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ORIGINAL RESEARCH PAPER

Cash-flow based valuation of pension liabilities

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Abstract This paper presents a computational framework for cash-flow based valuation of insurance liabilities in incomplete markets. It accounts for the risks associated with both insurance claims and investment returns until maturity in accordance with modern principles of asset-liability management. The valuation framework is market consistent in the sense that it takes into account the investment opportunities available to the insurer at the time of valuation. The framework is easily adapted to different lines of insurance and it can effectively employ advanced tools for strategic portfolio management. As an application, we value the insurance portfolio of the Finnish private sector occupational pension system where the liabilities extend over 82 years.

1 Introduction

In the absence of liquid markets for insurance obligations their pricing should be based on the cash-flows associated with the settlement of the obligations until maturity; see e.g. (International Association of Insurance Supervisors, 2007, Structure element 5) or Article 75(1) of the Solvency II-directive on the valuation of liabilities. When valuing long term pension liabilities, significant risks are

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associated both with the claims as well as with investment returns that affect the sufficiency of the capital reserved for covering the claims. According to modern risk management principles the value of insurance liabilities should reflect both the underwriting risks as well as investment risks until the amortization of the liabilities.

Due to significant uncertainties and the incompleteness of financial markets, pension insurance liabilities cannot be fully hedged or full hedging may amount to unreasonable costs. A more pragmatic approach is to define the value of pension liabilities as the minimal capital required to cover the claims at an acceptable level of risk when the capital is prudently invested in financial instruments available in the market. Such a value depends essentially on the following subjective factors:

1. **Probability distribution**: The description of future development of investment returns and the insurance claims both of which involve significant uncertainties.
2. **Risk preferences**: The level of risk at which the assets should cover the liabilities. Instead of simple confidence levels, one could use risk measures that better support risk management.
3. **Hedging strategy**: The strategy according to which the given capital is invested in financial markets. Adjusting the investment strategy to the liabilities may allow for a reduction of the required initial capital.

The significance of market expectations is usually well understood but risks are often ignored or they are accounted for by heuristic adjustments. The description of the underlying risk factors should also account for possible dependencies between investment returns and the claims. Such dependencies are instrumental in pricing and hedging of insurance liabilities. When valuing risky cash-flows, the effects of (more or less subjective) risk preferences cannot be avoided. The choice of the investment strategy, on the other hand, reflects the insurer's expertise in producing the cash-flows associated with its insurance portfolio. Coming up with an appropriate investment strategy is one of the most important functions of an insurance company. A valuation framework should also be market consistent by taking into account the investment opportunities available to the insurer at the time of valuation. Interaction of the probability distribution, the investment strategy and the risk preferences highlights the need for an integrated valuation framework, that incorporates all the relevant features in a flexible way according to the views of the insurer.

This paper presents a computational framework based on the above principles for valuation of insurance liabilities in incomplete markets. The framework is easily adapted to different lines of insurance (and financial instruments in general) and different market conditions and it can effectively employ advanced tools for strategic portfolio management. The interplay between liability valuation and asset management has been recently studied in different settings in Artzner et al. [2] and Pennanen [17]. Complementary to the mathematical aspects studied there, the present paper focuses on the computational side of the problem.

We apply the framework to the valuation of the insurance portfolio of the Finnish private sector occupational pension system where the claims extend over 82 years. We illustrate the importance of appropriate recognition of both the investment and underwriting risks as well as the specification of risk preferences in the valuation. The effects of the investment strategy are demonstrated by applying different well known and widely studied parametric investment strategies as insurance portfolio hedges and by using an adaptive optimization technique developed for long term risk management. In our case study, the optimization results in over 15% decrease in capital requirements when compared to the best found parametric hedging strategy.

The rest of this paper is organized as follows. Section 2 reviews the valuation of insurance liabilities in a deterministic environment and outlines our case study on the Finnish occupational pension system. Section 3 introduces uncertainties and risk preferences into the framework. Section 4 presents the fully fledged market consistent valuation framework, where the return distribution of the invested capital can be adjusted by investment decisions to better suit the risk profile of the insurance claims and the given risk preferences.

2 Valuation of liabilities in a deterministic world

Throughout this paper c_t will denote the aggregate claims associated with an existing insurance portfolio payable in period $[t-1, t]$. We assume that the liabilities will amortize after a finite time so that the last claim will be paid at time T.

