Graded Algebras with Nil Neutral Component

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Abstract

Let $\mathcal{F}$ be an arbitrary field, G be an arbitrary group and $\mathcal{F}\langle X^G\rangle$ be the free associative algebra over $\mathcal{F}$ generated by a countable infinite set $X^G = \bigcup_{g\in G} X_g$ where $X_g = \{x_1^{(g)}, x_2^{(g)}, \dots\}$ and $g \in G$, $X_g \cap X_h = \varnothing$ if $g \neq h$. The indeterminates of X_g are said to be homogeneous of degree g. Given a monomial $m = x_{i_1}^{(g_1)} x_{i_2}^{(g_2)} \cdots x_{i_s}^{(g_s)} \in \mathcal{F}\langle X^G\rangle$, the *homogeneous degree* of m, denoted by $\deg(m)$, is defined by $g_1g_2\cdots g_s$. Therefore, it is natural to write $\mathcal{F}\langle X^G\rangle = \bigoplus_{g\in G}\mathcal{F}_g$, where $\mathcal{F}_g$ is the subspace of the algebra $\mathcal{F}\langle X^G\rangle$ generated by all the monomials having homogeneous degree g. It is easy to check that $\mathcal{F}_g\mathcal{F}_h \subseteq \mathcal{F}_{gh}$ for all $g, h \in G$. The above decomposition into direct sum makes $\mathcal{F}\langle X^G\rangle$ a G-grading algebra. Hence, $\mathcal{F}\langle X^G\rangle$ is the free G-graded associative algebra generated by the sets X_g, $g \in G$.

Now, let $\mathcal{A}$ be an algebra over $\mathcal{F}$ with a G-grading Γ on $\mathcal{A}$, i.e., $\Gamma : \mathcal{A} = \bigoplus_{g\in G}\mathcal{A}_g$ with $\mathcal{A}_g$ subspace of $\mathcal{A}$ and $\mathcal{A}_g\mathcal{A}_h \subseteq \mathcal{A}_{gh}$ for all $g, h \in G$. We say that $\mathcal{A}$ is an associative GPI-algebra over $\mathcal{F}$ (or simply GPI-algebra) if there exists a nonzero $f = f(x_1^{(g_1)}, x_2^{(g_2)}, \dots, x_n^{(g_n)}) \in \mathcal{F}\langle X^G\rangle$ such that $f(a_1, a_2, \dots, a_n) = 0$ for all $a_1 \in \mathcal{A}_{g_1}, a_2 \in \mathcal{A}_{g_2}, \dots, a_n \in \mathcal{A}_{g_n}$. In this case, we write $f \equiv_G 0$ in $\mathcal{A}$ and we say that f is a G-graded polynomial identity of $\mathcal{A}$. We denote by $T^G(\mathcal{A})$ the set of all G-graded identities of $\mathcal{A}$. In other words, $T^G(\mathcal{A}) = \{f \in \mathcal{F}\langle X^G\rangle : f \equiv_G 0 \text{ in } \mathcal{A}\}$. It is easy to check that $T^G(\mathcal{A})$ is a G-graded ideal of $\mathcal{F}\langle X^G\rangle$ invariant by G-endomorphisms of $\mathcal{F}\langle X^G\rangle$, called GT-ideal of G-graded identities of $\mathcal{A}$. Consider $\mathrm{Supp}(\Gamma) = \{g_1, \dots, g_d\}$ finite, where $\mathrm{Supp}(\Gamma) = \{g \in G : \mathcal{A}_g \neq 0\}$. For each $i = 1, 2, \dots,$ put $x_i = \sum_{j=1}^d x_i^{(g_j)}$. Let $\mathcal{F}\langle X\rangle$ be the free associative algebra generated by set $X = \{x_1, x_2, \dots\}$. Consider the set $T(\mathcal{A}) \subseteq \mathcal{F}\langle X\rangle$ of polynomial (ordinary) identities of $\mathcal{A}$, i.e., $T(\mathcal{A}) = \{f \in \mathcal{F}\langle X\rangle : f \equiv 0 \text{ in } \mathcal{A}\}$. We have that $T(\mathcal{A})$ is an ideal of $\mathcal{F}\langle X\rangle$ invariant by its endomorphisms, called T-ideal of identities of $\mathcal{A}$. Note that $T(\mathcal{A}) \subseteq T^G(\mathcal{A})$.

