NON-LINEAR ANALYTIC DIFFERENTIAL EQUATIONS AND THEIR INVARIANTS

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Abstract. We study the classification of analytic differential equations in $(\mathbb{C}^2, 0)$ with finitely many separatrices. We give a complete set of analytic invariants determining the analytic type of a generic case (generalized curves). We define the notion of simple resolution foliation. With some conditions on the holonomy and the separatrix set we study the topological and analytic types of generalized curves with simple resolution.

Contents

1. Introduction

1.0.1. *Origins.* The problem of the local classification of differential equation of the form $Adx + Bdy = 0$ in two variables has been studied by various mathematicians — since the end of the nineteen century — as C. A. Briot, J. C. Bouquet, H. Dulac, H. Poincaré, I. Bendixson, G. D. Birckhoff, C. L. Siegel, A. D. Brjuno *et Al.* One of the main tools to develop this problem has been the blow-up method. Indeed, it was introduced by L. Kronecker and independently by M. Noether ([7]), as a systematic method to identify invariants of algebraic curves, and then classify such curves. On the other hand, for every foliation $\mathcal{F}$ in $(\mathbb{C}^2, 0)$ one may consider the set of

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invariant analytic curves passing through the origin, called the **separatrices**, and denoted by $Sep(\mathcal{F})$. Hence it is natural to seek a generalization of the above method to analytically classify differential equations. In fact, after the works of I. Bendixson ([2],[3]), F. Dumortier ([19]), and A. Seidenberg ([36]) it was obtained a generalization, for singular holomorphic foliations, of the resolution theorem of M. Noether for curves, opening a new perspective in the study of differential equations. Indeed, in the middle of the seventies, René Thom proposed the following:

Conjecture 1. *If $Sep(\mathcal{F})$ has a finite number of components then the analytic type and the monodromy of $Sep(\mathcal{F})$ may determine $\mathcal{F}$ up to conjugacy class.*

In [28],[29], and [30] it is proved that the conjecture has an affirmative answer in case we have non-nilpotent linear part of the vector field defining the foliation. But, in [32] it is proved that the conjecture is not true in general, by introducing an analytic invariant called *vanishing holonomy*. Since this time this question is known as *Thom's problem*. In [17] the results of [32] are generalized, classifying a Zariski open subset of the nilpotent singularities, in terms of the vanishing holonomy (now with the name of *projective holonomy*). In these lines we only treat the case we have just non-degenerated singularities along the minimal resolution, the remaining cases shall be considered later.

1.1. **Basic definitions and notations.** A germ of singular foliation, say $(\mathcal{F} : \omega = 0)$, in $(\mathbb{C}^2, 0)$ of codimension 1 is, roughly, the set of integral curves of a given germ of 1-form $\omega \in \Omega^1(\mathbb{C}^2, 0)$ which may be assumed to have just an isolated singularity at the origin. Let $Diff(\mathbb{C}^k, 0)$ (respect. $Homeo(\mathbb{C}^k, 0)$) be the group of germs of analytic diffeomorphisms (respect. homeomorphisms) of $(\mathbb{C}^k, 0)$ fixing the origin. We say that two germs of foliations $(\mathcal{F}_j : \omega_j = 0)$, in $(\mathbb{C}^2, 0)$, $j = 1, 2$, are analytically (respect. topologically) conjugated if there is $\Phi \in Diff(\mathbb{C}^2, 0)$ (respect. $Homeo(\mathbb{C}^2, 0)$), such that Φ sends leaves of $\mathcal{F}_1$ in leaves of $\mathcal{F}_2$. We say that $h_j \in Diff(\mathbb{C}, 0)$ (respect. $Homeo(\mathbb{C}, 0)$) $j = 1, 2$, are analytically (respect. topologically) conjugated if there is $\phi \in Diff(\mathbb{C}, 0)$ such that $\phi_*(h_1) := \phi \circ h_1 \circ \phi^{-1} = h_2$. We denote by $Iso(\mathcal{F})$ the isotropy group of the germ of foliation $(\mathcal{F} : \omega = 0)$ in $(\mathbb{C}^2, 0)$, given by

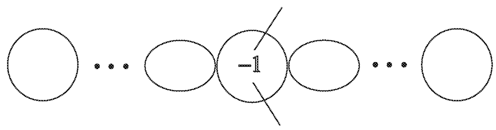
$$Iso(\mathcal{F}) = \{\phi \in Diff(\mathbb{C}^2, 0) : \phi^*\omega \wedge \omega = 0\}$$

Further, let us denote the *Hopf bundle of Chern class $-k$* by $\mathbb{H}(-k) : \big(p_{(k)} : \mathcal{H}(-k) \to \mathbb{CP}(1)\big)$, or just by its total space $\mathbb{H}(-k)$.

Recall from the theory of Algebraic Curves that whether $\pi : (\widetilde{X}, D) \longrightarrow (\mathbb{C}^2, 0)$ is a map resulting from the iteration of finite number of blow-ups, with exceptional curve $D = \pi^{-1}(0)$, whose irreducible components are D_j, $j = 1 \cdots n$, with Chern classes $-k_j$. Then a suitable neighborhood of D in $\widetilde{X}$ results from pasting together suitable neighborhoods of the zero sections of $\mathbb{H}(-k_j)$. Then denote by $\widetilde{\mathcal{F}}$ the unique extension of $\pi^*(\mathcal{F})$ whose singular set has codimension greater or equal to 2 (c.f. [28]) and Thus, for each Hopf bundle component $p : \mathcal{H}_j \to D_j$ of a given resolution, we shall denote by $\widetilde{\mathcal{F}}_j$, the germ of foliation in $(\mathcal{H}_j, D_j)$ induced by the restriction of $\widetilde{\mathcal{F}}$, and call it the $j^{\textbf{th}}$ **Hopf component** of the resolution. Further we shall denote by $\widetilde{\mathcal{F}}_{i,j}$ the restriction of $\widetilde{\mathcal{F}}$ to a neigborhood of the corner $t_{ij} := D_i \cap D_j$. The "strict transform" of $Sep(\mathcal{F})$ at $D_j \subset \mathcal{H}_j$, that is the set of local separatrices transversal to the zero section of $\mathcal{H}_j$, namely $Sep(\widetilde{\mathcal{F}}_j) = \overline{(\pi^* Sep(\mathcal{F})) \mid_{\mathcal{H}_j} \setminus D_j}$, will be called the $j^{\textbf{th}}$ **Hopf component** of $\pi^*(Sep(\mathcal{F}))$.

Let $\mathbb{H} : (p : \mathcal{H} \to D \simeq \mathbb{CP}(1))$ be a Hopf bundle, and $\mathcal{F}$ a foliation defined in $\mathcal{H}$. Then one says that $\mathcal{F}$ is **non-dicritical** if $\mathbb{CP}(1)$ is an invariant set of the foliation, and **dicritical** otherwise. In the former case, the holonomy of $\mathcal{F}$ with respect to a section Σ transversal to D is called the **projective holonomy** of $\mathcal{F}$, and denoted by $Hol_\Sigma(\mathcal{F}, D)$, furthermore one says that it is **solved** whether it has just *reduced* singularities (see [28]). Moreover let $\widetilde{\mathcal{F}}^i$, $i = 1, 2$, be two germs of singular non-dicritical foliation at $\mathbb{CP}(1) \subset \mathcal{H}$, without saddle-nodes and $\varphi \in \mathbb{PGL}(2, \mathbb{C})$ (respect. $Homeo(\mathbb{CP}(1))$) be an isomorphism between their sets of singular points $\{t_j^i\}_{j=1}^n$, that is $\varphi(t_j^1) = t_j^2$. Further let $t_0^1 \in \mathbb{CP}(1)$ be a regular point of $\widetilde{\mathcal{F}}^1$, $t_0^2 = \varphi(t_0^1)$, and denote by h_γ^i the holonomy of a path $\gamma \in \pi_1(\mathbb{CP}(1)\backslash\{t_j^i\}_{j=1}^n, t_0^i)$, with respect to sections Σ_i, $i = 1, 2$ transversal to $\mathbb{CP}(1)$. Then one says that the projective holonomies of these foliations have an analytic (respect. topological) **conjugacy subordinated to** φ whether there is $\phi \in Diff(\mathbb{C}, 0)$ (respect. $Homeo(\mathbb{C}, 0)$) such that $\phi_*(h_\gamma^1) = h_{\varphi_*\gamma}^2$, for every $\gamma \in \pi_1(\mathbb{CP}(1)\backslash\{t_j^1\}_{j=1}^n, t_0^1)$. Moreover we set $Diff(\mathcal{F}^1, \mathcal{F}^2) = \{\Phi \in Diff(\mathcal{H}, \mathbb{CP}(1)) : \Phi_*(\mathcal{F}^1) = \mathcal{F}^2, \Phi|_{Sing(\mathcal{F}^1)} = \varphi|_{Sing(\mathcal{F}^1)}\}$. In particular we denote the isotropy group of $\mathcal{F}$ by $Iso(\mathcal{F}) = Diff(\mathcal{F}, \mathcal{F})$.

Now recall that a *generalized curve* is a germ of singular foliation in $(\mathbb{C}^2, 0)$ which does not have any saddle-node or dicritical components along the minimal resolution ([14]). Further one says that a germ of non-reduced singular foliation $\mathcal{F}$ has **simple resolution** if its minimal resolution has only non-dicritical components and its exceptional divisor has only one projective line with three or more singularities (for instance, see the figure below), which will be called **principal projective line** of the exceptional divisor; its holonomy is called the **projective holonomy** of the foliation.



A simple resolution foliation

Later on we will prove that the resolution of such singularities is always like in the figure above.

1.2. **Statement of the main results.** First let us note that all the nilpotent singularities studied in [32] and [17], have simple resolution, so it is natural to seek a topological classification for such germs of foliations. We have the following result in this direction.

Theorem 1. *Let $\mathcal{F}$ and $\mathcal{F}'$ be simple resolution generalized curves with order-one-topologically equivalent trees of singularities, namely $\varphi = (\varphi_i)_{i=1}^k : T_s \longrightarrow T_s'$. Further suppose that $\mathcal{F}$ and $\mathcal{F}'$ satisfy one of the following conditions:*

1. *$Hol_\Sigma(\mathcal{F},)$ and $Hol_{\Sigma'}(\mathcal{F}', D')$ are rigid;*
2. *Each singularity is in the Poincaré domain or is Siegel resonant.*

Then they are topologically equivalent if their projective holonomies have an orientation preserving conjugacy subordinated to φ.

Now let us consider the analytic case. First recall that in the works of Cerveau and Moussu ([32] and [17]), the main idea is to obtain normal forms for nilpotent singularities, in order to construct a Milnor fibration, transversal to the foliation away from the non-principal projective lines of the exceptional divisor of the minimal resolution; which contains the separatrices as fibers. In a more general setting this construction is no more possible as it is remarked in [4]. There it is outlined an example of a foliation solved after one blow-up with the separatrices given by five distinct curves: four lines which are fibers of the Hopf bundle and a parabola. In this case, there will never be a fibration containing these curves as fibers, and hence we cannot use the technics of path lifting (see [28]) in order to obtain a conjugacy. In fact foliations with these resolution up to order one ([17]) cannot have their analytic type determined by the projective holonomy, in general. On the other hand, we realize that the minimal resolution only busy itself with the work of simplify the local singularities which appear along the resolution ([36]), but for our purpose we have to look to the behavior of the separatrices after a resolution. Indeed, we will say that a resolution $\pi : (M, D) \longrightarrow (\mathbb{C}^2, 0)$ of a generalized curve is rectifier whether for each non-dicritical

Hopf component $\widetilde{\mathcal{F}}_j$, $Sep(\widetilde{\mathcal{F}}_j)$ is contained in the fibers of a fibration f_j transversal to $\widetilde{\mathcal{F}}_j$, away from $Sep(\widetilde{\mathcal{F}}_j)$.

Proposition 1. *Every generalized curve has a rectifier resolution.*

Hence we shall say that two germs of foliations in $(\mathbb{C}^2, 0)$ are component -wise isomorphic if they have isomorphic minimal rectifier resolutions, namely foliations $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{F}}'$ such that $\widetilde{\mathcal{F}}_j$ and $\widetilde{\mathcal{F}}'_j$ are isomorphic for all j. Then fix a foliation $\mathcal{F}^{\mathrm{o}}$, component-wise isomorphic to $\mathcal{F}$, such that in the coordinates given by a minimal rectifier resolution[1], $Sep(\widetilde{\mathcal{F}}^{\mathrm{o}}_j)$ is contained in the fibers of $\mathbb{H}_j$ (this foliation does exist by [27]), which shall be called a **fixed model** for $\mathcal{F}$, and consider the elements $\Phi_j \in Diff(\widetilde{\mathcal{F}}_j, \widetilde{\mathcal{F}}^{\mathrm{o}}_j)$, which shall be called a **projective chart** for the j^{th} component of the fixed model. Then it is straightforward that for each component $\widetilde{\mathcal{F}}_j$ and each fixed model component $\widetilde{\mathcal{F}}^{\mathrm{o}}_j$, there exists only one projective chart, up to left composition by an element of $Iso(\widetilde{\mathcal{F}}^{\mathrm{o}}_j(U_j))$, where U_j is a neighborhood of D_j. in $\mathcal{H}_j$. So, consider the sheaf of non-abelian groups $\Lambda^{\mathrm{o}} := Iso(\widetilde{\mathcal{F}}^{\mathrm{o}})$, then we say that $\mathcal{U} := \cup U_j$ is a good covering for Λ^{o} whether U_j are neighborhoods of $D_j \subset \mathcal{H}_j$. Therefore, consider the first cohomology set $H^1(\mathcal{U}, \Lambda^{\mathrm{o}})$ associated to the good covering $\mathcal{U}$, and set $H^1(D, \Lambda^{\mathrm{o}})$ as the direct limit of $H^1(\mathcal{U}, \Lambda^{\mathrm{o}})$ for the good coverings of $\widetilde{\mathcal{F}}$, associated to $D = \cup D_j$, the exceptional divisor of the given minimal rectifier resolution of $\mathcal{F}$.

