Heterogeneity in Risk Preferences leads to Stochastic Volatility

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Abstract

This paper studies the price processes of a claim on terminal endowment and of a claim on firm book value when the underlying variables follow a bivariate geometric Brownian motion. If the state-price process is multiplicatively separable into time and endowment functions, our main result shows that firm (endowment) price volatility is stochastic (state-dependent) if, and only if, the endowment function is not a power function. In a pure exchange economy populated by two agents with CRRA preferences we confirm the separability and show furthermore that firm (endowment) price volatility is stochastic (state-dependent) if, and only if, both agents are heterogeneous in risk-preferences.

Keywords

stochastic volatility, state-dependent volatility, heterogeneity

JEL Classification

G12, G13

1 Introduction

Volatility is a major risk parameter. It is well-documented that stock volatility is stochastic[1], i.e. it evolves randomly over time. Our paper is motivated by results from two strands of the literature: first of all, starting with Heston (1993), the continuous-time option pricing literature showed that stochastic volatility of the underlying stock price process leads to the so-called implied volatility "smile." Second, Merton (1973) and Rubinstein (1976) proved within a multi-period model that the Black-Scholes formula claims for call options written on aggregate endowment and firm book value, respectively, when CRRA agents are homogeneous in their risk-preferences; however, Benninga and Mayshar (2000) later showed that heterogeneity in CRRA leads to the "smile" pattern. Our goal is to join these two strands of the literature and study if heterogeneity in risk preferences leads to stochastic volatility.

Our paper studies a continuous-time version of the discrete-time economy of Rubinstein (1976). We study the price process of a claim on the terminal aggregate endowment and the shadow price process of a claim on terminal firm value in a complete markets economy. The underlying price processes of aggregate endowment and firm value follow a two-dimensional geometric Brownian, i.e. *volatility of the underlying economic variables* is neither state-dependent nor stochastic, it is *constant*. The aggregate endowment drives the state-price process (pricing kernel) and enters into the price process of both claims.

Our analysis proceeds in two steps. In the first step we relate stochastic volatility to properties of the state-price process (a.k.a. pricing kernel). In particular, we show that volatility of the endowment claim is state-independent if, and only if, the state-price process can be expressed as a power function of the underlying endowment; analogously, volatility of the firm claim is not stochastic if and only if the state-price process can be expressed as a power function of the underlying endowment. In a second step we study an economy populated by two agents with CRRA preferences, and relate the power function property to homogeneity/heterogeneity in risk-preferences. This allows us to conclude that heterogeneity in risk preferences leads to stochastic

[1] In stochastic volatility models, volatility is driven by a second stochastic process, in addition to the stochastic process that drives stock returns; it is different from time-dependent and state-dependent volatility.

volatility.

Our paper relates to studies of the state-price process. Chabi-Yo et al. (2007) analyzed state-price processes in the presence of latent variables to explain the risk aversion puzzle. In a static framework, Brennan (1979) studied if the Black-Scholes formula holds for call options on the aggregate endowment and found that the state-price process must be a power function in the aggregate endowment. Our paper is in continuous-time and provides the link between the state-price process and state-dependent volatility of the endowment claim; in addition we study the volatility process of another claim, the claim on firm book value and links the power function property to stochastic volatility of the price process.

We provide interesting implications for the structure of the representative agent. It is well known, see Rubinstein (1974) and Constantinides (1982), that a representative agent exists who supports the equilibrium stock price dynamics; yet, typically, this simplifies economic analysis only if its functional structure is unknown. When volatility is stochastic, our analysis implies that heterogeneous CRRA agents *cannot* aggregate to a representative agent with CRRA preferences, because these preferences would yield a state-price process that is a power function. Our paper thereby highlights the importance of being careful with assumptions about the utility functions for representative agents.

More important, our paper adds to our understanding of the stochastic nature of volatility. Most of the literature attributes stochastic volatility to the stochastic nature of dividends in representative agent economies, see, e.g. the recent paper by Bhamra and Uppal (2011) and references therein, or to the stochastic nature of the information process (Admati (1985), Brock and Hommes (1997)). Our paper, provides an alternative explanation related to heterogeneity[2]: here, we attribute stochastic volatility to the aggregation of risk-preferences. Most related is Weinbaum (2009): he looked at the price process of what we call the endowment claim and found

[2]Our paper is related to Cochrane et al. (2008), who study a two-tree extension of Lucas (1978). In a representative agent economy, they derive interesting dynamics of the aggregate consumption claim by modeling the price processes of two stocks as geometric Brownian motion; the aggregate amount of the consumption good is the sum of the two trees and will *not* follow a geometric Brownian motion. While this leads to stochastic volatility, it is is driven by the stochastic nature of the share of each stock in the aggregate endowment and not by agent's risk preferences.

that volatility is not constant. Our analysis confirms his results but we also delve deeper looking at the firm claims; we show that stochastic volatility of the firm claim is due to heterogeneity in agent's risk preferences.

In the next section we introduce firm and endowment claims and study properties of the state-price process that lead to state-dependent (stochastic volatility) of the claim price processes. In the third section we then characterize the state-price process in an exchange economy where agents have CRRA preferences and link stochastic volatility to heterogeneity in agent's risk preferences. The paper concludes with section 5. All proofs are postponed to the appendix.

2 Volatility Processes and State-price Functions

This section characterizes state-price processes (a.k.a. pricing kernels) that lead to time-varying or stochastic volatility; for this we introduce in subsection 2.1 so-called state-price functions. Throughout this paper we are interested in two claims on the consumption good; in this section we will analyze them separately in the subsections 2.2 and 2.3.

2.1 State-price Functions

There is a single perishable consumption good and all units are expressed in terms of that good. Time 0 is today; at time $T > 0$ all economic activity ceases. There is a long tradition to study price processes in terms of aggregate consumption; typically, a representative agent with an exogenously specified utility function is invoked and the aggregate consumption stream is exogenously specified. Within such analysis there is no distinction between aggregate endowment and aggregate consumption. In this section[3], however, we do not want to depend on a specification of agents' preferences or consumption. Therefore, we only specify exogenously the so-called aggregated endowment process Y: we assume $Y_0 = 1$ and that it evolves over time $0 \leq t \leq T$ according to the stochastic differential equation

$$dY_t = \sigma_Y Y_t dW_t^Y; \tag{1}$$

[3]In the next section we will model an economy and take a look at consumption; however, at this stage, there is no need to relate to consumption.

here, $\sigma_Y > 0$ is a constant and $(W_t^Y)_t$ denotes a standard Wiener process on a suitable probability space $(\Omega, \mathcal{F}, P)$. The current endowment is observable at all times. The purpose of this paper is to study volatility processes; for simplicity, we do not permit any drift in the endowment process.

We define for later use throughout this paper random variables Z_{tT}^Y $(0 \leq t \leq T)$ and a time function ψ_Y by

$$\ln Z_{tT}^Y = -\frac{\sigma_Y^2}{2}(T-t) + \sigma_Y \sqrt{T-t}U \text{ and } \psi_Y(t) = \exp(\sigma_Y^2(T-t)); \tag{2}$$

here, U is a standard normal random variable. Note that $Y_T \stackrel{d}{=} Y_t Z_{tT}^Y$, where $\stackrel{d}{=}$ denotes equality in distribution.

We assume that the market is (dynamically) complete in claims on the aggregate consumption stream. This is warranted, e.g., when a riskfree security and a claim on the endowment can be traded. Then, it is well known, see e.g. Duffie (2001) that a *state-price process* $(M_t)_{0 \leq \tau \leq T}$ exists with the property that the time t price of a claim to any Markovian payout process $(\Pi_\tau)_{t \leq \tau \leq T}$ of the consumption good is

$$\frac{1}{M_t} E\left[\int_t^T \Pi_\tau M_\tau d\tau \middle| \Pi_t, Y_t \right]. \tag{3}$$

Throughout this paper we study the following class of state-price processes that allows us to separate the impact of time and aggregate endowment on state-price processes:

Definition 1 (Time-separable State-price Process) *A* state-price function *is a strictly positive function m on the positive real line with the property that P-a.s. $M_t = \gamma(t) \cdot m(Y_t)$ for all $0 \leq t \leq T$ for a suitable, strictly positive function γ on the time interval $[0, T]$, and subject to the following technical restrictions:*

1. *m is four times continuously differentiable.*

2. *the function $(t, y) \mapsto E\left[m(y \cdot Z_{tT}^Y)\right]$ is finite, continuously differentiable in time t, continuously differentiable up to third order in y and continuously differentiable in the cross derivative of first order in y and up to third order in t. Furthermore, for all these derivatives we can exchange differentiation and expectation.*

Throughout this section we assume that the state-price process leads to a (time-separable) state-price function m in the sense of this definition; the differentiability restrictions here are necessary to characterize price dynamics based on fundamentals and/or to analyze volatility processes.

For later reference we denote by v_B the elasticity of the function $y \mapsto E[m(yZ_{tT}^Y)]$ and by v_m the elasticity of the state-price function m, all w.r.t. the aggregate endowment; i.e. we set

$$v_B(t,y) = \frac{E\left[m'(y \cdot Z_{tT}^Y) \cdot (y \cdot Z_{tT}^Y)\right]}{E[m(y \cdot Z_{tT}^Y)]}, v_m(y) = \frac{m'(y)y}{m(y)}. \tag{4}$$

These two elasticities are well-defined based on definition 1.

2.2 Endowment Price

It is common in financial economics to study the price of a claim to one unit of the aggregate consumption *stream* at all times. We could do so and represent the price of such a claim as the total value of simpler claims to aggregate endowment that all pay one unit at a single specified time and nothing at all other times. However, this would considerably complicate our analysis and discussion; instead, for simplicity of exposition we study only those simpler claims.

