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HUYGENS' PRINCIPLE FOR DIRAC OPERATORS

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"Thus, our actual physical world,
in which acoustic and electromagnetic
signals are the basis of communication
seems to be singled out among other
mathematically conceivable models by
intrinsic simplicity and harmony."

Richard Courant and David Hilbert

À Gabriela,
ao Guilherme
e ao Bernardo
(que está aprendendo a somar).

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Resumo

O assunto desta tese é o princípio de Huygens no sentido estrito de Hadamard para operadores de Dirac. Por tal propriedade, entendemos que a solução do problema de valor inicial depende apenas das condições iniciais na interseção do cone de luz passado (e não do interior deste cone).

Uma interessante conexão entre as soluções racionais da hierarquia de Korteweg-de Vries e a propriedade de Huygens foi descoberta como conseqüência dos trabalhos independentes de Lagnese e Stellmacher e de Adler e Moser. Esta ligação foi reforçada por trabalhos de Y. Berest nos anos 90. A equação da KdV é apenas uma pequena amostra da teoria de sistemas completamente integráveis em dimensão infinita, posteriormente estendidos imensamente por Ablowitz, Kaup, Newell e Segur (AKNS) e por Zakharov e Shabat.

Neste trabalho, estabelecemos uma conexão entre as soluções racionais da hierarquia AKNS de equações integráveis não-lineares e operadores de Dirac que satisfazem a propriedade de Huygens. Nós também caracterizamos os operadores do tipo Huygens em 1+1 e 3+1 dimensões.

Palavras-chave: princípio de Huygens, operadores de Dirac, Integrabilidade.

Abstract

This work is concerned with Huygens' property in Hadarmard's strict sense for Dirac operators. By such property we mean that the solutions of the initial value problem depend only on the intersection of the light cone with the initial data manifold, and not on the interior part of the light cone.

A fascinating connection between the rational solutions of the Korteweg-de Vries hierarchy and Huygens property was discovered as a consequence of the independent works of Lagnese & Stellmacher and Adler & Moser. This link was further strenghened by the results of Y. Berest in 90's. The KdV equation was the tip of the iceberg for the rich theory of infinite dimensional completely integrable systems, which was later extended tremendously by Ablowitz, Kaup, Newell & Segur (AKNS) and Zakharov & Shabat.

In this work we establish a connection between the rational solutions of the AKNS hierarchy of nonlinear integrable equations and Dirac operators satisfying Huygens' property. We also characterize Huygens' operators in 1+1 dimensions and 3+1 under certain assumptions.

Key-words: Huygens' principle, Dirac operators, Integrability

Preface

The main purpose of this thesis is to show a connection between two apparently unrelated topics: Huygens' principle for Dirac Operators and AKNS systems.

Huygens' principle, in Hadamard's strict sense, is a property of hyperbolic partial differential equations. By such property we mean that the solutions of the initial value problem depend only on the intersection of the light cone with the initial data manifold, and not on the interior part of the light cone. This is a crucial property for the meaningful propagation of information.

Ablowitz, Kaup, Newell & Segur (AKNS) systems consists of a hierarchy of integrable equations that generalizes the Korteweg-de Vries (KdV), the modified Korteweg-de Vries (mKdV) and the cubic Schrödinger equations, among others.

The aforementioned connection can be summarized as follows: for a large class of rational solutions of the AKNS, we present a way of constructing a Dirac operator of Huygens' type. This extends for Dirac Operators the now classical results of Lagnese and Stellmacher.

This work is structured as follows:

- In Chapter 1, we present the historical development of Huygens' principle and review some important results from the literature that will be useful in the current work. Particular attention is given to the study of the wave operator and its relation to other branches of Mathematics. A recent conjecture by Berest about the full classification of second order hyperbolic operators that satisfy Huygens' property is discussed.

- In Chapter 2, we present the basic tools for the study of Huygens' principle in the case of Dirac operators. The results of this chapter, albeit original, are simple generalizations of known results for the case of second order operators. The key point is the introduction of what we call Dirac kernels.

- In Chapter 3, we examine Huygens' principle for Dirac operators in 1 and 3 space dimensions. For 1 dimension we prove that the only Huygens' operator (modulo certain set of equivalences) is the trivial one. In the 3 dimensional case, we study a certain restricted class of potentials and obtain for them the full classification of Huygens' operators.

- In Chapter 4, we start with a short review of integrable systems. We first prove that rational solutions of the mKdV are directly linked to Huygens' potentials for Dirac operators. We also point out an interesting phenomena, which is the reduction of the minimal number of space dimensions. We conclude by proving that the rational solutions of the AKNS hierarchy are related to the Huygens' property.

- In Chapter 5, we analyze the results obtained, in particular those of Chapter 4, and conclude with some possible directions for future research.

For the mental confort of the reader, one remark should be made: the terms "Huygens' principle," "Huygens' property" and "Huygens' type" are used interchangeably. These terms, strictly speaking, are properties of differential operators, but are sometimes used to refer to its solutions. We hope this causes no damage.

Fabio Chalub
Rio de Janeiro, July 6, 2001.

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CHAPTER 1

Introduction

1.1 Huygens' Principle

By 1690 two different theories were disputing the nature of light: for Newton [1], light was composed by tiny particles propagating along straight lines; whereas Huygens believed that light was a mechanical wave propagating in the æther, the hypothetical medium that filled all the space [2].

Huygens' theory was particularly useful to explain diffraction ("the strange refraction of the Iceland crystal"), but had some remarkable difficulties to explain the apparent propagation of the light along straight lines. To overcome this difficulty, he wrote his masterpiece "Treatise on Light" [3] where he formulated the principle that bears his name.

In a very beautiful introduction to some mathematical aspects of Huygens' principle [4], Duistermaat discussed his surprise when he saw that Huygens used no partial differential equations nor differential calculus to formulate his principle. Actually, all his constructions are purely geometrical.

In more modern terms (in a formulation essentially due to Fresnel [5]) we say that each point in the wave front acts as a secondary source of wave and all these waves interfere to produce the new wave front.

Despite the large use of the term "Huygens' principle" in the physical community, its precise mathematical formulation is seldom made clear. In 1923, the French mathematician J. Hadamard, in a series of lectures at Yale University [6], formulated, in mathematical terms, three different meanings of "Huygens' principle" he found in the literature of his time[1]. These were:

[1]Following Hadamard, we call the wave velocity ω.

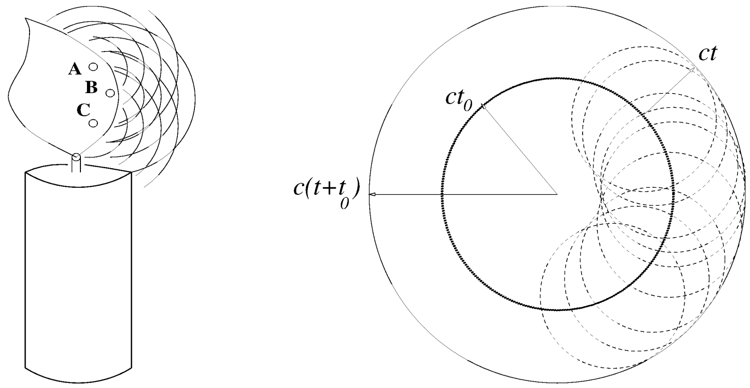

Figure 1.1: Wave fronts propagating from a candle (left). Huygens' construction of a wave front from the previous one (right).

(A) – Major premise:

"The action of phenomena produced at the instant $t = 0$ on the state of matter at the later time $t = t_0$ takes place by the mediation of every intermediate instant $t = t'$, i.e. (assuming $0 < t' < t_0$), in order to find out what takes place for $t = t_0$, we can deduce from the state at $t = 0$ the state at $t = t'$ and, from the latter, the required state at $t = t_0$."

(B) – Minor premise:

"If, at the instant $t = 0$ — or more exactly throughout a short interval $-\epsilon \leq t \leq 0$ — we produce a luminous disturbance localized in the immediate neighborhood of 0, the effect of it will be, for $t = t'$, localized in the immediate neighborhood of the surface of the sphere with center O and radius $\omega t'$: that is, it will be localized in a very thin spherical shell with center O including the aforesaid sphere."

(C) – Conclusion:

"In order to calculate the effect of our initial luminous phenomenon produced at O at $t = 0$, we may replace it by a proper system of disturbance taking place at $t = t'$ and distributed over the surface of the sphere with center 0 and radius $\omega t'$."

Hadamard calls the major premise a "truism." This, however, does not mean that it is of no interest: "For the geometer does not dislike truism" — he said. Proposition **(A)** is directly linked to (semi-)group properties of propagators. Actually, the formula

$$\psi(t', \vec{x}') = \int d^3\vec{x}\; G(t', \vec{x}'; t, \vec{x})\psi(t, \vec{x}) \; ,$$

describing how to obtain the wave amplitude ψ in some space-time point $(t', \vec{x}')$ from the amplitude in a generic point $(t, \vec{x})$ and the **propagator** $G(t', \vec{x}'; t, \vec{x})$ from $(t, \vec{x})$ to $(t', \vec{x}')$ is know in the Physical literature as Huygens' principle[2] (see [7]).

Proposition **(C)** is a property of many operators and was subject of investigation first by Kirckhoff and later by Volterra. See [6] and references therein.

Proposition **(B)** was the main interest of Hadamard, and the totality of this thesis is dedicated to investigate this property in a certain class of operators to be defined.

As Huygens' principle was originally formulated for light propagation, a physical phenomena modeled by the wave equation, it is an interesting idea to investigate Huygens' principle in Hadamard's minor premise sense in this case.

The wave equation is a partial differential equation in $n+1$ variables, $(x^0, x^1, \cdots, x^n)$, where x^0 is taken as *time*, and some times will be called t, and the other variables are the *space* variables. Explicitly, it is an equation for a function u such that

$$\frac{\partial^2 u}{(\partial x^0)^2} - \frac{\partial^2 u}{(\partial x^1)^2} - \cdots - \frac{\partial^2 u}{(\partial x^n)^2} = f \ ,$$

where f is known as the *source*[3]. The above equation is known as wave equation in n space variables.

We define the **D'Alembertian** operator as

$$\Box_n \stackrel{\text{def}}{=} \frac{\partial^2}{(\partial x^0)^2} - \sum_{i=1}^{n} \frac{\partial^2}{(\partial x^i)^2} \ . \tag{1.1}$$

So, the wave equation is written

$$\Box_n u = f \ .$$

Whenever possible we will omit the sub-index n. From now on we abbreviate

$$\partial_\mu = \frac{\partial}{\partial x^\mu} \qquad \mu = 0, 1, \cdots, n \ .$$

For a general metric tensor $g^{\mu\nu}(x)$ we define the light cone with vertex in y as

$$\mathcal{C}(y) = \left\{ x | \sum_{\mu,\nu=0}^{n} g_{\mu\nu}(x)(x^\mu - y^\mu)(x^\nu - y^\nu) = 0 \right\} \ ,$$

where the matrix $g_{\mu\nu}$ is the inverse of $g^{\mu\nu}$. The forward (respectively, backward) light cone, $\mathcal{C}^+(y)$ (respectively, $\mathcal{C}^-(y)$) is the sub-set of $\mathcal{C}(y)$ such that $x^0 > y^0$ (respectively, $x^0 < y^0$).

[2] We shall adopt the notation $\vec{x}$ whenever we are referring to the space components.

[3] With these definitions we take the velocity of propagation c equals 1. If we do not want to fix the value of c, we change in the above and the following expression the value of x^0 (or t) by cx^0 (or ct).

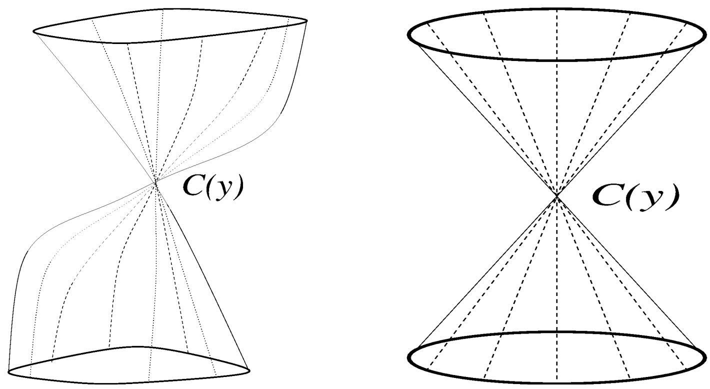

Figure 1.2: Light cone for a general pseudo-Riemannian metric with signature equals to $[+,-,\cdots,-]$ (left) and for a Minkowskian metric (right).

The *interior* of the light cone is defined as

$$\mathcal{J}(y) = \left\{ x | \sum_{\mu,\nu=0}^{n} g_{\mu\nu}(x)(x^\mu - y^\mu)(x^\nu - y^\nu) > 0 \right\} .$$

The interior of the forward (respectively, backward) light cone, $\mathcal{J}^+(y)$ (respectively, $\mathcal{J}^-(y)$) is the sub-set of $\mathcal{J}(y)$ such that $x^0 > y^0$ (respectively, $x^0 < y^0$). Obviously $\partial\mathcal{J}^\pm(y) = \mathcal{C}^\pm(y) \cup \{y\}$.

We will rewrite Huygens' principle in two different but equivalent ways, both definitions equivalent to Hadamard's minor premise.

Huygens' principle – Cauchy problem:

Consider Cauchy's initial value problem, for the hyperbolic[4] second-order differential operator $\mathcal{L}$.

$$\begin{cases} \mathcal{L}\psi = 0 \ , \\ \psi\Big|_{x^0=0} = f \ , \\ \partial_0\psi\Big|_{x^0=0} = g \ . \end{cases} \tag{1.2}$$

We say that the operator $\mathcal{L}$ (and, by extension, its solution ψ) possesses Huygens' property (or obeys Huygens' principle) if the domain of dependence of the solution ψ in an arbitrary point in space-time y, with $y^0 > 0$, depends on the initial manifold $x^0 = 0$ only on its intersection with the backward light cone with vertex in y.

[4]By *hyperbolic* we mean that its principal part, given by $\sum_{\mu,\nu} g^{\mu\nu}(x)\partial_\mu\partial_\nu$, is such that the quadratic form $\sum_{\mu,\nu} g_{\mu\nu}(x)x^\mu x^\nu$ has signature $[+,-,\cdots,-]$ $\forall x \in \mathbb{R}^{n+1}$ ($g_{\mu\nu}$ is the inverse matrix of $g^{\mu\nu}$).