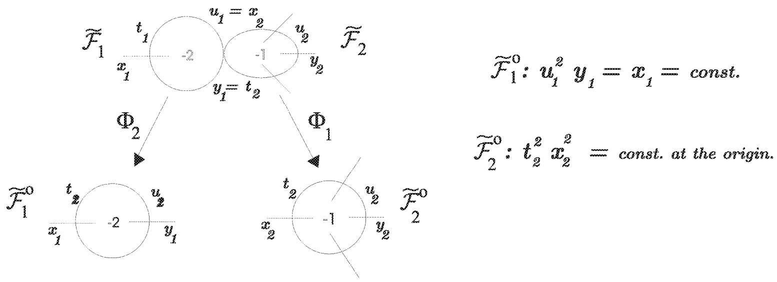


Figure 1.2: A fixed model for $\sum^{\mathrm{c},1}_{d(y^3+x^6)}$

Notation 1. *We denote as* $\sum^{c}_{\omega}$ *(respect.* $\sum^{c,1}_{\omega}$*) the set of germs of foliations which has a minimal rectifier resolution component-wise analytically equivalent (respect. up to order one) to a minimal rectifier resolution of* $(\mathcal{F} : \omega = 0)$. *Further we denote as* $\sum^{c,1}_{\omega,f}$ *the subset of* $\sum^{c,1}_{\omega}$ *whose separatrix set has the same analytic type of the curve* $f^{-1}(0)$.

Thus we define the map

$$\begin{array}{ccc} \sum^{\mathrm{c}}_{\omega} & \xrightarrow{\Theta} & H^1(D, \Lambda^{\mathrm{o}}) \\ \mathcal{F} & \mapsto & (\Phi_{i,j}) \end{array}$$

where $\Phi_{i,j} := \Phi_i \circ \Phi_j^{-1}$. Note that Θ does not depends on the fixed models up to component-wise conjugacy class. Then, as a consequence of path-lifting construction, one has that

Theorem 2. *Let* $\mathcal{F}, \mathcal{F}'$ *be two germs of generalized curves in* $(\mathbb{C}^2, 0)$, *and* $\widetilde{\mathcal{F}}$, $\widetilde{\mathcal{F}}'$ *respectivly minimal rectifier resolutions with equivalent trees. Then* $\mathcal{F}, \mathcal{F}'$ *are analytically equivalent if, and only if they satisfy:*

i) $\mathcal{F}, \mathcal{F}' \in \sum^{c,1}_{\omega}$ *for some* 1*-form* $\omega \in \Omega^1(\mathbb{C}^2, 0)$ *(holomorphic one-forms). In particular we denote by* $\varphi = (\varphi_j) : \Gamma_s \longrightarrow \Gamma'_s$ *the component-wise isomorphism between the trees of singularities of their minimal rectifier resolution.*

[1]Notice that a foliation may has more than one minimal rectifier resolution

ii) *For each Hopf components* $\widetilde{\mathcal{F}}_j$ *and* $\widetilde{\mathcal{F}}'_j$, *their projective holonomies have an analytic conjugacy subordinated to* φ.
iii) $\Theta(\mathcal{F}) = \Theta(\mathcal{F}')$, *for one (and every) fixed model of* $\sum_{\omega}^{c}$.

Now let $(\mathcal{F} : \omega = 0)$ be a generalized curve with the same resolution of $y^2 + x^n = 0$ (that is with isomorphic trees of points), then recall that in [17], $\sum_{\omega}^{c,1}$ is classified by its projective holonomy. In fact, using Weierstrass preparation theorem, one can prove that for any $\mathcal{F} \in \sum_{\omega}^{c,1}$, the analytic type of $Sep(\mathcal{F})$ is the same of $y^2 + x^n = 0$, that is $\sum_{\omega}^{c,1} = \sum_{\omega,y^2+x^n}^{c,1}$. But in general this is not true, for instance consider the curves $y^3 = x^7$, and $y^3 = x^7(1 + x^{\frac{1}{3}})^3$. In fact they are topologically equivalent (and thus have the same resolution) but not analytically equivalent (see for instance [9]). Therefore, in general it is not true that $\sum_{\omega}^{c,1} = \sum_{\omega,f}^{c,1}$, for $(\mathcal{F} : \omega = 0)$ with the same resolution of $f^{-1}(0)$. This justifies the hypothesis of the following:

Theorem 3. *Let* $f(x, y) = \prod_{j=1}^{d}(y^p - \lambda_j x^q)$, *where* $1 < p < q \in N^*$, *g.c.d.*$(p, q) = 1$, *and* $\lambda_j \in \mathbb{C}^*$. *Then* $\Theta(\mathcal{F})$ *is trivial in* $H^1(D, \Lambda^o)$ *for every* $\mathcal{F} \in \sum_{\omega,f}^{c,1}$. *In other words, any two generalized curves of* $\sum_{\omega,f}^{c,1}$, *are analytically conjugated if, and only if their projective holonomies have a conjugacy subordinated to a isomorphism between the set of singular points of their principal projective lines.*

Finally we want to point out that theorem 1, is proved in 4.2, proposition 1 is proved in 5, theorem 2 is proved in 6, and theorem 3 is proved in 7.

In a future work we shall study, under mild conditions, the finiteness of $H^1(D, \Lambda^o)$ related to the given cohomology and show the role of developments in this general setting. In particular we shall show some examples for which the cohomology is non-trivial.

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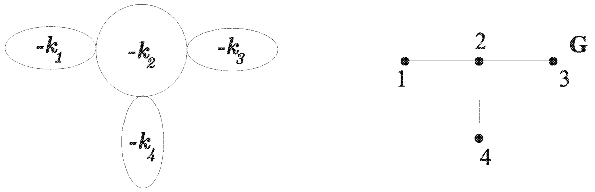
2. Non-dicritical foliations with one blow-up resolution

We investigate some necessary conditions to obtain an order one determined reduced singularities tree, over a projective line with negative Chern class, in a minimal resolution.

A **tree of projective lines** is an embedding of a connected and simply connected chain of projective lines, intercepting transversely in a complex surface (two dimensional complex analytic manifold) with two projective lines in each intersection, made by the pasting of Hopf bundles whose zero sections are the projective lines itself. A **tree of points** is any tree of projective lines in which are discriminated a finite number of points. Finally a **tree of singularities** is a tree of points such that in each point is specified a germ of singular foliation.

Note that the above nomenclature has a natural motivation. In fact, as it is well know from the literature (see [9] for instance) that we can assign to each projective line a point and to each interception an edge in other to form a tree (as in graph theory).



Hence we use the following notations for a tree of projective lines, a tree of points and a tree of singularities respectively

$$T_{PL} = \left\{G, \{k_i\}_{i=1}^{k}\right\}$$

$$T_p = \left\{G, \{k_i\}_{i=1}^{k}, \{t_{ij}\}_{j=1}^{l_i}\right\}$$

$$T_s = \left\{G, \{k_i\}_{i=1}^{k}, \{t_{ij}\}_{j=1}^{l_i}, \{\omega_{ij}\}_{j=1}^{l_i}\right\}$$

where G is a tree, $k_i \in \mathbb{N}$, $t_{ij} \in D_i$, and D_i is the zero section of the Hopf bundle $\mathcal{H}_i$ with Chern class $-k_i$, and $\omega_{ij} = 0$ is a germ of singular holomorphic foliation at t_{ij}.

Definition 1. *One says that two trees of projective lines, say* $T_{PL} = \left\{G, \{k_i\}_{i=1}^{k}\right\}$ *and* $T'_{PL} = \left\{G', \{k'_i\}_{i=1}^{k'}\right\}$ *are isomorphic if* G *and* G' *are isomorphic (as graphs) and* $k_i = k_i{}'$*. Moreover, two trees of points* $T_p = \left\{G, \{k_i\}_{i=1}^{k}, \{t_{ij}\}_{j=1}^{l_i}\right\}$ *and* $T'_p = \left\{G', \{k'_i\}_{i=1}^{k\prime}, \{t'_{ij}\}_{j=1}^{l'_i}\right\}$ *are analytically (respect. topologically) component-wise isomorphic whether their trees of projective lines are isomorphic,* $l_i = l'_i$ *and there exist analytic diffeomorphisms (respect. homeomorphisms)* $\varphi_i \in Diff(D_i)$ *(respect.* $\in Homeo(D_i)$*) such that* $\varphi_i(t_{ij}) = t'_{ij}$*. Finally we say that two trees of singularities* $T_s = \left\{G, \{k_i\}_{i=1}^{k}, \{t_{ij}\}_{j=1}^{l_i}, \{\omega_{ij}\}_{j=1}^{l_i}\right\}$ *and* $T'_s = \left\{G', \{k'_i\}_{i=1}^{k\prime}, \{t'_{ij}\}_{j=1}^{l'_i}, \{\omega'_{ij}\}_{j=1}^{l'_i}\right\}$ *are analytically (respect. topologically) component-wise isomorphic up to first order whether their trees of points are analytically (respect. topologically) component-wise isomorphic and* $J^1(\omega_{ij}) = J^1(\omega'_{ij})$*, that is* ω_{ij} *and* ω'_{ij} *have the same linear part.*

Definition 2. *One says that one tree of singularities* T_s *is* ***realizable****, whether there exists a singular foliation whose minimal resolution has* T_s *as tree of singularities.*

Remark 1. *:*

(1) *Note that a realizable tree of singularities may be realized in two distinct germs of foliations, whose minimal resolution trees have analytically (respect. topologically) equivalent up to first order trees of singularities, but are not necessarily equivalent (for instance see* [6]*);*
(2) *In the topological case the isomorphism of trees of points is just a matter of isomorphism of tree of projective lines and cardinality of the singularities in each component. On the other hand the analytic case is more subtle if we consider trees with more then three singularities in some of its projective lines.*

Now, note that by the separatrix theorem of Camacho-Sad ([11]), every non-dicritical singular foliation which is solved by one blow-up, must have at least one singularity (this result can be verified by straightforward accounts as well).

Let $\omega = A(x,y)dy + B(x,y)dx$ where $A = \sum_{\nu \geq 0} A_\nu$, $B = \sum_{\nu \geq 0} B_\nu$; A_ν and $B_\nu \in \mathbb{C}[x,y]$ are homogeneous polynomials of degree ν. Then the q-jet of ω is given by:

$$J^q\omega = \sum_{j=0}^{q} \left(A_q dy + B_q dx\right),$$

hence the first non-zero jet of ω is given by $J^{q_0}\omega$ where $q_0 = \min\{q \in \mathbb{Z}_+ \mid J^q\omega \neq 0\}$.

Now let us see how the blow-up works explicitly. First, recall that we may find bundle charts for $\mathbb{H}(-1)$, namely $(t,x) \in \mathbb{C}^2$ and $(u,y) \in \mathbb{C}^2$, with $\left\{ \begin{array}{c} u = 1/t; \\ y = tx. \end{array} \right.$, such that the foliation will be regular at $(u,y) = (0,0)$. Hereafter we will call these two charts, affine charts of the origin and the affine chart of the infinity respectively for (t,x) and (u,y). Let $(\mathcal{F} : \omega = 0)$, $\omega = Ady + Bdx$ where $A = \sum_{\nu \geq m} A_\nu$ and $B = \sum_{\nu \geq n} B_\nu$, A_ν, $B_\nu \in \mathbb{C}[x,y]$, be homogeneous polynomials of degree ν. Then given a tree of singularities:

$$T_s := \left\{ \begin{array}{c} (t_j, \omega_j) \in \mathbb{C} \times \Omega^1(\mathbb{C}, 0) | t_j \neq \infty, \omega_j(t,x) = (\mu_j t + \alpha_j x)dx + \nu_j x dt, \\ \lambda_j := -\nu_j/\mu_j \notin \mathbb{Q}_+, \, j = 1 \dots q, \sum_{j=1}^q \lambda_j = -1 \end{array} \right.$$

defined over a projective line, then we will obtain necessary conditions over (ν, μ, t) for the realization of this tree and moreover, conditions over A and B for the resolution of $\mathcal{F} : (\omega = 0)$ has $\mathcal{A}$ as singularities tree. In fact applying a blow-up over ω one has that:

$$\begin{aligned} \pi^*\omega(t,x) &= A(x,tx)(tdx + xdt) + B(x,tx)dx \\ &= (B(x,tx) + tA(x,tx))dx + xA(x,tx)dt \\ &= \sum_{\nu \geq p} (x^\nu B_\nu(1,t) + tx^\nu A_\nu(1,t))dx + \sum_{\nu \geq p} x^{\nu+1} A_\nu(1,t)dt \end{aligned}$$

where $p = \min(m,n)$.