In particular, here we study only a claim to the aggregate endowment that pays exactly one unit of the aggregate endowment and only at time T. (Nothing is paid before time T.) We refer to the price of this claim to the terminal endowment as the *endowment price* and denote it $S_Y = (S_{Yt})_{0 \le t \le T}$. It can be written as a function S_Y of time t and current aggregate endowment Y_t, i.e. $S_{Yt} = S_Y(t, F_t, Y_t)$; the pricing property of the state-price process, equation (3), implies that

$$S_Y(t,y) = y \frac{\gamma(T)}{\gamma(t)} \frac{E\left[m(y \cdot Z_{tT}^Y) Z_{tT}^Y\right]}{m(y)}. \tag{5}$$

For later reference we denote by v_Y the elasticity of the endowment price (function) w.r.t. current aggregate endowment, i.e. we set

$$v_Y(t,y) \quad = \quad \frac{\partial S_Y}{\partial y} \frac{y}{S_Y}, \tag{6}$$

Definition 1 implies that the function S_Y is differentiable such that the price elasticity v_Y is well defined. Using Itô's lemma we prove:

Theorem 2 *The endowment price dynamics is* $dS_{Yt} = \mu_Y(t, Y_t)S_{Yt}dt + \tilde{\sigma}_Y(t, Y_t)S_{Yt}dW_t^Y$ *for a suitable function* μ_Y *and for a function* $\tilde{\sigma}_Y$ *defined by setting*

$$\tilde{\sigma}_Y(t, y) = \sigma_Y \cdot \upsilon_Y(t, y). \tag{7}$$

Because W^Y is a standard Wiener process, the function $\tilde{\sigma}_Y(t, y)$ captures what is commonly referred as volatility of the endowment price processes. We see that it is driven by υ_Y, the elasticity of the endowment price; therefore, this will be our main object of study in this subsection[4].

The literature associates *state-dependent volatility* with the stochastic movement of volatility over time as a function of the underlying asset value; it differentiates this from so-called *time-dependent volatility* where volatility is a function of time, *only*. The time-dependence of υ_Y may lead to time-varying volatility, but state-dependency will show up only if υ_Y is *not* constant as a function of y. This leads us naturally to the following definition:

Definition 3 *We say that volatility of the endowment price process is* state-dependent*, if there is a time point* $0 \le t < T$ *and an open interval* (ν_l, ν_u) *such that the function* $\upsilon_Y(t, \cdot)$ *is not constant on the interval* (ν_l, ν_u)*. Otherwise, we say that volatility of the endowment price process is* state-independent*.*

This definition means that for state-independent volatility the function υ_Y and thus $\tilde{\sigma}_Y$ can only depend on time t. (While both functions could potentially depend on time, we see below that they are constant in time, too.) Before analyzing the link between the state-price function and volatility, it helps to understand the impact of the price system:

Proposition 4 *At all times* $0 < t < T$ *and all* $y > 0$ *we have* $\upsilon_Y(t, y) = 1 + \upsilon_B(t, \psi_Y(t) \cdot y) - \upsilon_m(y)$*.*

Proposition 4 shows that we can decompose the driving force υ_Y of endowment price volatility into two effects that relate to the pricing system: the first is due to the elasticity υ_B of the function $y \mapsto E[m(yZ_{tT}^Y)]$ and captures changes in the future price system; the second is due to the elasticity υ_m of the state-price function m and captures changes in the current price system (the state-price function).

[4]We do not present the drift term μ_Y since our paper focuses on volatility.

Assume that m is a power function on the entire positive real line, i.e. there exist constants α, β such that the state-price function is $m(y) = \alpha y^\beta$ for all $y > 0$. Then we calculate $m'(y) = \alpha\beta y^{\beta-1}$, that $E\left[m'(y \cdot Z_{tT}^Y) \cdot (y \cdot Z_{tT}^Y)\right] = \alpha\beta y^\beta E\left[(Z_{tT}^Y)^\beta\right]$, and that $E[m(y \cdot Z_{tT}^Y)] = y^\beta E\left[(Z_{tT}^Y)^\beta\right]$. Therefore, power state-price functions may lead to state-price processes that fulfill the technical restrictions of definition 1; also, we find $v_B(t,y) = v_m(t,y) = \beta$ such that $v_Y = 1$ for all t, y. Then, volatility is not only state-independent but also time-invariant: $\tilde{\sigma}_Y = 2\sigma_Y$.

We can confirm this using well-known formulas for lognormal random variables:

$$S_Y(t,y) \quad = \quad y\frac{\gamma(T)}{\gamma(T)}E\left[\left(Z_{tT}^Y\right)^{1+\beta}\right] = y\frac{\gamma(T)}{\gamma(T)}\exp\left(\beta(\beta+1)\frac{\sigma_Y^2}{2}(T-t)\right). \tag{8}$$

This shows that the endowment price function is linear in the aggregate endowment; this confirms our above result that its volatility is σ_Y and that it is neither time-varying nor state-dependent.

Since the impact of the future pricing system $y \mapsto E[m(yZ_{tT}^Y)]$ is via an expectation taken over a general function m, we cannot analyze v_B easily. However, the following proposition permits us to gain intuition into the driving forces behind v_Y:

Proposition 5 *At any point in time, we can write $v_B(\psi_Y \cdot y) = v_m(y) + \frac{\partial v_m}{\partial y}(y) \cdot y\sigma_Y^2(T-t)$, up to terms of order higher than quadratic in σ_Y.*

This proposition together with proposition 4 tells us how volatility varies with changes in the endowment: the driving force of volatility then reads $v_Y = 1 + \frac{\partial v_m}{\partial y}(y) \cdot y\sigma_Y^2(T-t)$, up to terms of order higher than quadratic in σ_Y. As a sufficient condition, this suggests that volatility will be state-dependent on an interval (ν_l, ν_u) if the first derivative $\frac{\partial v_m}{\partial y}$ of the elasticity of the state-price function is either *not* zero on that interval or proportional to $1/y$, i.e. if the elasticity of the state-price function is *not* constant on the interval (ν_l, ν_u). This, however, is the case (only) for state-price functions that are power functions in the endowment on that interval. Overall, this suggests that volatility will be (locally) state-dependent if and only if the state-price function is (locally) not a power function.

It is important to stress that proposition 5 is a (second order) expansion in endowment volatility σ_Y and that we only looked at a sufficient condition. It helps to gain intuition, but does not imply rigorously the link between stochastic volatility and state-price functions that are power functions in the endowment. This will be done in the following proposition:

Theorem 6 *If endowment price volatility is state-independent, then m is a power function, i.e. there exist constants α, β such that the state-price function is $m(y) = \alpha y^{\beta}$ for all $y > 0$.*

Overall we conclude based on Theorem 6 and our above discussion of the impact of power state-price functions:

Theorem 7 *Endowment price volatility is state-independent if and only if the state-price function m is a power function.*

2.3 Firm Price

This subsection studies a claim on an asset that is (partially) correlated with the aggregate endowment. We refer to the underlying asset as the firm book value and denote it by F. Today's book value is $F_0 = 1$; book value F evolve over time $0 \leq t \leq T$ according to the stochastic differential equation

$$dF_t = \sigma_F F_t dW_t^F, \tag{9}$$

where $(W_t^F)_t$ is standard Wiener processes on the probability space $(\Omega, \mathcal{F}, P)$ and $\sigma_F > 0$ is a constant. The instantaneous correlation between the Wiener processes W^F and W^Y is constant and we denote it by $-1 \leq \kappa \leq 1$. The current book value F is observable at all times. The purpose of this paper is to study volatility processes; for simplicity, we do not permit any drift in the process of firm book value.

A claim that pays a stream of the firm book value at all times could be valued using the state-price process. However, as discussed in the previous subsection, this would considerably complicate our analysis without adding new insights beyond an analysis of a claim that only pays at a single point in time. For simplicity we study only a simple claim that pays only at time T the book value of a firm; nothing is paid before time T. We refer to the *shadow price* of this claim on firm book value as the *firm price* and denote it $S_F = (S_{Ft})_{0 \leq t \leq T}$.

For future reference throughout the remainder of this paper we define random variables Z_{tT}^F $(0 \le t \le T)$ and a time function ψ_F by

$$\ln Z_{tT}^F = -\frac{\sigma_F^2}{2}(T-t) + \sigma_F\sqrt{T-t}V, \text{ and } \psi_F(t) = \exp(\kappa\sigma_Y\sigma_F(T-t)); \tag{10}$$

here, V is a standard normal random variable that has correlation κ with the random variable U introduced in equation (10). Note that $F_T \stackrel{d}{=} F_t Z_{tT}^F$, where $\stackrel{d}{=}$ continues to denote equality in distribution.

The firm price is a function of current time t, book value F_t and endowment Y_t, i.e. $S_{Ft} = S_F(t, F_t, Y_t)$. The pricing property of the state-price process, equation (3), implies that

$$S_F(t,f,y) = f\frac{\gamma(T)}{\gamma(t)}\frac{E\left[m(y \cdot Z_{tT}^Y)Z_{tT}^F\right]}{m(y)}. \tag{11}$$

In addition to the elasticities that we studied earlier, we denote by v_F the elasticity of the firm price w.r.t. the current aggregate endowment, i.e.

$$v_F(t,f,y) = \frac{\partial S_F}{\partial y}\frac{y}{S_F}. \tag{12}$$

Definition 1 implies that the function S_F is differentiable such that the price elasticity v_F is well defined. Equation (11) shows that $\frac{\partial S_F}{\partial f}$ is linear in f; therefore, v_F does not depend on current book value f and for simplicity of exposition we drop the dependence of v_F on firm (book) value f throughout this paper. Using Itô's lemma we prove:

Theorem 8 *A standard Wiener process $\tilde{W}^F$ and a function μ_F exists with the property that the dynamics of the firm price process is $dS_{Ft} = \mu_F(t, F_t, Y_t)S_{Ft}dt + \tilde{\sigma}_F(t, Y_t)S_{Ft}d\tilde{W}_t^F$; here, we used a function $\tilde{\sigma}_F$ defined by setting*

$$\tilde{\sigma}_F(t,y) = \sqrt{\sigma_F^2 + \big(\sigma_Y v_F(t,y)\big)^2 + 2\kappa\sigma_F\sigma_Y v_F(t,y)}. \tag{13}$$

The term $\tilde{\sigma}_F(t, Y_t)$ captures what is commonly referred as volatility, since $\tilde{W}^F$ is a standard Wiener processes. Note that volatility is a (stochastic) process driven by the aggregate endowment. By equation (7) the firm volatility process $\tilde{\sigma}_F$ depends on two terms: the volatility of book value enters through a constant level σ_F, because firm price S_F depends linearly on firm book value, see equation (5). The second contribution to volatility will be from the endowment via $\sigma_Y \cdot v_F(y)$, because endowment enters (potentially non-linearly) into firm price S_F, see equation (5). (There is also a correlation term, since firm book value and aggregate endowment are correlated.) Throughout we see that v_F essentially drives firm volatility; therefore, this will be our main object of study in the remainder of this section.