For purposes of comparison, we begin with a simplified framework where both the claims and investment returns are deterministic. More realistic models will be developed in the following sections. Assume that any amount of money invested in financial markets will return R_t over the period $[t-1, t]$ for $t = 1, \ldots, T$. The initial capital required to cover all the insurance claims c_t is obtained by solving V_0 from the system of equations

$$\begin{aligned} V_t &= R_t V_{t-1} - c_t \quad t = 1, \ldots, T, \\ V_T &= 0. \end{aligned} \tag{1}$$

In this deterministic setting, the resulting value V_0 corresponds to Article 75(1) of the Solvency II-directive on the valuation of liabilities. Furthermore, in the deterministic setting, the solution to the above system can be written as

$$V_0 = \sum_{t=1}^{T} \frac{c_t}{\prod_{s=1}^{t} R_s}. \tag{2}$$

This corresponds to the traditional actuarial present value of insurance liabilities; see e.g. Bowers et al. [6]. We emphasize that the validity of (2) depends essentially on the claims c_t and the investment returns R_t being deterministic. When moving to more realistic settings with uncertainty in claims and/or returns, one has to go back to the budget constraints in (1); see Sects. 3 and 4.

Remark 1 [Solvency II] If the investment returns R_t are defined according to

$$\prod_{s=1}^{t} R_s = \exp(tY_t),$$

where Y_t is the value of riskless yield curve at maturity t (so that $R_s = \exp(f_s)$, where f_s is the forward rate over period $[s-1, s]$), (2) becomes

$$V_0 = \sum_{t=1}^{T} \frac{c_t}{e^{tY_t}}.$$

If c_t are set equal to the expectations of the future cash-flows, (2) becomes the "best estimate" specified in Article 77(2) of the Solvency II-directive. However, yield curves have been developed for the valuation of bond portfolios and they are poorly suited for the valuation of insurance liabilities with uncertain future claims. In some cases, the "best estimate" can be considerably larger that more realistic valuations of insurance liabilities. For example, the value of a European call-option is (nearly) independent of the expected growth rate of the underlying index while the "best estimate" of a call-option grows without a bound when the expected growth rate increases. In such situations, the "risk margin" defined according to Article 77(3) would have to be negative.[1] The valuation technique presented in the following sections corresponds to the General provisions of Article 76 better than the sum of the "best estimate" and the "risk margin" described in Article 77.

2.1 Case study: Finnish private sector pension liabilities

We will study the insurance portfolio of the Finnish private sector occupational pension system. The yearly claims c_t consist of aggregate old age, disability and unemployment pension benefits that have accrued by the end of 2008 and become payable during year t. These payments form the majority (80%) of the total pension expenditure of the Finnish private sector occupational pension system. The liabilities are of the defined benefit type and they depend on the development of wage and consumer price indices. Figure 1 depicts the forecasted annual total pension expenditure. The forecast is based on the assumption of constant annual wage increases of 3.8%, annual inflation of 2.0% and the current accrued pension rights and Finnish mortality tables according to which all the liabilities will be amortized in $T = 82$ years.

Setting the annual investment return to $R_t = 1.06$, as in Biström et al. [3] the value of the pension liabilities given by (2) is $V_0 = 207.7$ billion euros. This is the minimum capital that would, in a deterministic world, suffice to cover the future pension payments associated with the accrued pension rights. The erosion of capital over time $(V_t)_{t=0}^{T}$ is depicted in Figure 2.

According to Savela the Finnish private sector pension providers (pension insurance companies and pension funds) have approximately $W_0 = 72.1$ billion euros capital as of the end of 2008. The funds have accumulated from contributions and investment returns and they will be used to cover part of the accrued pension rights. The rest will be covered on the pay-as-you-go basis. In this deterministic setting the *solvency ratio* of the Finnish private sector occupational pension system equals

[1] Some studies have suggested the use of so called "risk neutral measures" (see e.g. Wüthrich et al. [21]) but in realistic market models, the specification of a risk neutral measure can be a difficult task. Even if feasible, one would still face the choice among the infinitely many risk neutral measures that exist in incomplete markets.