But recall that $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ have isolated singularities. Then, as we are considering the non-dicritical case — that is $A_p(x,y)y + B_p(x,y)x \neq 0$ (see [28]), we have that $\widetilde{\mathcal{F}}$ can be given by:

$$\begin{aligned} \frac{\pi^*\omega(t,x)}{x^p} &= \sum_{\nu \geq p} (x^{\nu-p} B_\nu(1,t) + tx^{\nu-p} A_\nu(1,t))dx + \sum_{\nu \geq p} x^{\nu-p+1} A_\nu(1,t)dt \\ &= P_p(t)dx + xA_p(1,t)dt + \sum_{\nu \geq 1} x^\nu P_{\nu+p}(t)dx + x^{\nu+1} A_{\nu+p}(1,t)dt, \end{aligned}$$

where $P_\nu(t) = B_\nu(1,t) + tA_\nu(1,t)$. But note that, as we are assuming that the origin of $\mathcal{H}(-1)$ is a singularity of $\widetilde{\mathcal{F}}$, and that there is no singularity at ∞. then $B_p(1,0) = 0$ and by analogous accounts in the affine chart of the infinity one has that $A_p(0,1) \neq 0$. Therefore, if we let $A_p(x,y) = \sum_{j=0}^p a_j x^{p-j} y^j$ and $B_p(x,y) = \sum_{j=0}^p b_j x^{p-j} y^j$ then we have that $b_0 = 0$ and $a_p \neq 0$, in particular P_p has degree $p+1$. Furthermore

$$\begin{aligned} P_p(t) &= \sum_{l=1}^p b_l t^l + \sum_{l=0}^p a_l t^{l+1} \\ &= \sum_{l=0}^{p-1} (a_l + b_{l+1}) t^{l+1} + a_p t^{p+1} \end{aligned}$$

Now, in order to study the relations in the other singularities one has to perform the change of coordinates:

$$(s,x) := (t - t_j, x),$$

so obtaining after elementary calculations

$$P_p(s + t_j) = P_p(t_j) + P_p'(t_j)s + O_2$$

and

$$A_p(1, s + t_j) = \sum_{l=0}^p a_l (s + t_j)^l = \sum_{l=0}^p a_l {t_j}^l + O_1$$

thus obtaining

$$\begin{aligned}\widetilde{\omega}(s,x) &= \sum_{\nu\geq 0} x^{\nu} P_{\nu+p}(s+t_j)dx + x^{\nu+1}A_{\nu+p}(1,s+t_j)dt \\ &= P_p(s+t_j)dx + xA_p(1,s+t_j)dt + xP_{p+1}(s+t_j)dx + O_2 \\ &= (P_p(s+t_j) + xP_{p+1}(s+t_j))dx + xA_p(1,s+t_j)dt + O_2 \\ &= (P_p(t_j) + P_p'(t_j)s + xP_{p+1}(t_j))dx + \sum_{l=0}^{p} a_l {t_j}^l x dt + O_2,\end{aligned}$$

Hence, as our foliation has T_s as singularities tree, then:

$$P_p(t) = \sum_{l=0}^{p-1}(a_l + b_{l+1})t^{l+1} + a_p t^{p+1} = at^{n_1}\prod_{j=2}^{q}(t-t_j)^{n_j},\ a\in\mathbb{C}^*\,; \tag{2.1}$$

$$\mu_j = \lambda\sum_{l=0}^{p-1}(l+1)(a_l+b_{l+1}){t_j}^l + (p+1)a_p {t_j}^{p+1} = \lambda P_p'(t_j)\,; \tag{2.2}$$

$$\nu_j = \lambda A_p(1,t_j) = \lambda\sum_{l=0}^{p} a_l t_j^l\,; \tag{2.3}$$

$$\alpha_j = \lambda P_{p+1}(t_j),\ j = 1\ldots q\,,\ \lambda\in\mathbb{C}^*. \tag{2.4}$$

Note that in (2.1), the last equality comes from the fact that $\{t_j\}_{j=1}^{q}$ are the only singularities of $\widetilde{\mathcal{F}}$. In order to apply the above calculations we recall the:

Definition 3 ([14]). *A **generalized curve** is a germ of singular foliation whose minimal resolution has only non-dicritical components at the divisor, and no saddle-nodes as singularities.*

Remark 2. *In* [14] *it was proved that the generalized curves have the same resolution of their separatrices.*

Lemma 1. *Let $\mathcal{F}$ be a generalized curve. Then along its minimal resolution it will never appear a projective line with at most two singularities in a projective line with Chern class -1.*

Proof. First recall that we may assume that the foliation is singular at the origin and regular at ∞; of a given a pair of affine bundle charts as above. Recall that in this case $a_p \neq 0$. Now let us consider the case that there is just one singularity, then according to (2.1) and (2.2), we have that $p = 0$ and this Hopf component of the solved foliation must come from the blow-up of a regular foliation. Now let us consider the case we have just two singularities, then by (2.1) ,(2.2) and (2.3), one has that, $p = 1$ and $b_1 \neq -a_0 \neq 0$. Hence $(\mathcal{F} : \omega = 0)$ has non-degenerated linear part. Therefore as it must not be reduced, then we just have to verify that it cannot have positive rational eigenvalues ratio. In fact let $\omega(x,y) = x(1+a(x,y))dy - \lambda y(1+b(x,y))dx$, where $\lambda = \frac{p}{q} \in \mathbb{Q}^+$ and $a(0,0) = b(0,0) = 0$. Then after one blow-up we obtain $(\widetilde{\mathcal{F}} : \widetilde{\omega} = 0)$ where

$$\begin{aligned}\widetilde{\omega}(t,x) &= ((1-\lambda)t + ta(x,tx) - \lambda tb(x,tx))dx + x(1+a(x,tx))dt,\\ \widetilde{\omega}(u,y) &= ((1-\lambda)u + ua(u,uy) - \lambda ub(uy,y))dy - \lambda y(1+b(uy,y))du.\end{aligned}$$

Now we have to consider two distinct cases. First whether $p = q$ or not. In the former case we have that $p = 1 = q$, and it will appear two saddle nodes which is a contradiction by hypothesis.

In the later case we have that the foliation is not solved as

$$\lambda_1 = \frac{-1}{1-\lambda} = \frac{q}{p-q}$$
$$\lambda_2 = \frac{\lambda}{1-\lambda} = \frac{p}{q-p}.$$

□

Remark 3. *This is not true, in general, for non-dicritical foliations. In fact there are examples of non-dicritical foliations whose minimal resolution presents a Hopf component with Chern class -1, and with just two singularities: a saddle-node and a non-degenerated singularity at the corner (*[31]*).*

3. Hopf bundles and projective holonomy

We describe the invariants that determine the analytic type of "rectified" singular foliations defined in the neighborhood of the zero section of a Hopf bundle, without degenerated singularities.

First let us note that if two singularities are analytically conjugated, then they have isomorphic trees of singular points along its minimal resolution foliation, so if we look to each Hopf bundle we see that isomorphic points have their local holonomy generators conjugated by a global conjugacy. Moreover, note that, as we will see soon, every non-dicritical foliation has a "rectifier" resolution, that is, there is a resolution such that the local separatrices are contained in a fibration "transversal" to the zero section of the Hopf bundle. To clarify these ideas we need the following:

Definition 4. *Let $\mathbb{H} : (p : \mathcal{H} \to \mathbb{CP}(1))$ be the Hopf bundle, $\mathcal{F}$ a germ of singular solved non-dicritical foliation at $\mathbb{CP}(1)$, without saddle nodes. Then we say that a germ of holomorphic map at $\mathbb{CP}(1)$, $f : (\mathcal{H}, \mathbb{CP}(1)) \to (\mathbb{CP}(1), Id|_{\mathbb{CP}(1)})$ is a **transversal fibration** to $\mathcal{F}$, if it satisfies:*

1. *f is a retraction, that is f is a submersion and $f|_{\mathbb{CP}(1)} = Id|_{\mathbb{CP}(1)}$;*
2. *the fiber $f^{-1}(t_j)$ is a separatrix of $\mathcal{F}$, for each $t_j \in Sing(\mathcal{F})$;*
3. *$f^{-1}(t)$ is transversal to $\mathcal{F}$, for every point $t \in \mathcal{F} \backslash Sing(\mathcal{F})$.*

Remark 4. *We can make an analogous definition as above for the C^r case, just by asking f to be of class C^r.*

Note that each transversal fibration as above, defines a non-singular foliation transversal to the zero section of $\mathbb{H}$, given by the level curves of f. In particular the radial foliation, say $\mathcal{R}$, can be viewed as the fibration:

Notation 2. $f_{\mathcal{R}} = p : \mathcal{H} \to \mathbb{CP}(1)$.

Now consider a Hopf bundle $\mathbb{H} : (p : \mathcal{H} \to \mathbb{CP}(1))$ and a germ of singular solved foliation $\mathcal{F}$ in $\mathbb{CP}(1)$, without saddle nodes, a transversal fibration to $\mathcal{F}$, namely $f : (\mathcal{H}, \mathbb{CP}(1)) \to \mathbb{CP}(1)$ and $t_0 \in \mathbb{CP}(1) \backslash Sing(\mathcal{F})$, a regular point of $\mathcal{F}$. Hence, by the path lifting construction, the projective holonomy $Hol(\mathcal{F}, f^{-1}(t))$ is completely determined by $Hol(\mathcal{F}, f^{-1}(t_0))$, with $t, t_0 \in \mathbb{CP}(1) \backslash Sing(\mathcal{F})$. Such a holonomy will be called the **projective holonomy** of $\mathcal{F}$ with respect to f.

Notation 3. $Hol_{f)}(\mathcal{F}, \mathbb{CP}(1)$.

Whether there is no doubt about the fibration we only talk about the projective holonomy of the foliation and denote it by $Hol_{\mathbb{CP}(1)}(\mathcal{F})$.

Definition 5. *Let $\mathbb{H} : (p : \mathcal{H} \to \mathbb{CP}(1))$ be a Hopf bundle over $\mathbb{CP}(1)$, $\mathcal{F}, \mathcal{F}_o$ germs of singular solved non-dicritical foliations at $\mathbb{CP}(1)$ in $\mathcal{H}$, without saddle-nodes, with an isomorphism, say φ, between their trees of singularities, such that f is transversal to $\mathcal{F}, \mathcal{F}_o$, and $f_{\mathcal{R}}$ is transversal to $\mathcal{F}_o$, where f is a fibration in $\mathbb{H}$. Then we set*

$$Diff_{\mathcal{F},\mathcal{F}_o}(\mathcal{H}, \mathbb{CP}(1)) = \{\Phi \in Diff(\mathcal{H}, \mathbb{CP}(1)) :\, \Phi_*(\mathcal{F}) = \mathcal{F}_o,\, \Phi|_{Sing(\mathcal{F})} = \varphi|_{Sing(\mathcal{F})}\}.$$

And call

$$Iso(\mathcal{F}_o) := \{\Phi \in Diff_{\mathcal{F}_o,\mathcal{F}_o}(\mathcal{H}, \mathbb{CP}(1)) :\, \Phi|_{Sing(\mathcal{F})} = Id\}$$

*the **isotropy group** of the foliation $\mathcal{F}_o$.*

Definition 6. *Let $\mathbb{H} : (p : \mathcal{H} \to \mathbb{CP}(1))$ be a Hopf bundle, $\mathcal{F}_i$, $i = 1, 2$, be two germs of singular solved non-dicritical foliations, without saddle nodes, at $\mathbb{CP}(1) \subset \mathcal{H}$, with the respective analytically equivalent trees of singular points $\{t^i_j\}^n_{j=1}$, by $\varphi \in Diff(\mathbb{CP}(1))$, that is $\varphi(t^1_j) = t^2_j$. Let f_i , $i = 1, 2$, be two analytic fibrations such that f_i is transversal to $\mathcal{F}_i$. Further let $t^1_0 \in \mathbb{CP}(1)$ be a regular point of $\mathcal{F}_1$, $t^2_0 = \varphi(t^1_0)$, and denote by h^i_γ the holonomy of a path $\gamma_i \in \pi_1(\mathbb{CP}(1)\backslash\{t^i_j\}^n_{j=1}, t^i_0)$ with respect to f_i, $i = 1, 2$. Then one says that the projective holonomies of these foliations have a conjugacy **subordinated** to φ if there is $\phi \in Diff(\mathbb{C}, 0)$ such that $\phi_*(h^1_\gamma) = h^2_{\varphi_*\gamma}$, for every $\gamma \in \pi_1(\mathbb{CP}(1)\backslash\{t^1_j\}^n_{j=1}, t^1_0)$.*

Remark 5. *We can make an analogous definition as above for the continuous case, just by asking $\varphi \in Homeo(\mathbb{CP}(1))$ — that is, a homeomorphism of the Riemann sphere — and the fibrations f^i, $i = 1 \cdots 2$ to be of class C^r.*

Proposition 2. *Let $\mathbb{H} : (p : \mathcal{H} \to \mathbb{CP}(1))$ be a Hopf bundle, $\mathcal{F}_i$, $i = 1, 2$, be two foliations defined in $\mathcal{H}$, as in the above definition. Then $\mathcal{F}^1$ and $\mathcal{F}^2$ are analytic conjugated if, and only if their projective holonomies have a conjugacy subordinated to $\varphi \in Diff(\mathbb{CP}(1))$.*

Proof. The necessary part is straightforward as was remarked before the above definition. Hence let us see the sufficient part. In fact consider the trees of singular points $\{t^i_j\}^n_{j=1}$ and the regular points $t^i_0 \in \mathbb{CP}(1)$ as in the above definition and let us suppose that there is $\phi \in Diff(\mathbb{C}, 0)$ such that $\phi \circ (h^1_j) \circ \phi^{-1} = h^2_j$, for $j = 1 \cdots n$. Then we define the map $\Phi : \mathcal{F}\backslash \bigcup^n_{j=1} f_1^{-1}(t^1_j) \longrightarrow \mathcal{F}'\backslash \bigcup^n_{j=1} f_2^{-1}(t^2_j)$, by:

$$\Phi(t, x) := \Phi_t(x) := h^2_t \circ \phi \circ (h^1_t)^{-1}(x),$$

where $x \in f_1^{-1}(t)$, $h^i_t : f_i^{-1}(t_0) \longrightarrow f_i^{-1}(t)$ are the holonomy maps obtained by path lifting a curve connecting t_o to t in the leaves of $\mathcal{F}^i$. Note that this map does not depends on the chosen base curves, as ϕ conjugates the elements of the respective projective holonomy of $\mathcal{F}^1$ and $\mathcal{F}^2$. Moreover Φ is holomorphic by (complex) ODE theory ([26]) and by Hartogs's theorem, since it is holomorphic in each variable. Further by [30] and [28], one can extend the diffeomorphisms to the local separatrices, in a neighborhood of $\mathbb{CP}(1)$ projective line, as desired. □

4. Simple resolution foliations

We define what is a simple resolution foliation, and explain its importance (at our level of comprehension of differential equations). Further we describe some relations between the generalized curves, their holonomies and separatrices. We study the geometric properties of its exceptional divisors and describe some of its topological properties.

