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GENERIC WEBS ON THE COMPLEX PROJECTIVE PLANE

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To my parents Joseph
and Dorothy with love,
admiration and thanks.

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Introdução

Uma 'web' é uma coleção de d-folheações em posição geral e com a mesma codimensão. O nome Francês e Alemão para o conceito é, respectivamente, 'tissu' e 'Gewebe'. 'Webs' têm sido estudadas por vários autores do ponto de vista local e com várias técnicas. Akivis e Shelekhon [1] utilizam técnicas da Geometria Diferencial, Hénaut [14] técnicas de D-módulos e Alcides e Nakai [22] técnicas de Geometria Analítica. Tais estudos incluem linearização de 'webs', curvatura de 'webs', posto de 'webs', 'webs' hexagonais, etc.

Na tese, estudamos 'webs' do ponto de vista global. A idéia principal desse trabalho é generalizar, se possível, os trabalhos e resultados conhecidos para folheações (1-web) em $\mathbb{C}P^2$ para 'webs'.

O estudo de 'webs' singulares é motivado pela equação diferencial implícita de primeira ordem na forma

$$F(x, y, p) = 0, \qquad p = \frac{dy}{dx} \qquad (*)$$

onde F é uma polinômio de grau d em p. Nos pontos $(x_0, y_0, p_0) \in \mathbb{C}^3$ onde a derivada parcial $F_p = \frac{\partial F}{\partial p} \neq 0$, podemos reescrever a equação acima na forma

$$p = \frac{dy}{dx} = f_j(x, y), \qquad j = 1, \ldots, d,$$

pelo teorema das funções implícitas, onde f_j são funções holomorfas numa vizinhança (x_0, y_0). Portanto temos d-curvas passando por (x_0, y_0). No ponto onde $F = F_p = 0$, não esperamos boas propriedades como acima em geral. Vários estudos foram feitos [ver [27], [8] [2], [9], [29]] para dar a forma local normal das curvas integrais de $(*)$ numa vizinhança de tais pontos.

No capítulo 1, definimos d-'webs' como dadas por

$$a_0(x,y)dy^d + a_1(x,y)dy^{d-1}dx + \cdots + a_d(x,y)dx^d = 0$$

onde $a_j's$ são polinômios. A noção de grau de uma 'web' é dada pelo número de tangências de uma reta genérica com as curvas integrais da 'web'. Sobre esse fato, damos condições para que a reta no infinito sejam não invariante pelo d-'web'. Damos então a forma de uma d-'web' de grau n em $\mathbb{C}P^2$. Isto é,

Proposição 1.4.2 Uma d-'web' singular, $\mathcal{W}_d$ de grau n em $\mathbb{C}P^2$ pode ser expressa numa carta afim $(x,y) \in \mathbb{C}^2$ por uma equação diferencial da forma

$$\sum_{j=0}^{d} a_j(x,y)dx^j dy^{d-j} + \sum_{j=1}^{d}\sum_{k=0}^{d-j}(xdy - ydx)^j dx^k dy^{d-j-k} g_{jk}(x,y) = 0 \qquad (*)$$

onde

a) g_{jk} são polinômios homogêneos de grau n(ou $g_{jk} \equiv 0$.)

b) $a_j's$ são polinômios de grau $\leq n$ para $j = 0, \ldots, d$.

c) a reta no infinito, $\mathcal{L}_\infty$, é invariante por $\mathcal{W}_d$ se e somente se $g_{d0} \equiv 0$.

Segue da proposição anterior uma estrutura do espaço projetiva para o conjunto de d-'webs' em $\mathbb{C}P^2$, onde $d \in \mathbb{N}, d \geq 0$.

Colocando $p = \frac{dy}{dx}$ na forma $(*)$, temos uma função $F : \mathbb{C}^3 \longrightarrow \mathbb{C}$. Portanto observamos neste capítulo que dado uma d-'web' em $\mathbb{C}P^2$ podemos associar a ela uma superfície $S_{\mathcal{W}_d} = F^{-1}(0) \subset \mathbb{P}T^*\mathbb{C}P^2$ e uma folheação $\mathcal{F}_{\mathcal{W}_d}$ em $S_{\mathcal{W}_d}$ cujas folhas, quando projetadas por $\pi : \mathbb{P}T^*\mathbb{C}P^2 \longrightarrow \mathbb{C}P^2$, dão a

d-'web' em $\mathbb{C}P^2$.
Como uma corolario do proposição acima, damos uma outra forma da uma d-web em $\mathbb{C}P^2$ de grau n :

Corolario 1.4.2 Uma d-'web' singular, $\mathcal{W}_d$ de grau n em $\mathbb{C}P^2$ tambem pode ser expressa em coordenadas homogêneas $(x, y, z) \in \mathbb{C}^3$ por

$$\sum_{j=0}^{d} \sum_{k=0}^{d-j} A_{jk}(x, y, z)\alpha^j \beta^k \gamma^{d-j-k} = 0 \qquad (**)$$

onde

a) A_{jk} são polinômios homogêneos de grau n.

b) $\alpha = xdy - ydx, \beta = zdx - xdz, \gamma = ydz - zdy$, com $x\gamma + y\beta + z\alpha = 0$.

Agora colocando $G(x, y, z, \alpha, \beta, \gamma)$ como na forma $(**)$ temos uma função $G : \mathbb{C}^3 \times \mathbb{C}^3 \longrightarrow \mathbb{C}$ e a superfície associada ao d-'web' é dada por

$$S_{\mathcal{W}_d} = \{G = 0\} \subset X = \{x\gamma + y\beta + z\alpha = 0\} \subset \mathbb{C}P^2 \times \mathbb{C}P^2.$$

Observamos que X é isomorfa ao $\mathbb{P}T^*\mathbb{C}P^2$ e a função $G(x, y, z, \alpha, \beta, \gamma)$ é simétrica no seguinte sentido: dada uma d-'web' de grau n, então trocando as posições de (x, y, z) e (α, β, γ) temos uma n-'web' de grau d.

É importante observar que a escritura $(**)$ não é única. Isto é $G_1(x, y, z, \alpha, \beta, \gamma)$ e $G_2(x, y, z, \alpha, \beta, \gamma)$ defina a mesma d-'web' em $\mathbb{C}P^2$ se e somente se $\Big(G(x, y, z, \alpha, \beta, \gamma) - \lambda G_2(x, y, z, \alpha, \beta, \gamma)\Big)$ é divisível por $x\gamma + y\beta + z\alpha$, para alguma $\lambda \in \mathbb{C}^*$.

No capítulo 2 observamos que em seu estudo de funções do plano no plano, Whitney [28] provou que qualquer destas funções pode ser aproximada por uma outra com singularidades particularmente simples. Aplicando esta filosofia para $\pi|_{S_{\mathcal{W}_d}} : S_{\mathcal{W}_d} \longrightarrow \mathbb{C}^2$ temos as seguintes propriedades:

(1) $S_{\mathcal{W}_d}$ é liso

(2) o conjunto crítico de π é liso

(3) as únicas singularidades de π são nós e cúspides

(4) as singularidades da curva discriminate são nós e cúspides.

Dizemos que uma d-'web' in $\mathbb{C}P^2$ é genérica se ela tem as propriedades (1)-(4). Nesse sentido provamos:

Teorema 2.1.1 O conjunto dos d-'webs' de grau n, em $\mathbb{C}P^2$ com as propriedades (1)-(4) acima é um subconjunto aberto e denso no espaço dos d-'webs'.

Finalmente no último capítulo, damos o número de cúspides e nós da projeção e também o número de singularidades da folheação genérica $\mathcal{F}_{\mathcal{W}_d}$ em $S_{\mathcal{W}_d}$, em termos de d e do grau da web n. Então provamos:

Teorema 3.3.1 Seja $\mathcal{W}_d$ uma d-'web' genérica em $\mathbb{C}P^2$ de grau n. Então o número de cúspides κ e de nós δ da projeção $\pi : S_{\mathcal{W}_d} \longrightarrow \mathbb{C}P^2$ são dados por

$$\kappa = 3(d-2)(n^2+nd+d)$$
$$\delta = \tfrac{1}{2}(d-2)(d-3)(3d+d^2+4nd+4n^2)$$

Teorema 3.3.2 Seja $\mathcal{W}_d$ uma d-web genérica em $\mathbb{C}P^2$ de grau n tal que a folheação assosciada $\mathcal{F}_{\mathcal{W}_d}$ em $S_{\mathcal{W}_d}$ tem somente singularidades isoladas. Então o número total das singularidades de $\mathcal{F}_{\mathcal{W}_d}$ é dado por

$$\sum_{p\in S_{\mathcal{W}_d}} mult(\mathcal{F}_{\mathcal{W}_d}, p) = 3nd^2 + 3n^2d - d^2 - n^2 - 4nd + 3n + 3d$$

No Teorema 3.3.2 observamos que o resultado é simétrico em n e d, isto é,

$$\sum_{p\in S_{\mathcal{W}_d}} mult(\mathcal{F}_{\mathcal{W}_d}, p) = H(n,d) = H(d,n).$$

Damos também a fórmula da soma dos índices de Baum-Bott para a folheação $\mathcal{F}_{\mathcal{W}_d}$ em $S_{\mathcal{W}_d}$. Isto é

Proposição 3.2.1 Seja $\mathcal{W}_d$ uma d-web genérica em $\mathbb{C}P^2$ de grau n. Então

$$\sum_{p \in S_{\mathcal{W}_d}} BB(\mathcal{F}_{\mathcal{W}_d}, p) = 3(n+d)$$

onde BB representa o índice de Baum-Bott. Comparamos os resultados no Teorema 3.3.2 e na Proposição 3.2.1 com o caso $d = 1$. Isto é colocando $d = 1$ no Teorema 3.3.2 e na Proposição 3.2. temos os seguintes :

$$\sum_{p \in S_{\mathcal{W}_1}} mult(\mathcal{F}_{\mathcal{W}_1}, p) = 2(n^2 + n + 1) \quad (1)$$

$$\sum_{p \in S_{\mathcal{W}_1}} BB(\mathcal{F}_{\mathcal{W}_1}, p) = 3(n+1) \quad (2)$$

Mas observamos que quando $d = 1$, a 1-web, é dada por

$$a_0(x,y)dy + a_1(x,y)dx + (xdy - ydx)g_{10}(x,y) = 0$$

isto é, uma folheção $\tilde{\mathcal{F}}$ em $\mathbb{C}P^2$ de grau n na carta afim (x, y), $\pi|_{S_{\mathcal{W}_1}} : S_{\mathcal{W}_1} \longrightarrow \mathbb{C}^2$ representa a explosão uma vez das $(n^2 + n + 1)$ singularidades genéricas de $\tilde{\mathcal{F}}$ e a superfície $S_{\mathcal{W}_1}$ é obtida após a explosão. Como a explosão da cada singularidade dá mais 2 singularidades em $S_{\mathcal{W}_1}$ temos que o número total das singularidades de $\mathcal{F} = (pi|_{S_{\mathcal{W}_d}})^* \tilde{\mathcal{F}}$ em $S_{\mathcal{W}_1}$ é $2(n^2 + n + 1)$ coincidindo com (1).
Seja $N_{\tilde{\mathcal{F}}}$ e $N_{\mathcal{F}} = (pi|_{S_{\mathcal{W}_d}})^*(N_{\tilde{\mathcal{F}}})$ os fibrados normias (ver [5]) de $\tilde{\mathcal{F}}$ e $\mathcal{F}$ respectivamente, então

$$c_1(N_{\mathcal{F}}) = c_1(N_{\tilde{\mathcal{F}}}) + \sum_{j=1}^{(n^2+n+1)} D_j$$

onde $D_j's$ são os divisores obtidos após os explosões. Portanto

$$\begin{aligned} \Big[c_1(N_{\mathcal{F}})\Big]^2 &= \Big[c_1(N_{\tilde{\mathcal{F}}})\Big]^2 + \sum_{j=1}^{(n^2+n+1)} D_j^2 \\ &= (n+2)^2 - (n^2+n+1) \\ &= 3(n+1) \end{aligned}$$

Coincidindo com (2).

Introduction

A web is a collection of d-foliations in general position of the same codimension. The French and German names for the concept are respectively 'tissu' and 'Gewebe'.Webs have been studied by various authors from the local point of view and using various techniques.Akivis and Shelekhon [1] used techniques of differential geometry, Hénaut [14] techniques of D-modules and Alcides and Nakai techniques of Analytic Geometry. Such studies include linearization of webs, curvature of webs, rank of webs, hexagonal webs,etc.

In this thesis, we study webs from a global point of view. The main idea of this work is to generalise, if possible, the works and results known for foliations on $\mathbb{C}P^2$ to webs.

The study on singular webs is motivated by the first order implicit differential equation

$$F(x, y, p) = 0, \qquad p = \frac{dy}{dx} \qquad (*)$$

where F is a polynomial of degree d in p. At points $(x_0, y_0, p_0) \in \mathbb{C}^3$ where the partial derivative $F_p = \frac{\partial F}{\partial p} \neq 0$, we can locally rewrite the above equation in the form

$$p = \frac{dy}{dx} = f_j(x, y), \qquad j = 1, \ldots, d$$

by the implicit function theorem , where f_j are holomorphic functions in a neighborhood of (x_0, y_0). Hence we have integral d-curves passing through (x_0, y_0). At a point where $F = F_p = 0$,we cannot expect good properties as the above in general. Various studies have been made [see [27], [8] [2], [9], [29]] to give the local normal forms of integral curves of $(*)$ in a neighborhood of such points.

In chapter 1, we define algebraic d-webs in $\mathbb{C}P^2$ as given by

$$a_0(x,y)dy^d + a_1(x,y)dy^{d-1}dx + \cdots + a_d(x,y)dx^d = 0$$

where $a_j^{\prime}s$ are polynomials. The notion of a degree of a web is given as been the number of tangencies of a generic line with the integral curves of the webs. Based on this definition, we give conditions by which the line at infinity is not invariant by the d-web. We give then the form of a $d-$web of degree n in $\mathbb{C}P^2$. That is:

Proposition 1.4.2 A singular d-web, $\mathcal{W}_d$ of degree n , in $\mathbb{C}P^2$ can be expressed in the affine chart $(x,y) \in \mathbb{C}^2$ by a differential equation of the form

$$\sum_{j=0}^{d} a_j(x,y)dx^j dy^{d-j} + \sum_{j=1}^{d}\sum_{k=0}^{d-j}(xdy - ydx)^j dx^k dy^{d-j-k} g_{jk}(x,y) = 0 \qquad (*)$$

where

a) g_{jk} are homogeneous polynomials of degree n (or $g_{jk} \equiv 0$)

b) $a_j^{\prime}s$ are polynomials of degree $\deg a_j \leq n$ for $j = 0, \ldots, d$.

c) the line at infinity, $\mathcal{L}_\infty$, is invariant by $\mathcal{W}_d$ if and only if $g_{d0} \equiv 0$.

It follows from the proposition above that we have a projective struture for the set of d-webs in $\mathbb{C}P^2$, where $d \in \mathbb{N}, d \geq 0$.

Putting $p = \frac{dy}{dx}$ in the form $(*)$, we obtain a function $F : \mathbb{C}^3 \longrightarrow \mathbb{C}$. Thus we observe in this chapter that given a d-web in $\mathbb{C}P^2$ we can associate to it a surface $S_{\mathcal{W}_d} = F^{-1}(0) \subset \mathbb{P}T^*\mathbb{C}P^2$ and a foliation $\mathcal{F}_{\mathcal{W}_d}$ on $S_{\mathcal{W}_d}$ whose leaves, when projected by $\pi : \mathbb{P}T^*\mathbb{C}P^2 \longrightarrow \mathbb{C}P^2$, gives the d-web in $\mathbb{C}P^2$.