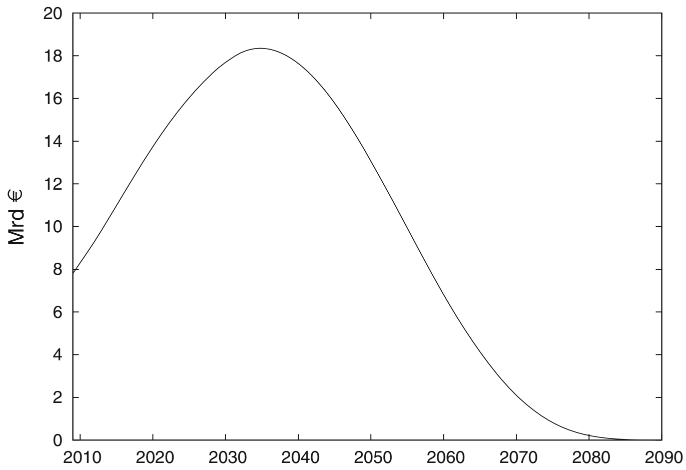

Fig. 1 Evolution of the projected aggregate claim payments associated with the accrued old age, disability and unemployment pension benefits

$$\frac{W_0}{V_0} = \frac{72.1}{207.7} = 0.35.$$

The valuation based on (2) is very sensitive with respect to the assumed parameter values for the inflation, wage growth and especially the investment return. The sensitivity of V_0 with respect to the anticipated annual investment return is depicted in Fig. 3. The value depends nonlinearly of the expected return and a mere percentage point reduction in the expected annual rate of return of 6%, increases the capital requirement by roughly 20%, or 40 billion euros. This highlights the need for a risk sensitive framework for liability valuation.

3 Valuation under uncertainty

In this section we extend the liability valuation technique to allow uncertainties in both the claims c_t and investment returns R_t. The investment returns are still assumed exogenous so that investment decisions have no effect on them. The role of asset management will be studied in Sect. 4.

In an uncertain environment and in the absence of liquid markets for the liabilities, there is always a risk that in some future scenarios the return on invested capital is insufficient to cover the claims. On the other hand, it may very well happen that the investment returns exceed the claims. The valuation of insurance liabilities must therefore take into account the risk preferences of the insurer. In what follows, the claims c_t and the investment returns R_t are modeled as random variables on a probability space $(\Omega, \mathcal{F}, P)$. The specified probability distribution should reflect the views of the insurer concerning the future development of the underlying risk factors.

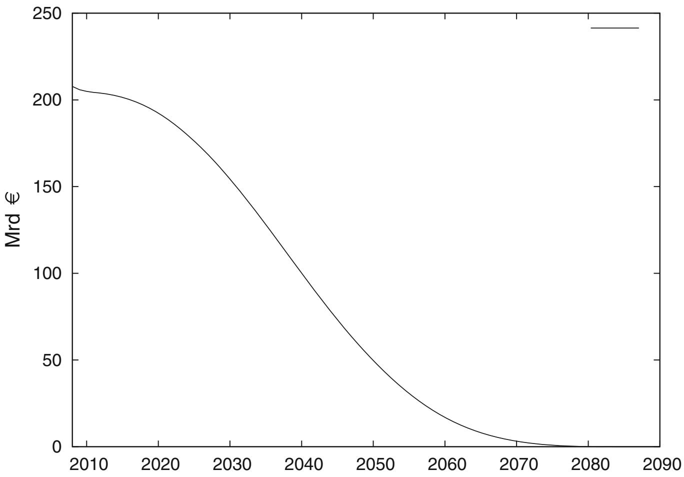

Fig. 2 Development of wealth

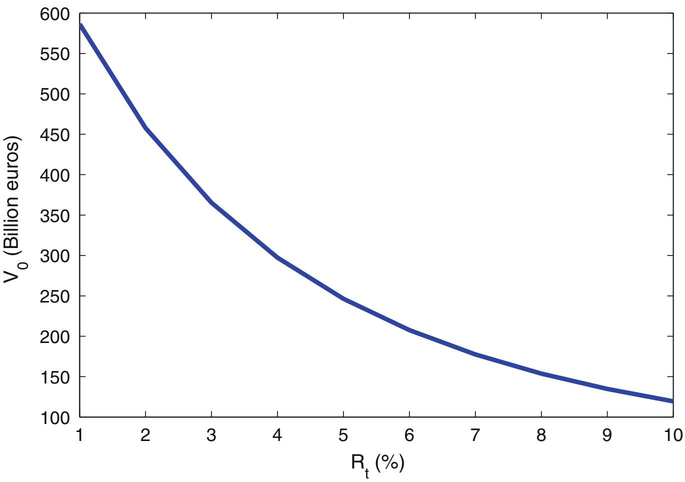

Fig. 3 The value of liabilities V_0 as a function of the assumed annual investment return R_t in the deterministic setting