The central problem in the study of graded algebras is to obtain non-graded (ordinary) properties from the analysis of gradings assumed for a given algebra and vice versa. In this sense, given a graded algebra, we try to determine relationships between their graded identities and their non-graded identities. Let $\mathcal{A} = \bigoplus_{g\in G}\mathcal{A}_g$ be a G-graded algebra, G is a finite group with neutral element e. In [2], Bergen and Cohen showed that if $\mathcal{A}_e$ is a PI-algebra, then $\mathcal{A}$ is also a PI-algebra. In this work, it is not exhibited, in the general case, a bound for the degree of the polynomial identity satisfied by $\mathcal{A}$. On the other hand, in [1], Bahturin, Giambruno and Riley produced the same results, but, in addition, their results produced a bound for the minimal degree of the polynomial identity satisfied by $\mathcal{A}$. Namely, the following result was shown:

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Theorem (Theorem 5.3, [1])**.** *Let $\mathcal{F}$ be an arbitrary field and G be a finite group. Suppose that $\mathcal{A}$ is a G-graded associative $\mathcal{F}$-algebra such that $\mathcal{A}_e$ satisfies a polynomial identity of degree d. Then $\mathcal{A}$ satisfies a polynomial identity of degree n, where n is any integer satisfying the inequality*

$$\frac{|G|^n(|G|d-1)^{2n}}{(|G|d-1)!} < n!.$$

In particular, if n is the least integer such that $e|G|(|G|d-1)^2 \leq n$, then $\mathcal{A}$ satisfies a polynomial identity of degree n, where e is the base of the natural logarithm.

In this work, we have analyzed a concrete case of the statements in the previous theorem. We have studied an important class of algebras: Nilpotent Algebras. Therefore, our goal is to present some results that are direct implications of the case "$\mathcal{A}_e$ is Nilpotent" or "$\mathcal{A}_e$ is Nil". In this sense, we have given some upper bounds for $\mathrm{nd}(\mathcal{A})$, nilpotency index of $\mathcal{A}$.

In [6] and [4], Nagata and Higman, respectively, proved that, under suitable conditions, any nil algebra is also nilpotent algebra. First, in [6], Nagata proved the validity of the result over a field of characteristic zero. Posteriorly, in [4], Higman established the result in the general case. In this way, we have some natural questions: how to characterize a G-graded algebra whose neutral component is Nil/Nilpotent? Does Nil neutral Component imply Nilpotent algebra? If so, what are the possible limits for the nilpotency index? In what follows, we discuss these issues in detail.

Let $\mathcal{A}$ be a Nilpotent algebra. We define the *Nilpotency index* of $\mathcal{A}$, denoted by $\mathrm{nd}(\mathcal{A})$, as the smallest number $d \in \mathbb{N}$ such that $\mathcal{A}^d = \{0\}$. Analogously, if $\mathcal{A}$ is a Nil algebra of bounded index, we define the *Nil index* of $\mathcal{A}$, denoted by $\mathrm{nd}_{Nil}(\mathcal{A})$, as the smallest number $r \in \mathbb{N}$ such that $a^r = 0$ for any $a \in \mathcal{A}$. Consequently, any nilpotent algebra is still a nil algebra. Therefore, for any $\mathcal{A}$ Nilpotent algebra, $\mathrm{nd}_{Nil}(\mathcal{A}) \leq \mathrm{nd}(\mathcal{A})$.

From now on, unless otherwise stated, we denote by $\mathcal{A}$ a G-graded $\mathcal{F}$-algebra with G-grading given by $\Gamma : \mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$, where G is a group and $\mathcal{F}$ is a field, both arbitrary.

Proposition 1. *Suppose* $\mathrm{Supp}(\Gamma)$ *is finite. If $A_e = \{0\}$, then $\mathcal{A}^{d+1} = \{0\}$.*

In the proof of the previous proposition, we have used arguments in combinatorial methods. Evidently, the same techniques used above can be extended for the case "$\mathcal{A}_e$ is Nil of bounded index and central in $\mathcal{A}$".

