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Groups of germ of analytic diffeomorphisms

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Contents

1 Introduction

The study of germs of holomorphic diffeomorphisms and finitely generated groups of analytic germs of diffeomorphisms fixing the origin in one complex variable was started in the XIX century, and has been intensively studied by mathematicians in the past century. Such groups appear naturally when we study the holonomy group of some leaf of codimension one holomorphic foliation.

From the Poincaré-Siegel linearization theorem, (See [2] or [14]) it follows the study of local topology and the analytical and topological classification of diffeomorphisms of $(\mathbb{C}^n, 0)$ having linear part in the Poincaré domain, or in the Siegel domain that satisfy the Brjuno condition. The resonant case in one dimension is well known too, the local topology and topological classification was giving in Camacho [3] and Camacho and Sad [5]. Moreover the analytical classification in the resonant case is due to Écalle [9], Voronin [25], Martinet and Ramis [16] and Malgrange [17].

The topological classification in dimension 2 in the partially hyperbolic case with resonances has been studied by Canille [6] and the topological behavior in dimension ≥ 2 in the partial hyperbolic case by Ueda [23]. In the case, tangent to the identity, Hakim [13] and Abate [1], have shown the existence of parabolic attractive points.

In addition, there is an almost complete analytic classification of the group of one dimensional germs when we assume some algebraic hypothesis as finiteness (See [18]), abelian, solvable (see [7] and [19]). See [11] for a complete survey of this classification.

In this work, we deal with germs of diffeomorphisms in $(\mathbb{C}^2, 0)$ and the finitely generated group of germs.

In section 2 we give the definitions and preliminary results. In section 3 we study the finite groups of diffeomorphisms. We prove a generalization of Mattei-Moussu topological criteria about finiteness of a group,

Theorem 3.2 *Let $F \in \mathrm{Diff}(\mathbb{C}^2, 0)$. The group generated by F is finite if and only if there exists a neighborhood V of 0, such that $|\mathcal{O}_V(F, X)| < \infty$ for all $X \in V$ and F leaves invariant infinite analytic varieties at 0.*

We show a relationship between finite groups and the existence of a complete set of first integrals.

Theorem 3.5 *Let $\mathcal{G} < \mathrm{Diff}(\mathbb{C}^n, 0)$. $\mathcal{G}$ is a finite group if and only if there exist $F_1, \ldots, F_n$ germs of holomorphic first integrals such that 0 is an isolated point of $F_1^{-1}(0) \cap \cdots \cap F_n^{-1}(0)$.*

Finally, we compare a topological conjugacy class of some finite diffeomorphisms with their analytical conjugacy class, i.e. we construct the moduli space (topological vs analytical) of the diffeomorphisms that are conjugate with some finite order diffeomorphism,

Theorem 3.6 *Let $F \in \mathrm{Diff}(\mathbb{C}^n, 0)$ such that $F^N = Id$. Then*

$$\frac{\mathcal{H}_{top}(F)}{\mathcal{H}_{hol}(F)} \simeq \pm \frac{SL(n,\mathbb{Z})}{SL_A(n,\mathbb{Z})},$$

where $A = DF(0) = \begin{pmatrix} e^{2\pi i \frac{p_1}{N}} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{2\pi i \frac{p_n}{N}} \end{pmatrix}$, *$SL(n,\mathbb{Z})$ is the special linear group of $n \times n$ matrices over the ring $\mathbb{Z}$ and*

$$SL_A(n,\mathbb{Z}) = \left\{ B \in SL(n,\mathbb{Z}) | (B - I) \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix} \in N\mathbb{Z}^n \right\}.$$

In particular, we prove that the moduli space topological vs C^∞ is trivial. In section 4 we deal with the linearizable groups. Proposition 4.3 shows that in a special case the topological linearization implies the analytical linearization. In addition, we prove a generalization of theorem 3.6 in this special case. Moreover, we show that a group of diffeomorphisms is linearizable if and only if there exists a vector field with radial first jet invariant by the group-action, i.e.

Theorem 4.2 *A group $\mathcal{G} \subset \mathrm{Diff}(\mathbb{C}^n, 0)$ is analytically linearizable if and only if there exists a vector field $\mathcal{X} = \vec{R} + \cdots$, where $\vec{R}$ is a radial vector field such $\mathcal{X}$ is invariant for every $F \in \mathcal{G}$, i.e. $F^*\mathcal{X} = \mathcal{X}$.*

In section 5 we study the groups of diffeomorphisms supposing that they have some algebraic structure. We prove that if $\mathcal{G} \subset \mathrm{Diff}(\mathbb{C}^2, 0)$ is a solvable group then its 7^{th} commutator subgroup is trivial. Furthermore, we characterize the abelian subgroup of diffeomorphisms tangent to the identity, and in the case when the group contains a dicritic diffeomorphism, i.e. the group contains a diffeomorphism $F(X) = X + F_{k+1}(X) + \cdots$ where $F_{k+1}(X) = f(X)X$ and f is a homogeneus polynomial of degree k, we prove that the group is a subgroup of a one parameter group. We write $\mathrm{Diff}_1(\mathbb{C}^2, 0)$ to denote the group of diffeomorphisms tangent to the identity at $0 \in \mathbb{C}^2$.

Theorem 5.5 *Let $\mathcal{G} < \mathrm{Diff}_1(\mathbb{C}^2, 0)$ be abelian group, and $F \in \mathcal{G}$ a dicritic diffeomorphism. Suppose that* $\exp(\mathfrak{f})(x,y) = F(x,y)$ *where*

$$\mathfrak{f} = (f(x,y)x + p_{k+2}(x,y) + \cdots)\frac{\partial}{\partial x} + (f(x,y)y + q_{k+2}(x,y) + \cdots)\frac{\partial}{\partial y},$$

$f(x,y)$ is a homogeneous polynomial of degree k and g.c.d$(f, xq_{k+2}(x,y) - yp_{k+2}(x,y)) = 1$, then

$$\mathcal{G} < \langle \exp(t\mathfrak{f})(x,y) | t \in \mathbb{C} \rangle.$$

In section 6 we show that the Milnor number of the diffeomorphisms tangent to the identity is a topological invariant. Furthermore, we show that this number is determined by a finite jet of the diffeomorphism.
In the section 7 we analyze the behavior of the orbits of a diffeomorphism tangent to the identity. We prove a generalization of the one dimensional flower theorem to two dimensional dicritic diffeomorphisms, i.e.

Theorem 7.3 *Let $F : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ be a dicritic diffeomorphism fixing zero, i.e. F can be represented by a convergent series*

$$F(x,y) = \begin{pmatrix} x + xp_k(x,y) + p_{k+2}(x,y) + \cdots \\ y + yp_k(x,y) + q_{k+2}(x,y) + \cdots \end{pmatrix},$$

and $\tilde{F} = \Pi^* F : (\tilde{\mathbb{C}}^2, D) \to (\tilde{\mathbb{C}}^2, D)$ be the continuous extension of the diffeomorphism after making the blow-up in $(0,0)$. Then there exist open sets $U^+, U^- \subset \tilde{\mathbb{C}}^2$ such that

1. *$\overline{U^+ \cup U^-}$ is a neighborhood of $D \setminus \{(1:v) \in D | p_k(1,v) = 0\}$.*

2. *For all* $P \in U^+$, *the sequence* $\{\tilde{F}^n(P)\}_{n\in\mathbb{N}}$ *converge and* $\lim_{n\to\infty} \tilde{F}^n(P) \in D$.

3. *For all* $P \in U^-$, *the sequence* $\{\tilde{F}^{-n}(P)\}_{n\in\mathbb{N}}$ *converge and* $\lim_{n\to\infty} \tilde{F}^{-n}(P) \in D$.

An equivalent local theorem is proven in the non dicritic case.
Finally, in section 8 we show the formal classification of diffeomorphisms tangent to the identity using the notion of the semiformal conjugacy. We show that a representative diffeomorphism found using semiformal conjugacy and a cocycle determine its formal conjugacy class.
In the case of dicritic diffeomorphisms, we find a rational function that it is going to play an equivalent role to the residue in a one dimension diffeomorphism.

Theorem 8.3 *Let* $\tilde{F} \in \widehat{\mathrm{Diff}}_1(\mathbb{C}^2, 0)$ *be dicritic diffeomorphism and* $F(x,v) = \begin{pmatrix} x + x^{k+1}p(v) + x^{k+2}(\cdots) \\ v + x^{k+1}(\cdots) \end{pmatrix}$ *be the continuous extension of the diffeomorphism after making the blow-up in* $(0,0)$. *Then there exists a unique rational function* $q(v)$ *such that* F *is semiformally conjugate to*

$$G_F = \begin{pmatrix} x + x^{k+1}p(v) + x^{2k+1}q(v) \\ v \end{pmatrix}$$

in $\overline{\mathbb{C}} \setminus \{p(v) = 0\}$. *In addition,* $q(v) = \frac{s(v)}{p(v)^{2k+1}}$ *where* $s(v)$ *is a polynomial of degree* $2k + 2 + 2k\partial(p(v))$.

2 Preliminaries

Let $\widehat{\mathrm{Diff}}(\mathbb{C}^n, 0)$ denote the group of n-dimensional formal diffeomorphisms fixing zero, i.e.

$$\widehat{\mathrm{Diff}}(\mathbb{C}^n, 0) = \{H(X) = AX + P_2(X) + \cdots | A \in Gl(n, \mathbb{C}), P_i \in \mathbb{C}^n[[X]]_i\},$$

where $X = (x_1, \ldots, x_n)$ and $\mathbb{C}^n[[X]]_i$ is the set of n-dimensional vectors with coefficients homogeneous polynomials of degree i.

Let $\mathrm{Diff}(\mathbb{C}^n,0) \subset \widehat{\mathrm{Diff}}(\mathbb{C}^n,0)$ denote the pseudo-group of germs of holomorphic diffeomorphisms, i.e. the power series that represent every $H \in \mathrm{Diff}(\mathbb{C}^n,0)$ convergent in some neighborhood of $0 \in (\mathbb{C}^n,0)$.
For each $j \geq 2$, let $\widehat{\mathrm{Diff}}_j(\mathbb{C}^n,0)$ (resp. $\mathrm{Diff}_j(\mathbb{C}^n,0)$) denote the subgroup of formal (resp. analytic) diffeomorphisms j-flat, i.e. $F(X) \in \widehat{\mathrm{Diff}}_j(\mathbb{C}^n,0)$ (resp. $F(X) \in \mathrm{Diff}_j(\mathbb{C}^n,0)$) then $F(X) = X + P_j(X) + \cdots$.
The same way define $\hat{\chi}_j(\mathbb{C}^n,0)$ the Lie algebra of formal vector fields of $\mathbb{C}^n$, j-flat in $0 \in \mathbb{C}^n$, i.e.

$$\hat{\chi}_j(\mathbb{C}^n,0) = \{F_1(X)\frac{\partial}{\partial x_1} + \cdots + F_n(X)\frac{\partial}{\partial x_n} | \text{ where } F_k \in \bigoplus_{i=j}^{\infty} \mathbb{C}[[X]]_i\}.$$

Proposition 2.1 *The exponential map* $\exp : \hat{\chi}_j(\mathbb{C}^n,0) \to \widehat{\mathrm{Diff}}_j(\mathbb{C}^n,0)$ *is a bijection for every* $j \geq 2$*.*

Proof: The proof follows from a straightforward generalization of the proposition 1.1. in [16]. □

Notice that $\hat{\chi}_j(\mathbb{C}^n,0)$ is the formal Lie algebra associate to the formal Lie group $\widehat{\mathrm{Diff}}_j(\mathbb{C}^n,0)$. It is not difficult to prove that if the vector field is holomorphic, then the associated diffeomorphism is also holomorphic. The converse is false in general.