As a corollary of the above proposition, we give another form of a d-web in $\mathbb{C}P^2$ of degree n as:

Corollary 1.4.2 A singular d-web, $\mathcal{W}_d$ of degree n, in $\mathbb{C}P^2$ can also be expressed in homogeneous coordinates $(x, y, z) \in \mathbb{C}^3$ by

$$\sum_{j=0}^{d} \sum_{k=0}^{d-j} A_{jk}(x, y, z)\alpha^j \beta^k \gamma^{d-j-k} = 0 \qquad (**)$$

where

a) A_{jk} are homogeneous polynomials of degree n.

b) $\alpha = xdy - ydx, \beta = zdx - xdz, \gamma = ydz - zdy$, with $x\gamma + y\beta + z\alpha = 0$.

Now putting $G(x, y, z, \alpha, \beta, \gamma)$ as the form $(**)$ we obtain a function $G : \mathbb{C}^3 \times \mathbb{C}^3 \longrightarrow \mathbb{C}$ and the surface associated to the d-web is given by

$$S_{\mathcal{W}_d} = \{G = 0\} \subset X = \{x\gamma + y\beta + z\alpha = 0\} \subset \mathbb{C}P^2 \times \mathbb{C}P^2.$$

We observe that X is isomorphic to $\mathbb{P}T^*\mathbb{C}P^2$ and the function $G(x, y, z, \alpha, \beta, \gamma)$ is symmetric in the following sense: suppose we have a d-web of degree n, then interchanging the positions of (x, y, z) and (α, β, γ) we obtain an n-web of degree d.

It is important to observe that the form $(**)$ is not unique. That is $G_1(x, y, z, \alpha, \beta, \gamma)$ and $G_2(x, y, z, \alpha, \beta, \gamma)$ define the same d-web in $\mathbb{C}P^2$ if and only if $\Big(G_1(x, y, z, \alpha, \beta, \gamma) - \lambda G_2(x, y, z, \alpha, \beta, \gamma)\Big)$ is divisible by $x\gamma + y\beta + z\alpha$, for some $\lambda \in \mathbb{C}^*$.

In chapter 2 we observe that in his study of mappings of the plane to the plane, Whitney [28], has shown that any such map can be approximated by a map having particularly simple singularities. Applying this philosophy to $\pi|_{S_{\mathcal{W}_d}} : S_{\mathcal{W}_d} \longrightarrow \mathbb{C}^2$ we have the following properties:

(1) $S_{\mathcal{W}_d}$ is smooth

(2) the critical set of π is smooth

(3) the only singularities of π are folds and ordinary cusps

(4) the only singularities of the discriminant curve are nodes and ordinary cusps.

We call any d-web in $\mathbb{C}P^2$ a generic web if it has properties (1) through (4). In this sense we prove:

Theorem 2.1.1 The set of d-webs of degree n, in $\mathbb{C}P^2$ with properties (1) through (4) above is an open dense subset in the space of d-webs.

Finally in the last chapter, we give the number of cusps and nodes of the projection and also the number of singularities of the generic foliation $\mathcal{F}_{\mathcal{W}_d}$ in $S_{\mathcal{W}_d}$, in terms of d and the degree of the web n. Thus we prove:

Theorem 3.3.1 Let $\mathcal{W}_d$ be a generic d-web in $\mathbb{C}P^2$ of degree n. Then the number of cusps κ and nodes δ of the projection $\pi : S_{\mathcal{W}_d} \longrightarrow \mathbb{C}P^2$, are given by

$$\kappa = 3(d-2)(n^2+nd+d)$$
$$\delta = \tfrac{1}{2}(d-2)(d-3)(3d+d^2+4nd+4n^2)$$

Theorem 3.3.2 Let $\mathcal{W}_d$ be a generic d-web in $\mathbb{C}P^2$ of degree n such the associated foliation $\mathcal{F}_{\mathcal{W}_d}$ in $S_{\mathcal{W}_d}$ has only isolated singularities. Then the total number of singularities of $\mathcal{F}_{\mathcal{W}_d}$ is given by

$$\sum_{p\in S_{\mathcal{W}_d}} mult(\mathcal{F}_{\mathcal{W}_d}, p) = 3nd^2 + 3n^2d - d^2 - n^2 - 4nd + 3n + 3d$$

In Theorem 3.3.2, we observe that the result is symmetric in n and d, that is,

$$\sum_{p \in S_{\mathcal{W}_d}} mult(\mathcal{F}_{\mathcal{W}_d}, p) = H(n,d) = H(d,n).$$

We give also a formula for the sum of the indices of Baum-Bott for the foliation $\mathcal{F}_{\mathcal{W}_d}$ on $S_{\mathcal{W}_d}$. That is

Proposition 3.2.1 Let $\mathcal{W}_d$ be a generic d-web in $\mathbb{C}P^2$ of degree n.Then

$$\sum_{p \in S_{\mathcal{W}_d}} BB(\mathcal{F}_{\mathcal{W}_d}, p) = 3(n+d)$$

where BB represents the Baum-Bott index.

We compare the results in Theorem 3.3.2 and Proposition 3.2.1 in the case $d = 1$. That is putting $d = 1$ in Theorem 3.3.2 and Proposition 3.2.1 we get the following:

$$\sum_{p \in S_{\mathcal{W}_1}} mult(\mathcal{F}_{\mathcal{W}_1}, p) = 2(n^2 + n + 1) \tag{1}$$

$$\sum_{p \in S_{\mathcal{W}_1}} BB(\mathcal{F}_{\mathcal{W}_1}, p) = 3(n+1) \qquad (2).$$

But we observe that when $d = 1$, the 1-web is given by

$$a_0(x,y)dy + a_1(x,y)dx + (xdy - ydx)g_{10}(x,y) = 0$$

that is a foliation $\tilde{\mathcal{F}}$ in $\mathbb{C}P^2$ of degree n in the affine coordinates (x, y), $\pi|_{S_{\mathcal{W}_1}} : S_{\mathcal{W}_1} \longrightarrow \mathbb{C}^2$ represents the blow-up once of the $(n^2 + n + 1)$ generic singularities of $\tilde{\mathcal{F}}$ and the surface $S_{\mathcal{W}_1}$ is obtained after the blow-up. Since the blow-up of each singularity gives rise to 2 other singularities in $S_{\mathcal{W}_1}$, we have that the total number of singularities of $\mathcal{F} = (\pi|_{S_{\mathcal{W}_1}})^* \tilde{F}$ in $S_{\mathcal{W}_1}$ is $2(n^2 + n + 1)$ coinciding with (1).

Let $N_{\tilde{\mathcal{F}}}$ and $N_{\mathcal{F}} = (\pi|_{S_{\mathcal{W}_1}})^*(N_{\tilde{\mathcal{F}}})$ be the normal line bundles (see [5]) of $\tilde{\mathcal{F}}$ and $\mathcal{F}$ respectively, then

$$c_1(N_{\mathcal{F}}) = c_1(N_{\tilde{\mathcal{F}}}) + \sum_{j=1}^{n^2+n+1} D_j$$

where $D_j's$ are the divisors obtained after the blow-ups. Hence

$$\begin{aligned}
\Big[c_1(N_{\mathcal{F}})\Big]^2 &= \Big[c_1(N_{\tilde{\mathcal{F}}})\Big]^2 + \sum_{j=1}^{n^2+n+1} D_j^2 \\
&= (n+2)^2 - (n^2+n+1) \\
&= 3(n+1)
\end{aligned}$$

Coinciding with (2).

Chapter 1

Singular d-webs

1.1 Non Singular d-webs in $\mathbb{C}^2$

Definition 1.1.1. *A local non-singular d-web $\mathcal{W}_d$ in $(\mathbb{C}^2, 0)$ is defined by d foliations $\mathcal{F}_j$, j=1,...,d of analytic, smooth complex curves of $(\mathbb{C}^2, 0)$ in general position. That is, the leaves passing through $0 \in \mathbb{C}^2$ are given by $F_i(x, y) = const.$ where*

$$F_i : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$$

holomorphic, $F_i(0) = 0$ then $dF_i(0) \wedge dF_j(0) \neq 0$ for all $1 \leq i < j \leq d$.

In particular, each F_i is a submersion(i.e $dF_i(0) \neq 0$). We denote a non-singular d-web by $\mathcal{W}_d = [\mathcal{F}_1, \ldots, \mathcal{F}_d]$.

Example 1.1.1.

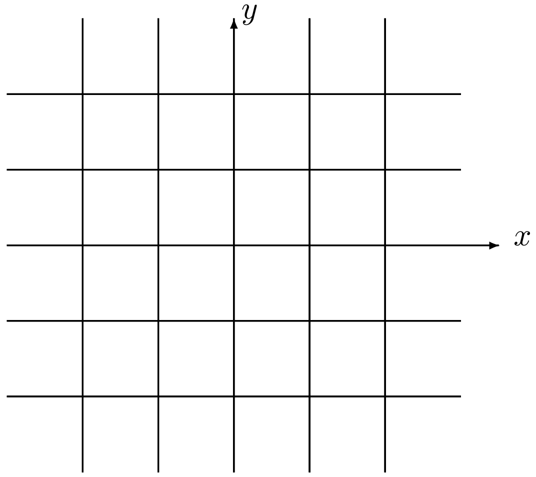
Let (x,y) be a system of coordinates in $U \subset \mathbb{C}^2$ and $\mathcal{F}_j, j = 1, \ldots, d$ be the foliations defined respectively by $y + \alpha_j x = const$ with the α_j two by two distinct, then $[\mathcal{F}_1, \ldots, \mathcal{F}_d]$ defines a non-singular d-web in U.

Definition 1.1.2. *1) A non-singular d-web $\mathcal{W}_d = [\mathcal{F}_1, \ldots, \mathcal{F}_d]$ in $(\mathbb{C}^2, 0)$ is linear if all the leaves of the foliations $\mathcal{F}_j$ are straight lines.*

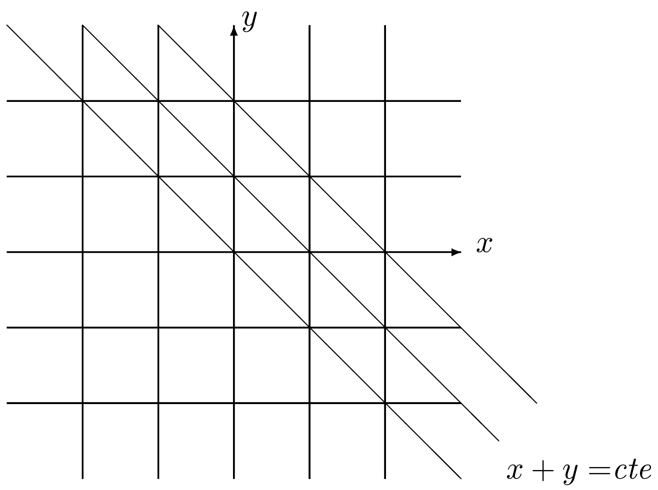
2)Two non-singular d-webs $\mathcal{W}_d = [\mathcal{F}_1, \ldots, \mathcal{F}_d]$ and $\mathcal{W}'_d = [\mathcal{F}'_1, \ldots, \mathcal{F}'_d]$ in $(\mathbb{C}^2, 0)$ are locally equivalent (or conjugate) if there exists a local isomorphism $\Phi : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}^2, 0)$ such that $\Phi^* \mathcal{F}'_j = \mathcal{F}_j$ j=1,..., d.

3)A non- singular d-web $\mathcal{W}_d$ is linearizable if it is conjugate to a d-web linear.

Example 1.1.2. *1)Any 2-web is linearizable, that is locally conjugated to the 2-web x=const, y=const.*



2)If a 3-web is linearizable, then is locally conjugated to the 3-web x=const, y=const and x+y=const.



1.2 Singular d-webs in $(\mathbb{C}^2, 0)$

The study of singular d-webs in U a neighborhood of $(\mathbb{C}^2, 0)$ is motivated by the first order implicit differential equation

$$F(x, y, p) = a_0(x, y)p^d + a_1(x, y)p^{d-1} + \cdots + a_d(x, y) = 0 \qquad (*)$$

where $p = \frac{dy}{dx}$ and the coefficients a_j are germs of holomorphic functions in U with $a_0 \not\equiv 0$.

Consider locally in $\mathbb{P}T^*U$, the surface

$$S = \{(x, y, p) \in \mathbb{P}T^*U; F(x, y, p) = 0\}$$

and let $\alpha = dy - pdx$.
Here $\mathbb{P}T^*U$ is the projective cotangent space of U whose elements are tangent lines in the tangent space of U.
$\mathbb{P}T^*U$ is covered by two coordinates patches having coordinates (x, y, p) or $(x, y, q), q = \frac{1}{p}$.

The 1-form α defines on S a foliation $\mathcal{F}(\alpha)$ whose leaves, when projected on the plane (x, y) by the natural projection $\pi|_S : S \longrightarrow \mathbb{C}^2$, $\pi(x, y, p) = (x, y)$, gives origin to the solutions of the d-web in U.
The foliation $\mathcal{F}(\alpha)$ induced by α on S is also defined by a vector field $\mathcal{Z}$, meromorphic on $\mathbb{P}T^*U$ and tangent to S. This vector field satisfies $i_{\mathcal{Z}}(\alpha) = 0$, $i_{\mathcal{Z}}(dF) = 0$ along S where $i_{\mathcal{Z}}$ denotes the interior product and it's given by

$$\mathcal{Z} = F_p \frac{\partial}{\partial x} + pF_p \frac{\partial}{\partial y} - (F_x + pF_y) \frac{\partial}{\partial p}$$

restricted to S.

Definition 1.2.1. *A germ of a singular d-web $\mathcal{W}_d$ in $(\mathbb{C}^2, 0)$ is given by the triple $(S_{\mathcal{W}_d}, \pi|_{S_{\mathcal{W}_d}}, \mathcal{F}_{\mathcal{W}_d})$ where $S_{\mathcal{W}_d}$ is the surface in $\mathbb{P}T^*\mathbb{C}^2$ given by the equation $F(x, y, p) = \sum_{j=0}^{d} a_j(x, y)p^j$, $\pi|_{S_{\mathcal{W}_d}} : S_{\mathcal{W}_d} \longrightarrow \mathbb{C}^2$, $\pi(x, y, p) = (x, y)$*

the natural projection and $\mathcal{F}_{\mathcal{W}_d}$ *the foliation on* $S_{\mathcal{W}_d}$ *given by* $\alpha = dy - pdx$ *or by the vector field* $\mathcal{Z} = F_p\frac{\partial}{\partial x} + pF_p\frac{\partial}{\partial y} - (F_x + pF_y)\frac{\partial}{\partial p}$ *restricted to* $S_{\mathcal{W}_d}$.

Definition 1.2.2. [4] The set of critical points of $\pi|_{S_{\mathcal{W}_d}} : S_{\mathcal{W}_d} \longrightarrow \mathbb{C}^2$ is called the criminant of F and is given by the equations $F = F_p = 0$. The set of critical values of the projection is called the discriminant of F, and is obtained by eliminating p from the equations $F = F_p = 0$. We denote the discriminant curve by $\triangle(x, y)$.

Remark 1.2.1. The discriminant curve is also given as the p-resultant of F and F_p, denoted by $\mathrm{Res}_p(F, F_p)$. That is if

$$
\begin{aligned}
F(x,y,p) &= a_0(x,y)p^d + a_1(x,y)p^{d-1} + \cdots + a_d(x,y)\\
F_p(x,y,p) &= pa_0(x,y)p^{d-1} + (d-1)a_1(x,y)p^{d-2} + \cdots + a_{d-1}(x,y)
\end{aligned}
$$

then $\mathrm{Res}_p(F, F_p)$ is the (2d-1) by (2d-1) determinant

$$
\begin{vmatrix}
a_0 & a_1 & \cdots & a_{d-1} & a_d & & & \\
 & a_0 & a_1 & \cdots & a_{d-1} & a_d & & \\
 & & \ddots & & & & & \\
 & & a_0 & a_1 & & \cdots & a_{d-1} & a_d \\
da_0 & (d-1)a_1 & \cdots & a_{d-1} & & & & \\
 & da_0 & (d-1)a_1 & \cdots & a_{d-1} & & & \\
 & & & \ddots & & & & \\
 & & & da_0 & (d-1)a_1 & & \cdots & a_{d-1}
\end{vmatrix}
$$

with zeros in all blank spaces.