Now, using arguments of Theory of Groups, we have established more general results than the previous proposition. Obviously, we can apply the same methods of Theory of Group used in Theorem 1 to prove the Proposition 1, but we have decided not to do because it is more natural with combinatorial methods.

Theorem 1. *Assume that $\mathcal{A}_e$ is a Nil algebra of bounded index,* $\mathrm{nd}_{Nil}(\mathcal{A}_e) = s > 1$. *The following statements are true:*

i) For any $g \in G$, $\left(x_1^{(g)}x_2^{(g)}\cdots x_{k_g}^{(g)}\right)^s \equiv_G 0$ in $\mathcal{A}$, where $k_g = \min\{d, \deg(g)\}$. In other words, $\mathcal{A}_g^{k_g} \subseteq \mathcal{A}_e$ for any $g \in G$;

ii) If $\mathcal{A}_e$ is nilpotent, then $\mathcal{A}$ is also nilpotent with $s \leq \mathrm{nd}(\mathcal{A}) \leq rd$, where $r = \mathrm{nd}(\mathcal{A}_e)$;

iii) If $\mathcal{A}_e$ is commutative and finitely generated by n elements, then $\mathcal{A}$ is nilpotent with $s \leq \mathrm{nd}(\mathcal{A}) \leq ((s-1)n+1)d$;

Below, we present the Nagata-Higman Theorem. In suitable conditions, it ensures the equivalence between nil and nilpotent algebras. Besides, an upper bound is given to the nilpotency index, depending only on the nil index of algebra. In 1953, Nagata proved that any Nil algebra over a field of characteristic zero is Nilpotent. After, in 1956, Higman generalized the result of Nagata to any field. Posteriorly, it was discovered that this result was published in [3], in 1943, by Dubnov and Ivanov.

Theorem 2 (Nagata-Higman Theorem, [6, 4])**.** *Let $\mathcal{A}$ be a associative algebra over a field $\mathcal{F}$. Assume* $\mathrm{char}(\mathcal{F}) = p$*. Suppose $x^n \equiv 0$ in $\mathcal{A}$. If $p = 0$ or $n < p$, then $x_1 x_2 \cdots x_{2^n-1} \equiv 0$ in $\mathcal{A}$.*

In [5], Kuzmin exhibited a lower bound for the nilpotency index of a nil algebra of bounded index $\mathcal{A}$ over a field of characteristic zero. He showed that $\mathrm{nd}(\mathcal{A}) \geq \frac{n(n+1)}{2}$, where $n = \mathrm{nd}_{Nil}(\mathcal{A})$. After it, in [7], Razmyslov proposed a smaller estimate than that given by Higman in [4], and in [8] he himself exhibited a proof. This result follows.

Theorem 3 (Theorem 33.1, [8])**.** *In any associative algebra over a field of characteristic zero in which the identity $y^n \equiv 0$ is valid, the identity $x_1 x_2 \cdots x_{n^2} \equiv 0$ is valid.*

Finally, we conclude deducing an immediate consequence of Theorem 1 and the previous theorem. Therefore, we can generalize Theorem 1.

Theorem 4. *Let $\mathcal{A}$ be a associative algebra over a field $\mathcal{F}$ with a G-grading of finite support,* $\mathrm{char}(\mathcal{F}) = p$*. Suppose $\mathcal{A}_e$ is nil algebra of bounded index $s = \mathrm{nd}_{Nil}(\mathcal{A}_e)$. If $p = 0$ or $p > s$, then $\mathcal{A}$ is a nilpotent algebra. In addition, if* $\mathrm{char}(\mathcal{F}) = p > 0$*, then* $\mathrm{nd}(\mathcal{A}) \leq d(2^s - 1)$*, and if* $\mathrm{char}(\mathcal{F}) = 0$*, then* $\mathrm{nd}(\mathcal{A}) \leq ds^2$*.*

Note that the upper bound for nilpotency index obtained in Theorem 1 is still smaller than the limitation given by Theorem 4. Nevertheless, in Theorem 4, $\mathcal{A}_e$ is not necessarily commutative. Thus, it follows that Theorem 4 is a more general result than Theorem 1.

Keywords: G-graded associative algebra, GPI-algebra, graded identities, nil algebra, nilpotent algebra, nilpotency index.

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