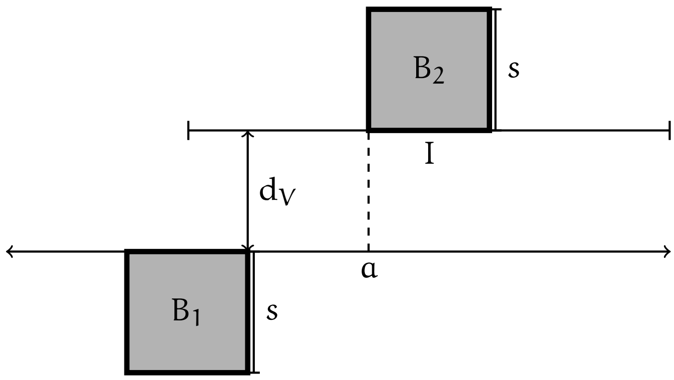

Figure 3.1: The boxes B_1, B_2 and the interval I.

Let $I = [-\lceil 2\Gamma_+ s + d_V \rceil - s/2, s/2 + \lceil 2\Gamma_+ s + d_V \rceil]$ and define the event

$$E = \left\{ \begin{array}{c} \text{some particle of } \eta \text{ is outside I} \\ \text{at time } d_V \text{ and enters the box } B_2 \end{array} \right\}. \tag{3.1.16}$$

If $d_V \geqslant C_6$, we can use the coupling of Proposition 3.1.7 with the interval I. Define the bad event for the coupling

$$F = \left\{ \begin{array}{c} \text{there exists } x \in I \\ \text{such that } \eta_{d_V}(x) > \bar{\eta}_{d_V}(x) \end{array} \right\}. \tag{3.1.17}$$

Markov's property, Proposition 3.1.7 and the fact that the functions f_1 and f_2 are non-decreasing can be used to obtain

$$\begin{aligned} \mathbb{E}_\rho[f_1 f_2] &\leqslant \mathbb{E}_\rho[f_1 \mathbb{E}_\rho[f_2|\eta_0]] \\ &\leqslant \mathbb{E}_\rho[f_1 \mathbb{E}[f_2(\eta)(\mathbf{1}_{E^c \cap F^c} + \mathbf{1}_E + \mathbf{1}_F)|\eta_0]] \\ &\leqslant \mathbb{E}_\rho[f_1 \mathbb{E}[f_2(\bar{\eta})\mathbf{1}_{E^c \cap F^c}|\eta_0]] + \mathbb{P}[E] + \mathbb{P}[F] \\ &\leqslant \mathbb{E}_\rho[f_1]\mathbb{E}_{\rho+\epsilon}[f_2] + \mathbb{P}[E] + \mathbb{P}[F]. \end{aligned} \tag{3.1.18}$$

Proposition 3.1.7 implies, by possibly increasing constants, that

$$\mathbb{P}[F] \leqslant c_{10}\, d_V (d_V + s) e^{-c_{10}^{-1} \epsilon^2 d_V^{1/4}}, \tag{3.1.19}$$

It remains to bound the probability of E. Here, we apply the same ideas from the proof of Lemma 3.1.3. We use symmetry and the fact that, in order for a particle that is at site $y + \lceil 2\Gamma_+ s + d_V \rceil$ at time d_V to enter B_2, it is necessary for it to jump at least $\lceil 2\Gamma_+ s + d_V \rceil$ times before time $d_V + s$. Since the number of jumps a particle performs between times d_V and $d_V + s$ is bounded by a Poisson$(\Gamma_+ s)$ random variable, we obtain

$$\begin{aligned}
\mathbb{P}[E] &\leqslant 2 \sum_{y \geqslant 0} \mathbb{P}[\eta_0(y + \lceil 2\Gamma_+ s + d_V \rceil) \geqslant c_7 \rho + d_V + 1 + y] \\
&\qquad + 2 \sum_{y \geqslant 0} (c_7 \rho + d_V + 1 + y) \mathbb{P}[\text{Poisson}(\Gamma_+ s) \geqslant y + \lceil 2\Gamma_+ s + d_V \rceil] \\
&\leqslant 2 \sum_{y \geqslant 0} e^{-y - d_V} + (c_7 \rho + d_V + 1 + y) e^{-y - d_V} \\
&\leqslant c(d_V + 1) e^{-d_V}.
\end{aligned} \tag{3.1.20}$$

Combining Equations (3.1.18), (3.1.19), (3.1.20), and possibly changing constants concludes the proof. □

3.1.4 *The coupling*

This subsection is devoted to the construction of the coupling stated in Proposition 3.1.7. The construction is similar to the one used for the exclusion process. However, since the number of particles in each site is not necessarily bounded, we need to be more careful in the estimates. We begin this section with an informal description of the coupling and then proceed to the estimates we need.

Fix two initial independent configurations $\eta_0 \sim \mu_\rho$ and $\bar{\eta}_0 \sim \mu_{\rho+\epsilon}$. The strategy is to match the particles of the configuration η_0 to particles of the configuration $\bar{\eta}_0$. Once this matching is constructed, we set the joint evolution of the pair $(\eta_s, \bar{\eta}_s)$.

The pair evolves in such a way that, if two matched particles stay at any time at the same site, they keep moving together. This will help to assure that $\eta_t(x) \leqslant \bar{\eta}_t(x)$, for every $x \in I$, with high probability.

The correct construction of the matching is important to ensure that each pair meets fast enough with large probability. This is done by restricting the distance between two particles that are matched.

For the evolution, we use the matching and two independent copies of the graphical construction presented in Subsection 3.1.2. This will help to evolve both processes in a way that particles that have met their pairs do not disturb the particles that still did not and hence do not decrease the probability of this event.

However, this construction is not enough, since, as we will see, the decay of the probability that two matched particles do not meet is related with the probability that a random walk does not reach zero. We improve this bound by remaking the matching at some fixed times, allowing particles to have new pairs and new chances to meet.

We now begin the construction of the coupling.

Remark 3.1.8. All constants that appear in the remaining of this subsection are uniformly bounded for any $\rho \in [0, \rho_+]$ and may depend also on Γ_- and Γ_+. We will omit these dependencies.

The first step is to fix an interval H that contains $I = [a, b]$. In our case, we set $H = [a - \lceil 3\Gamma_+ t \rceil, b + \lceil 3\Gamma_+ t \rceil]$. Now, split H into a collection of subintervals $(I_j)_{j=1}^N$. We will assume that all intervals I_j have the same size $L = \lfloor t^{1/4} \rfloor$. It is possible to ensure this if we increase the size of H by at most L. Besides, the number of intervals N is clearly bounded by $|H|$.

For any configuration $\bar{\eta}$, denote by $\sigma_j(\bar{\eta}) = \sum_{x \in I_j} \bar{\eta}(x)$ the number of particles of $\bar{\eta}$ inside the interval I_j. We have the following claim.

Claim 3.1.9. *If* $\eta \sim \mu_\rho$ *and* $\bar{\eta} \sim \mu_{\rho+\epsilon}$, *with* $\epsilon \in (0, 1]$, *then*

$$\mathbb{P}[\exists\, j \leqslant N : \sigma_j(\eta) > \sigma_j(\bar{\eta})] \leqslant 2N e^{-c_{11} \epsilon^2 t^{1/4}}, \tag{3.1.21}$$

even if the configurations are not independent.

Proof. It follows directly from Proposition 3.1.2. $\square$

We now sample independently two configurations $\eta_0 \sim \mu_\rho$ and $\bar{\eta}_0 \sim \mu_{\rho+\epsilon}$ and assume that the event in (3.1.21) does not hold.

The next step is to match the two configurations inside each of the intervals of the partition. In this matching, each particle of the configuration η_0 that lies inside the interval I_j will be paired to a particle of the configuration $\bar{\eta}_0$ that is inside the same interval, but this construction is done in a special way. In the first step, for each site $x \in I_j$, we match the largest number possible of particles of η_0 at x to particles of $\bar{\eta}_0$ that are at the same site (see Figure 3.2). Once this is done we can finish. There are many ways to match the remaining particles in a deterministic way. We fix an arbitrary one from now on.

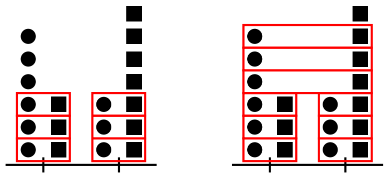

Figure 3.2: The construction of a matching between two configurations. Balls represent the process η and squares represent the configuration $\bar{\eta}$. First, pair as many particles of η to particles of $\bar{\eta}$ that are at the same site as possible, and then complete the construction in an arbitrary deterministic way.

Once we have this matching, it is time to set the evolution. We proceed as follows. Let $\mathscr{P}_1 = (\mathcal{P}_1(x))_{x\in\mathbb{Z}}$ and $\mathscr{P}_2 = (\mathcal{P}_2(x))_{x\in\mathbb{Z}}$ be two independent copies of the graphical construction described in Subsection 3.1.2. We use the clocks from $\mathscr{P}_1$ to evolve the process $(\bar{\eta}_s)_{s\geqslant 0}$. The process $(\eta_s)_{s\geqslant 0}$ will alternate between both constructions: If a particle of η has met its pair, it uses the clocks from $\mathscr{P}_1$. Otherwise, it moves with the graphical construction $\mathscr{P}_2$.

Observe however that, if a particle always jumps to the top of its new pile, then it is not necessarily true that particles that meet jump together. This is fixed by updating the matching after each jump. If a pair of matched particles jumps together, they will land at the bottom of the pile. Whenever a particle jumps alone, it will look for its matching particle at the next pile: If the particle and its pair are at the same site, they will both move to the bottom of the pile. Otherwise, the new particle will land on the top of its new corresponding pile.

This construction ensures that, if two particles have met, they remain together, and allows for pairs of particles that did not meet to do so.

Since the process $(\bar{\eta}_s)_{s\geqslant 0}$ follows the original graphical construction up to changing heights of particles in the piles, it clearly behaves like a zero range process. It remains to prove that the same is true for the process $(\eta_s)_{s\geqslant 0}$, stated as a claim and proved in Appendix D.

Claim 3.1.10. *The process* $(\eta_s)_{s\geqslant 0}$ *is a zero range process.*

We now have the main part of the coupling, we work the details in order to obtain the bound in (3.1.15). We introduce the *coupling times* $(t_k)_{k=0}^{\lfloor t^{1/4} \rfloor}$ defined by $t_k = kt^{3/4}$. At these times, the matching is remade preserving the couples already formed. This procedure will help the particles that still have not found their couples by giving them new pairs that are hopefully closer to them than their old partners were.

We now need to bound the probability that some particle that lies inside the interval I at time t did not find a couple during the time interval $[0, t]$. Let

$$A = \left\{ \begin{array}{c} \text{there exists a particle from } \eta \text{ that} \\ \text{is inside I at time t and did not find} \\ \text{a couple in any of its attempts} \end{array} \right\}. \tag{3.1.22}$$

To bound the probability of A, we begin by bounding the probability of some bad events. The first event we introduce is related to the possibility that some particle that ends up in the interval I at time t does not find a couple because it is outside the interval H at some time where the matching is remade. We consider

$$B = \left\{ \begin{array}{c} \text{there exists a particle from } \eta \\ \text{that spends time outside H} \\ \text{and is inside I at time t} \end{array} \right\}. \tag{3.1.23}$$

The second event deals with the possibility that, for some coupling time, it is not possible to construct the matching. Recall that $\sigma_j(\bar{\eta}) = \sum_{x\in I_j} \bar{\eta}(x)$ and define the event

$$C = \left\{ \begin{array}{c} \text{there exist a coupling time } t_k \text{ and } j \in [N] \\ \text{such that } \sigma_j(\eta_{t_k}) \geqslant \sigma_j(\bar{\eta}_{t_k}) \end{array} \right\}. \tag{3.1.24}$$

The bound in the probability of C follows from Claim 3.1.9 and union bound. We obtain

$$\mathbb{P}[C] \leqslant 2(t^{1/4}+1)Ne^{-c_{11}\varepsilon^2 t^{1/4}}. \tag{3.1.25}$$

The bound on the probability of B is more delicate, and we state it as a claim.

Claim 3.1.11. *There exists a constant* $c_{12} > 0$ *such that, if* t *is large enough,*

$$\mathbb{P}[B] \leqslant c_{12}t^2 e^{-c_{12}^{-1}t}. \tag{3.1.26}$$

Proof. Denote by x the leftmost site at the right of H. By symmetry, we only need to bound the probability that there exists a particle that spends some time at x and is inside I at time t.

To bound the probability of B, let $A(t,t)$ as in Lemma 3.1.3 and consider

$$\tilde{A} = \left\{ \begin{array}{c} \eta_s(x) \geqslant A(t,t), \\ \text{for some } s \in [0,t] \end{array} \right\}, \tag{3.1.27}$$

and

$$\tilde{B} = \left\{ \begin{array}{c} \text{more than } 3\Gamma_+ A(t,t)t \text{ clocks} \\ \text{ring at } x \text{ before time } t \end{array} \right\}. \tag{3.1.28}$$

Since the number of jumps a fixed particle performs before time t is bounded by a Poisson random variable with mean $\Gamma_+ t$, union bounds gives

$$\begin{aligned} \mathbb{P}[B] &\leqslant 2\left(\mathbb{P}[\tilde{A}] + \mathbb{P}[\tilde{B}\cap\tilde{A}^c] + 3\Gamma_+A(t,t)t\mathbb{P}[\text{Poisson}(\Gamma_+ t) \geqslant 3\Gamma_+ t]\right) \\ &\leqslant 2\left(\mathbb{P}[\tilde{A}] + \mathbb{P}[\tilde{B}\cap\tilde{A}^c] + 3\Gamma_+A(t,t)te^{-\Gamma_+ t}\right). \end{aligned} \tag{3.1.29}$$

It remains to bound the probability of the events $\tilde{A}$ and $\tilde{B}\cap\tilde{A}^c$. For the later, observe that, in $\tilde{A}^c$, the number of clocks that ring at site x before time t is dominated by a Poisson random variable with mean $\Gamma_+A(t,t)t$. This implies

$$\mathbb{P}[\tilde{B}\cap\tilde{A}^c] \leqslant \mathbb{P}[\text{Poisson}(\Gamma_+A(t,t)t) \geqslant 3\Gamma_+A(t,t)t] \leqslant e^{-\Gamma_+ t}. \tag{3.1.30}$$

A bound on the probability of $\tilde{A}$ is obtained in Lemma 3.1.3. Combining Equations (3.1.10), (3.1.29), (3.1.30) and increasing, if necessary, the value of t, we conclude the claim. □

Assume we are in the event $B^c \cap C^c$. The next step is to bound the probability that a fixed particle that lies inside I at time t do not find a couple.

First, observe that, since particles of both process move faster than random walks with jump rate Γ_-, the probability that two particles do not meet between two coupling times is at most the probability that a random walk with jump rate $2\Gamma_-$ and starting somewhere in the interval $[0, L]$ do not reach zero before time t_1. Since the initial distance between the pair is at most L, if $(X_s)_{s\geqslant 0}$ is a random walk that jumps with rate one, standard heat-kernel bounds allows us to estimate

$$\begin{aligned}
\mathbb{P}\left[\begin{array}{c}\text{a fixed pair matched of particles}\\ \text{do not meet before time } t^{3/4}\end{array}\right] &\leqslant \max_{0\leqslant k\leqslant L} \mathbb{P}_k\left[\inf_{u\leqslant 2\Gamma_- t^{3/4}} X_u > 0\right]\\
&= \max_{0\leqslant k\leqslant L} \mathbb{P}_0\left[\sup_{u\leqslant 2\Gamma_- t^{3/4}} X_u < k\right] = \mathbb{P}_0\left[\sup_{u\leqslant 2\Gamma_- t^{3/4}} X_u < L\right]\\
&= 1 - \mathbb{P}_0\left[\sup_{u\leqslant 2\Gamma_- t^{3/4}} X_u \geqslant L\right]\\
&\leqslant 1 - 2\mathbb{P}_0\left[X_{2\Gamma_- t^{3/4}} > L\right] = \mathbb{P}_0\left[|X_{2\Gamma_- t^{3/4}}| \leqslant L\right]\\
&= \sum_{k=-L}^{L} \mathbb{P}_0\left[X_{2\Gamma_- t^{3/4}} = k\right] \leqslant \frac{C(2L+1)}{\sqrt{2\Gamma_- t^{3/4}}} \leqslant t^{-1/16},
\end{aligned} \tag{3.1.31}$$

if t is large enough.

Since we are assuming we are in the event $B^c \cap C^c$, we bound

$$\mathbb{P}\left[\begin{array}{c}\text{a particle that is inside } I\\ \text{at time } t \text{ do not meet}\\ \text{any of its pairs}, B^c \cap C^c\end{array}\right] \leqslant t^{-\frac{1}{16}\frac{t^{\frac{1}{4}}}{2}} \leqslant e^{-\frac{1}{32}t^{\frac{1}{4}}\log t}. \tag{3.1.32}$$

The last step is to bound the number of particles inside I at time t. We choose c_7 as in (3.1.8) and bound

$$\mathbb{P}\left[\begin{array}{c}\text{there is more than } c_7\rho|I| + t\\ \text{particles from } \eta\\ \text{inside } I \text{ at time } t\end{array}\right] \leqslant \left[\frac{Z(eR^{-1}(\rho))}{Z(R^{-1}(\rho))}e^{-c_7\rho}\right]^{|I|} e^{-t} \leqslant e^{-t}. \tag{3.1.33}$$

Finally, combining Equations (3.1.25), (3.1.26), (3.1.32) and (3.1.33), we obtain

$$\begin{aligned}\mathbb{P}[A] &\leqslant \mathbb{P}[B] + \mathbb{P}[C] + \mathbb{P}\begin{bmatrix}\text{there is more than } c_7\rho|I| + t \\ \text{particles from } \eta \\ \text{inside } I \text{ at time } t\end{bmatrix} \\ &\quad + (c_7\rho|I| + t)\mathbb{P}\begin{bmatrix}\text{a particle that is inside } I \\ \text{at time } t \text{ do not meet} \\ \text{any of its pairs, } B^c \cap C^c\end{bmatrix} \\ &\leqslant c_6 t(|I| + t)e^{-c_6^{-1}\varepsilon^2 t^{1/4}},\end{aligned} \tag{3.1.34}$$

for some large enough c_6 and all t large. This finishes the proof of Proposition 3.1.7.

3.2 THE INFECTION PROCESS

Now that we concluded the decoupling for the zero range process, we consider our infection process. We first precisely define our model and prove some preliminary results.

Given the initial configuration η_0 for the zero range process with density ρ, define the set of infected particles ξ_0 as

$$\xi_0(x) = \begin{cases}\eta_0(x), & \text{if } x \leqslant 0, \\ 0, & \text{if } x > 0.\end{cases} \tag{3.2.1}$$

Let $\zeta_0 = \eta_0 - \xi_0$ be the collection of healthy particles. The process $\xi + \zeta$ evolves as a zero range process with rate function g. Besides, a healthy particle becomes immediately infected when it shares a site with some already infected particle.

Observe that this construction satisfies

$$\min\{\xi_t(x), \zeta_t(x)\} = 0 \text{ for all } x \in \mathbb{Z} \text{ and } t \geqslant 0. \tag{3.2.2}$$

This means that, in any non-empty site, either all particles are healthy or all particles are infected.

Define the front of the infection wave as

$$r_t = \sup\{x : \xi_t(x) > 0\}. \tag{3.2.3}$$

We now prove some preliminary lemmas regarding the behavior of r_t. These estimates are uniform over compact sets of positive densities. For the remaining of the section, we fix $0 < \rho_- < \rho_+ < \infty$.

First, we prove a crude estimate saying that it is unlikely for r_t to travel a distance of order t^2 in time t. Let $A(t,t)$ be as in Lemma 3.1.3 and observe that there exists a positive constant such that $3\Gamma_+ A(t,t) \leqslant c_{13} t$, for $t \geqslant 1$ and $\rho \in [\rho_-, \rho_+]$.

Lemma 3.2.1. *There exists a positive constant* c_{14} *such that*

$$\mathbb{P}_\rho \left[\sup_{0 \leqslant s \leqslant t} \{r_s\} - r_0 \geqslant c_{13} t^2 \right] \leqslant c_{14} e^{-c_{14}^{-1} t}, \tag{3.2.4}$$

for all $t \geqslant 0$ *and all* $\rho \in [\rho_-, \rho_+]$.

Proof. By increasing the value of the constant c_{14}, we may assume $t \geqslant 1$.

Write $J = [r_0, r_0 + c_{13} t^2]$ and observe that, in the event of the statement, either there exists $x \in J$ such that

$$\eta_s(x) \geqslant A(t,t), \text{ for some } s \leqslant t, \tag{3.2.5}$$

or this does not happen and, in order for the infection to cross J, it must travel through a region that is not dense in particles. This allows us to bound the number of jumps the front of the wave infection can make. We obtain, by possibly increasing the values of the constants,

$$\begin{aligned} \mathbb{P}_\rho \left[\sup_{0 \leqslant s \leqslant t} \{r_s\} - r_0 \geqslant c_{13} t^2 \right] &\leqslant c_{13} t^2 \mathbb{P}_\rho \left[\begin{array}{c} \eta_s(0) \geqslant A(t,t), \\ \text{for some } s \in [0,t] \end{array} \right] \\ &\quad + \mathbb{P}\left[\text{Poisson}(\Gamma_+ t A(t,t)) \geqslant c_{13} t^2 \right] \\ &\leqslant c_{14} (t^2+1) e^{-c_8^{-1} t} + e^{-t} \leqslant c_{14} e^{-c_{14}^{-1} t}, \end{aligned} \tag{3.2.6}$$

and the statement follows. □

Our next lemma is similar to the last one, but we consider a slightly different event, illustrated in Figure 3.3.