The literature associates state-dependent volatility with the stochastic movement that is driven by the underlying asset value. The term v_F does not depend on firm book value and so the firm price cannot exhibit state-dependent volatility.

The literature associates stochastic volatility with the stochastic movement of volatility over time that is *not* driven by the underlying asset, i.e. it must be driven by a second stochastic process; it differentiates this from time-dependent volatility where volatility is *only* a function of time. The time-dependence of v_F may lead to time-varying volatility, but firm price volatility is varying stochastically only if v_F is *not* constant in y. To define *stochastic volatility* we ignore time-dependence of v_F and look instead only at its dependence on the aggregated endowment. This leads us naturally to the following definition of stochastic volatility:

Definition 9 [5] *We say that volatility of the firm price process is* stochastic*, if there is a time point* $0 \leq t < T$ *and an open interval* (ν_l, ν_u) *such that the function* $v_F(t, \cdot)$ *is not constant on the interval* (ν_l, ν_u)*. Otherwise, we say that firm volatility is* not stochastic*.*

This definition means that for volatility that is not stochastic, the elasticity v_F and thus the volatility $\tilde{\sigma}_F$ can only depend on time t. Before analyzing the link between the state-price function and volatility, it helps to understand the impact of the price system:

Proposition 10 *For all times* $0 < t < T$ *and* $y > 0$ *we have* $v_F(t, y) = v_B(t, \psi_F(t) \cdot y) - v_m(y)$*.*

As in the previous subsection, proposition 4 shows that we can decompose v_F, the driving force of firm volatility, into two effects that relate to the pricing system, trading off changes

[5] This is slightly different from the previous subsection, where the dependence on y lead to state-dependent volatility. Previously, the endowment process was driving also the payout from the endowment claim which could only enter as a state-dependency; here, however, the firm price is driven by the book value process, such that the endowment process can drive a separate volatility process.

in the future price system against those in the current price system. Note that the elasticities v_F studied here and v_Y studied in the previous subsection have a similar structure: they both use the elasticity v_B; the only difference is that the y dependence enters through $\psi_Y(t) \cdot y$ and $\psi_F(t) \cdot y$, respectively, and that v_Y has an additional constant 1 that does not show up in v_F.

Assume that m is a power function on the entire positive real line, i.e. there exist constants α, β such that the state-price function is $m(y) = \alpha y^\beta$ for all $y > 0$. The calculations of the previous subsection show that m may lead to state-price processes that fulfill the technical restrictions of definition 1 and that $v_B(t,y) = v_m(t,y) = \beta$ such that $v_F = 0$ for all t, y. This means that volatility is not stochastic, that it is state-independent and also time-invariant: $\tilde{\sigma}_F = \sigma_F$ for all t, y. We can confirm this using well-known formulas for (bivariate) lognormal variables:

$$S_F(t,f,y) = f\frac{\gamma(T)}{\gamma(T)} E\left[Z_{tT}^F \left(Z_{tT}^Y\right)^\beta\right] = f\frac{\gamma(T)}{\gamma(T)} \exp\left(\left(\frac{\sigma_Y^2}{2}\beta(\rho+1) + 2\beta\kappa\sigma_F\sigma_Y\right)(T-t)\right);$$

this is independent of the aggregate endowment and driven by firm book value $f = F_t$ such that its volatility is σ_F.

We would like to show the converse, i.e. that volatility is stochastic if the state-price function is not of power type. Unfortunately, due to the expectation, v_B cannot be analyzed in closed-form. However, the following proposition permits us to gain intuition into the driving forces behind v_F:

Proposition 11 *At any point in time, we can write $v_B(\psi_F \cdot y) = v_m(y) + \frac{\partial v_m}{\partial y}(y) \cdot y\kappa\sigma_F\sigma_Y(T-t)$, up to terms of order higher than quadratic in σ_Y.*

This proposition 11 together with proposition 10 tells us how volatility varies with changes in the endowment: the driving force of volatility then reads $v_F = \frac{\partial v_m}{\partial y}(y) \cdot y\kappa\sigma_F\sigma_Y(T-t)$, up to terms of order higher than quadratic in σ_Y. As a sufficient condition, this suggests that volatility will be stochastic on an interval (ν_l, ν_u), if the first derivative $\frac{\partial v_m}{\partial y}$ of the elasticity of the state-price function is either *not* zero on that interval or proportional to $1/y$, i.e. if the elasticity of the state-price function is *not* constant on the interval (ν_l, ν_u). This, however, is the case (only) for state-price functions that are power functions in the endowment. Overall, this suggests that volatility will be stochastic if and only if the state-price function is a power function.

It is important to stress that proposition 11 is a (second order) expansion in endowment volatility σ_Y and that we only looked at a sufficient condition. It helps to gain intuition, but does not imply rigorously the link between stochastic volatility and state-price functions that are power functions in the endowment. This will be done in the following proposition:

Theorem 12 *If firm price volatility is not stochastic, then m is a power function, i.e. there exist constants α, β such that the state-price function is $m(y) = \alpha y^\beta$ for all $y > 0$.*

Overall we conclude based on Theorem 12 and our above discussion of power state-price functions:

Theorem 13 *Firm price volatility is not stochastic if and only if the state-price function m is a power function.*

3 Stochastic Volatility and Power Utility

The previous section characterized the link between the state-price function and stochastic volatility; it did not consider what determines the state-price function economically. To close this gap, this section studies a pure exchange economy populated by two agents with power utility. This allows us to link the power function property of state-price functions to homogeneity in agent's risk preferences. Throughout, we continue to study the firm and endowment claim of the previous section; in particular we continue to assume complete markets as well as the dynamics for aggregate endowment and firm book value of equations (1) and (9).

3.1 The Economy

We study a continuous-time pure exchange economy populated by two agents $i = 1, 2$. Both agents receive at all times $0 \leq t \leq T$ an equal share $\frac{1}{2}$ of the aggregate endowment stream[6], i.e.

[6]This assumption is to simplify the exposition; it affects the budget equation but does not affect our results qualitatively.

each agent receives at all times the endowment stream $\frac{Y_t}{2}$. Throughout the remainder of this paper we assume that both agents have preferences with constant relative risk-aversion $\rho_i > 0$ (CRRA preferences, a.k.a. power utility), i.e.

$$u_i(c) = \frac{c^{1-\rho_i}}{1-\rho_i}.$$

(We follow the convention that an agent with $\rho_i = 1$ has log utility preferences, $u_i = \ln$.)

We say a consumption process is *budget-feasible* if it can be financed by selling the agent's endowment stream and purchasing the consumption process. Because the market is (dynamically) complete, agent $i = 1, 2$ can implement any strictly positive consumption stream $c_i = (c_{it})_t$ that is budget-feasible; she chooses the one that maximizes her time-separable utility

$$E\left[\int_0^T e^{-\gamma t} u_i(c_{it}) dt\right]. \tag{14}$$

Here, $\gamma > 0$ denotes the time-preference parameter. Our focus in this paper is on the impact of heterogeneity in risk-preferences on the stochastic nature of stock volatility. Therefore, we assume both agents have identical *time-preference* parameters. We will compare the economy where agents are homogeneous in their risk-preferences ($\rho_1 = \rho_2$) with the economy where their risk-preferences are heterogeneous ($\rho_1 \neq \rho_2$).

We adopt the concept of a :

Definition 14 (Rational Expectations Equilibrium) *An equilibrium consists of consumption processes* $(c_{it})_{0 \le t \le T}$ *for both agents* $i = 1, 2$ *which maximize their utility s.t. the budget feasibility condition, and clear the market at all times in all states, i.e. for all* $0 \le t \le T$*:* $c_{1t} + c_{2t} = Y_t$*, P-a.s.*

It is well known, see, e.g. Duffie (2001) that the first agent's equilibrium consumption process defines the (unique) *state-price process* M with

$$M_t = e^{-\gamma t} u_1'(c_{1t}) = e^{-\gamma t} c_{1t}^{-\rho_1}, \tag{15}$$

and that a positive constant λ exists with the property that for all $0 \le t \le T$, P-a.s.,

$$\lambda = \frac{u_2'(c_{2t})}{u_1'(c_{1t})}. \tag{16}$$

For future reference we denote by ω the *(market) clearing function*

$$\omega(c) = \lambda^{-1/\rho_2} \cdot c^{\rho_1/\rho_2} + c \tag{17}$$

on the positive real line. Whatever agents' risk-preferences, the function ω is a strictly increasing, infinitely often differentiable function defined on the positive real line that maps into the positive real line with $\omega(0) = 0$ and $\lim_{x\to\infty} \omega(x) = \infty$. This implies that the function ω has a unique, infinitely often differentiable, inverse φ on the positive real line; it describes the *sharing rule* of the aggregate endowment. Equation (16) implies $c_{2t} = \lambda^{-1/\rho_2} \cdot c_{1t}^{\rho_1/\rho_2}$; the market clearing condition then reads for all $0 \leq t \leq T$:

$$Y_t = c_{1t} + c_{2t} = c_{1t} + \lambda^{-1/\rho_2} \cdot c_{1t}^{\rho_1/\rho_2} = \omega(c_{1t}), \text{ i.e. } c_{1t} = \varphi(Y_t) \ P\text{-a.s.} \tag{18}$$

We then define a function m by

$$m = \varphi^{-\rho_1}, \text{ so that } M_t = e^{-\gamma t} m(Y_t), \tag{19}$$

according to equation (15). Note that m is strictly positive and separable in time t and endowment y. Appendix B checks the technical conditions in definition 1:

Proposition 15 *m is a state-price function.*

This proposition, together with equation (19) links our pure-exchange economy to our characterization of the state-price function in the previous section.