Definition 2.1 $\mathcal{H},\mathcal{G} \subset \widehat{\mathrm{Diff}}(\mathbb{C}^n,0)$ *are called* **formally (analytically) conjugate** *if there exists* $g \in \widehat{\mathrm{Diff}}(\mathbb{C}^n,0)$ *(*$g \in \mathrm{Diff}(\mathbb{C}^n,0)$*) such that* $g \circ \mathcal{H} \circ g^{-1} = \mathcal{G}$*.*

Definition 2.2 $\mathcal{H},\mathcal{G} \subset \mathrm{Diff}(\mathbb{C}^n,0)$ *are called* **topologically conjugate** *if there exists* $t : \mathcal{H} \to \mathcal{G}$ *bijective group homomorphism and* $g : U \to g(U)$ *local homeomorphism at 0, such that* $F \circ g = g \circ t(F)$ *for all* $F \in \mathcal{H}$ *in some neighborhood of 0.*

In particular, some group $\mathcal{G}$ is called linearizable if there exists g such that $g \circ \mathcal{G} \circ g^{-1}$ is a linear group action. Moreover, if g is a diffeomorphism formal, analytic or continuous then the group is called formally, analytically or topologically linearizable respectively. Let $\Lambda_{\mathcal{G}}$ denote the set $\{DG(0)|G \in \mathcal{G}\}$. To understand when a diffeomorphism is formally and analytically linearizable we need the following definition

Definition 2.3 *Let $G \in \text{Diff}(\mathbb{C}^n, 0)$. G is called* **resonant** *if the eigenvalues of $G'(0)$, $\lambda := (\lambda_1, \dots, \lambda_n)$, satisfies some relation like*

$$\lambda^m - \lambda_j = \lambda_1^{m_1} \cdots \lambda_n^{m_n} - \lambda_j = 0$$

for some $j = 1, \dots, n$ and $m \in \mathbb{N}^n$ where $|m| = m_1 + \cdots + m_n \geq 2$. Otherwise, it is called non resonant.

Definition 2.4 *Let $A \in Gl(n, \mathbb{C})$ and suppose that A is not resonant. We say that A satisfies the Brjuno condition if*

$$\sum_{k=0}^{\infty} 2^{-k} \log(\Omega^{-1}(2^{k+1})) < +\infty$$

where $\Omega(k) = \inf_{\substack{2 \leq |j| \leq k \\ 1 \leq i \leq n}} |\lambda^j - \lambda_i|$.

Theorem 2.1 *Let $G \in \text{Diff}(\mathbb{C}^n, 0)$, where $G(X) = AX + \cdots$, A diagonalizable, non resonant. Then G is formally linearizable. In addition, if G satisfies the Brjuno condition, then this linearization is in fact holomorphic.*

Notice that in the case $n = 1$, we can write $A = \exp(2\pi i\alpha)$. The non resonance condition is equivalent to say $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.
Let $\alpha = [a_0 : a_1 : a_2 : \cdots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}}$ be the continued fraction of α and $\left\{\frac{p_n}{q_n}\right\}_{n \in \mathbb{N}}$ the convergent of α. The Brjuno condition is equivalent to say

$$\sum_{j=0}^{\infty} q_n^{-1} \log q_{n+1} < +\infty.$$

The most common proofs of this theorem use marjorant series or rapid iteration methods and the so called KAM techniques.
When the diffeomorphism is not linearizable, the analytic classification is very complicated. In the one dimensional case we have 2 cases: The resonant case, i.e. when $f(x) = \lambda x + \cdots$ where $\lambda^n = 1$ for some $n \in \mathbb{Z}^+$, and the Liouvillian case, i.e. $\lambda = e^{2\pi i\alpha}$ where α does not satisfies the Brjuno condition.
In the Liouvillian case, there exists an analytic germ that is not equivalent to its linear part, see Perez Marco [22].

In the resonant case is easy to see that $f(x) = g \circ h(x)$ where g and h commute, $g^n = id$ and h is tangent to the identity. Notice that g and h are in general, formal diffeomorphisms.

Theorem 2.2 (Camacho [3]) *Let $h(x) = x + ax^{k+1} + \cdots$, then h is topologically conjugate with $x \mapsto x + x^{k+1}$.*

Thus, the topological classification is very simple. In the same way the formal classification is simple

Theorem 2.3 *Let $h(x) = x + ax^{k+1} + \cdots$, then there exists ρ such that h is formally conjugate with $f_{a,k,\rho}(z) = \exp(a\frac{z^{k+1}}{1+\rho z^k}\frac{\partial}{\partial z})$.*

$\rho(h)$ is called the residue of h and $\rho(h) = \dfrac{1}{2\pi i}\oint \dfrac{1}{h(x)-x}dx$. In particular, h is formally conjugate with $x \mapsto x + x^{k+1} + \rho x^{2k+1}$.
In addition, there exists a unique g_h tangent to the identity such that $g_h^* h = g_h \circ h \circ g_h^{-1} = \exp(a\chi_{k,\rho})$ where $\chi_{k,\rho} = \frac{z^{k+1}}{1+\rho z^k}\frac{\partial}{\partial z}$. This g_h is called the **normalizer transformation** of h.
It is clear that if two diffeomorphisms are analytically conjugate, in particular they are formally. Thus, fixing the formal class, the analytic classification can be obtained by several ways, using theory of resurgent function as Écalle, or using quasi conformal conjugacy as Voronin, or using the theory of resummation as Martinet and Ramis. Notice that this classification depends on two facts: first, the construction of sectorial normalizer transformation, and second, the characteristic cocycles obtained from that transformation, in fact

Theorem 2.4 *Let h be a diffeomorphism tangent to the identity, formally conjugate with $f_{a,k,\rho}$, and g_h its normalizer transformation. Then for every $\theta \in S^1$ there exists a unique germ $g_{h,\theta}$ in $(S_k(\theta), 0)$, where*

$$S_k(\theta) = \{x \in \mathbb{C} |\ |\arg(x) - \theta| < \frac{\pi}{k}\},$$

such that

1. *The asymptotic development of $g_{h,\theta}$ in 0 is g_h.*
2. *$g_{h,\theta}^* h = f_{a,k,\rho}|_{(S_k(\theta),0)}$.*

Theorem 2.5 *The collection of germs*

$$C_j(h) = g_{h,\frac{2\pi}{k}(j+1)} \circ g^{-1}_{h,\frac{2\pi}{k}j} : \left(S_{\frac{k}{2}}(\frac{\pi(2j+1)}{2}),0\right) \to \left(S_{\frac{k}{2}}(\frac{\pi(2j+1)}{2}),0\right)$$

where $j = 0, \ldots, k-1$ *define the analytic conjugacy class of* h.

2.1 Classification of one dimensional groups

Suppose that $G \subset \mathrm{Diff}(\mathbb{C}, 0)$ is a finitely generated group. If we suppose some algebraic properties both the topological behavior and the analytic classification are known. If $|G| < \infty$, then G is analytically conjugate with a rotation. In the case G commutative

Theorem 2.6 *Let* G *be a commutative non finite group of germ of holomorphic diffeomorphism of* $(\mathbb{C}, 0)$ *and* $\Lambda_G = \{g'(0) \in \mathbb{C} | g \in G\}$. *Then*

1. *If* Λ_G *is finite, i.e.* $\Lambda_G = \langle e^{\frac{2\pi i}{k}} \rangle$, G *is formally conjugate to some subgroup of*
$$\langle a \exp(b\chi_{k,\rho}) | a \in \Lambda_G, b \in \mathbb{C} \rangle$$
where ρ *is a fixed complex number.*

2. *If* $a \in A$, *where* $|a| \neq 1$ *or* a *is a Brjuno number, i.e.* $a = e^{2\pi i\alpha}$, *where* α *satisfies the Brjuno condition, then* G *is analytically conjugate to* A. *Otherwise* G *is formally conjugate with* A.

Theorem 2.7 *Let* G *be a solvable non commutative group of germs of diffeomorphism of* $(\mathbb{C}, 0)$. *Then* G *is formally conjugate to*

$$\langle a \exp(b\chi(z)) | a \in \mathbb{C}^*, b \in \mathbb{C} \rangle$$

where $\chi = z^{k+1}\frac{\partial}{\partial z}$. *In addition,*

1. *If* G_0, *the subgroup of* G *of the elements tangent to the identity, is non isomorphic to* $\mathbb{Z}$ *then this conjugacy is analytic.*

2. *If* G_0 *is isomorphic to* $\mathbb{Z}$ *then* $|A| = n < \infty$ *and* $k = \frac{n}{2}, 3\frac{n}{2}, \ldots$. *Furthermore,* G *is generate by* f *and* g, *where* f *is a* k*-flat element with* $\rho(f) = \frac{k+1}{2}$, g *with a linear term* λ, $\lambda^{2k} = -1$ *and* $f \circ g = g \circ f^{(-1)}$. *In this case the group* G *is called* **exceptional**.

When the group G is non solvable, the topological classification is the same as the analytic classification.

Definition 2.5 $\mathcal{G} \subset \mathrm{Diff}(\mathbb{C}^n, 0)$ *is called* **rigid** *if for all* $\mathcal{H} \subset \mathrm{Diff}(\mathbb{C}^n, 0)$*, such that* $\mathcal{H}, \mathcal{G}$ *are topologically conjugate then they are analytically conjugates.*

Theorem 2.8 (Il'yashenko)

1. *Suppose that* G *is non commutative but solvable and* Λ_G *is dense in* $\mathbb{C}$*, them* G *is rigid.*
2. *Suppose that* G *and* G' *are topologically conjugate and* G *is non solvable. Let* h *be the homeomorphism that conjugates* G *and* G'*. If* h *is orientation preserving then* h *is holomorphic.*

Theorem 2.9 (Cerveau, Moussu [7]) *Suppose that* G *and* G' *are non commutative, non exceptional and formally conjugates. Then the formal conjugacy is convergent.*

3 Finite Groups

The finite group of germs of holomorphic diffeomorphisms in one dimension appears naturally in the study of the holonomy group of the local foliation with holomorphic first integral around a singular point.
Other question that relates finite groups with holomorphic foliation is the question posed by Haefliger: when a compact foliation is a stable foliation? i.e. the leaf space is Hausdorff?. Partial answers to this problem arose in the works of Eptein, Edwards-Millet-Sullivan, Kaup, Holmann, etc. The stability of complex 1-codimension holomorphic foliation on a complex spaces has been investigated by Kaup and Holmann. One of the equivalent condition for stability is finiteness of holonomy groups of the leaves. In this section we give a criteria for finiteness of finitely generated subgroups of $\mathrm{Diff}(\mathbb{C}^2, 0)$, using some topological and analytic properties. In addition, we find a relationship between a finite group and a complete set of first integrals. Last, we construct a moduli space of every finite groups that are topologically conjugate with one generator finite group.