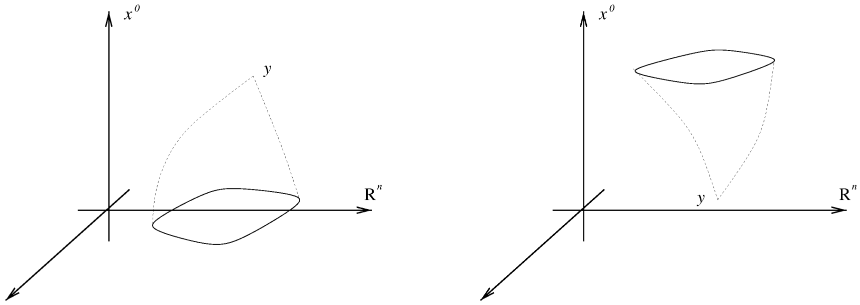

Figure 1.3: Light-cone with vertex in y. In the left — Cauchy problem — the solution of the Cauchy problem should depend on the initial manifold $x^0 = 0$ only in the intersection of the backward light-cone. In the right — fundamental solution — the fundamental solution of $\mathcal{L}$ associated to a Dirac-delta distribution supported in y should have support in the forward light-cone.

Huygens' principle – Fundamental solution:

Now let us consider the fundamental solution (also known as Green's function) of the operator $\mathcal{L}$

$$\mathcal{L}\phi = \delta_y \ , \tag{1.3}$$

where δ_y is the Dirac-delta distribution supported in y.

We say that $\mathcal{L}$ (and, by extension, its fundamental solution ϕ) possesses Huygens' property (or obeys Huygens' principle) if, for every y, ϕ is supported in the forward light cone with vertex in y.

It is a non trivial fact that both definition are equivalent. The proof is consequence of Duhamel's principle. See [8].

A minute's thought reveals that such property is crucial for the meaningful transmission of information. For an elementary account, see P. Günther's delightful paper [9].

Hadamard's problem consists in classifying all second-order hyperbolic differential operators $\mathcal{L}$ that possesses Huygens' property, up to some set of transformations that trivially preserves Huygens' principle: change of coordinates and left and right multiplication of the operator by non-singular smooth functions. Since Hadamard's time it is known that if $\mathcal{L} = \Box_n$, then Huygens' property holds if, and only if, the number of spatial dimensions n is odd and greater than 1. In particular, flat-landers are not fond of music concerts! This result will be fully addressed in the next section.

Before finishing this section we will directly address Huygens' principle for $n = 1, 2$ and 3, in the Cauchy problem case.

Let's consider the solution of Cauchy problem (1.2) for $\mathcal{L} = \Box_1$. It is easy to see that the solution is given by

$$\psi(x,t) = \frac{1}{2}[f(x+t) + f(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} g(\xi)d\xi \ .$$

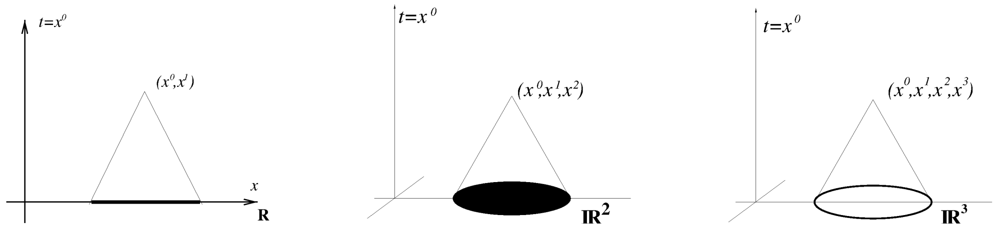

Figure 1.4: Dependence of the initial condition for the D'Alembertian in dimensions 1,2 and 3.

Due to the presence of this integral, we conclude that $\Box_1$ does not possesses Huygens' property.

Now, let's consider $\mathcal{L} = \Box_3$. The solution of (1.2) is obtained by the method of spherical means [8] and is

$$\psi(t,\vec{x}) = \frac{1}{4\pi t}\int_{|\vec{y}-\vec{x}|=t} g(\vec{y})dS_y + \frac{\partial}{\partial t}\left(\frac{1}{4\pi t}\int_{|\vec{y}-\vec{x}|=t} f(\vec{y})dS_y\right) .$$

From the above expression we readily see that in three space dimensions the wave operator obeys Huygens' principle.

In order to obtain the solution for $n = 2$, we use the method know as *Hadamard's method of descent* [8, 10]. The solution is

$$\psi(x^1,x^2,t) = \frac{1}{2\pi}\iint_{r<t} \frac{g(y^1,y^2)}{t^2-r^2}dy^1dy^2 + \frac{\partial}{\partial t}\left\{\frac{1}{2\pi}\iint_{r<t} \frac{f(y^1,y^2)}{t^2-r^2}dy^1dy^2\right\} ,$$

where

$$r = \sqrt{(x^1-y^1)^2+(x^2-y^2)^2} .$$

Again, looking to the solution, we see that $\Box_2$ does not obey Huygens' principle. These results justify the epigraph of this thesis, extracted from [11]:

> "Thus, our actual physical world, in which acoustic and electromagnetic signals are the basis of communication seems to be singled out among other mathematically conceivable models by intrinsic simplicity and harmony."

For others values of n we can prove, by the same way above, that $\Box_n$ obeys Huygens' principle if, and only if, n is odd. See [10].

1.2 Fundamental Solutions for the Wave Operator

In this section we will sketch the proof that $\Box_n$ is a Huygens operator if, and only if, n is odd and greater than 1. For the details, the reader is referred to Folland's paper [12].

Following [12], we classify all fundamental solutions invariant by the orthocronous Lorentz group. The Lorentz group $O(1,n)$ is the subgroup of the group of invertible matrices $GL(n+1,\mathbb{R}^{n+1})$ that preserves the quadratic form given by

$$\lambda^2 = (x^0)^2 - \sum_{i=1}^{n}(x^i)^2 \ . \tag{1.4}$$

The orthocronous Lorentz group $O^+(1,n)$ is the connected component of $O(1,n)$ containing the identity, i.e., that preserves the direction of time. The D'Alembertian is clearly invariant by the action of this group.

For positive values of the real part of α we can define the Riesz kernel[5]

$$\Lambda^\alpha = N(\alpha)\lambda^\alpha \ , \tag{1.5}$$

where the normalization is given by

$$N(\alpha) = \left[2^{\alpha+n}\pi^{(n-1)/2}\Gamma(\frac{\alpha+n+1}{2})\Gamma(\frac{1}{2}\alpha+1)\right]^{-1} \ . \tag{1.6}$$

$\Gamma(\cdot)$ is the Gamma function.

$N(\alpha)$ obeys the important recursion rule

$$(\alpha+2)(\alpha+n+1)N(\alpha+2) = N(\alpha) \ . \tag{1.7}$$

With the help of preceding formula, one proves [12]

$$\Box\Lambda^{\alpha+2} = \Lambda^\alpha \ .$$

So, we can extend the definition of Riesz kernels for all complex values of α.

From equation (1.6) we see that $N(\alpha) = 0$ when $\alpha = -2, -4, -6, \cdots$ or $\alpha = -n-1, -n-3, -n-5, \cdots$. These are the poles of the Gamma function in equation (1.6). In this case Λ^α is supported where λ^α diverges, i.e., in the light cone (for the real part of α negative).

By an analytic continuation of Λ^α to the region $\Re(\alpha) > -n-2$, one finds that $\Lambda^{-n-1} = \delta_0$, so Λ^{-n+1} is the fundamental solution of $\Box_n$. Λ^α is an analytic family of distributions, i.e., for every fixed φ smooth and compactly supported, $\varphi \in C_0^\infty$, the function $\alpha \to \int \Lambda^\alpha \varphi$ is analytic.

From the aforementioned values of $N(\alpha)$ we conclude that the fundamental solution of $\Box_n$ is supported on the light cone (and, consequently, $\Box_n$ has the Huygens property) if, and only if, n is odd and greater than 1.

For a fundamental solution with the Dirac-delta measure supported out of the origin we just need to translate the solution, as the D'Alembertian is invariant under translation.

[5]Note that α is an exponent in λ and an index in Λ.

1.3 Huygens' Principle in the Second Half of the XXth Century

General second order hyperbolic differential operators

$$\mathcal{L} = \sum_{\mu,\nu=0}^{n} g^{\mu\nu}\partial_\mu\partial_\nu + \sum_{\mu=0}^{n} A^\mu\partial_\mu + C \ ,$$

where $g^{\mu\nu}$ (the metric tensor), A^μ and C are functions of x, have been studied by many authors.

For the the validity of Huygens' principle in 3+1 space-time, obeying Einstein's equation in vacuum $R_{\mu\nu} = 0$, where $R_{\mu\nu}$ is the Ricci tensor, see articles [13, 14, 15, 16]. The validity of Huygens' principle for ultra-hyperbolic operators[6], specifically in the case of Minkowski's generalized (p, q) metric, was studied in [17, 18]. Some important background papers are [19, 20, 21]. Validity of Huygens' property seems to be related to Painlevé's property, defined in [22], as is observed by [23] and a precise relation between both is given in [24]. Huygens' equations in spaces with non-trivial conformal groups are studied in [23, 25]. Important surveys are [23, 26, 27, 28, 29]. The interested reader should also consult the compendium [30].

The theory of Huygens' principle for non-linear differential equation is yet to be explored, but an interesting example of a non-linear equation which possesses Huygens' property is given by

$$\Box_3\Psi + (\partial_0\Psi)^2 - \sum_{i=1}^{3}(\partial_i\Psi)^2 = 0 \ .$$

It can be obtained from $\Box_3\psi = 0$ by setting $\psi = 1 - \exp\Psi$. See [23].

In the upcoming subsections we study operators in the form $\mathcal{L} = \Box + u(x)$. We will now develop the tools to approach this problem.

In order to obtain the fundamental solution of the operator $\Box + u$, we shall develop a series expansion in Riesz kernels, which were defined in equation (1.5)

$$\Phi = \sum_{k=0}^{\infty} \Lambda^{-n+1+2k} w_k \ . \tag{1.8}$$

Applying the operator $\Box + u$ and equating to $\Lambda^{-n-1} = \delta_y$ we find the following recursion:

$$\begin{aligned} w_0 &= 1 \ , \\ w_k + \frac{1}{k}\sum_{\mu=0}^{n}(x^\mu - y^\mu)\partial_\mu w_k &= -(\Box + u)w_{k-1} \ . \end{aligned}$$

This recursion is known as Hadamard's recursion. Its termination with respect to k is equivalent to the validity of Huygens' principle for large enough n, as can be

[6] As *ultra-hyperbolic* we mean an operator whose signature is given by $[+, +, \cdots, +, -, -, \cdots, -]$.

readily seen from the properties of Riesz kernels stated in the preceding section. The termination requirement implies a highly nonlinear condition over u, which makes it very difficult to find examples of Huygens' potentials.

The conjecture that every Huygens' operator should be trivially equivalent (in the sense defined in Section 1.1) to the D'Alembertian became known in the literature by **Hadamard's conjecture** [11], despite the fact that it is doubtful that Hadamard really stated this conjecture [9]. Furthermore, the conjecture is, in general, false.

For $n = 3$, however, this conjecture is true, as can be readily seen by the fact that Hadamard's expansion (1.8) should be of the form $\Phi = \Lambda^{-2}$, so there is no freedom to add a potential.

For $n = 5$, Stellmacher found in 1953 a counterexample [31, 32], generalized by himself in collaboration with Lagnese to an entire hierarchy of operators. See [33, 34, 35, 36]. We will start studying these examples in the next subsection.

1.3.1. Lagnese & Stellmacher

Following an idea that seems to be originated in Geometry, by Darboux [37], we consider the following factorization[7]

$$\partial_t^2 = \left(\partial_t + \frac{1}{t}\right)\left(\partial_t - \frac{1}{t}\right) .$$

Now, invert the factors, to find a new operator in the variable t:

$$\left(\partial_t - \frac{1}{t}\right)\left(\partial_t + \frac{1}{t}\right) = \partial_t^2 - \frac{2}{t^2} .$$

The central idea is to show that this new operator is also a Huygens' one for odd $n \geq 5$, as can be seen from its fundamental solution

$$\left(\Box - \frac{2}{t^2}\right)\left(\Lambda^{-n+1} + \frac{2}{tt_0}\Lambda^{-n+3}\right) = \delta_y .$$

It is easy to see that this process can be carried on in order to find a full hierarchy of Huygens operators. Explicitly, if we have a Huygens operator $\mathcal{L}$ and find a pair of operator $\tilde{\mathcal{L}}$ and ℓ (called a *intertwining* operator) such that the relation

$$\ell\mathcal{L} = \tilde{\mathcal{L}}\ell$$

holds, then $\tilde{\mathcal{L}}$ is a Huygens operator. This will be subject of Lemma 6. Lagnese and Stellmacher, in [36], specialized to the study of operators of the form

$$\mathcal{L} = \Box + c(t) = \partial_t^2 + c(t) - \Delta ,$$

[7] We stress that the Darboux transformation can be performed in any variable.

where $c(t)$ is continuous in some interval I. We choose $\mu(t)$ as one of the non-zero solutions[8] of

$$\mu_{tt} + c(t)\mu = 0$$

on an interval $I' \subset I$ and define the linear operators

$$l = \partial_t - \frac{\mu_t}{\mu} \ , \quad l^* = -\partial_t - \frac{\mu_t}{\mu} \ ,$$

where l^* is the formal adjoint of l and the sub-index t denotes derivative with respect to the variable t. We write

$$\mathcal{L} = -\Delta - l^* l \ .$$

As l commutes with Δ we find that

$$l\mathcal{L} = \tilde{\mathcal{L}} l \ ,$$

where

$$\tilde{\mathcal{L}} = -\Delta - l l^* \ .$$

We easily see that

$$l l^* = -\partial_t^2 - \frac{\mu_{tt}}{\mu} + \frac{2\mu_t^2}{\mu^2} \ .$$

We define

$$\tilde{c}(t) = \frac{\mu_{tt}}{\mu} - \frac{2\mu_t^2}{\mu^2} \ ,$$

and the new operator can be written

$$\tilde{\mathcal{L}} = \Box + \tilde{c}(t) \ .$$

The process of decomposing an operator into a product of two (formally) adjoint operators and interchanging the order, producing a new one, is called **Darboux transformation**. See [38].