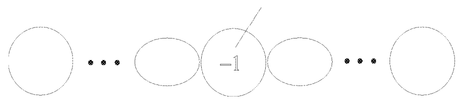
4.1. **The divisor's geometry of a simple resolution foliation.** We determine the geometry of the exceptional divisor of a generalized curve with simple resolution.

Definition 7. *One says that a germ of non-reduced singular foliation $\mathcal{F}$ has* ***simple resolution*** *if its minimal resolution has only non-dicritical components and its exceptional divisor has only one projective line with three or more singularities, which will be called* ***principal projective line*** *of the exceptional divisor; its holonomy is called the* ***projective holonomy*** *of the foliation.*

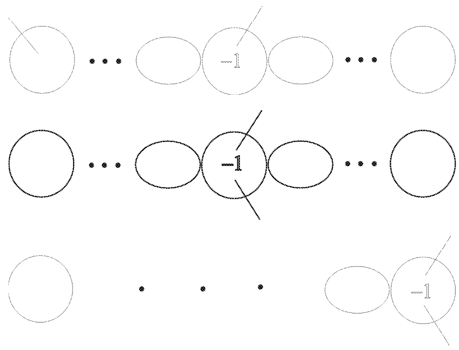
Now recall that a singularity has a *linear chain* as minimal resolution whether it has at most two corners at each projective line of the exceptional divisor.

Proposition 3. *Let $\mathcal{F}$ be a simple resolution generalized curve. Then $\mathcal{F}$ has a linear chain as minimal resolution with the Chern class of its principal projective line equal to -1. In particular, according to the number of irreducible components, namely $n = \#\{$of irreducible components of $Sep(\mathcal{F})\}$, one has the following possible "schemes" of resolution are:*

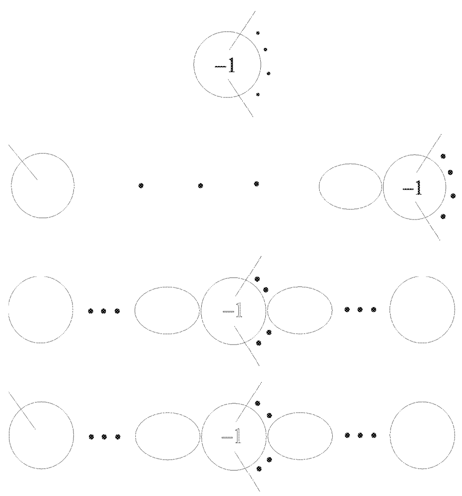
(1) $n = 1$.

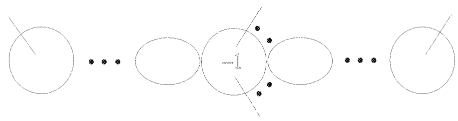


(2) $n = 2$.



(3) $n \geq 3$.





Proof. First let us consider the case where $\mathcal{F}$ has just one irreducible separatrix, and recall that for generalized curves the irreducible components of $Sep(\mathcal{F})$ are one to one with the non-corner singularities of the minimal resolution. By lemma 1 it will never be a one blow-up resolution foliation. Hence, we just have to consider the following two distinct cases:

(1) Along the minimal resolution the singularity will never be a corner at any step. In this case the foliation has a linear chain, with the reduced singularity crossing the principal projective line with just one corner singularity. Again by applying lemma 1, this is a contradiction.
(2) Along the minimal resolution the singularity corresponding to the separatrix will be a corner at some step. In this case we have after some blow-ups, the singularity will leave the corner in an (obviously) non-corner singularity. The projective line intercepted by this non-corner singularity has two corner singularities. If this non-corner singularity is reduced, we are done. Otherwise we have to repeat the process, and as $\mathcal{F}$ has simple resolution, this new singularity must be in the former case, that is, it has a resolution with linear chain, with the singularity intercepting transversely a projective line with just one (corner) singularity. But this is a contradiction by lemma 1.

For the case $n \geq 2$, the result follows by similar reasonings, again by lemma 1 and the definition of simple resolution foliation. □

Notice that the simple resolution foliations appears in [17] as a generic case for the nilpotent singularities. In fact, as we shall see, this is the most general case in which the holonomy of a Hopf component can determine the analytic or topological type of the foliation.

4.2. **Topological classification.** We will see that the topological type of some simple resolution generalized curves (with some conditions on the singularities after the resolution) are determined by their trees of singular points and their projective holonomy.

First let us recall that a subgroup of $Diff(\mathbb{C},0)$ is **rigid** when there exists a bijection between the topological conjugacy classes and holomorphic conjugacy classes ([18]).

Example 1. *Let $G \subset Diff(\mathbb{C},0)$ and $J^1G \subset \mathbb{C}^*$ be the group of multipliers of G (*[20]*) satisfying one of the following conditions:*

(1) *G is non-solvable (*[35], [33]*);*
(2) *G is non-abelian and J^1G is dense in $\mathbb{C}^*$, S^1 or $\mathbb{R}^*$ (*[16], [18]*);*
(3) *G is non-solvable and J^1G is non-real (*[17]*).*

Then G is rigid.

Now let us take a look in the relation between the topological equivalence of linear singularities and topological conjugacy of its holonomies. First, let us recall that for a germ of singular foliation $(\mathcal{F} : \omega = 0)$ with first jet given by $J^1\omega(x,y) = xdy - \lambda ydx$, such that $\lambda \in \mathbb{R}$, one has that λ is a topological invariant ([23]).

Lemma 2. *Let $(\mathcal{F}_j : \omega_j = 0)$ be given by $\omega_j(x_j,y_j) = x_jdy_j - \lambda_jy_jdx_j$, $\lambda_j \in \mathbb{C}\backslash(\mathbb{R}_- \cup \mathbb{Q}_+)$, where $\lambda_1 = \lambda_2$ whether $\lambda_j \in \mathbb{R}_+\backslash\mathbb{Q}_+$, and let $\Sigma_j = \{(x_j,y_j) \in \mathbb{C}^2 : x_j = 1\}$, for $j = 1,2$. Further let $\langle h_j\rangle = Hol(\mathcal{F}_j,(y=0),\Sigma_j)$, and suppose that there is an orientation preserving germ of homeomorphism $\phi : (\Sigma_1,0) \to (\Sigma_2,0)$ such that $h_2 = \phi \circ h_1 \circ \phi^{-1}$. Then there exists a leaf preserving homeomorphism $\Phi \in Homeo(\mathbb{C}^2,0)$, such that $\Phi|_{\Sigma_1} = \phi$.*

Proof. We are going first to extend ϕ to a homeomorphism $\phi : \big(S^1\times\mathbb{D}_r\big)\cup\big(\mathbb{D}_r\times S^1\big)\to\big(S^1\times\mathbb{D}_r\big)\cup\big(\mathbb{D}_r\times S^1\big)$ for some $r>0$, as in this case we are done. In fact, there is a real flow inside the leaves, for which the origin is an attractor, for instance given by

$$\xi_{j,t}(x_j,y_j)=(e^{\mu_j it}x_j,e^{\upsilon_j it}y_j)$$

such that $\mu_j,\nu_j\notin\mathbb{R}$, and $\frac{\upsilon_j}{\mu_j}=\lambda_j$ (see [13]). Now let us verify how we can do the former extension. Indeed we have to distinguish two different cases:

(1) $\lambda:=\lambda_1=\lambda_2\in\mathbb{I}_+=\mathbb{R}_+\backslash\mathbb{Q}_+$. That is a positive irrational number. In this case, as it is well known ([1]), the conjugacy ϕ has the form $\phi(y)=g(|y|)\cdot y$, where g is a continuous map (in fact, this is a consequence of the fact that the closure of the orbits of the holonomies are circles). Now recall that the leaves of the foliations are given by the complex flows $\varphi_j(T,x_j,y_j)=(e^{iT}x_j,e^{\lambda iT}y_j)$, for $T\in\mathbb{C}$. So, if we consider the real flow inside the leaves of $\mathcal{F}_j$ and in $S^1\times\mathbb{D}$, given by $\varphi_{j,t}(x_j,y_j)=(e^{it}x_j,e^{\lambda it}y_j)$, $t\in\mathbb{R}$, we can extend ϕ to the solid torus $S^1\times\mathbb{D}$ using these real flows by

$$\phi(x_1,y_1)=\varphi_{2,t(x_1,y_1)}\circ\phi\circ\varphi_{1,-t(x_1,y_1)}(x_1,y_1)$$

where $t=t(x_1,y_1)$ is given by the equation $(x_1,y_1)=\varphi_{1,t}(1,y)=(e^{it},e^{\lambda it}y)$. Explicitly, we have that

$$\begin{aligned}\phi(x_1,y_1)&=\varphi_{2,t(x_1,y_1)}\circ\phi(1,e^{-\lambda it}y_1)\\&=\varphi_{2,t(x_1,y_1)}(1,g(|y_1|)\cdot e^{-\lambda it}y_1)\\&=(e^{it},g(|y_1|)\cdot e^{(\lambda-\lambda)it}y_1)\\&=(x_1,g(|y_1|)\cdot y_1).\end{aligned}$$

Note that by construction, this extension is leaf-preserving. Now let us extend ϕ to the solid torus $\mathbb{D}\times S^1$. First note that in $S^1\times S^1$ we have that $\phi(x_1,y_1)=(x_1,g(1)\cdot y_1)$. Now, as $\lambda_j\in\mathbb{I}_+$ we have that the closure of the intersection of the leaves with the solid torus $\mathbb{D}\times S^1$ are the torus $T_{r,1}=\{|x_1|=r\le 1,|y_1|=1\}$ and $T_{r,|g(1)|}=\{|x_2|=r\le 1,|y_2|=|g(1)|\}$. Now extend ϕ to a homeomorphism between the disks $\mathbb{D}^1=\{|x_1|\le 1,y_1=1\}$ and $\mathbb{D}^2=\{|x_2|\le 1,y_2=g(1)\}$, with the same expression as in $S^1\times S^1$ and consider the real flows contained in the leaves of $\mathcal{F}_j$ and the solid torus $\mathbb{D}\times S^1$, given by $\psi_{j,t}(x_j,y_j)=\varphi_j(\frac{t}{\lambda_j},x_j,y_j)=(e^{i\frac{t}{\lambda}}x_j,e^{it}y_j)$, then extend ϕ to $\mathbb{D}\times S^1$ by

$$\phi(x_1,y_1)=\psi_{2,t(x_1,y_1)}\circ\phi\circ\psi_{1,-t(x_1,y_1)}(x_1,y_1)$$

where $t=t(x_1,y_1)$ is given by the equation $(x_1,y_1)=\psi_{1,t}(x,1)=(e^{i\frac{t}{\lambda}}x,e^{it})$. Explicitly, we have that

$$\begin{aligned}\phi(x_1,y_1)&=\psi_{2,t(x_1,y_1)}\circ\phi(e^{-i\frac{t}{\lambda}}x_1,1)\\&=\psi_{2,t(x_1,y_1)}(e^{-i\frac{t}{\lambda}}x_1,g(1))\\&=(e^{it(\frac{1}{\lambda}-\frac{1}{\lambda})}x_1,g(1)\cdot e^{it})\\&=(x_1,g(1)\cdot y_1).\end{aligned}$$

(2) $\lambda_1,\lambda_2\in\mathbb{C}\backslash\mathbb{R}$. In this case, as the conjugacy is orientation preserving, we may suppose without loss of generality that their holonomies are at the same time attractors ([38]). Hence, since the holonomies are linear maps, by an iteration argument we may suppose that the conjugacy is defined not only in the unitary disk but in the whole complex plane. Again,

if we consider the real flow inside the leaves of $\mathcal{F}_j$ given by $\varphi_{j,t}(x_j, y_j) = (e^{it}x_j, e^{\lambda_j it}y_j)$, $t \in \mathbb{R}$, we can extend ϕ to the solid torus $S^1 \times \mathbb{C}$ using these real flows by

$$\phi(x_1, y_1) = \varphi_{2,t(x_1,y_1)} \circ \phi \circ \varphi_{1,-t(x_1,y_1)}(x_1, y_1)$$