3.2 Volatility with General CRRA Preferences

When agents have identical risk aversion parameters $\rho_1 = \rho_2 = \rho$ (homogeneous agents), equations (17, 19) read

$$\omega(c) = (1 + \lambda^{-1/\rho}) \cdot c, \varphi(y) = \frac{1}{1 + \lambda^{-1/\rho}} y, \text{ and } m(y) = (1 + \lambda^{-1/\rho})^{\rho} y^{-\rho}.$$

The sharing rule φ is linear, a well-known result for homogeneous CRRA preferences, see, e.g. Magill and Quinzii (1996). Because m is a power function of the aggregate endowment stream, we recall that our earlier analysis in the previous section showed $v_Y = 1, v_F = 0$ such that $\tilde{\sigma}_Y(t, y) = \sigma_Y, \tilde{\sigma}_F(t, y) = \sigma_F$. This proves:

Theorem 16 *If both agents have identical CRRA preferences ($\rho_1 = \rho_2$), firm price volatility and endowment price volatility are both constant across time and states.*

Finally, we look at the heterogeneous agent case ($\rho_1 \neq \rho_2$). When ρ_1, ρ_2 take any positive value, there are no closed-form solutions for the sharing rule, in general. However, the appendix proves that the converse of theorem 16 is also true and concludes:

Theorem 17 *For CRRA agents with identical time-preference parameters, endowment (firm) price volatility is state-dependent (stochastic) if and only if agents have heterogeneous risk-preferences.*

This is our main result in this paper: it relates properties of the volatility (process) to heterogeneity in agent's risk preferences.

3.3 Volatility in a Heterogeneous Agent Economy with Closed-form Sharing Rule

A closed-form sharing rule is available when the market-clearing function is quadratic[7], i.e. when $\rho_1/\rho_2 = 2$ or equivalently $\rho_1 = 2\rho_2$; throughout this subsection we adopt this assumption and vary ρ_2. Equation (17) then has a unique inverse

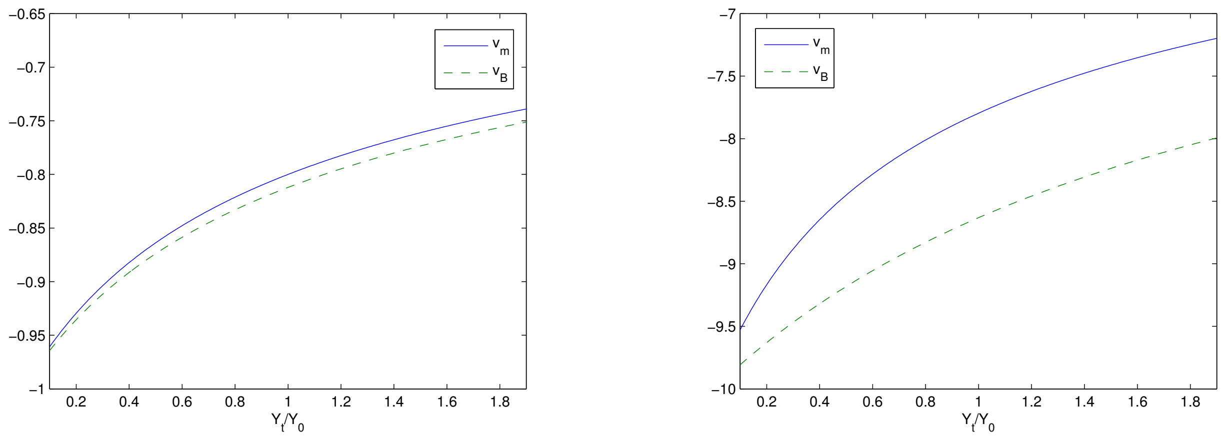
$$\varphi(y) = \frac{1}{2}\left(\lambda^{\frac{1}{2\rho_2}}\sqrt{\lambda^{\frac{1}{\rho_2}} + 4y} - \lambda^{\frac{1}{\rho_2}}\right). \tag{20}$$

This implies

$$m(y) = 2^{2\rho_2}\left(\lambda^{\frac{1}{2\rho_2}}\sqrt{\lambda^{\frac{1}{\rho_2}} + 4y} - \lambda^{\frac{1}{\rho_2}}\right)^{-2\rho_2}, \tag{21}$$

i.e. the state-price function is *not* a power function. Based on Theorems 7 and 13 we conclude that endowment (firm) price volatility is state-dependent (stochastic). Furthermore, based on equation (21) we calculate

$$v_m(y) = -\frac{4\rho_2 y}{4y + \lambda^{\frac{1}{\rho_2}} - \lambda^{\frac{1}{2\rho_2}}\sqrt{4y + \lambda^{\frac{1}{\rho_2}}}}.$$



(a) $\rho_1 = 1, \rho_2 = 1/2$

(b) $\rho_1 = 10, \rho_2 = 5$

Figure 1: Elasticities of the price system.

While we cannot calculate in closed-form the elasticity v_B we can calculate it using a numerical integration; for *illustration*we use $T-t = 10, \sigma_Y = 0.1, \sigma_F = 0.5, \kappa = 0.9$. Figures 1, 2 and 3 show the elasticities of the price system, the elasticities of the two claims and the resulting volatility, respectively. All are plotted as function of current (time t) aggregated endowment in relation to initial aggregate endowment. We compare two cases: $\rho_1 = 1, \rho_2 = 1/2$ (left-hand plots) with $\rho_1 = 10, \rho_2 = 5$ (right-hand plots). We chose $\lambda = 1.5$ in the first case and $\lambda = 20$ in the second case[8].

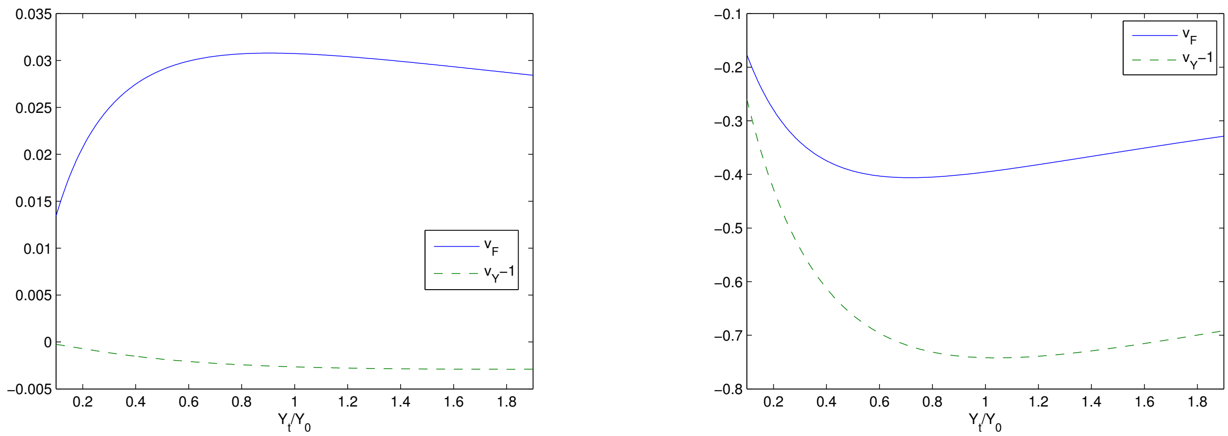
Figure 1 shows the elasticities of the price system. We see in both plots that v_B is structurally similar to v_m: both are increasing and appear concave in y. The curvature of v_m is stronger than that of v_B; this can be seen particularly well in the right-hand plot. This as expected because the expectation in the numerator and denominator of equation (4) should lead to a smoothing such that the curvature is reduced[9].

This needs to be distinguished from the elasticities with homogeneous agents ($\rho = \rho_1 = \rho_2$):

[7]Wang (1996) noted this and pointed out that closed-form solutions are available also for market clearing functions that are third and for fourth order polynomials, i.e. $\rho_1/\rho_2 = 3$ or 4. He looked at the link between the aggregate consumption stream and short-term interest rates, but was *not* interested in the volatility dynamics.

[8]The parameter choice of the underlying economic variables does not affect the plots qualitatively. The shadow price of the budget equation λ affects the location of the hump and was chosen to have it at Y_0.

[9]We do not show this here, but our numerical scheme confirms that a reduction in the endowment volatility σ_Y leads to v_B becoming more and more like v_m; at $\sigma_Y = 0$ both coincide.