Proposition 3.1 *Let $\mathcal{H}$ be a finite subgroup of $\widehat{\mathrm{Diff}}(\mathbb{C}^n,0)$ (resp. $\mathrm{Diff}(\mathbb{C}^n,0)$) then $\mathcal{H}$ is formally (analytic) linearizable, and it is isomorphic to a finite subgroup of $Gl(n,\mathbb{C})$.*

Proof: Let $g(X) = \sum_{H\in\mathcal{H}} (H'(0))^{-1}H(X)$. Note that g is a diffeomorphism because $g'(0) = \#(\mathcal{H})I$, and moreover for all $F \in \mathcal{H}$

$$\begin{aligned} g\circ F(X) &= \sum_{H\in\mathcal{H}} (H'(0))^{-1}H\circ F(X) \\ &= F'(0)\sum_{H\in\mathcal{H}} ((H\circ F)'(0))^{-1}H\circ F(X) \\ &= F'(0)g(X) \end{aligned}$$

Thus $g\circ F\circ g^{-1}(X) = F'(0)X$. In fact, we obtain a injective groups homomorphism

$$\begin{aligned} &\mathcal{H} \xrightarrow{\Lambda} Gl(n,\mathbb{C}) \\ &F \longrightarrow (g\circ F\circ g^{-1})'(0). \end{aligned}$$ □

Denote by $\Lambda_{\mathcal{H}} \subset Gl(n,\mathbb{C})$ the group of linear parts of the diffeomorphisms in $\mathcal{H}$, and $p = \#(\mathcal{H})$, then for all matrix $A \in \Lambda_{\mathcal{H}}$, $A^p = I$. We claim that A is diagonalizable, in fact, for the Jordan canonical form theorem there exists $B \in Gl(n,\mathbb{C})$ such that $BAB^{-1} = D + N$ where D is a diagonal matrix and N is such that $ND = DN$ and $N^l = 0$ for some $l \in \mathbb{N}$, but $A^p = I$ then $(D + N)^p = I$. To prove the claim we need the following lemma

Lemma 3.1 *Let l be the least integer such that $N^l = 0$, then $I, N, \ldots, N^{l-1}$ are $\mathbb{C}$-linearly independent.*

Proof: Let $a_mN^m + \cdots + a_{l-1}N^{l-1} = 0$ be a linear combination with $a_m \neq 0$ $m \geq 0$, multiply by N^{l-m-1} we obtain that $a_mN^{l-1} = 0$, and for the minimality of l we conclude that $a_m = 0$, it is a contradiction. □

We can suppose, that each block of the Jordan canonical form is of the form $D = \lambda I$, thus $(D+N)^p = (\lambda I+N)^p = I$, therefore $(\lambda^p-1)I+p\lambda^{p-1}N+ \cdots + N^p = 0$, for the lemma we have that $\lambda^p = 1$ and $N = 0$.

One necessary formal condition for $F \in \widehat{\mathrm{Diff}}(\mathbb{C}^n,0)$ in order to have $F^{[m+1]}(X) = X$ is that F has no resonant term, in fact if we suppose $F(X) = AX+P_k(X)+ \cdots$ where $A = Diag(\lambda_j)$ and $(Diag(\lambda_j))^{m+1} = I$ then we have that

$$F^{[m+1]} = X + (A^mP_k(X) + A^{m-1}P_k(AX) + \cdots + P_k(A^mX)) + \cdots.$$

Suppose $P_k(X) = (f_1(X), \dots, f_n(X))^t$, where each f_j is a homogeneous polynomial of degree k we have that

$$\lambda_j^m f_j(X) + \lambda_j^{m-1} f_j(\lambda \cdot X) + \cdots + f_j(\lambda^m \cdot X) = 0, \text{ where } \lambda^m X = (\lambda_1^m x_1, \dots, \lambda_n^m x_n)$$

i.e., coefficient to coefficient has to be zero, but $f_j(X) = \sum_{\substack{\alpha\in\mathbb{N}^n \\ |\alpha|=k}} a_{j,\alpha} X^\alpha$, then the coefficient of X^α in that sum is

$$a_{j,\alpha}(\lambda_j^m + \lambda_j^{m-1}(\lambda)^\alpha + \cdots + \lambda_j(\lambda^{m-1})^\alpha + (\lambda^m)^\alpha) = \begin{cases} (m+1) a_{j,\alpha} \lambda_j^m & \text{if } \lambda^\alpha = \lambda_j \\ 0 & \text{if } \lambda^\alpha \neq \lambda_j. \end{cases}$$

In general we have

Theorem 3.1 (Brjuno [14]) *Let $F \in \mathrm{Diff}(\mathbb{C}^n, 0)$ be a diffeomorphism such that $DF(0) = diag(\lambda_1, \dots, \lambda_n)$ is ressonant. Then there exists a formal conjugation h such that*

$$h^* F(X) = \begin{pmatrix} \lambda_1 x_1 + \sum_{m\in S_1} a_{1m} X^m \\ \vdots \\ \lambda_n x_n + \sum_{m\in S_n} a_{nm} X^m \end{pmatrix}$$

where $S_j = \{m \in \mathbb{N}^n | \ \lambda^m - \lambda_j = 0\}$.

In order to show a topological criteria to know when a diffeomorphism is finite we need the following modification of the Lewowicz lemma.

Lemma 3.2 *Let M, $0 \in M$, be a complex analytic variety of $\mathbb{C}^n$ and K a connected component of 0 in $\overline{B}_r(0) \cap M$. Suppose that f is a homeomorphism from K to $f(K) \subset M$ such that $f(0) = 0$. Then there exists $x \in \partial K$ such that the number of iterations $f^m(x) \in K$ is infinity.*

Proof: Denote by $\overline{\mu} = \mu|_K$ and $\mu = \mu|_{\overset{\circ}{K}}$ the number of iteration in K and $\overset{\circ}{K}$. It is easy to see that $\overline{\mu}$ is upper semicontinuous, μ is under semicontinuous and $\overline{\mu}(x) \geq \mu(x)$ for all $x \in \overset{\circ}{K}$. Suppose by contradiction that $\overline{\mu}(x) < \infty$ for all $x \in \partial K$, therefore exists $n \in \mathbb{N}$ such that $\overline{\mu}(x) < n$ for all $x \in \partial K$. Let $A = \{x \in K | \overline{\mu}(x) < n\} \supset \partial K$ and $B = \{x \in \overset{\circ}{K} | \mu(x) \geq n\} \ni 0$ open set, and $A \cap B = \emptyset$ since $\overline{\mu}(x) \geq \mu(x)$.
Using the fact that K is a connected set, there exists $x_0 \in K \setminus (A \cup B)$ i.e $\overline{\mu}(x_0) \geq n > \mu(x_0)$, then the orbit of x_0 intersects the border of K, which is

a contradiction since $\partial K \subset A$ implies $x_0 \in A$. $\square$

Let $f : U \to f(U)$ be a homeomorphism with $f(0) = 0$ and $x \in U$. We denote by $\mathcal{O}_U(f, x)$ the f-orbit of x that do not leave U, i.e. $y \in \mathcal{O}_U(f, x)$ if and only if $\{x, f(x), \ldots, y = f^{[k]}(x)\} \subset U$ or $\{x, f^{[-1]}(x), \ldots, y = f^{[-k]}(x)\} \subset U$ for some $k \in \mathbb{N}$.

Theorem 3.2 *Let $F \in \mathrm{Diff}(\mathbb{C}^2, 0)$. The group generated by F is finite if and only if there exists a neighborhood V of 0, such that $|\mathcal{O}_V(F, X)| < \infty$ for all $X \in V$ and F leaves invariant infinite analytic varieties at 0.*

Proof: ($\Rightarrow$) Let $N = \#\langle F \rangle$ and $h \in \mathrm{Diff}(\mathbb{C}^2, 0)$ such that $h \circ F \circ h^{-1}(x, y) = (\lambda_1 x, \lambda_2 y)$ where $\lambda_1^N = \lambda_2^N = 1$. It is clear than $|\mathcal{O}(F, X)| \leq N$ for all X in the domain of F, and $M_c = \{h(x, y) | x^N - cy^N = 0\}$ is a complex analytic variety invariant by F for all $c \in \mathbb{C}$.
($\Leftarrow$) Without loss of generality we suppose that $V = \overline{B}_r(0)$ where $F(V)$ and $F^{-1}(V)$ are well defined. Let M be a F-invariant complex analytic variety and K_M the connected component of 0 in $M \cap V$. Let $A_1 = K$, $A_{j+1} = K \cap F^{-1}(A_j)$ and C_n the connected component of 0 in A_n. It is clear, by construction that A_n is the set of point of K_M with n or more iterates in K_M. Moreover, since A_n is compact and C_n is compact and connected, it follows that $C_M = \cap_{j=1}^{\infty} C_n$ is compact and connected too, and therefore $C_M = \{0\}$ or C_M is non enumerable.
We claim that $C_M \cap \partial K \neq \emptyset$ and then this is non enumerate. In fact, if $C_M \cap \partial K = \emptyset$ there exists j such that $C_j \cap \partial K = \emptyset$. Let B compact connected neighborhood of C_j such that $(A_j \setminus C_j) \cap B = \emptyset$, therefore for all $X \in \partial B$ we have $\mathcal{O}_K(F, X) < N$, that is a contradiction by the lemma.
In particular, $C = C_{\mathbb{C}^2}$ is a set of point with infinite orbits in V and therefore every point in C is periodic. If we denote $D_n = \{X \in C | F^{n!}(X) = X\}$, it is clear that D_n is a close set and $D_n \subset D_{n+1}$, moreover $C = \cup_{n=1}^{\infty} D_n$, then exists $n \in \mathbb{N}$ such that $C = D_n$. Let $G = F^{n!}$ where it is well defined, observe that C is in the domain U of G and $C \subset \{X \in U | G(X) = X\} = L$. Since L is a complex analytic variety of U that contain C then its dimension is 1 or 2. The case $\dim L = 1$ is impossible because $C_M \subset C \subset L$ for all M analytic variety F-invariant, contradicting that fact that $\mathcal{O}_2$ is Noetherian ring. In the case $\dim L = 2$ follows that $F^{n!}(X) = X$ for all $X \in U$, therefore $\langle F \rangle$ is finite. $\square$

This theorem can be extended to $\mathbb{C}^n$ if we suppose that there exist infinitely many one dimensional invariant analytic varieties at 0 in general position.

Definition 3.1 *Let $\mathcal{G} < \mathrm{Diff}(\mathbb{C}^n, 0)$ be a group of germs of diffeomorphisms.*

- *a) $\mathcal{G}$ is called* **periodic** *if every element $F \in \mathcal{G}$ has finite order.*
- *b) $\mathcal{G}$ is called* **locally finite** *if every subgroup finitely generated is finite.*
- *c) $\mathcal{G}$ is called* **a group of exponent** *d, if $F^{[d]} = id$, for every element $F \in \mathcal{G}$.*

Proposition 3.2 *Let $\mathcal{G}$ be a periodic group, then the homomorphism*

$$\begin{aligned} \mathcal{G} &\xrightarrow{\Lambda} Gl(n, \mathbb{C}) \\ G &\longmapsto G'(0), \end{aligned}$$

is injective.