Lemma 3.2.2. *For any* $t \geqslant 0$,

$$\mathbb{P}_\rho \left[r_0 - \inf_{0 \leqslant s \leqslant t} \{r_s\} \geqslant (2\Gamma_+ + 1) t \right] \leqslant e^{-t}. \tag{3.2.7}$$

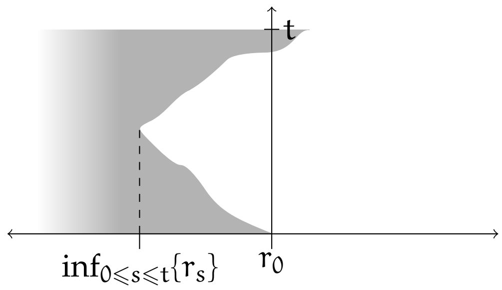

Figure 3.3: The infimum considered in Lemma 3.2.2.

Proof. Simply notice that, on the event above, it is necessary that the first particle on r_0 jumps more than $(2\Gamma_+ + 1)t$ times before time t. This gives the bound

$$\mathbb{P}_\rho\left[r_0 - \inf_{0\leqslant s\leqslant t}\{r_s\} \geqslant (2\Gamma_+ + 1)t\right] \leqslant \mathbb{P}_\rho[\text{Poisson}(\Gamma_+ t) \geqslant (2\Gamma_+ + 1)t] \leqslant e^{-t}, \tag{3.2.8}$$

and the proof is complete. □

We can also bound the probability that the front of the infection has a big displacement to the right.

Lemma 3.2.3. *There exists a positive constant c_{15} such that*

$$\mathbb{P}_\rho\left[\sup_{s\leqslant t}\{r_s\} - r_t \geqslant (2\Gamma_+ + 1)t\right] \leqslant c_{15} e^{-c_{15}^{-1} t}, \tag{3.2.9}$$

for all $\rho \in [\rho_-, \rho_+]$.

Figure 3.4 helps to illustrate the event in Lemma 3.2.3.

Proof. Let B denote the event in the statement of the lemma, write $I = [-(2\Gamma_+ + 2)t, c_{13}t^2]$ and notice that

$$\begin{aligned}\mathbb{P}_\rho[B] \leqslant \mathbb{P}_\rho\left[r_0 \notin [-t, 0]\right] + \mathbb{P}_\rho\left[\inf_{s\leqslant t} r_s \leqslant -(2\Gamma_+ + 2)t, r_0 \geqslant -t\right] \\ + \mathbb{P}_\rho\left[\sup_{s\leqslant t} r_s \geqslant c_{13}t^2\right] + \mathbb{P}_\rho\left[B, r_s \in I,\ \text{for all } s \leqslant t\right].\end{aligned} \tag{3.2.10}$$

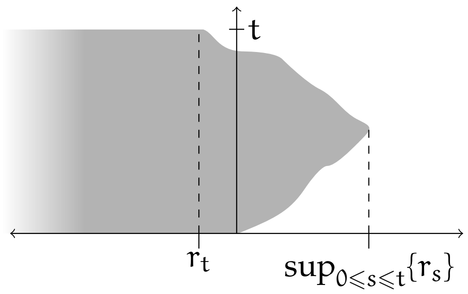

Figure 3.4: The supremum in the event considered in Lemma 3.2.3.

Combining Lemmas 3.2.1 and 3.2.2, we easily obtain that

$$\mathbb{P}_\rho[B] \leqslant ce^{-c^{-1}t} + \mathbb{P}_\rho\left[B, r_s \in I, \text{ for all } s \leqslant t\right]. \tag{3.2.11}$$

To bound the last probability of the last event above, observe that, if it holds, then either there exists some particle from outside $H = [-(5\Gamma_+ + 2)t, c_{13}t^2 + 3\Gamma_+ t]$ enters the interval I before time t, or some particle that starts inside H jumps many times before time t. Using the same strategy as in Lemma 3.1.3, concentration of the number of particles inside H and the fact that each particle jumps at most Poisson$(\Gamma_+ t)$ times before time t we obtain

$$\begin{aligned}\mathbb{P}_\rho\left[B, r_s \in I, \text{ for all } s \leqslant t\right] \leqslant \mathbb{P}_\rho &\left[\begin{array}{c}\text{some particle that starts outside}\\ I \text{ enters } H \text{ before time } t\end{array}\right]\\ &+ \mathbb{P}_\rho\left[\begin{array}{c}\text{some particle inside } I \text{ jumps more}\\ \text{than } (2\Gamma_+ + 1)t \text{ times before time } t\end{array}\right]\\ &\leqslant c(t^2+t+1)e^{-c^{-1}t}.\end{aligned} \tag{3.2.12}$$

Combining the last expression above with (3.2.10) and]3.2.11 completes the proof. □

To finish this section, we introduce the space-time translated infection process. Fix $m = (x,t) \in \mathbb{Z} \times [0,\infty)$ and define the collection of infected particles as

$$\xi_0^m(y) = \begin{cases}\eta_t(y), & \text{if } y \leqslant x,\\ 0, & \text{if } y > x.\end{cases} \tag{3.2.13}$$

As before, $\zeta_0^m = \eta_t - \xi_0^m$ denotes the collection of healthy particles. The evolution of the infection is the same, and the front of the infection wave is

$$r_s(m) = \sup\{y \in \mathbb{Z} : \xi_s^m(y) > 0\}. \tag{3.2.14}$$

3.3 FINITE VELOCITY

We now begin a more in depth study of our infection process. This section aims to prove Theorem 1.1.3. We split the discussion in three subsections. The first subsection contains some notation we will need to develop our multiscale renormalisation, which can be found in Subsection 3.3.2. Subsection 3.3.3 contains the proof of Theorem 1.1.3.

3.3.1 *The box notation*

We begin by introducing the sequence of scales $(L_k)_{k\in\mathbb{N}_0}$ as

$$L_0 = 100 \qquad \text{and} \qquad L_{k+1} = L_k^3. \tag{3.3.1}$$

We will also write $\ell_k = \lfloor L_k^{1/2} \rfloor$.

For $k \in \mathbb{N}_0$, define the box

$$B_k = [-\ell_k L_k^2, \ell_k L_k^2] \times [0, L_k], \tag{3.3.2}$$

and, for $m \in \mathbb{Z} \times L_k\mathbb{N}_0$, let $B_k(m)$ denote the translated box $B_k(m) = m + B_k$.

Define also the sequence of velocities

$$v_0 = v > 0 \qquad \text{and} \qquad v_{k+1} = v_k + \frac{1}{(k+1)^2}, \tag{3.3.3}$$

where v is a positive value that will be chosen afterwards to be sufficiently large.

We want to bound the probability of the events where r_t travels fast to the right. However, the continuous time nature of the process implies that events of this form do not have a bounded support. Therefore, we will

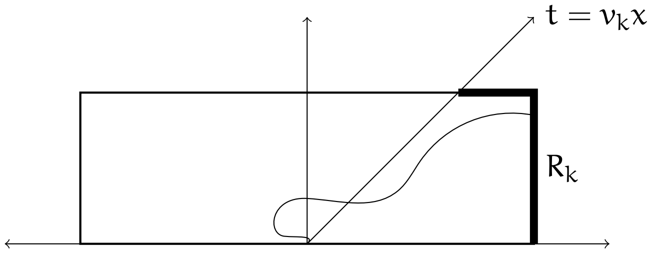

Figure 3.5: The box B_k, the set R_k and the event E_k.

introduce a well chosen event that treats the possibility that either r_t leaves the box B_k before time L_k or it is far to the right at time L_k. For $k \in \mathbb{N}_0$, define the set

$$R_k = \{\ell_k L_k^2\} \times [0, L_k] \cup [v_k L_k, \ell_k L_k^2] \times \{L_k\}. \tag{3.3.4}$$

Figure 3.5 contains a representation of B_k and R_k. For $m \in \mathbb{Z} \times L_k \mathbb{N}_0$, define $R_k(m) = m + R_k$.

The event we consider is defined as follows. For $m = (x, sL_k) \in \mathbb{Z} \times L_k \mathbb{N}_0$, consider

$$E_k(m) = \left\{ \begin{array}{l} r_0(m) = x \text{ and } (r_t(m))_{t>0} \text{ first touches} \\ \text{the boundary of } B_k(m) \text{ in } R_k(m) \end{array} \right\}. \tag{3.3.5}$$

See Figure 3.5 for a representation of the event E_k. Observe that the events $E_k(m)$ are non-decreasing and have support in $B_k(m)$. When $m = (0, 0)$, we will omit it and denote $E_k(0, 0)$ simply by E_k.

We introduce the sequence of densities. Fix $\rho_0 > 0$ and define

$$\rho_k = \rho_{k+1}(1 - L_k^{-1/16}). \tag{3.3.6}$$

The sequence $(\rho_k)_{k \in \mathbb{N}_0}$ is decreasing and $\rho_\infty = \lim \rho_k$ is positive.

Define, for $m \in \mathbb{Z} \times L_k \mathbb{N}_0$, the probability of the bad events as

$$p_k = \mathbb{P}_{\rho_k}[E_k(m)]. \tag{3.3.7}$$

By translation invariance, the probability above does not depend on the value of m.

Remark 3.3.1. Even though p_k also depends on the value of ν_k which are determined by the fixed value of $\nu_0 = \nu$, we omit these dependencies.

We also introduce the event

$$D_k(m) = \{r_t(m) \in B_k(m), \text{ for all } t \in [0, L_k]\}. \tag{3.3.8}$$

Lemmas 3.2.1 and 3.2.2 imply that, given $0 < \rho_- < \rho_+$, there exists $c_{16} > 0$ such that

$$\mathbb{P}_\rho[D_k^c] \leqslant c_{16} e^{-c_{16}^{-1} L_k}, \tag{3.3.9}$$

for all $k \in \mathbb{N}_0$ and $\rho \in [\rho_-, \rho_+]$.

Finally, let M_k denote the set of values m for which the translated box $B_k(m)$ still intersects the larger box B_{k+1}, more precisely,

$$M_k = \{m \in \mathbb{Z} \times L_k\mathbb{N}_0 : B_k(m) \cap B_{k+1} \neq \emptyset\}, \tag{3.3.10}$$

and observe that

$$|M_k| \leqslant c_{17} L_{k+1}^4. \tag{3.3.11}$$

3.3.2 *Estimates on* p_k

Our next step is to prove that p_k decreases very fast when ν_0 is chosen large enough. This is done in three lemmas, proved in this subsection.

The first lemma we prove is a recursive inequality that relates p_k to p_{k+1}.

Lemma 3.3.2. *There exists* k_0 *such that, for all choice of* ν_0 *and* $k \geqslant k_0$,

$$p_{k+1} \leqslant c_{18} L_{k+1}^{28} [p_k^4 + e^{-c_{18}^{-1} \rho_\infty^2 L_k^{1/8}}]. \tag{3.3.12}$$

Proof. Fix $k_0 \in \mathbb{N}_0$ such that, for all $k \geqslant k_0$,

$$\frac{1}{6(k+1)^2} > \frac{1}{L_k^{1/2}}. \tag{3.3.13}$$

Fix $k \geqslant k_0$ and assume we are in the event $E_{k+1} \cap D_{k+1}$. We claim that

either $D_k(m)^c$ holds for some $m \in M_k$ or there are seven elements $m_i = (x_i, s_i) \in M_k, 1 \leqslant i \leqslant 7,$ with $s_i \neq s_j$, if $i \neq j$, such that $E_k(m)$ holds.

The proof follows by contradiction. Assume we are in the event $E_{k+1} \cap D_{k+1}$, that $D_k(m)$ holds for all $m \in M_k$, and that $E_k(m)$ holds for at most six values of $m \in M_k$ with different time coordinates.

Observe that, if $E_k(m) \cap D_k(m)$ holds, $r_t(m)$ has a maximum displacement of $\ell_k L_k^2$ before time L_k. Thus, we have

$$\begin{aligned} r_{L_{k+1}} - r_0 &= \sum_{j=0}^{L_k^2-1} r_{L_k}(r_{jL_k}) - r_{jL_k} \\ &\leqslant 6L_k^{5/2} + (L_k^2 - 6)\nu_k L_k \\ &\leqslant 6L_{k+1}\left(\frac{1}{L_k^{1/2}} - \frac{1}{6(k+1)^2}\right) + L_{k+1}\nu_{k+1} \\ &< L_{k+1}\nu_{k+1}. \end{aligned} \tag{3.3.14}$$

This implies that we are in $E_{k+1}^c \cup D_{k+1}^c$, a contradiction.

Thus, on the event $E_{k+1} \cap D_{k+1}$, either some $D_k(m)^c$ with $m \in M_k$ occurs, or there are seven elements $m_i = (x_i, s_i) \in M_k$, $1 \leqslant i \leqslant 7$, with $s_i \neq s_j$, if $i \neq j$, such that $E_k(m_i)$ occurs.

Assume we are in the last case described above and that $L_k \leqslant s_{i+2} - s_i \leqslant L_{k+1}$, for $1 \leqslant i \leqslant 5$.

We now apply Theorem 3.1.4 considering the event $E_k(m_1)$ and the intersection $\cap_{3\leqslant i\leqslant 7} E_k(m_i)$, as in Figure 3.6: We can use boxes of side length $5\ell_{k+1}L_{k+1}$. Set $\epsilon = \frac{1}{3}(\rho_k - \rho_{k+1}) = \rho_{k+1}\frac{L_k^{-1/16}}{3}$ and estimate

$$\begin{aligned} \mathbb{P}_{\rho_{k+1}}\left[\bigcap_{i=1}^{7} E_k(m_i)\right] &\leqslant \mathbb{P}_{\rho_{k+1}+\epsilon}\left[E_k(m_1)\right]\mathbb{P}_{\rho_{k+1}+\epsilon}\left[\bigcap_{i=3}^{7} E_k(m_i)\right] \\ &\qquad + c_9 L_{k+1}^3 e^{-c_9^{-1}\rho_\infty^2 L_k^{1/8}} \\ &\leqslant \mathbb{P}_{\rho_k}\left[E_k(m_1)\right]\mathbb{P}_{\rho_{k+1}+\epsilon}\left[\bigcap_{i=3}^{7} E_k(m_i)\right] + c_9 L_{k+1}^3 e^{-c_9^{-1}\rho_\infty^2 L_k^{1/8}}. \end{aligned} \tag{3.3.15}$$

We apply Theorem 3.1.4 two more times: In the first use, we consider the events $E_k(m_3)$ and $\bigcap_{i=5}^{7} E_k(m_i)$. The last time uses the events $E_k(m_5)$ and $E_k(m_7)$. These computations yield the bound

$$\mathbb{P}_{\rho_{k+1}}\left[\bigcap_{i=1}^{7} E_k(m_i)\right] \leqslant \mathbb{P}_{\rho_k}[E_k]^4 + 3c_9 L_{k+1}^3 e^{-c_9^{-1}\rho_\infty^2 L_k^{1/8}}. \tag{3.3.16}$$

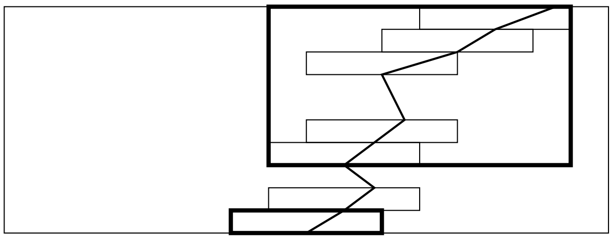

Figure 3.6: The boxes in the cascading event and the supports of the functions in the first application of Theorem 3.1.4.

By changing constants, it is easy to conclude that

$$\begin{aligned} p_{k+1} &\leqslant \mathbb{P}_{\rho_{k+1}}[E_{k+1} \cap D_{k+1}] + \mathbb{P}_{\rho_{k+1}}[D_{k+1}^c] \\ &\leqslant |M_k|^7 (\mathbb{P}_{\rho_k}[E_k]^4 + 3c_9 L_{k+1}^3 e^{-c_9^{-1}\rho_\infty^2 L_k^{1/8}}) \\ &\quad + |M_k| \mathbb{P}_{\rho_{k+1}}[D_k^c] + \mathbb{P}_{\rho_{k+1}}[D_{k+1}^c] \\ &\leqslant c_{18} L_{k+1}^{28} [p_k^4 + e^{-c_{18}^{-1}\rho_\infty^2 L_k^{1/8}}], \end{aligned} \tag{3.3.17}$$

and the statement follows. ☐

We now prove a recursive estimate on p_k.

Lemma 3.3.3. *There exists* $k_1 \geqslant k_0$ *such that, for* $k \geqslant k_1$ *and any choice of* ν_0*, if*

$$p_k \leqslant e^{-\log^{5/4} L_k}, \tag{3.3.18}$$

then

$$p_{k+1} \leqslant e^{-\log^{5/4} L_{k+1}}. \tag{3.3.19}$$

Proof. Observe that $3^{5/4} \leqslant 4$.

Assume that (3.3.18) holds for some $k \geqslant k_0$. Recall that $L_{k+1} = L_k^3$ and use Lemma 3.3.2 to conclude that

$$\begin{aligned} e^{\log^{5/4} L_{k+1}} p_{k+1} &\leqslant c_{18} L_{k+1}^{28} [p_k^4 + e^{-c_{18}^{-1}\rho_\infty^2 L_k^{1/8}}] e^{\log^{5/4} L_{k+1}} \\ &\leqslant c_{18} L_{k+1}^{28} [e^{-4\log^{5/4} L_k} + e^{-c_{18}^{-1}\rho_\infty^2 L_k^{1/8}}] e^{3^{5/4}\log^{5/4} L_k} \\ &\leqslant c_{18} L_{k+1}^{28} [e^{(-4+3^{5/4})\log^{5/4} L_k} + e^{-c_{18}^{-1}\rho_\infty^2 L_k^{1/8} + 3^{5/4}\log^{5/4} L_k}]. \end{aligned} \tag{3.3.20}$$

Now simply choose $k_1 \geqslant k_0$ such that, if $k \geqslant k_1$, then

$$c_{18} L_{k+1}^{28} [e^{(-4+3^{5/4}) \log^{5/4} L_k} + e^{-c_{18}^{-1} \rho_\infty^2 L_k^{1/8} + 3^{5/4} \log^{5/4} L_k}] < 1.$$

This concludes the proof. □

The last step is to verify that, if v_0 is chosen large enough, (3.3.18) holds for some $k \geqslant k_1$.

Lemma 3.3.4. *There exist v_0 and $k_2 \geqslant k_1$ such that $p_{k_2} \leqslant e^{-\log^{5/4} L_{k_2}}$.*

Proof. For $k \geqslant k_2$, set $\tilde{v}_k = \ell_k L_k$, and observe that for this velocity,

$$E_k \subset \left\{ \sup_{s \leqslant L_k} \{r_s\} - r_0 \geqslant c_{13} L_k^2 \right\}. \tag{3.3.21}$$

Now, Lemma 3.2.1 implies that

$$p_k(\tilde{v}_k) \leqslant \mathbb{P}_{\rho_k} \left[\sup_{s \leqslant L_k} \{r_s\} - r_0 \geqslant c_{13} L_k^2 \right] \leqslant c_{14} e^{-c_{14}^{-1} L_k}. \tag{3.3.22}$$

Increasing the value of k if necessary gives the desired bound. Now simply choose the corresponding value of v_0 according to (3.3.3). □

3.3.3 *Proof of Theorem 1.1.3*

In this section, we conclude the proof of Theorem 1.1.3. We use the multiscale renormalisation scheme developed in the last subsection to prove that r_t has finite speed.

Define the space-time cone

$$\mathscr{H}_{v,L} = \{(x,t) \in \mathbb{Z} \times \mathbb{R}_+ : x \geqslant tv + L\}. \tag{3.3.23}$$

We will prove that the probability that $r_t \in \mathscr{H}_{v,L}$, for some $t \geqslant 0$, decays fast with L when v is large enough. We already have information about r_t for the times L_k. All that is necessary now is to interpolate between these times.