The repetition of this process produces a full hierarchy of Huygens' operators,

$$\Box_n + u_k(x^0) \ . \tag{1.9}$$

Lagnese and Stellmacher proved in [36] that each iteration of the Darboux transformation increases in two the minimum dimension where the above operator has the Huygens' property. Actually, the operator defined in equation (1.9) has the Huygens' property if, and only if, n is odd and $n \geq 2k + 3$.

[8]There are two non-zero solutions, but as this recursive procedure is carried on, one of the solutions comes for free from the previous equation. We are interested in the other linearly independent solution.

The first members of the hierarchy, modulo trivial transformations, are

$$\begin{aligned} u_0 &= 0 \ , \\ u_1 &= -\frac{2}{t^2} \ , \\ u_2 &= -\frac{6t(2+t^3)}{(1-t^3)^2} \ , \\ u_3 &= -\frac{2[1+18\kappa t - 3\kappa^2 t(50-75t^3-2t^9)]}{[t+\kappa(5+5t^3-t^6)]^2} \ . \end{aligned}$$

We remark that u_3 has a free parameter κ.

It seems that R. Schimming was the first to remark [39, 40] that these solutions are exactly the same ones found by Adler and Moser [41] in the framework of the rational solutions of the KdV hierarchy. The KdV hierarchy and others integrable systems are explained, in a very synthetic way, in Section 4.1.

The potentials u_k can be written as

$$u_k(t) = 2\frac{d^2}{dt^2}\log \mathcal{P}_k(t) \ ,$$

where $\mathcal{P}_k$ is known as the $k-$ th Adler-Moser polynomials[9]. They are defined by the following recursion rule

$$\mathcal{P}'_{k+1}\mathcal{P}_{k-1} - \mathcal{P}'_{k-1}\mathcal{P}_{k+1} = (2k+1)\mathcal{P}_k^2 \ , \quad \mathcal{P}_0(t) = 1 \ , \quad \mathcal{P}_1(t) = t \ .$$

Changing the variable t for x in these polynomials and performing a global change of sign, $-u_k(x)$ becomes rational solutions of the KdV equation

$$\dot{u} = u_{xxx} - 6uu_x \ ,$$

when given the right time dependence. This is an important and unexpected link between integrability theory and Huygens' principle. This is subject of extensive investigation (see, for example, [28] and references therein) and will be the main subject of this thesis. For a generalization of the above result to non-Minkowskian metric, see [23].

1.3.2. Berest's Conjecture

In a very extensive and detailed recent survey [28], Berest presents all known (up to date) Huygens' operators of the form $\Box + u(x)$ (many of them previously found in the cited references) and shows that these examples can be unified in a single way of

[9]The first time these polynomials appear in the literature seem to be in reference [42], due to J. Burchnall and T. Chaudy.

construction. We will shortly review this result and afterwards, following Berest, we will make some additional comments.

The first example given is related to the Adler-Moser polynomials, as was explained in previous section.

The second family of examples is related to Coxeter groups. This family was discovered by Berest and Veselov [43, 44], and this result shall be understood as a generalization from Lagnese and Stellmacher family of examples.

A Coxeter group is a reflection group first introduced to the study of symmetries in crystals. Let us briefly define it, following [28]:

The *Coxeter root system* is defined as a finite set of nonzero vectors $\mathcal{R} \subset \mathbb{R}^n$ which satisfies the following properties:

i - the set $\mathcal{R}$ spans $\mathbb{R}^n$;

ii - the reflection operator[10]

$$s_\alpha(x) = x - 2\frac{\langle \alpha, x \rangle}{\langle \alpha, \alpha \rangle}\alpha \ , \quad x \in \mathbb{R}^n \ ,$$

associated with any vector $\alpha \in \mathcal{R}$ preserves the system $\mathcal{R}$: $s_\alpha(\mathcal{R}) \subset \mathcal{R}$.

We define an arbitrary function $k : \mathcal{R} \to \mathbb{Z}, \alpha \mapsto k_\alpha$ invariant under the action of the group generated by the reflections s_α (called the *Coxeter group*). This function is called *multiplicity function*.

For a generic hyperplane in $\mathbb{R}^n$ we also define the subset of positive roots $\mathcal{R}_+ \subset \mathcal{R}$ as the subset that includes all vectors in $\mathcal{R}$ in the same half-space of the generic hyperplane.

Berest states and proves that the operator

$$\square_n + \sum_{\alpha \in \mathcal{R}_+} \frac{k_\alpha(k_\alpha + 1)\langle \alpha, \alpha \rangle}{\langle \alpha, x \rangle^2} \ ,$$

has the Huygens' property for a large enough n.

The last family of examples characterized by Berest is directed linked to *the problem of wave families.* Let

$$\mathcal{L} \stackrel{\text{def}}{=} \square_n + \sum_{\mu=0}^{n} A^\mu \partial_\mu + C \ ,$$

[10]In the following, the notation $\langle \cdot, \cdot \rangle$ should be interpreted as the inner product in $\mathbb{R}^n$.

in a domain[11] Ω. One defines a *spherical progressive wave* as

$$\phi(x) = \sum_{\nu=0}^{M} h_\nu(x) F^{(\nu)}(\alpha(x)) \ ,$$

where $h_\nu(x)$ is a smooth function and $F^{(\nu)}$ are successive derivatives of some distribution F. Then, if $\mathcal{L}$ is a Huygens operator, there exists a spherical wave family of order $M \geq (n-3)/2$ such that

$$\mathcal{L}\phi = 0 \ .$$

This result was known from [45].

In order to unify all these examples in one theoretical framework, we define the concept of *adjoint* by the iterative procedure

$$\begin{aligned} \mathrm{ad}_{\mathcal{L},\mathcal{L}_0}[\theta(x)] &\stackrel{\mathrm{def}}{=} \mathcal{L} \circ \theta - \theta \circ \mathcal{L}_0 \ , \\ \mathrm{ad}^N_{\mathcal{L},\mathcal{L}_0}[\theta(x)] &= \mathrm{ad}_{\mathcal{L},\mathcal{L}_0}\left[\mathrm{ad}^{N-1}_{\mathcal{L},\mathcal{L}_0}[\theta(x)]\right] \ , \quad N > 1 \ . \end{aligned}$$

After that, one introduces the concept of N-gauge equivalence. The operators $\mathcal{L}$ and $\mathcal{L}_0$ are N-gauge equivalent if there exists a smooth nonzero function $\theta(x) \in C^\infty(\Omega)$ such that

$$\mathrm{ad}^N_{\mathcal{L},\mathcal{L}_0}[\theta(x)] \equiv 0 \ , \quad N \geq 1 \ .$$

Studying all the above mentioned examples and other examples (in these cases the metric is given by a plane wave metric), Berest stated the (still open) conjecture:

Conjecture 1. (Berest's conjecture) *Any Huygens' operator in a Minkowski space is N-gauge equivalent to the ordinary wave operator $\mathcal{L}_0 = \Box_n$.*

In their masterpiece [11], R. Courant and D. Hilbert wrote about the (so-called) Hadamard's conjecture:

> "Examples to the contrary show that this conjecture cannot be completely true in this form, although it is highly plausible that somehow it is essentially correct."

They were essentially guessing that the inclusion of a new kind of "trivial transformation" should be enough to make the statement of Hadamard's conjecture correct. In this case, the inclusion of this new "trivial transformation" should be Berest's ad.

[11] There are some technicalities in the definition of Ω: it should be a *causal domain*, i.e., an open, connected set such that any two points x and y are connected by a unique geodesic (in the metric induced by the principal part of $\mathcal{L}$ — in this case, the Minkowski metric) in Ω and a set formed by the intersection of the forward and backward half conoids induced by the metric with vertices in x and y is compact in Ω unless it is empty.

1.3.3. Bispectrality

The *bispectral problem* was discussed in the work of Duistermaat and Grünbaum [46]. An interesting introduction can be found in [38]. Here we will simply state the problem and sketch why is it important to the study of Huygens' operators.

Consider the following problem: a signal is to be transmitted by a physical channel. Consider, for simplicity, that the channel is accurate only for certain range of frequencies. The signal is also limited in time.

Usually, we say that for the transmission of the signal we would change from the real to the momentum space (by a Fourier transform), transmit the spectrum of frequencies, and invert the Fourier transform. Taking in consideration the above limitations, all these steps are imprecise, i.e., the Fourier transform and its inverse are not integrals over $\mathbb{R}$, but integrals in a certain finite range and the transmitted signal should be multiplied by a cut-off function, representing the band of the spectrum were the transmission is accurate.

Comparing the initial and final signals, a natural question is posed: how good is the transmitted information in comparison with the initial one?

This question leads to the study of the commutativity of certain families of differential and integral operators. This inspired Grünbaum to pose the following problem:

Find all instances of differential operators $L(x, \partial_x)$ such that there exists a family of eigenfunctions $\varphi(x, \lambda)$ satisfying simultaneously the equation

$$L(x, \partial_x)\varphi = \lambda x \ ,$$

and a differential equation in the spectral parameter of the form

$$B(\lambda, \partial_\lambda)\varphi = \theta(x)\varphi \ .$$

Definition: *Let $L(x, \partial_x)$ and $B(k, \partial_k)$ be differential operators of positive order and $\theta(x)$ a smooth function independent of k. The triple (L, B, θ) is called bispectral if there exists a family $\{\varphi(x, k)\}$ such that*

$$L\varphi = \lambda(k)\varphi \ ,$$

for some non-constant smooth function $\lambda(k)$ and

$$B\varphi = \theta(x)\varphi \ .$$

In the language of the previous subsection, Duistermaat and Grünbaum proved that if L has the bispectral property (i.e., there exists a bispectral triple (L, B, θ)), then L is N-gauge related to itself, i.e.,

$$\mathrm{ad}^N_{L,L}[\theta(x)] \equiv 0 \ .$$

If L is a Schrödinger operator, it follwos from [46] that this condition is also sufficient.

It is a remarkable fact that the rational solutions of the KdV hierarchy are bispectral potentials for Schrödinger operators and possesses the Huygens' property.

CHAPTER 2

Huygens' Principle for Dirac Operators: Preliminaries

In this chapter we will develop the tools to attack the problem of Huygens' principle for Dirac operators. In Section 2.1 we define the Clifford Algebras and Dirac matrices which are fundamental to the definition of Dirac operators in Section 2.2. In this section we address the Huygens' principle directed to the case under study. In the sequel (Section 2.3) we classify free Dirac operators of Huygens' type. In Section 2.4 we define a certain set of transformation that trivially preserves the validity of Huygens' principle. This allows one to rule out uninteresting examples of Huygens' operators. Finally, in Section 2.5, we state and prove the results that will be instrumental in the remainder of the thesis.

2.1 Clifford Algebras

Let $\{\gamma^\mu\}$, $\mu = 0, \cdots, n$ be a set of elements in an associative algebra[1] over the field of real numbers such that

$$\{\gamma^\mu, \gamma^\nu\} \stackrel{\text{def}}{=} \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} I \ , \tag{2.1}$$

where $g^{\mu\nu} = \text{diag}[1, -1, -1, \cdots, -1]$ is the Minkowski tensor in $n+1$ dimensions, I is the identity in $n+1$ dimensions and shall be henceforth omitted.

The *Clifford algebra* generated by these elements is the set of all linear combinations of all products in the form

$$(\gamma^0)^{m_0}(\gamma^1)^{m_1} \cdots (\gamma^n)^{m_n} \ , \tag{2.2}$$

[1]An *algebra* over a certain field is a vector space over this field with a bilinear multiplication. If this multiplication is associative, we call it an *associative algebra*.

where $m_\mu = 0$ or $m_\mu = 1$. When all $m_\mu = 0$ the above product should be interpreted as the identity. If n is odd the Clifford Algebra has maximum dimension 2^{n+1}. So, all the products in the form (2.2) are linearly independent. See [47, 48, 49].

If n is odd we can define a matrix $\bar{\gamma}$ such that

$$\{\bar{\gamma}, \gamma^\mu\} = 0 \quad \mu = 0, \cdots, n \tag{2.3}$$

and

$$\bar{\gamma}^2 = 1 \; . \tag{2.4}$$

We remark that such matrix cannot be defined for n even. Relations (2.3) and (2.4) uniquely define, modulo a sign,

$$\bar{\gamma} = (-1)^{(n-1)/4} \gamma^0 \gamma^1 \cdots \gamma^n \; .$$

A particular representation of this algebra can be obtained for the $2^N \times 2^N$ complex matrices, where $N = (n+1)/2$. These $n+1$ matrices are called Dirac matrices. Due to Pauli's fundamental theorem [50, 51] all the possible representations are conjugate.

In some parts of this text a particular representation shall be used. In most cases we will use the Pauli-Dirac representation (extended to $n+1$ dimensions):

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \qquad \bar{\gamma} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \; .$$

where the σ^i's are $N/2 \times N/2$ matrices. In dimension 3+1, the σ^i's are the well celebrated Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and $\bar{\gamma}$ is known as γ^5.

In Section 4.3, so as to compare our work with the literature, we shall use the Weyl representation, where

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix}, \qquad \bar{\gamma} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \; .$$

The conjugation matrix between Pauli-Dirac and Weyl representations is given by

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} .$$

2.2 Huygens' Principle Revisited

Dirac operators appear in Mathematical Physics in the framework of the construction of a relativistic electron theory. In a celebrated paper [52], Dirac defined a Lorentz-covariant Hamiltonian, i.e, a Hamiltoniant invarian by the action of the Lorentz group $O(1,n)$, for the Schrödinger equation (and consequently first-order in space variables, as Schrödinger equation is first order in time) whose square gives the Klein-Gordon operator $\Box + m^2$.

Let $g^{\mu\nu} = \text{diag}[1, -1, \cdots, -1]$ denote the Minkowski tensor. Associated to $g^{\mu\nu}$ we can construct a Clifford Algebra, as in the previous section. We define Dirac operators as

$$\partial\!\!\!/ + v$$

where $\partial\!\!\!/$ is a short-hand notation for

$$\partial\!\!\!/ \stackrel{\text{def}}{=} \sum_{\mu=0}^{n} \gamma^\mu \frac{\partial}{\partial x^\mu}.$$

The attentive reader may ask about the complex unit i that appears on the side of Dirac operators $\partial\!\!\!/$ in many textbooks. As we will shortly see this is immaterial.