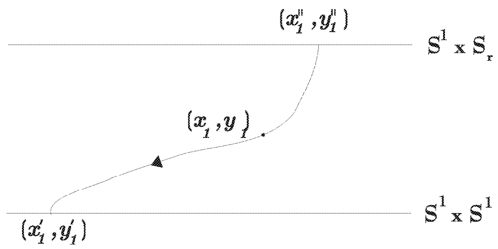
where $t = t(x_1, y_1)$ is given by the equation $(x_1, y_1) = \varphi_{1,t}(1, y) = (e^{it}, e^{\lambda_1 it}y)$. Explicitly, we have that

$$\begin{aligned}\phi(x_1, y_1) &= \varphi_{2,t(x_1,y_1)}(1, \phi(e^{-\lambda_1 it}y_1)) \\ &= (e^{it}, e^{\lambda_2 it}\phi(e^{-\lambda_1 it}y_1)) \\ &= (x_1, e^{\lambda_2 it}\phi(e^{-\lambda_1 it}y_1))\end{aligned}$$

Note that, by construction, this extension is leaf-preserving. But now note that in general, this extension of ϕ will not carry torus in torus, as the original homeomorphism ϕ may not have to carry circles in circles, in general. so we have to use a topological trick to overcome this difficult. In fact, using real flows in the leaves, we extend the homeomorphism $\phi : S^1 \times \mathbb{D} \longrightarrow \phi(S^1 \times \mathbb{D})$ to a homeomorphism $\phi : S^1 \times \mathbb{D}_r \longrightarrow S^1 \times \mathbb{D}_r$ for $r >> 1$. First let us note that for any point $(x_j, y_j) \in S^1 \times \mathbb{C}$ there is only one point in the intersection of the torus $S^1 \times S_r$, where $S_r = \partial\mathbb{D}_r$ and the trajectory of $\varphi_{j,t}$ by (x_j, y_j), as we do not have singularities in the solid torus $S^1 \times \mathbb{C}$. Similarly, for any point $(x_2, y_2) \in S^1 \times \mathbb{C}$ there is only one point in the intersection of the topological torus $\phi(S^1 \times S^1)$. Now given $(x_1', y_1') \in S^1 \times S^1$, and $(x_j'', y_j'') \in S^1 \times S_r$ in the same orbit of the flow $\varphi_{1,t}$, then let $t_1(x_1', y_1')$ be the time such that $(x_1'', y_1'') = \varphi_{1,-t_1(x_1',y_1')}(x_1', y_1')$. Similarly, given $(x_2', y_2') \in \phi(S^1 \times S^1)$, and $(x_2'', y_2'') \in S^1 \times S_r$ in the same orbit of the flow $\varphi_{2,t}$, then let $t_2(x_2', y_2')$ be the time such that $(x_2'', y_2'') = \varphi_{2,-t_2(x_2',y_2')}(x_2', y_2')$. Then from ODE theory we have that t_j is a continuous function of (x_j', y_j'). Further if (x_1, y_1) belongs to the same orbit of (x_1', y_1'), then it is a continuous function $(x_1, y_1) = F(x_1', y_1')$. Now let $s : S^1 \times S^1 \longrightarrow [0, 1]$ be the continuous function given the ratio between the time the flow spends to leave (x_1, y_1) and to reach (x_1', y_1'), and the time $t_1(x_1', y_1')$, that is $(x_1, y_1) = \varphi_{1,-st_1(x_1',y_1')}(x_1', y_1')$. Then we can extend the homeomorphism $\phi : S^1 \times \mathbb{D} \longrightarrow \phi(S^1 \times \mathbb{D})$ by

$$\phi(x_1, y_1) = \varphi_{2,-st_2(x_2',y_2')} \circ \phi \circ \varphi_{1,st_1(x_1',y_1')}(x_1, y_1)$$

where $(x_2', y_2') = \phi(x_1', y_1')$ and $\varphi_{1,-st_1(x_1',y_1')}(x_1, y_1) = (x_1', y_1')$.



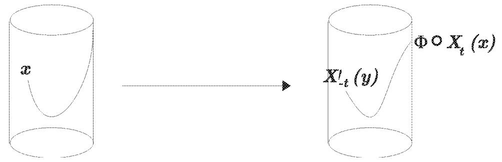
Now we extend this map to the solid torus $\mathbb{D}_r \times S^1$. First, consider the real flow contained in the leaves of $\mathcal{F}_j$ and in the solid torus $\mathbb{D}_r \times S^1$, given by $\psi_{j,t}(x_j, y_j) = \varphi_j(\frac{t}{\lambda_j}, x_j, y_j) = (e^{i\frac{t}{\lambda_j}}x_j, e^{it}y_j)$, and two disks D^j, topologically transversal to the flow $\psi_{j,t}$. Now consider the continuous map $g_j : S_r \times S^1 \to D^j$ induced by the flow $\psi_{j,t}$, namely $g_j(x, y) = \psi_{j,t_j(x,y)}(x, y) \in D^j$, with $t_j(x_j, y_j) > 0$ being the less real number with this property. Further let $K_j = g_j(S_r \times S^1)$, and fix a fundamental domain for the holonomy map h_j in K_j, namely $U_j \subset K_j$, then we have that $g_j|_{(S_r \times S^1) \setminus \partial D^j} : \left(S_r \times S^1\right) \setminus \partial D^j \to \overset{\circ}{U}_j$ is a homeomorphism. Again, as the holonomies are attractors, then by an iteration argument we may extend ϕ to $\phi : (S_r \times S^1) \cup D^1 \longrightarrow \big(S_r \times S^1\big) \cup D^2$ by

$$\phi(x_1, y_1) = h_2^n \circ g_2 \circ \phi \circ g_1^{-1} \circ h_1^{-n}(x_1, y_1)$$

where n is the number of iterations we need to have $h_1^{-n}(x_1, y) \in U_j$, and in particular if $h_1^{-n}(x_1, y) \in \partial U_j$, we assume that $h_1^{-n}(x_1, y) \in \partial U_j \cap S_r \times S^1$. Now let us extend ϕ to $\mathbb{D}_r \times S^1$. Now consider a point $(x_j, y_j) \in \mathbb{D}_r \times S^1$ and the following real times: the less $t_j(x_j, y_j), s_j(x_j, y_j) > 0$ such that $\psi_{j,t_j(x_j,y_j)}(x_j, y_j), \psi_{1,-s_j(x_j,y_j)}(x_j, y_j) \in D^j$ (note that $u_j(x_j, y_j) = t_j(x_j, y_j) + s_j(x_j, y_j)$ is bounded, as $\mathbb{D}_r \times S^1$ is compact and at the limit circle $\{0\} \times S^1$ it is equal to 2π). Now we extend the map $\phi : \big(S_r \cup D^1\big) \times S^1 \longrightarrow \big(S_r \cup D^2\big) \times S^1$ by

$$\phi(x_1, y_1) = \psi_{2,\frac{t_1(x_1,y_1)}{u_1(x_1,y_1)} u_2(x_2,y_2)} \circ \phi \circ \psi_{1,-t_1(x_1,y_1)}(x_1, y_1).$$

□



Remark 6. *:*

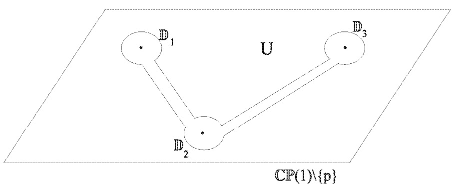
(1) *First note that by the topology of their orbits, an irrational rotation and an attractor cannot be topological conjugated, and that two irrational rotations, namely $h_j(z) = \exp(2\pi i \lambda_j)$, $\lambda_j \in \mathbb{I}_+$, are topological conjugated if, and only if $\lambda_2 = \lambda_1 \bmod \mathbb{Z}$ ([1]). Hence, in the above lemma, $\lambda_1 \in \mathbb{I}_+$ if, and only if $\lambda_2 \in \mathbb{I}_+$ and condition $\lambda_2 = \lambda_1$ is natural in case $\lambda_1, \lambda_2 \in \mathbb{I}_+$, as we have:* $\begin{cases} \lambda_2 = \lambda_1 \bmod \mathbb{Z}, \\ \frac{1}{\lambda_2} = \frac{1}{\lambda_1} \bmod \mathbb{Z}. \end{cases}$ *;*

(2) *Any two attractors are topologically equivalent ([38]), so there is no conditions in this case.*

Now we have the necessary tools to the:

Proof of theorem 1. First let us extend the holonomy conjugacy to a neighborhood of the principal projective line, say $\mathbb{P}_l$.

For the first case — that is, when the holonomy is rigid — consider the principal projective line of the exceptional divisor of the minimal resolution of $\mathcal{F}$, and an open subset, say U, of it given by the union of non-intercepting disks around the singularities and bands joining them, such that it fattens out a simple curve (see figure below).

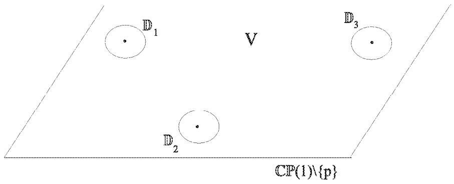


Now consider an holomorphic line bundle over $U_j = U \cap \mathcal{H}_j$, containing $Sep(\widetilde{\mathcal{F}}_j)$ as fibers — in fact it does exists as U_j is Stein — then by a bump function argument we can continuously extend this fibration to $\mathcal{H}_j$. Analogously, we can do the same for $\mathcal{F}'$. Now recall that the fibrations are holomorphic over U_l, hence as the holonomies are rigid, we may assume that their conjugacy is holomorphic. Then by path lifting (see proposition 2) we obtain a topological conjugacy between

$\mathcal{F}$ and $\mathcal{F}'$ away from $Sep(\widetilde{\mathcal{F}}_l)$, which is analytic over $p^{-1}(U_l)\setminus Sep(\widetilde{\mathcal{F}}_l)$. Now by [30] and [28] we obtain a conjugacy between $\mathcal{F}$ and $\mathcal{F}'$ along the whole of $\mathcal{H}_l$.

Now as the foliations have simple resolution foliation, then the projective holonomy of $\widetilde{\mathcal{F}}_j$, for $j \neq l$, is determined by the local holonomy of each corner, which are analytic conjugated. Applying the same reasonings as before — except for rigidity, as the holonomies conjugacy is already analytic — to each Hopf component, we can continuously extend the conjugacy to a neighborhood of the whole divisor. Finally after blowing-down this conjugacy we obtain the desired result.

For the second case — that is when the singularities are in the Poincaré domain or when they are Siegel resonant — consider an open subset $V \subset \mathbb{P}_l$, given by the complement of the union of non-intercepting disks around the singularities, say $\{\mathbb{D}_j\}_{j=1}^n$(see figure below).



Then, by a bump function argument, we can construct a C^∞ subbundle $\mathcal{H}_l|_V$, with V as zero section. Then by path lifting arguments, as before, we can obtain a conjugacy between $\mathcal{F}|_{V\times\mathbb{D}}$ and $\mathcal{F}'|_{V\times\mathbb{D}}$ (here $\mathbb{D}$ represents the unity disk), namely $\Phi : \mathcal{H}_l|_{V\times\mathbb{D}} \longrightarrow \mathcal{H}'_l|_{V\times\mathbb{D}}$. Now by lemma 2 and [12] we can extend this conjugacy to some neighborhoods of the singularities.

Finally let us extend this conjugacy to a neighborhood of the other projective lines. Indeed, as the foliations have simple resolution, then the projective holonomy of $\widetilde{\mathcal{F}}_j$, for $j \neq l$, is determined by the local holonomy of each corner. Hence, applying the same reasonings as before, to each Hopf component we can continuously extend the conjugacy to a neighborhood of the whole divisor. Finally after blowing-down this conjugacy we obtain the desired result. □

Remark 7. *In* [18] *it is proved the converse of the above result for some foliations with one-blow-up resolution. The general case is already open.*

5. Generalized curves and rectifier resolutions

Notice that, in order to classify a generalized curve by its minimal resolution, we arrive at the problem of relating the holonomies of each projective line of the divisor, which points towards the problem of describing the virtual holonomy ([15]) of one (or possibly more) projective lines of the exceptional divisor. Hence we have to consider the geometry of the divisor to classify, by the blow-up method, the generalized curves.

Definition 8. *A resolution $\pi : (M, D) \longrightarrow (\mathbb{C}^2, 0)$ of a generalized curve $\mathcal{F}$ is called rectifier whether all the Hopf components, namely $Sep(\widetilde{\mathcal{F}}_j)$, are contained in the fibers of a fibration f_j transversal to $\widetilde{\mathcal{F}}_j$.*

Recall that the aim of the above definition is to determine equivalence of "models" of singular foliations in $-k$-class Hopf bundles, by its projective holonomy group.

Example 2. *Any simple resolution foliation with at most four singularities at the principal projective line is rectifiable. In fact this is just a matter of apply proposition 7 (in appendix A).*

Notice that the above statement is not true in general for five singularities or more (see proposition 8).