Fix v_0 and k_2 as in Lemma 3.3.4 and define

$$\bar{v} = v_\infty = \lim_{k\to\infty} v_k. \tag{3.3.24}$$

Define the events $\bar{E}_k(m)$ as in (3.3.5) but with v_k replaced by $\bar{v}$. Observe that we have

$$\mathbb{P}_{\rho_\infty}[\bar{E}_k(m)] \leqslant \mathbb{P}_{\rho_k}[E_k(m)] \leqslant e^{-\log^{5/4} L_k}, \qquad \text{for all } k \geqslant k_3. \tag{3.3.25}$$

Proof of Theorem 1.1.3. Notice that, if

$$\mathbb{P}_{\rho_\infty}\left[\begin{array}{c} r_t \in \mathscr{H}_{\bar{v},L}, \text{ for some } t \geqslant 0, \\ \text{and } r_0 = 0 \end{array}\right] \leqslant c_1 e^{-c_1^{-1}\log^{5/4} L}, \tag{3.3.26}$$

then

$$\begin{aligned}\mathbb{P}_{\rho_\infty}\left[\begin{array}{c} r_t \in \mathscr{H}_{\bar{v},L}, \\ \text{for some } t \geqslant 0 \end{array}\right] &\leqslant \sum_{y=0}^{\infty} \mathbb{P}_{\rho_\infty}\left[\begin{array}{c} r_t \in \mathscr{H}_{\bar{v},L}, \text{ for some } t \geqslant 0, \\ \text{and } r_0 = -y \end{array}\right] \\ &\leqslant \sum_{y=0}^{\infty} c_1 e^{-c_1^{-1}\log^{5/4}(L+y)} \leqslant c_1 e^{-c_1^{-1}\log^{5/4} L}.\end{aligned} \tag{3.3.27}$$

Hence, we may condition on the event $\{r_0 = 0\}$.

By changing constants, we may assume that $L \geqslant L_{k_2}$. Choose $\tilde{k} \geqslant k_2$ such that

$$L_{\tilde{k}} \leqslant L < L_{\tilde{k}+1}. \tag{3.3.28}$$

For $m = (x, s) \in \mathbb{Z} \times L_k\mathbb{N}_0$, we define the event where $r(m)$ does not travel very far in time L_k, more precisely,

$$H_k(m) = \left\{\begin{array}{c} \sup_{0\leqslant t\leqslant L_k} r_t(m) - x \leqslant (\bar{v}+1)L_k \\ \text{and} \\ x - \inf_{0\leqslant t\leqslant L_k} r_t(m) \leqslant 2(\Gamma_+ + 1)L_k \end{array}\right\}, \tag{3.3.29}$$

and observe that Lemmas 3.2.3 and 3.2.2 imply, by possibly changing the value of the constant and increasing the value of $\bar{v}$,

$$\begin{aligned}\mathbb{P}_{\rho_\infty}[\bar{E}_k^c \cap H_k^c] \leqslant{}& \sum_{j=0}^{L_k-1} \mathbb{P}_{\rho_\infty}[r_0 = -j, E_k^c(-j,0) \cap H_k^c] + \mathbb{P}[E_k(-j,0)] \\ &+ \mathbb{P}_{\rho_\infty}[r_0 \leqslant -L_k] \\ \leqslant{}& c_{15} L_k e^{-\log^{5/4} L_k}.\end{aligned} \tag{3.3.30}$$

We will define an event where r_t is well-behaved. Recall (3.3.10) and consider

$$\tilde{B}_{\tilde{k}} = \bigcap_{k \geqslant \tilde{k}} \bigcap_{m \in M_k} \bar{E}_k(m)^c \cap H_k(m). \tag{3.3.31}$$

In the event above, we have bounds for r_t at the times L_k and we also know that the front does not travel far away during the time intervals of length L_k.

Observe that Equations (3.3.25) and (3.3.30) imply that

$$\begin{aligned} \mathbb{P}_{\rho_\infty}[\tilde{B}^c_{\tilde{k}}] &\leqslant \sum_{k \geqslant \tilde{k}} \sum_{m \in M_k} \mathbb{P}_{\rho_\infty}[\bar{E}_k(m)] + \mathbb{P}_{\rho_\infty}[\bar{E}_k(m)^c \cap H_k(m)^c] \\ &\leqslant c_{19} \sum_{k \geqslant \tilde{k}} L_k^{14} e^{-\log^{5/4} L_k} \leqslant c_{19} L_{\tilde{k}}^{14} e^{-\log^{5/4} L_{\tilde{k}}} \\ &\leqslant c_{19} L^{14} e^{-c_{19}^{-1} \log^{5/4} L}, \end{aligned} \tag{3.3.32}$$

where the tail bound in the second line above is proved in an analogous way as Lemma D.1 of [26].

We now study the event $\tilde{B}_{\tilde{k}}$. Consider

$$J_{\tilde{k}} = \bigcup_{k \geqslant \tilde{k}} \bigcup_{\ell=0}^{L_{k+1}/L_k} \{\ell L_k\}. \tag{3.3.33}$$

We claim that, on $\tilde{B}_{\tilde{k}} \cap \{r_0 = 0\}$,

$$r_t \leqslant \bar{v} t, \qquad \text{for all } t \in J_{\tilde{k}}. \tag{3.3.34}$$

To see why this is true, fix $k \geqslant \tilde{k}$ and use induction on ℓ. The claim is clearly true for $\ell = 0$. Suppose it is true for some $\ell < L_{k+1}/L_k$. Observe that, since we are in $H_k(m)$, for $m \in M_k$, $(r_{\ell L_k}, \ell L_k)$ belongs to B_{k+1}. Using that $E_k(r_{\ell L_k})^c$ holds, we have

$$\begin{aligned} r_{(\ell+1)L_k} &= \left(r_{L_k}(r_{\ell L_k}) - r_{\ell L_k}\right) + r_{\ell L_k} \\ &\leqslant \bar{v} L_k + \bar{v} \ell L_k = \bar{v}(\ell+1) L_k. \end{aligned} \tag{3.3.35}$$

It remains to interpolate the relation in (3.3.34) for the remaining values of $t \geqslant 0$.

Initially, consider $t \geqslant L$. Let κ be the smallest $k \geqslant \tilde{k}$ such that

$$\ell L_\kappa \leqslant t < (\ell+1)L_\kappa, \qquad \text{for some } \ell < \frac{L_{\kappa+1}}{L_\kappa}. \tag{3.3.36}$$

Let $\bar{\ell}$ denote the unique value of ℓ and observe that $\bar{\ell} \geqslant 1$.

We compute

$$\begin{aligned} r_t &= \left(r_{t-\bar{\ell}L_\kappa}(r_{\bar{\ell}L_\kappa}) - r_{\bar{\ell}L_\kappa}\right) + r_{\bar{\ell}L_\kappa} \\ &\leqslant (\bar{v}+1)L_\kappa + \bar{v}\bar{\ell}L_\kappa \leqslant (2\bar{v}+1)t. \end{aligned} \tag{3.3.37}$$

We now consider $t \leqslant L$. Observe that, on $\tilde{B}_{\tilde{k}} \cap \{r_0 = 0\}$, we have $r_L \leqslant (2\bar{v}+1)L$. Lemma 3.2.3 implies

$$\begin{aligned} &\mathbb{P}_{\rho_\infty}\left[\sup_{s \leqslant L} r_s \geqslant 2(\bar{v}+\Gamma_+ +1)L, \tilde{B}_{\tilde{k}} \cap \{r_0 = 0\}\right] \\ &\qquad\qquad \leqslant \mathbb{P}_{\rho_\infty}\left[r_L - \sup_{s\leqslant L} r_s \geqslant (2\Gamma_+ +1)L\right] \leqslant c_{15}e^{-c_{15}L}. \end{aligned} \tag{3.3.38}$$

Combining the last expression above with (3.3.32), we obtain

$$\mathbb{P}_{\rho_\infty}\left[\begin{array}{c} r_t \in \mathscr{H}_{2\bar{v}+1,2(\bar{v}+\Gamma_++1)L}, \\ \text{for some } t \geqslant 0, \text{ and } r_0 = 0 \end{array}\right] \leqslant c_2 e^{-c_2^{-1}\log^{5/4} L}. \tag{3.3.39}$$

By changing constants, the proof is complete. □

3.4 POSITIVE VELOCITY

This section contains the proof of Theorem 1.1.4. This proof is also based on multiscale renormalisation, but we need to use a different approach to the problem, since the events in the renormalisation are not so well-behaved as in the proof of Theorem 1.1.3.

3.4.1 *Simultaneous decoupling*

For the proof of positive velocity, it is not possible to apply the decoupling stated in Therorem 3.1.4 for the class of events we consider in the renormalisaton. In this subsection we provide a stronger version the the decoupling.

For $\rho < \rho'$, we construct the measure $\mathbb{P}_{\rho,\rho'}$ in the following way. Begin with two initial configurations of the zero range process that satisfy $\eta_0(x) \leqslant \eta'_0(x)$ (this can be done using the usual monotone coupling) and use one copy of the graphical construction presented in Subsection 3.1.2 to evolve both processes at the same time. Whenever a particle jumps, it goes on top of its respective pile and particles of η are also seen as particles of the process η'.

The probability measure $\mathbb{P}_{\rho,\rho'}$ provides the construction of two zero range processes, η and η', with respective densities ρ and ρ' and that satisfy $\eta_t(x) \leqslant \eta'_t(x)$, for all $(x,t) \in \mathbb{Z} \times \mathbb{R}_+$.

We prove a decoupling for the collection of measures $\mathbb{P}_{\rho,\rho'}$.

Proposition 3.4.1. *Fix $0 < \rho_- < \rho_+$. There exist positive constants c_{20} and C_7 such that, for any two boxes B_1 and B_2 with side-length s that satisfy*

$$d_V = d_V(B_1, B_2) \geqslant C_7, \tag{3.4.1}$$

and any two functions $f_1(\eta,\eta')$ and $f_2(\eta,\eta')$ satisfying

1. *f_i is supported in B_i;*
2. *$0 \leqslant f_i(\eta,\eta') \leqslant 1$ almost surely;*
3. *f_i is non-increasing in η and non-decreasing in η';*

we have the following. For any $\rho \leqslant \rho' \in [\rho_-\rho_+]$ and $\epsilon \in (0,1]$ such that $\rho - \epsilon \geqslant \rho_-$,

$$\mathbb{E}_{\rho,\rho'}[f_1 f_2] \leqslant \mathbb{E}_{\rho-\epsilon,\rho'+\epsilon}[f_1]\mathbb{E}_{\rho-\epsilon,\rho'+\epsilon}[f_2] + c_{20}\, d_V(d_V + s + 1)e^{-c_{20}^{-1}\epsilon^2 d_V^{1/4}}. \tag{3.4.2}$$

The proof follows exactly the same steps of the proof of Theorem 3.1.4. The existence of a coupling with the same characteristics of the one in Proposition 3.1.7 is guaranteed by the next result.

Proposition 3.4.2. *Fix $0 < \rho_- < \rho_+$. There exist positive constants c_{21} and C_8 such that, for any $t \geqslant C_8$, interval $I \subset \mathbb{R}$, densities $\rho \leqslant \rho' \in [\rho_-, \rho_+]$ and $\epsilon \in (0,1]$ such that $\rho - \epsilon \geqslant \rho_-$, there exists a coupling between two pairs of zero range processes $(\eta_s, \eta'_s)_{s\geqslant 0}$ and $(\bar\eta_s, \bar\eta'_s)_{s\geqslant 0}$ such that*

1. $(\eta_s, \eta'_s)_{s \geqslant 0}$ *is distributed as* $\mathbb{P}_{\rho,\rho'}$ *and* $(\bar\eta_s, \bar\eta'_s)_{s \geqslant 0}$ *is distributed as* $\mathbb{P}_{\rho-\epsilon,\rho+\epsilon}$*;*
2. $(\bar\eta_s, \bar\eta'_s)_{s \geqslant 0}$ *is independent from* (η_0, η'_0)*;*
3. $$\mathbb{P}\left[\begin{array}{c}\textit{there exists } x \in I \textit{ such that}\\ \eta_t(x) < \bar\eta_t(x) \textit{ or } \eta'_t(x) > \bar\eta'_t(x)\end{array}\right] \leqslant c_{21} t(|I| + t)e^{-c_{21}^{-1}\epsilon^2 t^{1/4}}. \quad (3.4.3)$$

The construction of the coupling stated in the proposition above is similar to the one in Proposition 3.1.7. Hence, in the proof presented here we only point out the main differences between the constructions.

Proof. We want to couple two pairs $(\eta_s, \eta'_s)_{s \geqslant 0}$ and $(\bar\eta_s, \bar\eta'_s)_{s \geqslant 0}$ with respective densities (ρ, ρ') and $(\rho - \epsilon, \rho' + \epsilon)$. We also start with two independent pairs of configurations and two copies of the graphical construction of Subsection 3.1.2, $\mathscr{P}_1 = (\mathcal{P}_1(x))_{x \in \mathbb{Z}}$ and $\mathscr{P}_2 = (\mathcal{P}_2(x))_{x \in \mathbb{Z}}$.

The pair $(\bar\eta_s, \bar\eta'_s)_{s \geqslant 0}$ will evolve with the second copy of the graphical construction $\mathscr{P}_2$. It remains to set the evolution of $(\eta_s, \eta'_s)_{s \geqslant 0}$. This coupling also uses the pairing between the configuration and the coupling times. For the first half of the coupling times, we only pair $\bar\eta$ to η and use the evolution of the coupling from Proposition 3.1.7.

For the second half of the coupling times, the matching includes the particles from the processes η' and $\bar\eta'$. This part of the coupling is a little bit more delicate, due to the existence of the particles from η and $\bar\eta$. Whenever a particle jumps to a new site, we may need to perform a change of the matching. We update the pairing to obey the rule that particles from η (resp., $\bar\eta$) are always below particles of η' (resp., $\bar\eta'$). When a particle jumps, it goes to its correct place in the new pile. If it meets its pair or it is a particle from η or $\bar\eta$, we update the matching just by changing the heights of the matched particles to obey that particles that already meet stay together. Figure 3.7 gives an example where an update is necessary.

It is easy to verify that all the estimates in the proof from Proposition 3.1.7 remain valid in this case, up to a change of constants. □

3.4.2 *The box notation*

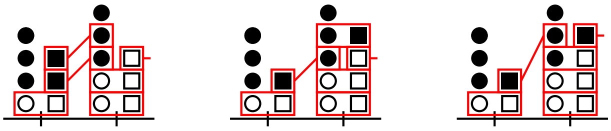

Figure 3.7: A pairing where an update is necessary. Notice that, after the jump, in order to obey that particles from density ρ stay always below particles from the configuration with density ρ', we change the pairing in the pile.

We now begin to introduce the notation for the proof of Theorem 1.1.4. Some notation has already been introduced in Subsection 3.3.1 and we recall it here too.

In this section we write $I_k = \left[-\frac{\ell_k}{4}L_k^2, \frac{\ell_k}{4}L_k^2\right]$ and, for $m = (x, sL_k) \in \mathbb{Z} \times L_k\mathbb{N}_0$, let $I_k(m) = x + I_k$.

We say that a path $\gamma : [0, L_k] \to \mathbb{Z}$ is η-allowed (for the scale k) if

1. $\gamma(0) = 0$;
2. $\gamma(t) \in I_k$, for all $t \in [0, L_k]$;
3. γ is a nearest-neighbor path;
4. γ only moves when a particle of η jumps from that site.

Being η-allowed is a non-decreasing property. This means that if γ is η-allowed and $\eta \preccurlyeq \tilde{\eta}$, then γ is also $\tilde{\eta}$-allowed.

With high probability, the front of the infection, r_t, is an η-allowed path. In order to prove that it moves to the right with positive speed, we will verify that it shares a site with two or more particles a positive proportion of time. In these times, r_t has a drift to the right. However, instead of investigating directly these times, we introduce a quantity that measures the amount of time a path is within distance R from at least two particles. For $R > 0$, $t > 0$ and a càdlàg path $\gamma : [0, t] \to \mathbb{Z}$,

$$V_\eta^{R,t}(\gamma) = \frac{1}{t}\left|\left\{s \in [0, t] : \sum_{x=-\gamma(s)-R}^{\gamma(s)+R} \eta_s(x) \geqslant 2\right\}\right|. \tag{3.4.4}$$

The bad event we are interested in here deals with the existence of a η-allowed path γ with $V^{R,L_k}_{\eta'}(\gamma)$ small.

Observe that, if $\eta \preccurlyeq \tilde{\eta}$, then $V^{R,k}_{\eta}(\gamma) \leqslant V^{R,k}_{\tilde{\eta}}(\gamma)$. This will allow us to use the stronger version of the zero range process decoupling, Proposition 3.4.1.

In a similar flavor of (3.3.3), we introduce the sequence

$$\epsilon_0 = \epsilon > 0 \qquad \text{and} \qquad \epsilon_{k+1} = \epsilon_k \left(1 - \frac{1}{(k+1)^2}\right). \tag{3.4.5}$$

Observe that the sequence above is non-increasing and $\epsilon_\infty = \lim \epsilon_k$ is positive. Consider the sequence of events

$$F^R_k = \left\{ (\eta, \eta') : \begin{array}{c} \text{there exists a path } \gamma \text{ that is} \\ \eta'\text{-allowed and } V^{R,L_k}_{\eta}(\gamma) \leqslant \epsilon_k \end{array} \right\} \tag{3.4.6}$$

The events F^R_k are non-increasing in η and non-decreasing in η'. Besides, when $R \leqslant \frac{3\ell_k}{4} L_k$ the event F^R_k has support in B_k.

For some fixed $\rho_0 > 0$, recall we defined the sequence $(\rho_k)_{k \in \mathbb{N}_0}$ in (3.3.6) by setting $\rho_k = \rho_{k+1}(1 - L_k^{-1/16})$. We set $\rho'_0 = \rho_0$ and define

$$\rho'_{k+1} = \rho'_k(1 + L_k^{-1/16}). \tag{3.4.7}$$

In this case, $(\rho'_k)_{k \in \mathbb{N}_0}$ is increasing and $\rho'_\infty = \lim \rho'_k$ exists and is finite.

Finally, define the probabilities

$$q_k = \mathbb{P}_{\rho_k, \rho'_k}[F^R_k]. \tag{3.4.8}$$

3.4.3 *Estimates on* q_k

We now focus on the bounds of q_k. This will be done in a similar way as in Subsection 3.3.2, and hence some proofs are omitted.

The first thing we need to do is to relate the properties of being η'-allowed for different scales. We prove a lemma that bounds the probability of the following event

$$G_k = \left\{ \begin{array}{c} \text{all paths } \gamma \text{ that are } \eta\text{-allowed for the escale} \\ k+1 \text{ do not leave } I_k \text{ before time } L_k \end{array} \right\}. \tag{3.4.9}$$

Lemma 3.4.3. *There exists a positive constant* c_{22} *such that, for all* $\rho \in [\rho_\infty, \rho'_\infty]$ *and* $k \geqslant 0$*, we have*

$$\mathbb{P}_\rho[G_k^c] \leqslant c_{22} e^{-c_{22}^{-1} L_k}. \tag{3.4.10}$$

Proof. We consider two special paths that are η-allowed in the scale $k+1$: γ_+, that always jumps to the right, and γ_-, that always jumps to the left. Observe that, for all k large,

$$\begin{aligned}\mathbb{P}_\rho[G_k^c] &\leqslant \mathbb{P}_\rho\left[\begin{array}{c}\gamma_+ \text{ or } \gamma_- \text{ leaves } I_k\\ \text{before time } L_k\end{array}\right]\\ &\leqslant L_k^3 \mathbb{P}_\rho\left[\begin{array}{c}\eta_s(0) \geqslant A(L_k, L_k),\\ \text{for some } s \in [0, L_k]\end{array}\right]\\ &\qquad + 2\mathbb{P}\left[\text{Poisson}(\Gamma_+ L_k A(L_k, L_k)) \geqslant \frac{\ell_k}{4} L_k^2\right]\\ &\leqslant c_{14}(L_k^3 + 1)e^{-c_8^{-1} L_k} + e^{-t} \leqslant c_{22} e^{-c_{22}^{-1} L_k}.\end{aligned} \tag{3.4.11}$$

By possibly increasing the value of c_{22}, we obtain that the estimate above is true for all $k \geqslant 0$ and conclude the proof. □

For $m \in \mathbb{Z} \times L_k\mathbb{N}_0$, if we define the translation

$$G_k(m) = \left\{\begin{array}{c}\text{all paths } \gamma \text{ that are } \eta\text{-allowed for scale } k+1 \text{ and touch } m\\ \text{satisfy that } \gamma|_{[sL_k, (s+1)L_k]} \text{ does not leave } I_k(m)\end{array}\right\}, \tag{3.4.12}$$

we easily obtain the bound $\mathbb{P}_\rho[G_k(m)] \leqslant \mathbb{P}_\rho[G_k]$.

We now focus on the probabilities q_k. As before, the first step is to obtain a recursive inequality that relates q_k and q_{k+1}.

Lemma 3.4.4. *There exists* k_0 *such that, for all* $k \geqslant k_0$ *and* $1 \leqslant R \leqslant \frac{3\ell_k}{4} L_k$,

$$q_{k+1} \leqslant c_{23} L_{k+1}^{28} [q_k^4 + e^{-c_{23}^{-1} \rho_\infty^2 L_k^{1/8}}]. \tag{3.4.13}$$

Proof. The proof very similar as the one of Lemma 3.3.2, but we use the stronger version of the decoupling in this case. Here, we only prove that the events F_k^R are cascading.

Choose $k_0 \in \mathbb{N}_0$ such that, for all $k \geqslant k_0$

$$\frac{1}{6(k+1)^2} \geqslant \frac{1}{L_k^2}. \tag{3.4.14}$$

Fix $k \geqslant k_0$, a value $1 \leqslant R \leqslant \frac{\ell_k}{2} L_k$ and assume we are in F^R_{k+1}. We claim that

either $G_k(m)^c$ holds for some $m \in M_k$ or there are seven elements $m_i = (x_i, s_i) \in M_k, 1 \leqslant i \leqslant 7,$ with $s_i \neq s_j$, if $i \neq j$, such that $F^R_k(m)$ holds.