Following the above principles, the value of the liabilities can be defined as the solution to the optimization problem

$$\begin{array}{ll} \text{minimize} & V_0 \\ \text{subject to} & V_t = R_t V_{t-1} - c_t \quad t = 1, \ldots, T, \quad P\text{-a.s.} \\ & V_T \in \mathcal{A}, \end{array} \tag{3}$$

where the first equation is required to hold along almost every scenario (i.e. with probability one) and $\mathcal{A}$ is a given *acceptance set* that specifies the insurer's preferences over the random terminal wealth. Modeling the claims c_t and returns R_t

as random variables, implies that the corresponding terminal wealth V_T is indeed random. Unlike in the deterministic case of Sect. 2, we can no longer write down a simple discounting formula for the value of liabilities V_0. Instead, numerical techniques are required; see the case study below.

The most conservative specification of acceptable terminal positions is

$$\mathcal{A} = \{V_T | V_T \geq 0 \text{ almost surely}\}.$$

In financial terms, this corresponds to *superhedging* of the insurance claims. In many situations, the probability distribution of the return and claim processes is such that superhedging would require an unreasonable amount of initial capital. More practical notions of acceptance can be defined in terms of a *risk measure* ρ as

$$\mathcal{A} = \{V_T | \rho(V_T) \leq 0\}. \quad (4)$$

In fact, there is a one-to-one correspondence between acceptance sets and risk measures; see e.g. Artzner et al. [1]. Well-studied examples of risk measures include the *Value at risk* [14] and *Conditional Value at Risk* [19][2]. For example, the initial capital V_0 corresponding to Value at Risk at level δ would be sufficient to cover the pension claims until full amortization with a probability of δ. Another natural definition of acceptance is

$$\mathcal{A} = \{V_T | Eu(V_T) \geq u(0)\},$$

where E denotes expectation and u is a given utility function. This corresponds to the *zero utility principle* studied e.g. in Bühlmann [7]. The associated risk measure is given by

$$\rho_u(V) = \inf\{\alpha | Eu(V + \alpha) \geq u(0)\}.$$

In the case of exponential utility, this becomes the well-known *entropic risk measure* We refer the reader to Föllmer and Schied [9] or Rockafellar for general treatment of risk measures.

Defined according to (3), the minimum amount V_0 of initial capital needed to cover the claims depends essentially on

1. **Probability distribution**: Decision maker's views regarding the uncertain future development of the claims $c = (c_t)_{t=0}^T$ and the investment returns $R = (R_t)_{t=0}^T$.
2. **Risk preferences**: The acceptance set $\mathcal{A}$ that defines the acceptable terminal positions of the insurer.

Both factors are subjective. They describe how the insurer perceives the uncertain future. Even in deterministic models, the required amount of initial capital depends on how the investment returns and insurance claims are assumed to develop. In reality, exact matching of investment returns with the uncertain claims is impossible

[2] The Value at Risk at confidence level $\delta \in [0, 1]$, $V@R_\delta(V_T)$ is defined as the negative of the $(1 - \delta)$-quantile of V_T. The Conditional Value at Risk at confidence level δ is defined as the conditional expectation $CV@R_\delta(V_T) = -E[V_T \mid V_T \leq -V@R_\delta(V_T)]$.

so the value of liabilities necessarily depends on the insurer's risk tolerances with respect to the random terminal position.

3.1 Case study: Finnish private sector pension liabilities

We modify the case study outlined in Sect. 2 by allowing the claims and the investment returns to be random. A detailed description of the stochastic model for the claims can be found in Hilli et al. [11]. Figure 4 displays the median and 95% confidence interval of the cash-flows c_t associated with the pension rights accrued by the end of 2008. The investment returns are modeled as a simple log-normal process

$$\ln R_t = \mu + \sigma \varepsilon_t,$$

where ε_t are iid standard normal and the parameters μ and σ are chosen so that the annualized rate of return has a mean and standard deviation of 6%.

We will value the pension liabilities according to (3). In order to illustrate the effect of risk tolerances, we will specify the acceptance set $\mathcal{A}$ according to (4) with two risk measures, namely $V@R_\delta$ and $CV@R_\delta$ with varying confidence levels δ. After the specification of the risk measure and the probability distribution of the relevant risk factors, the valuation of the liabilities can be carried out numerically by generating a finite number N of scenarios of investment returns R_t and claims c_t over $t = 1, \ldots, T$ and by solving the corresponding discretized version of system (3) by a simple line search. In the present study, we use $N = 200000$ scenarios.