It is easy to see that

$$\partial\!\!\!/^2 = \Box \ . \tag{2.5}$$

This equation is the central ideal in Dirac's construction. Its validity is a requirement that leads naturally to Clifford algebras, as explained in the previous section. In this sense, we say that the Dirac operator is the *square-root* of the wave operator. The naïve idea of taking a scalar operator whose square is the wave operator leads to pseudo-differential operators, that have the uncomfortable characteristics of being non-local. On the other hand, if we enlarge the representation (taking matrix operators), as Dirac did, we still have local operators.

Unless explicitly indicated, from now on we shall adopt the following useful conventions:

- Every time a Greek index is repeated, the summation from 0 to n is implied;
- Every time a Latin index is repeated, the summation from 1 to n is implied;
- We abbreviate
$$\partial_\mu \stackrel{\text{def}}{=} \frac{\partial}{\partial x^\mu};$$
- To raise and lower indexes we contract with the metric tensor:
$$a^\mu = g^{\mu\nu} a_\nu \ .$$

A fundamental solution for Dirac operators is the solution of

$$(\partial\!\!\!/ + v)\Psi = \delta_y ,$$

where δ_y denotes the Dirac-delta distribution supported at an arbitrary point y in space-time.

We shall say that a Dirac operator $\partial\!\!\!/+v$ (and by extension, its fundamental solution) obeys Huygens' principle if Ψ satisfies

$$\text{supp } \Psi \subset \mathcal{C}(y) ,$$

for all $y \in \mathbb{R}^n$, where

$$\mathcal{C}(y) = \{x|\lambda = 0\} ,$$

is the light cone with vertex in y and λ is the geodesic distance from x to the point y:

$$\lambda = \sqrt{(x^0 - y^0)^2 - \sum_{i=1}^{n}(x^i - y^i)^2} = \sqrt{(x^\mu - y^\mu)(x_\mu - y_\mu)} ,$$

as in equation (1.4).

The restriction of orthocronicity, as we did in the wave operator case, is too strong in the case of Dirac operators: Dirac operators are not invariant when time is reverted $t \mapsto -t$ (unlike the wave operator), because they are first order in time. We have forward and backward propagations simultaneously. In fact, this was the reason for the discovery of the positron [7, 50].

This fact is not a serious problem, as we could easily define a cut-off function and restrict our study to the forward light cone. However, we will not follow this line here.

2.3 Free Dirac Operators

From equation (2.5) we can see that $\partial\!\!\!/\Lambda^\alpha$ is a fundamental solution of the free Dirac operator $\partial\!\!\!/$ whenever Λ^α is a fundamental solution of wave operator.

This allows the definition of a family of distributions[2]

$$\Theta^\alpha \stackrel{\text{def}}{=} \partial\!\!\!/\Lambda^\alpha , \tag{2.6}$$

called Dirac kernels, defined in the same way the Riesz kernels Λ^α were defined in Section 1.2.

Riesz kernels Λ^α and Dirac kernels Θ^α have many important properties:

i - $\Box\Lambda^\alpha = \Lambda^{\alpha-2}$;

[2]From now on, Riesz kernels are multiplied by the identity matrix I, but we will omit it.

ii - $\Box\Theta^\alpha = \Theta^{\alpha-2}$;

iii - $\partial\!\!\!/\Theta^\alpha = \Lambda^{\alpha-2}$;

iv - $\partial\!\!\!/^{-k}\Theta^{-n+1} = \Theta^{-n+1+k}$ if k is even;

v - $\partial\!\!\!/^{-k}\Theta^{-n+1} = \Lambda^{-n+k}$ if k is odd;

vi - $\partial_\mu\Lambda^\alpha = \frac{\Lambda^{\alpha-2}}{\alpha+n-1}(x_\mu - y_\mu)$, $\alpha \neq -n+1$;

vii - if $f \in C^1(\mathbb{R}^{n+1})$ and $\alpha \neq -n+1$ then

$$\gamma^\mu\Theta^\alpha\partial_\mu f = \frac{2}{\alpha+n-1}\Lambda^{\alpha-2}(x^\mu - y^\mu)\partial_\mu f - \Theta^\alpha \partial\!\!\!/ f \ .$$

The three first properties are elementary consequences of the definitions of Dirac and Riesz kernels and equation (2.5). Properties (iv) and (v) are consequences of the three first. With the help of equation (1.7) we prove (vi).

Let's define

$$\Gamma \stackrel{\text{def}}{=} \gamma^\mu(x_\mu - y_\mu) \ .$$

We have that

$$\Gamma^2 = \lambda^2 \ , \tag{2.7}$$

so Γ is a square-root of λ^2 (in the same sense $\partial\!\!\!/$ is the square-root of $\Box$).

From property (vi) we have that, for $\alpha \neq -n+1$

$$\Theta^\alpha = \partial\!\!\!/\Lambda^\alpha = \frac{\Lambda^{\alpha-2}}{\alpha+n-1}\Gamma \ . \tag{2.8}$$

With this formula in mind and considering the definition of Clifford Algebras (2.1), we prove property (vii).

We are now ready to determine which free Dirac operators satisfy Huygens' property.

Theorem 1. *The Lorentz-invariant fundamental solution of the free Dirac equation in dimension n, Θ^{-n+1}, obeys Huygens' principle if, and only if, n is odd.*

Proof. With the help of equation (2.8) we can easily prove that (for $\alpha \neq -n+1$)

$$\partial\!\!\!/[\Lambda^\alpha(\log\lambda)\Gamma] = (\alpha+n+1)\Lambda^\alpha\log\lambda + \Lambda^\alpha \ .$$

Since Λ^α is an analytic family of distributions we can take the limit $\alpha \to -n-1$ to get

$$\partial\!\!\!/[\Lambda^{-n-1}(\log\lambda)\Gamma] = \Lambda^{-n-1} \ .$$

Because Λ^{-n-1} is the Dirac-delta distribution (in a space-time of dimension $n+1$), we conclude that

$$\Theta^{-n+1} = \Lambda^{-n-1}(\log \lambda)\Gamma \ ,$$

is the fundamental solution of $\partial\!\!\!/$. Since $\Lambda^{-n-1} = N(-n-1)\lambda^{-n-1}$ and $N(-n-1) = 0$, Θ^{-n+1} is null wherever $\lambda^{-n-1} \log \lambda\Gamma$ is finite, i.e., outside the (surface of the) light cone. □

An important point to be stressed is that in one dimension, the free Dirac operator obeys Huygens' principle, unlike the wave operator. This point will be explored in Section 4.2. In higher dimensions they have the same behavior.

2.4 Trivial Transformations

In this section, we define some transformations that send a Dirac operator into another Dirac operator. Those transformations trivially preserve the validity of Huygens' principle, so potentials generated by those are uninteresting. We define trivial transformations strongly inspired by Hadamard's original definition for wave operators.

i - Change of coordinates:

$\tilde{x}^\mu = f^\mu(x^0, \cdots, x^n),\ \mu = 0, \cdots, n$, with $\det\big(\partial_\mu f^\nu\big)_{\mu,\nu=0,\cdots,n} \neq 0$.

In this case, the Dirac operator is transformed according to

$$\partial\!\!\!/ \mapsto (\partial\!\!\!/ f^\mu)\partial_\mu \ .$$

ii - Left multiplication:

Now we take $\mathcal{D} \mapsto \bar{\mathcal{D}} = \lambda(x)\mathcal{D}$ and $\Psi \mapsto \bar{\Psi} = \Psi\lambda(y)^{-1}$, where $\lambda(x)$ is an everywhere non-singular matrix, $\lambda(x) \in C^1(\mathbb{R}^{N\times N})$.

In order to see that $\bar{\Psi}$ is the fundamental solution of $\bar{\mathcal{D}}$ we write:

$$\bar{\mathcal{D}}\bar{\Psi} = \lambda(x)\mathcal{D}\Psi\lambda(y)^{-1} = \lambda(x)\delta_y\lambda(y)^{-1} = \delta_y \ .$$

iii - Factor transformations:

Let ρ be a non-singular matrix of the form

$$\rho = \rho_\phi I + \rho_\mu\gamma^\mu + \bar{\rho}_\mu\gamma^\mu\bar{\gamma} + \bar{\rho}\bar{\gamma} \ ,$$

where ρ_ϕ, ρ_μ, $\bar{\rho}_\mu$ and $\bar{\rho}$ are differentiable functions. The factor transformation consists in transforming $\mathcal{D} \mapsto \bar{\mathcal{D}} = \rho(x)\mathcal{D}\rho(x)^{-1}$ and $\Psi \mapsto \bar{\Psi} = \rho(x)\Psi\rho(y)^{-1}$.

The second transformation fully justifies the omission of the complex unit i following Dirac matrices.

Some authors use the expression "gauge transformations" instead of "factor transformations". To avoid confusion with the use of the term "gauge" in electromagnetism, we shall adopt the Hadamard's original term.

It is important to stress that the transformations defined above *include the change of the electromagnetic gauge*, $A_\mu \to A_\mu + \partial_\mu \theta$, where θ defines the new (electromagnetic) gauge.

$$\gamma^\mu(\partial_\mu + iA_\mu + i\partial_\mu\theta) = e^{-i\theta}\gamma^\mu(\partial_\mu + iA_\mu)e^{i\theta} \ .$$

In this case the fundamental solution is transformed

$$\bar{\Psi} = e^{-i\theta(x)}\Psi e^{i\theta(y)} \ .$$

As we should expect, the change of electromagnetic gauge does not interfere in the validity of the Huygens' principle. We will address this question again in Chapter 3.

2.5 Hadamard Expansions

Adapting Hadamard's seminal idea for the wave operator, we look for a series expansion for the fundamental solution Ψ. Here, Ψ denotes the solution of

$$(\partial\!\!\!/ + v)\Psi = \delta_y \ . \tag{2.9}$$

We look for series expansion of Ψ:

$$\Psi = \sum_{k=0}^{\infty} \partial\!\!\!/^{-k}\Theta^{\alpha_0}u_k = \sum_{k=0}^{\infty}\left\{\Theta^{\alpha_0+2k}u_{2k} + \Lambda^{\alpha_0+2k}u_{2k+1}\right\} \ , \tag{2.10}$$

where $u_k = u_k(x,y)$ is a matrix coefficient and α_0 should be taken equal to $-n+1$.

From now on we will restrict ourselves to the case

$$v = aI + a_\mu\gamma^\mu + \bar{a}\bar{\gamma} \ .$$

Just to fix the notation, a is called the *scalar potential*, a_μ are the *electromagnetic potentials*, and $\bar{a}$ is the *pseudo-scalar potential.* This is a general potential in the 1+1 case. In this thesis only potentials of this form are to be treated.

As v and $\Theta^{\alpha+2k}$ are matrices, we need to study how to commute these two objects. From equation (2.8), we conclude that

$$v\Theta^\alpha = \frac{2}{\alpha+n-1}\Lambda^{\alpha-2}(x^\mu - y^\mu)a_\mu + \Theta^\alpha v^* \ , \tag{2.11}$$

where

$$v^* \stackrel{\text{def}}{=} aI - a_\mu \gamma^\mu - \bar{a}\bar{\gamma} \ . \tag{2.12}$$

Applying $\not\partial + v$ to Ψ we find

$$(\not\partial + v)\Psi = \Lambda^{\alpha_0 - 2}\Big[u_0 + \frac{2}{\alpha_0 + n - 1}(x^\mu - y^\mu)(\partial_\mu + a_\mu)u_0\Big] +$$

$$+ \sum_{k=0}^{\infty} \Theta^{\alpha_0 + 2k}\Big[-\not\partial u_{2k} + u_{2k+1} + v^* u_{2k}\Big] +$$

$$+ \sum_{k=1}^{\infty} \Lambda^{\alpha_0 + 2k - 2}\Big[u_{2k} + \frac{2}{\alpha_0 + 2k + n - 1}(x^\mu - y^\mu)(\partial_\mu + a_\mu)u_{2k} + \not\partial u_{2k-1} + v u_{2k-1}\Big] \ .$$

Equating $(\not\partial + v)\Psi = \Lambda^{\alpha_0 - 2}$ we find **Hadamard's recursion**:

$$\begin{array}{rcll} u_0 + \frac{2}{\alpha_0+n-1}(x^\mu - y^\mu)(\partial_\mu + a_\mu)u_0 & = & 1 & \alpha_0 \to -n+1; \\ u_{2k+1} & = & (\not\partial - v^*)u_{2k} & k = 0, 1, \cdots ; \\ u_{2k} + \frac{1}{k}(x^\mu - y^\mu)(\partial_\mu + a_\mu)u_{2k} & = & -(\not\partial + v)u_{2k-1} & k = 1, 2, \cdots . \end{array} \tag{2.13}$$

Theorem 2. *If $v = aI + \bar{a}\bar{\gamma}$, Ψ is given by equation (2.10), and satisfies $(\not\partial + v)\Psi = \delta_y$, then the coefficients in the expansion (2.13) are uniquely determined by the normalization conditions $u_0(y, y) = 1$ and by the imposing regularity of u_k in the vicinity of the vertex of the light cone, i.e., when $x \to y$.*

Remark: This theorem remains true if we include electromagnetic terms in the potential, but this proof has to wait until Subsection 3.3.1.

Proof. If $a_\mu = 0$ we easily conclude that $u_0 = 1$. The uniqueness for odd terms is obvious. For even terms, suppose that u^1 and u^2 are solutions of the third equation in the system (2.13). So, their difference $\bar{u} = u^1 - u^2$ is solution of

$$\bar{u} + \frac{1}{k}(x^\mu - y^\mu)\partial_\mu \bar{u} = 0 \ .$$

Consider a ray emanating from the vertex of the light cone y and remember that λ measures the geodesic distance between x and y. Studying the behavior of $\bar{u}$ along this ray we find that $\bar{u}$ obeys the equation

$$\bar{u}(\lambda) + \frac{1}{k}\lambda \bar{u}'(\lambda) = 0$$

The only non-trivial solution of this equation is given by $\bar{u} \propto \lambda^{-k}$, that is not well-behaved near $\lambda = 0$. So $\bar{u} = 0$ and the theorem is proved. □

Theorem 3. *A Dirac operator in the form $\not{\partial} + v$ possesses Huygens' property in odd spatial dimension n if, and only if, its Hadamard series truncates at n, i.e, $u_k = 0$ for $k \geq n$.*

Proof. If the series truncates, we simply need to take n high enough in order to guarantee that all Riesz and Dirac kernels in the expansion have Huygens' property. In the other direction, we take the Hadamard's series of the operator $\not{\partial} + v$, and from its uniqueness, all the kernels should be of Huygens' type. □

We state two more results before characterizing Dirac operators.