Now we verify the main result of this section. But first:

Remark 8. *After blowing-up a reduced singularity we shall obtain a singular foliation defined in a Hopf bundle with Chern class -1, with two reduced singularities. In particular if the former singularity is non-degenerated then the new two singularities are non-degenerated too.*

Furthermore we have that:

Lemma 3. *Let $\mathbb{H}(-k) : \big(p_{(k)} : \mathcal{H}(-k) \to \mathbb{CP}(1)\big)$ be a Hopf bundle of class $-k$, and $(\widetilde{\mathcal{F}} : \widetilde{\omega} = 0)$ be a germ of solved non-dicritical singular foliation in $\mathbb{CP}(1) \subset \mathcal{H}(-k)$. Then, after a blow-up in each of its m reduced singularities, say $\{t_j\}_{j=1}^m$, $\mathcal{H}(-k)$ turns out to be $\mathcal{H}(-k-m)$ and the Hopf component of the transform of $\widetilde{\mathcal{F}}$, namely $\widetilde{\widetilde{\mathcal{F}}} = \pi^*(\widetilde{\mathcal{F}})$, has its separatrix' Hopf component contained in fibers of $\mathbb{H}(-k-m) : \big(p_{(k+m)} : \mathcal{H}(-k-m) \to \mathbb{CP}(1)\big)$, where $p_{(k+m)}|_{\mathcal{H}(-k-m)\setminus\cup_{j=1}^m \pi^{-1}(t_j)} = \pi^{-1} \circ p_{(k)} \circ \pi$, with π being the proper map obtained by the composition of the m blow-ups.*

Proof. The first statement is well known from the algebraic curves theory, hence let us verify the second. Suppose that $\widetilde{\mathcal{F}}$ has $0 \leq n < m$, of its singularities with their local separatrices contained in the fibers of $\mathbb{H}(-k)$. Now for simplicity, let us suppose that the $(n+1)^{\text{th}}$ singularity is at the origin of the affine chart, then we have that

$$\widetilde{\omega}(t,x) = (t-t_1)\cdots(t-t_n)a(t,x)dx + xb(t,x)dt,$$

where $a(t,x) \in \mathfrak{M}$ (the maximal ideal of $\mathcal{O}$), and $t_{n+1} = 0$. Hence, after one blow-up at the origin one obtains

$$\begin{aligned}\widetilde{\omega}(t,\xi t) &= (t-t_1)\cdots(t-t_n)a(t,\xi t)(\xi dt + td\xi) + \xi tb(t,\xi t)dt \\ &= \xi[(t-t_1)\cdots(t-t_n)a(t,\xi t) + tb(t,\xi t)]dt + t(t-t_1)\cdots(t-t_n)a(t,\xi t)d\xi,\end{aligned}$$

and then

$$\widetilde{\widetilde{\omega}}(t,\xi) = \xi[(t-t_1)\cdots(t-t_n)\frac{a(t,\xi t)}{t} + b(t,\xi t)]dt + t(t-t_1)\cdots(t-t_n)\frac{a(t,\xi t)}{t}d\xi,$$

where $\frac{a(t,\xi t)}{t} \in \mathcal{O}$, and the result follows. □

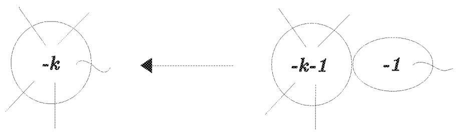


Figure 5: A rectifier blow-up

Proof of proposition 1. In fact, consider the minimal resolution of a generalized curve $\mathcal{F}$, say $\widetilde{\mathcal{F}}$. Then each Hopf component of $\mathcal{F}$, say $\mathcal{F}_j$, has reduced singularities, whose local separatrices are transversal to the invariant projective line D_j (zero section of the Hopf bundle $\mathcal{H}_j$) are given by the Hopf irreducible components of $Sep(\mathcal{F})$, say $Sep(\widetilde{\mathcal{F}}_j)$. Now suppose that $Sep(\widetilde{\mathcal{F}}_j)$ has m irreducible components, with $n < m$ of them being non-corner singularities. Then, by lemma 3, after we blow-up each of these n singularities, we obtain a Hopf bundle with Chern class $c(\mathcal{H}_j) - n$ with only corner singularities, say $\mathcal{H}_{j,0}$, and n Hopf bundle of Chern class -1, say $\mathcal{H}_{j,s}$, $s = 1 \cdots n$ intercepting the previous one by normal crossing at corner singularities (see figure 5). Let us denote the Hopf components of $Sep(\mathcal{F})$ in the new resolution by $Sep(\widetilde{\mathcal{F}}_{j,s}) \in \mathcal{H}_{j,s}$, $s = 0 \cdots n$. Therefore, by lemma 3, the new Hopf component $Sep(\widetilde{\mathcal{F}}_{j,0})$ is rectifiable, as it is composed by the fibers of some radial fibration $\mathcal{R}$ in $\mathbb{H}_{j,0}$, with the base points at the singularities. Moreover, again by remark 8 and proposition 7 (in appendix A) each Hopf component $Sep(\widetilde{\mathcal{F}}_{j,s})$, $s = 1 \cdots n$ of $Sep(\mathcal{F})$ in the new resolution is rectifiable. □

Remark 9. *Note that similarly to the case of an ordinary resolution, we can talk about the a minimal rectifier resolution*

6. Analytic classification

We analytically classify the generalized curves by the trees of singularities of one of their minimal rectifier resolutions, the projective holonomies of the Hopf components of these resolutions and some analytical cocycles that appear as obstructions to extend component-wise conjugacies.

6.1. **Component-wise isomorphisms.** We study the problem of finding conditions to determine whether two component-wise isomorphic germs of foliations up to first order are indeed component-wise isomorphic, that is determine the module space $\sum_{\omega}^{c,1} \Big/ (\sum_{\omega}^{c} \cap \sum_{\omega}^{c,1})$ (recall notation 1) for a minimal rectifier resolution of $(\mathcal{F} : \omega = 0)$. Finally we verify the uniqueness of the ambient surface for a minimal rectifier resolution of any element of $\sum_{\omega}^{c,1}$.

First, recall that $\sum_{\omega}^{c}$ (respect. $\sum_{\omega}^{c,1}$) is the set of germs of foliations which has equivalent minimal rectifier resolution is component-wise analytically equivalent (respect. up to order one) to a minimal rectifier resolution of $(\mathcal{F} : \omega = 0)$. Further we denote as $\sum_{\omega,f}^{c,1}$ the subset of $\sum_{\omega}^{c,1}$ whose separatrix set has the same analytic type of the curve $f^{-1}(0)$.

Now as a straightforward consequence of proposition 2 we have that:

Proposition 4. *Let $\mathcal{F}, \mathcal{F}'$ belong to the same class in $\in \sum_{\omega}^{c,1}$. Then they belong to the same class in $\sum_{\omega}^{c}$ if, and only if the projective holonomies of the Hopf components of one of their minimal rectifier resolutions are conjugated.*

Now, note that once we have two germs of foliations in $\sum_{\omega}^{c}$ we want in some sense to verify under what conditions they are in fact globally holomorphically conjugated. For this sake, it is necessary first to verify indeed that minimal rectifier resolution with equivalent trees of any element of $\sum_{\omega}^{c}$ must be contained in the same surface (up to biholomorphism).

Definition 9. *We say that a complex surface is resolution-like, whether it is obtained by a holomorphic pasting of Hopf bundles with negative Chern class, in such a way that the union of their zero sections become a tree of projective lines isomorphic to the exceptional divisor of the composition of a finite numbers of blowing-up process applied to $(\mathbb{C}^2, 0)$.*

Of course, this definition is given in such a way that every resolution surface of some singularity became resolution-like. In fact, any resolution-like surface is biholomorphic to the resolution surface of some singularity.

Proposition 5. *Let M_j, $j = 1, 2$ be two resolution-like surfaces with isomorphic trees of projective lines D_j, $j = 1, 2$. Then (M_1, D_1) is biholomorphic to (M_2, D_2).*

In order to prove the proposition we need the following results about line-bundles.

Theorem 4 (Grauert [24])**.** *Let S be a complex surface and C be a rational curve with negative Chern class. Then there are neighborhoods U and V of C, respectively in S and $N(C; S)$ (the normal bundle of C in S) and a biholomorphism $\Psi : U \to V$ sending C in the zero section of $N(C; S)$.*

Theorem 5 (Grothendieck [25])**.** *Two complex line bundles over the Riemann sphere have the same Chern class if, and only if they are biholomorphic.*

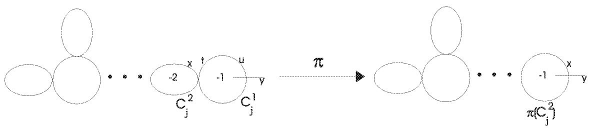
Proof of the proposition 5. The proof is done by induction in the number of projective lines in the chains. In fact if the chains are composed by just one projective line then, the result follows by the theorems of Grauert and Grothendieck. Now let us suppose that the result is true for all chains composed with n (≥ 1) projective lines, and let D_j have $n+1$ projective lines. Then, by hypothesis D_j has two intercepting projective lines, namely C_j^1 and C_j^2, with Chern classes given respectively by -1 and -2. Hence applying Grauert's and Grothendieck's theorems one obtains that a neighborhood of each curve is biholomorphic to a neighborhood of the zero section of the Hopf bundle with respective Chern class. Hence we can blow-down a neighborhood of the curve C_j^1 obtaining yet an analytic surface defined in a neighborhood of Riemann sphere, say $\pi(C_j^2)$ — where π stands for the blow-down (see figure below). In fact $\pi(C_j^2)$ has Chern class -1. For, recall that a neighborhood of C_2 in the surface is biholomorphic to the Hopf bundle of class -2, hence one can construct a global meromorphic section for this line bundle without zeros an with just one pole of order two at the corner (for instance, in affine coordinates one may have $x = 1$ and $u = \frac{1}{u^2}$, where $u = 0$ is the corner point). Now if we look to the affine charts (t, x), (u, y), $u = 1/t$ and $y = tx$, of a neighborhood of C_1 in the surface (with the corner at $t = 0$), then we obtain that the meromorphic section for C_2 is locally given by:

$$t = \frac{a_{-2}}{x^2} + \frac{a_{-1}}{x} + a_0 + a_1 x + \cdots$$

where $a_{-2} \neq 0$. Hence after blowing-down C_1 (see figure below), one obtains a global meromorphic section for $\pi(C_2)$, without zeros and with just one simple pole given by

$$y = \frac{a_{-2}}{x} + a_{-1} + a_0 x + \cdots$$

and thus $\pi(C_2)$ has Chern class -1, as it was claimed.



Now, by induction one has that a biholomorphism $\Phi : (\pi(M_1), \pi(D_1)) \longrightarrow (\pi(M_2), \pi(D_2))$, leaving $\pi(C_1^1)$ in $\pi(C_2^1)$. Finally, by Riemann's extension theorem ([26]) the map Φ can be lifted to a biholomorphism $\widetilde{\Phi} : (M_1, D_1) \longrightarrow (M_2, D_2)$. □

6.2. **Analytic cocycles.** We define a characteristic class for germs of generalized curves, which are the remaining elements to determine their analytic classification. In order to do this we use strongly the concept of non-abelian cohomology. For the reader which is not acquainted with this subject we refer [22] and [29].

First let $(\mathcal{F} : \omega = 0)$, then fix a foliation $\mathcal{F}^{\text{o}}$ component-wise isomorphic to $\mathcal{F}$, such that in the coordinates given by a given minimal rectifier resolution, $Sep(\widetilde{\mathcal{F}}_j^{\text{o}})$ is contained in the fibers of $\mathbb{H}_j$ (this foliation does exist by [27]), which shall be called a **fixed model** for $\mathcal{F}$, and consider the elements $\Phi_j \in Diff(\widetilde{\mathcal{F}}_j, \widetilde{\mathcal{F}}_j^{\text{o}})$, which shall be called a **projective chart** for the j^{th} component of the fixed model. Then it is straightforward that:

Lemma 4. *For each $\widetilde{\mathcal{F}}_j = \widetilde{\mathcal{F}}\Big|_{\mathcal{H}_j, \mathbb{CP}(1)}$ and each fixed model $\widetilde{\mathcal{F}}_j^o$, there exists only one projective chart, up to left composition by an element of $Iso(\widetilde{\mathcal{F}}_j^o)$.*

So, consider the sheaf of non-abelian groups $\Lambda^{\text{o}} := Iso(\widetilde{\mathcal{F}}^{\text{o}})$, then we say that $\mathcal{U} := \cup U_j$ is a good covering for Λ^{o} whether U_j are neighborhoods of $D_j \subset \mathcal{H}_j$. Therefore, consider the first cohomology set $H^1(\mathcal{U}, \Lambda^{\text{o}})$ associated to the good covering $\mathcal{U}$, and set $H^1(D, \Lambda^{\text{o}})$ as the direct limit of $H^1(\mathcal{U}, \Lambda^{\text{o}})$ for the good coverings of $\widetilde{\mathcal{F}}$, associated to $D = \cup D_j$, the exceptional divisor of the given minimal rectifier resolution of $\mathcal{F}$. Hence by proposition 5 the map

$$\begin{array}{ccc} \sum_{\omega}^{c} & \xrightarrow{\Theta} & H^1(D,\Lambda^{o}) \\ \mathcal{F} & \mapsto & (\Phi_{i,j}) \end{array}$$

where $\Phi_{i,j} := \Phi_i \circ \Phi_j^{-1}$, is well defined and onto. Note that Θ does not depends on the fixed models up to component-wise conjugacy class.

Note that by the definition of the fixed model we have that $\widetilde{\omega}_j^{o}(u_j, y_j) = \widetilde{\omega}_{j+1}^{o}(t_{j+1}, x_{j+1})$, where $(\widetilde{\mathcal{F}}_l^{o} : \widetilde{\omega}_l^{o} = 0)$, $l = \{1 \cdots k\}$ (see figure 1.2 in §1.2).