Table 1 displays the value of liabilities V_0 at various confidence levels for $V@R$ and $CV@R$. Table 2 gives the solvency ratios W_0/V_0 corresponding to the $W_0 = 72.1$ billion of the Finnish pension system at the beginning of 2008 and the liability values V_0 of Table 1. The effect of the choice of a risk measure under

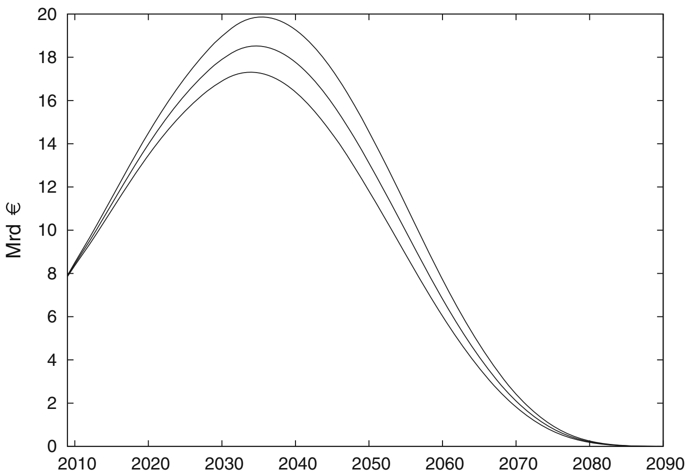

Fig. 4 Evolution of the claims associated with the accrued old age, disability and unemployment pension benefits

Table 1 Value of the pension liabilities V_0 (billion euros) with varying confidence levels

		Confidence level				
		95%	90%	85%	80%	66%
Risk measure	*V@R*	289	271	259	250	232
	CV@R	305	288	276	268	252

Table 2 Solvency ratios W_0/V_0 with varying confidence levels

		Confidence level				
		95%	90%	85%	80%	66%
Risk measure	*V@R*	24.9	26.6	27.9	28.9	31.1
	CV@R	23.6	25.1	26.1	26.7	28.7

the modeled uncertainties can be clearly seen in the results. The 66% confidence level in our problem with $T = 82$ years until amortization of the liabilities roughly corresponds to the 99.5% one-year confidence level required in Article 101(3) of Solvency II. Indeed, $0.995^{82} \approx 0.66$, in accordance with Article 122(1).

Figure 5 displays the evolution of the median and the 34%- and 66%-quantiles of $(V_t)_{t=0}^{T}$ when the initial wealth is set according to the risk measure $\rho(V_T) = V@R_{66\%}(V_T)$. As expected, the 34%:th percentile of the distribution of V_T equals zero after all the pension claims have been paid off.

4 Market consistent valuation

In real markets where an insurer has multiple investment opportunities, the chosen investment strategy plays a significant role in the determination of the sufficiency of the capital reserved for covering the liabilities. Choosing an investment strategy whose returns conform to the cash-flows associated with the liabilities, it may be possible to lower the initial amount. Such an approach is sometimes referred to as liability driven investment. The construction of an appropriate hedging strategy for an insurance portfolio is one of the most important tasks of an insurance company.

The same principle is one of the cornerstones of modern finance theory. For example, the classical Black–Scholes option pricing formula gives the initial capital required for exact replication of cash-flows of a call option in complete markets when the capital is invested according to the so called delta-hedging strategy [5]. Contrary to the Black–Scholes model, the exact replication (or superhedging) of financial instruments is often impossible or it may become prohibitively expensive. In practice, the hedging of insurance contracts always involves the risk that the return on invested capital might be insufficient to cover the claims. On the other hand, the earned investment returns may exceed the claims. Given the inherent uncertainties in the resulting net cash-flow, the choice of a hedging/investment strategy must be based on the risk preferences of the insurer.