Once again, the proof follows by contradiction. Assume we are in the event F^R_{k+1}, that $G_k(m)$ holds for all $m \in M_k$, and that $F^R_k(m)$ holds for at most six values of $m \in M_k$ with different time coordinates.

Observe that, if F^R_{k+1} holds, there exists an η'_{k+1}-allowed path γ with $V^{R,L_{k+1}}_{\eta_{k+1}}(\gamma) \leqslant \epsilon_{k+1}$. Besides, for all but at most six values of $0 \leqslant s \leqslant L_k^2$, the path $\gamma_s = \gamma|_{[sL_k,(s+1)L_k]}$ is η'_k-allowed and $V^{R,L_k}_{\eta_k}(\gamma_s) > \epsilon_k$. Observe now that

$$
\begin{aligned}
V^{R,L_{k+1}}_{\eta_{k+1}}(\gamma) &= \frac{L_k}{L_{k+1}} \sum_{s=0}^{L_k^2-1} V^{R,L_k}_{\eta_{k+1}}(\gamma_s) \\
&\geqslant \frac{L_k}{L_{k+1}} \sum_{s=0}^{L_k^2-1} V^{R,L_k}_{\eta_k}(\gamma_s) \\
&> \frac{L_k}{L_{k+1}} \epsilon_k (L_k^2 - 6) \\
&\geqslant \epsilon_k \left(1 - \frac{6}{L_k^2}\right) \geqslant \epsilon_{k+1},
\end{aligned}
\tag{3.4.15}
$$

a contradiction. □

Observe that Lemma 3.3.3 is also valid for the quantities q_k and the proof remains the same. We now prove an analogous of Lemma 3.3.4: We will verify that, if ϵ_0 is small enough and R and k are large enough, then we have the correct decay.

Lemma 3.4.5. *There exists* R, ϵ_0 *and* $k_3 \geqslant k_2$ *such that* $q_{k_3} \leqslant e^{-\log^{5/4} L_{k_3}}$.

Proof. First we compute

$$
\begin{aligned}
\mathbb{P}_\rho\left[\sum_{x \in I_k} \eta_0(x) \leqslant 1\right] &\leqslant \mathbb{P}_\rho\left[\begin{array}{c}\text{there exists } x \in I_k \text{ such that } \eta_0(y) = 0,\\ \text{for all } y \in I_k \setminus \{x\}\end{array}\right] \\
&\leqslant L_k^3 \mathbb{P}_\rho[\eta_0(0) = 0]^{L_k^2} \leqslant e^{-cL_k}.
\end{aligned}
\tag{3.4.16}
$$

Now define, for $R_k = \frac{\ell_k}{2} L_k^2$,

$$\tilde{F}_k = \left\{ (\eta, \eta') : \begin{array}{c} \text{there exists a path } \gamma \text{ that is} \\ \eta'\text{-allowed and } V_\eta^{R_k, L_k}(\gamma) = 0 \end{array} \right\}, \tag{3.4.17}$$

and observe that

$$\mathbb{P}_{\rho_k, \rho'_k}[\tilde{F}_k] \leqslant \mathbb{P}_{\rho_k}\left[\sum_{x \in I_k} \eta_0(x) \leqslant 1 \right] \leqslant e^{-cL_k}. \tag{3.4.18}$$

Fix $k_3 \geqslant k_2$ such that $2e^{-cL_{k_3}} \leqslant e^{-\log^{5/4} L_{k_3}}$. Since $\lim_{\epsilon_{k_3} \to 0} \mathbb{P}_{\rho_{k_3}, \rho'_{k_3}}[F_{k_3}^{R_{k_3}}] = \mathbb{P}_{\rho_{k_3}, \rho'_{k_3}}[\tilde{F}_{k_3}]$, we can choose ϵ_{k_3} such that $\mathbb{P}_{\rho_{k_3}, \rho'_{k_3}}[F_{k_3}^{R_{k_3}}] \leqslant \mathbb{P}_{\rho_{k_3}, \rho'_{k_3}}[\tilde{F}_{k_3}] + e^{-cL_{k_3}}$ and conclude that

$$\mathbb{P}_{\rho_{k_3}, \rho'_{k_3}}[F_{k_3}^{R_{k_3}}] \leqslant \mathbb{P}_{\rho_{k_3}, \rho'_{k_3}}[\tilde{F}_{k_3}] + e^{-cL_{k_3}} \leqslant 2e^{-cL_{k_3}}. \tag{3.4.19}$$

This concludes the proof with $R = R_{k_3} = \frac{\ell_{k_3}}{2} L_{k_3}^2$ and the suitable choice of ϵ_0. □

3.4.4 *Proof of Theorem 1.1.4*

We now turn to the proof that r_t travels to the right with positive velocity. The renormalisation developed in Subsection 3.3.2 does not give us this information for the sequence of times L_k. Our first goal is to obtain bounds for these times. With it, we use a concatenation argument similar to the one used in the proof of Theorem 1.1.3 to conclude.

We begin by introducing the zero-mean martingale

$$M_t = r_t - r_0 - \int_0^t \frac{1}{2} g(\eta_s(r_s)) \mathbb{1}_{\{\eta_s(r_s) \geqslant 2\}} \, ds, \tag{3.4.20}$$

and stating a concentration estimate for it.

Proposition 3.4.6. *For every $\delta > 0$, there exists a positive constant c_{24} that depends also on $\rho > 0$ such that, for all k,*

$$\mathbb{P}_\rho[|M_{L_k}| \geqslant \delta L_k] \leqslant c_{24} e^{-c_{24}^{-1} L_k^{1/8}}. \tag{3.4.21}$$

We postpone the proof of this proposition to the Appendix. With it, we can study the behavior of r_t at the times L_k. Since we know that M_{L_k} is concentrated around its mean, in order to verify that r_{L_k} drifts to the right it suffices to study the integral term in (3.4.20).

Proposition 3.4.7. *There exists* $k_4 \geqslant k_3$ *and* $\delta > 0$ *such that, for all* $k \geqslant k_4$,

$$\mathbb{P}_{\rho_\infty}[r_{L_k} \leqslant \delta L_k, \textit{ and } r_0 = 0] \leqslant 4e^{-\log^{5/4} L_k}. \tag{3.4.22}$$

The idea of the proof is to use that, with high probability, the path r_t is η-allowed. Therefore, for a positive fraction of times, there are more than two particles close to it. Using this fact, we will prove that there is a positive fraction of times for which two particles are on top of the front, producing a drift to the right.

Proof. Begin by introducing the event

$$\bar{G}_k = \left\{ \sup_{0 \leqslant t \leqslant L_k} |r_t| \geqslant \frac{\ell_k}{4} L_k^2 \right\}, \tag{3.4.23}$$

and notice that, by Lemmas 3.2.1 and 3.2.2,

$$\mathbb{P}_{\rho_\infty}\left[\begin{array}{c} (r_t)_{0 \leqslant t \leqslant L_k} \text{ is not } \eta\text{-allowed for} \\ \text{the scale } k \text{ and } r_0 = 0 \end{array} \right] \leqslant \mathbb{P}_{\rho_\infty}[\bar{G}_k, r_0 = 0] \leqslant ce^{-c^{-1} L_k}, \tag{3.4.24}$$

for some positive constant c.

By possibly increasing the value of k_3, we obtain, for $k \geqslant k_3$,

$$\mathbb{P}_{\rho_\infty}[V_\eta^{R,L_k}(r_t) \leqslant \epsilon_\infty, r_0 = 0] \leqslant q_k + \mathbb{P}_{\rho_\infty}[\bar{G}_k, r_0 = 0] \leqslant 2e^{-\log^{5/4} L_k}. \tag{3.4.25}$$

We now claim that, if, for some time $t \in [0, L_k]$, we have $\sum_{x=r_t-R}^{r_t+R} \eta_t(x) \geqslant 2$, then there exists a positive probability that

$$|\{s \in [t, t+1] : \eta_s(r_s) \geqslant 2\}| \geqslant \frac{1}{2}. \tag{3.4.26}$$

One way to see why this is true is the following. Consider initially the stopping time $\tau = \inf\{s \geqslant t : \eta_s(r_s) \geqslant 2\}$. Since, at time t, there exists at least one more particle that is within distance R from r_t, we have that $\mathbb{P}_{\rho_\infty}[\tau < t + 1/2] > 0$ uniformly. When this happens, there is also a positive probability that no particle in r_τ moves between times r_τ and $r_{\tau+1/2}$: It suffices that no particle in a large interval around r_τ moves and no particle from outside this interval reaches r_τ before time $\tau + 1/2$.

This implies that, conditioned on (η_t, r_t), the indicator function of the event in (3.4.26) stochastically dominates a random variable X with positive expectation and that assumes only the values zero and one. Define $\delta = \frac{\epsilon_\infty}{4}\mathbb{E}_{\rho_\infty}[X]$.

We now investigate the event $\{V_\eta^{R,L_k}(r_t) \geqslant \epsilon_\infty\}$. In it, there exists a sequence of times $(t_i)_{i\in[N]}$, $N = \lfloor \frac{\epsilon_\infty}{2} L_k \rfloor$, such that $|t_i - t_j| \geqslant 2$, for $i \neq j$, and $\sum_{x=r_{t_i}-R}^{r_{t_i}+R} \eta_{t_i}(x) \geqslant 2$, for all $i \in [N]$. These times allow us to estimate

$$\begin{aligned} \mathbb{P}_{\rho_\infty}[V_\eta^{0,L_k}(r_t) \leqslant \delta/2, r_0 = 0] \leqslant \mathbb{P}_\infty\left[X_1 + X_2 + \cdots + X_N \leqslant \delta L_k\right] \\ + \mathbb{P}_\rho[V_\eta^{R,L_k}(r_t) \leqslant \epsilon_\infty, r_0 = 0], \end{aligned} \tag{3.4.27}$$

where $(X_i)_{i\in[N]}$ are i.i.d. copies of X.

Standard concentration bounds for $(X_i)_{i\in[N]}$ and Equation (3.4.25) imply

$$\mathbb{P}_{\rho_\infty}[V_\eta^{0,L_k}(r_t) \leqslant \delta/2, r_0 = 0] \leqslant 3e^{-\log^{5/4} L_k}. \tag{3.4.28}$$

Notice that, if $V_\eta^{0,L_k}(r_t) \geqslant \delta/2$, then

$$\int_0^t \frac{1}{2} g(\eta_s(r_s)) \mathbb{1}_{\{\eta_s(r_s)\geqslant 2\}}\, ds \geqslant \frac{\delta}{4} g(2) L_k. \tag{3.4.29}$$

Define $\delta' = \frac{\delta}{8} g(2)$ and use Proposition 3.4.6 to conclude that

$$\begin{aligned} \mathbb{P}_{\rho_\infty}[r_{L_k} \leqslant \delta' L_k, r_0 = 0] &\leqslant \mathbb{P}_{\rho_\infty}[V_\eta^{0,L_k}(r_t) \leqslant \delta, r_0 = 0] + \mathbb{P}_{\rho_\infty}[|M_{L_k}| \geqslant \delta' L_k] \\ &\leqslant 4e^{-\log^{5/4} L_k}. \end{aligned} \tag{3.4.30}$$

□

We are now ready to conclude the proof of Theorem 1.1.4. The last step is a concatenation argument similar to the one used in the last section to conclude Theorem 1.1.3. For this reason, we provide just a sketch of the proof.

Proof of Theorem 1.1.4. We may assume that $L \geqslant 2L_{k_4+2}$. Choose $\bar{k} \geqslant k_4$ such that

$$2L_{\bar{k}+2} \leqslant L < 2L_{\bar{k}+3}. \tag{3.4.31}$$

For $m = (x, s) \in \mathbb{Z} \times L_k\mathbb{N}_0$, define the events

$$\bar{E}_k(m) = \{r_{L_k} - x \leqslant \delta L_k \text{ and } r_0(m) = x\} \tag{3.4.32}$$

where δ is given by Proposition 3.4.7. Consider also

$$\bar{H}_k(m) = \left\{x - \inf_{0 \leqslant t \leqslant L_k} r_t(m) \leqslant 2(\Gamma_+ + 1)L_k\right\}. \tag{3.4.33}$$

Finally, define

$$A = \left\{ \begin{array}{c} r_t \geqslant \nu_+ t + L, \\ \text{for some } t \geqslant 0 \end{array} \right\}, \tag{3.4.34}$$

where ν_+ is given by Theorem 1.1.3 and is such that (1.1.7) holds.

Define the set of indices

$$\bar{M}_k = \{m \in \mathbb{Z} \times L_k\mathbb{N}_0 : B_k(m) \cap B_{k+2} \neq \emptyset\}, \tag{3.4.35}$$

and consider the event

$$\bar{B}_{\bar{k}} = A \cap \bigcap_{k \geqslant \bar{k}} \bigcap_{m \in \bar{M}_k} \bar{E}_k(m)^c \cap \bar{H}_k(m). \tag{3.4.36}$$

Proposition 3.4.7, Lemma 3.2.2 and Theorem 1.1.3 imply, by possibly changing constants, that

$$\mathbb{P}_{\rho_\infty}[\bar{B}_{\bar{k}}] \leqslant c_{25} e^{-c_{25}^{-1} \log^{5/4} L}. \tag{3.4.37}$$

Similarly to the proof of Theorem 1.1.3, define

$$\bar{J}_{\bar{k}} = \bigcup_{k \geqslant \bar{k}} \bigcup_{\ell=0}^{L_{k+2}/L_k} \{\ell L_k\}. \tag{3.4.38}$$

On the event $\bar{B}_{\bar{k}} \cap \{r_0 = 0\}$, induction implies that

$$r_t \geqslant \delta t, \qquad \text{for all } t \in J_{\bar{k}}. \tag{3.4.39}$$

We now interpolate for the remaining values of t. Consider initially $t \geqslant L$. Let κ be the smallest $k \geqslant \bar{k}$ such that

$$\ell L_\kappa \leqslant t < (\ell+1)L_\kappa, \qquad \text{for some } \ell < \frac{L_{\kappa+2}}{L_\kappa}. \tag{3.4.40}$$

Let $\bar{\ell}$ denote the unique value of ℓ and observe that $\ell \geqslant L_\kappa/L_{\kappa-1}$. This easily implies, by increasing the value of L if necessary,

$$r_t \geqslant \delta\ell L_\kappa - (2\Gamma_+ + 1)L_\kappa \geqslant \frac{\delta}{2}t. \tag{3.4.41}$$

The interpolation for the values $t \leqslant L$ is done in the same way as in the proof of Theorem 1.1.3 and we omit it here. □

Finally, as an application, we prove a decoupling for the zero range process considering functions of the space-time configurations whose supports are far away in space. Recall the definition of the vertical distance (3.1.12) and consider the horizontal distance between the boxes B_1 and B_2

$$d_H = \inf\{|x-y| : (x,t) \in B_1 \text{ and } (y,s) \in B_2\}. \tag{3.4.42}$$

Proposition 3.4.8. *Fix $0 < \rho_- \leqslant \rho_+ < \infty$. There exist positive constants C_9, C_{10} and c_{26} such that, for any two square boxes B_1 and B_2 of side-length s satisfying*

$$d_H \geqslant C_9(s + d_V) + C_{10}, \tag{3.4.43}$$

and any two functions of the space-time $f_1, f_2 : \mathbb{N}_0^{\mathbb{Z}\times\mathbb{R}_+} \to [0,1]$ with respective supports in B_1 and B_2, we have, for any $\rho \in [\rho_-, \rho_+]$,

$$\mathbb{E}_\rho[f_1 f_2] \leqslant \mathbb{E}_\rho[f_1]\mathbb{E}_\rho[f_2] + c_9 e^{-c_9^{-1}\log^{5/4} d_H} \tag{3.4.44}$$

Proof. Observe first that we may increase the side-length of both boxes by at most $d_V + s$ and assume the boxes have the form

$$B_1 = [-s, 0] \times [0, s],$$
$$B_2 = [d_H, d_H + s] \times [0, s].$$

Figure 3.8 can be used as a reference.

We now verify that, with high probability, the outcomes of f_1 and f_2 are determined by disjoint parts of the graphical construction in the space-time.

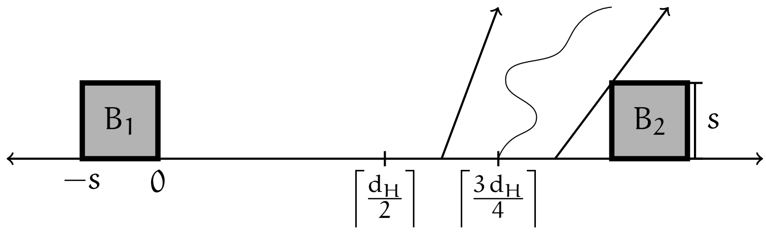

Figure 3.8: The boxes B_1 and B_2. Notice also the infection process $r_t(\lceil 3d_H/4 \rceil, 0)$ and the lines that bound the evolution of the front.

Consider initially $r_t(\lceil 3d_H/4 \rceil, 0)$. Observe that, if f_2 is not determined by the graphical construction restricted to $(\lceil d_H/2 \rceil, \infty) \times [0, s]$ (and the initial configuration restricted to $(\lceil d_H/2 \rceil, \infty)$), then the infection $r_t(\lceil 3d_H/4 \rceil, 0)$ touches either B_2 or the line $y = \lceil d_H/2 \rceil$. On the other hand, if we consider the reflected infection $\tilde{r}_t(\lceil d_H/4 \rceil, 0)$, that starts with the right half-axis infected and travels to the left, we obtain a similar statement for B_1: The outcome of f_1 is not determined by the graphical construction restricted to $(-\infty, \lceil d_H/2 \rceil) \times [0, s]$ if, and only if, the reflected infection reaches either B_1 or the line $y = \lceil d_H/2 \rceil$. Besides, the graphical construction is independent in disjoint subsets of the space-time.

Let A be the event where $r_t(\lceil 3d_H/4 \rceil, 0)$ touches either B_2 or the line $y = \lceil d_H/2 \rceil$, and denote by $\tilde{A}$ the respective event with the infection $\tilde{r}_t(\lceil d_H/4 \rceil, 0)$ and the box B_1. If we choose C_9 and C_{10} large enough, we can use Theorems 1.1.3 and 3.1.4 to bound

$$\mathbb{P}_\rho[A] \leqslant c e^{-c^{-1} \log^{5/4} d_H}. \tag{3.4.45}$$

By symmetry, the same is true for $\mathbb{P}_\rho[\tilde{A}]$. We now can bound

$$\begin{aligned} \mathbb{E}_\rho[f_1 f_2] &\leqslant \mathbb{E}_\rho[f_1 f_2 \mathbf{1}_{A^c \cap \tilde{A}^c}] + \mathbb{P}_\rho[A] + \mathbb{P}_\rho[\tilde{A}] \\ &\leqslant \mathbb{E}[f_1]\mathbb{E}_\rho[f_2] + 2(\mathbb{P}_\rho[A] + \mathbb{P}_\rho[\tilde{A}]) \; . \\ &\leqslant \mathbb{E}[f_1]\mathbb{E}_\rho[f_2] + c e^{-c^{-1} \log^{5/4} d_H} \end{aligned} \tag{3.4.46}$$

The proof is complete. □

4

NOISE SENSITIVITY AND VORONOI PERCOLATION

In this chapter, we change subjects and focus on the second part of this thesis. We now work on a different class of problems: Noise sensitivity for Voronoi percolation. We develop the tools that allow us to prove Theorems 1.2.1 and 1.2.2.

Tools and techniques from the analysis of Boolean functions will be central in the remainder the chapter. We shall in Section 4.1 begin with a brief review of these, centering on the use of randomized algorithms and their revealment. In Section 4.2 we outline the discretization method developed in [2], which will allow for these techniques to be applied in the setting of Voronoi percolation. In Section 4.3 we describe an algorithm that will be used to prove Theorem 1.2.1, and estimate its revealment. The proof of Theorem 1.2.1 is then given in Section 4.4, and Sections 4.5 and 4.6 are dedicated to study the effect of alternative perturbations, and to prove Theorem 1.2.2.

4.1 ANALYSIS OF BOOLEAN FUNCTIONS

In the analysis of Boolean functions, discrete Fourier techniques have become an indispensable tool. Although phenomena such as sharp thresholds and noise sensitivity can be directly linked to the spectrum of the Fourier-Walsh decomposition of a Boolean function, it is often a very challenging task to obtain precise estimates on the spectrum itself. A range of techniques have therefore been developed in order to relate such phenomena to notions such as influence of variables and revealment of algorithms, which are typically more tractable quantities to estimate.

In this section, we review some results connecting influences and revealment to threshold behavior and noise sensitivity. We shall avoid the discussion of Fourier techniques, that lie behind several of the results we describe, and refer the reader to the books [23] and [36] for a more extensive treatment.