Theorem 4. *Trivial transformations do not change the validity of Huygens' principle.*

Proof. In Section 2.4 we studied how the fundamental solution of the Dirac operator changes whenever we trivially transform the operator. In the first case we simply change the coordinates of the solution, so the order of the series is not changed. In the second case, we multiply the Hadamard coefficients on the right by the matrix $\lambda(y)^{-1}$. For the third transformation we see that

$$\rho\Theta^\alpha = \Theta^\alpha \rho^* + \frac{2}{\alpha + n - 1}\Lambda^{\alpha-2}(\rho^\mu I + \bar{\rho}^\mu \bar{\gamma})(x_\mu - y_\mu) \ ,$$

where $\rho^* \stackrel{\text{def}}{=} \rho I_\phi - \rho_\mu \gamma^\mu + \bar{\rho}_\mu \gamma^\mu \bar{\gamma} - \bar{\rho}\bar{\gamma}$. Thus, if Ψ is a terminating series, so is $\bar{\Psi}$. □

Remark: It is very tempting to generalize the third trivial transformation to arbitrary matrices. In order to do that, it is necessary to understand how to exchange the order of ρ and Θ. As far as we see, there is no canonical way to do that.

Theorem 5. *Change of representation of Dirac matrices do not affect Huygens' property.*

Proof. Due to Pauli's Fundamental Theorem [50, 51] all the representations are equivalent. □

Remark: Given $v = a(x^0)I + \bar{a}(x^0)\bar{\gamma}$, the operators $\not{\partial} + v$, $\not{\partial} - v$, $\not{\partial} - v^*$ and $\not{\partial} + v^*$ are trivially equivalent.

Proof. The identity

$$-\bar{\gamma}(\not{\partial} + v)\bar{\gamma} = \not{\partial} - v \ ,$$

proves the first equivalence. For the second one, we define

$$M = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ ,$$

such that $M = -i\gamma^0\bar{\gamma}$ in the Pauli-Dirac representation and thus $M^{-1} = M$. A simple calculation proves that

$$M\gamma^0 M^{-1} = -\gamma^0 \ , \quad M\gamma^i M^{-1} = \gamma^i \ \text{ and } \ M\bar{\gamma}M^{-1} = -\bar{\gamma} \ .$$

and $M(\not{\partial}+v)M^{-1} = -\gamma^0\partial_0 + \gamma^i\partial_i + v^*$. Proceeding with a spatial inversion, $x^i \to -x^i$, the last operator changes[3] to $-\not{\partial} + v^*$.

Other equivalences are elementary consequences of these two. □

As a final example let's study the validity of Huygens' principle for the *massive* Dirac operator, i.e., $\not{\partial} + m$, when $m \neq 0$ is a constant.

Example: Consider $v = m$. The solution of the recursion (2.13) is given by $u_k = (-m)^k$, so, by Theorem 3, $\not{\partial} + m$ never obeys Huygens' principle, if $m \neq 0$.

[3] The transformations we just performed defines a *space-time reversion.* Since a is a scalar, it remains invariant under this kind of transformation, but $\bar{a}$, a pseudo-scalar, changes sign.

CHAPTER 3

Huygens' Principle in 1 and 3 Spatial Dimensions

In this chapter we will find the first non-trivial examples of Huygens' type Dirac operators. The main goal is to seek as many as possible examples of Huygens' operators in one and three dimensions. For the one dimensional case we show in the Section 3.1 that the only Huygens' potentials are the trivial ones. The first non-trivial example will be found in Section 3.2. This result will be extended in Section 3.3, where we also make some physical considerations.

3.1 Huygens' Principle in 1 Dimension

Using the tools developed in the preceding chapter, we start classifying Huygens potentials. The simplest case is, of course, $n = 1$.

Theorem 6. *If $n = 1$, modulo trivial transformations, the only Huygens' operator is the free Dirac operator $\not\partial$.*

Proof. For $n = 1$ we represent Dirac matrices as

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

It is straightforward to verify that

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} ,$$

where $g^{\mu\nu} = \mathrm{diag}[-1, 1]$. We also have

$$\bar{\gamma} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

So every 2×2 matrix v can be written

$$v = aI + a_0\gamma^0 + a_1\gamma^1 + \bar{a}\bar{\gamma} \ .$$

If we multiply the equation for u_0 in (2.13) by $\alpha_0 + n - 1$ and take the limit $\alpha_0 \to -n+1$ we get

$$(x^\mu - y^\mu)\partial_\mu u_0 = -(x^\mu - y^\mu)a_\mu$$

whose solution, imposing $u_0(y, y) = 1$ is given by

$$u_0(x, y) = \exp\Big(-\int_0^1 (x^\mu - y^\mu)a_\mu\big(y + s(x-y)\big)ds\Big) \ . \tag{3.1}$$

From now on we abbreviate

$$\xi(s) \stackrel{\text{def}}{=} y + s(x-y) \ . \tag{3.2}$$

The next term in the recursion, which must vanish, is

$$u_1(x, y) = \Big[-aI + \gamma^\mu\big(a_\mu - \partial_\mu \int_0^1 (x^\nu - y^\nu)a_\nu(\xi(s))ds\big) + \bar{a}\bar{\gamma}\Big]u_0(x, y) \ .$$

Since u_0 is a non-zero scalar, we can equate the term inside the brackets to 0 and find

$$\begin{aligned} a_\mu &= \partial_\mu \int (x^\nu - y^\nu)a_\nu(\xi(s))ds \ ; \\ a &= 0 \ ; \\ \bar{a} &= 0 \ . \end{aligned}$$

Defining

$$\theta(x, y) \stackrel{\text{def}}{=} \int_0^1 (x^\mu - y^\mu)a_\mu(\xi(s))ds \ , \tag{3.3}$$

it is easy to see that

$$e^{-\theta}\partial\!\!\!/ e^\theta = \partial\!\!\!/ + a_\mu\gamma^\mu \ ,$$

so $\partial\!\!\!/ + v$ is trivially equivalent to the free Dirac operator. □

3.2 Huygens' Principle in 3 Dimensions: The Scalar Case

The main goal of this section is to prove the following:

Theorem 7. *The only scalar potential v for which $\partial\!\!\!/ + v$ is a Huygens' operator, apart from the null potential and modulo trivial transformations, in 3+1 dimensions is $v(x^0, \cdots, x^3) = 1/x^0$.*

Proof. The fundamental solution looks like

$$\Psi = \Theta^{-2} + \Lambda^{-2} u_1 + \Theta^0 u_2 \ .$$

So, the Hadamard recursion (considering $v^* = v$) is:

$$\begin{aligned} u_0 &= 1 \ ; \\ u_1 &= (\partial\!\!\!/ - v) u_0 = -v \ ; \\ u_2 + (x^\mu - y^\mu)\partial_\mu u_2 &= -(\partial\!\!\!/ + v) u_1 = \partial\!\!\!/ v + v^2 \ ; & (3.4) \\ 0 = u_3 &= (\partial\!\!\!/ - v) u_2 \Longrightarrow \partial\!\!\!/ u_2 = v u_2 \ . & (3.5) \end{aligned}$$

The theorem is then reduced to finding v such that, together with some u_2, obeys the last two relations. We will study the compatibility relations between (3.4) and (3.5).

We apply the Dirac operator to equation (3.4) and get

$$2\partial\!\!\!/ u_2 + (x^\mu - y^\mu)\partial_\mu \partial\!\!\!/ u_2 = \partial\!\!\!/ [\partial\!\!\!/ v + v^2] \ .$$

With the help of equation (3.5) this can be re-written

$$[v + (x^\mu - y^\mu)\partial_\mu v] u_2 = \Box v + (\partial\!\!\!/ v) v - v^3 \ . \tag{3.6}$$

In the last equation u_2 is a function of x and y, while $v = v(x)$ (the dependence of v on x will be omitted, while u_2 will be explicitly indicated whenever it is being evaluated at (x, x)). Evaluation of equation (3.6) at $y = x$ gives:

$$v u_2(x, x) = \Box v + (\partial\!\!\!/ v) v - v^3 \ . \tag{3.7}$$

This is a necessary condition for v to be a Huygens' potential.

Another necessary condition can be obtained differentiating equation (3.6) with respect to y^ν. We denote

$$\tilde{\partial}_\nu \stackrel{\text{def}}{=} \frac{\partial}{\partial y^\nu} \ . \tag{3.8}$$

Applying $\tilde{\partial}_\nu$ and evaluating equation (3.6) at (x, x) we find

$$v \tilde{\partial}_\nu u_2(x, x) = (\partial_\nu v) u_2(x, x) \qquad \nu = 0, \cdots, n \ . \tag{3.9}$$

From equation (3.4) we get

$$u_2(x, x) = \partial\!\!\!/ v + v^2 \ .$$

Differentiating (3.6) with respect to ∂_ν

$$2\partial_\nu u_2 + (x^\mu - y^\mu)\partial_\mu \partial_\nu u_2 = \partial_\nu (\partial\!\!\!/ v + v^2) \qquad \nu = 0, \cdots, n \ .$$

Equation (3.4) differentiated with respect to $\tilde{\partial}_\nu$ and evaluated at (x, x) reads

$$\tilde{\partial}_\nu u_2(x,x) = \partial_\nu u_2(x,x) = \frac{1}{2}\partial_\nu(\not{\partial} v + v^2) \quad \nu = 0, \cdots, n \ .$$

Substituting this result in equation (3.9) yields

$$\frac{1}{2} v \partial_\nu(\not{\partial} v + v^2) = (\partial_\nu v)(\not{\partial} v + v^2) \ ,$$

which can be re-written

$$\gamma^\mu[(\partial_\mu \partial_\nu v)v - 2(\partial_\mu v)(\partial_\nu v)] = 0 \quad \nu = 0, \cdots, n \ .$$

As the Dirac matrices are linearly independent, this equation is equivalent to

$$\partial_\mu \partial_\nu \left(\frac{1}{v} \right) = 0 \quad \mu, \nu = 0, \cdots, n \ ,$$

whose solution is

$$v = (k_\mu x^\mu + \tilde{k})^{-1} \ . \tag{3.10}$$

The constant $\tilde{k}$ can be incorporated in any variable, by means of a trivial transformations that do not change the principal part of Dirac's operator.

Restrictions in the vector k_μ can be obtained substituting this solution into equation (3.7). Using that $\partial_\mu v = -k_\mu v^2$, we get

$$k^\mu k_\mu = 1 \ .$$

We now proceed with a Lorentz-covariant change of coordinates (which does not change the principal part of a Dirac operator), in such a way that the time axis points in the direction of k^μ, so the above operator is trivially equivalent to $\not{\partial} + 1/x^0$.

As these calculations give *necessary* conditions to get a Huygens potential, we shall explicitly verify that the recursion terminates after three steps.

We easily find that

$$\begin{aligned} u_0 &= 1 \ , \\ u_1 &= -\frac{1}{x^0} \ , \\ u_2 &= \frac{I - \gamma^0}{x^0 y^0} \ , \\ u_3 &= 0 \ . \end{aligned}$$

This concludes the proof. □

Remark: The attentive reader will notice that the proof above is slightly different of the proof in [53]. Both proofs are correct, but the one presented here seems to be simpler.

3.3 Huygens' Principle in an Electromagnetic Field

In this section we shall consider potentials of the form $v = a_\mu \gamma^\mu$. In the last subsection the result of the preceding section will be considered and a theorem about potentials of the form $v = aI + a_\mu \gamma^\mu$ will be proved.

Potentials of the form $a_\mu \gamma^\mu$ are called electromagnetic potentials because they correspond to including an electromagnetic field in the physical situation under consideration, except for a multiplication by the complex unit i. If the physical potential is given by the "four-vector" A_μ the new momentum is given by $i\partial_\mu - A_\mu$, so the "physical case," in the expression of v, is given by purely imaginary components.

If we write the potential as a 1-form, which is standard [54], $\mathbf{a} = a_\mu dx^\mu$, the electromagnetic field is given by its exterior derivative

$$\mathbf{F} = d\mathbf{a} \ .$$

The 2-form $\mathbf{F}$ can be viewed as a tensor,

$$F_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu \ . \tag{3.11}$$

The "canonical way" to write its components, modulo personal preferences of different authors (which changes signs, or columns), is [55, 56]:

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -Ez \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix} \ .$$

Electromagnetic fields obey Maxwell's equation, which in vacuum are written

$$\partial^\mu F_{\nu\mu} = j_\nu \qquad \partial_{[\mu} F_{\nu\rho]} = 0 \ . \tag{3.12}$$

In these equations j_ν is the density of electrical charge and current, and the notation $[\cdot]$ means that we sum over all the permutation of the indexes in brackets, taking in consideration the parity of the permutation[1]. The second equation is automatically obeyed, as long as $F_{\mu\nu}$ is obtained from a potential. Inside matter the second one remains valid (which implies, from Poincaré's lemma that, locally, the electromagnetic field can be obtained from a potential).

In Classical Physics the field at each point is observable, while the potentials are considered just a technique for calculating such field, having no separate reality[2]. This means that adding an exact form $d\theta$ to the potential should not change the field (which,

[1] For example: $B_{[\mu\nu]} = B_{\mu\nu} - B_{\nu\mu}$, $B_{[\mu}C_{\nu]} = B_\mu C_\nu - B_\nu C_\mu$ and $B_{[\mu}C_{\nu\rho]} = B_\mu C_{\nu\rho} + B_\nu C_{\rho\mu} + B_\rho C_{\mu\nu} - B_\mu C_{\rho\nu} - B_\nu C_{\mu\rho} - B_\rho C_{\nu\mu}$.