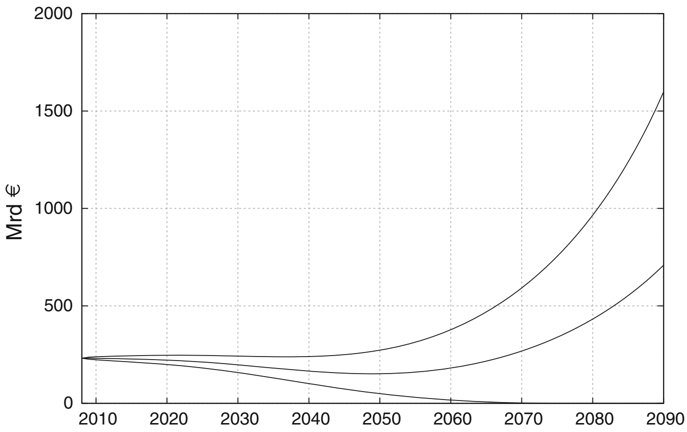

Fig. 5 The development of the 34, 50 and 66%-quantiles of $(V_t)_{t=0}^T$ when the initial capital corresponds to $V@R_{66\%}$

The most important factors affecting the value of liabilities are

1. **Probability distribution**: see Sect. 3,
2. **Risk preferences**; see Sect. 3,
3. **Hedging strategy**: The investment strategy, according to which the given capital is invested in financial markets.

The choice of an asset management strategy plays an important role in competitive insurance markets. The lower the initial capital an insurer needs for hedging its liabilities, the lower its costs of producing insurance contracts, or alternatively, the more it can distribute capital back to its shareholders.

Consider a strategic asset-liability management problem, where wealth can be diversified each year among a finite set J of available asset classes. Denote by $R_{t,j}$ the (total) return on class $j \in J$ during period $[t-1,t]$. The amount of wealth $h_{t,j}$ invested in class $j \in J$ in the beginning of year t can react to all available information at time t, but the decision is not allowed to depend on information that will be revealed after time t. The dynamic investment strategy $h = (h_t)_{t=0}^T$, where h_t is $\mathbb{R}^J$-valued random vector, is thus adapted to the available information. Mathematically, for each t, the portfolio h_t is $\mathcal{F}_t$-measurable, where $\mathcal{F}_t \subset \mathcal{F}$ is the sigma-field generated by the information observable by time t; (see e.g. [9], Sect. 5.1).

An insurer may also have investments in illiquid financial assets without well functioning secondary markets (e.g. reinsurance or private placement bonds). Even when liquid secondary markets do exist, the market values of some instruments may not reflect their value in hedging an insurance portfolio. In other words, the market values of some financial instruments may deviate substantially from their value to the insurer. This may be the case with some hedging instruments such as mortality linked bonds or equity- and interest rate derivatives. We will denote by $\bar{J}$ the set of assets that the insurer plans to hold to maturity.

The investment strategy of the insurer thus consists of a dynamic trading strategy $h = (h_t)_{t=0}^T$ in the liquid assets J and a static allocation $\bar{h} \in \mathbb{R}^{\bar{J}}$ in $\bar{J}$. We assume that short positions are allowed only in the money market investments which will be indexed by $0 \in J$. If an investment strategy $(h, \bar{h})$ satisfies

$$\begin{aligned} &\sum_{j\in J} h_{t,j} + c_t \leq \sum_{j\in J} R_{t,j} h_{t-1,j} + \sum_{j\in \bar{J}} \bar{R}_{t,j} \bar{h}_j \quad t = 1, \ldots, T, \\ &h_{t,j}, \bar{h}_{\bar{j}} \geq 0 \quad j \in J \setminus \{0\}, \bar{j} \in \bar{J}, \\ &\sum_{j\in J} h_{T,j} \in \mathcal{A}, \end{aligned}$$

then the initial capital

$$V_0 = \sum_{j\in J} h_{0,j} + \sum_{j\in \bar{J}} \bar{h}_j$$

covers the pension claims $c = (c_t)_{t=1}^T$ at the level of risk specified by the acceptance set $\mathcal{A}$. Here $\bar{R}_{t,j}$ denotes the annual cash return (e.g. coupon payments or option payouts) per invested capital in asset class $j \in \bar{J}$.

Due to the uncertain nature of the liabilities and asset returns, it is generally impossible to guarantee that the total wealth remains nonnegative in all scenarios. If short-selling was prohibited also in the money market account, the above system would become infeasible in general. We have thus assumed that if wealth becomes negative in some time and state, the insurer (the owners of the insurance company) is required to provide the missing funds.

The dependence structure between the random claims c and the investment returns R and $\bar{R}$ is an important factor in valuation of liabilities. It largely determines how well the investment strategy can be adapted to the insurance liabilities. For example, if one of the asset classes $j \in \bar{J}$ corresponds e.g. to re-insurance, its return process $\bar{R}_j$ may be completely determined by the claims c. Such an instrument might be a good ingredient in a hedging strategy (depending on the price, of course, since $\bar{R}_{t,j}$ was defined as the return per invested capital).