4.1.1 *Influence of variables*

Let $f : \{0,1\}^n \to \{0,1\}$ be a Boolean function. The **influence** of bit $k \in [n]$ for f is defined as

$$\mathrm{Inf}_k^p(f) = \mathrm{Inf}_k^p(f,[n]) := \mathbb{P}_p[f(\omega) \neq f(\sigma_k \omega)], \tag{4.1.1}$$

where σ_k is the operator that changes ω at position k from ω_k to $1-\omega_k$. Recall that a Boolean function is called monotone if $f(\omega') \geqslant f(\omega)$ whenever $\omega'_k \geqslant \omega_k$ for each $k \in [n]$. It is well-known that many monotone Boolean functions exhibit a threshold phenomenon, where the probability $\mathbb{P}_p[f=1]$ increases from close to 0 to close to 1 in a narrow window – the threshold window. The central role of influences in the understanding of this phenomenon is emphasized by the Margulis-Russo formula. It says that, for any monotone function $f : \{0,1\}^n \to \{0,1\}$,

$$\frac{d}{dp}\mathbb{P}_p[f=1] = \sum_{k=1}^{n} \mathrm{Inf}_k^p(f). \tag{4.1.2}$$

Russo's approximate 0-1 law [42] gives the first general condition for the existence of a threshold. Russo showed that, if, for every $\epsilon > 0$, there exists $\delta > 0$ such that if $\mathrm{Inf}_k^p(f) \leqslant \delta$ uniformly in k and p, then $\mathbb{P}_p(f=1)$ transitions from below ϵ to above $1-\epsilon$ in a window of width at most ϵ. Later works [28, 21, 48] have obtained a more precise formulation of Russo's theorem that allows one to get a quantitative bound on the width of the threshold window.

Influences are likewise fundamentally connected to the notion of noise sensitivity. The BKS Theorem, due to Benjamini, Kalai and Schramm [12], says that a sufficient condition for a sequence $(f_n)_{n\geqslant 1}$ of Boolean functions to be noise sensitive at level p is that

$$\sum_{k=1}^{n} \mathrm{Inf}_k^p(f_n)^2 \to 0 \quad \text{as } n \to \infty. \tag{4.1.3}$$

For monotone functions this condition is also necessary.

4.1.2 *Revealment of algorithms*

A (randomized) algorithm is a rule that queries the bits of $\omega \in \{0,1\}^n$ in a random order, which is allowed to depend on what has been seen so far, and outputs either 0 or 1. An algorithm is said to **determine** f if its output equals $f(\omega)$ for each $\omega \in \{0,1\}^n$. The **revealment** of an algorithm $\mathcal{A}$ with respect to $K \subseteq [n]$ is defined as

$$\delta_p(\mathcal{A}, K) := \max_{k \in K} \mathbb{P}_p[\mathcal{A} \text{ queries bit } k]. \tag{4.1.4}$$

In order to verify when the condition in (4.1.3) is satisfied, Benjamini, Kalai and Schramm [12] devised a method involving algorithms. This method was developed further in later work by Schramm and Steif [43]. In essence, this method shows that a sequence of functions is noise sensitive if there exists (a sequence of) algorithms that determines f_n without being likely to query any specific bit. The next proposition, due to Schramm and Steif [43], gives an explicit formulation of this last statement.

Proposition 4.1.1. *Let $\mathcal{A}$ be an algorithm that determines the function $f : \{0,1\}^n \to \{0,1\}$. Then, for every $m \geqslant 1$, we have*

$$\mathbb{E}_p[f(\omega)f(\omega^\epsilon)] - \mathbb{E}_p[f(\omega)]^2 \leqslant e^{-\epsilon m} + m^2 \delta_p(\mathcal{A}, [n]),$$

Since the correlation is non-negative, it is immediate from the proposition above that a sequence $(f_n)_{n \geqslant 1}$ is noise sensitive if there exists an algorithm $\mathcal{A}$ determining f_n with revealment tending to zero. Moreover, if $\delta_p(\mathcal{A}, [n])$ decays polynomially fast, then the sequence $(f_n)_{n \geqslant 1}$ has positive noise sensitivity exponent.

Randomized algorithms are related to influences and threshold phenomena via the following inequality, due to O'Donnell, Saks, Schramm and Servedio [37].

Proposition 4.1.2. *Let $f : \{0,1\}^n \to \{0,1\}$ be a Boolean function and $\mathcal{A}$ an algorithm that determines f. Then, for every $p \in (0,1)$, we have*

$$\mathrm{Var}_p(f) \leqslant p(1-p) \sum_{k \in [n]} \delta_p(\mathcal{A}, k) \, \mathrm{Inf}_k^p(f). \tag{4.1.5}$$

The above inequality implies, in particular, that

$$\mathrm{Var}_p(f) \leqslant \frac{1}{4}\delta_p(\mathcal{A},[n])\sum_{k=1}^{n}\mathrm{Inf}_k^p(f),$$

and hence, together with the Margulis-Russo formula, one concludes that monotone Boolean functions satisfy the inequality

$$\frac{d}{dp}\mathbb{P}_p[f=1] \geqslant 4\frac{\mathrm{Var}_p(f)}{\delta_p(\mathcal{A},[n])}.$$

We will, via the study of algorithms, be able to obtain polynomial bounds on the width of the threshold window of certain Boolean functions, where methods based on influences would give logarithmic bounds, see e.g. [21].

Somewhat less standard is the following upper bound on the sum of influences in terms of the revealment: For every function $f:\{0,1\}^n \to \{0,1\}$ that is monotone in each coordinate we have

$$\sum_{k\in[n]}\mathrm{Inf}_k^p(f) \leqslant \sqrt{n\sum_{k\in[n]}\mathrm{Inf}_k^p(f)^2} \leqslant \frac{1}{p(1-p)}\sqrt{n\,\delta_p(\mathcal{A},[n])}. \tag{4.1.6}$$

The former of the two inequalities is immediate from Cauchy-Schwarz inequality, whereas the latter follows from (a variant of) the Schramm-Steif revealment Theorem. Although we are not aware of an application of this kind, this inequality provides a way to obtain a lower bound on the width of the threshold window for monotone Boolean functions that is sharper than the elementary lower bound of order $1/\sqrt{n}$.

4.2 CONTINUUM TO DISCRETE

We now begin to set the stage for the proof of Theorem 1.2.1. Our approach will be based on a method developed in [2], and revisited in [5], that allows one to reduce the continuum problem at hand to its discrete counterpart via a two-stage construction of the continuum process.

Recall that $\mathbb{P}_{n,p}$ denotes the distribution of a Poisson point process in Ω with intensity measure $n\lambda_S \otimes [p\delta_1 + (1-p)\delta_0]$. Fix an integer $k \geqslant 2$ and choose $\eta_k \in \Omega$ distributed as $\mathbb{P}_{kn,p}$. Let η be obtained from η_k by independently including each point of η_k with probability $1/k$. Notice that

η is distributed according to $\mathbb{P}_{n,p}$, and that conditional on η_k, we may consider η as an element in $\{0,1\}^{\eta_k}$ chosen according to $\mathbb{P}_{1/k}$.

Recall the notation $(\eta, \eta(\epsilon))$ for a pair of configurations in Ω distributed according to $\mathbb{P}_{n,p}$, where the latter is an ϵ-perturbation of the former. The two-stage construction gives an alternative way to obtain a pair of configurations (η, η^{ϵ}) where, conditional on η_k, the latter is obtained by an ϵ-perturbation of the former seen as elements in $\{0,1\}^{\eta_k}$. Using the fact that η and $\eta_k \setminus \eta$ are independent, it is for $\epsilon' \leqslant 1 - 1/k$ and $\epsilon = \epsilon'/(1 - 1/k)$ straightforward to verify that $(\eta, \eta(\epsilon'))$ and (η, η^{ϵ}) are equal in distribution.

The two-stage construction thus leads us to the identity

$$\begin{aligned} \mathbb{E}_{n,1/2}\big[f_R(\eta)f_R(\eta(\epsilon'))\big] - \mathbb{E}_{n,1/2}\big[f_R(\eta)\big]^2 \; &= \; \mathrm{Var}_{kn,1/2}\left(\mathbb{E}\left[f_R(\eta)|\eta_k\right]\right) \\ &+ \mathbb{E}_{kn,1/2}\big[\mathbb{E}[f_R(\eta)f_R(\eta^{\epsilon})|\eta_k] - \mathbb{E}[f_R(\eta)|\eta_k]^2\big]. \end{aligned} \tag{4.2.1}$$

In order to prove that f_R is noise sensitive it will thus suffice to prove that each term in the right-hand side of (4.2.1) is small for large n. To prove that the variance term, for fixed k, tends to zero as n tends to infinity turns out to be equivalent to the original problem. To see this, let η' and η'' be obtained independently from η_k by keeping each point with probability $1/k$. Then, for $\epsilon' = 1 - 1/k$ the joint law of (η', η'') equals that of $(\eta, \eta(\epsilon'))$, and hence[1]

$$\begin{aligned} &\mathbb{E}_{n,1/2}\big[f_R(\eta)f_R(\eta(\epsilon'))\big] - \mathbb{E}_{n,1/2}\big[f_R(\eta)\big]^2 \\ &\quad = \mathbb{E}_{kn,1/2}\big[\mathbb{E}[f_R(\eta')f_R(\eta'')|\eta_k]\big] - \mathbb{E}_{kn,1/2}\big[\mathbb{E}[f_R(\eta')|\eta_k]\big]^2 \\ &\quad = \mathrm{Var}_{kn,1/2}\big(\mathbb{E}[f_R(\eta')|\eta_k]\big). \end{aligned} \tag{4.2.2}$$

However, we shall in Lemma 4.2.1 see that the expression in (4.2.2) tends to zero as $k \to \infty$. The goal will then be to show that, for large k, conditional on η_k, the function $f_R : \{0,1\}^{\eta_k} \to \{0,1\}$ is noise sensitive in the sense of (1.2.1), with high probability.

In a similar manner we shall rely on the two-stage construction in order to prove that f_R has a sharp threshold at $p = 1/2$. The construction here will have to be slightly different, since we now want to vary the color of certain points and not their presence. We will thus let $\bar{\eta}_k$ denote the projection of η_k to S, and instead aim to show that $\mathbb{P}[f_R(\eta) = 1|\bar{\eta}_k]$ grows from 0 to 1 in a narrow interval around $p = 1/2$, with high probability. A first step in both

[1] Here, $k > 1$ does not have to be an integer.

these instances is obtained in the following lemma, which has its origins in [2], although the proof we present here is taken from [5].

Lemma 4.2.1. *For every integer* $k \geqslant 2$ *and* $p \in (0,1)$ *we have*

$$\mathrm{Var}_{kn,p}\left(\mathbb{E}\left[f_R(\eta)|\eta_k\right]\right) \leqslant \frac{1}{k}.$$

Proof. It all boils down to use a suitable construction for the pair (η_k, η). Consider k independent copies $\eta^{(1)}, \eta^{(2)}, \ldots, \eta^{(k)}$ of η, and let κ be chosen uniformly in $[k]$. We then observe that

$$\begin{aligned}\mathrm{Var}_{kn,p}\left(\mathbb{E}\left[f_R(\eta)|\eta_k\right]\right) &\leqslant \mathrm{Var}_{n,p}\left(\mathbb{E}\left[f_R\left(\eta^{(\kappa)}\right)\Big|(\eta^{(i)})_{i=1}^k\right]\right)\\ &= \mathrm{Var}_{n,p}\left(\frac{1}{k}\sum_{i=1}^k f_R\left(\eta^{(i)}\right)\right).\end{aligned}$$

The lemma then follows from the independence of the $\eta^{(i)}$. □

As an easy corollary of the lemma above we obtain the following.

Lemma 4.2.2. *For every rectangle* $R \subseteq S$ *there exists* k_0*, depending only on the aspect ratio of* R*, such that, if* $k \geqslant k_0$*, then we have, for all large* n*, that*

$$\mathbb{P}_{kn,1/2}\Big[\mathbb{P}\left[f_R(\eta) = 1|\eta_k\right] \notin [c_3/2, 1 - c_3/2]\Big] \leqslant \frac{1}{\sqrt{k}},$$

where c_3 *is the constant in* (1.2.2).

Proof. Chebyshev's inequality and Lemma 4.2.1 imply that

$$\begin{aligned}&\mathbb{P}_{kn,1/2}\Big[\mathbb{P}\left[f_R(\eta) = 1|\eta_k\right] \notin [c_3/2, 1 - c_3/2]\Big]\\ &\qquad\leqslant \mathbb{P}\left[\Big|\mathbb{P}\left[f_R(\eta) = 1|\eta_k\right] - \mathbb{P}_{n,1/2}[f_R = 1]\Big| \geqslant \frac{c_3}{2}\right]\\ &\qquad\leqslant \frac{4}{c_3^2 k} \leqslant \frac{1}{\sqrt{k}},\end{aligned}$$

for k and n large enough. □

Remark 4.2.3. Notice that if, for some $k \geqslant 2$, we have

$$\lim_{n\to\infty} \mathrm{Var}_{kn,1/2}\left(\mathbb{E}[f_R(\eta)\,|\,\eta_k]\right) = 0,$$

then the conclusion in Lemma 4.2.2 strengthens to

$$\lim_{n\to\infty} \mathbb{P}_{kn,1/2}\left[\mathbb{P}\left[f_R(\eta) = 1|\,\eta_k\right] \notin [c_3/2, 1 - c_3/2]\right] = 0.$$

4.3 AN ALGORITHM WITH LOW REVEALMENT

In this section, we continue to work towards a proof of Theorem 1.2.1. We will adopt the two-stage construction introduced in the previous section, and devise an algorithm which, conditional on the denser set of points η_k, determines the outcome of $f_R(\eta)$ by querying points of η_k whether they are contained in the sparser set η. We then proceed to show that this algorithm has low revealment, which in the next section will allow us to deduce that f_R is noise sensitive and has a threshold at $p = 1/2$.

4.3.1 *The algorithm*

In this subsection we describe the algorithm. Loosely speaking, it will explore the square S until it has discovered all blue components that touch a randomly selected vertical line through R. This is achieved by querying points close to the vertical line first, and then proceeding to points that are close to already explored blue components connected to the vertical line. Since we cannot tell the Voronoi tessellation of η by just observing η_k, we will only gain information about the actual tiling locally as we go. To contour this difficulty, we will split S into boxes on a mesoscopic scale (see Figure 4.1), so that by querying all points within such a box we will correctly determine the tiling within that box with high probability, apart from close to the boundary. That is, by further dividing each box into nine sub-boxes the thus learn the tiling of η correctly within the centre box with high probability.

If the algorithm discovers a blue component that touches both left and right sides of R, then there is a horizontal blue crossing of R. If not, then there is a vertical red crossing. The reason the algorithm has low revealment is that a given point is both unlikely to be close to the randomly located vertical line, and unlikely to be connected far by a blue path.

The rest of this section will be dedicated to confirming these claims. We first give a more precise description of our algorithm, see Algorithm 4.3.1. Recall that Ω is the collection of finite subsets of $S \times \{0, 1\}$.

Lemma 4.3.1. *Algorithm 4.3.1 determines the outcome of* f_R *almost surely.*

Proof. Observe that if there exists a horizontal blue crossing of R, then it necessarily crosses every vertical line through R. Hence, it suffices to

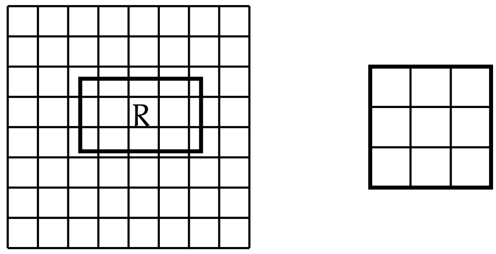

Figure 4.1: The unit square divided into smaller squares at a mesoscopic scale. When all points of η_k in a sub-square are queried, then the tiling within the center box in a further division into nine sub-boxes is correctly determined with high probability.

Algorithm 4.3.1 (Existence of a horizontal blue crossing)

1: **Input:** $\eta_k \in \Omega$, $\eta \in \{0,1\}^{\eta_k}$ and $R = [a,b] \times [c,d] \subseteq S$.
2: Choose a point x_0 uniformly in the mid third of the interval $[a,b]$.
3: Consider a lattice in S with mesh size $m = 1/\lceil n^{1/4} \rceil$, and divide each cell in this lattice into nine equally sized subcells.
4: Query points in all cells of the lattice that intersect $R \cap \{x = x_0\}$ and their neighbouring cells. Declare the examined cells *explored*, and each explored cell *safe* if also the eight cells that surround it are explored.
5: If any of the cells explored so far contains an empty subcell, then query all points of η_k. Otherwise, proceed and explore all cells that share an edge with a safe cell and are connected to the line $\{x = x_0\}$ by a blue component inside the safe region. Explore also any cell neighbouring to these cells and declare an explored cell which is surrounded by explored cells safe.
6: Repeat Step 5 until all connected blue components inside R that intersect $\{x = x_0\}$ are discovered. If there is a connected blue component inside R that connects $\{x = a\}$ to $\{x = b\}$, return 1. Otherwise, return 0.

---

verify that given η_k the algorithm correctly determines all connected blue components of η inside R that intersect the random vertical line $\{x = x_0\}$.

If the algorithm queries all points of η_k then this is trivially true. If not, then all we need to verify is that for each safe cell, i.e., a cell which is explored along with its eight surrounding neighbours, we have determined the tiling within. This is indeed the case since if no neighbouring cell has an empty subcell, then no point outside the safe cell and its eight neighbours can affect the tiling inside the safe cell. □

Now that we have an algorithm that determines f_R, we need to bound its revealment. Since the algorithm only reveals the configuration inside cells of a mesoscopic lattice, we consider each such cell individually and bound the revealment of every point inside it at once. This is done in the next two subsections.

4.3.2 *One-arm estimates*

For a point in η_k to be queried by the algorithm above one of the following three things would have to occur: Either there is a subcell of some cell of the lattice which does not contain any point of η, or it is contained in a cell 'close' to the random vertical line through R, or it is in a cell located 'far' from the line, but there exists a connected blue path in η connecting the vertical line with one of the eight cells that surround that cell. In this subsection we shall bound the probability of the third of these possibilities.

Let $m = \lceil n^{1/4} \rceil^{-1}$ as before, and partition S into squares of side length m. The precise choice of m is irrelevant as long as $n^{-1/2} \ll m \ll 1$. Let $C \subseteq S$ be a cell in this lattice, and let C' be the square of side length $3m$ centered at C. We define $\mathrm{Arm}(C)$ as the event that there exists a blue path that connects C' to the boundary of the square of side $\sqrt{m}$ centered at C.

Proposition 4.3.2. *There exists $\delta > 0$ such that, for every $\gamma > 0$, we can find $k_0 \geqslant 1$ so that, for $k \geqslant k_0$, $p \leqslant 1/2$, and all large n, depending on k, we have*

$$\mathbb{P}_{kn,p}\Big[\mathbb{P}\left[\eta \in \mathrm{Arm}(C) \,|\, \eta_k\right] > n^{-\delta}\Big] < n^{-\gamma}.$$

Estimates of this type have previously been obtained in [2, 3, 5], and the proof presented here will be similar, although different in some details. It will suffice to consider the critical case $p = 1/2$ due to monotonicity. As a first step, we prove a lemma that bounds the probability that a configuration contains a large cell. Let

$$E := \left\{\text{some cell of } \eta \text{ has radius larger than } n^{-1/3}\right\}. \tag{4.3.1}$$

Lemma 4.3.3. *There exists* $c_{27} > 0$ *such that, for all* $n \geqslant 1$, *we have*

$$\mathbb{P}_{n,1/2}[E] \leqslant \exp(-c_{27} n^{1/3}). \tag{4.3.2}$$

Proof. We split the unit square S into boxes of side length $(10\lceil n^{1/3} \rceil)^{-1}$. Notice that for E to occur it is necessary for the intersection of η with at least one of these about $100n^{2/3}$ boxes to be empty. For each individual box this occurs with probability at most $\exp(-0.01n \cdot n^{-2/3})$. Via the union bound we conclude that

$$\mathbb{P}_{n,1/2}[E] \leqslant 100\, n^{2/3} \exp(-0.01n^{1/3}),$$

as required. □

Proof of Proposition 4.3.2. Fix a cell $C \subseteq S$ of side length $m = \lceil n^{1/4} \rceil^{-1}$. For every integer $j \geqslant 0$, denote by A_j the square annulus centered around C, with inner side-length $4^j m$ and outer side-length $3 \cdot 4^j m$. Let O_j be the event that there is *not* a blue path connecting the inner and outer boundary of A_j. That is, O_j is the even that there is a red path in A_j that disconnects any blue component touching C from the exterior of A_j. Observe that, in order for the event $\mathrm{Arm}(C)$ to occur, O_j cannot occur for integers j in the set

$$J := \{ j \in \mathbb{N} : m \leqslant 4^j m \leqslant m^{1/2} \}.$$

Let E be the event in (4.3.1), and let A'_j denote the set of points within distance $m/3$ of A_j. We note that, on E^c, the events O_j are determined by the restriction of η to A'_j, which we shall denote $\eta^{(j)}$. That is, if $g_j : \Omega \to \{0,1\}$ denotes the indicator of O_j, then

$$\mathbf{1}_{E^c} \cdot g_j(\eta) = \mathbf{1}_{E^c} \cdot g_j(\eta^{(j)}). \tag{4.3.3}$$

Moreover, since the sets A'_j are disjoint the configurations $\eta^{(j)}$ are independent. Since O_j cannot occur for any $j \in J$ in case that $\mathrm{Arm}(C)$ occurs, it follows that

$$\begin{aligned} \mathbb{P}[\mathrm{Arm}(C) \,|\, \eta_k] &\leqslant \mathbb{P}[E \,|\, \eta_k] + \mathbb{P}\Big[E^c \cap \bigcap_{j \in J} O_j^c \,\Big|\, \eta_k\Big] \\ &\leqslant \mathbb{P}[E \,|\, \eta_k] + \prod_{j \in J} \mathbb{P}\big[g_j(\eta^{(j)}) = 0 \,|\, \eta_k\big] \\ &\leqslant \mathbb{P}[E \,|\, \eta_k] + \prod_{j \in J} \Big(\mathbb{P}[O_j^c \,|\, \eta_k] + \mathbb{P}[E \,|\, \eta_k]\Big). \end{aligned} \tag{4.3.4}$$

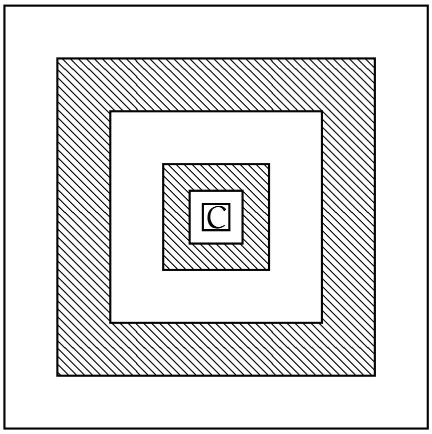

Figure 4.2: The square C, surrounded by a larger square with side length $m^{1/2}$. The dashed annuli represent the sets A_j. Notice that, if there is a blue path from C to the boundary of the square, none of the annuli can contain a red circuit.