Definition 1.2.3. [8] The singular set of the d-web $\mathcal{W}_d$ denoted by $Sing(\mathcal{W}_d)$ is given by

- the zeros of the vector field $\mathcal{Z}$,i.e $F = F_p = F_x + pF_y = 0$
- singularities of the projection $\pi|_{S_{\mathcal{W}_d}} : S_{\mathcal{W}_d} \longrightarrow \mathbb{C}^2$.

Remark 1.2.2.

a) *The singular set of* $\mathcal{W}_d$ *is contained in the criminant.*

b) *We will call* $S_{\mathcal{W}_d}$ *the surface associated to the d-web and* α *the canonical 1-form of the d-web.*

c) *We rewrite* $F(x,y,p) = \sum_{j=0}^{d} a_j(x,y)p^{d-j}$ *as*

$$\Omega = \sum_{j=0}^{d} a_j(x,y)(dx)^j(dy)^{d-j}.$$

d) *The leaves of a d-web are obtained as follows: Let the d-web be given by*

$$\sum_{j=0}^{d} a_j(x,y)(dx)^j(dy)^{d-j} = 0.$$

Take $p = (x,y) \in \mathbb{C}^2$ *at which* $\triangle(x,y) \neq 0$ *and let*

$$t \xrightarrow{\varphi} \Big(x(t), y(t)\Big)$$

be a holomorphic mapping on some disk $D(0,r)$ *such that* $\varphi(0){=}p$ *and*

$$\sum_{j=0}^{d} a_j\Big(x(t), y(t)\Big)x'(t)^j y'(t)^{d-j} = 0$$

for all $t \in D(0,r)$. *The germs of leaves through* p *are the images under all possible* φ *as before of an eventually smaller neighborhood of 0.*

e)

$$\Omega = \sum_{j=0}^{d} a_j(x,y)(dx)^j(dy)^{d-j} = 0$$

and

$$\Omega' = \sum_{j=0}^{d} a'_j(x,y)(dx)^j(dy)^{d-j} = 0$$

define the same d-web in $\mathbb{C}^2$ if and only if $\Omega = f\Omega'$ for some meromorphic function

$$f : \mathbb{C}^2 \to \mathbb{C}$$

which is nowhere zero. The importance of this property lies in the fact that it allows one to extend the d-web from $\mathbb{C}^2$ to the complex projective plane $\mathbb{C}P^2$.

Example 1.2.1.

$$x^4 dy^2 + (y^2 - 1)dx^2 = 0$$

defines a 2-web $\mathcal{W}_2$ in $\mathbb{C}^2$ with discriminant curve

$$\triangle(x, y) = \{(x, y); x(y^2 - 1) = 0\}$$

and the leaves through $(x_0, y_0) \in \mathbb{C}^2 \backslash \triangle$ is given by

$$t \xrightarrow{\varphi} \Big(t, sin\Big[\pm\frac{1}{t} \pm \frac{1}{x_0} + sin^{-1} y_0\Big]\Big).$$

1.3 Singular d-webs on $\mathbb{C}P^2$

The d-web, $\mathcal{W}_d$, in $\mathbb{C}^2$ extends to $\mathbb{C}P^2$ as follows: The projective space $\mathbb{C}P^2$ has three affine coordinates $(x, y), (u, v), (r, s)$ related by the transition functions: $\phi(u, v) = \left(\frac{1}{u}, \frac{v}{u}\right) = (x, y)$ and $\psi(r, s) = \left(\frac{s}{r}, \frac{1}{r}\right) = (x, y)$.

Consider

$$\Omega(x, y, dx, dy) = \sum_{j=0}^{d} a_j(x, y)(dx)^j (dy)^{d-j}$$

where $a_j's$ are polynomials and $n = \max\{\deg a_j, j = 0, \ldots, d\}$ on $\mathbb{C}^2$ and its corresponding singular d-web $\mathcal{W}_d$ as in 1.2. Using the coordinates map ϕ one can transport $\mathcal{W}_d$ to $(u,v)\in\mathbb{C}^2$. To this end,write

$$\begin{aligned}\tilde{\Omega}(u,v,du,dv) &:= (\phi^*\Omega)(u,v,du,dv)\\ &= \sum_{j=0}^{d} a_j(\frac{1}{u},\frac{v}{u})(d(\frac{1}{u}))^j(d(\frac{v}{u}))^{d-j}\\ &= \sum_{j=0}^{d}\frac{a_j(\frac{1}{u},\frac{v}{u})}{u^{2d}}(-du)^j(udv-vdu)^{d-j}\\ &= \frac{1}{u^{2d+n}}\sum_{j=0}^{d}u^{d-j}b_j(u,v)(du)^j(dv)^{d-j}.\end{aligned}$$

where

$$b_j(u,v)=(-1)^j u^n\sum_{k=0}^{j}\binom{d-k}{d-j}a_k\Big(\frac{1}{u},\frac{v}{u}\Big)v^{j-k}$$

and $j=0,1,2,\ldots,d$.

Multiplying the last expression of $\tilde{\Omega}$ by u^{2d+n} in order to cancel the pole, we obtain $\Omega' := u^{2d+n}\tilde{\Omega}$ on $(u,v)\in\mathbb{C}^2$. Then the two webs $\mathcal{W}'_d$ and $\tilde{\mathcal{W}}_d$ are then identical on $\{(u,v)\in\mathbb{C}^2 : u\neq 0\}$ by Remark 1.2.2.(e); however, $\mathcal{W}'_d$ which is defined on all of $(u,v)\in\mathbb{C}^2$ is a well-defined extension of $\tilde{\mathcal{W}}_d$.

In a similar way,$\mathcal{W}_d$ can be transported to the affine chart (r,s) by ψ to obtain a web $\mathcal{W}''_d$ induced by Ω'' on $(r,s)\in\mathbb{C}^2$.

It follows from the above construction that in each affine chart, W_d is given by the integral curves of the following

In $(x,y)\in\mathbb{C}^2$;

$$\Omega=\sum_{j=0}^{d}a_j(x,y)(dx)^j(dy)^{d-j}$$

In $(u,v)\in\mathbb{C}^2$;

$$\Omega'=\sum_{j=0}^{d}u^{d-j}b_j(u,v)(du)^j(dv)^{d-j}$$

In $(r,s)\in\mathbb{C}^2$;

$$\Omega'' = \sum_{j=0}^{d} r^{d-j} c_j(r,s)(dr)^j (ds)^{d-j}$$

where

$$b_j(u,v) = (-1)^j u^n \sum_{k=0}^{j} \binom{d-k}{d-j} a_k\Big(\frac{1}{u},\frac{v}{u}\Big) v^{j-k}$$

$$c_j(r,s) = (-1)^j u^n \sum_{k=0}^{j} \binom{d-k}{d-j} a_{d-k}\Big(\frac{s}{r},\frac{1}{r}\Big) s^{j-k}$$

and $\quad j = 0,1,2,\ldots,d.$

This shows that $\mathcal{W}_d$ is a singular d-web in $\mathbb{C}P^2$ and may have singularities on the line at infinity $\mathcal{L}_\infty = \mathbb{C}P^2\backslash\mathbb{C}^2$.

1.4 Degree of a d-web

Let $\mathcal{W}_d$ be a singular d-web in $\mathbb{C}P^2$ and $\mathcal{L}\subset\mathbb{C}P^2$ be a generic line. We say that $p\in\mathcal{L}$ is a tangency point of $\mathcal{W}_d$ with $\mathcal{L}$, if the tangents of some of the leaves of $\mathcal{W}_d$ through p with $\mathcal{L}$ coincide at p. This can be expressed as follows: let $\mathcal{W}_d$ be given by

$$\Omega = \sum_{j=0}^{d} a_j(x,y)(dx)^j(dy)^{d-j} = 0$$

in an affine chart $U_0 = (x,y)\in\mathbb{C}^2$ and $\mathcal{L}\cap U_0$ be parametrized by $\varphi(t) = (x_0+\alpha t, y_0+\beta t)$ where $p=(x_0,y_0)=\varphi(0)$. Then p is a tangency point of $\mathcal{W}_d$ with $\mathcal{L}$ if and only if $t=0$ is a root of the polynomial

$$h_{\mathcal{L}}(t) = \sum_{j=0}^{d} a_j(\varphi(t))\alpha^j\beta^{d-j}.$$

The multiplicity of tangency of $\mathcal{W}_d$ with $\mathcal{L}$ at p is, by definition, the multiplicity of $t=0$ as a root of $h_{\mathcal{L}}$. We denote this number by $m(\mathcal{W}_d,\mathcal{L};p)$.

Remark 1.4.1. $m(\mathcal{W}_d, \mathcal{L}; p)$ does not depend on the parametrization of $\mathcal{L} \cap U_0$[1] and is invariant under analytic change of variables. Hence we can define the total number of tangencies of $\mathcal{W}_d$ with $\mathcal{L}$ as

$$m(\mathcal{W}_d, \mathcal{L}) := \sum_{p \in \mathcal{L}} m(\mathcal{W}_d, \mathcal{L}; p)$$

where $m(\mathcal{W}_d, \mathcal{L}; p) = 0$ if $p \in \mathcal{L}$ is not a tangency point.

Example 1.4.1. 1) Consider the 3-web

$$dy^3 + dx^3 = 0.$$

Let $p \in \mathcal{L}$ be the generic line $y = \alpha x + \beta$. Then $h_{\mathcal{L}}(x) = \alpha^3 + 1$ and $m(\mathcal{W}_3, \mathcal{L} : p) = 0$ and p is not a tangency point if $\alpha^3 \neq -1$. If $\alpha^3 = -1$ then all points of $\mathcal{L}$ are tangency points, which means that the line is a leaf of the web.

2) Consider the 3-web

$$dy^3 + 12xdydx^2 + 6ydx^3 = 0.$$

Let $p \in \mathcal{L}$ be the generic line $y = \alpha x + \beta$. Then

$$h_{\mathcal{L}}(x) = \alpha^3 + 12\alpha x + 6\alpha x + 6\beta$$

and

$$p = \left(x = \frac{-\alpha^3 - 6\beta}{18\alpha}, \alpha x + \beta\right)$$

is a tangency point of $\mathcal{W}_3$ with multiplicity 1. Hence $m(\mathcal{W}_3, \mathcal{L} : p) = 1$.

Proposition 1.4.1. *Let $\mathcal{W}_d$ be a singular d-web in $\mathbb{C}P^2$. Suppose that $\mathcal{W}_d$ is expressed in the affine chart $(x, y) \in U_0$ by the polynomial 'd-form'*

$$\sum_{j=0}^{d} a_j(x, y)(dx)^j(dy)^{d-j} = 0$$

[1]if the parametrization has a non-zero derivative at p

where $\max\{\deg a_j, j = 1, \ldots, d\} \leq deg(a_0) = n$ *where* $n > d$.
Let

$$a_j = \sum_{k=0}^{n} a_{j,k}$$

$j = 0, \ldots, d$, *where* $a_{j,k}$ *is the homogeneous part of degree* k *of* a_j.
Then the line at infinity $\mathcal{L}_\infty = \mathbb{C}P^2 \backslash U_0$, *is* *<u>not invariant</u>* *by* $\mathcal{W}_d$ *if and only if the following* $\frac{d(d+1)}{2}$ *equations* $(*)$

$$(*) \begin{cases} \sum_{j=0}^{d} x^j y^{d-j} a_{j,n-k} \equiv 0, & k = 0, \ldots, d-1. \\ \sum_{j=0}^{d-1} \binom{d-j}{1} x^j y^{d-1-j} a_{j,n-k} \equiv 0, & k = 0, \ldots, d-2. \\ \sum_{j=0}^{d-2} \binom{d-j}{2} x^j y^{d-2-j} a_{j,n-k} \equiv 0, & k = 0, \ldots, d-3. \\ \vdots & \\ \sum_{j=0}^{2} \binom{d-j}{d-2} x^j y^{2-j} a_{j,n-k} \equiv 0, & k = 0, 1. \\ \sum_{j=0}^{1} \binom{d-j}{d-1} x^j y^{1-j} a_{j,n} \equiv 0. & \end{cases}$$

are all identically zero.

<u>Proof:</u>

Consider first the change of variables

$$x = \frac{1}{u}, y = \frac{v}{u}$$

where $\{u = 0\}$ represents $\mathcal{L}_\infty$ in the affine chart U_0. We saw in section 1.3 that this change of variables transforms the equation

$$\sum_{j=0}^{d} a_j(x, y)(dx)^j (dy)^{d-j} = 0$$

into

$$u^d b_0(u,v)(dv)^d + \sum_{j=1}^{d} u^{d-j} b_j(u,v)(du)^j (dv)^{d-j} = 0 \qquad (**)$$

where b_j are polynomials defined by

$$b_j(u,v) = (-1)^j u^n \sum_{k=0}^{j} \binom{d-k}{d-j} a_k\Big(\frac{1}{u}, \frac{v}{u}\Big) v^{j-k}$$

and $\quad j = 0, 1, 2, \ldots, d.$

If

$$a_k = \sum_{s=0}^{n} a_{k,s} \quad ; \quad k = 0, \ldots, d$$

where $a_{k,s}$ is the homogeneous part of degree s of a_k, then we write

$$b_j(u,v) = f_{j,0}(v)u^n + f_{j,1}(v)u^{n-1} + \cdots + f_{j,n}(v)$$

where

$$f_{j,s} = (-1)^j \sum_{k=0}^{j} \binom{d-k}{d-j} a_{k,s}(1,v) v^{j-k}$$

and

$$j = 1, 2, \ldots, d; \quad s = 0, 1, \ldots, n.$$

If

$$f_{j,n-s} \equiv 0 \quad \text{for} \begin{cases} j = 1 & s = 0 \\ j = 2 & s = 0, 1 \\ j = 3 & s = 0, 1, 2 \\ \vdots & \\ j = d & s = 0, 1, 2, \ldots, d-1 \end{cases}$$

That is

$$\begin{cases} \sum_{k=0}^{1} \binom{d-k}{d-1} a_{k,n}(1,v)v^{1-k} \equiv 0 \\ \sum_{k=0}^{2} \binom{d-k}{d-2} a_{k,n-s}(1,v)v^{2-k} \equiv 0, & s = 0,1 \\ \sum_{k=0}^{3} \binom{d-k}{d-3} a_{k,n-s}(1,v)v^{3-k} \equiv 0, & s = 0,1,2 \\ \vdots \\ \sum_{k=0}^{d-1} \binom{d-k}{1} a_{k,n-s}(1,v)v^{d-1-k} \equiv 0, & s = 0,1,\dots,d-2 \\ \sum_{k=0}^{d} a_{k,n-s}(1,v)v^{d-k} \equiv 0, & s = 0,1,\dots,d-1 \end{cases}$$

or going back to the affine chart $(x, y) \in U_0$ as the $\frac{d(d+1)}{2}$ equations $(*)$ given in lemma, then we can rewrite $b_j(u, v)$ as

$$b_j(u,v) = u^j \tilde{b}_j(u,v),\ j = 1,\dots,d$$

where $\tilde{b}_j$ is a polynomial.
Hence $(**)$ can be rewritten as

$$u^d \left[b_0(u,v)dv^d + \sum_{j=1}^{d} \tilde{b}_j(u,v)du^j dv^{d-j} \right] = 0$$

or dividing by u^d, as

$$b_0(u,v)dv^d + \sum_{j=1}^{d} \tilde{b}_j(u,v)du^j dv^{d-j} = 0.$$

Observe now that,u is not a factor of b_0 since deg $a_0 = n$. Hence

$$f_{0,n} = a_{0,n} \not\equiv\ 0.$$

This implies that $\mathcal{L}_\infty = \{u = 0\}$ is not invariant by $\mathcal{W}_d$.
On the other hand, if the $\frac{d(d+1)}{2}$ equations $(*)$ listed in the proposition are not all identically zero, then clearly $\{u = 0\}$ is an invariant solution of $\mathcal{W}_d$.
Hence the proof.

$\square$

Corollary 1.4.1. *Let $\mathcal{W}_d$ be a singular d-web in $\mathbb{C}P^2$.*
a)If $\mathcal{L}_1$ and $\mathcal{L}_2$ are projective lines, which are not algebraic solutions of $\mathcal{W}_d$, then $m(\mathcal{W}_d, \mathcal{L}_1) = m(\mathcal{W}_d, \mathcal{L}_2)$.