Proof: In fact, suppose by contradiction that there exists $F \in \mathcal{G}$ such that $\Lambda(F(X)) = I$, i.e., $F(X) = X + P_k(X) + \cdots$, then, by straightforward calculation we have that $F^{[r]}(X) = X + rP_k(X) + \cdots \neq Id$, therefore, the unique element of $\mathcal{G}$ tangent to the identity is itself. □

Corollary 3.1 *Let $\mathcal{G} < \mathrm{Diff}(\mathbb{C}^n, 0)$ such that, every element $F \in \mathcal{G}$ has finite order, then $\mathcal{G}_1 \leq \mathcal{G}$ the subgroup of element tangent to the identity is trivial, i.e., $\mathcal{G}_1 = \{id\}$.*

Proposition 3.3 *Let $\mathcal{G} < \mathrm{Diff}(\mathbb{C}^n, 0)$*

- *a) If there exists $p \in \mathbb{N}^*$ such that, for every $F \in \mathcal{G}$, $F^p = id$, then $\mathcal{G}$ is finite.*
- *b) If $\mathcal{G}$ is periodic then every finitely generated subgroup is finite.*

For the proposition above we only need to prove that the item a) and b) are true for the group $\Lambda_{\mathcal{G}} < Gl(n, \mathbb{C})$. This fact follows from the next theorems

Theorem 3.3 (Burnside) *If $G < Gl(n, \mathbb{C})$ is a group with finite exponent m, then $|G| < m^{n^3}$.*

Proof: Since the trace of g is the sum of the eigenvalues of g and they are m-th roots of unity, we have that trace(g) can take m^n or less values. In the first case we are going to suppose that there exist $g_1, \dots g_{n^2}$ in G linearly independent over $\mathbb{C}$, i.e. $M_{n\times n}(\mathbb{C}) = g_1\mathbb{C} \oplus \cdots \oplus g_{n^2}\mathbb{C}$. Let

$$\begin{array}{cccc} Tr: & G & \longrightarrow & \mathbb{C}^{n^2} \\ & g & \mapsto & (tr(g_1 g), tr(g_2 g), \dots, tr(g_{n^2} g)) \end{array},$$

where $tr(g)$ is the trace of g. Observe that Tr is one to one. In fact, if $Tr(h_1) = Tr(h_2)$ then $tr(g_i h_1) = tr(g_i h_2)$ for $i = 1, \dots n^2$. Since $g_1, \dots, g_{n^2}$ generate $M_{n\times n}(\mathbb{C})$ and tr is a linear map we have that $tr(ah_1) = tr(ah_2)$ for every $a \in M_{n\times n}(\mathbb{C})$, therefore $h_1 = h_2$. But $|Tr(G)| \le |tr(G)|^{n^2} \le m^{n^3}$, this conclude the proof of the theorem. □

Theorem 3.4 (Schur) *If $G < Gl(n, \mathbb{C})$ is a periodic group then G is locally finite.*

Proof: Suppose G is finitely generated. By the Burnside theorem we only need to prove that G has finite exponent. Let $g_1, \dots g_l$ generate G and K the subfield of $\mathbb{C}$ generated by the entries of these matrices, then $G < Gl_n(K)$. Since K is finitely generated, there exists a finite transcendence basis $x_1, \dots, x_m$ of K over $\mathbb{Q}$. Let E the field generated by $x_1, \dots, x_m$. From the definition $[K : E] = d < \infty$. It is easy to prove that $K \simeq Gl_d(E)$ and then G is isomorphic to a subgroup H of $Gl_{nd}(E)$.
For each $g \in H$, let $F_g(X) \in E[X]$ the monic minimal polynomial for g. Since g has finite order, the zeros of $F_g(X)$ are all roots of 1. Thus the coefficients of $F_g(X)$ are sums of roots of the unity and hence algebraic integers. By the definition of E, the set of algebraic numbers in E is $\mathbb{Q}$, then the coefficient of $F_g(X)$ are integers. Moreover, if $F_g(X) = \sum_{j=0}^{m} a_j X^j$, it is clear that $m \le nd$ and $|a_j| < \binom{nd}{j}$. Hence there are only a finite number of different polynomial in the set $\{F_g(X) |\ g \in H\}$. Since two elements g_1 and g_2 has the same minimal polynomial implies that they have the same order, we conclude that there are a finite numbers of orders and then H has a finite exponent. □

Corollary 3.2 *Let $\mathcal{F}$ be a regular codimension two foliation of M^n and $L \subset M$ be a compact leaf such that*

- *No other leaf accumulate at L.*
- *There exist infinitely many codimension one subvarieties N invariant by $\mathcal{F}$, such that $L \subset N \subset M$.*

Then

1. *The holonomy group of the leaf L is finite.*
2. *There exists a local first integral in some neighborhood of L.*

Definition 3.2 *Let $\mathcal{G} < \mathrm{Diff}(\mathbb{C}^n, 0)$. A function $F : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is called a first integral if $F \circ G(X) = F(X)$ for all $G \in \mathcal{G}$ in some neighborhood of 0.*

Theorem 3.5 *Let $\mathcal{G} < \mathrm{Diff}(\mathbb{C}^n, 0)$. $\mathcal{G}$ is a finite group if and only if there exist $F_1, \ldots, F_n$ germs of holomorphic first integrals such that 0 is an isolated point of $F_1^{-1}(0) \cap \cdots \cap F_n^{-1}(0)$.*

Proof: ($\Rightarrow$) If $\mathcal{G}$ is a finite group, there exists a holomorphic change of coordinates h such that for all $G \in \mathcal{G}$, $h \circ G \circ h^{-1}(X) = G'(0)X$, i.e. $h \circ \mathcal{G} \circ h^{-1} = \Lambda_{\mathcal{G}}$. Claim: there exist an invertible matrix A such that for each choice of vectors $v_1, \ldots, v_n$, where v_j is a j row of arbitrary element of $A \cdot \Lambda_{\mathcal{G}}$, they are linearly independent.
Proof of the claim. We are going to construct the matrix $A = (a_1, \ldots, a_n)^t$ row by row. Let $a_1 = e_1 = (1, 0 \ldots, 0)$. Suppose that we have already constructed the rows $a_1, \ldots a_{j-1}$ and the matrix $A_{j-1} = (a_1, \ldots, a_{j-1}, e_j, \ldots, e_n)^t$ is a lower-triangular invertible matrix.
Let $S = \{(v_1, \ldots, v_{j-1})|$ where v_k is a k-row of arbitrary matrix in $A_{j-1}\Lambda_{\mathcal{G}}\}$. Notice that the set $U = \bigcup_S \mathbb{C}v_1 \oplus \cdots \oplus \mathbb{C}v_{j-1}$ is a finite union of $(j-1)$-dimensional subspaces of $\mathbb{C}^n$. Then for each element $B \in A_{j-1}\Lambda_{\mathcal{G}}$ the set

$$U_B = \{a \in \mathbb{C}^n |\ a = (b_1, \ldots, b_j, 0, \ldots, 0),\ a \cdot B \notin U,\ b_j \neq 0\}$$

is a dense open subset of $\mathbb{C}^j \times \vec{0}$, it follows that

$$U_j = \bigcap_{B \in A_{j-1}\Lambda_{\mathcal{G}}} U_B \neq \emptyset,$$

then choosing a_j like an arbitrary element of U_j, we conclude the claim.

Returning with the proof of the theorem, see that

$$\tilde{F}_i(X) = \prod_{G \in \Lambda_{\mathcal{G}}} \pi_j(A \cdot G(X))$$

is a holomorphic first integral of $\Lambda_{\mathcal{G}}$, where π_i is a projection in the i coordinate. Moreover, from the claim, 0 is the unique solution of the systems of equations $\{\tilde{F}_i(X) = 0\}_{i=1,\ldots,n}$. Now, it is easy to see that $F_i = \tilde{F}_i \circ h$ $(i = 1, \ldots, n)$, are holomorphic first integrals of $\mathcal{G}$ and 0 is an isolated point of $F_1^{-1}(0) \cap \cdots \cap F_n^{-1}(0)$.
($\Leftarrow$) Let $G \in \mathcal{G}$, notice that for all points p near to 0, we have that

$$\begin{aligned}\#\{G^j(p)\}_{j\in\mathbb{Z}} &\leq \#(F_1^{-1}(F_1(p)) \cap \cdots \cap F_n^{-1}(F_n(p)) \\ &\leq \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle F_1, \ldots, F_n \rangle} = m < \infty\end{aligned}$$

then $G^{m!}(X) = X$ and from the Burnside theorem it follows that $\mathcal{G}$ is a finite group. □

3.1 Moduli space

For all $F \in \mathrm{Diff}(\mathbb{C}^n, 0)$, let

$$\mathcal{H}_{top}(F) = \{G \in \mathrm{Diff}(\mathbb{C}^n, 0) | G \text{ is topologically conjugate with } F\}$$

and

$$\mathcal{H}_{hol}(F) = \{G \in \mathrm{Diff}(\mathbb{C}^n, 0) | G \text{ is holomorphically conjugate with } F\}.$$

Theorem 3.6 *Let $F \in \mathrm{Diff}(\mathbb{C}^n, 0)$ such that $F^N = Id$. Then*

$$\frac{\mathcal{H}_{top}(F)}{\mathcal{H}_{hol}(F)} \simeq \pm \frac{SL(n,\mathbb{Z})}{SL_A(n,\mathbb{Z})},$$

where $SL(n,\mathbb{Z})$ is the special linear group of $n \times n$ matrices over the ring $\mathbb{Z}$, $A = DF(0) = \begin{pmatrix} e^{2\pi i \frac{p_1}{N}} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{2\pi i \frac{p_n}{N}} \end{pmatrix}$ *and*

$$SL_A(n,\mathbb{Z}) = \left\{ B \in SL(n,\mathbb{Z}) | (B - I) \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix} \in N\mathbb{Z}^n \right\}.$$

Proof: We only prove in the case $n = 2$, because the general case is similar. Suppose that $G \in \mathrm{Diff}(\mathbb{C}^2, 0)$ is topologically conjugate with F, i.e. there exists $H : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ local homeomorphism such that $F \circ H = H \circ G$ in some neighborhood of 0. Since F has finite order, we can suppose, making a holomorphic change of coordinates, that F is a linear diagonal transformation, i.e. $F(x, y) = (e^{2\pi i \frac{p_1}{N}} x, e^{2\pi i \frac{p_2}{N}} y)$ where $g.c.d(p_1, p_2, N) = 1$ and $0 \le p_1, p_2 < N$. It is clear that the order of $G = H \circ F \circ H^{-1}$ is N too, then, we can suppose in the same way that $G(x, y) = (e^{2\pi i \frac{r_1}{N}} x, e^{2\pi i \frac{r_2}{N}} y)$.
Let $h := H|_{S^1 \times S^1}$, $f = F|_{S^1 \times S^1}$ and $g = G|_{S^1 \times S^1}$. If $h = (h_1, h_2)$, since the norm of h_1 and h_2 are invariant by F and G, we can suppose without loss of generality that $h(S^1 \times S^1) = S^1 \times S^1$ and then the following diagram commute

$$\begin{array}{ccc} S^1 \times S^1 & \xrightarrow{h} & S^1 \times S^1 \\ \downarrow^f & & \downarrow^g \\ S^1 \times S^1 & \xrightarrow{h} & S^1 \times S^1. \end{array}$$