The investment strategy $(h, \bar{h})$ can be interpreted as a "replicating portfolio" for the insurance claims c. In practice, exact replication is usually impossible so that part of the liabilities always remain unhedged. This residual is represented by the terminal position $\sum_{j\in J} h_{T,j}$ of the insurer. It is often suggested that the value of liabilities should be the sum of the market value of the replicating portfolio plus the value of the unhedged part (this was taken as the definition of "market consistency" e.g. in Ernst and Young [8]). Our valuation procedure does exactly this. We look for replicating portfolios so that the value of the unhedged part is *zero* in terms of the risk measure

$$\rho(V_T) = \inf\{\alpha | V_T + \alpha \in \mathcal{A}\}.$$

The search for the minimum amount of capital needed to cover the liabilities leads to the optimization problem

$$
\begin{array}{lll}
\text{minimize} & \sum_{j\in J} h_{0,j} + \sum_{j\in \bar{J}} \bar{h}_j \quad \text{over} \quad h\in\mathcal{N}, \bar{h}\in\mathbb{R}^{\bar{J}} & \\
\text{subject to} & \sum_{j\in J} h_{t,j} + c_t \le \sum_{j\in J} R_{t,j} h_{t-1,j} + \sum_{j\in\bar{J}} \bar{R}_{t,j}\bar{h}_j \quad t=1,\ldots,T, & \\
& h_{t,j}, \bar{h}_{\bar{j}} \ge 0 \quad j\in J\setminus\{0\}, \bar{j}\in\bar{J}, & \\
& \sum_{j\in J} h_{T,j} \in \mathcal{A}, &
\end{array} \tag{5}
$$

where $\mathcal{N}$ denotes the $\mathbb{R}^J$-valued portfolio processes adapted to the filtration $(\mathcal{F}_t)_{t=0}^T$. The above problem cannot be solved analytically, except in some simple special cases. In practice, one has to rely on expert knowledge of the problem or on numerical approximation schemes or both.

When $\mathcal{A} = \{V_T | \rho(V_T) \le 0\}$ for an appropriate risk measure ρ, one can approach the solutions of (5) by applying the optimization procedure developed in Koivu and Pennanen [15] to the problem

$$
\begin{array}{ll}
\text{minimize} & \rho\left(\sum_{j\in J} h_{T,j}\right) \quad \text{over} \quad h\in\mathcal{N}, \bar{h}\in\mathbb{R}^{\bar{J}} \\
\text{subject to} & \sum_{j\in J} h_{0,j} + \sum_{j\in\bar{J}} \bar{h}_j \le w \\
& \sum_{j\in J} h_{t,j} + c_t \le \sum_{j\in J} R_{t,j} h_{t-1,j} + \sum_{j\in\bar{J}} \bar{R}_{t,j}\bar{h}_j \quad t=1,\ldots,T, \\
& h_{t,j}, \bar{h}_{\bar{j}} \ge 0 \quad j\in J\setminus\{0\}, \bar{j}\in\bar{J},
\end{array} \tag{6}
$$

for varying values of initial wealth $w \in \mathbb{R}$. Under mild assumptions on the risk measure and the return processes, we have that if w is such that the optimum value of (6) is zero then w is the optimum value of (5) and the optimal investment strategies of (6) are optimal also in (5). If ρ is the Conditional Value at Risk with given confidence level, then the procedure presented in Hilli et al. [12] is directly applicable to (6).

4.1 Case study: Finnish private sector pension liabilities

On the strategic level, the assets of a typical Finnish pension fund are usually allocated in interest rate, equity and real estate funds. In our numerical study, the liquid assets are modeled accordingly. A more detailed breakdown of the available asset classes is given in Table 3. The stochastic model for asset returns used in this study is described in Koivu et al. [16]; Hilli et al. [10]. Table 3 gives the median and the 90% confidence intervals of the annualized rates of return of the liquid asset classes. The illiquid instruments consist of fixed coupon bonds with a 4% annual coupon rate and maturities of 10, 20 and 30 years. The prices of the illiquid bonds were computed by discounting the cash-flows with swap rates. The pension claims are computed as in Sect. 3.