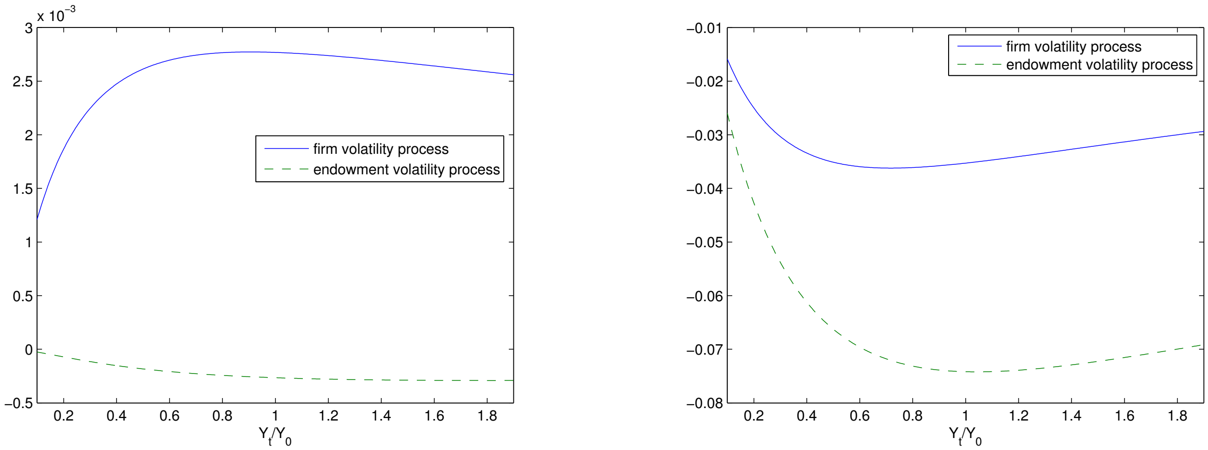
(a) $\rho_1 = 1, \rho_2 = 1/2$ (b) $\rho_1 = 10, \rho_2 = 5$

Figure 2: Price elasticities of the claims.

In the previous subsection we found that the state-price function m is then a power function of the aggregate endowment with power $-\rho$; based on the results in the previous section we then conclude that for homogeneous agents the elasticities of the price system are both constant and equal to $-\rho$.

The structural properties of the price elasticities $v_Y(y) = 1 + v_B(\psi_Y(t)y) - v_m(y)$ and $v_F(y) = 1 + v_B(\psi_F(t)y) - v_m(y)$ determine whether endowment volatility is state-dependent, respectively if firm volatility is stochastic, see Theorem 2. Note that the function v_B is evaluated here at $\psi_Y(t)y$, respectively at $\psi_F(t)y$; this means that the function that enters is shifted to the right compared to figure 1. Based on this figure it is hard to see whether the price elasticities of the claims are constant in the y.

Figure 2 presents the price elasticities of both claims. We show here $v_Y - 1$ instead of v_Y to have scales of similar sizes. We see that none of them is constant in y. This suggests that endowment volatility is state-dependent and that firm volatility is stochastic. This coincides with our analysis based on Theorem 7: volatility is stochastic because the state-price function in equation (21) is not a power function. Figure 3 presents the volatility as a function of aggregate endowment and shows the same picture.



(a) $\rho_1 = 1, \rho_2 = 1/2$ (b) $\rho_1 = 10, \rho_2 = 5$

Figure 3: Volatility as a function of endowment for both claims.

4 Conclusion

This paper studied state-price processes that can be represented as the product of a time function and a function in the aggregate endowment; we referred to the latter as the state-price function. We analyzed an economy populated by two agents with identical time-preference parameters but potentially heterogeneous CRRA preferences; aggregate endowment and firm (book) value followed a bivariate geometric Brownian motion. We found that the state-price process can be represented through a state-price function. In addition, we showed that the firm price process exhibits stochastic volatility if the state-price function is not a power function, i.e. if agents are heterogeneous; volatility is constant if the state-price function is a power function, i.e. if agents are homogeneous.

Appendix

A State-price Process

Throughout this appendix we use the random variables Z_{tT}^{Y}, Z_{tT}^{F} of equations (2, 10).

Lemma A1 *For a state-price function m, $y > 0$ and $0 < t < T$ we have*

$$\begin{aligned} E\left[m(y \cdot Z_{tT}^Y) \cdot Z_{tT}^F\right] &= \theta_F(t) E[m(\psi_F(t) y Z_{tT}^Y)] \\ \text{and } E\left[m(y \cdot Z_{tT}^Y) \cdot Z_{tT}^Y\right] &= \theta_Y(t) E[m(\psi_Y(t) y Z_{tT}^Y)], \end{aligned}$$

for suitable, infinitely often differentiable time functions θ_F, θ_Y that depend only on the distributional properties $\sigma_F, \sigma_Y, \kappa$.

Proof of Lemma A1. The random variable Z_{tT}^F fulfills the properties of a Radon-Nikodyn density and can be used to define a new probability measure Q^F. Under this new probability measure, the random variable $\ln Z_{tT}^Y$ has mean $-\sigma_Y^2(T-t) + \kappa\sigma_Y\sigma_F(T-t)$. This means that we can treat Z_{tT}^Y as $\psi_F(t) Z_{tT}^Y$ under the original measure and implies the statement for the firm variable. The statement for the endowment variable follows similarly. ■

Lemma A2 *The functions $(t, y) \mapsto E\left[m(y \cdot Z_{tT}^Y) Z_{tT}^Y\right]$ and $(t, y) \mapsto E\left[m(y \cdot Z_{tT}^Y) Z_{tT}^F\right]$ are finite, continuously differentiable in time t, continuously differentiable up to third order in y and continuously differentiable in the cross derivative of first order in y and up to third order in t. Furthermore, for all these derivatives we can exchange differentiation and expectation.*

Proof of Lemma A2. Based on Lemma A1 we can rewrite both functions in such a way, that it is sufficient to prove the statements for the functions $(t, y) \mapsto E\left[m(\psi_Y(t) y Z_{tT}^Y)\right]$ and $(t, y) \mapsto E\left[m(\psi_F(t) y Z_{tT}^Y)\right]$. However, this follows directly from definition 1. ■

Lemma A3 *Assume there are constants α, β such that the state-price function m fulfills for all $y > 0$:*

$$\frac{m''(y) y^2 + \alpha m'(y) y}{m(y)} = \beta. \tag{A2}$$

Then, m is a power function in y.

Proof of Lemma A3. We choose a parameter $\gamma = -\alpha/2$ and define a function ξ by setting $\xi(y) = m(y) y^{-\gamma}$ for $y > 0$. We then calculate $m(y) = \xi(y) y^\gamma$, $m'(y) = \xi'(y) y^\gamma + \xi(y)\gamma y^{\gamma-1}$ and

$m''(y) = \xi''(y)y^\gamma + 2\xi'(y)\gamma y^{\gamma-1} + \xi(y)\gamma(\gamma-1)y^{\gamma-2}$. Plugging this into (A2) proves that ξ fulfills for all $y > 0$:

$$\xi''(y)y^2 = \xi(y)(\beta - \gamma^2).$$

This implies that ξ is a power function and, therefore, that m is a power function. ■

Proof of Theorem 2. Equation (5) together with definition 1 implies that the endowment price function S_Y is continuously differentiable in t and twice continuously differentiable in y. This ensures sufficient differentiability to apply Itô's lemma; it follows that for a suitable drift function μ_Y the endowment price dynamics is

$$dS_{Yt} = \mu_Y(t, Y_t)S_{Yt}dt + \frac{\partial S_Y}{\partial y}\sigma_Y Y_t dW_t^Y = \mu_Y(t, Y_t)S_{Yt}dt + \tilde{\sigma}_Y(t, Y_t)S_{Yt}dW_t^Y,$$

where we set

$$\tilde{\sigma}_Y(t, y) = \frac{y}{S_Y}\frac{\partial S_Y}{\partial y}\sigma_Y = \sigma_Y \upsilon_Y.$$

This ends the proof. ■

Proof of Propositions 4. Based on Lemma A1 the endowment price function of equation (5) can be written as

$$S_Y(t, f, y) = \theta_{1Y}(t)\frac{\gamma(T)}{\gamma(t)}\frac{E[m(y\psi_Y(t)Z_{tT}^Y)]y}{m(y)}.$$

We then calculate

$$\begin{aligned}\frac{\partial S_Y}{\partial y} &= \theta_{1Y}(t)\frac{\gamma(T)}{\gamma(t)} \\ &\cdot \frac{\left(E\left[m'(y\psi_Y(t)Z_{tT}^Y)\psi_Y(t)Z_{tT}^Y\right] + E[m(y\psi_Y(t)Z_{tT}^Y)]\right)m(y) - E\left[m(y\psi_Y(t)Z_{tT}^Y)\right]ym'(y)}{m^2(y)}.\end{aligned}$$

(Lemma A1 ensures we can exchange integration and differentiation in y.) Plugging this into equation (6) and using the above characterization of S_Y proves the representation of υ_Y. ■

Proof of Propositions 5. The proof of the similar statement for the firm price elasticity (proposition 11) will be shown below; the statement for the endowment price elasticity follows analogous to that one, replacing $\kappa\sigma_F$ by σ_Y; note that the analysis for the next higher order term will then show it to be of order $\sigma_Y^2\kappa\sigma_F = \sigma_Y^{5/2}$. ■

Proof of Theorem 6. The proof of the similar statement for the firm price volatility (Theorem 12) will be shown below; the statement for the endowment price volatility follows analogous to that one, setting $\kappa = 1$ and replacing σ_F by σ_Y. (Based on the structure of υ_Y and υ_F in propositions 4 and 10 this will become clear from the proof of Theorem 12 below.) ■

Proof of Theorem 8. Equation (11) implies that the firm price function S_F is twice continuously differentiable in f; definition 1 implies that it is continuously differentiable in t and twice continuously differentiable in y. This ensures sufficient differentiability to apply Itô's lemma; it follows that for a suitable drift function μ_F the firm price dynamics is

$$dS_{Ft} = \mu_F(t, F_t, Y_t)S_{Ft}dt + \frac{\partial S_F}{\partial f}\sigma_F F_t dW_t^F + \frac{\partial S_F}{\partial y}\sigma_Y Y_t dW_t^Y.$$