Let $\Pi : \mathbb{R}^2 \to S^1 \times S^1, \quad (z_1, z_2) \mapsto (e^{2\pi i z_1}, e^{2\pi i z_2})$ be a universal covering of $S^1 \times S^1$, $A(x, y) = (x, y) + (\frac{p_1}{N}, \frac{p_2}{N})$, $B(x, y) = (x, y) + (\frac{r_1}{N}, \frac{r_2}{N})$ lifting of f and g respectively and C lifting of h such that $C(0, 0) = (0, 0)$. It is clear that

$$\Pi(C \circ A(z_1, z_2)) = \Pi(B \circ C(z_1, z_2)),$$

and therefore $C \circ A(z_1, z_2) - B \circ C(z_1, z_2) \in \mathbb{Z}^2$.
Claim: there exist vectors v_1, v_2 such that the net $\mathbb{Z}v_1 + \mathbb{Z}v_2$ is invariant for the A-action and the translations $(z_1, z_2) \mapsto (z_1 + 1, z_2)$ and $(z_1, z_2) \mapsto (z_1, z_2 + 1)$. Moreover, if V is the parallelogram $V = \{\theta_1 v_1 + \theta_2 v_2 | 0 \le \theta_1, \theta_2 < 1\}$ then $\Pi|_V : V \to \Pi(V) = W$ is injective and

$$S^1 \times S^1 = \bigsqcup_{j=0}^{N-1} f^j(W).$$

In fact, let be $m = g.c.d(N, p_2)$, $v_1 = (\frac{1}{m}, 0)$ and $v_2 = (\frac{a}{N}, \frac{m}{N})$ where a is the solution of $\frac{p_2}{m} a \equiv p_1 \pmod{\frac{N}{m}}$ with $0 \le a < \frac{N}{m}$. It is easy to check that

$$A(0, 0) = bv_1 + \frac{p_2}{m} v_2 \quad \text{ where } b = \frac{p_1 m - a p_2}{N} \in \mathbb{Z},$$

$(1, 0) = m v_1$ and $(0, 1) = -a v_1 + \frac{N}{m} v_2$. In addition, $V \subset [0, 1)^2$ then $\Pi|_V$ is injective.

Now, we only need to prove that $S^1 \times S^1 = \bigsqcup_{j=0}^{N-1} f^j(W)$. First we are going to prove that this is a disjoint union. In fact if $f^j(W) \cap f^{k+j}(W) \neq \emptyset$ then $W \cap f^k(W) \neq \emptyset$.
Let $(x_0, y_0) \in W$ such that $f^k(x_0, y_0) \in W$. If $(x_0, y_0) = \Pi(\theta_1 v_1 + \theta_2 v_2)$ where $0 \leq \theta_1, \theta_2 < 1$, then $A^k(\theta_1 v_1 + \theta_2 v_2) = (bk + \theta_1)v_1 + (\frac{kp_2}{m} + \theta_2)v_2$, it follows that there exist integers s, t such that $A^k(\theta_1 v_1 + \theta_2 v_2) + (s, t) \in V$ i.e.

$$0 \leq bk + \theta_1 + ms - at < 1, \quad 0 \leq \frac{kp_2}{m} + \theta_2 + \frac{N}{m}t < 1,$$

it is equivalent to say

$$bk + ms - at = 0, \quad \frac{kp_2}{m} + \frac{N}{m}t = 0.$$

For the second equation we know that k has to be a multiple of $\frac{N}{m}$, i.e. $k = k'\frac{N}{m}$ and then $t = -k'\frac{p_2}{m}$. Replacing in the first equation we obtain $p_1 k' + ms = 0$, but p_1 and m have not common factors different that 1, it follows that $m|k'$ and k is a multiple of N, i.e. k has to be 0. Now, it is easy to see that the union is all $S^1 \times S^1$ since the area of V is $\frac{1}{N}$.
Now, since $\mathbb{Z}v_1 + \mathbb{Z}v_2$ is invariant by the A-action, and the translations $(z_1, z_2) \mapsto (z_1 + 1, z_2)$ and $(z_1, z_2) \mapsto (z_1, z_2 + 1)$, then $\mathbb{Z}C(v_1) + \mathbb{Z}C(v_2)$ is invariant for the B-action and the translations $(z_1, z_2) \mapsto (z_1 + 1, z_2)$ and $(z_1, z_2) \mapsto (z_1, z_2 + 1)$. In addition, from the continuity of h it is clear that $C(n_1 v_1 + n_2 v_2) = n_1 C(v_1) + n_2 C(v_2)$, i.e. $C|_{\mathbb{Z}v_1 + \mathbb{Z}v_2}$ is a linear transformation, and since $[0, 1)^2$ is a fundamental domain of $S^1 \times S^1$ then $C([0, 1)^2)$ is also a fundamental domain. Notice that $C(1, 0) = (m_{11}, m_{21}) \in \mathbb{Z}^2$ and $C(0, 1) = (m_{12}, m_{22}) \in \mathbb{Z}^2$, let $M = (m_{ij})$ be the representation of $C|_{\mathbb{Z}v_1 + \mathbb{Z}v_2}$ in the canonical base, it is easy to see that $M \in \pm SL(2, \mathbb{Z})$ since the area of $C([0, 1)^2)$ is one, and since

$$\begin{aligned} M(A(0,0)) - B(M(0,0)) &= M(\frac{p_1}{N}, \frac{p_2}{N}) - (\frac{r_1}{N}, \frac{r_2}{N}) \\ &= (\frac{p_1}{N}m_{11} + \frac{p_2}{N}m_{12} - \frac{r_1}{N}, \frac{p_1}{N}m_{21} + \frac{p_2}{N}m_{22} - \frac{r_2}{N}) \in \mathbb{Z}^2, \end{aligned}$$

we conclude that (r_1, r_2) is determinated by one element

$$[M] \pmod{N} \in \pm \frac{SL(2, \mathbb{Z})}{SL_A(2, \mathbb{Z})}.$$

Finally, for each element $M \in \pm\frac{SL(2,\mathbb{Z})}{SL_A(2,\mathbb{Z})}$ it is easy to see that a linear holomorphic diffeomorphism G

$$(x,y) \mapsto (e^{2\pi i\frac{r_1}{N}}x, e^{2\pi i\frac{r_2}{N}}y) \quad \text{where} \quad \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} \equiv M \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \pmod N,$$

and the C^∞- diffeomorphism H

$$H(x,y) = (|x|e^{m_{11}\arg(x)+m_{12}\arg(y)}, |y|e^{m_{21}\arg(x)+m_{22}\arg(y)})$$

satisfies $H \circ F = G \circ H$ (here we suppose F linear). □

Corollary 3.3 *Let $F, G \in \mathrm{Diff}(\mathbb{C}^n, 0)$ such that they are topologically conjugates and $F^N = I$, then they are C^∞ conjugate.*

Proof: Let h_1, h_2 holomorphism diffeomorphisms such that $F_1 = h_1 \circ F \circ h^{-1}$ and $G_1 = h_1 \circ G \circ h^{-1}$ are linear transformations. From the theorem there exists a matrix $M = (m_{ij}) \in \pm SL(n,\mathbb{Z})$ such that the C^∞-diffeomorphism $H(X) = \left(|x_1|e^{\sum_{j=1}^n m_{1j}\arg(x_j)}, \ldots, |x_n|e^{\sum_{j=1}^n m_{nj}\arg(x_j)}\right)$ conjugate F_1 and G_1. Therefore $h_2^{-1} \circ H \circ h_1$ conjugate F and G. □

4 Linearizable groups of diffeomorphisms

Proposition 4.1 *Let $\mathcal{G} < \mathrm{Diff}(\mathbb{C}^n, 0)$ be an abelian subgroup of analytic diffeomorphism germs. There exists a linear change of coordinates such that $DG(0)$ is a upper triangular matrix for all $G \in \mathcal{G}$. Moreover, if $\mathcal{G}$ has an element F such that the eigenvalues of $DF(0)$ are different, then we can find a linear change of coordinates such that for all $G \in \mathcal{G}$, $DG(0)$ is a diagonal matrix.*

Proof: See [20]. □

Proposition 4.2 *Let $\mathcal{G} < \mathrm{Diff}(\mathbb{C}^n, 0)$ be an abelian subgroup of analytic diffeomorphism germs. Suppose $\mathcal{G}$ has a element G linearizable, non resonant, then there exists h, formal diffeomorphism, that linearize every element of $\mathcal{G}$.*

Proof: We can suppose without loss of generality that $AX \in \mathcal{G}$ where AX is non resonant. We have to show that each element of $\mathcal{G}$ is linear. In fact, let $F \in \mathcal{G}$ be an arbitrary element. We know $F(AX) = AF(X)$. Rewrite F like $F(X) = \sum_{k=1}^{\infty} F_k(X)$ where $F_k(X)$ is vector of homogeneous polynomials of degree k, and comparing terms of the same degree, we have that $F_k(AX) = AF_k(X)$ for all $k \in \mathbb{N}^*$. We claim that this relation imply $F_k \equiv 0$ for $k \geq 2$. Let $v_1, \dots, v_n$ be a bases of eigenvectors of A, i.e. $Av_j = \lambda_j v_j$, such bases exists because the eigenvalues are different. By the fact that $\{v_j\}$ is bases and F_k is homogeneous of degree k,

$$F_k(v_1 y_1 + \cdots + v_n y_n) = \sum_{l=1}^{n} p_l(y_1, \dots, y_n) v_l,$$

where p_l is homogeneous polynomial of degree k for all $l = 1, \dots n$. Thus by the commutativity with A

$$F_k(A(v_1 y_1 + \cdots + v_n y_n)) = AF_k(v_1 y_1 + \cdots + v_n y_n)$$
$$\sum_{l=1}^{n} p_l(\lambda_1 y_1, \dots, \lambda_n y_n) v_l = \sum_{l=1}^{n} p_l(y_1, \dots, y_n) \lambda_l v_l,$$

$p_l(\lambda_1 y_1, \dots, \lambda_n y_n) = \lambda_l p_l(y_1, \dots, y_n)$ for $l = 1, \dots, n$. Since this is a polynomial equality, comparing monomial to monomial, if $a_m y^m$ is a monomial of p_l, $m \in \mathbb{N}^n, |m| = k$ then $a_m \lambda^m y^m = a_m \lambda_l y_m$, but A is non resonant, therefore $a_m = 0$ and $p_l \equiv 0$ for all $l = 1, \dots, n$. □

Proposition 4.3 *Let $F \in \mathrm{Diff}(\mathbb{C}^n, 0)$ such that $A = DF(0)$ is a diagonalizable matrix and the eigenvalues have norm 1. Then F is analytically linearizable if and only if F is topologically linearizable.*