[2] This changes dramatically in Quantum Mechanics, as can be seen by the Aharonov-Bohm effect. See, for example [50].

by now, should be obvious from the fact that $d^2{=}0$). In terms of its components, this means to change from a_μ to $a'_\mu = a_\mu + \partial_\mu \theta$. As the Huygens' property is a physically sound problem, the validity or not of the Huygens' principle cannot depend on the particular choice of the electromagnetic gauge[3,4].

The change of the gauge is a particular kind of trivial transformation, as defined in Section 2.4. We see immediately

$$\partial\!\!\!/ + (a_\mu + \partial_\mu \theta)\gamma^\mu = e^{-\theta}(\partial\!\!\!/ + a_\mu \gamma^\mu)e^{\theta} \ ,$$

so by Theorem 4 the change of the gauge does not change the validity of the Huygens' principle.

Thus, the invariance of Huygens' principle under gauge transformations can be concluded by two distinct reasonings, coming from two different points of view, historically unrelated. We point out that, as far as we know, the fact that the change of electromagnetic gauge is a particular case of trivial transformations as defined by Hadamard was never mentioned in the literature.

After these remarks we are ready to state:

Theorem 8. *If a Dirac operator $\partial\!\!\!/ + a_\mu \gamma^\mu$ obeys Huygens' property for real or purely imaginary potentials a_μ in 3+1 dimensions, then it is equivalent to the free operator $\partial\!\!\!/$.*

The proof of this theorem will be the subject of the next two subsections. Subsection 3.3.3 is concerned with conjoining the results of the two previous ones.

3.3.1. The Choice of the Gauge

In Section 3.1, when we started solving Hadamard's recursion, u_0 was obtained in equation (3.1). In the present case, where $a = \bar{a} = 0$ and the dimension is higher, the solution is the same, as can be easily verified. So, we should proceed with the calculation. However, the expressions for u_1, u_2 and u_3 (that should be equal 0) become more and more complicated, and the calculations turn to be a cumbersome task.

[3] The reader may ask why are we considering a classical field if Dirac equation is a quantum equation. The answer is quite simple: we are in the first quantization, where the wave function is already quantized, but not the field. This procedure is a common one (see, for example, the comment in the end of the first Chapter of [50]).

[4] To understand the idea that Huygens' principle is physically observable we should understand the meaning of Ψ, the solution of Dirac equation. Following Section 1.7.3 of [57], we say that for every Borel-measurable set $\mathcal{B} \in \mathbb{R}^3$ the probability of finding the particle in $\mathcal{B}$ is given by $\langle \Psi, E(\mathcal{B})\Psi \rangle$, where $E(\mathcal{B})$ is a projection operator in $\mathcal{B}$ and the inner product should be interpreted as Hilbert space inner product where the Dirac operator is acting and the solution Ψ lives. In simpler words, we have a non-zero probability of finding the particle only on the support of the wave-function.

Everything can be made easier if a good gauge is selected. Inspired by the 1+1 case, let's define

$$f_\mu(x,y) \stackrel{\text{def}}{=} a_\mu(x) - \partial_\mu \int_0^1 (x^\nu - y^\nu) a_\nu(\xi(s)) ds \ ,$$

where $\xi(s) = y + s(x-y)$ was defined in equation (3.2). The first remark is that

$$\partial_\mu f_\nu - \partial_\nu f_\mu = \partial_\mu a_\nu - \partial_\nu a_\mu \ ,$$

which shows that **f** and **a** are gauge equivalent. Explicitly, the change of gauge is given by

$$\not\partial + f_\mu \gamma^\mu = e^\theta (\not\partial + a_\mu \gamma^\mu) e^{-\theta} \ ,$$

where $\theta = \theta(x,y)$ is given by equation (3.3).

There is one important difference between the trivial transformation we just made and the ones defined in Section 2.4: now, θ depends on x **and** y. So, we need to prove:

Lemma 1. *The Dirac operator $\not\partial + a_\mu(x)\gamma^\mu$ is Huygens' if, and only if, $\not\partial + f_\mu(x,y)\gamma^\mu$ is Huygens'* in the sense of its fundamental solution *for each $y \in \mathbb{R}^4$*.

Proof. From the two different (and equivalent) meanings of Huygens' principle (Cauchy problem and fundamental solution), only the second one makes sense for $\not\partial + f_\mu\gamma^\mu$, since f_μ depends on y. This is not a big problem, however. Let y be fixed. By the same argument of Section 2.4 we prove that $\not\partial + a_\mu\gamma^\mu$ is Huygens (in the sense of its fundamental solution) whenever $\not\partial + f_\mu\gamma^\mu$ is (in the same sense). This procedure is carried on *for each y*, and after scanning all $y \in \mathbb{R}^4$ we finish the lemma. □

From now on, the calculations will be performed in the above defined new gauge.

3.3.2. Solving the Recursion

From the fundamental solution transformation rules, if we apply to the operator a trivial transformation, it is immediate that $u_0 = 1$. However, it will be instructive to explicitly get u_0 in this new gauge.

The functions f_μ can be re-written as

$$f_\mu(x,y) = \int_0^1 (x^\nu - y^\nu) F_{\nu\mu}(\xi(s)) s \, ds \ , \tag{3.13}$$

and, so, by the anti-symmetry of the tensor $F_{\mu\nu}$ we have

$$(x^\mu - y^\mu) f_\mu(x,y) = 0 \qquad \forall x, y \in \mathbb{R}^4 \ . \tag{3.14}$$

If we look at equation (2.11), it is clear that this new gauge is exactly the one that cancels the term in the Riesz kernel.

Since Hadamard's recursion stops at the second term, equation (2.13) becomes

$$\begin{aligned}
u_0 + \frac{2}{\alpha_0 + n - 1}(x^\mu - y^\mu)\partial_\mu u_0 &= 1 \ , \\
u_1 = (\partial\!\!\!/ + f_\mu \gamma^\mu)1 &= \gamma^\mu f_\mu \stackrel{\text{def}}{=} f \ . \\
u_2 + (x^\mu - y^\mu)\partial_\mu u_2 &= -(\partial\!\!\!/ + f)f \ , \\
(\partial\!\!\!/ + f)u_2 &= 0 \ .
\end{aligned} \tag{3.15}$$

The first equation can be easily solved to get $u_0 = 1$.

In this gauge and with help of Lemma 1, we easily extend Theorems 2 and 3 to the electromagnetic case.

We shall study compatibility conditions between the last two equations. We start applying Dirac operator to the first:

$$2\partial\!\!\!/ u_2 + (x^\mu - y^\mu)\partial_\mu \partial\!\!\!/ u_2 = -\partial\!\!\!/(\partial\!\!\!/ + f)f \ .$$

With the second condition this can be written

$$2fu_2 + (x^\mu - y^\mu)(\partial_\mu f)u_2 + f(x^\mu - y^\mu)\partial_\mu u_2 = \partial\!\!\!/(\partial\!\!\!/ + f)f \ .$$

Applying ∂_ν to equation (3.14) we find that $(x^\mu - y^\mu)\partial_\nu f_\mu = -f_\nu$ which allows one to write the equation

$$(x^\mu - y^\mu)\partial_\mu f = \gamma^\mu (x^\nu - y^\nu)F_{\nu\mu} - f \ .$$

Finally we write

$$\gamma^\mu (x^\nu - y^\nu)F_{\nu\mu} u_2 = (\partial\!\!\!/ + f)(\partial\!\!\!/ + f)f \ . \tag{3.16}$$

We proceed as we did in Section 3.2. The RHS of last equation is

$$(\partial\!\!\!/ + f)(\partial\!\!\!/ + f)f = \Box f + \partial\!\!\!/ f^2 + f\partial\!\!\!/ f + f^3 \ .$$

For $x = y$, it is immediate to see that $f(x,y) = 0$. So, the only non-zero term in $x = y$ in the RHS is $\Box f$. Actually,

$$\Box f_\mu = \partial^\rho \partial_\rho \int_0^1 (x^\nu - y^\nu)F_{\nu\mu}(\xi) s ds = 2\int_0^1 \partial^\rho F_{\rho\mu}(\xi) s^2 ds + \int_0^1 \Box F_{\nu\mu}(\xi) s^3 ds \ .$$

Letting $y \to x$

$$\Box f(x,x) = \frac{2}{3}\gamma^\mu \partial^\nu F_{\nu\mu}(x) \ .$$

As the LHS of equation (3.16) is zero at (x,x), we conclude that

$$\partial^\nu F_{\mu\nu} = 0 \ , \qquad \mu = 0, \cdots, 3 \ , \tag{3.17}$$

is a necessary condition for the operator $\partial\!\!\!/ + a_\mu \gamma^\mu$ to be of Huygens type. From Maxwell's equations (3.12), this is interpreted as the absence of sources.

To get a second necessary condition, we apply $\tilde{\partial}_\eta$ (as defined in equation (3.8)) to the LHS of equation (3.16) and evaluate it at (x, x). We get

$$\begin{aligned} \tilde{\partial}_\eta \Box f(x,x) &= \frac{1}{6}\gamma^\mu \partial_\eta \partial^\rho F_{\rho\mu} - \frac{1}{4}\gamma^\mu \Box F_{\eta\mu} \ ; \\ \tilde{\partial}_\eta \partial\!\!\!/ f(x,x)^2 &= 2\gamma^\mu (\partial_\mu f_\nu)(x,x)(\tilde{\partial}_\eta f^\nu)(x,x) \ ; \\ \tilde{\partial}_\eta (f \partial\!\!\!/ f)(x,x) &= (\tilde{\partial}_\eta f)(x,x)(\partial\!\!\!/ f)(x,x) \ ; \\ \tilde{\partial}_\eta f(x,x)^3 &= 0 \ . \end{aligned}$$

Taking into account equations (3.13) and (3.17) the above system can be re-written as

$$\tilde{\partial}_\eta (\partial\!\!\!/ + f)^2 f(x,x) = \frac{1}{2}\gamma^\mu (F^2)_{\mu\nu} - \frac{1}{4}\gamma^\mu \gamma^\nu \gamma^\rho F_{\eta\mu} F_{\nu\rho} \ ,$$

where $(F^2)_{\mu\nu}$ indicates the (μ, ν) component of the tensor given by the square of the matrix F, $(F^2)_{\mu\nu} = F_{\mu\rho} F^\rho{}_\nu$.

The RHS of equation (3.16) gives

$$\tilde{\partial}_\eta [\gamma^\mu (x^\nu - y^\nu) F_{\nu\mu} u_2](x,x) = -\gamma^\mu F_{\eta\mu} u_2(x,x) \ ,$$

and $u_2(x,x)$ can be easily obtained from equation (3.15):

$$u_2(x,x) = -(\partial\!\!\!/ + f) f(x,x) = \frac{1}{2}\gamma^\mu \gamma^\nu F_{\mu\nu} \ .$$

Gathering all this information in a unique equation

$$\frac{1}{2}\gamma^\mu (F^2)_{\mu\eta} - \frac{3}{4}\gamma^\mu \gamma^\nu \gamma^\rho F_{\eta\mu} F_{\nu\rho} = 0 \ .$$

In the Clifford base this becomes

$$\gamma^\mu (F^2)_{\mu\eta} + \frac{3}{4} \sum_{\mu<\nu<\rho} \gamma^\mu \gamma^\nu \gamma^\rho F_{\eta[\mu} F_{\nu\rho]} = 0 \ ,$$

where, as in Section 3.3, $[\cdot]$ means that we sum over all the permutation of μ, ν and ρ taking in consideration the parity of the permutation.

From the independence of the generators of the Clifford algebra, we conclude that

$$(F^2)_{\mu\nu} = 0 \qquad \text{and} \qquad F_{\eta[\mu} F_{\nu\rho]} = 0 \qquad \forall \mu, \nu, \rho, \eta = 0, \cdots, 3 \ .$$

Let's write (F^2) componentwise:

$$(F^2)_{\mu\nu} = F_{\mu\rho} F^\rho{}_\nu =$$

$$= \begin{pmatrix} E_x^2 + E_y^2 + E_z^2 & E_zB_y - E_yB_z & E_xB_z - E_zB_x & E_yB_x - E_xB_y \\ E_zB_y - E_yB_z & B_y^2 + B_z^2 - E_x^2 & -E_xE_y - B_xB_y & -E_xE_z - B_xB_z \\ E_xB_z - E_zB_x & -E_xE_y - B_xB_y & B_x^2 + B_z^2 - E_y^2 & -E_yE_z - B_yB_z \\ E_yB_x - E_xB_y & -E_xE_z - B_xB_z & -E_yE_z - B_yB_z & B_x^2 + B_y - E_z^2 \end{pmatrix} .$$

From the condition $F^2 = 0$ and the hypothesis that all the potentials are real or purely imaginary (so are the fields), we conclude that $E_x = E_y = E_z = B_x = B_y = B_z = 0$, so $F_{\mu\nu} = 0$ and $\mathbf{a}$ is gauge equivalent to the zero field. This finishes the proof of Theorem 8. □

Remark: If we relax the hypothesis that the potential is real or purely imaginary, then we can find Huygens potentials. One family of examples is given by

$$a_0 = -a_3 = \int_{\mathbb{R}} \varepsilon(\omega) e^{i\omega(x^3 - x^0)}(x^1 - ix^2) d\omega \quad a_1 = a_2 = 0 ,$$

where $\varepsilon(\omega)$ is any function such that $\varepsilon(\omega) = \varepsilon^*(-\omega)$ and * denotes complex conjugation.

3.3.3. The Main Theorem

In this subsection we will conjoint both results to study the case when $v = aI + a_\mu \gamma^\mu$, with a_μ real or purely immaginary.