We define the process $\tilde{W}^F$ and a function $\tilde{\sigma}_F$ by setting

$$d\tilde{W}_t^F = \frac{1}{\tilde{\sigma}_F \cdot S_{Ft}}\left(\frac{\partial S_F}{\partial f}\sigma_F F_t dW_t^F + \frac{\partial S_F}{\partial y}\sigma_Y Y_t dW_t^Y\right) \text{ and} \tag{A3}$$

$$\tilde{\sigma}_F(t, f, y) = \frac{1}{S_F}\sqrt{\left(\frac{\partial S_F}{\partial f}\sigma_F f\right)^2 + 2\kappa\frac{\partial S_F}{\partial f}\sigma_F f\frac{\partial S_F}{\partial y}\sigma_Y y + \left(\frac{\partial S_F}{\partial y}\sigma_Y y\right)^2}. \tag{A4}$$

It is straightforward to check that the process $\tilde{W}^F$ has independent increments as well as that these are conditionally normal distributed with a mean of zero and a variance equal to the time increment; therefore, $\tilde{W}^F$ describes a standard Wiener process. Based on equation (5) we calculate $\frac{\partial S_F}{\partial f} = \frac{S_F}{f}$; using the definition of the elasticity υ_F, see equation (12) proves the statements. ■

Proof of Proposition 10. Lemma A1 proves that the firm price function of equation (11) can be written as

$$S_F(t, f, y) = \theta_{1F}(t) f \frac{\gamma(T)}{\gamma(t)} \frac{E[m(y\psi_F(t)Z_{tT}^Y)]}{m(y)}.$$

Based on this representation for every y we calculate

$$\frac{\partial S_F}{\partial y} = \theta_{1F}(t) f \frac{\gamma(T)}{\gamma(t)} \frac{E\left[m'(y\psi_F(t)Z_{tT}^Y)\psi_F(t)Z_{tT}^Y\right] m(y) - E\left[m(y\psi_F(t)Z_{tT}^Y)\right] m'(y)}{m^2(y)}.$$

Note that we can exchange integration and differentiation in y because of Lemma A1. Plugging this into equation (12) and using the above characterization of S_F proves the representation of υ_F. ■

Proof of Proposition 11. We define a random variable

$$\phi(\sigma_Y, y) = y \exp\left(\kappa\sigma_F\sigma_Y(T-t) - \frac{\sigma_Y^2}{2}(T-t) + \sigma_Y\sqrt{T-t}U\right).$$

where U is the standard normal random variable introduced in equation (2). We calculate

$$\begin{aligned}\frac{\partial\phi}{\partial\sigma_Y} &= \phi\cdot\left(\kappa\sigma_F(T-t)-\sigma_Y(T-t)+\sqrt{T-t}U\right),\\ \text{such that } \phi(\sigma_Y=0) &= y,\ \frac{\partial\phi}{\partial\sigma}(\sigma_Y=0)=y\left(\kappa\sigma_F(T-t)+\sqrt{T-t}U\right).\end{aligned}$$

We then define functions ξ_1,ξ_2 and ζ_1,ζ_2 by

$$\begin{aligned}\xi_1(\sigma_Y,y) &= m'\left(\phi(\sigma_Y)\right)\phi(\sigma_Y),\xi_2(\sigma_Y,y)=m\left(\phi(\sigma_Y)\right),\\ \text{and } \zeta_1(\sigma_Y,y) &= E[\xi_1(\sigma_Y,y)],\zeta_2(\sigma_Y,y)=E[\xi_2(\sigma_Y,y)].\end{aligned}$$

Our idea is to study the asymptotic behavior of $v_B(t,\psi_F(t)y)$ in the parameter σ_Y near 0. Note that $y\psi_F(t)Z^Y_{tT}=\phi$, such that $v_B(t,\psi_F(t)y)=\zeta_1(\sigma_Y,y)/\zeta_1(\sigma_Y,y)$. The properties of the functions ζ_1,ζ_2 that are stated in Lemma A2 w.r.t. to time t derivatives clearly carry over to σ_Y derivatives. Therefore, we can interchange first and second order σ_Y differentiation in ζ_1, ζ_2 and expectation, i.e. for $i=1,2$:

$$\frac{\partial^i\zeta_1}{\partial\sigma_Y^i}(\sigma,y)=E\left[\frac{\partial^i\xi_1}{\partial\sigma_Y^i}(\sigma,y)\right],\frac{\partial^i\zeta_2}{\partial\sigma_Y^i}(\sigma,y)=E\left[\frac{\partial^i\xi_2}{\partial\sigma_Y^i}(\sigma,y)\right].$$

Furthermore, we can apply Taylor's Theorem; it tells us that

$$\zeta_1(\sigma_Y,y) = \zeta_1(0,y)+\frac{\partial\zeta_1}{\partial\sigma_Y}(0,y)\sigma_Y,\quad \zeta_2(\sigma_Y,y)=\zeta_2(0,y)+\frac{\partial\zeta_2}{\partial\sigma_Y}(0,y)\sigma_Y,$$

both up to terms of order higher than 1 in σ_Y. This expansion implies

$$\frac{\zeta_1(\sigma_Y,y)}{\zeta_2(\sigma_Y,y)} = \frac{\zeta_1(0,y)}{\zeta_2(0,y)}+\left(\frac{\frac{\partial\zeta_1}{\partial\sigma_Y}(0,y)}{\zeta_2(0,y)}-\frac{\zeta_1(0,y)\frac{\partial\zeta_2}{\partial\sigma_Y}(0,y)}{\zeta_2^2(0,y)}\right)\sigma_Y, \tag{A5}$$

up to terms of order higher than 1 in σ_Y. For further analysis we then calculate

$$\frac{\partial\zeta_1}{\partial\sigma_Y} = E\left[m''\left(\phi\right)\phi\frac{\partial\phi}{\partial\sigma_Y}+m'(\phi)\frac{\partial\phi}{\partial\sigma_Y}\right],\ \text{and}\ \frac{\partial\zeta_2}{\partial\sigma_Y}=E\left[m'(\phi)\frac{\partial\phi}{\partial\sigma_Y}\right].$$

At $\sigma_Y=0$ we have

$$\begin{aligned}\zeta_1(0,y) &= m'(y)y,\frac{\partial\zeta_1}{\partial\sigma}(0,y)=\left(m''(y)y^2+m'(y)y\right)\kappa\sigma_F(T-t),\\ \zeta_2(0,y) &= m(y),\frac{\partial\zeta_2}{\partial\sigma}(0,y)=m'(y)y\kappa\sigma_F(T-t).\end{aligned}$$

Using equation (A5) this shows

$$\begin{aligned}\frac{\zeta_1(\sigma_Y,y)}{\zeta_2(\sigma_Y,y)} &= \frac{m'(y)y}{m(y)} + \left(\frac{m''(y)y^2+m'(y)y}{m(y)} - \frac{(m'(y)y)^2}{m^2(y)}\right)\kappa\sigma_F\sigma_Y(T-t),\\ &= \frac{m'(y)y}{m(y)} + \frac{\partial}{\partial y}\left(\frac{m'(y)y}{m(y)}\right) y\kappa\sigma_F\sigma_Y(T-t),\end{aligned}$$

which implies the statement. ■

Proof of Theorem 12. In this proof let us assume that firm volatility is not stochastic. For our analysis we define a random variable

$$\phi(\sigma,y,U) = y\exp\left(\kappa\frac{\sigma_F}{\sigma_Y}\sigma^2 - \frac{\sigma^2}{2} + \sigma U\right),$$

where U is the standard normal random variable introduced in equation (2). We calculate

$$\frac{\partial\phi}{\partial\sigma} = \phi\cdot\left(2\kappa\frac{\sigma_F}{\sigma_Y}\sigma - \sigma + U\right),\ \frac{\partial^2\phi}{\partial\sigma^2} = \phi\cdot\left(2\kappa\frac{\sigma_F}{\sigma_Y}\sigma - \sigma + U\right)^2 + \phi\cdot\left(2\kappa\frac{\sigma_F}{\sigma_Y} - 1\right),$$

such that

$$\phi(\sigma=0) = y,\ \frac{\partial\phi}{\partial\sigma}(\sigma=0) = yU,\ \text{and}\ \frac{\partial^2\phi}{\partial\sigma^2}(\sigma=0) = y\left(U^2 + 2\kappa\frac{\sigma_F}{\sigma_Y} - 1\right).$$

We then define random variables ξ_1,ξ_2 and functions ζ_1,ζ_2 by

$$\xi_1(\sigma,y) = m'(\phi(\sigma))\,\phi(\sigma), \xi_2(\sigma,y) = m(\phi(\sigma)),\ \text{and}\ \zeta_1(\sigma,y) = E[\xi_1(\sigma,y)], \zeta_2(\sigma,y) = E[\xi_2(\sigma,y)].$$