Proof: Suppose that F is topologically linearizable and A is a diagonal matrix, i.e. there exists a homeomorphism $h : U \to h(U)$ where U is a neighborhood of $0 \in \mathbb{C}^n$, such that $h \circ F(X) = Ah(X)$ where $X \in F^{-1}(U) = W$.
Claim: there exists a neighborhood $V \subset W$ of $0 \in \mathbb{C}^n$ such that $F(V) = V$. In fact, let $r > 0$ such that $B(0, r) \subset h(U) \cap W$ and $V = h^{-1}(B(0, r))$, it is clear that

$$F(V) = h^{-1}(Ah(V)) = h^{-1}(A(B(0, r))) = h^{-1}(B(0, r)) = V.$$

Since $H_m(X) = \frac{1}{m}\sum_{j=0}^{m-1} A^{-j}F^j(X)$ is a normal family of holomorphic diffeomorphisms defined from V to the smaller Reinhardt domain that contains V, then

$$H(X) = \lim_{m\to\infty} \frac{1}{m}\sum_{j=0}^{m-1} A^{-j}F^j(X)$$

is holomorphic in V, with $H'(0) = Id$, it follows that H is a local diffeomorphism at $0 \in \mathbb{C}^n$. Moreover, notice that

$$H_m(F(X)) = AH_m(X) + \frac{1}{m}(A^{m-1}F^m(X) - X)$$

and $\lim_{m\to\infty} \frac{|A^{m-1}F^m(X)-X|}{m} = 0$. We conclude therefore that $H(F(X)) = AH(X)$ for all $X \in V$. □

Notice that in the case when F has finite order, the moduli space topological vs analytical is very simple. This is a particular case when the eigenvalues have norm 1. In fact,

Theorem 4.1 *Let be $F \in \mathrm{Diff}(\mathbb{C}^n, 0)$, where $A = DF(0)$ is a diagonalizable matrix with norm 1 eigenvalue, i.e. $A = diag(e^{2\pi i\lambda_1}, \ldots, e^{2\pi i\lambda_n})$ where $\lambda_j \in \mathbb{R}$. Suppose that F is topologically linearizable. Then*

$$\frac{\mathcal{H}_{top}(F)}{\mathcal{H}_{hol}(F)} \simeq \pm\frac{SL(n,\mathbb{Z})}{SL_A(n,\mathbb{Z})}$$

where

$$SL_A(n,\mathbb{Z}) = \{B \in SL(n,\mathbb{Z}) | (B - I)\lambda \in \mathbb{Z}^n\}.$$

Proof: From the proposition 4.3, we know that every element of $\mathcal{H}_{top}(F)$ is analytically linearizable. Let $G \in \mathcal{H}_{top}(F)$, H local homeomorphism such that $F \circ H = H \circ G$ and C be a lifting of $H|_{S^1\times\cdots\times S^1}$ as in the theorem 3.6. Observe that $C|_{\mathbb{Z}^n} : \mathbb{Z}^n \to \mathbb{Z}^n$ can be represented as a linear transformation, and since the volume of $C([0,1]^n)$ is 1, then $C|_{Z^n} = M \in \pm SL(n,\mathbb{Z})$. Therefore, $C(Z) = M([\![Z]\!]) + \theta(\{Z\})$, where $[\![Z]\!] = ([\![z_1]\!], \ldots, [\![z_n]\!])$, $\{Z\} = Z - [\![Z]\!]$ and $\theta = C|_{[0,1)^n}$. In addition,

$$C(Z) + \mu + C(Z + \lambda) \in \mathbb{Z}^n$$

where $\mu = (\mu_1, \ldots, \mu_n)$ and $DG(0) = diag(e^{2\pi i\mu_1}, \ldots, e^{2\pi i\mu_n})$. In particular, making $Z = 0$ we have

$$C(j\lambda) - j\mu \in \mathbb{Z}^n, \quad \text{for all } j \in \mathbb{Z}$$

and replacing μ by $\mu + m$ for some $m \in \mathbb{Z}^n$, we can suppose that $C(j\lambda) = j\mu$. It follows that

$$M([\![j\lambda]\!]) - [\![j\mu]\!] = \{j\mu\} - \theta(\{j\lambda\}).$$

From the continuity of θ and using that there exists a positive integers increasing sequence $\{a_n\}_{n\in\mathbb{N}}$ such that $\{a_n\lambda\} \underset{n\to\infty}{\longrightarrow} 0$ and $\{a_n\mu\} \underset{n\to\infty}{\longrightarrow} 0$, we have that $M([\![a_n\lambda]\!]) = [\![a_n\mu]\!]$ for all $n \gg 0$, therefore

$$M\lambda = \lim_{n\to\infty} \frac{1}{a_n} M([\![a_n\lambda]\!]) = \lim_{n\to\infty} \frac{1}{a_n} [\![a_n\mu]\!] = \mu.$$

Thus, if we have two homeomorphism that conjuge F and G, each one has associated a matrix $M_1, M_2 \in SL(n, \mathbb{Z})$ such that $M_1\lambda - \mu \in \mathbb{Z}^n$ and $M_2\lambda - \mu \in \mathbb{Z}^n$, then $(M_1 M_2^{-1} - I)\lambda \in \mathbb{Z}^n$, as we want to prove. □

Now observe that the radical vector field is invariant by the action of every linear diffeomorphism. This fact characterizes every linearizable group

Theorem 4.2 *$\mathcal{G} \subset \text{Diff}(\mathbb{C}^n, 0)$ is a group analytically linearizable if and only if there exists a vector field $\mathcal{X} = \vec{R} + \cdots$, where $\vec{R}$ is a radial vector field such $\mathcal{X}$ is invariant for every $F \in \mathcal{G}$, i.e. $F^*\mathcal{X} = \mathcal{X}$.*

Proof: ($\Rightarrow$) Suppose that $\mathcal{G}$ is linearizable, i.e. there exists $g : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that $g^{-1} \circ \mathcal{G} \circ g = \{DF(0)Y | F \in \mathcal{G}\}$. Since $(AY)^*\vec{R} = \vec{R}$ for all $A \in Gl(2, \mathbb{C})$, in particular for every element $F \in \mathcal{G}$ we have

$$\begin{aligned}\vec{R} &= (g^{-1} \circ F \circ g)^*\vec{R} = D(g^{-1} \circ F \circ g)_{(g^{-1} \circ F^{-1} \circ g(Y))} \cdot \vec{R}(g^{-1} \circ F^{-1} \circ g(Y)) \\ &= Dg^{-1}_{(g(Y))} \cdot DF_{(F^{-1}(g(Y))} \cdot Dg_{((g^{-1} \circ F^{-1} \circ g(Y))} \cdot (g^{-1} \circ F^{-1} \circ g(Y))\end{aligned}$$

Replacing $X = g(Y)$ and multiplying by $Dg_{g^{-1}(Y)}$ we have that

$$DF_{(F^{-1}(X)} \cdot Dg_{((g^{-1} \circ F^{-1}(X))} \cdot (g^{-1} \circ F^{-1}(X)) = Dg_{g^{-1}(Y)} g^{-1}(X),$$

i.e. denoting $\mathcal{X} = Dg_{g^{-1}(X)} g^{-1}(X)$ we have that $F^*\mathcal{X} = \mathcal{X}$. It is easy to see that $\mathcal{X} = \vec{R} + \cdots$.

($\Leftarrow$) Now suppose that $F^*\mathcal{X} = \mathcal{X}$ for every element $F \in \mathcal{G}$, where $\mathcal{X} = \vec{R} + \cdots$. Since every eigenvalue of the linear part of $\mathcal{X}$ is 1, then $\mathcal{X}$ is in the Poincaré domain without resonances, therefore there exists a analytic diffeomorphism $g : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that $g^*\mathcal{X} = \vec{R}$, i.e. $\mathcal{X} = (Dg_{(X)})^{-1} \cdot g(X)$.
We claim that $g \circ F \circ g^{-1}(Y) = DF(0)Y$ for every $F \in \mathcal{G}$. In fact, from the same procedure as before we can observe that

$$(g \circ F \circ g^{-1})^* \vec{R} = \vec{R}.$$

Now, if we suppose that $g \circ F \circ g^{-1} = AX + P_l(X) + P_{l+1}(X) + \cdots$, where $P_j(X)$ is a polynomial vector field of degree j, then it is easy to see using the Euler equality that

$$(g \circ F \circ g^{-1})^* \vec{R} = AX + lP_l(X) + (l+1)P_{l+1}(X) + \cdots,$$

and therefore $P_j(X) \equiv 0$ for every $j \geq 2$. $\square$

5 Groups with some algebraic structure

In this section we are going to study the groups of diffeomorphisms, when they have some additional algebraic structure as abelian, nilpotent or solvable.
Let $\mathcal{G} < \mathrm{Diff}(\mathbb{C}^n, 0)$. The upper central series

$$Z_0 = \{id\} \subset Z_1(\mathcal{G}) \subset \cdots \subset Z_n(\mathcal{G}) \subset \cdots$$

of $\mathcal{G}$ is defined inductively where $Z_{j+1}(\mathcal{G})/Z_j(\mathcal{G})$ is the center of $\mathcal{G}/Z_j(\mathcal{G})$. The group $Z_j(\mathcal{G})$ is called the **j-th hypercenter** of $\mathcal{G}$. $\mathcal{G}$ is called **nilpotent** if $\mathcal{G} = Z_l(\mathcal{G})$ for some l. The smallest l for which $\mathcal{G} = Z_l(\mathcal{G})$ is the *nilpotency class* of $\mathcal{G}$.
In the same way for any subalgebra $\mathcal{L} \subset \hat{\chi}(\mathbb{C}^n, 0)$, we define $Z_j[\mathcal{L}]$ the jth hypercenter of $\mathcal{L}$ inductively such that $Z_{j+1}(\mathcal{L})/Z_j(\mathcal{L})$ is the center of $\mathcal{L}/Z_j(\mathcal{L})$ and we say that $\mathcal{L}$ is nilpotent of class l if l is the smallest integer such that $Z_l(\mathcal{L}) = \mathcal{L}$.
The lower central series

$$Z^0(\mathcal{G}) = \mathcal{G} \supset Z^1(\mathcal{G}) \supset \cdots \supset Z^j(\mathcal{G}) \supset \cdots$$

of $\mathcal{G}$ is also defined inductively like $Z^{j+1}(\mathcal{G}) = [\mathcal{G}, Z^j(\mathcal{G})]$ where $[\mathcal{G}, Z^j(\mathcal{G})]$ is the commutator subgroup of $\mathcal{G}$ and $Z^j(\mathcal{G})$, i.e. the group generated from the elements $\{f \circ g \circ f^{-1} \circ g^{-1} | f \in \mathcal{G},\ g \in Z^j(\mathcal{G})\}$. It is known that $\mathcal{G}$ is nilpotent of class l if and only if $Z^l(\mathcal{G}) = \{id\}$.
Let $\mathcal{G}$ be a group generated by $\{g_1, \ldots, g_i, \ldots\}$. Define $S_0 = \mathcal{G}$ and for all $j \in \mathbb{N}$ define S_{j+1} the set of commutators $[g, h] = ghg^{-1}h^{-1}$ with $g \in S_0$ and $h \in S_j$. It is known that if S_k is the identity element of $\mathcal{G}$ for some integer k then $\mathcal{G}$ is a nilpotent group of height k.
The commutator series

$$\mathcal{G}^0 = \mathcal{G} \supset \mathcal{G}^1 \supset \cdots \supset \mathcal{G}^j \supset \cdots$$