In the numerical study we evaluated 529 parametric dynamic investment strategies with varying investment styles. The strategies are based on the well-known and widely applied *buy and hold*, *fixed proportion* and *constant proportion portfolio insurance* trading rules. In buy and hold strategies the initial asset allocation is held over time without rebalancing. In fixed-proportion strategies the asset allocation is rebalanced, in each period, to fixed portfolio weights. In constant

Table 3 Quantiles of the annualized rates of return (%) of the liquid asset classes

	5%	50%	95%
Money market	2.9	3.6	4.4
Bonds	−0.6	4.4	10.8
Nordic equities	−26.8	7.8	58.2
European equities	−17.9	6.7	38.6
US equities	−19.7	6.7	41.7
Asian equities	−22.9	7.7	50.6
Real estate	−17.4	6.2	36.5

proportion portfolio insurance-strategies the portfolio weights are adjusted according to the "cushion" which is defined as the difference of the total assets and a rough estimate of the value of liabilities calculated using a deterministic model much as in Sect. 2. The larger the cushion, the higher the weight of risky assets; see Black and Perold [4]. As in Hilli et al. [12] all strategies were modified to accommodate for claim payments and for the possibility that the wealth becomes negative prior to maturity so that the budget constraints in (6) will be satisfied. In case the earned investment returns are insufficient to cover the liabilities it is assumed that the insurer borrows the required funds from the money market.

For each of the 529 strategies, we computed the required initial capital when the capital is invested according to the given strategy and the risk is measured with $CV@R_\delta$. The first row of Table 4 gives lowest of the 529 numbers for different values of δ. We then applied the procedure developed in Koivu and Pennanen [15] and Hilli et al. [12] to (6) in order to find an optimal diversification among the 529 basis strategies that leads to the lowest possible initial capital. The corresponding values are given on the last row of Table 4. All the computations were based on approximating the probability distribution of the risk factors $(R, \bar{R}, c)$ by a sample of 200,000 scenarios. In order to avoid bias in the case of optimized strategy, the optimization was based on an independent set of 100,000 scenarios.

The optimally diversified strategies significantly reduce the required capital compared to the individual strategies. The optimization adjusts the return distribution of the invested capital in order to better suit the uncertain insurance claims and the given risk preferences.

The solvency ratios W/V_0 corresponding to the values of liabilities V_0 in Table 4 and the $W_0 = 72.1$ billion euros of the Finnish pension system at the beginning of 2008 is are given Table 5. Again, the row labeled "best basis" gives the solvency ratios obtained with the individual basis strategy that achieves the lowest value of V_0 among all the evaluated 529 basis strategies.

Table 4 Value of the pension liabilities V_0 (billion euros) with varying investment strategies and confidence levels

	Confidence level				
	95%	90%	85%	80%	66%
Best basis	296	284	273	261	239
Optimization	288	271	254	236	202

Table 5 Solvency ratios (%) with varying investment strategies and confidence levels

	Confidence level				
	95%	90%	85%	80%	66%
Best basis	24.3	25.4	26.4	27.6	30.1
Optimization	25.0	26.6	28.3	30.5	35.6

5 Conclusions

This paper develops a computational framework for cash-flow based valuation of pension liabilities in incomplete markets. In accordance with general principles of e.g. International Association of Insurance Supervisors [13] and Solvency II-framework, the value of liabilities is defined as the minimum initial capital an insurer would require in order to take on the liabilities. The framework is built on three fundamental factors that are known to affect the value that economic agents assign to uncertain future cash-flows: the probability distribution of the underlying risk factors, the risk preferences and the hedging strategy of the insurer. While highly subjective, these factors strongly affect the value of an insurance portfolio to an insurer. Using an example from the Finnish private sector occupational pension system this paper illustrates the effects of the above factors on liability valuation.

Although we have presented the valuation procedure from the point of view of the insurer, the procedure can be used for regulatory purposes as well. In this case, the probability distribution and the risk preferences should conform to the views of the supervisor. In this context, our procedure can be interpreted as a computational implementation of market consistent valuation frameworks where the value of liabilities is the sum of the market value of a “replicating portfolio” plus the value of the unhedged part. In our procedure, the replicating portfolio is required to be such that the value of the unhedged part is zero in the sense of a given risk measure.

The developed approach leaves ample possibilities for future research. An interesting question is how uncertainties in mortality forecasts affect the value of pension liabilities. Also, it would be interesting to study the role of alternative investment classes in valuation and hedging of pension liabilities. The presented framework can readily accommodate such extensions.

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