Introduce the events

$$D := \{\mathbb{P}[E \mid \eta_k] \geqslant 1/n\} \quad \text{and} \quad D_j := \left\{\mathbb{P}[O_j \mid \eta_k] \leqslant c_3^4/32 - 2/n\right\},$$

and let D^* denote the event that D_j occurs for at least half the indices in J. From (4.3.4) we conclude that on $(D^* \cup D)^c$ there exists $\delta > 0$ such that

$$\mathbb{P}[\mathrm{Arm}(C)|\eta_k] \leqslant 1/n + \left[(1 - c_3^4/32) + 3/n\right]^{|J|/2} \leqslant n^{-\delta}.$$

It remains to bound the probability that either D or D^* occurs. By Markov's inequality and Lemma 4.3.3,

$$\mathbb{P}_{kn,1/2}[D] \leqslant n\,\mathbb{P}_{n,1/2}[E] \leqslant n \cdot \exp(-c_{27} n^{1/3}). \tag{4.3.5}$$

Since $nm^2 \gg 1$ and the annulus A_j is the union of four rectangles with sides $3 \cdot 4^j m$ and $4^j m$, it follows from Lemma 4.2.2 and Harris' inequality that

$$\mathbb{P}_{kn,1/2}\left[\mathbb{P}[O_j|\eta_k] \leqslant c_3^4/16\right] \leqslant 4k^{-1/2}. \tag{4.3.6}$$

Here, it is important to observe that the bound is independent of the chosen annulus. Indeed, if the annulus is not entirely contained in S, then it would only be harder for blue to reach its outer boundary from within.

We then observe that

$$\mathbb{P}_{kn,1/2}[D \cup D^*] \leqslant \mathbb{P}_{kn,1/2}[D] + 2^{|J|/2} \sup_I \mathbb{P}_{kn,1/2}\Big[D^c \cap \bigcap_{j\in I} D_j\Big], \qquad (4.3.7)$$

where the supremum above is taken over all subsets of J with at least |J|/2 elements. Repeated use of (4.3.3) shows that

$$\begin{aligned}
\mathbb{P}_{kn,1/2}\Big[D^c \cap \bigcap_{j\in I} D_j\Big] &\leqslant \mathbb{P}_{kn,1/2}\Big[\bigcap_{j\in I}\Big\{\mathbb{P}\Big[g_j(\eta^{(j)}) = 1 \,|\, \eta_k\Big] \leqslant \frac{c_3^4}{16} - 1/n\Big\}\Big] \\
&\leqslant \prod_{j\in I} \mathbb{P}_{kn,1/2}\Big[\mathbb{P}\Big[g_j(\eta^{(j)}) = 1 \,|\, \eta_k\Big] \leqslant \frac{c_3^4}{16} - 1/n\Big] \\
&\leqslant \prod_{j\in I}\Big(\mathbb{P}_{kn,1/2}[D] + \mathbb{P}_{kn,1/2}\Big[\mathbb{P}[O_j|\eta_k] \leqslant \frac{c_3^4}{16}\Big]\Big)
\end{aligned}$$

Hence, combined with the estimates in (4.3.5)-(4.3.7) we conclude that

$$\mathbb{P}_{kn,1/2}[D \cup D^*] \leqslant n \cdot \exp(-c_{27} n^{1/3}) + 2^{|J|/2}\big[n \cdot \exp(-c_{27} n^{1/3}) + 4k^{-1/2}\big]^{|J|/2}.$$

Since $|J| = \Omega(\log n)$ we may for every $\gamma > 0$ choose k large so that the above estimate is bounded by $n^{-\gamma}$ for all large n. □

4.3.3 *Revealment of the algorithm*

Now that we have the one-arm estimate, we can bound the revealment of our algorithm. We recall that a point in η_k may be queried if the $m \times m$ cell in which it belongs is either 'close' to the random vertical line through R, or 'far' but connected by a blue path to that line, or if the algorithm at some point discovers a subcell of some $m \times m$ cell which is empty.

Proposition 4.3.4. *Let $\mathcal{A}$ denote Algorithm 4.3.1. There exist $\delta > 0$ and $k_0 \geqslant 1$ such that, for every $k \geqslant k_0$, $p \leqslant 1/2$, and all large n, we have*

$$\mathbb{P}_{kn,p}\big[\delta_{1/k}(\mathcal{A}, \eta_k) > n^{-\delta}\big] < n^{-50}.$$

Proof. As before we partition the unit square S into cells of side length m, and split each cell C into nine further subcells. Let G be the event that each such subcell contains a point of η, and let

$$B := \big\{\mathbb{P}[G^c|\eta_k] > 1/n\big\}.$$

Markov's inequality then gives that, for large n,

$$\mathbb{P}_{kn,p}[B] \leqslant n\,\mathbb{P}_{n,p}[G^c] \leqslant n \cdot 9m^{-2}\exp(-9^{-}1nm^2) < \frac{1}{2}n^{-50}.$$

Next we fix $\gamma = 100$ and let $\delta > 0$ and $k_0 \geqslant 1$ be as in Proposition 4.3.2. Let B' denote the event that for some $m \times m$ cell C, we have $\mathbb{P}[\mathrm{Arm}(C)|\eta_k] > n^{-\delta}$. The union bound and Proposition 4.3.2 then gives that for large n

$$\mathbb{P}_{kn,p}[B'] \leqslant m^{-2} \max_{C \subseteq S} \mathbb{P}_{kn,p}\left[\mathbb{P}[\mathrm{Arm}(C)|\eta_k] > n^{-\delta}\right] < \frac{1}{2}n^{-50}.$$

For a given $m \times m$ cell C we let D_C be the event that C is within distance $2\sqrt{m}$ of the random line through R. The probability of D_C is independent of η_k, and one can obtain an upper bound of order $\sqrt{m}$, uniformly in C.

For a point of η_k to be queried there has either to exist a subcell of some $m \times m$ cell that is empty, or the point must lie in a cell C within distance $2\sqrt{m}$ of the randomly chosen vertical line through R, or $\mathrm{Arm}(C)$ has to occur. The revealment of $\mathcal{A}$ thus has to satisfy

$$\delta_{1/k}(\mathcal{A}, \eta_k) \leqslant \max_{C \subseteq S} \Big(\mathbb{P}[G^c|\eta_k] + \mathbb{P}[D|\eta_k] + \mathbb{P}[\mathrm{Arm}(C)|\eta_k]\Big),$$

which restricted to the event $(B \cap B')^c$ is at most $n^{-1} + n^{-1/8} + n^{-\delta}$. □

We may analogously to the algorithm $\mathcal{A}$ define an algorithm $\mathcal{A}'$ which looks for a vertical red crossing of R. By symmetry it follows that, for $p \geqslant 1/2$,

$$\mathbb{P}_{kn,p}\left[\delta_{1/k}(\mathcal{A}', \eta_k) > n^{-\delta}\right] < n^{-50}.$$

4.4 NOISE SENSITIVITY AND THE THRESHOLD WINDOW

This section is devoted to the proof of Theorem 1.2.1. The proof is divided in two parts. First, we prove that Voronoi percolation is noise sensitive, with a positive noise sensitivity exponent. Then we bound the width of the threshold window. Throughout the section we work with the two-stage construction of the random Voronoi configuration, as described in Section 4.2.

Proof of Theorem 1.2.1 (first part). Due to Equation (4.2.1) and Lemma 4.2.1 it will suffice, for the first part of the theorem, to show that for some $\gamma > 0$ and all large k we have

$$\mathbb{E}_{kn,1/2}\Big[\mathbb{E}[f_R(\eta)f_R(\eta^{\epsilon_n})|\eta_k] - \mathbb{E}[f_R(\eta)|\eta_k]^2\Big] \to 0 \quad \text{as } n \to \infty, \tag{4.4.1}$$

where $\epsilon_n = n^{-\gamma}$.

Let $\mathcal{A}$ be the algorithm in Algorithm 4.3.1. The Schramm-Steif revealment Theorem (Proposition 4.1.1) gives that, for almost every η_k and $m \geqslant 1$, we have

$$\mathbb{E}[f_R(\eta)f_R(\eta^{\epsilon_n})|\eta_k] - \mathbb{E}[f_R(\eta)|\eta_k]^2 \leqslant \exp(-\epsilon_n m) + m^2\delta_{1/2}(\mathcal{A},\eta_k).$$

Let $\delta > 0$ be as in Proposition 4.3.4, and let B_n denote the event that $\delta_{1/k}(\mathcal{A},\eta_k) > n^{-\delta}$. Then $\mathbb{P}_{kn,1/2}[B_n] < n^{-50}$, and consequently

$$\begin{aligned}
\mathbb{E}_{kn,1/2}&\Big[\mathbb{E}[f_R(\eta)f_R(\eta^{\epsilon_n})|\eta_k] - \mathbb{E}[f_R(\eta)|\eta_k]^2\Big] \\
&\leqslant n^{-50} + \exp(-\epsilon_n m) + m^2\mathbb{E}_{kn,1/2}[\delta_{1/k}(\mathcal{A},\eta_k)\mathbf{1}_{B_n^c}] \\
&\leqslant n^{-50} + \exp(-\epsilon_n m) + m^2 n^{-\delta}.
\end{aligned}$$

Hence, (4.4.1) holds with $\gamma = \delta/3$ and $n^{\delta/3} \ll m \ll n^{\delta/2}$, which concludes the proof of the first part of Theorem 1.2.1. $\square$

We proceed with the proof of the second part.

Proof of Theorem 1.2.1 (second part). Given $\eta_k \in \Omega$ we shall with $\bar{\eta}_k$ denote its projection onto S. We first note that by dominated convergence we have

$$\frac{d}{dp}\mathbb{P}_{n,p}[f_R = 1] = \mathbb{E}_{n,p}\Big[\frac{d}{dp}\mathbb{P}[f_R(\eta) = 1|\bar{\eta}_k]\Big], \tag{4.4.2}$$

since the rate at which $\mathbb{P}[f_R(\eta) = 1|\bar{\eta}_k]$ may increase as p varies is bounded by the number of variables $|\bar{\eta}_k|$ affected by p. Moreover, given $\bar{\eta}_k$, we may think of η as an element in $\{0,1\}^{\bar{\eta}_k} \times \{0,1\}^{\bar{\eta}_k}$, where the first half of the coordinates determine 'color' and the second half determine 'presence' in the final configuration. The Margulis-Russo formula then gives that

$$\frac{d}{dp}\mathbb{P}[f_R(\eta) = 1\,|\,\bar{\eta}_k] = \sum_{x\in\bar{\eta}_k} \mathbb{P}\left[\begin{array}{c} x \text{ is present and its color} \\ \text{is pivotal for } f_R \end{array} \,\middle|\, \bar{\eta}_k\right]$$

almost surely. Since a blue point is better than no point, and no point is better than a red point, it follows that switching presence rather than color of a point is less likely to affect the outcome of f_R. Consequently, the derivative is bounded from below by the sum

$$\sum_{x\in\overline{\eta}_k} \mathbb{P}\big[x \text{ is present and its presence is pivotal for } f_R \,\big|\, \overline{\eta}_k\big].$$

Each term in the above expression can be rewritten as $\frac{1}{k}\mathbb{E}[\mathrm{Inf}_x^{1/k}(f_R,\eta_k)|\overline{\eta}_k]$, where the factor $1/k$ comes from the probability of being present. Hence, Equation (4.4.2) and the OSSS inequality (Proposition 4.1.2) together give that

$$\frac{d}{dp}\mathbb{P}[f_R(\eta)=1] \geqslant \frac{1}{k}\mathbb{E}\Big[\sum_{x\in\eta_k}\mathrm{Inf}_x^{1/k}(f_R,\eta_k)\Big] \geqslant \frac{4}{k}\,\mathbb{E}\left[\frac{\mathrm{Var}(f_R|\eta_k)}{\delta_{1/k}(\mathcal{A},\eta_k)}\right]. \qquad (4.4.3)$$

Fix $\epsilon > 0$ and let $I_\epsilon = I_\epsilon(n)$ denote the set of points $p \in [0,1]$ for which $\mathbb{P}_{n,p}[f_R = 1] \in (\epsilon, 1-\epsilon)$. By monotonicity I_ϵ is an interval, and for small ϵ the interval contains the point $1/2$. Consequently, to complete the proof it will suffice to show that there exists $\gamma > 0$ such that $|I_\epsilon| \leqslant n^{-\gamma}$ for all $\epsilon > 0$.

Let $\mathcal{A}$ be the algorithm in Algorithm 4.3.1, and $\mathcal{A}'$ be the analogously defined algorithm that looks for a vertical red crossing of R. We introduce the events

$$A := \big\{\mathbb{P}[f_R(\eta)=1|\eta_k] \in (\epsilon/2, 1-\epsilon/2)\big\},$$
$$B := \big\{\min\{\delta_{1/k}(\mathcal{A},\eta_k), \delta_{1/k}(\mathcal{A}',\eta_k)\} < n^{-\delta}\big\},$$

with which (4.4.3) reduces to

$$\frac{d}{dp}\mathbb{P}_{n,p}[f_R=1] > \epsilon^2 k^{-1} n^{\delta}\, \mathbb{P}_{n,p}[A\cap B]. \qquad (4.4.4)$$

Next we fix $k \geqslant 16/\epsilon^2$. By Chebyshev's inequality and Lemma 4.2.1 we then have, for all $p \in I_\epsilon$, that

$$\mathbb{P}_{n,p}[A^c] \leqslant (2/\epsilon)^2\,\mathrm{Var}_{n,p}\big(\mathbb{P}[f_R(\eta)=1|\eta_k]\big) \leqslant 4/(\epsilon^2 k) \leqslant 1/4.$$

By increasing k if necessary, Proposition 4.3.4 gives that $\mathbb{P}_{n,p}[B^c] \leqslant 1/4$ for all $p \in [0,1]$ and n large. Integrating over I_ϵ in (4.4.4) thus leads to the bound

$$1 \geqslant \int_{I_\epsilon} \frac{d}{dp}\mathbb{P}_{n,p}[f_R=1]\,dp \geqslant \frac{1}{2}\epsilon^2 k^{-1} n^{\delta}|I_\epsilon|,$$

and hence that $|I_\epsilon| \leqslant 2k/(\epsilon^2 n^\delta)$. Since $\epsilon > 0$ was arbitrary, the theorem follows with $\gamma = \delta/2$. □

We can also study the behavior of $f_R(\eta)$ for fixed values of k.

Proposition 4.4.1. *For every* $p \in [0,1]$ *and* $k > 1$, *not necessarily an integer,*

$$\lim_{n\to\infty} \mathrm{Var}_{kn,p}\left(\mathbb{E}\left[f_R(\eta)|\eta_k\right]\right) = 0.$$

Besides, there exists $\delta > 0$ *such that for all* $p \in [0,1]$, $k > 1$ *and all large* n

$$\mathbb{P}_{kn,p}\left[\delta_{1/k}(\mathcal{A}, \eta_k) > n^{-\delta}\right] < n^{-50}.$$

Proof. For $p = 1/2$ the first statement of the proposition is immediate from (4.2.2) and the first statement of Theorem 1.2.1. For $p \neq 1/2$ it is a trivial consequence of the second statement of Theorem 1.2.1.

As for the second statement, it is necessary to go through the arguments in Section 4.3 again, and notice that the only place where k needs to be large is in (4.3.6). Due to the first part of this proposition, we may modify Lemma 4.2.2, as pointed out in Remark 4.2.3, to obtain that the probability in (4.3.6) is small for every $k > 1$ and n large. □

4.5 SQUARE-ROOT STABILITY

In Section 4.4, we concluded the proof of Theorem 1.2.1, and the remainder of this chapter will aim to establish Theorem 1.2.2. The first step in this direction is to establish a result that roughly states that f_R is stable with respect to perturbations that act independently and uniformly on each of the two colors and change at most order square-root of the points.

Throughout this section we shall use the notation $\xi := \{x \in S : (x,0) \in \eta\}$ and $\zeta := \{x \in S : (x,1) \in \eta\}$ to denote the set of red and blue points respectively, and identify η with the pair (ξ, ζ) when appropriate.

Proposition 4.5.1. *Let* $\eta' = (\xi', \zeta')$ *and* $\eta = (\xi, \zeta)$ *be a pair of configurations in* Ω, *chosen according to* $\mathbb{P}_{n,1/2}$, *and whose joint law satisfies the following properties, stated only for the* ξ*-coordinates:*

(i) *Given* ξ, *the distribution of* $\xi \cap \xi'$ *is invariant by permutations of* ξ, *and, conditioned on its size, the set* $\xi' \setminus \xi$ *is formed by independently and uniformly distributed points in* S.

(ii) *For every* $\delta > 0$, *there exists a constant* C *such that, for all large* n,

$$\mathbb{P}_{n,1/2}\big[|\xi' \triangle \xi| > C\sqrt{n}\big] < \delta, \tag{4.5.1}$$

where $\xi' \triangle \xi$ *is the symmetric difference between the two sets.*

If, in addition, the pairs (ζ, ζ') *and* (ξ, ξ') *are independent, then, for any rectangle* $R \subseteq S$, *we have*

$$\mathbb{P}_{n,1/2}\big[f_R(\zeta, \xi) \neq f_R(\zeta', \xi')\big] \to 0 \quad \textit{as } n \to \infty.$$

The square-root scale that figures in the theorem is meaningful in the sense that $\sqrt{n}$ is an upper bound on the derivative of a monotone Boolean function on n bits. Consequently, the threshold window cannot have a width smaller than $1/\sqrt{n}$, and noise sensitive monotone functions have a window that is strictly wider (cf. (4.1.6)). Hence, a uniform perturbation that involve order $\sqrt{n}$ bits is therefore too small to affect the outcome of the function.

The above heuristic has been made precise in the setting of Boolean functions in a paper by Broman, Garban and Steif [17, Lemma 6.1]. We shall prove Proposition 4.5.1 via a suitable two-stage construction in which a version of the result from [17] can be applied.

Lemma 4.5.2. *Let* $A_1, A_2, \ldots, A_k$ *be a partition of* $[n]$, *and let* (ω, ω^*) *be a pair of configurations in* $\{0,1\}^n$ *with law* $\overline{\mathbb{P}}$ *satisfying the following properties:*

(i) *there exists* $c > 0$ *such that* $|A_i| \geqslant cn$, *for all* $i = 1, 2, \ldots, k$;
(ii) ω *and* ω^* *are under* $\overline{\mathbb{P}}$ *uniformly distributed in* $\{0,1\}^n$;
(iii) $\overline{\mathbb{P}}$ *is invariant under all permutations* π *of* $[n]$ *such that* $\pi(A_i) = A_i$, *for all* $i = 1, 2, \ldots, k$;
(iv) *for every* $\delta > 0$ *there exists a constant* C *such that, for all large* n *and all* $i = 1, 2, \ldots, k$;

$$\overline{\mathbb{P}}\big[d_{A_i}(\omega, \omega^*) > C\sqrt{|A_i|}\big] < \delta,$$

where $d_{A_i}(\omega, \omega^*) := \sum_{j \in A_i} |\omega(j) - \omega^*(j)|$.

Then, for every $\epsilon > 0$, *there exists a constant* $\tilde{C}$ *such that, for all large* n *and any function* $f : \{0,1\}^n \to \{0,1\}$, *we have*

$$\overline{\mathbb{P}}\big[f(\omega) \neq f(\omega^*)\big] < \epsilon + \frac{\tilde{C}}{\sqrt{n}} \sum_{k \in [n]} \mathrm{Inf}_k^{1/2}(f). \tag{4.5.2}$$

Combined with (4.1.6) the bound in (4.5.2) may be expressed in terms of the sum of influences squared or the revealment of algorithms.