From this assertion we can define the degree of $\mathcal{W}_d$, $deg(\mathcal{W}_d)$ as $m(\mathcal{W}_d, \mathcal{L})$ where $\mathcal{L}$ is a projective line which is not an algebraic solution of $\mathcal{W}_d$.
b)Suppose that $\mathcal{W}_d$ is expressed in the affine chart $(x, y) \in U_0$ by the polynomial 'd-form'

$$\sum_{j=0}^{d} a_j(x, y)(dx)^j(dy)^{d-j} = 0$$

where $\max\{\deg a_j, j = 1, \ldots, d\} \leq deg(a_0) = n$ *where $n > d$.*
Then $n - d \leq deg(\mathcal{W}_d) \leq n$.
Moreover, $deg(\mathcal{W}_d) = n - d \Leftrightarrow \mathcal{L}_\infty$ is not invariant by $\mathcal{W}_d$.

Proof:
Let $\mathcal{L}$ be a projective line, which is not invariant by $\mathcal{W}_d$. After a linear change of variables, we can suppose that $\mathcal{L} \cap U_0 = \{y = 0\}$, in the affine chart U_0. In this case

$$h_{\mathcal{L}}(t) = a_d(t, 0),$$

hence

$$m(\mathcal{W}_d, \mathcal{L}) = deg(a_d(t, 0)) + m(\mathcal{W}_d, \mathcal{L} \cap \mathcal{L}_\infty).$$

On the other hand, if we take the change of coordinates

$$x = \frac{1}{u}, y = \frac{v}{u},$$

then the d-web is given by

$$\sum_{j=0}^{d} u^{d-j} b_j(u,v)(du)^j(dv)^{d-j} = 0 \qquad (*)$$

where

$$b_j(u,v) = (-1)^j u^n \sum_{k=0}^{j} \binom{d-k}{d-j} a_k(\frac{1}{u}, \frac{v}{u}) v^{j-k}$$

and a parametrization of $\mathcal{L}$ in the new system is

$$v = 0, u = s = \frac{1}{t}$$

and

$$\mathcal{L} \cap \mathcal{L}_\infty = \{v = u = 0\}.$$

Suppose $(*)$ is divisible by u^m, $\quad m = 0, \ldots, d$. Then the d-web is given in the (u, v) chart by

$$\sum_{j=0}^{d} u^{d-m-j} b_j(u,v)(du)^j(dv)^{d-j} = 0 \quad ; \quad m = 0, \ldots, d.$$

Hence $m(\mathcal{W}_d, \mathcal{L}; \mathcal{L} \cap \mathcal{L}_\infty)$ is the multiplicity of $s = 0$ as a root of

$$\tilde{h}_{\mathcal{L}}(s) = s^{-m} b_d(s, 0) = s^{n-m} a_d\Big(\frac{1}{s}, 0\Big) \quad ; \quad m = 0, \ldots, d$$

which is equal to $n - m - \deg a_d \quad ; \quad m = 0, \ldots, d.$
Hence

$$\begin{aligned} m(\mathcal{W}_d, \mathcal{L}) &= deg(a_d(t,0)) + m(\mathcal{W}_d, \mathcal{L} \cap \mathcal{L}_\infty) \\ &= n - m \quad ; \quad m = 0, \ldots, d. \end{aligned}$$

Since $\mathcal{L}$ is arbitrary, this proves a) and b).
The last part of b) follows from Proposition 1.4.1.

□

Proposition 1.4.2. *A singular d-web, $\mathcal{W}_d$ of degree n , in $\mathbb{C}P^2$ can be expressed in the affine chart $(x,y) \in \mathbb{C}^2$ by a differential equation of the form*

$$\sum_{j=0}^{d} a_j(x,y)dx^j dy^{d-j} + \sum_{j=1}^{d}\sum_{k=0}^{d-j}(xdy - ydx)^j dx^k dy^{d-j-k} g_{jk}(x,y) = 0 \qquad (*)$$

where

a) g_{jk} are homogeneous polynomials of degree n(or $g_{jk} \equiv 0$).

b) $a_j^{'}s$ are polynomials of degree $\deg a_j \leq n$ for $j = 0, \ldots, d$.

c) $\mathcal{L}_\infty$ is invariant by $\mathcal{W}_d$ if and only if the $g_{d0} \equiv 0$.

Proof:

Let the d-web be given in the affine chart $(x,y) \in U_0$ by

$$\sum_{j=0}^{d} a_j(x,y)dx^j dy^{d-j} = 0$$

where

$$\max\{\deg a_j, j = 1, \cdots, d\} \leq n.$$

If $deg(\mathcal{W}_d) = n - d$, then we have the $\frac{d(d+1)}{2}$ equations $(*)$ listed in Proposition 1.4.1. Solving for the $a_{j,k}$ for $k = n-d+1, \ldots, n$ in the $\frac{d(d+1)}{2}$ equations we have :

$$a_{j,k}(x,y) = \sum_{l=0}^{n-k}(-1)^{j-l}\binom{k-n+d}{j-l} x^{k-n+d-j+l} y^{j-l} g_{l,k}(x,y)$$

for $j = 0, \ldots, d$; $k = n-d+1, \ldots, n$ and where $g_{l,k}$ are homogenous polynomials of degree $n - d$.

Now

$$\begin{aligned}\sum_{j=0}^{d} a_j dx^j dy^{d-j} &= \sum_{j=0}^{d}\sum_{k=0}^{n-d} a_{j,k}(x,y) dx^j dy^{d-j} + \sum_{j=0}^{d}\sum_{k=n-d+1}^{n} a_{j,k}(x,y) dx^j dy^{d-j} \\ &= \sum_{j=0}^{d} \tilde{a}_j(x,y) dx^j dy^{d-j} + \sum_{j=0}^{d}\sum_{k=n-d+1}^{n} a_{j,k}(x,y) dx^j dy^{d-j}\end{aligned}$$

where $\deg \tilde{a}_j(x,y) \leq (n-d)$.

Substituting the values of $a_{j,k}$, the second expression on the right-hand side of the equation above becomes:

$$\sum_{j=0}^{d}\sum_{k=n-d+1}^{n} a^{!}_{j,k}(x,y) dx^j dy^{d-j} =$$

$$\begin{aligned}&= \sum_{j=0}^{d}\sum_{k=n-d+1}^{n}\sum_{l=0}^{n-k} (-1)^{j-l} \binom{k-n+d}{j-l} x^{k-n+d-j+l} y^{j-l} dx^j dy^{d-j} g_{l,k}(x,y) \\ &= \sum_{j=0}^{d}\sum_{s=1}^{d}\sum_{l=0}^{d-s} (-1)^{j-l} \binom{s}{j-l} x^{s-j+l} y^{j-l} dx^j dy^{d-j} g_{l,s}(x,y)\end{aligned}$$

making the substitution $s = k-n+d$

$$\begin{aligned}&= \sum_{s=1}^{d}\sum_{l=0}^{d-s} dx^l dy^{d-l-s} g_{l,s}(x,y) \sum_{j=0}^{d} (-1)^{j-l} \binom{s}{j-l} x^{s-j+l} y^{j-l} dx^{j-l} dy^{s-j+l} \\ &= \sum_{s=1}^{d}\sum_{l=0}^{d-s} dx^l dy^{d-l-s} g_{l,s}(x,y) \sum_{j=l}^{s+l} \binom{s}{j-l} (-ydx)^{j-l} (xdy)^{s-j+l} \\ &= \sum_{s=1}^{d}\sum_{l=0}^{d-s} dx^l dy^{d-l-s} g_{l,s}(x,y) \sum_{m=0}^{s} \binom{s}{m} (-ydx)^{m} (xdy)^{s-m}\end{aligned}$$

since $\binom{s}{j-l} \equiv 0$ for $s < j-l < 0$ and making the change of variable $m = j-l$. Hence

$$\sum_{j=0}^{d}\sum_{k=n-d+1}^{n} a_{j,k}(x,y) dx^j dy^{d-j} = \sum_{s=1}^{d}\sum_{l=0}^{d-s} (xdy - ydx)^s dx^l dy^{d-l-s} g_{l,s}(x,y).$$

Therefore

$$\sum_{j=0}^{d} a_j(x,y)dx^j dy^{d-j} = \sum_{j=0}^{d} \tilde{a}_j(x,y)dx^j dy^{d-j} + \sum_{j=0}^{d-1}\sum_{k=1}^{d-j}(xdy-ydx)^j g_{jk} dx^j dy^{d-k-j}.$$

Hence the proposition.

□

Corollary 1.4.2. *A singular d-web, $\mathcal{W}_d$ of degree n, in $\mathbb{C}P^2$ can also be expressed in homogenous coordinates $(x,y,z) \in \mathbb{C}^3$ by*

$$G(x,y,z,\alpha,\beta,\gamma) := \sum_{j=0}^{d}\sum_{k=0}^{d-j} A_{jk}(x,y,z)\alpha^j\beta^k\gamma^{d-j-k} = 0 \qquad (**)$$

where

a) $A_{j,k}$ are homogenous polynomials of degree n.

b) $\alpha = xdy - ydx, \beta = zdx - xdz, \gamma = ydz - zdy$, with $x\gamma + y\beta + z\alpha = 0$.

Proof:

Let

$$x = \frac{X}{Z} \longrightarrow dx = \frac{ZdX - XdZ}{Z^2}$$
$$y = \frac{Y}{Z} \longrightarrow dy = \frac{ZdY - YdZ}{Z^2}$$

Substituting in $(*)$ of Proposition 1.4.2. we have

$$\sum_{j=0}^{d} a_j\Big(\frac{X}{Z},\frac{Y}{Z}\Big)\frac{(ZdX - XdZ)^j(ZdY - YdZ)^{d-j}}{Z^{2d}} +$$

$$+\sum_{j=1}^{d}\sum_{k=0}^{d-j}\frac{(XdY - YdX)^j(ZdX - XdZ)^k(ZdY - YdZ)^{d-j-k}}{Z^{2d}} g_{jk}\Big(\frac{X}{Z},\frac{Y}{Z}\Big) = 0$$

Putting

$$\alpha = XdY - YdX$$

$$\beta = ZdX - XdZ$$

$$\gamma = YdZ - ZdY$$

and eliminating the pole Z^{2d+n} gives

$$\sum_{j=0}^{d}\sum_{k=0}^{d-j} A_{jk}(X,Y,Z)\alpha^j\beta^k\gamma^{d-j-k}$$

where A_{jk} are homogeneous polynomials of degree n and $z\alpha + y\beta + x\gamma = 0$.

Hence the proposition.

□

Remark 1.4.2. The form $(**)$ of the d-web in the corollary above is <u>not</u> unique. That is

$$G_1(x,y,z,\alpha,\beta,\gamma) := \sum_{j=0}^{d}\sum_{k=0}^{d-j} A^1_{jk}(x,y,z)\alpha^j\beta^k\gamma^{d-j-k} = 0$$

and

$$G_2(x,y,z,\alpha,\beta,\gamma) := \sum_{j=0}^{d}\sum_{k=0}^{d-j} A^2_{jk}(x,y,z)\alpha^j\beta^k\gamma^{d-j-k} = 0$$

where $z\alpha + y\beta + x\gamma = 0$ defines the same d-web of degree n in $\mathbb{C}P^2$ if and only if

$$G_1(x,y,z,\alpha,\beta,\gamma) - \lambda G_2(x,y,z,\alpha,\beta,\gamma)$$

is divisible by $z\alpha + y\beta + x\gamma$, for some $\lambda \in \mathbb{C}^*$.

Remark 1.4.3. We denote by $\mathcal{W}(2,d,n)$ the space of d-webs of degree n in $\mathbb{C}P^2$. By the Proposition 1.4.2. $\mathcal{W}(2,d,n)$ can be parametrized using polynomials $a_0, a_1, \dots, a_d$ and g_{jk} for

$$\begin{cases} j = 1; k = 0, 1, \dots, d-1 \\ j = 2; k = 0, 1, \dots, d-2 \\ \vdots \\ j = d-1; k = 0, 1 \\ j = d; k = 0. \end{cases}$$

Since by remark 1.2.19(b), Ω and Ω' defines the same d-web if and only if there exists a non-zero constant λ such that $\Omega' = \lambda\Omega$, each $W(2, d, n)$ can be topologized in the natural way: a neighborhood of $\mathcal{W} \in \mathcal{W}(2, d, n)$ consists of all d-webs of degree n whose defining polynomials have coefficients close to that of $\mathcal{W}$, up to multiplication by a non-zero constant.

Therefore $\mathcal{W}(2, d, n)$ may be considered as an open subset of the complex projective space $\mathbb{C}P(N)$, where

$$\begin{aligned} N &= (d+1)\left(\sum_{k=1}^{n+1} k\right) + \frac{d(d+1)(n+1)}{2} - 1 \\ &= \frac{(d+1)(n+1)(n+d+2) - 2}{2}. \end{aligned}$$

Corollary 1.4.3. *The set $\mathcal{W}(2, d, n)$ of all d-webs on $\mathbb{C}P^2$ of degree n is an open, connected and dense subset of the complex projective space $\mathbb{C}P(N)$, where*

$$N = \frac{(d+1)(n+1)(n+d+2) - 2}{2}.$$

Chapter 2

Generic Properties Of Webs in $\mathbb{C}P^2$

2.1 Some Generic Properties

Consider a d-web $\mathcal{W}_d$ of degree n in $\mathbb{C}P^2$, given in an affine chart by the triple $(S_{\mathcal{W}_d}, \pi|_{S_{\mathcal{W}_d}}, \mathcal{F}_{\mathcal{W}_d})$ where $S_{\mathcal{W}_d}$ is the surface in $\mathbb{P}T^*\mathbb{C}P^2$ given by the equation

$$F(x,y,p) = \sum_{j=0}^{d} a_j(x,y)p^{d-j} + \sum_{j=1}^{d}\sum_{k=0}^{d-j}(xp-y)^j p^{d-j-k} g_{jk}(x,y),$$

$\pi|_{S_{\mathcal{W}_d}} : S_{\mathcal{W}_d} \longrightarrow \mathbb{C}P^2$ is the restriction of $\pi(x,y,p) = (x,y)$ the natural projection to $S_{\mathcal{W}_d}$ and $\mathcal{F}_{\mathcal{W}_d}$ the foliation on S_{W_d} given by the restriction of $\alpha = dy - pdx$ to $S_{\mathcal{W}_d}$ or by the vector field $\mathcal{Z} = F_p\frac{\partial}{\partial x} + pF_p\frac{\partial}{\partial y} - (F_x + pF_y)\frac{\partial}{\partial p}$ restricted to $S_{\mathcal{W}_d}$.

Remark 2.1.1. The surface $S_{\mathcal{W}_d}$ can be covered by 6 affine charts $\{U_j\}_{j=1}^{6}$, see chapter 3.

Definition 2.1.1. *The* <u>*ramification curve*</u>*, denoted by* R*, of* $\pi|S_{\mathcal{W}_d} : S_{\mathcal{W}_d} \longrightarrow \mathbb{C}P^2$ *is given by the* $\{(x, y, p)|F = F_p = 0\}$ *and the* <u>*discriminant curve*</u>*, denoted by* $\triangle(x, y)$*, is obtained by eliminating* p *from the equations* $F = F_p = 0$.

We will consider d-webs, $\mathcal{W}_d$, satisfying the following properties:

(i) zero is a regular value of F(in particular, $S_{\mathcal{W}_d}$ is smooth)

(ii) the ramification curve is smooth

(iii) the only singularities of the projection are ordinary folds and ordinary cusps for $d \geq 3$

(iv) the only singularities of the discriminant curve are nodes and ordinary cusps

Definition 2.1.2. [8] a) A point of $S_{\mathcal{W}_d}$ where $F = F_p = 0$ but $F_{pp} \neq 0$ is called a fold point of the projection. The projection of a fold point is a regular point of the discriminant.
b)A point of $S_{\mathcal{W}_d}$ where $F = F_p = F_{pp} = 0$, $F_{ppp} \neq 0$ and $F_x F_{py} - F_y F_{px} \neq 0$ is called an ordinary cusp point of the projection and it's projection gives an ordinary cusp point of the discriminant.