Our idea is to study the asymptotic behavior of $v_B(t,\psi_F(t)y)$ in the parameter t near T; instead of doing this, we find it more convenient to change variables to $\sigma = \sigma_Y\sqrt{T-t}$ and study σ near 0. Throughout, we denote $o(\cdot)$ the Landau symbol and use $o(\sigma^2)$ to express terms that are of order higher than σ^2, i.e. when divided by σ^2 those terms vanish in the limit of σ to zero. Note that our change-in-variable means that $\psi_F(t) = \exp(\kappa\sigma_F\sigma_Y(T-t)) = \exp(\kappa\frac{\sigma_F}{\sigma_Y}\sigma^2)$ and $y\psi_F(t)Z^Y_{tT} = \phi$, such that $v_B(t,\psi_F(t)y) = \zeta_1(\sigma,y)/\zeta_1(\sigma,y)$. The properties of the functions ζ_1,ζ_2 that are stated in Lemma A2 w.r.t. to time t derivatives clearly carry over to σ derivatives. Therefore we can apply Taylor's Theorem; it tells us that

$$\begin{aligned}\zeta_1(\sigma,y) &= \zeta_1(0,y) + \frac{\partial\zeta_1}{\partial\sigma}(0,y)\sigma + \frac{\partial^2\zeta_1}{\partial\sigma^2}(0,y)\sigma^2 + o(\sigma^2),\\ \zeta_2(\sigma,y) &= \zeta_2(0,y) + \frac{\partial\zeta_2}{\partial\sigma}(0,y)\sigma + \frac{\partial^2\zeta_2}{\partial\sigma^2}(0,y)\sigma^2 + o(\sigma^2),\end{aligned}$$

where the upper bound $o(\sigma^2)$ holds uniformly in y on any compact, strictly positive interval. This expansion implies

$$\begin{aligned}
&\frac{\zeta_1(\sigma,y)}{\zeta_2(\sigma,y)} \qquad\qquad (A6)\\
=\ & \frac{\zeta_1(0,y)}{\zeta_2(0,y)} + \left(\frac{\frac{\partial\zeta_1}{\partial\sigma}(0,y)}{\zeta_2(0,y)} - \frac{\zeta_1(0,y)\frac{\partial\zeta_2}{\partial\sigma}(0,y)}{\zeta_2^2(0,y)}\right)\sigma\\
+\ & \left(\frac{\frac{\partial^2\zeta_1}{\partial\sigma^2}(0,y)}{\zeta_2(0,y)} - \frac{\zeta_1(0,y)\frac{\partial^2\zeta_2}{\partial\sigma^2}(0,y)+\frac{\partial\zeta_1}{\partial\sigma}(0,y)\frac{\partial\zeta_2}{\partial\sigma}(0,y)}{\zeta_2^2(0,y)} + 2\frac{\zeta_1(0,y)\left(\frac{\partial\zeta_2}{\partial\sigma}(0,y)\right)^2}{\zeta_2^3(0,y)}\right)\sigma^2 + o(\sigma^2),
\end{aligned}$$

uniformly in y on any compact, strictly positive interval $[\nu_l,\nu_u]$ ($\nu_l < \nu_u$). Lemma A2 implies that we can interchange first and second order σ differentiation in ζ_1, ζ_2 and expectation, i.e. for $i=1,2$:

$$\frac{\partial^i\zeta_1}{\partial\sigma^i}(\sigma,y) = E\left[\frac{\partial^i\xi_1}{\partial\sigma^i}(\sigma,y)\right], \frac{\partial^i\zeta_2}{\partial\sigma^i}(\sigma,y) = E\left[\frac{\partial^i\xi_2}{\partial\sigma^i}(\sigma,y)\right].$$

For further analysis we then calculate

$$\begin{aligned}
\frac{\partial\zeta_1}{\partial\sigma} &= E\Big[m''(\phi)\,\phi\frac{\partial\phi}{\partial\sigma} + m'(\phi)\frac{\partial\phi}{\partial\sigma}\Big],\\
\frac{\partial^2\zeta_1}{\partial\sigma^2} &= E\Big[m'''(\phi)\,\phi\left(\frac{\partial\phi}{\partial\sigma}\right)^2 + 2m''(\phi)\left(\frac{\partial\phi}{\partial\sigma}\right)^2 + m''(\phi)\phi\frac{\partial^2\phi}{\partial\sigma^2} + m'(\phi)\frac{\partial^2\phi}{\partial\sigma^2}\Big],
\end{aligned}$$

and

$$\frac{\partial\zeta_2}{\partial\sigma} = E\left[m'(\phi)\frac{\partial\phi}{\partial\sigma}\right],\ \frac{\partial^2\zeta_2}{\partial\sigma^2} = E\left[m''(\phi)\left(\frac{\partial\phi}{\partial\sigma}\right)^2 + m'(\phi)\frac{\partial^2\phi}{\partial\sigma^2}\right].$$

At $\sigma = 0$ we have

$$\begin{aligned}
\zeta_1(0,y) &= m'(y)y, \frac{\partial\zeta_1}{\partial\sigma}(0,y) = 0, \frac{\partial^2\zeta_1}{\partial\sigma^2}(0,y) = m'''(y)y^3 + m''(y)y^2\left(2+2\kappa\frac{\sigma_F}{\sigma_Y}\right) + m'(y)y2\kappa\frac{\sigma_F}{\sigma_Y},\\
\zeta_2(0,y) &= m(y), \frac{\partial\zeta_2}{\partial\sigma}(0,y) = 0, \frac{\partial^2\zeta_2}{\partial\sigma^2}(0,y) = m''(y)y^2 + m'(y)y2\kappa\frac{\sigma_F}{\sigma_Y}.
\end{aligned}$$

This implies using equation (A6) that

$$\begin{aligned}
&\frac{\zeta_1(\sigma,y)}{\zeta_2(\sigma,y)} - \frac{m'(y)y}{m(y)} \qquad\qquad (A7)\\
=\ & \left(\frac{m'''(y)y^3 + 2m''(y)y^2\left(1+\kappa\frac{\sigma_F}{\sigma_Y}\right) + m'(y)y2\kappa\frac{\sigma_F}{\sigma_Y}}{m(y)} - \frac{m''(y)y^2 + m'(y)y2\kappa\frac{\sigma_F}{\sigma_Y}}{m^2(y)}m'(y)y\right)\sigma^2,\\
=\ & \frac{\partial}{\partial y}\left(\frac{m''(y)y^2 + m'(y)y2\kappa\frac{\sigma_F}{\sigma_Y}}{m(y)}\right)\cdot y\cdot\sigma^2,
\end{aligned}$$

up to $o(\sigma^2)$, uniformly in y on any compact, strictly positive interval $[\nu_l, \nu_u]$ ($\nu_l < \nu_u$). By assumption, volatility is not stochastic; according to Proposition 4 this means that

$$\frac{\zeta_1(\sigma, y)}{\zeta_2(\sigma, y)} - \frac{m'(y)y}{m(y)}$$

does not depend on y for all positive σ, in particular for all positive σ near zero. This implies that the second-order term in σ^2 of equation (A7) does not depend on y, i.e. that on any compact, strictly positive interval $[\nu_l, \nu_u]$ ($\nu_l < \nu_u$) the function

$$\frac{m''(y)y^2 + m'(y)y2\kappa\frac{\sigma_F}{\sigma_Y}}{m(y)}$$

must be equal to a constant in y. According to Lemma A3, m must be a power function. This ends our proof. ■

B Preferences

Throughout this appendix we study the properties of the function m defined in equation (19). We use the random variables Z_{tT}^Y, Z_{tT}^F defined in equations (2, 10).

Lemma B1 *The function m is infinitely often differentiable. We define the function ξ on the positive real line by setting $\xi(y) = \varphi(y) + \lambda^{-1/\rho_2}\frac{\rho_1}{\rho_2}(y - \varphi(y))$. For suitable constants α_{ni} the first four derivatives ($n = 1, 2, 3, 4$) of the state-price function w.r.t. y are given by:*

$$m^{(n)}(c) \quad = \quad m(y)\sum_{i=0}^{n-1}\alpha_{ni}\frac{(y-\varphi(y))^i}{\xi^{n+i}(y)}.$$

Proof of Lemma B1. Because the ω function is infinitely often differentiable, its inverse φ is also infinitely often differentiable. Therefore, the state-price function m is infinitely often differentiable.

Throughout this proof we use the equalities $\varphi(y) = c, \omega(c) = y$ to simplify the presentation. The definition of the market clearing function ω and its inverse in equation (18) implies $y = \varphi(y) + \lambda^{-1/\rho_2}(\varphi(y))^{\rho_1/\rho_2}$, i.e. $\lambda^{1/\rho_2}(y - \varphi(y)) = (\varphi(y))^{\rho_1/\rho_2} = c^{\rho_1/\rho_2}$. Using this we calculate inductively the first four derivatives of the market clearing function ω ($n = 2, 3, 4$):

$$\begin{aligned}\omega'(c) &= 1 + \lambda^{-1/\rho_2} \frac{\rho_1}{\rho_2} c^{\frac{\rho_1}{\rho_2}-1} = \frac{\xi(y)}{\varphi(y)}, \\ \text{and } \omega^{(n)}(c) &= \beta_n c^{\frac{\rho_1}{\rho_2}-2} = \beta_n \lambda^{1/\rho_2} \frac{y - \varphi(y)}{(\varphi(y))^2},\end{aligned}$$

where β_n are suitable constants.

Based on Faà di Bruno's formula, i.e. applying the chain rule with implicit functions, we find successively the following expressions for derivatives of the sharing rule:

$$\begin{aligned}\varphi'(y) &= 1/\omega'(\varphi(y)), \varphi''(y) = -\frac{\omega''(\varphi(y)}{(\omega'(\varphi(y)))^3}, \varphi'''(y) = -\frac{\omega'''(\varphi(y)}{(\omega'(\varphi(y)))^4} + 3\frac{(\omega''(\varphi(y)))^2}{(\omega'(\varphi(y)))^5}, \\ \varphi''''(y) &= -\frac{\omega''''(\varphi(y)}{(\omega'(\varphi(y)))^5} + 10\frac{\omega''(\varphi(y))\omega'''(\varphi(y))}{(\omega'(\varphi(y)))^6} - 15\frac{(\omega''(\varphi(y)))^3}{(\omega'(\varphi(y)))^7}.\end{aligned}$$

This allows us to write the first four derivatives of the sharing function using suitable constants γ_{ni} ($i = 1, \ldots, n$, $n = 2, 3, 4$):

$$\varphi'(y) = \gamma_{n1} \frac{\varphi(y)}{\xi(y)}, \varphi^{(n)}(y) = \varphi(y) \sum_{i=1}^{n-1} \alpha_{ni} \frac{(y - \varphi(y))^i}{\xi^{n+i}(y)}.$$