is defined inductively where $\mathcal{G}^j = [\mathcal{G}^{j-1}, \mathcal{G}^{j-1}]$ is the j**-th commutator subgroup**. $\mathcal{G}$ is called **solvable**, if there exists a positive integer l such that $\mathcal{G}^l = \{Id\}$. It is obvious that every nilpotent group is solvable.
Denoting $\mathcal{G}_1 \lhd \mathcal{G}$ the normal subgroup of the diffeomorphisms tangent to the identity, it is easy to see that

$$\mathcal{G}/\mathcal{G}_1 \sim \Lambda_{\mathcal{G}} = \{DG(0) | G \in \mathcal{G}\},$$

therefore $\mathcal{G}$ is solvable if and only if $\Lambda_{\mathcal{G}}$ is solvable and $\mathcal{G}_1$ is solvable.
In addition, if $\mathcal{G}$ is solvable, and let $\{id\} = \mathcal{G}^l \lhd \mathcal{G}^{l-1} \lhd \cdots \lhd \mathcal{G}^1 \lhd \mathcal{G}$ the resolution string, then using the group homomorphism Λ we obtain a new resolution string

$$\begin{array}{ccccccccc} \mathcal{G}^l & \lhd & \mathcal{G}^{l-1} & \lhd & \cdots & \lhd & \mathcal{G}^1 & \lhd & \mathcal{G} \\ \downarrow^{\Lambda} & & \downarrow^{\Lambda} & & \vdots & & \downarrow^{\Lambda} & & \downarrow^{\Lambda} \\ G^l & \lhd & G^{l-1} & \lhd & \cdots & \lhd & G^1 & \lhd & G \end{array}$$

where $G^j = \Lambda(\mathcal{G}^j)$. Denote by $height(\mathcal{G}) = l$ the height of the resolution string of $\mathcal{G}$, then it is clear $height(G) \leq height(\mathcal{G})$. Note that in the case where $G < Gl(n, \mathbb{C})$ is a linear solvable group, it is known that $height(G)$ is limited by a function that only depends on n, $\rho(n) \leq 2n$ (Zassenhaus). In fact,

Theorem 5.1 (Newman)

$$\rho(n) = \begin{cases} 1 & \textit{if } n = 1 \\ 4 & \textit{if } n = 2 \\ 9 + \tau((n-2)/8) & \textit{if } n \in \{6, 7, 17, 59, 60, 61, 62, 63, 64, 65\} \\ 10 + \tau((n-2)/8) & \textit{in other case.} \end{cases}$$

where τ is a function over the positive rational numbers like

$$\tau(q) = \begin{cases} 5a(q) & \textit{where } 9^{a(q)} \le q < 16 \cdot 9^{a(q)-1} \\ 5a(q)+1 & \textit{where } 16 \cdot 9^{a(q)-1} \le q < 3 \cdot 9^{a(q)} \\ 5a(q)+2 & \textit{where } 3 \cdot 9^{a(q)} \le q < 4 \cdot 9^{a(q)} \\ 5a(q) & \textit{where } 4 \cdot 9^{a(q)} \le q < 64 \cdot 9^{a(q)-1} \\ 5a(q) & \textit{where } 64 \cdot 9^{a(q)} \le q < 9^{a(q)+1} \end{cases}$$

where $a(q) = [\![\log_9 q]\!]$.

Let $\mathcal{G}_j = \{G \in \mathcal{G}_1 | \text{where } G \text{ is } k\text{-flat}, k > j\}$. It is easy to see that

$$\cdots \lhd \mathcal{G}_3 \lhd \mathcal{G}_2 \lhd \mathcal{G}_1$$

is a normal series and $\mathcal{G}_j/\mathcal{G}_{j+1}$ are abelian groups.
If $\mathcal{G}_1 \cap \widehat{\text{Diff}}_k(\mathbb{C}^n, 0) = \{id\}$ for some $k \in \mathbb{N}$ the following lemma show that $\mathcal{G}_1$ is solvable, and provide a necessary condition in order to two diffeomorphisms tangent to the identity commute.

Lemma 5.1 *Let* $F \in \widehat{\text{Diff}}_{r+1}(\mathbb{C}^n, 0)$ *and* $G \in \widehat{\text{Diff}}_{s+1}(\mathbb{C}^n, 0)$*, then*

$$F(G(X)) - G(F(X)) = \nabla F_{r+1}(X) \cdot G_{s+1}(X) - \nabla G_{s+1}(X) \cdot F_{r+1}(X) + O(|X|^{r+s+2}).$$

In particular, $[F, G] \in \widehat{\text{Diff}}_{r+s+1}(\mathbb{C}^n, 0)$

Proof: Let $F(X) = X + \sum_{k=r}^{r+s} F_{k+1}(X) + O(|X|^{r+s+2})$ and $G(X) = X + \sum_{k=s}^{r+s} G_{j+1}(X) + O(|X|^{r+s+2})$, then

$$\begin{aligned} F(G(X)) &= X + \sum_{k=s}^{r+s} G_{j+1}(X) + O(|X|^{r+s+2}) + \\ &\quad + \sum_{k=r}^{r+s} F_{k+1}\Big(X + \sum_{k=s}^{r+s} G_{j+1}(X) + O(|X|^{r+s+2})\Big) + O(|X|^{r+s+2}) \\ &= X + \sum_{k=s}^{r+s} G_{j+1}(X) + \sum_{k=r}^{r+s}\Big(F_{k+1}(X) + \nabla F_{k+1}(X) \cdot G_{s+1}(X) \\ &\qquad + O(|X|^{k+s+2})\Big) + O(|X|^{r+s+2}) \\ &= X + \sum_{k=s}^{r+s} G_{j+1}(X) + \sum_{k=r}^{r+s} F_{k+1}(X) + \nabla F_{r+1}(X) \cdot G_{s+1}(X) + O(|X|^{r+s+2}) \end{aligned} \tag{1}$$

In the same way we have

$$G(F(X)) = X+\sum_{k=s}^{r+s} G_{j+1}(X)+\sum_{k=r}^{r+s} F_{k+1}(X)+\nabla G_{s+1}(X)\cdot F_{r+1}(X)+O(|X|^{r+s+2}). \tag{2}$$

The lemma follows subtracting the equations (1) and (2). □

Corollary 5.1 *Let $\mathcal{G}$ be a group of diffeomorphisms and suppose that there exists $l \in \mathbb{N}$ such that $\mathcal{G} \cap \mathrm{Diff}_l(\mathbb{C}^n, 0) = \{id\}$. Then $\mathcal{G}$ is solvable if and only if $\mathcal{G}^{\rho(n)} \subset \mathrm{Diff}_1(\mathbb{C}^n, 0)$, where $\rho(n)$ is the Newman function.*

This is a particular case of the Epstein & Thurston theorem. In fact, it is clear that every finite generated group $\mathcal{G} = \langle G_1, \dots, G_l\rangle \subset \mathrm{Diff}_1(\mathbb{C}^n, 0)$ is a discrete subgroup of the formal Lie Group

$$\overline{\mathcal{G}} = \left\langle f_1^{[t_1]} \circ \cdots \circ f_k^{[t_k]} | t_j \in \mathbb{R} \text{ and } f_j \in \{G_1, \dots, G_l\} \right\rangle.$$

In general the dimension of this Lie group is not finite. In the case that $\overline{\mathcal{G}}$ is finite dimensional we have

Theorem 5.2 (Epstein & Thurston) *Let $\overline{\mathcal{G}}$ be a finite dimensional Lie group as above, then*

1. *If $\mathcal{G}$ is nilpotent, then height$(\overline{\mathcal{G}}) \le n$.*
2. *If $\mathcal{G}$ is solvable, then height$(\overline{\mathcal{G}}) \le n+1$.*

Theorem 5.3 *Let $\mathcal{G}$ be a solvable subgroup of $\widehat{\mathrm{Diff}}(\mathbb{C}^n, 0)$, then $\mathcal{G}^{\rho(n)+1}$ is a nilpotent group.*

Proof: Observe that If $\mathcal{G}$ is a solvable group, then $\mathcal{H} = \mathcal{G}^{\rho(n)} \subset \mathrm{Diff}_1(\mathbb{C}^n, 0)$ is a solvable group. Let $\mathcal{L}$ be the algebra associate to the group H by the exp function. For every integer $i \ge 0$ denote $\mathrm{VF}_i = \frac{\hat{\chi}(\mathbb{C}^n,0)}{\hat{\chi}_i(\mathbb{C}^n,0)}$. Notice that for all $j > i$ there exists a natural projection $\mathrm{VF}_j \to \mathrm{VF}_i$, therefore we can think of $\hat{\chi}(\mathbb{C}^n, 0)$ as a projective limit $\lim\limits_{j\to\infty} \mathrm{VF}_j$, moreover VF_j has a natural structure of finite dimensional complex algebraic linear algebra. Let $[\mathcal{L}]_i$ be the projection of $\mathcal{L}$ in VF_i, and $\overline{[\mathcal{L}]_i}$ the algebra of every linear combination . It follows that $\overline{[\mathcal{L}]_i}$ is a connected solvable Lie algebra of the same height as $\mathcal{L}$. Since $\overline{[\mathcal{L}]_i}$ is isomorphic to a linear Lie algebra, we know from the Lie-Kolchi theorem that they can be represented by upper triangular matrices, then $[\overline{[\mathcal{L}]_i}, \overline{[\mathcal{L}]_i}]$ is represented by nilpotent triangular matrices for every $i \in \mathbb{N}$. It follows that $\mathcal{L}^1 = \lim_{j\to\infty} [\mathcal{L}^1]_j$ is a nilpotent algebra and then $\mathcal{H}^1$ is a nilpotent group.□

Corollary 5.2 *Let $\mathcal{G}$ a solvable subgroup of $\widehat{\mathrm{Diff}}_1(\mathbb{C}^n, 0)$, then $\mathcal{G}^1 = [\mathcal{G}, \mathcal{G}]$ is a nilpotent group.*

From the proposition 2.1 is clear that if $\mathcal{G} \subset \mathrm{Diff}_1(\mathbb{C}^n, 0)$ is solvable (nilpotent) if and only if the algebra associate to $\mathcal{G}$ by the exp function has to be solvable (nilpotent). In particular in dimension 2 we have

Proposition 5.1 *Every nilpotent subalgebra $\mathcal{L}$ of $\hat{\chi}(\mathbb{C}^2, 0)$ is metabelian.*

Proof: Let $\mathcal{R}$ be the center of $\mathcal{L}$. Since $\mathcal{R}$ is non trivial then $\mathcal{R} \otimes \hat{K}(\mathbb{C}^2)$ is a vector space of dimension 1 or 2 over $\hat{K}(\mathbb{C}^2)$, where $\hat{K}(\mathbb{C}^2)$ is the fraction field of $\mathcal{O}(\mathbb{C}^2)$.
In the case when the dimension is 2, there exist formal fields $\mathfrak{f}$ and $\mathfrak{g}$ linearly independent over $\hat{K}(\mathbb{C}^2)$, thus every element $\mathfrak{h} \in \mathcal{L}$, can be written as $\mathfrak{h} = u\mathfrak{f} + v\mathfrak{g}$, and since $\mathfrak{f}, \mathfrak{g} \in \mathcal{R}$ follows $\mathfrak{f}(u) = \mathfrak{f}(v) = \mathfrak{g}(u) = \mathfrak{g}(v) = 0$, i.e. u and v are constants, therefore $\mathcal{L}$ is abelian algebra.
If the dimension of the center is 1, let $\mathfrak{f}$ be a non-trivial element of $\mathcal{R}$, and $\mathcal{S} = \hat{K}(\mathbb{C}^2)\mathfrak{f} \cap \mathcal{L}$, is clear that $\mathcal{R} \subset \mathcal{S}$ is an abelian subalgebra of $\mathcal{L}$. In the case $\mathcal{L} = \mathcal{S}$ we have nothing to proof. Otherwise, since $\mathcal{S}$ is an ideal of $\mathcal{L}$, let $\mathfrak{g}$ be an element of $\mathcal{L}$ such that its image at $\mathcal{L}/\mathcal{S}$ is in the center of $\mathcal{L}/\mathcal{S}$. In the same way every element of $\mathcal{L}$ is of the form $u\mathfrak{f}+v\mathfrak{g}$ where $\mathfrak{f}(u) = \mathfrak{f}(v) = 0$, moreover since $[u\mathfrak{f} + v\mathfrak{g}, \mathfrak{f}] \in \mathcal{S}$, i.e. $\mathfrak{g}(v) = 0$, follows that v is constant, and then $\mathcal{L}$ is a metabelian algebra. □

Corollary 5.3 *Let $\mathcal{G}$ a subgroup of $\mathrm{Diff}(\mathbb{C}^2, 0)$, then*

a) If $\mathcal{G}$ is solvable then $\mathcal{G}^7 = \{id\}$.

b) If $\mathcal{G}$ is nilpotent then $\mathcal{G}^6 = \{id\}$.