Remark 2.1.2. It can be proved that the following conditions are equivalent:
a) $F(z) = F_p(z) = F_{pp}(z) = 0; F_{ppp} \neq 0$ and $F_x(z)F_{py}(z) - F_y(z)F_{px}(z) \neq 0$.
b) z is a regular point of the the map $(F, F_p, F_{pp}) : \mathbb{C}^3 \longrightarrow \mathbb{C}^3$.
c) The discriminant curve has a local branch at $\pi(z)$ which is a cusp.

Definition 2.1.3. [28]
a) At an ordinary fold, $\pi|S_{\mathcal{W}_d} : S_{\mathcal{W}_d} \longrightarrow \mathbb{C}P^2$ is isomorphic to a mapping

$$\begin{cases} X = x \\ Y = y^2 \end{cases}$$

b) At an ordinary cusp, $\pi|S_{\mathcal{W}_d} : S_{\mathcal{W}_d} \longrightarrow \mathbb{C}P^2$ is isomorphic to a mapping

$$\begin{cases} X = x \\ Y = xy + y^3 \end{cases}$$

Definition 2.1.4. *A point of $\triangle$ is called a* <u>*node*</u> *or an* <u>*ordinary cusp*</u> *if it has a local equation isomorphic to $y^2 = x^2$ or $y^2 = x^3$ respectively.*

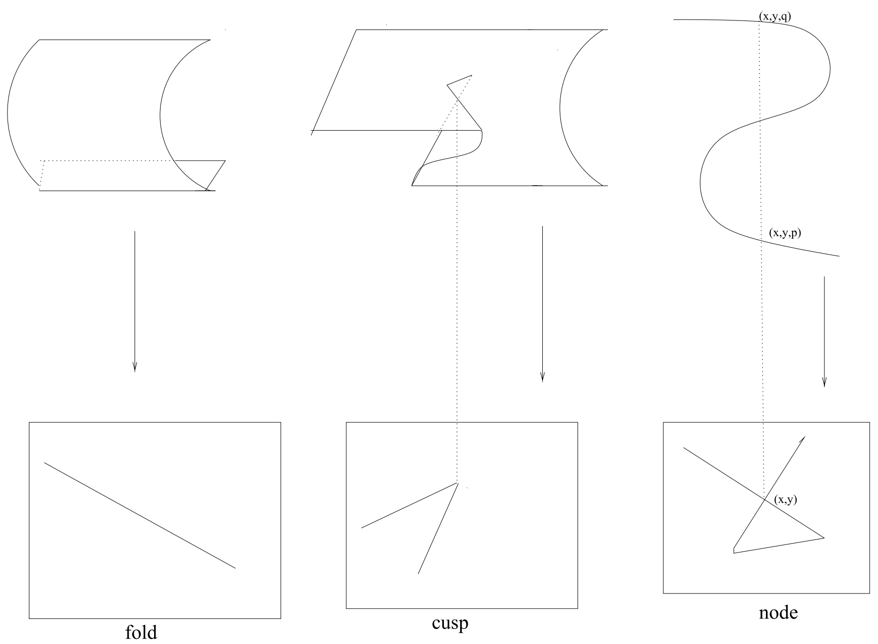


Figure 2.1:

We will prove in this section that the properties above are generic properties of d-webs. In this sense we define:

Definition 2.1.5. *By a* <u>*generic*</u> *d- web of degree n,* $\mathcal{W}_d$*, in* $\mathbb{C}P^2$ *we mean that properties (i),(ii),(iii) and (iv) above are satisfied.*

We adopt the notion of genericity as follows: A subset of $\mathcal{W}(2,n,d)$ is called generic if is an open and dense subset in $\mathcal{W}(2,n,d)$.

Theorem 2.1.1. *The set of d-webs of degree n with properties (i),(ii),(iii) and (iv) is Zariski open dense subset of* $\mathcal{W}(2,n,d)$

The theorem will be broken down into the following lemmas and the proofs will be given basically using the following transversality theorem in algebraic geometry see [26].

Theorem 2.1.2. *Let* $f : X \longrightarrow Z$ *and* $\pi : X \longrightarrow A$ *be proper morphisms (*C^∞ *functions) between smooth varieties(resp.* C^∞ *manifolds) and* W *be a smooth subvariety(resp. submanifold) of* Z*. Also assume that* π *is surjective and* f *is transverse to* W*, then there exists an open and dense subset* U *of* A *such that* $f|_{\pi^{-1}(\alpha)}$ *is transverse to* W *for every* $\alpha \in U$*.*

Lemma 2.1.1. *Generically* $S_{\mathcal{W}_d}$ *is smooth.*

<u>Proof:</u> Fix an affine chart $\mathbb{C}^3 \simeq U_1$. Let $F : U_1 \longrightarrow \mathbb{C}$ given by

$$F(x,y,p) = \sum_{j=0}^{d} a_j(x,y)p^{d-j} + \sum_{j=1}^{d}\sum_{k=0}^{d-j}(xp-y)^j p^{d-j-k} g_{jk}(x,y)$$

defines $S_{\mathcal{W}_d}$ in U_1, where $a_j's$ are polynomials of degree $\leq n$ and g_{jk} are homogenous of degree n.
Define $\tilde{F} : U_1 \times \mathcal{W}(2,n,d) \longrightarrow \mathbb{C}$ by

$$\tilde{F}(x,y,p,a,g) = \sum_{j=0}^{d} a_j(x,y)p^{d-j} + \sum_{j=1}^{d}\sum_{k=0}^{d-j}(xp-y)^j p^{d-j-k} g_{jk}(x,y)$$

where a and g represent the coefficients of the $a_j's$ and $g_{jk}'s$ respectively. and we have the following diagram

$$\begin{array}{ccc} U_1 \times \mathcal{W}(2,n,d) & \xrightarrow{\pi_1} & \mathcal{W}(2,n,d) \\ \tilde{F} \downarrow & & \\ \mathbb{C} & & \end{array}$$

Since

$$a_j = \sum_{s,k} a_{sk}^j x^s y^k$$

we have that $\frac{\partial \tilde{F}}{\partial a_{00}^d} = 1$.

Hence for any vector $(\bar{x}, \bar{y}, \bar{p}, \bar{a_0}, \bar{a_1}, \ldots, \bar{a_d}, \bar{g})$ we have that

$$D_{(x,y,p,a_0,a_1,\ldots,a_d,g)}\tilde{F}(\bar{x}, \bar{y}, \bar{p}, \bar{a_0}, \bar{a_1}, \ldots, \bar{a_d}, \bar{g}) \neq 0.$$

Hence 0 is a regular value of $\tilde{F}$. Therefore by Theorem 2.1.2 ,there exists an open dense subset $\mathcal{W}_1$ of $\mathcal{W}(2,n,d)$ such that 0 is regular value of F for every $(a_0, a_1, \ldots, a_d, g) \in \mathcal{W}_1$.
Similarily, by the same arguments we obtain open dense subsets $\mathcal{W}_2, \mathcal{W}_3, \mathcal{W}_4, \mathcal{W}_5, \mathcal{W}_6$ of $\mathcal{W}(2,n,d)$ in the other 5 charts.
Let $\mathcal{W} = \cap_{j=1}^6 \mathcal{W}_j$ and since the intersection of finite number of open dense subsets is an open dense subset, the lemma follows.

□

Remark 2.1.3. It follows from the proof that $\cap_{j=1}^6 \mathcal{W}_j$ is a Zariski open set.

Lemma 2.1.2. *Generically the singularities of the projection $\pi|S_{\mathcal{W}_d}$ are folds and ordinary cusps for $d \geq 3$. In the particular the ramification curve is smooth.*

Remark 2.1.4. *Under generic conditions,for d=2, it is not possible for the projection to exhibit a cusp singularity, since the projection to the (x, y) plane* *has 0,1,2 or infinitely many points in a fibre. Infact, if* $F = a(x,y)p^2 + b(x,y)p + c(x,y)$ *and the discriminant curve* $\triangle = b^2(x,y) + 4a(x,y)c(x,y)$ *exhibits a cusp at* (x_0, y_0)*, then it is possible to prove that* $a(x_0,y_0) = b(x_0,y_0) = c(x_0,y_0) = 0$ *which is not a generic condition.*

<u>Proof of Lemma 2.1.2:</u> Fix an affine chart $\mathbb{C}^3 \simeq U_1$ as above. Define

$$G_1 : U_1 \times \mathcal{W}(2,n,d) \longrightarrow \mathbb{C}^2$$

by

$$G_1(x,y,p,a,g) = (F, F_p)$$

and

$$G_2 : U_1 \times \mathcal{W}(2,n,d) \longrightarrow \mathbb{C}^3$$

by

$$G_2(x,y,p,a,g) = (F, F_p, F_{pp})$$

where a and g represent the coefficients of the $a_j's$ and $g_{jk}'s$ respectively and we have the following diagrams

$$\left\{ \begin{array}{ccc} U_1 \times \mathcal{W}(2,n,d) & \xrightarrow{\pi_1} & \mathcal{W}(2,n,d) \\ G_1 \downarrow & & \\ \mathbb{C}^2 & & \end{array} \right. \quad \left\{ \begin{array}{ccc} U_1 \times \mathcal{W}(2,n,d) & \xrightarrow{\pi_1} & \mathcal{W}(2,n,d) \\ G_2 \downarrow & & \\ \mathbb{C}^3 & & \end{array} \right.$$

Now since

$$a_j = \sum_{s,k} a^j_{sk} x^s y^k$$

we have that

$$det\Big(\frac{\partial^2 G_1}{\partial a^d_{00} \partial a^{d-1}_{00}}\Big) = det \begin{bmatrix} \frac{\partial F}{\partial a^d_{00}} & \frac{\partial F}{\partial a^{d-1}_{00}} \\ & \\ \frac{\partial F_p}{\partial a^d_{00}} & \frac{\partial F_p}{\partial a^{d-1}_{00}} \end{bmatrix} = det \begin{bmatrix} 1 & p \\ 0 & 1 \end{bmatrix} = 1.$$

and

$$det\Big(\frac{\partial^2 G_2}{\partial a_{00}^d \partial a_{00}^{d-1} \partial a_{00}^{d-2}}\Big) = det\begin{bmatrix} \frac{\partial F}{\partial a_{00}^d} & \frac{\partial F}{\partial a_{00}^{d-1}} & \frac{\partial F}{\partial a_{00}^{d-2}} \\ \\ \frac{\partial F_p}{\partial a_{00}^d} & \frac{\partial F_p}{\partial a_{00}^{d-1}} & \frac{\partial F_p}{\partial a_{00}^{d-2}} \\ \\ \frac{\partial F_{pp}}{\partial a_{00}^d} & \frac{\partial F_{pp}}{\partial a_{00}^{d-1}} & \frac{\partial F_{pp}}{\partial a_{00}^{d-2}} \end{bmatrix} = det\begin{bmatrix} 1 & p & p^2 \\ 0 & 1 & p \\ 0 & 0 & 1 \end{bmatrix} = 1.$$

Hence there exist a 2×2 matrix of DG_1 and a 3×3 matrix of DG_2 with determinants different from zero. This implies that $(0,0)$ and $(0,0,0)$ are regular values of G_1 and G_2 respectively.
Therefore by Theorem 2.1.2, exists open dense subsets $\mathcal{W}_1^1$ and $\mathcal{W}_1^2$ of $\mathcal{W}(2,n,d)$ such that $(0,0)$ and $(0,0,0)$ are regular values of G_1 and G_2 respectively for every $(a_0, a_1, \ldots, a_d, g) \in \mathcal{W}_1^j, j = 1,2$.
Similarily, by the same arguments we obtain open dense subsets $\mathcal{W}_2^j, \mathcal{W}_3^j, \mathcal{W}_4^j, \mathcal{W}_5^j, \mathcal{W}_6^j$; $j = 1,2$ of $\mathcal{W}(2,n,d)$ in the other 5 charts for the 2 maps G_1 and G_2 respectively.
Let $\mathcal{W}^j = \cap_{i=1}^6 \mathcal{W}_i^j, j = 1,2$ and since the intersection of finite number of open dense subsets is an open dense subset, the lemma follows.

□

Lemma 2.1.3. *Let*

$$\Psi : R \times R \longrightarrow \mathbb{C}^2 \times \mathbb{C}^2, \qquad \Psi(x,y) = (\pi(x), \pi(y))$$

where $R \subset S_{\mathcal{W}_d}$ is the ramification curve of π.
Let $D = \{(a,a) \in \mathbb{C}^2 \times \mathbb{C}^2\}$
If Ψ is transversal to D at (x_0, y_0), then the discriminant curve, $\triangle$, of π has a node at $\pi(x_0) = \pi(y_0)$.

Proof: Follows from the fact that

$$D\Psi(x_0,y_0)(v_1,v_2) = (D\pi(x_0)v_1, D\pi(y_0)v_2).$$

Lemma 2.1.4. *Generically Ψ is transversal to D.*

Proof:

Let

$$R = \{F = F_p = 0\} \subset \mathbb{C}^3 \times \mathcal{W}(2, n, d).$$

and

$$a_j = \sum_{s,k} a^j_{sk} x^s y^k$$

The tangent space of R is given by

$$TR = \left\{ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ \vdots \end{bmatrix} ; \begin{bmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} & \frac{\partial F}{\partial p} & \frac{\partial F}{\partial a^d_{00}} & \frac{\partial F}{\partial a^{d-1}_{00}} & \cdots \\ \frac{\partial F_p}{\partial x} & \frac{\partial F_p}{\partial y} & \frac{\partial F_p}{\partial p} & \frac{\partial F_p}{\partial a^d_{00}} & \frac{\partial F_p}{\partial a^{d-1}_{00}} & \cdots \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ \vdots \end{bmatrix} = 0 \right\}$$

$$= \left\{ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ \vdots \end{bmatrix} ; \begin{bmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} & 0 & 1 & p & \cdots \\ \frac{\partial F_p}{\partial x} & \frac{\partial F_p}{\partial y} & \frac{\partial F_p}{\partial p} & 0 & 1 & \cdots \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ \vdots \end{bmatrix} = 0 \right\}$$

Let

$$A \subset TR; A = \{v_6 = v_7 = \cdots = 0\}$$

The map

$$D\pi : A \longrightarrow \mathbb{C}^2; (v_1, v_2, v_3, v_4, v_5) \longrightarrow (v_1, v_2)$$

is surjective. Infact, given any $(v_1, v_2) \in \mathbb{C}^2$, we can always solve for v_3, v_4, v_5 in the system:

$$\frac{\partial F}{\partial x}v_1 + \frac{\partial F}{\partial y}v_2 + v_4 + pv_5 = 0$$
$$\frac{\partial F_p}{\partial x}v_1 + \frac{\partial F_p}{\partial y}v_2 + \frac{\partial F_p}{\partial p}v_3 + v_5 = 0$$

In particular

$$D\pi \times D\pi : A \times A \longrightarrow \mathbb{C}^2 \times \mathbb{C}^2$$

is surjective. Therefore the map Ψ is transversal to the diagonal D. Hence the lemma.

□

Remark 2.1.5. *Lemma 2.1.2 implies that all local singular branches of the discriminant curve are cusps. Lemma 2.1.3 implies that the singularities of the discriminant curve are either cusps or nodes.*

Proof of Theorem 2.1.1:

The proof follows froms Lemmas 2.1.1, 2.1.2, 2.1.3 and 2.1.4.

□

Chapter 3

Number of Singularities of Generic Webs in $\mathbb{C}P^2$

In this chapter, we will use analytic and algebro-geometric arguments to count the number of singularities of the contact 1-form α in $S_{\mathcal{W}_d}$ and also to calculate the number of nodes and cusps of the discriminant curve. We accomplish these objectives, by exploring the geometry of the space $\mathbb{P}T^*(\mathbb{C}P^2)$.

The number of singularities of α in $S_{\mathcal{W}_d}$ is obtained using technologies and results in [5].