Proof. The case $k = 1$ is the statement of Lemma 6.1 in [17] (with the additional hypothesis that ω^* is uniform in $\{0,1\}^n$). The remaining cases follows from induction on k.

Fix some $k \geqslant 2$, assume the result is true for all $j \leqslant k$ and fix a partition $A_1, A_2, \dots, A_{k+1}$. Denote by $\tilde{A} = [n] \setminus A_{k+1}$ and $\omega = (\omega_{\tilde{A}}, \omega_{A_{k+1}})$ for the restrictions of ω to the sets $\tilde{A}$ and A_{k+1}. Observe that

$$\begin{aligned}\overline{\mathbb{P}}\big[f(\omega_{\tilde{A}}, \omega_{A_{k+1}}) \neq f_R(\omega^*_{\tilde{A}}, \omega^*_{A_{k+1}})\big] \leqslant{}& \overline{\mathbb{P}}\big[f(\omega_{\tilde{A}}, \omega_{A_{k+1}}) \neq f_R(\omega^*_{\tilde{A}}, \omega_{A_{k+1}})\big] \\ &+ \overline{\mathbb{P}}\big[f(\omega^*_{\tilde{A}}, \omega_{A_{k+1}}) \neq f_R(\omega^*_{\tilde{A}}, \omega^*_{A_{k+1}})\big].\end{aligned} \tag{4.5.3}$$

To bound the first probability in the last expression above, we apply the induction hypothesis conditioned on $\omega_{A_{k+1}}$ and use that $\omega_{A_{k+1}}$ is uniformly distributed in $\{0,1\}^{A_{k+1}}$ to obtain

$$\overline{\mathbb{P}}\big[f(\omega_{\tilde{A}}, \omega_{A_{k+1}}) \neq f_R(\omega^*_{\tilde{A}}, \omega_{A_{k+1}})\big] \leqslant \epsilon + \frac{\tilde{C}}{\sqrt{|\tilde{A}|}} \sum_{k \in \tilde{A}} \mathrm{Inf}_k^{1/2}(f). \tag{4.5.4}$$

Analogous computations for the last term in (4.5.3) concludes the proof. □

We now focus on the proof of Proposition 4.5.1.

Proof of Proposition 4.5.1. The first step of the proof is to find a suitable construction of the pairs (ζ, ζ') and (ξ, ξ'). Since the perturbation acts independently on the two colors, this construction can be done separately.

For this purpose, let $M = |\xi' \cap \xi|$ and $N = |\xi' \setminus \xi|$. Let ξ_2 be a Poisson point process on S with intensity measure $n\lambda_S$, and let ξ and $\bar{\xi}$ be uniformly chosen subsets of ξ_2. Given $|\xi|$, sample the pair (M, N) according to the right conditional law. Next, choose uniformly a subset $\xi^A \subseteq \xi$ of size M and let ξ^B be a uniformly chosen subset of $\xi_2 \setminus \xi$ of size $\min\{N, |\xi_2 \setminus \xi|\}$. Besides, let ξ^C be a collection of N independent and uniformly chosen points of the square S. Now set

$$\xi'' := \begin{cases} \xi^A \cup \xi^B, & \text{if } N \leqslant |\xi_2 \setminus \xi|, \\ \bar{\xi}, & \text{if } N > |\xi_2 \setminus \xi|, \end{cases}$$

and

$$\xi''' := \begin{cases} \xi^A \cup \xi^B, & \text{if } N \leqslant |\xi_2 \setminus \xi|, \\ \xi^A \cup \xi^C, & \text{if } N > |\xi_2 \setminus \xi|. \end{cases}$$

Construct the collection $(\zeta, \zeta'', \zeta''')$ analogously, and note that (ζ, ζ''') and (ξ, ξ''') have the correct joint distribution.

In the next step, we note that $\xi''' = \xi''$ with probability tending to 1. To see this, fix $\epsilon > 0$ and notice that N and $|\xi_2 \setminus \xi|$ are independent, and that the latter is Poisson with parameter $n/2$. Then, by assumption *(ii)* we have

$$\mathbb{P}\big[N > |\xi_2 \setminus \xi|\big] \leqslant \mathbb{P}[N > n/4] + \mathbb{P}\big[|\xi_2 \setminus \xi| \leqslant n/4\big] \leqslant \epsilon$$

for all large n. The above construction thus gives, for large n, that

$$\mathbb{P}_{n,p}\big[f_R(\xi, \zeta) \neq f_R(\xi', \zeta')\big] \leqslant 2\epsilon + \mathbb{P}\big[f_R(\xi, \zeta) \neq f_R(\xi'', \zeta'')\big]. \tag{4.5.5}$$

Conditional on (ζ_2, ξ_2) the pairs (ζ, ζ'') and (ξ, ξ'') can be thought of as pairs of elements in $\{0,1\}^{\zeta_2}$ and $\{0,1\}^{\xi_2}$ respectively. The last step of the proof will thus be to apply Lemma 4.5.2 to bound the last probability above. In preparation for this, set $\delta_m := \epsilon 2^{-2m}$ and let C_m be the constant in hypothesis *(ii)* that corresponds to δ_m. Let

$$B_1 := \big\{\mathbb{P}\big[|\xi \triangle \xi''| > C_m\sqrt{n} \,\big|\, \xi_2\big] \geqslant 2^{-m} \text{ for some } m \geqslant 1\big\}.$$

Clearly $|\xi \triangle \xi''|$ is equal to $|\xi \triangle \xi'''|$ on the event where $N \leqslant |\xi_2 \setminus \xi|$. Hence, the union bound and Markov's inequality give, for large n, that

$$\begin{aligned} \mathbb{P}[B_1] &\leqslant \mathbb{P}[B_1, N > |\xi_2 \setminus \xi|] + \mathbb{P}[B_1, N \leqslant |\xi_2 \setminus \xi|] \\ &\leqslant \mathbb{P}[N > |\xi_2 \setminus \xi|] + \mathbb{P}\left[\begin{array}{c} |\xi \triangle \xi'''| > C_m\sqrt{n} \,\big|\, \xi_2\big] \geqslant 2^{-m} \\ \text{for some } m \geqslant 1 \text{ and } N \leqslant |\xi_2 \setminus \xi| \end{array}\right] \\ &\leqslant \epsilon + \sum_{m \geqslant 1} 2^m \, \mathbb{P}\big[|\xi \triangle \xi'''| > C_m\sqrt{n}\big] \leqslant \epsilon + \sum_{m \geqslant 1} \epsilon 2^{-m} \leqslant 2\epsilon. \end{aligned} \tag{4.5.6}$$

Let also $B_2 := \{|\xi_2| \notin [n/2, 2n]\}$ and define the analogous events $\tilde{B}_1$ and $\tilde{B}_2$ to the collection ζ_2. On the event $G := (B_1 \cup B_2 \cup \tilde{B}_1 \cup \tilde{B}_2)^c$, Lemma 4.5.2 combined with (4.5.2) can be applied and it gives that, for large n,

$$\begin{aligned} \mathbb{P}\big[f_R(\xi, \zeta) \neq f_R(\xi'', \zeta'')\big] &\leqslant 6\epsilon + \mathbb{E}\big[\mathbb{P}\big[f_R(\xi, \zeta) \neq f_R(\xi'', \zeta'')\big|\zeta_2, \xi_2\big]\mathbf{1}_G\big] \\ &\leqslant 6\epsilon + C\,\mathbb{E}\Bigg[\frac{1}{\sqrt{|\eta_2|}} \sum_{x \in \eta_2} \mathrm{Inf}_x^{1/2}(f_R, \eta_2)\Bigg]. \end{aligned}$$

By combining the last equation above with (4.5.5) and (4.1.6) we obtain

$$\mathbb{P}_{n,1/2}\big[f_R(\xi,\zeta) \neq f_R(\xi',\zeta')\big] \leqslant 8\epsilon + C\,\mathbb{E}_{n,1/2}\Big[\sqrt{\delta_{1/k}(\mathcal{A},\eta_2)}\Big],$$

which by Proposition 4.4.1 is no larger than 9ϵ when n is large. Since $\epsilon > 0$ was arbitrary, the proof is complete. □

4.6 CONSERVATIVE DYNAMICS AND RELATED TOPICS

This final section is devoted to different perturbations in our model.

Thinning and sprinkling. We begin with a comment on nonconservative and time dependent dynamics. We saw in Section 4.4 that sensitivity with respect to thinning a configuration uniformly is equivalent to the usual concept of noise sensitivity. We here complement that observation by showing that the same is true for sprinkling.

Let $\eta \in \Omega$ be chosen according to $\mathbb{P}_{(1-\epsilon)n,1/2}$, and let η' and η'' be independent configurations chosen according to $\mathbb{P}_{\epsilon n,1/2}$. Then the joint law of $(\eta\cup\eta',\eta\cup\eta'')$ equals that of $(\eta,\eta(\epsilon))$, and

$$\begin{aligned}
&\mathbb{E}_{n,1/2}\big[f_R(\eta)f_R(\eta(\epsilon))\big] - \mathbb{E}_{n,1/2}\big[f_R(\eta)\big]^2 \\
&\quad = \mathbb{E}\Big[\mathbb{E}\big[f_R(\eta\cup\eta')f_R(\eta\cup\eta'')\big|\eta\big]\Big] - \mathbb{E}\Big[\mathbb{E}\big[f_R(\eta\cup\eta')\big|\eta\big]\Big]^2 \\
&\quad = \mathrm{Var}\Big(\mathbb{E}\big[f_R(\eta\cup\eta')\big|\eta\big]\Big).
\end{aligned}$$

Hence, being sensitive with respect to an ϵ-sprinkling is equivalent to being noise sensitive, and thus follows from Theorem 1.2.1. That the same holds for an ϵ-thinning was seen already in Section 4.4.

Perturbing the colors. We shall briefly describe the results in [3], and explain how they imply that the crossing function is sensitive with respect to re-randomizing a small proportion of the colors of the points. That is, if η' is obtained from $\eta \in \Omega$ by resampling the second coordinate of each point $(x,u) \in \eta$ independently and uniformly with probability $\epsilon > 0$, then

$$\mathbb{E}_{n,1/2}\big[f_R(\eta)f_R(\eta')\big] - \mathbb{E}_{n,1/2}\big[f_R(\eta)\big]^2 \to 0 \quad \text{as } n\to\infty. \tag{4.6.1}$$

Given $\eta \in \Omega$, let $\overline{\eta}$ denote the projection onto S. Then,

$$\mathbb{E}_{n,1/2}\big[f_R(\eta)f_R(\eta')\big] - \mathbb{E}_{n,1/2}\big[f_R(\eta)\big]^2 = \mathbb{E}_{n,1/2}\Big[\mathbb{E}\big[f_R(\eta)f_R(\eta')\big|\overline{\eta}\big] - \mathbb{E}\big[f_R(\eta)\big|\overline{\eta}\big]\Big] + \mathrm{Var}_{n,1/2}\Big(\mathbb{E}\big[f_R(\eta)\big|\overline{\eta}\big]\Big).$$

In [3], the authors show that both expressions in the above right-hand side vanish as $n \to \infty$, and hence prove (4.6.1). That the variance term tends to zero shows that observing the tiling but not the coloring of a Voronoi configuration typically gives very little information about whether a coloring will typically produce a horizontal blue crossing or not, and confirms a conjecture of Benjamini, Kalai and Schramm [12]. The latter is essentially a statement of noise sensitivity of the crossing function in a quenched sense. One may show that noise sensitivity in the sense of (1.2.3) follows from that statement. However, we emphasize that the techniques used there are more restrictive than the techniques used here, as they are based on a color-switching trick. It is therefore motivated to present an alternative proof, as we have done here, that applies in a wide range of settings.

Perturbing the positions. We now turn to the proof of Theorem 1.2.2. The proof will be based on Proposition 4.5.1, which emphasizes a close relation to the exclusion sensitivity studied in [17].

Proof of Theorem 1.2.2. We shall show that the crossing function f_R is sensitive with respect to re-randomizing the positions of a small proportion of the points. This type of perturbation is conservative in the sense that the number of points of each color is kept constant. Our goal will be to construct the process in a suitable manner, and then apply Proposition 4.5.1.

As before we shall identify a configuration $\eta \in \Omega$ with a pair of configurations (ξ, ζ). Let $(X_i)_{i\geqslant 1}$ and $(Y_i)_{i\geqslant 1}$ be independent collections of independent and uniformly distributed points in S. In addition, let L, M and N be independent Poisson distributed random variables with parameters $(1-\epsilon)n/2$, $\epsilon n/2$ and $\epsilon n/2$, respectively. Next we define a triple (ξ', ξ'', ξ''') as

$$\begin{aligned} \xi' &:= \{X_1, X_2, \dots, X_{L+M}\}, \\ \xi'' &:= \{X_1, X_2, \dots, X_L\} \cup \{Y_1, Y_2, \dots, Y_N\}, \\ \xi''' &:= \{X_1, X_2, \dots, X_L\} \cup \{Y_1, Y_2, \dots, Y_M\}. \end{aligned} \tag{4.6.2}$$

Finally, we let $(\zeta', \zeta'', \zeta''')$ be an independent copy of (ξ', ξ'', ξ''').

Notice that the pair (η', η'') is distributed as the pair $(\eta, \eta(\epsilon))$ in (1.2.3), while the pair (η', η''') is distributed as the pair (η, η^*) in Theorem 1.2.2. We also notice that the pair (η'', η''') satisfy the conditions of Proposition 4.5.1. In particular, Chebyshev's inequality shows that for every $\delta > 0$ there exists C such that

$$\mathbb{P}\left[|\xi'' \triangle \xi'''| > C\sqrt{n}\right] = \mathbb{P}\left[|M - N| > C\sqrt{n}\right] \leqslant \frac{\mathrm{Var}(M - N)}{C^2 n} \leqslant \frac{\epsilon}{C^2} \leqslant \delta.$$

Consequently, Proposition 4.5.1 implies that

$$\mathbb{P}_{n,1/2}\left[f_R(\eta(\epsilon)) \neq f_R(\eta^*)\right] = \mathbb{P}\left[f_R(\eta'') \neq f_R(\eta''')\right] \to 0. \tag{4.6.3}$$

Finally, we obtain that

$$\begin{aligned}\left|\mathbb{E}_{n,1/2}\left[f_R(\eta) f_R(\eta^*)\right] - \mathbb{E}_{n,1/2}\left[f_R(\eta)\right]^2\right| &\leqslant \mathbb{P}_{n,1/2}\left[f_R(\eta(\epsilon)) \neq f_R(\eta^*)\right] \\ &+ \left|\mathbb{E}_{n,1/2}\left[f_R(\eta) f_R(\eta(\epsilon))\right] - \mathbb{E}_{n,1/2}\left[f_R(\eta)\right]^2\right|,\end{aligned}$$

which by Theorem 1.2.1 and (4.6.3) tend to zero as $n \to \infty$. □

Appendices

A

CONCENTRATION INEQUALITIES

Here we recall some results about concentration of measures. We begin by the classical result known as Azuma's inequality.

Theorem A.0.1 (Azuma's Inequality)**.** *Let* $\{X_k\}_{k=1}^n$ *be a collection of independent random variables and* $t > 0$*. Assume that there exist constants* $\{c_k\}_{k=1}^n$ *satisfying* $\mathbb{P}[|X_k| \leqslant c_k] = 1$ *for all* k*. Then*

1.
$$\mathbb{P}\left[\sum_{k=1}^n X_k - \mathbb{E}(X_k) \geqslant t\right] \leqslant \exp\left\{\frac{-t}{2\sum_{k=1}^n c_k^2}\right\};$$

2.
$$\mathbb{P}\left[\sum_{k=1}^n X_k - \mathbb{E}(X_k) \leqslant -t\right] \leqslant \exp\left\{\frac{-t}{2\sum_{k=1}^n c_k^2}\right\}.$$

This theorem also remains valid whent the sequence X_n is a martingale. We will not prove it here, since it can be found as Theorem 6.2 of [15]. This theorem implies a concentration bound for binomial random variables, that we state now as a corollary:

Corollary A.0.2. *If* X *is a random variable with distribution* $\text{Binomial}(n, p)$ *and* $t > 0$*:*

1.
$$\mathbb{P}\left[X - np \geqslant t\right] \leqslant \exp\left\{\frac{-t}{2n}\right\};$$

2.
$$\mathbb{P}\left[X - np \leqslant -t\right] \leqslant \exp\left\{\frac{-t}{2n}\right\}.$$

Our next objective is to prove concentration bounds for Poisson random variables. This is done in the next lemmas:

Lemma A.0.3. *Let* $\lambda > 0$*,* $t > 0$ *and* $X \sim \text{Poisson}(\lambda)$*. There exist constants* $c_{28} > 0$ *and* $c_{29} > 0$ *such that*

$$\mathbb{P}[X \geqslant 2\lambda + t] \leqslant e^{-c_{28}t},$$

and

$$\mathbb{P}[X \leqslant \lambda/3] \leqslant e^{-c_{29}\lambda}.$$

Moreover, the constants $c_{28} > 0$ *and* $c_{29} > 0$ *do not depend on* t *and* λ*.*

Proof. Let $c_{28} > 0$ such that $e^{c_{28}} - 1 = 2c_{28}$. Then, by Markov's inequality,

$$\begin{aligned}\mathbb{P}[X \geqslant 2\lambda + t] &= \mathbb{P}[e^{c_{28}X} \geqslant e^{c_{28}(2\lambda+t)}] \\ &\leqslant \exp\{\lambda(e^{c_{28}} - 1) - c_{28}(2\lambda + t)\} = e^{-c_{28}t}.\end{aligned}$$

For the second inequality, observe that if $\theta > 0$ then

$$\begin{aligned}\mathbb{P}[X \leqslant \lambda/3] &= \mathbb{P}[e^{-\theta X} \geqslant e^{-\theta\lambda/3}] \\ &\leqslant \exp\{\lambda(e^{-\theta} - 1) + \theta\lambda/3\} = e^{-\lambda(-e^{-\theta}+1-\frac{\theta}{3})}.\end{aligned}$$

If we take θ small enough, we have $c_{29} = -e^{-\theta} + 1 - \frac{\theta}{3} > 0$. □

Lemma A.0.4. *For all* $\lambda > 0$ *and* $X \sim \text{Poisson}(\lambda)$

$$\mathbb{P}[X \geqslant 3\lambda] \leqslant e^{-\lambda}.$$

Proof. Take $\theta > 0$ such that $3\theta = e^{\theta}$ and use the same computations as in the lemma above. □

Lemma A.0.5. *For all* $\lambda > 0$*,* $\mu \geqslant 3\lambda$ *and* $X \sim \text{Poisson}(\lambda)$

$$\mathbb{P}[X \geqslant \mu] \leqslant e^{-c_{30}(\lambda+\mu)},$$

for some positive constant c_{30}*.*

Proof. Simply observe that

$$\begin{aligned}\mathbb{P}[X \geqslant \mu] &= \mathbb{P}[e^{X} \geqslant e^{\mu}] \\ &\leqslant \exp\{\lambda(e - 1) - \mu\} \leqslant e^{-\lambda}e^{\mu(\frac{e}{3}-1)} \leqslant e^{-c_{30}(\lambda+\mu)}.\end{aligned}$$

□

B

HEAT KERNEL ESTIMATES

In this section we prove heat kernel estimates for the symmetric random walk on $\mathbb{Z}$.

The heat kernel is defined as

$$p_t(x,y) = \mathbb{P}_x[W_t = y],$$

where $(W_t)_{t\geqslant 0}$ is a continuous time simple symmetric random walk. It will be useful to consider also the discrete heat kernel, that is defined as

$$p_n(x,y) = \mathbb{P}_x[X_n = y],$$

where $(X_n)_{n\in\mathbb{N}}$ is a discrete time lazy symmetric random walk.

The next lemma gives us estimates in the discrete time case:

Lemma B.0.1. *There exists a constant* $c_{31} > 0$ *such that for all* $n \in \mathbb{N}$ *and* $x \in \mathbb{Z}$

$$p_n(0,x) \leqslant \frac{c_{31}}{\sqrt{n}}.$$

Proof. If we write $\{Z_n\}_{n\in\mathbb{N}}$ to the discrete time simple symmetric random walk, it is easy to see that $\left\{\frac{Z_{2n}}{2}\right\}_{n\in\mathbb{N}}$ is a discrete time lazy symmetric random walk. This implies that the lemma is a consequence of

$$\mathbb{P}_0[Z_{2n} = 2x] \leqslant \frac{c_{31}}{\sqrt{n}}.$$

Now we just need to count the number of paths that are in $2x$ at time $2n$. Assume $0 \leqslant x \leqslant n$ (by symmetry this extends to $-n \leqslant x \leqslant 0$, and this quantity is zero if $|x| > n$), and observe that the number of possible paths of the random walk that start at zero and is in $2x$ at time $2n$ is $\binom{2n}{n+x}$. With the aid of Stirling's approximation we estimate

$$p_n(0,x) = \mathbb{P}_0[Z_{2n} = 2x] = \frac{1}{2^n}\binom{2n}{n+x} \leqslant \frac{1}{2^n}\binom{2n}{n} = \frac{(2n)!}{2^n(n!)^2} \leqslant \frac{c_{31}}{\sqrt{n}}.$$

□

Now we get analogous bounds for continuous time random walks:

Proposition B.0.2. *For the continuous time random walk, there exists a constant* $c_{32} > 0$ *such that for every* $t \geqslant 0$ *and* $x \in \mathbb{Z}$

$$p_t(0,x) \leqslant \frac{c_{32}}{\sqrt{t}}.$$

Proof. We use a construction of the continuous time random walk with a Poisson process of rate 2 and a skeleton chain given by a lazy symmetric random walk. Let N_t be the number of jumps in the interval $[0,t]$, that has distribution Poisson$(2t)$. We use Lemmas A.0.3 and B.0.1 to get the estimates:

$$\begin{aligned} p_t(0,x) &\leqslant \mathbb{P}\left[N_t \leqslant 2t/3\right] + \sum_{k=\lceil 2t/3 \rceil}^{+\infty} \mathbb{P}[X_k = x\,,\, N_t = k] \\ &\leqslant e^{-2c_{29}t} + \sum_{k=\lceil 2t/3 \rceil}^{+\infty} \frac{c}{\sqrt{k}}\mathbb{P}[N_t = k] \leqslant e^{-2c_{29}t} + \sum_{k=\lceil 2t/3 \rceil}^{+\infty} \frac{c}{\sqrt{t}}\mathbb{P}[N_t = k] \\ &\leqslant e^{-2c_{29}t} + \frac{c}{\sqrt{t}} \leqslant \frac{c_{32}}{\sqrt{t}}. \end{aligned}$$

□

C

PROOF OF PROPOSITION 3.1.2

This section contains the proof of Proposition 3.1.2. We present only the proof of the first statement, since the second one is obtained in the same way.