Proposition 6. *Two generalized curves $\mathcal{F}, \mathcal{G} \in \sum_{\omega}^{c}$ are analytically equivalent if, and only if $\Theta(\mathcal{F}) = \Theta(\mathcal{G})$.*

Proof. Let $\Theta(\mathcal{F}) = (\Phi_1 \circ \Phi_2^{-1}, \cdots, \Phi_{k-1} \circ \Phi_k^{-1})$, and $\Theta(\mathcal{G}) = (\Psi_1 \circ \Psi_2^{-1}, \cdots, \Psi_{k-1} \circ \Psi_k^{-1})$. First let us verify the necessary part. Suppose that H is a global conjugation between $\mathcal{F}$ and $\mathcal{G}$, that is $H^*(\mathcal{G}) = \mathcal{F}$. Then, by lemma 4 $\Psi_j = \alpha_j \circ \Phi_j \circ H$ for some $\alpha_j \in Iso(\widetilde{\mathcal{F}}_j^{o})$, and thus

$$\begin{aligned} \Psi_{j-1} \circ \Psi_j^{-1} &= \alpha_{j-1} \circ \Phi_{j-1} \circ H \circ H^{-1} \circ \Phi_j^{-1} \circ \alpha_j^{-1} \\ &= \alpha_{j-1} \circ \Phi_{j-1} \circ \Phi_j^{-1} \circ \alpha_j^{-1}. \end{aligned}$$

Now let us see the sufficient part. First note that by hypothesis $\mathcal{F}$ and $\mathcal{G}$ have the same fixed models. So, if $(\Phi_1 \circ \Phi_2^{-1}, \cdots, \Phi_{k-1} \circ \Phi_k^{-1}) = \Theta(\mathcal{F}) = \Theta(\mathcal{G}) = (\Psi_1 \circ \Psi_2^{-1}, \cdots, \Psi_{k-1} \circ \Psi_k^{-1})$, then there is a collection (α_j), with $\alpha_j \in Iso(\widetilde{\mathcal{F}}_j^{o})$, such that $\Psi_{j-1} \circ \Psi_j^{-1} = \alpha_{j-1} \circ \Phi_{j-1} \circ \Phi_j^{-1} \circ \alpha_j^{-1}$. Hence, $(\alpha_{j-1} \circ \Phi_{j-1})^{-1} \circ \Psi_{j-1} = (\alpha_j \circ \Phi_j)^{-1} \circ \Psi_j$, so we can define a global conjugacy between them just by letting $H := (\alpha_j \circ \Phi_j)^{-1} \circ \Psi_j$, for $j = 1 \cdots k$. □

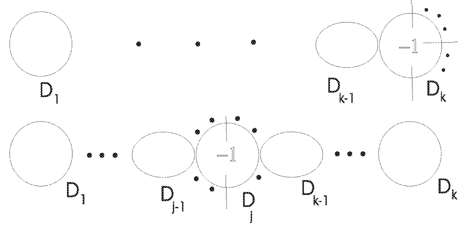
Hence, by the construction made in this section one obtains that theorem 2 follows from propositions 1, 4, and 6.

7. Application: Projective holonomy and some cusps

We classify the set $\sum_{\omega,f}^{c,1}$ for (reduced) $f(x,y) = \prod_{j=1}^{d}(y^p - \lambda_j x^q)$, where $1 < p < q \in N^*$, $g.c.d.(p,q) = 1$, and $\lambda_j \in \mathbb{C}^*$. In fact we show that their characteristic classes vanish and use the geometry of the exceptional divisor of the minimal resolution (which coincides with a minimal rectifier one) to show that the projective holonomy of the principal projective line classifies such generalized curves.

First let us investigate the geometry of the exceptional divisor of a minimal rectifier resolution of the Milnor fibrations $f(x,y) = \prod_{j=1}^{d}(y^p - \lambda_j x^q)$, where $p < q \in N^*$, $\lambda_j \in \mathbb{C}^*$, and $g.c.d.(p,q) = 1$. Then, it is not difficult to see that it has a simple resolution.

Lemma 5. *Let $\mathcal{F}$ be a foliation given by $(df = 0)$. Then $\mathcal{F}$ has a simple resolution with one of the following "schemes" of resolutions:*



Respectively whether $p = 1$ or not.

Proof. As we remarked, a generalized curve solves together with its separatrices. On the other hand, each irreducible curve $y^p - \lambda_j x^q = 0$ is a generic fiber of the fibration $\frac{y^p}{x^q} \equiv const$, so it solves together with the fibration. After one blow-up, one obtains:

$$t^p/x^{q-p} \equiv const,$$
$$u^q y^{q-p} \equiv const.$$

Hence, as $p < q$, we have a singularity with holomorphic first integral at infinity and a meromorphic first integral at the origin (as before). Going ahead with this process, then the Euclid's algorithm tells that the resolution stops when we blow up a radial foliation. In particular, if $p = 1$ then it is easy to see that the principal projective line is transversal to just one projective line of the divisor. Otherwise (that is, in case $p \neq 1$), as the singularity with meromorphic first integral "moves" to the "infinity", that is, it will appear in a corner singularity, then the principal projective line intercepts exactly two projective lines of the divisor. □

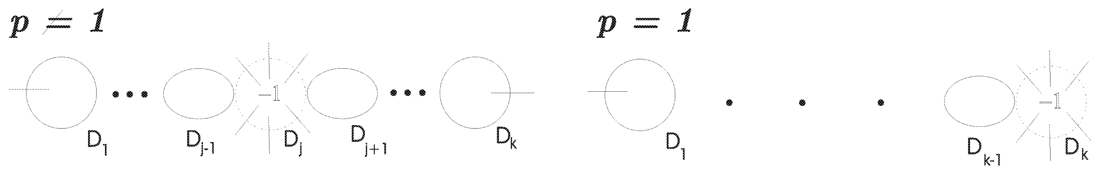


Figure 7

As an immediate consequence of the proof one has that:

Corollary 1. *Let* $\mathcal{F} \in \sum_{\omega,f}^{c,1}$*, with* $f(x,y) = \prod_{j=1}^{d}(y^p - \lambda_j x^q)$*, where* $1 < p < q \in N^*$*, g.c.d.*$(p,q) = 1$*, and* $\lambda_j \in \mathbb{C}^*$*. Then the number of singularities at the principal projective lines are given by*

$$\begin{cases} d+1 & \text{if } p = 1 \\ \\ d+2 & \text{otherwise} \end{cases}.$$

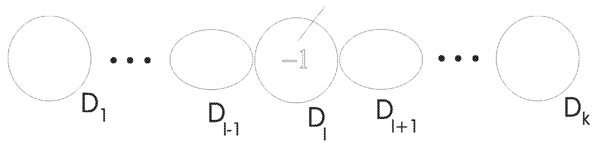
In particular if $f(x,y) = y^m + x^n$*, then this number is given by*

$$\begin{cases} m+1 & \text{if } m \text{ divides } n \\ \\ g.c.d.(m,n)+2 & \text{otherwise} \end{cases}.$$

Now, in other to prove theorem 3 we have to state some notation and preliminary results. Recall that $\mathcal{F} \in \sum_{\omega,f}^{c,1}$, with $f(x,y) = \prod_{j=1}^{d}(y^p - \lambda_j x^q)$, where $1 < p < q \in N^*$, $g.c.d.(p,q) = 1$, and $\lambda_j \in \mathbb{C}^*$. Then we order the first projective line to rise along the minimal (rectifier) resolution with 1 and increasingly with 2 the next one to intercept it — in the minimal (rectifier) resolution — and so on (see lemma 5 and the figure below). In particular we denote the principal projective line by D_l. As $D_1 \backslash Sing(\widetilde{\mathcal{F}}_1)$, is simply connected, so the holonomy of $\widetilde{\mathcal{F}}_1$ is trivial, hence, the germ of foliation $\widetilde{\mathcal{F}}_{1,2}$, determined by the restriction of $\widetilde{\mathcal{F}}$ to a neighborhood of $t_{1,2} := D_1 \cap D_2$, has a (local) holomorphic first integral. So the holonomy of $\widetilde{\mathcal{F}}_{1,2}$ with respect to D_2 is linearizable. Note that for $j \neq l$ we have that $\widetilde{\mathcal{F}}_j$ has at most two singularities, so if $l > 2$ then the projective holonomy of $\widetilde{\mathcal{F}}_2$ coincides with the local holonomy of $\widetilde{\mathcal{F}}_{1,2}$ with respect to D_2 and hence it is linearizable. On the other hand let $(\widetilde{\mathcal{F}}_j^{\mathrm{o}} : \widetilde{\omega}_j^{\mathrm{o}} = 0)$, $j \neq l$, have the (global) holomorphic first integral:

$$\text{(7.1)} \qquad \begin{cases} \widetilde{\omega}_j^{\mathrm{o}}(\tau,\xi) \wedge d(\tau^{p_j}\xi^{q_j}) = 0, \\ \widetilde{\omega}_j^{\mathrm{o}}(\upsilon,\zeta) \wedge d(\upsilon^{k_j q_j - p_j}\zeta^{q_j}) = 0. \end{cases}$$

where (τ, ξ), (υ, ζ) are affine charts for the Hopf bundle $\mathbb{H}_j(-k_j)$, and p_j, $q_j \in \mathbb{N}$ are relatively prime. Hence, by Camacho-Sad index theorem one has that $\widetilde{\mathcal{F}}_2$ and $\widetilde{\mathcal{F}}_2^{\circ}$ are isomorphic up to order one. In fact, as the projective holonomy of $\widetilde{\mathcal{F}}_2$ is linearizable, then proposition 2 guarantees that they are isomorphic. Applying the same arguments inductively we have that $\widetilde{\mathcal{F}}_j$ is isomorphic to $\widetilde{\mathcal{F}}_j^{\circ}$ for all $j \neq l$. In particular each Hopf component $\widetilde{\mathcal{F}}_j$ for all $j \neq l$ has a (global) holomorphic first integral.



Now note that, by the proof of lemma 5, we have that $\mathcal{F}$ solves together with any "generic" fiber of the "companion" fibration $\frac{y^p}{x^q} \equiv const$, that is $(\mathcal{G}, \eta = 0)$ where $\eta(x, y) = pxdy - qydx$. In particular the minimal (rectifier) resolution of $\mathcal{G}$ has the same tree of projective lines of any element of $\sum_{\omega,f}^{c,1}$ and contains its separatrices as fibers. Further, note that $\widetilde{\mathcal{G}}_l$ is a radial fibration, thus $\widetilde{\mathcal{G}}_{l-1}$ has just one singularity (see figure 7) and by similar arguments as above, each $\widetilde{\mathcal{G}}_j$, $j \neq l$, has a (global) holomorphic first integral of the form

$$\widetilde{\eta}(t, x) = d(t^{r_j} x^{s_j}),$$
$$\widetilde{\eta}(u, y) = d(u^{k_j s_j - r_j} y^{s_j}).$$

where (t, x), (u, y) are affine charts for the Hopf bundle $\mathbb{H}_j(-k_j)$, and p_j, $q_j \in \mathbb{N}$ are relatively prime. Applying the same reasoning to $j > l$, in case $p \neq 1$, one obtains the same result for all $j \neq l$. Now, comparing $\widetilde{\mathcal{F}}_j$ and $\widetilde{\mathcal{G}}_j$ (starting from $l - 1$ to 1) in view of Camacho-Sad's index theorem one obtain that $p_j \neq r_j$ and $q_j \neq s_j$. In particular we have that $p_j s_j - q_j r_j \neq 0$, for all $j \neq l$, since $g.c.d(p_j, q_j) = g.c.d.(r_j, s_j) = 1$.

Now notice that in the coordinate system (τ, ξ), (υ, ζ), as in (7.1), we have that

$$\tau = t \cdot U_1(t, x) \quad \xi = x \cdot U_2(t, x)$$
$$\upsilon = u \cdot V_1(u, y) \quad \zeta = y \cdot V_2(u, y)$$

where $U_j, V_j \in \mathcal{O}^*(\mathbb{C}, \mathbb{D}_\epsilon)$ ($\mathbb{D}_\epsilon$ is the disk of radius ϵ centered at the origin) and $U_j(t, 0) = V_j(u, 0) = 1$. Then we have that $V_1(u, y) = 1/U_1(1/u, u^{k_j} y)$, $V_2(u, y) = U_1(1/u, u^{k_j} y) \cdot U_2(1/u, u^{k_j} y)$. So the zero and polar sets of U_1 do not intercept $D_j \subset \mathcal{H}_j$. Hence we have that $U_1(t, x) = \sum_{m < \frac{n}{k_j}} a_{m,n} t^m x^n$. Similarly one has that $U_2(t, x) = \sum_{m < \frac{n}{k_j}} b_{m,n} t^m x^n$. In particular $\tau^{p_j} \xi^{q_j} = t^{p_j} x^{q_j} U(t, x)$, where $U(t, x) = \sum_{m < \frac{n}{k_j}} c_{m,n} t^m x^n \in \mathcal{O}^*(\mathbb{C}, \mathbb{D}_\epsilon)$, that is $V(u, y) := U(1/u, u^{k_j} y) \in \mathcal{O}^*(\mathbb{C}, \mathbb{D}_\epsilon)$.