The number of cusps of the discriminant curve is obtained using the technology of sheaf of jets of vector bundles and the number of nodes is obtained from the genus formula.

3.1 The Surface $S_{\mathcal{W}_d}$

The projective cotangent space of $\mathbb{C}P^2$, $\mathbb{P}T^*(\mathbb{C}P^2)$, can be covered by six affine coordinates $(x, y, p), (x, y, p_1), (u, v, q), (u, v, q_1), (r, s, t)$ and (r, s, t_1). These coordinates are related as follows:

$$\begin{cases} u = \frac{1}{x} \\ \\ v = \frac{y}{x} \\ \\ q = y - xp \end{cases} \quad \begin{cases} r = \frac{x}{y} \\ \\ s = \frac{1}{y} \\ \\ t = \frac{p}{xp-y} \end{cases} \quad \begin{cases} p_1 = \frac{1}{p} \\ \\ q_1 = \frac{1}{q} \\ \\ t_1 = \frac{1}{t} \end{cases}$$

where we observe that if (x, y, p) is an affine chart in $\mathbb{P}T^*\mathbb{C}P^2$, then (x, y) gives an affine coordinate in $\mathbb{C}P^2$ with (x, y, p) representing the line in $T_{(x,y)}\mathbb{C}P^2$ given by $dy - pdx = 0$.

We denote by $\mathcal{H}$ and ξ the divisors on $\mathbb{P}T^*(\mathbb{C}P^2)$ associated to the surfaces given by $\{y = \alpha x + \beta\}$ and $\{p = 0\}$ respectively on the first coordinate system (x, y, p). Note that $\mathcal{H} = \pi^*(H)$, where H denotes the divisor associated to a line in $\mathbb{C}P^2$.

We will write $\mathcal{H}_r = \mathcal{H}|_{S_{\mathcal{W}_d}} = j^*\mathcal{H}$ and $\xi_r = \xi|_{S_{\mathcal{W}_d}} = j^*\xi$, where $j :\hookrightarrow \mathbb{P}T^*\mathbb{C}P^2$, as the divisors of $\mathcal{H}$ and ξ restricted to $S_{\mathcal{W}_d}$.

Lemma 3.1.1. *Let $\mathcal{W}_d$ be a generic d-web of degree n in $\mathbb{C}P^2$.Then the* <u>*canonical line bundle*</u>*, $K_{S_{\mathcal{W}_d}}$, of the associated surface $S_{\mathcal{W}_d}$ is given by*

$$K_{S_{\mathcal{W}_d}} = -3\mathcal{H}_r + \mathcal{R}_r$$

where $\mathcal{R}_r = \mathcal{R}|_{S_{\mathcal{W}_d}}$ is the divisior of the ramification curve $R \subset S_{\mathcal{W}_d}$.

<u>**Proof:**</u>

Since $\pi|_{S_{\mathcal{W}_d}} : S_{\mathcal{W}_d} \longrightarrow \mathbb{C}P^2$ is branched along $\triangle$, choose a mermorphic 2-form ω on $\mathbb{C}P^2$ such that ω is holomorphic in a neighborhood of $\triangle \cap \mathbb{C}^2$ for some afine coordinate system $\mathbb{C}^2$ and $\omega(p) \neq 0 \quad \forall p \in \triangle$. Then away from R

the zeros and poles of $(\pi|_{S_{\mathcal{W}_d}})^*(\omega)$ correspond to those of ω. Therefore

$$\begin{aligned} K_{S_{\mathcal{W}_d}} &= \pi^*(K_{\mathbb{C}P^2}) + \mathcal{R}_r \\ &= -3\pi^*(H) + \mathcal{R}_r \\ &= -3\mathcal{H}_r + \mathcal{R}_r \end{aligned}$$

The divisor R_r appears because $R \subset S_{\mathcal{W}_d}$ is a locus of ordinary folds, so that it gives origin to zeroes of $(\pi|_{S_{\mathcal{W}_d}})^*(\omega)$. □

3.2 Some Facts About Projective Bundles

Let E be a complex vector bundle of rank r over a manifold X. We denote by $\mathbb{P}(E)$ the projective bundle of E whose fibres are the projective spaces derived from the fibres of E.
Let $\pi : \mathbb{P}(E) \longrightarrow X$ and $\pi^*(E)$ be the pull-back bundle of E to $\mathbb{P}(E)$. We have the following diagram.

$$\begin{array}{lccccc} z \in & \mathbb{P}(E) & \longleftarrow & \pi^*(E) & \hookleftarrow & \mathcal{O}_E(-1) \\ & \downarrow \pi & & \downarrow \pi & & \\ x \in & X & \longleftarrow & E & & \end{array}$$

If z is a point of $\mathbb{P}(E)$ with $\pi(z) = x$, the fibre of $\pi^*(E)$ at x is a copy of E_x and contains the 1-dimensional vector subspace which corresponds to the line of E_x giving rise to the point z in the fibre $\mathbb{P}(E)_x$. The aggregate of these lines is a line bundle over $\mathbb{P}(E)$ (a sub-bundle of $\pi^*(E)$) called the tautological bundle $\mathcal{O}_E(-1)$ of $\mathbb{P}(E)$. The dual bundle of the tautological bundle is denoted by $\mathcal{O}_E(1)$.
We have the following exact sequences, (see Appendix B of [10]):

$$0 \to \mathcal{O}_E(-1) \to \pi^*(E) \to Q \to 0$$

where Q is the quotient bundle $\pi^*(E)/O_E(-1)$.

$$0 \to \mathcal{O}_{\mathbb{P}(E)} \to \pi^*(E) \otimes \mathcal{O}_E(1) \to T_{\mathbb{P}(E)/X} \to 0 \tag{2}$$

where $T_{\mathbb{P}(E)/X}$ is the relative tangent bundle of $\mathbb{P}(E)$ over X.

The following proposition and it's proof can be found in [12, Page 606].

Proposition 3.2.1. *For X any compact oriented C^∞ manifold, $E \longrightarrow X$ any complex vector bundle of rank r, the cohomology ring $H^*(\mathbb{P}(E))$ is generated, as an $H^*(X)$-algebra, by the Chern class*

$$\eta = c_1(\mathcal{O}_E(1))$$

of the tautological bundle, with the single relation

$$\eta^r - c_1(E)\eta^{r-1} + c_2(E)\eta^{r-2} + \cdots + (-1)^{r-1}c_{r-1}(E)\eta + (-1)^r c_r(E) = 0.$$

Lemma 3.2.1. *The class, $[\alpha] := \mathcal{O}_{PT^*\mathbb{C}P^2}\Big((\alpha)_0 - (\alpha)_\infty\Big)$, of the contact form $\alpha = dy - pdx$ on $PT^*\mathbb{C}P^2$ is given by*

$$[\alpha] = -\mathcal{H} - \xi$$

where $(\alpha)_0$ and $(\alpha)_\infty$ represents the divisors of zeros and poles of α respectively.

<u>**Proof:**</u> Note that α doesn't vanish in any point of the affine coordinates (x, y, p). If we change to the other affine coordinates by the coordinate

changes given in (1), we get

$$\begin{cases} \frac{1}{p_1}(p_1 dy - dx) \\ \\ \frac{1}{u}(dv - qdu) \\ \\ \frac{1}{uq_1}(q_1 dv - du) \\ \\ \frac{1}{s(rt-s)}(ds - tdr) \\ \\ \frac{1}{s(r-t_1 s)}(t_1 ds - dr) \end{cases}$$

From this, we get that

$$\mathcal{O}_{PT^*\mathbb{C}P^2}\Big((\alpha)_0\Big) = 0$$

and

$$\mathcal{O}_{PT^*\mathbb{C}P^2}\Big((\alpha)_\infty\Big) = (u = 0) + (p_1 = 0) = \mathcal{H} + \xi.$$

Hence the lemma.

□

Remark 3.2.1. In particular, in our case where $E = T^*\mathbb{C}P^2$, we observe that the contact form $dy - pdx$ on $PT^*\mathbb{C}P^2$ represents the tautological bundle $\mathcal{O}_E(1)$ and if $\eta = c_1(\mathcal{O}_E(1))$, then from Lemma 3.2.1 we have that

$$\eta = \xi + \mathcal{H}.$$

Hence

$$\begin{aligned} H^*(\mathbb{P}(T^*\mathbb{C}P^2)) &= \frac{\mathbb{Z}[\eta, \mathcal{H}]}{\mathcal{H}^3 = 0 = \eta^2 - 3\eta\mathcal{H} + 3\mathcal{H}^2} \\ &= \frac{\mathbb{Z}[\xi, \mathcal{H}]}{\mathcal{H}^3 = 0 = \xi^2 - \xi\mathcal{H} + \mathcal{H}^2}. \end{aligned}$$

Lemma 3.2.2. *The following relations hold:*

a) $\xi^2\mathcal{H} = \mathcal{H}^2\xi = 1$

b) $\xi^3 = 0$

<u>Proof:</u> a) Since $\xi^2 - \xi\mathcal{H} + \mathcal{H}^2 = 0$ we have by multiplying by $\mathcal{H}$ that

$$\xi^2\mathcal{H} - \xi\mathcal{H}^2 + \mathcal{H}^3 = 0$$

which implies that $\xi^2\mathcal{H} = \mathcal{H}^2\xi$, since $\mathcal{H}^3 = 0$. But, since $\mathcal{H}^2$ represents a generic fiber of π we have that $\mathcal{H}^2\xi = 1$.
b) Again since $\xi^2 - \xi\mathcal{H} + \mathcal{H}^2 = 0$ we have by multiplying by ξ that

$$\xi^3 - \xi^2\mathcal{H} + \xi\mathcal{H}^2 = 0$$

which implies $\xi^3 = 0$ from a).

Lemma 3.2.3. *The following relations hold:*

a) $\mathcal{H}_r^2 = d$

b) $\xi_r^2 = n$

c) $\mathcal{H}_r.\xi_r = n + d$

where $\mathcal{A}^2$ *and* $(\mathcal{A}.\mathcal{B})$ *means the self intersection of the divisor* $\mathcal{A}$ *and intersection of the divisors* $\mathcal{A}$ *and* $\mathcal{B}$ *respectively.*

<u>Proof:</u> We take an affine coordinate system (x, y, p) such that $\mathcal{H}$ and ξ are given by the surfaces by $\{y = \alpha x + \beta\}$ and $\{p = 0\}$ respectively.

a)We look for the number of points of intersection of the generic hyperplanes $\{y = \alpha x + \beta\}$ and $\{y = \alpha' x + \beta'\}$ with the surface $S_{\mathcal{W}_d}$. Since this gives a generic fiber of π and π has degree d, we get $\mathcal{H}_r^2 = d$.

b)We look for the number of points of intersection of the hyperplanes $\{p = 0\}$ and $\{p = \epsilon\}$, where ϵ is a constant, with the surface $S_{\mathcal{W}_d}$, given by

$$\sum_{j=0}^{d} a_j(x,y)p^{d-j} + \sum_{j=1}^{d}\sum_{k=0}^{d-j}(xp-y)^j p^{d-j-k} g_{jk}(x,y) = 0$$

in the chart (x, y, p). These hyperplanes intersect in the (r, s, t) chart at points of the form $(r, 0, 0)$. Hence rewriting the equation of the surface $S_{\mathcal{W}_d}$ in the chart (r, s, t) and putting $s = p = 0$ gives the polynomial in r of degree n:

$$g_{d0}(1, r) = 0$$

Hence $\xi_r^2 = n$.

c)The intersection of the hyperplanes $\{y = \alpha x + \beta\}$ and $\{p = 0\}$ with $S_{\mathcal{W}_d}$ is given by the polynomial in x of degree $n + d$:

$$F(x, \alpha x + \beta, 0) = a_d(x, \alpha x + \beta) + \sum_{j=1}^{d}[-\alpha x - \beta]^j g_{j(d-j)}(x, \alpha x + \beta) = 0.$$

Hence $(\mathcal{H}_r).(\xi_r) = n + d$.

□

Lemma 3.2.4. *The class of the surface $S_{\mathcal{W}_d}$ in $PT^*\mathbb{C}P^2$ is given by*

$$\mathcal{L} := d\xi + n\mathcal{H}.$$

<u>Proof:</u>
Let the class of $S_{\mathcal{W}_d}$ in $PT^*\mathbb{C}P^2$ be given by

$$\mathcal{L} := a\xi + b\mathcal{H}.$$

Since

$$\mathcal{H}_r^2 = \mathcal{H}^2\mathcal{L} = d$$
$$\xi_r^2 = \xi^2\mathcal{L} = n$$

we have

$$a\mathcal{H}^2\xi + b\mathcal{H}^3 = d$$
$$a\xi^3 + b\mathcal{H}\xi^2 = n$$

But since

$$\xi^3 = 0 = \mathcal{H}^3, \xi^2\mathcal{H} = \xi\mathcal{H}^2 = 1,$$

we have that

$$a = d, b = n$$

Hence

$$\mathcal{L} := d\xi + n\mathcal{H}.$$

□

Lemma 3.2.5. *The class of the ramification curve $R \subset S_{\mathcal{W}_d}$ is given by*

$$\mathcal{R}_r = (n+1)\mathcal{H}_r + (d-2)\xi_r.$$

In particular we have the following relations:

a) $\mathcal{H}_r.\mathcal{R}_r = (d-1)(2n+d)$

b) $\xi_r.\mathcal{R}_r = n^2 + 2nd + d - n$

Moreover the degree of the discriminant curve , $\triangle$, in $\mathbb{C}P^2$, deg $(\triangle)$, is an even integer and is given by deg $(\triangle) = \mathcal{H}_r.\mathcal{R}_r = (d-1)(2n+d)$.

Proof:

Let $J^1\mathcal{L}$ denote the sheaf of the 1^{st} partial jets along the fibers of the line bundle $\mathcal{L}$ relative to the projection $\pi : \mathbb{P}T^*\mathbb{C}P^2 \longrightarrow \mathbb{C}P^2$. We have the following exact sequence (see Appendix):

$$0 \longrightarrow \Omega^1_{\mathbb{P}T^*\mathbb{C}P^2/\mathbb{C}P^2} \otimes \mathcal{L} \longrightarrow J^1\mathcal{L} \longrightarrow \mathcal{L} \longrightarrow 0$$

where $\Omega^1_{\mathbb{P}T^*\mathbb{C}P^2/\mathbb{C}P^2}$ denote the 1-forms of $\mathbb{P}T^*\mathbb{C}P^2$ relative to $\pi : \mathbb{P}T^*\mathbb{C}P^2 \longrightarrow \mathbb{C}P^2$ (essentially, the forms of the type $a(x, y, p)dp$).
Since $J^1\mathcal{L}$ has rank 2 and $R = \{F = F_p = 0\}$, the divisor $\mathcal{R}$ in $\mathbb{P}T^*\mathbb{C}P^2$ is given by the top Chern class of $J^1\mathcal{L}$. That is

$$\mathcal{R} = \Big(c_2(J^1\mathcal{L})\Big).$$

By the Whitney formula we get

$$c(J^1\mathcal{L}) = c\Big(\Omega^1_{\mathbb{P}T^*\mathbb{C}P^2/\mathbb{C}P^2} \otimes \mathcal{L}\Big) \times c(\mathcal{L})$$

where

$$c(E) = c_0(E) + c_1(E)t + c_2(E)t^2 + c_3(E)t^3 + \cdots + c_r(E)t^r,$$

r is the rank of the vector bundle E. Therefore

$$\begin{aligned} \mathcal{R} &= \Big(c_2(J^1\mathcal{L})\Big) \\ &= c_1\Big(\Omega^1_{\mathbb{P}T^*\mathbb{C}P^2/\mathbb{C}P^2} \otimes \mathcal{L}\Big) \times c_1(\mathcal{L}) \\ &= c_1\Big(\Omega^1_{\mathbb{P}T^*\mathbb{C}P^2/\mathbb{C}P^2} \otimes \mathcal{L}\Big)\Big[S_{\mathcal{W}_d}\Big] \end{aligned}$$

where $c_1(\mathcal{L}) = [S_{\mathcal{W}_d}]$ is the class of $S_{\mathcal{W}_d}$.
Therefore

$$\begin{aligned} R_r = \mathcal{R}|_{S_{\mathcal{W}_d}} = j^*\mathcal{R} = &= j_*\Big(c_1\Big((\Omega^1_{\mathbb{P}T^*\mathbb{C}P^2/\mathbb{C}P^2} \otimes \mathcal{L})|_{S_{\mathcal{W}_d}}\Big)\Big) \\ &= c_1\Big(\Omega^1_{\mathbb{P}T^*\mathbb{C}P^2/\mathbb{C}P^2}|_{S_{\mathcal{W}_d}}\Big) + c_1\Big(\mathcal{L}|_{S_{\mathcal{W}_d}}\Big) \end{aligned}$$

where $j : S_{\mathcal{W}_d} \hookrightarrow \mathbb{P}T^*\mathbb{C}P^2$ and we use the fact that
If $j : Y \hookrightarrow X$ and E a vector bundle on X, and $[Y]$ represents the class of Y in X, then

$$c_i(E)[Y] = j_*\Big(c_i(E|_Y)\Big).$$

We now calculate

$$c_1\Big(\Omega^1_{\mathbb{P}T^*\mathbb{C}P^2/\mathbb{C}P^2}\Big):$$

By [10, Appendix B.5.8. Page 435] we have the following exact sequence:

$$0 \to \mathcal{O}_{\mathbb{P}T^*\mathbb{C}P^2} \to \pi^*(T^*\mathbb{C}P^2)\otimes\mathcal{O}_{T^*\mathbb{C}P^2}(1) \to T_{\mathbb{P}T^*\mathbb{C}P^2/\mathbb{C}P^2} \to 0$$

where $T_{\mathbb{P}(T^*\mathbb{C}P^2)/\mathbb{C}P^2}$ is the relative tangent bundle of $\mathbb{P}T^*\mathbb{C}P^2$ over $\mathbb{C}P^2$.