Theorem 9. *Let a_μ be real or purely imaginary. If $\partial\!\!\!/ + aI + a_\mu\gamma^\mu$ possesses Huygens property in 3+1 dimensions, then it is trivially equivalent to the free Dirac operator $\partial\!\!\!/$ or to $\partial\!\!\!/ + 1/x^0$.*

Proof. We first change the gauge to $\tilde{v} = aI + f_\mu\gamma^\mu$ where f_μ is given by equation (3.13). Hadamard's recursion is

$$\begin{aligned} u_0 &= 1 ; \\ u_1 &= -\tilde{v}^* ; \\ u_2 + (x^\mu - y^\mu)\partial_\mu u_2 &= (\partial\!\!\!/ + \tilde{v})\tilde{v}^* ; \\ 0 &= (\partial\!\!\!/ - \tilde{v}^*)u_2 . \end{aligned}$$

With the same technique of the last subsection we get, from the second and the third equations

$$[\tilde{v}^* + (x^\mu - y^\mu)\partial_\mu \tilde{v}^*]u_2 = (\partial\!\!\!/ - \tilde{v}^*)(\partial\!\!\!/ + \tilde{v})\tilde{v}^* ,$$

which can be re-written as

$$[aI + (x^\mu - y^\mu)\partial_\mu a + \gamma^\mu(x^\nu - y^\nu)F_{\nu\mu}]u_2 = (\partial\!\!\!/ - \tilde{v}^*)(\partial\!\!\!/ + \tilde{v})\tilde{v}^* , \tag{3.18}$$

At the point (x, x) this equation reduces to

$$2a^3 - \Box a + \frac{2}{3}\gamma^\mu\partial^\nu F_{\mu\nu} - 2\sum_{\mu<\nu}\gamma^\mu\gamma^\nu a F_{\mu\nu} = 0 .$$

From the linear independence of the gamma matrices:

$$\begin{aligned} 2a^3 - \Box a &= 0 \; ; \\ \partial^\nu F_{\mu\nu} &= 0 \; ; \\ aF_{\mu\nu} &= 0 \; . \end{aligned}$$

The last equation implies that a is zero or $F_{\mu\nu}$ is zero at each point (the possibility of both being different from zero, but having disconnect support is irrelevant because Huygens' principle is a local property). In the first case we are in Theorem 8 and in the second we are in Theorem 7. $\Box$

CHAPTER 4

Huygens' Principle and Integrability

The main objective of this chapter is to relate Huygens' principle to integrability. Some links of these two different branches of mathematics were seen in the first chapter, for the case of the wave operator. Here, we will present these links in the case of Dirac operators.

Section 4.1 presents a short introduction to integrability, aimed at constructing the AKNS hierarchy. Section 4.2 shows a relation between the Huygens' principle and the rational solutions of the mKdV hierarchy, while Section 4.3 extends this relation to the AKNS system.

4.1 Few Words About Integrability: KdV, mKdV, AKNS & Friends

This section presents a short introduction to the world of integrability aiming specifically at applications in this thesis. Readers interested on a basic introduction to soliton theory and integrability should see [58, 59]. For an elementary introductory account see [60, 61], while a deeper overview can be found in [62, 63, 64, 65]. Geometrical aspects of integrable systems are discussed in [66, 67], while a good reference for its algebraic aspects, is [68]. Relations with symmetries are studied in [69]. A recent review is [70]. The relation of integrability and the Painlevé property is addressed in [22].

The Korteweg-de Vries equation (KdV) was introduced in fluid dynamics in 1895 [71]. Its form, modulo trivial transformations that do not change its properties, is

$$u_t = u_{xxx} - 6uu_x \ . \tag{4.1}$$

This equation models one-directional wave propagation with a special combination of non-linearity and dispersion, so it is useful in a large variety of models. One of its most remarkable properties is the presence of *solitons*. See [38, 59, 61, 72].

We say that a Hamiltonian equation is integrable if it presents an infinite number of independent conservation laws in involution with one another. An integrable equation never comes alone, but with a full set of equations, called an *integrable hierarchy*. There are many, but equivalent, ways of constructing the hierarchy. See [61, 62, 66]. Each equation of the hierarchy shares many properties with the other members.

The *modified Korteweg-de Vries*, or, in short, mKdV, which is closely related to the KdV, is given by:

$$v_t = v_{xxx} - 6v^2 v_x \ . \tag{4.2}$$

Solutions of these two equations are related by the Miura transformation

$$u = v_x \pm v^2 \ . \tag{4.3}$$

An important generalization of all these hierarchies is given by the Ablowitz, Kaup, Newell and Segur (AKNS) systems, see [73]. We will follow the construction presented in [74].

Let us consider two 2×2 matrices, P and R, depending on two variables, x and t and one spectral parameter k. Let us also assume that they depend polynomialy on k

$$P(x,t;k) = Jk + Q(x,t)$$

and

$$R(x,t;k) = Jk^n + R_1(x,t)k^{n-1} + \cdots + R_n(x,t) \ ,$$

where J is constant matrix. Suppose there exists a matrix valued function φ such that

$$\partial_x \varphi = P\varphi \quad \text{and} \quad \partial_t \varphi = R\varphi \ .$$

The compatibility condition between both P and R is written as a zero curvature condition

$$\partial_x R - \partial_t P + [R, P] = 0 \ ,$$

where $[\cdot,\cdot]$ is the commutator. If we write the expression for R and P in the above equation and equate the terms in k we find

$$\begin{aligned}
[J, Q] + [R_1, J] &= 0 \ , \\
\partial_x R_1 + [R_1, Q] + [R_2, J] &= 0 \ , \\
\partial_x R_2 + [R_2, Q] + [R_3, J] &= 0 \ , \\
&\vdots \\
\partial_x R_{n-1} + [R_{n-1}, Q] + [R_n, J] &= 0 \ , \\
\partial_t Q - \partial_x R_n + [Q, R] &= 0 \ ,
\end{aligned}$$

Now, let's consider the matrices

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} . \tag{4.4}$$

Impose

$$P = R^{(1)} = kH + qE + rF \ ,$$
$$R^{(n)} = \sum_{l=0}^{n} k^{n-l} R_l \ ,$$

where

$$R_0 = H \quad \text{and} \quad R_1 = qE + rF \ .$$

If we assign weight $i+1$ to $\partial_x^i q$ and to $\partial_x^i r$ the solutions $R_2, \cdots, R_n$ are unique. See [74, 75].

The n-th flow is given by

$$\partial_{t_n} R_1 - \partial_x R_n + [R_1, R_n] = 0 \ .,$$

which can be written as a system of two partial differential equations for q and r

$$\begin{aligned} q_{t_n} &= A_n(q, \cdots, \partial_x^n q; r, \cdots, \partial_x^n r) \ , \\ r_{t_n} &= B_n(q, \cdots, \partial_x^n q; r, \cdots, \partial_x^n r) \ . \end{aligned}$$

For $n = 1$ and $n = 2$ we find the linear flows

$$\begin{aligned} q_{t_0} &= 2q \ , & q_{t_1} &= q_x \ , \\ r_{t_0} &= -2r \ , & r_{t_1} &= r_x \ . \end{aligned}$$

For $n = 2$ and $n = 3$ we find the nontrivial flows

$$\begin{aligned} q_{t_2} &= -\tfrac{1}{2}(q_{xx} - 2q^2 r) \ , & q_{t_3} &= \tfrac{1}{4}(q_{xxx} - 6qrq_x) \ , \\ r_{t_2} &= \tfrac{1}{2}(r_{xx} - 2r^2 q) \ , & r_{t_3} &= \tfrac{1}{4}(r_{xxx} - 6qrr_x) \ . \end{aligned}$$

Proceeding with this calculation we get the so-called AKNS hierarchy. It is easy to see that, if we impose $q = \pm r$ we have the mKdV; if we impose $r = 1$, the equation for q is the KdV. In this sense the AKNS generalizes many integrable systems, including the cubic Schrödinger equation ($q = r^*$) and others.

The technique of Subsection 1.3.1 used to construct Huygens potentials for wave operators depending on just one variable, namely, the Darboux transformations was used also to construct rational solutions of the KdV hierarchy by Adler and Moser [41]. Now we will construct rational solutions of the AKNS hierarchy by matrix Darboux transformations and in the last section we will connect this construction with Huygens' potentials for Dirac operators. We keep on following [74].

Consider the eigenvalue problem

$$L\varphi \stackrel{\text{def}}{=} J^{-1}(\partial_x - Q)\varphi = k\varphi \ ,$$

and look for a transformation

$$\tilde{\varphi} = (k - A(x))\varphi \ ,$$

such that

$$\tilde{L}\tilde{\varphi} \stackrel{\text{def}}{=} J^{-1}(\partial_x - \tilde{Q})\tilde{\varphi} = k\varphi \ .$$

The necessary and sufficient conditions for this to happen is

$$\begin{aligned} \tilde{Q} &= Q + [J, A] \ , \\ \partial_x A &= [JA + Q, A] \ . \end{aligned} \tag{4.5}$$

We remark that the condition $A^2 = 0$ is preserved by the flow. Consider

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ .$$

The Darboux transformation process described above can be used to generate families of solutions to the AKNS hierarchy. Furthermore, if one starts form $Q = 0$ and apply successive Darboux transformations requiring $A^2 = 0$, it was shown by Zubelli (see [74]) that generically the Darboux iterates are of the form

$$Q(x) = q(x)E + r(x)F \ ,$$

and belong to the manifold of rational solutions of the AKNS hierarchy.

4.2 Huygens' Principle, Dirac Operators and mKdV: the Reduction of the Dimension

This sections concerns the first relation between Huygens' principle for Dirac operators and integrable systems. We follow reference [76].

Henceforth, for notational simplicity, we set the time variable $x^0 = t$ and $'$ will denote derivative with respect to t. Also, we identify the polynomial variable in the Adler-Moser polynomial $\mathcal{P}_k$, as defined in Section 1.3.1, with $x^0 = t$. Let $v_k = \partial_t \log(\mathcal{P}_{k+1}/\mathcal{P}_k)$ and $u_k = 2\partial_t^2 \log \mathcal{P}_k$. So,

$$(\partial\!\!\!/ + v_k)(\partial\!\!\!/ - v_k) = \Box - \gamma^0 v_k' - v_k^2 = \Box - \begin{pmatrix} v_k' + v_k^2 & 0 \\ 0 & -v_k' + v_k^2 \end{pmatrix} \tag{4.6}$$

It follows from the properties of the Adler-Moser polynomials that we have $u_k = -v_k' - v_k^2$ and $u_{k+1} = v_k' - v_k^2$. If we consider the fundamental solutions Φ_k such that $(\Box + u_k)\Phi_k = \delta_y$, the function

$$\Psi = (\partial\!\!\!/ - v_k)\begin{pmatrix}\Phi_k & 0\\ 0 & \Phi_{k+1}\end{pmatrix} \tag{4.7}$$

is the fundamental solutions of $\partial\!\!\!/ + v_k$. So, if Φ_k and Φ_{k+1} are Huygens in a certain dimension, then Ψ as defined above has this property in the same dimension.

We expect, from the above argument, that $\partial\!\!\!/ + v_k$ has the Huygens' property in the minimum dimension where $\Box + u_k$ and $\Box + u_{k+1}$ are both Huygens. However, an interesting fact happens and the dimension is reduced by two. We shall refer to this as the *reduction of the dimension.* This property holds for the free Dirac operator. In this case, it is Huygens for every odd dimension, unlike the wave operator, which is Huygens just from 3 on.

We are now ready to state:

Theorem 10. *Let $\mathcal{P}_k$ denote the k-th Adler-Moser polynomial and*

$$v_k(x_0) \stackrel{def}{=} \partial_0 \log(\mathcal{P}_{k+1}/\mathcal{P}_k)(x_0)\ .$$

Then, $\partial\!\!\!/ + v_k$ is Huygens in $n = 2k+3$ spatial dimensions for every $k \geq 0$.

Proof. Let us define $\mu_k = \mathcal{P}_{k+1}/\mathcal{P}_k$, then $v_k = \mu_k'/\mu_k$. It can be shown [41] that $u_k = -\mu_k''/\mu_k$. The operator $\Box + u_k$ is known to be Huygens in dimension $2k+3$. Let Φ_k be the fundamental solution of $\Box + u_k$. As Φ_k obeys Huygens' principle, its Hadamard expansion terminates. Let r_k be the last term in the expansion. If we are in the minimal dimension for the validity of this property, then r_k is the coefficient of the Λ^{-2} term. Let us study the behavior of this term:

$$(\Box + u_k)\Phi_k = (\Box + u_k)(\cdots + \Lambda^{-2} r_k) = \cdots + \Lambda^{-2}(\partial_t^2 + u_k) r_k\ .$$

As $(\Box + u_k)\Phi_k = \delta_y = \Lambda^{-n-1}$ and $n \geq 3$ we claim that $(\partial_t^2 + u_k) r_k = 0$.

Using the definition of u_k we write $(\partial_t + v_k)(\partial_t - v_k) = \partial_t^2 + u_k = (\partial_t - v_{k-1})(\partial_t + v_{k-1})$. This gives $r_k = \alpha_k \mu_k + \beta_k/\mu_{k-1}$.

Since μ_k is a ratio of two consecutive Adler-Moser polynomials, its asymptotic behavior, $t \to \infty$, is given by $\mu_k = \mathcal{O}(t^{k+1})$.

But the differential equation obeyed by r_k, when $t \to \infty$ (and remembering that $u_k = -\mu_k''/\mu_k \to 0$), is $(\partial_t^2) r_k \to 0$. In other words, $r_k = o(t^2)$ when $t \to \infty$. Thus, $r_k = \beta_k/\mu_{k-1}$.

We define

$$\Psi_k = (\partial\!\!\!/ - v_k)\begin{pmatrix}\Phi_k & 0\\ 0 & \Phi_{k+1}\end{pmatrix}\ .$$

The above properties yield that Ψ_k is the fundamental solution of $\not{\partial} + v_k$.