Lemma 6. *For all $j = 1 \cdots k$, there exists a projective chart $\Phi_j \in Diff_{\widetilde{\mathcal{F}}_j, \widetilde{\mathcal{F}}_j^{\circ}}(\mathcal{H}_j, \mathbb{CP}(1))$ such that $\Phi_j \in Iso(\widetilde{\mathcal{G}}_j)$. In particular, there are coordinates (t_1, x_1), and (u_1, y_1) for $\mathbb{H}_j$, such that the first integrals of $\widetilde{\mathcal{F}}_j$, and $\widetilde{\mathcal{G}}_j$ are given respectively by $t_1^{p_j} x_1^{q_j}$, $u_1^{k_j q_j - p_j} y_1^{q_j}$ and $t_1^{r_j} x_1^{s_j}$, $u_1^{k_j s_j - r_j} y_1^{s_j}$.*

Proof. First note that for the case $j = l$ the statement comes from proposition 2. Now let us consider the case $j \neq l$. In this case we first note that $\widetilde{\mathcal{F}}_j$ has first integrals of the form $t^{p_j} x^{q_j} U(t, x)$, where $U \in \mathcal{O}_2^*$. Hence if $\Phi_j^{-1}(t, x) = (a_j(t, x), b_j(t, x))$, then we have to find a solution for the system of equations

$$\begin{cases} a_j(t, x)^{p_j} b_j(t, x)_j{}^{q_j} = t^{p_j} x^{q_j} U(t, x) \\ a_j(t, x)^{r_j} b_j(t, x)^{s_j} = t^{r_j} x^{s_j} \end{cases}$$

which can be given in the affine chart by

$$a_j(t,x) = tU(t,x)^{\frac{s_j}{p_j s_j - q_j r_j}},$$
$$b_j(t,x) = xU(t,x)^{\frac{r_j}{q_j r_j - p_j s_j}}.$$

A straightforward calculation shows that the expression of Φ_j^{-1} in the chart of ∞, is given by

$$\Phi_j^{-1}(u,y) = (uV(u,y)^{\frac{s_j}{q_j r_j - p_j s_j}}, yV(u,y)^{\frac{r_j - k_j s_j}{q_j r_j - p_j s_j}})$$

where $V(u,y) := U(1/u, u^{k_j}y) \in \mathcal{O}^*(\mathbb{C}, \mathbb{D}_\epsilon)$, as it was remarked above. Finally $\Phi_j^{-1} \in Diff(\mathcal{H}_j, \mathbb{CP}(1))$ by the implicit function theorem and carry leaves of $\widetilde{\mathcal{F}}_j$ in leaves of $\widetilde{\mathcal{F}}_j^o$ by construction. □

Now let us denote $\widetilde{\mathcal{F}}_{j,j+1}$, and $\widetilde{\mathcal{G}}_{j,j+1}$ the germs of foliations defined in a neighborhood of the corner $t_{j,j+1} = D_i \cap D_j$, by the restriction of $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{G}}$., respectively.

Lemma 7. *Let $\Phi_{j-1,j} \in Iso(\widetilde{\mathcal{F}}^o_{j-1,j}) \cap Iso(\widetilde{\mathcal{G}}_{j-1,j})$ for $j = 2\cdots k$. Then $\Phi_{j-1,j}$ has a unique extension to $\Phi_j \in Iso(\widetilde{\mathcal{F}}^o_j)$*

Proof. First note that the uniqueness comes from the identity theorem. Then assuming that the corner is in the affine chart of ∞ of $\mathcal{H}_j$, we have $\Phi_{j-1,j}(t,x) = (a(t,x), b(t,x)) \in \mathcal{O}(\mathbb{D}^* \times \mathbb{D})$, where $\mathbb{D}$ is the unity disk. Now, as $\Phi_{j-1,j} \in Iso(\widetilde{\mathcal{F}}^o_{j-1,j}) \cap Iso(\widetilde{\mathcal{G}}_{j-1,j})$ then it satisfies the following system of equations.

$$\begin{cases} a(t,x)^{p_j} b(t,x)^{q_j} = t^{p_j} x^{q_j} \\ a(t,x)^{r_j} b(t,x)^{s_j} = t^{r_j} x^{s_j} \end{cases}$$

whose solutions are

$$a_j(t,x) = \alpha t,$$
$$b_j(t,x) = \beta x,$$

where α, $\frac{1}{\beta}$, are $(p_j s_j - q_j r_j)$-roots of unity. □

Finally we give the desired

Proof of theorem 3. For simplicity in the induction argument, we prove the statement in case the principal projective line is in the "edge" of the resolution, that is, it intercepts just one projective line. Let $\mathcal{F}, \mathcal{F}' \in \sum_{\omega,f}^{c,1}$, with projective charts $\Phi_j \in Diff_{\widetilde{\mathcal{F}}_j, \widetilde{\mathcal{F}}_j^o}(\mathcal{H}_j, \mathbb{CP}(1))$ and $\Phi'_j \in Diff_{\widetilde{\mathcal{F}}'_j, \widetilde{\mathcal{F}}_j^o}(\mathcal{H}_j, \mathbb{CP}(1))$ respectively, for $j = 1\cdots l-1$. Then by lemma 6 we may suppose that $\Phi_j, \Phi'_j \in Iso(\widetilde{\mathcal{G}}_j)$. Now we construct by a decreasing induction, a collection (α_j), with $\alpha_j \in Iso(\widetilde{\mathcal{F}}_j^o) \cap Iso(\widetilde{\mathcal{G}}_{j,j+1})$, such that $\Phi'_{j-1} \circ \Phi'^{-1}_j = \alpha_{j-1} \circ \Phi_{j-1} \circ \Phi_j^{-1} \circ \alpha_j{}^{-1}$. First let $\alpha_l = Id|_{Iso(\widetilde{\mathcal{F}}_l^o)}$ (here l stands for the principal projective line) and suppose that α_j $(j < l)$ is already defined, then let $\alpha_{j-1,j} := \Phi'_{j-1} \circ \Phi'^{-1}_j \circ \alpha_j \circ (\Phi_{j-1} \circ \Phi_j^{-1})^{-1}$. By construction, $\alpha_{j-1,j} \in Iso(\widetilde{\mathcal{F}}^o_{j-1,j}) \cap Iso(\widetilde{\mathcal{G}}_{j-1,j})$ and therefore, by lemma 7, it has an unique extension, namely $\alpha_{j-1} \in Iso(\widetilde{\mathcal{F}}^o_{j-1})$. Then we have that $\Phi'_{j-1} \circ \Phi'^{-1}_j = \alpha_{j-1} \circ \Phi_{j-1} \circ \Phi_j^{-1} \circ \alpha_j^{-1}$ for all $j = 1\cdots l-1$. The other case can be treated similarly, just differing by the induction arguments. □

8. Comments and open problems

First let us recall that in theorem 3 we show that in some cases the separatrix type may determine the characteristic classes $\Theta(\mathcal{F})$. On the other hand, this is not to be expected in the general case. Therefore it is natural to propose the following

Problem 1. *Construct two generalized curves $\mathcal{F}$ and $\mathcal{F}'$ which has component-wise equivalent minimal rectifier resolutions and have the same separatrices set, but with distinct classes, that is $\mathcal{F}, \mathcal{F}' \in \sum_{\omega}^{c}$, $Sep(\mathcal{F}) = Sep(\mathcal{F}')$, and $\Theta(\mathcal{F}) \neq \Theta(\mathcal{F}')$.*

Now let us recall that in [5], and [8] the nilpotent centers are classified. One of the main tools is the fact that the semi-Poincaré map is obtained by a relation between the generators of the projective holonomy of the principal projective line of the minimal resolution (which coincides with a minimal rectifier one). Indeed, this fact is used to show that such centers have a non-trivial involution. On the other hand, theorem 2 shows that a kind of relation as above — that is, depending just on the projective holonomy — is not to be expected for more general simple resolution foliations, even when there are few singularities in the principal projective line. In fact this semi-Poincaré map have to take account the characteristic classes $\Theta(\mathcal{F})$.

On the other hand we can see that for some particular case, $H^1(D, \Lambda^o)$ may be resumed to the calculation of the orbit space of the bi-action of $Iso(\widetilde{\mathcal{F}}_1) - Iso(\widetilde{\mathcal{F}}_2)$ in $Iso(\widetilde{\mathcal{F}}_{1,2})$, that is

$$Iso(\widetilde{\mathcal{F}}_1) \backslash Iso(\widetilde{\mathcal{F}}_{1,2}) / Iso(\widetilde{\mathcal{F}}_2)$$

as in the calculation of $\Theta(\mathcal{F})$ for $\sum_{d(y^3+x^6)}^{c}$ (see figure 1.2 in §1.2). This leads to the following:

Problem 2. *Let $G \subset Diff(\mathbb{C}^2, 0)$be a group and H_1 and H_2 be two subgroups of G. Hence as the composition turns G into a $(H_1 - H_2)$ bi-space, then describe the orbit space*

$$H_1 \backslash G / H_2.$$

In fact, the situation is very similar for any generalized curve, but this is the first step to be understood.

Further, remark that in [4] it is studied the conjugacy classes of $Iso(\mathcal{F})$ for $\mathcal{F}$ a germ of foliation in $(\mathbb{C}^2, 0)$. Hence, by the above remark, it is also natural to study the following:

Problem 3. *Let $\mathbb{H} : (p : \mathcal{H} \longrightarrow \mathbb{CP}(1))$ be a Hopf bundle and $\widetilde{\mathcal{F}}$ a non-dicritical solved foliation in $(\mathcal{H}, \mathbb{CP}(1))$. Then describe the conjugacy classes of $Iso(\widetilde{\mathcal{F}})$.*

Finally, we want to remark that the classes constructed in this work, were conceived in the same spirit of the characteristic classes of vector bundles ([22]), of resonant diffeomorphisms ([21]) and of saddle-nodes ([29]). In fact we define the "cellular" objects, — that is the most elementary geometrical objects of the solved foliation — to be the Hopf components of the foliation, and study the obstruction to extend any conjugacy between two of them. Similarly, in the bundle case, this natural cellular objects are the local trivial bundles. In the case of resonant diffeomorphisms and saddle-nodes these objects turn out to be restrictions of formal normal forms to sectorial open sets. Notice further that our case is the first one whose classification space is not a (direct) product of analytic spaces but a twisted one. This leads us to the study of infinite dimensional Lie groups.

In fact, as we shall show in a future work, for the remaining cases of Thom's problem, we have to redefine the equivalence classes for the projective holonomy. Indeed, even the dicritical singularities may be treated in this fashion, but changing the "cell" notion.

Appendix A. The five lines problem

We study the problem of simultaneous linearizing of germs of regular curves at the origin. These results are well known for the specialists, but by the absence of an explicit reference we prove it here.

Proposition 7. *Every germ of singular curve composed by at most four germs of non-singular irreducible analytic curves, non-tangent to each other, in* $(\mathbb{C}^2, 0)$ *can be linearized. That is, there exist a complex analytic diffeomorphism taking each irreducible component in its tangent direction line at the origin. In particular two of them are equivalent if, and only if they have isomorphic tree of points.*

Proof. First, as an application of the holomorphic implicit function theorem, two such curves may be supposed to be the axis (this may be easily verified directly, as in the case of three curves). Then for each curve $(y - g(x) = 0)$, with $g'(0) \neq 0 = g(0)$, we have that

$$\phi(x, y) = \left(x, y\left(\frac{g'(0) \cdot x}{g(x)} - 1\right)\right)$$

belongs to $Diff(\mathbb{C}^2, 0)$, preserves the axis, and takes the curve $(y - g(x) = 0)$ to its tangent direction line, at the origin: $(y - g'(0) \cdot x = 0)$. Now let us suppose that $\psi \in Diff(\mathbb{C}^2, 0)$ fixes three directions. As the components of the curve are non-tangent to each other, and are non-singular, after one blow-up the curve is solved. Hence, as $\mathbb{CP}(1)$ is 3-transitive by the group of its conformal transformation, then the three directions may be supposed to be $0, 1, \infty \in \mathbb{CP}(1)$. Therefore, for each curve $(y - g(x) = 0)$, with $g'(0) \neq 0 = g(0)$, we have that

$$\psi(x, y) = \left(x, y\left(1 + (y - x)\frac{g'(0) \cdot x - g(x)}{g(x)(g(x) - x)}\right)\right)$$

belongs to $Diff(\mathbb{C}^2, 0)$, keeps the axis and the line $(y - x = 0)$ invariant, and takes the curve $(y - g(x) = 0)$ to its tangent direction line, at the origin: $(y - g'(0) \cdot x = 0)$. □

Proposition 8. *In general, the germs of singular curve composed by at least five non-singular irreducible germs of analytic curves, non-tangent to each other, in* $(\mathbb{C}^2, 0)$ *cannot be linearized. In particular, such germs are not equivalent in general.*

Proof. Let $\phi \in Diff(\mathbb{C}^2, 0)$, keeping invariant four lines, which as we already note, may be assumed to be $(x = 0)$, $(y = 0)$, $(y - x = 0)$, $(y - kx = 0)$, where $k \neq 0, 1$ is determined by the cross ratio of the preserved directions, in $\mathbb{CP}(1)$. Now note that, as ϕ fixes four directions, then $\pi^*\phi|_{\mathbb{CP}(1)}$ fixes four points, and thus is the identity map, that is $J^1\phi = \lambda \cdot Id_{(\mathbb{C}^2,0)}$, $\lambda \in \mathbb{C}^*$. Further, as ϕ fixes the axis we have that

$$\phi(x, y) = (x(\lambda + A(x, y)), y(\lambda + B(x, y)))$$

On the other hand, note that ϕ fixes the four given lines and take another curve into its tangent direction line if, and only if

$$\psi(x, y) = (x, y(1 + C(x, y)))$$

where $1 + C(x, y) = \frac{\lambda + B(x,y)}{\lambda + A(x,y)}$; do as well. Further, as ψ fixes the lines $(y - x = 0)$, and $(y - kx = 0)$, then

$$\psi(x, y) = (x, y(1 + (y - x)(y - kx)E(x, y)))$$

where $E \in \mathcal{O}^2$. Finally it takes the curve $(x, g(x))$ into the line $(x, g\prime(0) \cdot x)$ if, and only if

$$g(x)[1 + (g(x) - x)(g(x) - kx)E(x, g(x))] = g'(0) \cdot x$$

and thus

$$E(x,g(x)) = \frac{g'(0)\cdot x - g(x)}{g(x)(g(x)-x)(g(x)-kx)}$$

Hence, as g may have non-null second order coefficient in his Taylor expansion, then $E(x,g(x))$ has a pole at the origin, which is a contradiction. □

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