Dualizing this sequence gives:

$$0 \longrightarrow \Omega^1_{\mathbb{P}T^*\mathbb{C}P^2/\mathbb{C}P^2} \longrightarrow \pi^*(T\mathbb{C}P^2)\otimes\mathcal{O}_{T^*\mathbb{C}P^2}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}T^*\mathbb{C}P^2} \longrightarrow 0$$

Therefore by the Whitney Formula

$$\begin{aligned}
c_1\Big(\Omega^1_{\mathbb{P}T^*\mathbb{C}P^2/\mathbb{C}P^2}\Big) &= c_1\Big(\pi^*(T\mathbb{C}P^2)\otimes\mathcal{O}_{T^*\mathbb{C}P^2}(-1)\Big)\\
&= c_1(\pi^*(T\mathbb{C}P^2)) - 2c_1(\mathcal{O}_{T^*\mathbb{C}P^2}(1))\\
&= 3\mathcal{H} - 2\eta\\
&= \mathcal{H} - 2\xi
\end{aligned}$$

since $c(\mathcal{O}_{\mathbb{P}T^*\mathbb{C}P^2}) = 1$ and $\eta = \mathcal{H}+\xi$. Hence

$$\begin{aligned}
\mathcal{R}_r &= c_1\Big(\Omega^1_{\mathbb{P}T^*\mathbb{C}P^2/\mathbb{C}P^2}|_{S_{\mathcal{W}_d}}\Big) + c_1\Big(\mathcal{L}|_{S_{\mathcal{W}_d}}\Big)\\
&= (\mathcal{H}_r - 2\xi_r) + (n\mathcal{H}_r + d\xi_r) = (n+1)\mathcal{H}_r + (d-2)\xi_r
\end{aligned}$$

In particular

$$\begin{aligned}
a)\mathcal{H}_r.\mathcal{R}_r &= (n+1)\mathcal{H}_r^2 + (d-2)\xi_r.\mathcal{H}_r\\
&= (n+1)d + (d-2)(n+d)\\
&= (d-1)(2n+d)
\end{aligned}$$

$$\begin{aligned} a)\xi_r.\mathcal{R}_r &= (n+1)\mathcal{H}_r.\xi_r + (d-2)\xi_r^2 \\ &= (n+1)(n+d) + n(d-2) \\ &= n^2 + 2nd + d - n \end{aligned}$$

The degree of $\triangle$ is computed by noting that $\pi|_R : R \longrightarrow \triangle$ is generically injective. Hence if H is a line in $\mathbb{C}P^2$ such that H cuts $\triangle$ transversally, then

$$\deg\,(\triangle) = \sharp(H \cap \triangle) = \sharp(\pi^* H \cap R) = \mathcal{H}_r.\mathcal{R}_r = (d-1)(2n+d).$$

Hence the lemma.

□

Lemma 3.2.6. *Let $\mathcal{W}_d$ be a generic d-web of degree n in $\mathbb{C}P^2$. Then the* <u>*Euler characteristic*</u> *of the associated surface $S_{\mathcal{W}_d}$ is given by*

$$\mathcal{X}(S_{\mathcal{W}_d}) = 6n^2 d + 6nd^2 - 14nd + d^2 + 6n - 8n^2 - \kappa$$

where κ is the number of cusps of the discriminant curve $\triangle$.

<u>**Proof:**</u>

Since the topological degree of π is d, according to [7] we have:

$$\mathcal{X}(S_{\mathcal{W}_d}) = d\Big[\mathcal{X}(\mathbb{C}P^2) - \mathcal{X}(\triangle)\Big] + \mathcal{X}(R \cup R_0) \qquad (*)$$

where $R \subset S_{\mathcal{W}_d}$ is the ramification curve and $R_0 = \overline{(\pi^{-1}(\triangle) - R)}$. Now, over a general smooth point of $\triangle$, the map π has (d-1) preimages, one fold point on R and (d-2) regular points on R_0. Over a node of $\triangle$, there lie two folds points on R, R_0 meets R once transversally at each fold point, and (d-4) regular points. Over a cusp of $\triangle$, R_0 meets R twice at the point of R lying over the cusp, and is smooth there; it otherwise has (d-3) regular points. In any case, over either a node or a cusp of $\triangle$, there are only (d-2) preimages, instead of the (d-1) preimages over a generic point of $\triangle$. Therefore

$$\mathcal{X}(R \cup R_0) = \mathcal{X}\Big(\pi^{-1}(\triangle)\Big) = (d-1)\mathcal{X}(\triangle) - (\delta + \kappa).$$

where δ is the number of nodes of $\triangle$. By the adjuction formula we have that

$$\begin{aligned}
-\mathcal{X}(R) &= K_{S_{\mathcal{W}_d}}.\mathcal{R}_r + \mathcal{R}_r^2 \\
&= (-3\mathcal{H}_r + \mathcal{R}_r).\mathcal{R}_r + \mathcal{R}_r^2 \\
&= -3\mathcal{H}_r.\mathcal{R}_r + 2\mathcal{R}_r^2 \\
&= -3(d-1)(2n+d) + 2\Big[(n+1)^2 d + 2(n+1)(d-2)(n+d) + n(d-2)^2\Big] \\
&= 6n^2 d + 6nd^2 - 14nd + d^2 - 3d + 6n - 8n^2
\end{aligned}$$

Since R and $\triangle$ differ, topologically, only over the nodes, we have that the Euler number of $\triangle$ is

$$\begin{aligned}
\mathcal{X}(\triangle) &= \mathcal{X}(R) - \delta \\
&= -6n^2 d - 6nd^2 + 14nd - d^2 + 3d - 6n + 8n^2 - \delta
\end{aligned}$$

Hence from (*) we have

$$\begin{aligned}
\mathcal{X}(S_{\mathcal{W}_d}) &= 3d - d\mathcal{X}(\triangle) + (d-1)\mathcal{X}(\triangle) - (\delta + \kappa) \\
&= 3d - \mathcal{X}(\triangle) - \delta - \kappa \\
&= 3d - (-6n^2 d - 6nd^2 + 14nd - d^2 + 3d - 6n + 8n^2 - \delta) - \delta - \kappa \\
&= 6n^2 d + 6nd^2 - 14nd + d^2 + 6n - 8n^2 - \kappa
\end{aligned}$$

□

3.3 Generic Singularities

Given a d-web in $\mathbb{C}P^2$, $(S_{\mathcal{W}_d}, \pi|_{S_{\mathcal{W}_d}}, \mathcal{F}(\alpha))$, we define the singular set of $\mathcal{W}_d$, $Sing(\mathcal{W}_d)$ as the zero set of the canonical 1-form $\alpha = dy - pdx$ on the criminant $\{F = F_p = 0\}$. This set is given:

- the set of zeros of the vector field $\mathcal{Z} = F_p\frac{\partial}{\partial x} + pF_p\frac{\partial}{\partial y} - (F_x + pF_y)\frac{\partial}{\partial p}$ called the contact singular points.

- the set of points satisfying $\{F = F_p = F_{pp} = 0\}$ called the contact regular points. These points correspond to the cusps of $\triangle$.

Definition 3.3.1. [8] *Let $\mathcal{W}_d$ be a generic d-web in $\mathbb{C}P^2$. A point $z = (x, y, p) \in Sing\mathcal{W}_d$ is generic singularity of $\mathcal{W}_d$ if it's one of the following types:*

a) z is isolated contact singular and is an ordinary fold point of $\pi|_{S_{\mathcal{W}_d}}$ i.e points where $F = F_p = F_x + pF_y = 0$ but $F_{pp} \neq 0$.

b) z is contact regular and is an ordinary cusp of $\pi|_{S_{\mathcal{W}_d}}$ i.e points where $F = F_p = F_{pp} = 0$ but $F_x + pF_y \neq 0, F_{ppp} \neq 0, F_xF_{py} - F_yF_{px} \neq 0$.

Definition 3.3.2. [4] *The multiplicity of a singular point is the maximum number of zeros it can split up into under deformation of the equation $F = 0$.*

We observe that the criminant R is a complete intersection, in the chart (x, y, p) defined by the pair (F, F_p).

Definition 3.3.3. [4, Proposition 2.7]

(a) The multiplicity of a contact singular point corresponding to an ordinary fold of the projection is given by

$$mult(\mathcal{F}(\alpha), q) := dim_{\mathbb{C}} \frac{\mathcal{O}_q}{< F, F_p, F_x + pF_y >}.$$

(b) The multiplicity of a contact regular point corresponding to a cusp of the projection is given by

$$\underline{mult}(\mathcal{W}_d, q) := dim_{\mathbb{C}} \frac{\mathcal{O}_q}{< F, F_p, F_{pp} >}.$$

Remark 3.3.1. *When the web is generic then $\underline{mult}(\mathcal{W}_d, q) = 1$ if $F(q) = F_p(q) = F_{pp}(q) = 0$, since 0 is a regular value of the map (F, F_p, F_{pp}).*

Lemma 3.3.1. *Let $\mathcal{W}_d$ be generic d-web of degree n in $\mathbb{C}P^2$ with isolated contact singularities. Then*

$$c_1(N^*_{\mathcal{F}}) = -\mathcal{H}_r - \xi_r$$

where $N^*_{\mathcal{F}}$ is the dual normal line bundle of the foliation $\mathcal{F}(\alpha)$.

Proof: According to [4, page 20], we have:

$$c_1(N^*_{\mathcal{F}}) = \mathcal{O}_{S_{\mathcal{W}_d}}\Big((\alpha)_0 - (\alpha)_\infty\Big).$$

Hence the lemma follows from Lemma 3.2.1.

□

Corollary 3.3.1. *The tangent line bundle, $T_{\mathcal{F}}$, of the foliation $\mathcal{F}(\alpha)$ is equivalent to a divisor*

$$\mathcal{D} = 2\mathcal{H}_r - \xi_r - \mathcal{R}_r.$$

Proof: Follows from the fact that(cf. [5, Page 20])

$$K_{S_{\mathcal{W}_d}} = c_1(N^*_{\mathcal{F}}) + c_1(T^*_{\mathcal{F}})$$

□

Theorem 3.3.1. *Let $\mathcal{W}_d$ be a generic d-web in $\mathbb{C}P^2$ of degree n. Then the number of cusps κ and nodes δ of the projection $\pi : S_{\mathcal{W}_d} \longrightarrow \mathbb{C}P^2$, are given by*

$$\kappa = 3(d-2)(n^2 + nd + d)$$
$$\delta = \tfrac{1}{2}(d-2)(d-3)(3d + d^2 + 4nd + 4n^2).$$

Proof: Note that

$$\begin{aligned} \kappa &= \sum_{p \in S_{\mathcal{W}_d}} \underline{mult}(\mathcal{W}_d, p) \\ &= [F = 0] \cap [F_p = 0] \cap [F_{pp} = 0] \end{aligned}$$

Let $J^2(\mathcal{L})$ denote the sheaf of 2^{nd} partial jets along the fibers of the line bundle $\mathcal{L}$ relative to the projection $\pi : \mathbb{P}T^*\mathbb{C}P^2 \longrightarrow \mathbb{C}P^2$. We have the following exact sequences (see Appendix):

$$0 \longrightarrow \left(\Omega^1_{\mathbb{P}T^*\mathbb{C}P^2/\mathbb{C}P^2}\right)^{\otimes^2} \otimes \mathcal{L} \longrightarrow J^2(\mathcal{L}) \longrightarrow J^1(\mathcal{L}) \longrightarrow 0$$

$$0 \longrightarrow \left(\Omega^1_{\mathbb{P}T^*\mathbb{C}P^2/\mathbb{C}P^2}\right) \otimes \mathcal{L} \longrightarrow J^1(\mathcal{L}) \longrightarrow \mathcal{L} \longrightarrow 0$$

where $\Omega^1_{\mathbb{P}T^*\mathbb{C}P^2/\mathbb{C}P^2}$ denote the 1-forms of $\mathbb{P}T^*\mathbb{C}P^2$ relative to $\mathbb{C}P^2$.
The number of cusps is the number of zeros of a section of the vector bundle $J^2(\mathcal{L})$. These zeros are given by the top Chern class of $J^2(\mathcal{L})$. Since $J^2(\mathcal{L})$ has rank 3 we have

$$\kappa = c_3\Big(J^2(\mathcal{L})\Big).$$

By the Whitney formula we get

$$\begin{aligned} c\Big(J^2(\mathcal{L})\Big) &= c\Big(J^1(\mathcal{L})\Big) \times c\Big(\Big(\Omega^1_{\mathbb{P}T^*\mathbb{C}P^2/\mathbb{C}P^2}\Big)^{\otimes^2} \otimes \mathcal{L}\Big) \\ &= c(\mathcal{L}) \times c\Big(\Omega^1_{\mathbb{P}T^*\mathbb{C}P^2/\mathbb{C}P^2} \otimes \mathcal{L}\Big) \times c\Big(\Big(\Omega^1_{\mathbb{P}T^*\mathbb{C}P^2/\mathbb{C}P^2}\Big)^{\otimes^2} \otimes \mathcal{L}\Big) \end{aligned}$$

where

$$c(E) = c_0(E) + c_1(E)t + c_2(E)t^2 + c_3(E)t^3 + \cdots + c_r(E)t^r,$$

r is the rank of the vector bundle E. Therefore

$$c_3\Big(J^2(\mathcal{L})\Big) = c_1(\mathcal{L}) \times c_1\Big(\Omega^1_{\mathbb{P}T^*\mathbb{C}P^2/\mathbb{C}P^2} \otimes \mathcal{L}\Big) \times c_1\Big(\Big(\Omega^1_{\mathbb{P}T^*\mathbb{C}P^2/\mathbb{C}P^2}\Big)^{\otimes^2} \otimes \mathcal{L}\Big)$$

and

$$c_1\Big(\Omega^1_{\mathbb{P}T^*\mathbb{C}P^2/\mathbb{C}P^2} \otimes \mathcal{L}\Big) = c_1\Big(\Omega^1_{\mathbb{P}T^*\mathbb{C}P^2/\mathbb{C}P^2}\Big) + c_1(\mathcal{L})$$

$$c_1\Big(\Big(\Omega^1_{\mathbb{P}T^*\mathbb{C}P^2/\mathbb{C}P^2}\Big)^{\otimes^2} \otimes \mathcal{L}\Big) = 2c_1\Big(\Omega^1_{\mathbb{P}T^*\mathbb{C}P^2/\mathbb{C}P^2}\Big) + c_1(\mathcal{L})$$