We shall study the last term in Ψ_k. As Φ_k and Φ_{k+1} act in the same spatial dimension n, and they obey Huygens' principle in this dimension, we chose the minimum n such that Φ_{k+1} is supported in the light cone, $n = 2k+5$, and in this case the last term in the expansion of Φ_k is $\Lambda^{-4} r_k$. In order to study the behavior of the last term in Ψ_k we write:

$$\Psi_k = (\not{\partial} - v_k)(\cdots + \Lambda^{-2} \begin{pmatrix} 0 & 0 \\ 0 & r_{k+1} \end{pmatrix}) =$$

$$= \cdots - \Lambda^{-2}(\partial_t + v_k) r_{k+1} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \cdots + \Lambda^{-2} \cdot 0 \ .$$

This equation shows that the last term in the Hadamard expansion of Ψ_k is Θ^{-2}. As Θ^0 has the Huygens' property, we can lower the dimension by two, to $n = 2k+3$. □

4.3 Further Relations: Huygens and AKNS

In this section we construct a full hierarchy of Huygens' potentials depending solely on the time variable x^0 directly related to the AKNS hierarchy defined in Section 4.1. For a potential v with scalar and pseudo-scalar components

$$v = a(x^0) I + \bar{a}(x^0) \bar{\gamma} \ ,$$

we define

$$Q = -a\gamma^0 - \bar{a}\gamma^0\bar{\gamma} \ .$$

We start proving the following lemmas:

Lemma 2. *The operators $\not{\partial} + v$ and $\bar{\gamma}\partial_0 - \gamma^0\bar{\gamma}\gamma^i\partial_i - \bar{\gamma}Q$ are trivially equivalent.*

Proof. $-\gamma^0\bar{\gamma}(\not{\partial} + v) = \bar{\gamma}\partial_0 - \gamma^0\bar{\gamma}\gamma^i\partial_i - \bar{\gamma}Q$. □

Lemma 3. *With the above definitions*

$$(\bar{\gamma}\partial_0 - A - \bar{\gamma}Q)(\bar{\gamma}\partial_0 - \bar{\gamma}Q) = (\bar{\gamma}\partial_0 - \bar{\gamma}\tilde{Q})(\bar{\gamma}\partial_0 - A - \bar{\gamma}Q)$$

$$\Updownarrow$$

$$\tilde{Q} = Q + [\bar{\gamma}, A] \quad \text{and} \quad \partial_0 A = [\bar{\gamma}A + Q, A] \ .$$

Proof. We just need to expand both sides and to compare the coefficients of the powers of ∂_0. □

Lemma 4. *Suppose $Q = -a\gamma^0 - \bar{a}\gamma^0\bar{\gamma}$. Then, $A = A_\phi I + A_0\gamma^0 + \bar{A}_0\gamma^0\bar{\gamma} + A_\phi\bar{\gamma}$ and $\tilde{Q} = -\tilde{a}\gamma^0 - \bar{\tilde{a}}\gamma^0\bar{\gamma}$.*

Proof. From the equation for $\partial_0 A$ given in Lemma 3 and the closedness of the algebra generated by I, γ^0, $\bar\gamma$ and $\gamma^0\bar\gamma$, we conclude the expression for A. From the trivial fact that $[\bar\gamma, A] = -2\bar{A}_0\gamma^0 - 2A_0\gamma^0\bar\gamma$, we conclude the second expression. □

Lemma 5. $(\bar\gamma\partial_0 - A - \bar\gamma Q)\gamma^0\bar\gamma\gamma^i\partial_i = \gamma^0\bar\gamma\gamma^i\partial_i(\bar\gamma\partial_0 - A - \bar\gamma Q)$.

Proof. We just need to remember that A and Q depend only on x^0 and, after a simple calculation, we get that I, γ^0, $\bar\gamma$ and $\gamma^0\bar\gamma$ commute with $\gamma^0\bar\gamma\gamma^i$. □

Theorem 11. *Let q and r be solutions of the AKNS constructed by rational Darboux transformations, as in Section 4.1, and the scalar and pseudo-scalar potentials be given by*

$$a(x^0) = -\frac{q(x^0) + r(x^0)}{2} \quad \text{and} \quad \bar{a}(x^0) = \frac{q(x^0) - r(x^0)}{2} ,$$

Then, Dirac operators of the form $\not\partial + v$, where $v = a(x^0)I + \bar{a}(x^0)\bar\gamma$ possesses Huygens' property.

Proof. We take the trivially equivalent operator given by Lemma 2, $\bar\gamma\partial_0 - \gamma^0\bar\gamma\gamma^i\partial_i - \bar\gamma Q$ and, with help of Lemmas 3 and 4, we show that $\bar\gamma\partial_0 - A - \bar\gamma Q$ is an intertwining operator between $\bar\gamma\partial_0 - \gamma^0\bar\gamma\gamma^i\partial_i - \bar\gamma Q$ and $\bar\gamma\partial_0 - \gamma^0\bar\gamma\gamma^i\partial_i - \bar\gamma\tilde{Q}$, i.e.

$$(\bar\gamma\partial_0 - \gamma^0\bar\gamma\gamma^i\partial_i - \bar\gamma\tilde{Q})(\bar\gamma\partial_0 - A - \bar\gamma Q) = (\bar\gamma\partial_0 - A - \bar\gamma Q)(\bar\gamma\partial_0 - \gamma^0\bar\gamma\gamma^i\partial_i - \bar\gamma Q) ,$$

if, and only if, the conditions in Lemma 3 are obeyed. In order to compare with the construction in Section 4.1 we just need to find a representation of the Clifford algebra, where $\bar\gamma = J$. This representation is the Weyl one, as defined in Section 2.1. In this case

$$Q = -a\gamma^0 - \bar{a}\gamma^0\bar\gamma = \begin{pmatrix} 0 & -a+\bar{a} \\ -a-\bar{a} & 0 \end{pmatrix} = (\bar{a} - a)E - (a + \bar{a})F .$$

So, comparing Q with the matrix Q defined in Section 4.1 we find $\bar{a} - a = q$ and $a + \bar{a} = -r$ and the theorem follows. □

We remark that Section 4.1 hypothesis, namely $A^2 = 0$, is used to compute the matrix $\tilde{Q}$.

Before focusing on some examples, let us relate fundamental solutions of two matrix Darboux related operators:

Lemma 6. *Let $\mathcal{L}$ and $\tilde{\mathcal{L}}$ be related by an intertwining operator ℓ, i.e. $\tilde{\mathcal{L}}\ell = \ell\mathcal{L}$ and Φ, $\tilde\Phi$ and ϕ be the fundamental solutions of $\mathcal{L}$, $\tilde{\mathcal{L}}$ and ℓ respectively. Then,*

$$\tilde\Phi(x, y) = \ell_x \int \Phi(x, z)\phi(z, y)dz .$$

Proof. To prove this, we compute

$$\begin{aligned}
\tilde{\mathcal{L}}_x \tilde{\Phi}(x,y) &= \tilde{\mathcal{L}}_x \ell_x \int \Phi(x,z)\phi(z,y)dz \\
&= \ell_x \int \mathcal{L}_x \Phi(x,z)\phi(z,y)dz \\
&= \ell_x \int \delta(x-z)\phi(z,y)dz \\
&= \ell_x \phi(x,y) \\
&= \delta(x-y) \ .
\end{aligned}$$

□

Examples: After performing one matrix Darboux transformation we have

$$\not{\partial} + \frac{1}{x^0}$$

whose solution is given in the Theorem 7.

One further matrix Darboux transformation and we get the operator

$$\not{\partial} + \frac{3\kappa_2 + 2(x^0)^3}{-3x^0\kappa_2 + (x^0)^4 + 3\kappa_1^2} I + \frac{6\kappa_1 x^0}{-3x^0\kappa_2 + (x^0)^4 + 3\kappa_1^2}\bar{\gamma} \ .$$

In this case, $u_4 \neq 0$ and $u_5 = 0$. Thus, this operator is of Huygens' type in $n = 5$.

CHAPTER 5

Conclusions and Perspectives

In this chapter we present, in Section 5.1, more examples of Huygens' operators, related to AKNS systems. In Section 5.2 we discuss the link between Huygens and integrability. We finish in Section 5.3 with a few words about possible continuations of the present work.

5.1 More and More Examples

After Theorem 11, a natural question that immediately arises concerns solutions of the AKNS hierarchy that cannot be obtained by Darboux matrix transformation. Following [77] let us explain how to obtain a large class of these solutions. We start by defining elementary Schur polynomials $q_j(y_1, \cdots, y_j)$, $j \in \mathbb{Z}$ as $q_j = 0$ whenever $j < 0$ and, for $j \geq 0$

$$\sum_{j\geq 0} \lambda^j q_j = \exp\Big(\sum_{j\geq 1} \lambda^j y_j\Big) \ .$$

Then we set

$$y_1 = t = x^0 \ , \quad y_j = -\frac{1}{2}t_2 \ , \ \cdots \ , \ y_j = \frac{1}{(-2)^{j-1}}t_j \ ,$$

and take the Wronskian W in the first variable x^0

$$\begin{aligned} \sigma_j^d &= \mathrm{W}[q_d, \cdots, q_{d-j+1}] \ , \\ \tau_j^d &= \mathrm{W}[q_d, \cdots, q_{d-j}] \ , \\ \rho_j^d &= -\mathrm{W}[q_d, \cdots, q_{d-j-1}] \ . \end{aligned}$$

For convenience, we extend our definition setting $\sigma_0^d = -1$, $\sigma_{-1}^d = 0$ and $\tau_{-1}^d = 1$. Then, for each d and each $j \leq d$, the functions of x^0, with the remaining values t_k interpreted as arbitrary constants,

$$q(x^0) = \frac{\sigma_j^d}{\tau_j^d} \, , \qquad r(x^0) = \frac{\rho_j^d}{\tau_j^d} \, , \tag{5.1}$$

are rational solutions of the AKNS.

The important point is that the operator

$$\not{\partial} - \frac{q(x^0) + r(x^0)}{2} I + \frac{q(x^0) - r(x^0)}{2} \bar{\gamma}$$

seems to possess Huygens' property. This is verified by extensive computational evidence. To be precise, we solved, using MAPLE, Hadamard's recursion, finding, for d ranging from 0 to 5 that the recursion always stops in the d-th element (i.e., $u_{d'} = 0 \; \forall d' > d$). As of the writing of the present work, this is a still open result and shall be addressed in near future[1].

As the order of the Hadamard series does not depend on j, but only on d, we could conjecture that the given operators are trivially equivalent (in the sense of Section 2.4). A candidate to realize the exact transformation is the Schlesinger transformation[2]. See [75, 77].

5.2 What is Important?

From the introduction, the main body of the thesis and specially from last section, it should be clear that much of the current importance of the study of Huygens' principle comes from its link with other branches of mathematics, specially with integrability. So, we cannot conclude this work without a word about it.

One very simple remark is that operators $\not{\partial} + v$ and $\not{\partial} + vI \cosh\theta + v\bar{\gamma} \sinh\theta$ are trivially equivalent, if v is a scalar. To see that we just need to multiply $\not{\partial} + v$ by $e^{\theta\bar{\gamma}}$ by both sides. The term $a^2 - \bar{a}^2$ remains invariant[3] and equals to v^2. Furthermore, $a^2 - \bar{a}^2 = vv^*$. So, at least for the cases studied in Section 4.2, vv^* equals the square of the solutions of the mKdV.

This is not surprise in view of Section 4.3, as we see that $vv^* = qr$ and the mKdV is the particular case of AKNS such that $q = r$. The important point is that the value of vv^* was the *guideline* to find the right way to combine q and r in order to obtain a Huygens' potential. In particular, before Theorem 11 was formulated in its final form, many examples (like solutions of the KdV, cubic Schrödinger) were found (including with the use of symbolic computation).

[1] We recently proved this result: If $q(x^0)$ and $r(x^0)$ are solutions of the AKNS given by 5.1, then $\not{\partial} - \frac{q+r}{2} I + \frac{q-r}{2} \bar{\gamma}$ has the Huygens' property in dimension $d+2$, if d is odd, or $d+3$ if d is even.

[2] This result was also recently proved.

[3] Attention: this is not invariant under trivial transformations.

If vv^* is really important to understand Huygens' principle and, particularly, its relation to integrability theory, potentials of the form $f(x)(I+\bar{\gamma})$ should produce large class of examples, as $vv^* = f(x)^2(I+\bar{\gamma})(I-\bar{\gamma}) = 0$. This is the case, as can be seen in the last theorem of this thesis.

Theorem 12. *If $f(x)$ is solution of $\Box^k f(x) = 0$ for any positive integer k, then $\partial\!\!\!/ + f(x)(I+\bar{\gamma})$ is a Huygens' operator for a sufficiently high dimension n.*

Proof. Write $u_k = (I + (-1)^k\bar{\gamma})\tilde{u}_k$, for $k \geq 1$. Solving Hadamard's recursion we see that this definition is consistent and that $\tilde{u}_k$ obeys Hadamard's recursion with $v = 0$, starting from $\tilde{u}_1 = -f$. We solve the recursion for $\tilde{u}_{2k}$ and find

$$\tilde{u}_{2k}(x,y) = -k\int_0^1 \partial\!\!\!/\tilde{u}_{2k-1}(y+s(x-y),y)s^{k-1}ds \ ,$$

To prove that we only need to know that

$$(x^\mu - y^\mu)\partial_\mu F(y+s(x-y)) = \frac{dF}{ds}(y+s(x-y))s$$

and integrate by parts. We find the next term

$$\tilde{u}_{2k+1} = \partial\!\!\!/\tilde{u}_{2k} = -k\int_0^1 \Box\tilde{u}_{2k-1}(y+s(x-y,y)s^k ds \ .$$

For the odd terms we write

$$\tilde{u}_{2k+1}(x,y) = (-1)^{k+1}k!\int_0^1\cdots\int_0^1 \Box^k f(\xi_k)s_1^k s_2^{k+1}\cdots s_k^{2k-1}ds_1\cdots ds_k \ , \tag{5.2}$$

where $\xi_i = y + s_i(\xi_{i-1} - y)$, and $\xi_0 = x$. □

5.3 Future Work

Many questions could be addressed as follow up of the present work. Here we will mention just a small sample of interesting ones to the author.

- Relations between Huygens' principle for Dirac operators and rational solutions of the AKNS system, as described in the beginning of this chapter.
- Relations of these examples and Berest's conjecture, which was presented in Subsection 1.3.2. This conjecture has to be changed to the matrix case. As far as we know there is no natural way to do this.
- Huygens' principle for higher-order differential operators. We can expect new links between Huygens' principle and integrable systems.

- Huygens' principle for other evolution equations, specially the ones with physical relevance. Wave equation is an equation for massless particles with zero spin. Dirac equation is for spin 1/2. Likewise, for spin 1, 3/2 and 2 the basic equations are, respectively, Maxwell's, Rarita-Schwinger's and Einstein's. As far as we know, these questions were never addressed in terms of relations between Huygens' principle and integrability (but they have been addressed in relation to the physical basis of Huygens' principle, see [26] and references therein).

- Huygens' principle for non-linear evolution equations.

- Huygens' principle for non-hyperbolic equations. In these cases we loose the physical appeal of Huygens' principle, but on the other hand, we can find new relations between Huygens and integrability. An instance of this phenomena was investigated in [78] for the wave equation.

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