Begin by observing that

$$\mathbb{E}_\rho[e^{\lambda X_1}] = \frac{Z(e^\lambda R^{-1}(\rho))}{Z(R^{-1}(\rho))}.$$

By independence, for $\lambda > 0$ we have

$$\begin{aligned}\mathbb{P}_\rho\left[\sum_{k=1}^{n} X_k \geqslant (\rho+\epsilon)n\right] &= \mathbb{P}_\rho\left[\exp\left\{\lambda\sum_{k=1}^{n} X_k\right\} \geqslant e^{\lambda(\rho+\epsilon)n}\right]\\ &\leqslant \left[\mathbb{E}_\rho[e^{\lambda X_1}]e^{-\lambda(\rho+\epsilon)}\right]^n\\ &\leqslant \left[\frac{Z(e^\lambda R^{-1}(\rho))}{Z(R^{-1}(\rho))}e^{-\lambda(\rho+\epsilon)}\right]^n.\end{aligned}$$

We now split the last term above and work with the function

$$f(\lambda) = \frac{Z(e^\lambda R^{-1}(\rho))}{Z(R^{-1}(\rho))}e^{-\lambda(\rho+\frac{\epsilon}{2})}.$$

Observe that $f(0) = 1$ and that

$$f'(\lambda) = e^{-\lambda(\rho+\frac{\epsilon}{2})}\frac{Z(e^\lambda R^{-1}(\rho))}{Z(R^{-1}(\rho))}\left[R(e^\lambda R^{-1}(\rho)) - \rho - \frac{\epsilon}{2}\right].$$

The function R is increasing. Besides, R' is continuous, hence R is Lipschitz continuous on the interval $[0, e\rho_+]$. Therefore, for $\lambda \leqslant 1$,

$$\begin{aligned} R(e^{\lambda} R^{-1}(\rho)) - \rho &= R(e^{\lambda} R^{-1}(\rho)) - R(R^{-1}(\rho)) \\ &\leqslant \tilde{c}(\rho_+) R^{-1}(\rho_+)(e^{\lambda} - 1) < \frac{\epsilon}{2}, \end{aligned}$$

for all $\lambda < \lambda_*(\epsilon) := \min\left\{\log\left(1 + \frac{\epsilon}{2\tilde{c}(\rho_+)R^{-1}(\rho_+)}\right), 1\right\}$. For such values of λ, $f'(\lambda) < 0$ and hence $f(\lambda) \leqslant 1$. Now, we just need to choose $c(\rho_+)$ such that $2c(\rho_+)\epsilon < \lambda_*(\epsilon)$ for all $\epsilon \leqslant 1$. This implies that we can bound

$$\mathbb{P}_{\rho}\left[\sum_{k=1}^{n} X_k \geqslant (\rho + \epsilon)n\right] \leqslant e^{-c(\rho_+)\epsilon^2 n}, \tag{C.1}$$

completing the proof.

D

PROOF OF CLAIM 3.1.10

Here we prove that the process $(\eta_s)_{s\geqslant 0}$ defined in Subsection 3.1.4 is indeed a zero range process.

The first step is to verify that this is the case when we have only finitely many particles. This follows easily, since we have a Markov chain in a countable state space: We simply wait until the first clock rings and make the necessary updates in the positions of the particles and the clocks being used.

The idea now is to split the integer lattice into parts that do not communicate between themselves until a small time t and study the process restricted to this partition.

A site $x \in \mathbb{Z}$ is an absorbing point up to time t if no particle of the process η leaves x before time t. This means that site x behaves as a trap until time t. We will now verify that the probability that a given site is an absorbing point up to time t is positive, for some choice of $t > 0$. By translation invariance, it suffices to consider $x = 0$.

Since we only need to lower bound the probability above, we will not prove sharp results. Notice that, if $[-K, K]$ contains no particle at time zero, then the origin is an absorbing point up to time t if no particle from outside $[-K, K]$ reaches the origin before time t. This implies we only need to choose K so that the probability that there exists a particle starting outside $[-K, K]$ reaches the origin before time t is strictly smaller than one.

For $K = \lfloor (6\Gamma_+ t + 3)(\rho + 1) \rfloor + 2$, we can use the same computations of (3.1.11) to conclude

$$\mathbb{P}_\rho \left[\begin{array}{c} \text{no particle from} \\ \text{outside } [-K, K] \text{ reaches} \\ \text{the origin before time } t \end{array} \right] \geqslant 1 - cte^{-\Gamma_+ t}, \tag{D.1}$$

for some constant c.

Now, we want that $[-K, K]$ is empty at time zero. This gives the lower bound

$$\mathbb{P}_\rho \left[\begin{array}{c} \text{the origin is an} \\ \text{absorbing point} \\ \text{up to time } t \end{array} \right] \geqslant (1 - cte^{-\Gamma_+ t}) \Big(\mathbb{P}_\rho[\eta_0(0) = 0] \Big)^{2K+1} > 0, \quad \text{(D.2)}$$

for t small enough.

Now that we know absorbing points occur with positive probability, Borel-Cantelli Lemma implies that it is possible to partition the integer lattice into random intervals such that the extreme points are absorbing up to time t.

Fixed this random partition, up to time t, in each of the intervals, we have finitely many particles and the result follows from the finite case. This proves that $(\eta_s)_{0 \leqslant s \leqslant t}$ is a zero range process. Now just repeat the process to all time intervals $[nt, (n+1)t)$.

E

PROOF OF PROPOSITION 3.4.6

In this section we study the martingale introduced in (3.4.20) and prove Proposition 3.4.6.

The first lemma we prove is a tail bound for the increments of this martingale.

Lemma E.0.1. *There exists a positive constant* c_{33} *that depends only on the density* $\rho > 0$ *such that, for all* $t \geqslant 0$ *and* $u \geqslant 0$*,*

$$\mathbb{P}_\rho[|M_{t+1} - M_t| \geqslant u] \leqslant c_{33} e^{-c_{33}^{-1} u^{1/2}}. \tag{E.1}$$

Proof. It is enough to consider $t = 0$ and, by increasing if necessary the value of c_{33}, we can also consider u large.

Consider the event

$$A = \left\{ \sup_{s \in [0,1]} |r_s - r_0| \geqslant \frac{u}{2} \right\}, \tag{E.2}$$

and observe that Lemmas 3.2.1 and 3.2.2 imply

$$\mathbb{P}_\rho[A] \leqslant c e^{-c^{-1} u^{1/2}}. \tag{E.3}$$

We also introduce the event

$$B = \left\{ \eta_s(x) \geqslant \frac{u}{\Gamma_+}, \text{ for some } (x, s) \in [-u, u] \times [0, 1] \right\}. \tag{E.4}$$

Union bound and Lemma 3.1.3 gives

$$\mathbb{P}_\rho[B] \leqslant c e^{-c^{-1} u}. \tag{E.5}$$

Finally, on $(A \cup B \cup \{r_0 \leqslant u/2\})^c$, using that $g(k) \leqslant \Gamma_+ k$, we obtain

$$\begin{aligned} |M_1 - M_0| &\leqslant |r_1 - r_0| + \left| \int_0^1 \frac{1}{2} g(\eta_s(r_s)) \mathbb{1}_{\eta_s(r_s) \geqslant 2} \, ds \right| \\ &\leqslant \frac{u}{2} + \frac{1}{2}\Gamma_+ \frac{u}{\Gamma_+} = u, \end{aligned} \tag{E.6}$$

and the proof is complete. □

Using the tail bound obtained above we can prove the concentration estimates for the martingale M_t. This proof follows the lines from [33].

Proof of Proposition 3.4.6. We investigate the martingale M_t restricted to the integer times. Denote $\mathcal{F}_n = \sigma(M_t : t \leqslant n)$ and $X_n = M_n - M_{n-1}$. Fix a positive integer k and define

$$Y_n = X_n \mathbb{1}_{\{|X_n| \leqslant L_k^{1/4}\}} - \mathbb{E}_\rho \left[X_n \mathbb{1}_{\{|X_n| \leqslant L_k^{1/4}\}} \,\middle|\, \mathcal{F}_{n-1} \right], \tag{E.7}$$

and

$$Y_n = X_n \mathbb{1}_{\{|X_n| > L_k^{1/4}\}} - \mathbb{E}_\rho \left[X_n \mathbb{1}_{\{|X_n| > L_k^{1/4}\}} \,\middle|\, \mathcal{F}_{n-1} \right]. \tag{E.8}$$

Observe that both Y_n and Z_n are martingale differences with respect to the filtration $(\mathcal{F}_n)_{n \geqslant 0}$ and that $Y_n + Z_n = X_n$.

We easily obtain that

$$\begin{aligned} \mathbb{P}_\rho[|M_{L_k}| \geqslant \delta L_k] &= \mathbb{P}_\rho \left[\left| \sum_{n=1}^{L_k} X_n \right| \geqslant \delta L_k \right] \\ &\leqslant \mathbb{P}_\rho \left[\left| \sum_{n=1}^{L_k} Y_n \right| \geqslant \frac{\delta}{2} L_k \right] + \mathbb{P}_\rho \left[\left| \sum_{n=1}^{L_k} Z_n \right| \geqslant \frac{\delta}{2} L_k \right]. \end{aligned} \tag{E.9}$$

We now focus on the two probabilities on the right hand side of the estimate above.

Notice that $|Y_n| \leqslant 2L_k^{1/4}$. Hence, Azuma's inequality implies

$$\mathbb{P}_\rho \left[\left| \sum_{n=1}^{L_k} Y_n \right| \geqslant \frac{\delta}{2} L_k \right] \leqslant 2e^{-\frac{\delta^2}{32} L_k^{1/2}}. \tag{E.10}$$

As for Z_n, observe initially that $F(u) = \mathbb{P}_\rho[|X_n| \geqslant u] \leqslant c_{33} e^{-c_{33}^{-1} u^{1/2}}$, according to Lemma E.0.1. We now bound

$$\begin{aligned}
\mathbb{E}_\rho[Z_n^2] &\leqslant \mathbb{E}_\rho\left[X_n^2 \mathbf{1}_{\{|X_n| > L_k^{1/4}\}}\right] \\
&= -\int_{L_k^{1/4}}^{\infty} x^2 dF(x) \\
&- \lim_{M\to\infty}\left(MF(M) - L_k^{1/2} F(L_k^{1/4}) - \int_{L_k^{1/4}}^{M} 2xF(x)dx\right) \qquad \text{(E.11)}\\
&\leqslant L_k^{1/2} c_{33} e^{-c_{33}^{-1} L_k^{1/8}} - \int_{L_k^{1/4}}^{M} 2c_{33} x e^{-c_{33}^{-1} x^{1/2}} dx \\
&\leqslant c L_k^{1/2} e^{-cL_k^{1/8}},
\end{aligned}$$

by possibly changing constants.

This implies that

$$\mathbb{P}_\rho\left[\left|\sum_{n=1}^{L_k} Z_n\right| \geqslant \frac{\delta}{2} L_k\right] \leqslant \frac{4}{\delta^2 L_k^2} \mathbb{E}\left[\left(\sum_{n=1}^{L_k} Z_n\right)^2\right] \leqslant c e^{-cL_k^{1/8}}. \qquad \text{(E.12)}$$

Combining Equations (E.9), (E.10) and (E.12) easily implies the proposition. □

BIBLIOGRAPHY

[1] Daniel Ahlberg and Rangel Baldasso. Noise sensitivity and voronoi percolation. *arXiv preprint arXiv:1708.03054*, 2017.

[2] Daniel Ahlberg, Erik Broman, Simon Griffiths, and Robert Morris. Noise sensitivity in continuum percolation. *Israel Journal of Mathematics*, 201(2):847–899, 2014.

[3] Daniel Ahlberg, Simon Griffiths, Robert Morris, and Vincent Tassion. Quenched voronoi percolation. *Advances in Mathematics*, 286:889–911, 2016.

[4] Daniel Ahlberg and Jeffrey E. Steif. Scaling limits for the threshold window: When does a monotone Boolean function flip its outcome? *Ann. Inst. Henri Poincaré Probab. Stat.*, to appear. With an appendix by Gábor Pete.

[5] Daniel Ahlberg, Vincent Tassion, and Augusto Teixeira. Sharpness of the phase transition for continuum percolation in $\mathbb{R}^2$. *arXiv preprint arXiv:1605.05926*, 2016.

[6] Oswaldo Alves, Fabio Machado, and Serguei Popov. Phase transition for the frog model. *Electronic Journal of Probability*, 7, 2002.

[7] Oswaldo Alves, Fabio Machado, Serguei Popov, and Krishnamurthi Ravishankar. Phase transition for the frog model. *Electronic Journal of Probability*, 7, 2002.

[8] Oswaldo S.M. Alves, Fabio P. Machado, and Serguei Popov. The shape theorem for the frog model. *The Annals of Applied Probability*, 12(2):533–546, 2002.

[9] Enrique Daniel Andjel. Invariant measures for the zero range process. *The Annals of Probability*, pages 525–547, 1982.

[10] Rangel Baldasso and Augusto Teixeira. Spread of an infection on the zero range process. *In preparation*.

[11] Rangel Baldasso and Augusto Teixeira. How can a clairvoyant particle escape the exclusion process? *arXiv preprint arXiv:1605.07575*, 2016.

[12] Itai Benjamini, Gil Kalai, and Oded Schramm. Noise sensitivity of boolean functions and applications to percolation. *Publications Mathématiques de l'Institut des Hautes Études Scientifiques*, 90(1):5–43, 1999.

[13] Jean Bérard and Alejandro F Ramírez. Large deviations of the front in a one-dimensional model of $X+Y \to 2X$. *The Annals of Probability*, pages 955–1018, 2010.

[14] Béla Bollobás and Oliver Riordan. The critical probability for random voronoi percolation in the plane is 1/2. *Probability theory and related fields*, 136(3):417–468, 2006.

[15] Stéphane Boucheron, Gábor Lugosi, and Pascal Massart. *Concentration inequalities: A nonasymptotic theory of independence*. OUP Oxford, 2013.

[16] Simon R Broadbent and John M Hammersley. Percolation processes: I. crystals and mazes. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 53, pages 629–641. Cambridge University Press, 1957.

[17] Erik I Broman, Christophe Garban, and Jeffrey E Steif. Exclusion sensitivity of boolean functions. *Probability theory and related fields*, 155(3-4):621–663, 2013.

[18] Francis Comets, Jeremy Quastel, and Alejandro Ramírez. Fluctuations of the front in a one dimensional model of $X+Y \to 2X$. *Transactions of the American Mathematical Society*, 361(11):6165–6189, 2009.

[19] Hugo Duminil-Copin, Aran Raoufi, and Vincent Tassion. Exponential decay of connection probabilities for subcritical Voronoi percolation in $\mathbb{R}^d$. Preprint, see *arXiv:1705.07978*.

[20] Paul Erdös and Alfréd Rényi. On the evolution of random graphs. *Publ. Math. Inst. Hung. Acad. Sci*, 5(1):17–60, 1960.

[21] Ehud Friedgut and Gil Kalai. Every monotone graph property has a sharp threshold. *Proceedings of the American mathematical Society*, 124(10):2993–3002, 1996.

[22] Christophe Garban, Gábor Pete, and Oded Schramm. The fourier spectrum of critical percolation. *Acta Mathematica*, 205(1):19–104, 2010.

[23] Christophe Garban and Jeffrey E Steif. *Noise sensitivity of Boolean functions and percolation*, volume 5. Cambridge University Press, 2014.

[24] Marco Antônio Giacomelli. On a one-dimensional model of infection spreading. *Brazilian Journal of Probability and Statistics*, 23(1):92–103, 2009.

[25] Geoffrey R Grimmett. Percolation (grundlehren der mathematischen wissenschaften). 2010.

[26] Marcelo Hilário, Frank Den Hollander, Vladas Sidoravicius, Renato Soares dos Santos, Augusto Teixeira, et al. Random walk on random walks. *Electronic Journal of Probability*, 20, 2015.

[27] Milton Jara, Gregorio Moreno, and Alejandro F Ramirez. Front propagation in an exclusion one-dimensional reactive dynamics. *Markov Processes and Related Fields*, 14(2):185–206, 2008.

[28] Jeff Kahn, Gil Kalai, and Nathan Linial. The influence of variables on boolean functions. In *Foundations of Computer Science, 1988., 29th Annual Symposium on*, pages 68–80. IEEE, 1988.

[29] George Kesidis, Takis Konstantopoulos, and Shashi Phoha. Surveillance coverage of sensor networks under a random mobility strategy. In *Sensors, 2003. Proceedings of IEEE*, volume 2, pages 961–965. IEEE, 2003.

[30] Harry Kesten. The critical probability of bond percolation on the square lattice equals $\frac{1}{2}$. *Comm. Math. Phys.*, 74(1):41–59, 1980.

[31] Harry Kesten and Vladas Sidoravicius. The spread of a rumor or infection in a moving population. *Annals of Probability*, pages 2402–2462, 2005.

[32] Claude Kipnis and Claudio Landim. *Scaling limits of interacting particle systems*, volume 320. Springer Science & Business Media, 2013.

[33] Emmanuel Lesigne and Dalibor Volnỳ. Large deviations for martingales. *Stochastic processes and their applications*, 96(1):143–159, 2001.

[34] Thomas M Liggett, Roberto H Schonmann, Alan M Stacey, et al. Domination by product measures. *The Annals of Probability*, 25(1):71–95, 1997.

[35] Ronald Meester and Rahul Roy. *Continuum percolation*, volume 119. Cambridge University Press, 1996.

[36] Ryan O'Donnell. *Analysis of boolean functions*. Cambridge University Press, 2014.

[37] Ryan O'Donnell, Michael Saks, Oded Schramm, and Rocco A Servedio. Every decision tree has an influential variable. In *Foundations of Computer Science, 2005. FOCS 2005. 46th Annual IEEE Symposium on*, pages 31–39. IEEE, 2005.

[38] Yuval Peres, Alistair Sinclair, Perla Sousi, and Alexandre Stauffer. Mobile geometric graphs: detection, coverage and percolation. In *Proceedings of the twenty-second annual ACM-SIAM symposium on Discrete Algorithms*, pages 412–428. SIAM, 2011.

[39] Serguei Popov and Augusto Teixeira. Soft local times and decoupling of random interlacements. *Journal of the European Mathematical Society*, 17(10):2545–2593, 2015.

[40] Serguei Yu Popov. Frogs and some other interacting random walks models. In *DRW*, pages 277–288, 2003.

[41] Alejandro F Ramírez and Vladas Sidoravicius. Asymptotic behavior of a stochastic combustion growth process. *Journal of the European Mathematical Society*, 6(3):293–334, 2004.

[42] Lucio Russo. An approximate zero-one law. *Z. Wahrsch. Verw. Gebiete*, 61(1):129–139, 1982.

[43] Oded Schramm and Jeffrey E Steif. Quantitative noise sensitivity and exceptional times for percolation. In *Selected Works of Oded Schramm*, pages 391–444. Springer, 2011.

[44] Vladas Sidoravicius and Alexandre Stauffer. Phase transition for finite-speed detection among moving particles. *Stochastic Processes and their Applications*, 125(1):362–370, 2015.

[45] Frank Spitzer. Interaction of markov processes. *Advances in Mathematics*, 5(2):246–290, 1970.

[46] Alexandre Stauffer. Space-time percolation and detection by mobile nodes. *The Annals of Applied Probability*, 25(5):2416–2461, 2015.

[47] Alain-Sol Sznitman. Vacant set of random interlacements and percolation. *Annals of mathematics*, pages 2039–2087, 2010.

[48] Michel Talagrand. On russo's approximate zero-one law. *The Annals of Probability*, pages 1576–1587, 1994.

[49] Vincent Tassion. Crossing probabilities for voronoi percolation. *The Annals of Probability*, 44(5):3385–3398, 2016.