We then write the derivatives of the state-price function $m = \varphi^{-\rho_1}$ using suitable constants δ_{ni}:

$$\begin{aligned}m' &= \delta_{11} \varphi^{-\rho_1-1} \varphi', m'' = \delta_{21} \varphi^{-\rho_1-2} (\varphi')^2 + \delta_{22} \varphi'', \\ m''' &= \delta_{31} \varphi^{-\rho_1-3} (\varphi')^3 + \delta_{32} \varphi^{-\rho_1-2} \varphi' \varphi'' + \delta_{33} \varphi^{-\rho_1-1} \varphi''' \\ m^{(4)} &= \delta_{41} \varphi^{-\rho_1-4} (\varphi')^4 + \delta_{42} \varphi^{-\rho_1-3} (\varphi')^2 \varphi'' + \delta_{43} \varphi^{-\rho_1-2} (\varphi'')^2 + \delta_{44} \varphi^{-\rho_1-2} \varphi' \varphi''' + \delta_{45} \varphi^{-\rho_1-1} \varphi^{(4)}.\end{aligned}$$

We replace the φ derivatives and use $m = \varphi^{-\rho_1}$ to get the stated result. ■

Lemma B2 *For any time $0 \le t < T$, $y > 0$, $E[m(y \cdot Z_{tT}^Y)]$ and $E[m^2(y \cdot Z_{tT}^Y)]$ are finite.*

Proof of Lemma B2. If both agents have the same risk-preferences, then $\varphi(y) = y/2$, $m(y \cdot Z_{tT}^Y) = y^{-\rho_1}(Z_{tT}^Y)^{-\rho_1}$ and $m^2(y \cdot Z_{tT}^Y) = y^{-2\rho_1}(Z_{tT}^Y)^{-2\rho_1}$. The stated results follow because any integer power of a lognormal random variable has finite expectation.

In the remainder of this proof we assume that agents are heterogeneous and, without loss of generality, that agents are ordered such that the first agent is more risk-averse than the second, i.e. $\rho_1 > \rho_2$.

We note first that when c tends to 0, the market clearing function $\omega(c)$ behaves asymptotically like the identity function; therefore, when $\tilde{y}$ tends to 0, its inverse φ also behaves asymptotically like the identity function. This implies that there is a positive constant α and a $\tilde{y}_0$, such that, for all $0 < \tilde{y} < \tilde{y}_0$, $\varphi(\tilde{y}) > \alpha\tilde{y}$ and so $m(\tilde{y}) < \alpha^{-\rho_1}\tilde{y}^{-\rho_1}$.

In addition, we note that φ is an increasing function which tends to infinity as $\tilde{y}$ tends to infinity. Therefore, there is a $\tilde{y}_1 > 0$ such that, for all $\tilde{y} > \tilde{y}_1$, $\varphi(\tilde{y}) > 1$ and so $m(\tilde{y}) < 1$.

Finally, we note that on the compact interval $\tilde{y}_0 \leq \tilde{y} \leq \tilde{y}_1$ the state-price function $m(\tilde{y})$ is continuous. This implies there exists an upper bound $\beta < \infty$ on that interval.

Because on these observations and $m > 0$, we conclude that

$$\begin{aligned} 0 &\leq E[m(y \cdot Z^Y_{tT})] \\ &= E[m(y \cdot Z^Y_{tT})1_{0<y \cdot Z^Y_{tT}<\tilde{y}_0}] + E[m(y \cdot Z^Y_{tT})1_{\tilde{y}_0 \leq y \cdot Z^Y_{tT} \leq \tilde{y}_1}] + E[m(y \cdot Z^Y_{tT})1_{\tilde{y}_1 < y \cdot Z^Y_{tT}}] \\ &\leq E[\alpha^{-\rho_1}(y \cdot Z^Y_{tT})^{-\rho_1}] + \beta + 1. \end{aligned}$$

The expectation term in the last line is finite, because the expectation of any integer power of a lognormal random variable is finite. Similarly, we can prove that $E[m^2(y \cdot Z^Y_{tT})]$ is finite. ■

Lemma B3 *Property 2 of definition 1 holds.*

Proof of Lemma B3. If both agents have the same risk-preferences, then $\varphi(y) = y/2$, $m(y \cdot Z^Y_{tT}) = y^{-\rho_1}(Z^Y_{tT})^{-\rho_1}$, such that: $E[m(y \cdot Z^Y_{tT})] = y^{-\rho_1}E[(Z^Y_{tT})^{-\rho_1}]$. The expectation is a well-known function that it is infinitely often differentiable in time t, in y and all cross-derivatives. The statement then follows directly.

In the remainder of this proof we assume that agents are heterogeneous and, without loss of generality, that agents are ordered such that the first agent is more risk-averse than the second, i.e. $\rho_1 > \rho_2$. We use the function ξ of Lemma B1. If $\lambda^{-1/\rho_2}\rho_1/\rho_2 \leq 1$ then we find that $\xi(y) \leq \varphi(y) + (y - \varphi(y)) = y$; if $\lambda^{-1/\rho_2}\rho_1/\rho_2 \geq 1$ then we find that $\xi(y) \leq \lambda^{-1/\rho_2}\rho_1/\rho_2\varphi(y) +$

$\lambda^{-1/\rho_2}\rho_1/\rho_2 \geq 1(y-\varphi(y)) = \lambda^{-1/\rho_2}\rho_1/\rho_2 y$. Overall, we have $\xi(y) \leq \max\{\lambda^{-1/\rho_2}\rho_1/\rho_2, 1\}y$. Furthermore, we know that the sharing rule is non-negative, i.e. $y - \varphi(y) < y$ for all $y > 0$. These two observations imply that for any non-negative integers i, j and all $y > 0$:

$$0 \leq m(y)\frac{(y-\varphi(y))^i}{(\xi(y))^j} \leq m(y)y^{i-j}. \tag{B1}$$

Based on equation (B1) we find for all non-negative integers i, j, k, l:

$$E\left[m(y \cdot Z_{tT}^Y)\frac{(y \cdot Z_{tT}^Y - \varphi(y \cdot Z_{tT}^Y))^i}{(\xi(y \cdot Z_{tT}^Y))^j}(Z_{tT}^Y)^k\left(-\frac{\sigma_Y^2}{2}t - \frac{\sigma_Y}{\sqrt{T-t}}U\right)^l\right] \tag{B2}$$

$$\leq E\left[m(y \cdot Z_{tT}^Y) \cdot (y \cdot Z_{tT}^Y)^{i-j+k}\left(-\frac{\sigma_Y^2}{2}t - \frac{\sigma_Y}{\sqrt{T-t}}U\right)^l\right]$$

$$\leq \left(E\left[m^2(y \cdot Z_{tT}^Y)\right]\right)^{1/2}\left(E\left[(y \cdot Z_{tT}^Y)^{2(i-j+k)}\left(-\frac{\sigma_Y^2}{2}t - \frac{\sigma_Y}{\sqrt{T-t}}U\right)^{2l}\right]\right)^{1/2}.$$

$$\leq \left(E\left[m^2(y \cdot Z_{tT}^Y)\right]\right)^{1/2}\left(E\left[(y \cdot Z_{tT}^Y)^{4(i-j+k)}\right]\right)^{1/4}\left(E\left[\left(-\frac{\sigma_Y^2}{2}t - \frac{\sigma_Y}{\sqrt{T-t}}U\right)^{4l}\right]\right)^{1/4} \tag{B3}$$

Lemma B2 shows that the expectation term in the first product term is finite. The other two product terms are finite because expectations of all powers exist for normal and lognormal distributed random variables. Note that the term inside the expectation of equation (B2) is positive; therefore its absolute value is Lebesgue-integrable. We calculate

$$\frac{\partial\left(m(yZ_{tT}^Y)\right)}{\partial t} = m'(yZ_{tT}^Y) \cdot Z_{tT}^Y \cdot \left(-\frac{\sigma_Y^2}{2}t - \frac{\sigma_Y}{\sqrt{T-t}}U\right). \tag{B4}$$

By Lemma B1 the first four derivatives of the state-price function can be bounded by linear combinations of terms on the right-hand side of equation (B1). Therefore, the first four derivatives of the state-price function can be bounded by linear combinations of terms on the right-hand side of equation (B1). The upper bound of equation (B3) then shows that there is a Lebesgue-integrable random variable that dominates these. The dominated convergence Theorem together with Fubini's Theorem shows that we can exchange integration and differentiation for the stated derivatives. The differentiability statements follow from there. ■

Proof of Proposition 15. The function m is strictly positive because the function φ is strictly positive. By its definition in equation (19) it is separable in time t and endowment y.

Lemma B1 tells us that m is four times continuously differentiable, i.e. it fulfills property 1. Property 2 has been shown in Lemma B3. ■

Proof of Theorem 17. We only prove that the converse of theorem 16 is true, i.e. we prove that if endowment (firm) volatility is state-independent (not stochastic), then both agents have identical risk-aversion parameters. The result follows directly from this.

For this, we assume that endowment (firm) volatility is state-independent (not stochastic). Theorem 6 implies that there are constants α, β such that the state-price function is $m(y) = \alpha y^{\beta}$. Also, we know that $m(y) = (\varphi(y))^{-\rho_1}$. Therefore,

$$\alpha^{-1/\rho_1} y^{-\beta/\rho_1} = \varphi(y), \text{ and } \lambda^{-1/\rho_2} \alpha^{-1/\rho_2} y^{-\beta/\rho_2} = \lambda^{-1/\rho_2} (\varphi(y))^{\rho_1/\rho_2},$$

which implies

$$y = \omega(\varphi(y)) = \varphi(y) + \lambda^{-1/\rho_2} (\varphi(y))^{\rho_1/\rho_2} = \alpha^{-1/\rho_1} y^{-\beta/\rho_1} + \lambda^{-1/\rho_2} \alpha^{-1/\rho_2} y^{-\beta/\rho_2}.$$

However this requires $\rho_1 = \rho_2$, i.e. agents are homogeneous. ■

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