But in the proof of Lemma 3.2.5., we have that

$$c_1\Big(\Omega^1_{\mathbb{P}T^*\mathbb{C}P^2/\mathbb{C}P^2}\Big) = \mathcal{H} - 2\xi.$$

Hence

$$\begin{aligned}
&c_1(\mathcal{L}) = n\mathcal{H} + d\xi \\
&c_1\Big(\Omega^1_{\mathbb{P}T^*\mathbb{C}P^2/\mathbb{C}P^2} \otimes \mathcal{L}\Big) = n\mathcal{H} + d\xi + \mathcal{H} - 2\xi = (n+1)\mathcal{H} + (d-2)\xi \\
&c_1\Big((\Omega^1_{\mathbb{P}T^*\mathbb{C}P^2/\mathbb{C}P^2})^{\otimes^2} \otimes \mathcal{L}\Big) = n\mathcal{H} + d\xi + 2(\mathcal{H} - 2\xi) = (n+2)\mathcal{H} + (d-4)\xi
\end{aligned}$$

Observing that $\mathcal{H}^3 = 0, \xi^3 = 0, \mathcal{H}^2\xi = 1, \mathcal{H}\xi^2 = 1$ we have that

$$\begin{aligned}
\kappa &= c_3\Big(J^2(\mathcal{L})\Big) \\
&= c_1(\mathcal{L}) \times c_1\Big(\Omega^1_{\mathbb{P}T^*\mathbb{C}P^2/\mathbb{C}P^2} \otimes \mathcal{L}\Big) \times c_1\Big(\Big(\Omega^1_{\mathbb{P}T^*\mathbb{C}P^2/\mathbb{C}P^2}\Big)^{\otimes^2} \otimes \mathcal{L}\Big) \\
&= \Big(n\mathcal{H} + d\xi\Big).\Big((n+1)\mathcal{H} + (d-2)\xi\Big).\Big((n+2)\mathcal{H} + (d-4)\xi\Big) \\
&= n(n+1)(d-4) + n(d-2)(n+2) + n(d-2)(d-4) + d(n+1)(n+2) + \\
&+ d(n+1)(d-4) + d(d-2)(n+2) \\
&= (n+2)(2nd - 2n - d + d^2) + (d-4)(n^2 - n + 2nd + d) \\
&= 3n^2d - 6nd + 3nd^2 - 6n^2 + 3d^2 - 6d \\
&= 3n\Big(n(d-2) + d(d-2)\Big) + 3d(d-2) \\
&= 3n(n+d)(d-2) + 3d(d-2) \\
&= 3(d-2)(n^2 + nd + d)
\end{aligned}$$

To calculate the number of nodes of the discriminant curve $\triangle$, we observe that

the genus of the ramification curve R, being a desingularization of the discriminant curve $\triangle$, is given by

$$g(R) = \frac{(b-1)(b-2)}{2} - (\delta + \kappa)$$

where b is the degree of $\triangle$ and δ and κ are the of number of nodes and cusps respectively of $\triangle$.
Therefore

$$\begin{aligned} 2\delta &= (b-1)(b-2) - 2\kappa - 2g(R) \\ &= b^2 - 3b - 2\kappa + \mathcal{X}(R) \end{aligned}$$

Hence substituting the values of

$$\begin{aligned} &b = \mathcal{H}_r.\mathcal{R}_r = (d-1)(2n+d) \\ &\kappa = 3(d-2)(n^2+nd+d) \\ &\mathcal{X}(R) = -6n^2d - 6nd^2 + 14nd - d^2 + 3d - 6n + 8n^2. \end{aligned}$$

we have

$$\begin{aligned} 2\delta &= (4n^2d^2 + 4nd^3 + d^4 - 8n^2d - 8nd^2 - 2d^3 + 4n^2 + 4nd + d^2) \\ &+ (-6nd - 3d^2 + 6n + 3d) + (-6n^2d - 6nd^2 - 6d^2 + 12n^2 + 12nd + 12d) \\ &+ (-6n^2d - 6nd^2 + 14nd - d^2 + 3d - 6n + 8n^2) \\ &= 4n^2d^2 + 4nd^3 + d^4 - 20n^2d - 20nd^2 - 2d^3 + 24n^2 + 24nd - 9d^2 + 18d \\ &= 4nd^2(n+d) - 20nd(n+d) + 24n(n+d) + (d^4 - 2d^3 - 9d^2 + 18d) \\ &= 4n(n+d)(d^2 - 5d + 6) + d(d-2)(d^2 - 9) \\ &= (d-2)(d-3)(4n^2 + 4nd + d^2 + 3d) \end{aligned}$$

Hence the theorem.

□

Theorem 3.3.2. *Let $\mathcal{W}_d$ be a generic d-web in $\mathbb{C}P^2$ of degree n such the associated foliation $\mathcal{F}_{\mathcal{W}_d}$ in $S_{\mathcal{W}_d}$ has only isolated singularities. Then the total number of singularities of $\mathcal{F}_{\mathcal{W}_d}$ is given by*

$$\sum_{p \in S_{\mathcal{W}_d}} mult(\mathcal{F}_{\mathcal{W}_d}, p) = 3nd^2 + 3n^2d - d^2 - n^2 - 4nd + 3n + 3d.$$

Proof:

From [5, Proposition 1, page 21], we have that

$$\sum_{p\in S_{\mathcal{W}_d}} mult(\mathcal{F}_{\mathcal{W}_d}, p) = \mathcal{X}(S_{\mathcal{W}_d}) + K_{S_{\mathcal{W}_d}}.c_1(N_{\mathcal{F}_{\mathcal{W}_d}}) + \Big[c_1(N_{\mathcal{F}_{\mathcal{W}_d}})\Big]^2$$

Now

$$\begin{aligned}
K_{S_{\mathcal{W}_d}}.c_1(N_{\mathcal{F}_{\mathcal{W}_d}}) &= (-3\mathcal{H}_r + \mathcal{R}_r).(\mathcal{H}_r + \xi_r) \\
&= -3\mathcal{H}_r^2 - 3\xi_r.\mathcal{H}_r + \xi_r.\mathcal{R}_r + \mathcal{H}_r.\mathcal{R}_r \\
&= -3d - 3(n+d) + n^2 + 2nd + d - n + (d-1)(2n+d) \\
&= n^2 + d^2 + 4nd - 6n - 6d \qquad (1)
\end{aligned}$$

and

$$\Big[c_1(N_{\mathcal{F}_{\mathcal{W}_d}})\Big]^2 = (\mathcal{H}_r+\xi_r)^2 = \mathcal{H}_r^2+2\mathcal{H}_r\xi_r+\xi_r^2 = 3(n+d) \qquad (2)$$

From Lemma 3.2.4 we that

$$\begin{aligned}
\mathcal{X}(S_{\mathcal{W}_d}) &= 6n^2d + 6nd^2 - 14nd + d^2 + 6n - 8n^2 - \kappa \\
&= 6n^2d + 6nd^2 - 14nd + d^2 + 6n - 8n^2 - 3(d-2)(n^2 + nd + d) \\
&= 3n^2d + 3nd^2 - 2d^2 - 7nd + 6n - 2n^2 + 6d \qquad (3)
\end{aligned}$$

Hence the theorem follows by adding (1), (2) and (3).

□

In [5, Chapter3], Brunella defines the Baum-Bott index $BB(\mathcal{F}, p)$ of a foliation $\mathcal{F}$ with isolated singularities on a surface and proves the following theorem:

Theorem 3.3.3. *Let $\mathcal{F}$ be a foliation on a compact surface X. Then*

$$\sum_{p\in Sing\mathcal{F}} BB(\mathcal{F}, p) = \Big[c_1(N_{\mathcal{F}})\Big]^2.$$

In particular we have the following:

Proposition 3.3.1. *Let $\mathcal{W}_d$ be a generic d-web in $\mathbb{C}P^2$ of degree n.Then*

$$\sum_{p \in S_{\mathcal{W}_d}} BB(\mathcal{F}_{\mathcal{W}_d}, p) = 3(n+d)$$

where BB represents the Baum-Bott index.

Proof: Follows from Theorem 3.3.3. and equation (2) of the proof of Theorem 3.3.2.
Hence the Proposition.

□

Remark 3.3.2. We observe here that when $d = 1$ and the singularities of the foliation are generic, the surface $S_{\mathcal{W}_1}$ is obtained as the blow-up of the $(n^2 + n + 1)$ singularities of the foliation $\tilde{\mathcal{F}}$ in $\mathbb{C}P^2$. Therefore

$$\Big[c_1(N_{\mathcal{F}})\Big] = \Big[c_1(N_{\tilde{\mathcal{F}}})\Big] + \sum_{j=1}^{(n^2+n+1)} D_j$$

where D_j are the divisors obtained after the blow-ups. Hence

$$\begin{aligned} \Big[c_1(N_{\mathcal{F}})\Big]^2 &= \Big[c_1(N_{\tilde{\mathcal{F}}})\Big]^2 + \sum_{j=1}^{(n^2+n+1)} D_j^2 \\ &= (n+2)^2 - (n^2+n+1) \\ &= 3(n+1) \end{aligned}$$

Coinciding with the result Proposition 3.2.1.
On the other hand, the number of singularities of $\tilde{\mathcal{F}}$ is $2(n^2 + n + 1)$, which coincides with the result of Theorem 3.3.2 if we put $d = 1$.

Appendix

We give a brief explanation of the vector bundles $J^2(\mathcal{L})$ and $J^1(\mathcal{L})$. We recommend the reader to see [11, Chapter 5] for more details.

<u>The Vector Bundle $J^2(\mathcal{L})$</u>

Let $\mathcal{U} = \{U_\alpha, \alpha \in \mathcal{A}\}$ be an open covering for $\mathbb{P}T^*\mathbb{C}P^2$.

The bundle $J^2(\mathcal{L})$ is given by the collection

$$\Big(U_\alpha; (f_\alpha, \frac{\partial f_\alpha}{\partial p_\alpha}, \frac{\partial^2 f_\alpha}{\partial p_\alpha^2})^T\Big), \alpha \in \mathcal{A}.$$

where $f_\alpha \in \mathcal{L}(U_\alpha)$, $(x_\alpha, y_\alpha, p_\alpha) \in U_\alpha$ and v^T means the transpose of the vector v.

Loosely speaking $J^2(\mathcal{L})$ represents the "Taylor series in the variable p to the 2^{nd} order". That is, a section $F \in H^0(\mathbb{P}T^*\mathbb{C}P^2, \mathcal{L})$ near a point $z = (0,0,0)$ in local coordinates has a local defining equation near z given by

$$F(z) + \frac{\partial F}{\partial p}(z)p + \frac{\partial^2 F}{\partial p^2}(z)p^2 + \ldots.$$

Let $l_{\alpha\beta}$ and $k_{\alpha\beta}$ be the transition functions of $\mathcal{L}$ and $\Omega^1_{\mathbb{P}T^*\mathbb{C}P^2/\mathbb{C}P^2}$ (essentially 1-forms of the type $a_\alpha(x_\alpha, y_\alpha, p_\alpha)dp_\alpha$ in U_α) respectively. Then if $U_\alpha \cap U_\beta \neq \phi$,

$$\begin{aligned} f_\alpha &= l_{\alpha\beta} f_\beta \\ \tfrac{\partial f_\alpha}{\partial p_\alpha} &= k_{\alpha\beta}\tfrac{\partial l_{\alpha\beta}}{\partial p_\beta} f_\beta + k_{\alpha\beta} l_{\alpha\beta}\tfrac{\partial f_\beta}{\partial p_\beta} \\ \tfrac{\partial^2 f_\alpha}{\partial p_\alpha^2} &= k^2_{\alpha\beta}\tfrac{\partial^2 l_{\alpha\beta}}{\partial p_\beta^2} f_\beta + 2k^2_{\alpha\beta}\tfrac{\partial l_{\alpha\beta}}{\partial p_\beta}\tfrac{\partial f_\beta}{\partial p_\beta} + k^2_{\alpha\beta} l_{\alpha\beta}\tfrac{\partial^2 f_\beta}{\partial p_\beta^2} \end{aligned}$$

Therefore

$$\begin{pmatrix} f_\alpha \\ \frac{\partial f_\alpha}{\partial p_\alpha} \\ \frac{\partial^2 f_\alpha}{\partial p_\alpha^2} \end{pmatrix} = \begin{pmatrix} l_{\alpha\beta} & 0 & 0 \\ k_{\alpha\beta}\frac{\partial l_{\alpha\beta}}{\partial p_\beta} & k_{\alpha\beta} l_{\alpha\beta} & 0 \\ k^2_{\alpha\beta}\frac{\partial^2 l_{\alpha\beta}}{\partial p_\beta^2} & 2k^2_{\alpha\beta}\frac{\partial l_{\alpha\beta}}{\partial p_\beta} & k^2_{\alpha\beta} l_{\alpha\beta} \end{pmatrix} \begin{pmatrix} f_\beta \\ \frac{\partial f_\beta}{\partial p_\beta} \\ \frac{\partial^2 f_\beta}{\partial p_\beta^2} \end{pmatrix}$$

Let

$$M_{\alpha\beta} = \begin{pmatrix} l_{\alpha\beta} & 0 & 0 \\ k_{\alpha\beta}\frac{\partial l_{\alpha\beta}}{\partial p_\beta} & k_{\alpha\beta} l_{\alpha\beta} & 0 \\ k^2_{\alpha\beta}\frac{\partial^2 l_{\alpha\beta}}{\partial p^2_\beta} & 2k^2_{\alpha\beta}\frac{\partial l_{\alpha\beta}}{\partial p_\beta} & k^2_{\alpha\beta} l_{\alpha\beta} \end{pmatrix}$$

It's easily checked that $\{M_{\alpha\beta}\}$ defines a cocyle.Hence $J^2(\mathcal{L})$ is a vector bundle of rank 3.

By the same arguments we get that $J^1(\mathcal{L})$ is a vector bundle of rank 2. Consider the map

$$\varphi : J^2(\mathcal{L}) \longrightarrow J^1(\mathcal{L})$$

given in the open set U_α by

$$(v_{\alpha,0}, v_{\alpha,1}, v_{\alpha,2})^T \longrightarrow (v_{\alpha,0}, v_{\alpha,1})^T.$$

This map is surjective and it's kernel is formed by all $(0, 0, v_{\alpha,2})^T \in J^2(\mathcal{L})$. If $U_\alpha \cap U_\beta \neq \phi$, we have that

$$(0, 0, v_{\alpha,2})^T = M_{\alpha\beta}(0, 0, v_{\beta,2})^T$$

That is

$$v_{\alpha,2} = k^2_{\alpha\beta} l_{\alpha\beta}.v_{\beta,2}$$

Hence $\ker(\varphi)$ may be identified with a line bundle isomorphic to

$$\left(\Omega^1_{\mathbb{P}T^*\mathbb{C}P^2/\mathbb{C}P^2}\right)^{\otimes^2} \otimes \mathcal{L}.$$

We therefore have the exact sequence

$$0 \longrightarrow \left(\Omega^1_{\mathbb{P}T^*\mathbb{C}P^2/\mathbb{C}P^2}\right)^{\otimes^2} \otimes \mathcal{L} \longrightarrow J^2\Big(\mathcal{L}\Big) \longrightarrow J^1\Big(\mathcal{L}\Big) \longrightarrow 0$$

By the same arguments for the map

$$\varphi' : J^1(\mathcal{L}) \longrightarrow \mathcal{L}$$

given in the open set U_α by

$$(v_{\alpha,0}, v_{\alpha,1})^T \longrightarrow v_{\alpha,0}.$$

we obtain the exact sequence

$$0 \longrightarrow \left(\Omega^1_{\mathbb{P}T^*\mathbb{C}P^2/\mathbb{C}P^2}\right) \otimes \mathcal{L} \longrightarrow J^1\left(\mathcal{L}\right) \longrightarrow \mathcal{L} \longrightarrow 0.$$

For further reading on jets bundles, we recommend the reader to see [11, Chapter 5].

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