Observe that *b*) is weaker that the Ghys theorem

Theorem 5.4 (Ghys) *Let $\mathcal{G}$ be a nilpotent subgroup of $\widehat{\mathrm{Diff}}(\mathbb{C}^2, 0)$, then $\mathcal{G}$ is a metabelian group.*

The proposition 5.1 gives a characterization of abelian subgroup of $\mathrm{Diff}_1(\mathbb{C}^2, 0)$.

Corollary 5.4 *If $\mathcal{G} < \mathrm{Diff}_1(\mathbb{C}^2, 0)$ be a abelian group, then one of the following items are true*

1. $\mathcal{G} < \left\langle \exp(N\mathcal{X}) \in \mathrm{Diff}_1(\mathbb{C}^2, 0) \middle| \begin{array}{c} N \text{ rational holomorphic function} \\ \text{such that } \mathcal{X}(N) = 0 \end{array} \right\rangle$ *where* $\exp(\mathcal{X}) \in \mathcal{G}$.
2. $\mathcal{G} < \langle F^{[t]} \circ G^{[s]} | t, s \in \mathbb{C} \rangle$, *where* $F, G \in \mathcal{G}$ *and* $[F, G] = Id$.

If F and G are two elements of an abelian group of diffeomorphisms tangent to the identity then from the lemma 5.1 we have

$$\nabla F_{r+1}(X) \cdot G_{s+1}(X) - \nabla G_{s+1}(X) \cdot F_{r+1}(X) \equiv 0. \tag{3}$$

It is clear that in the particular case when the dimension is 1 the equation (3) is equivalent to say $r = s$. In general, this is false in dimension > 1, for example $F(X) = \exp(\mathfrak{f})(X) \in \mathrm{Diff}_2(\mathbb{C}^2, 0)$ and $G(X) = \exp(\mathfrak{g})(X) \in \mathrm{Diff}_3(\mathbb{C}^2, 0)$, where

$$\mathfrak{f} = (x^2 + 3xy)\frac{\partial}{\partial x} + (3xy + y^2)\frac{\partial}{\partial y}$$

and

$$\mathfrak{g} = (3x^3 - 5x^2y + xy^2 + y^3)\frac{\partial}{\partial x} + (x^3 + x^2y - 5xy^2 + 3y^3)\frac{\partial}{\partial y}.$$

Since $[\mathfrak{f}, \mathfrak{g}] = 0$ it follows that $F(X)$ and $G(X)$ commute. In addition, F and G are holomorphic diffeomorphisms because $\mathfrak{f}$ and $\mathfrak{g}$ are holomorphic. In fact, we have

Proposition 5.2 *Let $F, G \in \mathrm{Diff}_1(\mathbb{C}^n, 0)$ be commuting diffeomorphism and $\mathfrak{f}$, $\mathfrak{g}$ be formal vector fields such that $F = \exp(\mathfrak{f})$ and $G = \exp(\mathfrak{g})$. Suppose that $\mathfrak{f} = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}$ and $\mathfrak{g} = R\frac{\partial}{\partial x} + S\frac{\partial}{\partial y}$ where P, Q, R, S are formal series. Then the formal variety $V := PS - RQ = 0$ is invariant by the actions of $\mathfrak{f}$ and $\mathfrak{g}$.*

Proof: Since $[\mathfrak{f}, \mathfrak{g}] = 0$ then

$$P\frac{\partial R}{\partial x} + Q\frac{\partial R}{\partial y} = R\frac{\partial P}{\partial x} + S\frac{\partial P}{\partial y} \text{ and } P\frac{\partial S}{\partial x} + Q\frac{\partial S}{\partial y} = R\frac{\partial Q}{\partial x} + S\frac{\partial Q}{\partial y}.$$

It follows that

$$\begin{aligned}\mathfrak{f}(V) &= P(S\frac{\partial P}{\partial x} + P\frac{\partial S}{\partial x} - Q\frac{\partial R}{\partial x} - R\frac{\partial Q}{\partial x}) + Q(S\frac{\partial P}{\partial y} + P\frac{\partial S}{\partial y} - Q\frac{\partial R}{\partial y} - R\frac{\partial Q}{\partial y}) \\ &= (PS - RQ)(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}) = V(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y})\end{aligned}$$

□

Definition 5.1 *Let $F \in \text{Diff}_{r+1}(\mathbb{C}^n, 0)$. F is called* **dicritic** *if $F(X) = X + F_{r+1}(X) + \cdots$, where $F_{r+1}(X) = f(X)X$ and f is a homogeneous polynomial of degree r.*

We are going to prove a generalization of the dimension one classification of abelian group for dicritic diffeomorphisms.

Proposition 5.3 *Let $F \in \widehat{\text{Diff}}_{r+1}(\mathbb{C}^n, 0)$ and $G \in \widehat{\text{Diff}}_{s+1}(\mathbb{C}^n, 0)$. Suppose that F is a dicritic diffeomorphism, and $F(G(X)) = G(F(X))$, then $r = s$ and G is also a dicritic diffeomorphism.*

Proof: For the lemma 5.1 we have

$$\begin{aligned}\nabla F_{r+1}(X)G_{s+1}(X) - \nabla G_{s+1}(X)F_{r+1}(X) &= \\ &= (f(X)I + (x_i\frac{\partial f}{\partial x_j})) \cdot G_{s+1}(X) - \nabla G_{s+1}(X) \cdot f(X)X \\ &= (f(X)I + (x_i\frac{\partial f}{\partial x_j})) \cdot G_{s+1}(X) - (s+1)f(X)G_{s+1}(X) \\ &= (-sf(X)I + (x_i\frac{\partial f}{\partial x_j})) \cdot G_{s+1}(X)\end{aligned} \tag{4}$$

From the identity $\det(aI + AB) = \det(aI + BA)$

$$\det(-sf(X)I + (x_i\frac{\partial f}{\partial x_j}))$$
$$= \begin{vmatrix} -sf(X)+x_1\frac{\partial f}{\partial x_1} & x_2\frac{\partial f}{\partial x_2} & \cdots & x_n\frac{\partial f}{\partial x_n} \\ x_1\frac{\partial f}{\partial x_1} & -sf(X)+x_2\frac{\partial f}{\partial x_2} & \cdots & x_n\frac{\partial f}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ x_1\frac{\partial f}{\partial x_1} & x_2\frac{\partial f}{\partial x_2} & \cdots & -sf(X)+x_n\frac{\partial f}{\partial x_n} \end{vmatrix}$$
$$= \begin{vmatrix} -sf(X) & 0 & \cdots & sf(X) \\ 0 & -sf(X) & \cdots & sf(X) \\ \vdots & \vdots & \ddots & \vdots \\ x_1\frac{\partial f}{\partial x_1} & x_2\frac{\partial f}{\partial x_2} & \cdots & -sf(X)+x_n\frac{\partial f}{\partial x_n} \end{vmatrix}$$
$$= \begin{vmatrix} -sf(X) & 0 & \cdots & 0 \\ 0 & -sf(X) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ x_1\frac{\partial f}{\partial x_1} & x_2\frac{\partial f}{\partial x_2} & \cdots & (r-s)f(X) \end{vmatrix}$$
$$= (-s)^{n-1}(r-s)(f(X))^n$$

So multiplying by the adjoint matrix of $-sf(X)I+(x_i\frac{\partial f}{\partial x_j})$ we obtain $(-s)^{n-1}(r-s)(f(X))^n G_{s+1} = 0$ and then $r = s$. Denoting $G_{r+1} = (g_1, \ldots, g_n)^t$, it is easy to see, multiplying the i row in (4) by x_n and subtracting the last row multiplied by x_i, that $-sf(X)x_n g_i + sf(X)x_i g_n = 0$. Therefore $x_n | g_n$. Defining $g = \frac{g_n}{x_n}$, we conclude $g_i = x_i g$ for all i. □

Theorem 5.5 *Let $\mathcal{G} < \mathrm{Diff}_1(\mathbb{C}^2, 0)$ be abelian group, and $F \in \mathcal{G}$ dicritic diffeomorphism. Suppose that $\exp(\mathfrak{f})(x,y) = F(x,y)$ where*

$$\mathfrak{f} = (f(x,y)x + p_{k+2}(x,y) + \cdots)\frac{\partial}{\partial x} + (f(x,y)y + q_{k+2}(x,y) + \cdots)\frac{\partial}{\partial y},$$

$f(x,y)$ is a homogeneous polynomial of degree k and g.c.d$(f, xq_{k+2}(x,y) - yp_{k+2}(x,y)) = 1$, then

$$\mathcal{G} < \langle \exp(t\mathfrak{f})(x,y) | t \in \mathbb{C} \rangle.$$

Proof: Let $G \in \mathcal{G}$, for the proposition 5.3, G is a $k+1$-flat dicritic diffeomorphism. Let $\mathfrak{g}$ such that $\exp(\mathfrak{g})(x,y) = G$, then

$$\mathfrak{g} = g(X)\vec{R} + (s_{k+2} + \cdots)\frac{\partial}{\partial x} + (t_{k+2} + \cdots)\frac{\partial}{\partial y},$$

where $\vec{R} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ and $X = (x,y)$. Since $[\mathfrak{f}, \mathfrak{g}] = 0$, each j-jet has to be zero. The $2k+2$-jet is $[f\vec{R}, s_{k+2}\frac{\partial}{\partial x} + t_{k+2}\frac{\partial}{\partial y}] + [p_{k+2}\frac{\partial}{\partial x} + q_{k+2}\frac{\partial}{\partial y}, g\vec{R}] = 0$, by a straightforward calculation we have

$$[f\vec{R}, s_{k+2}\frac{\partial}{\partial x} + t_{k+2}\frac{\partial}{\partial y}] = (k+1)f(s_{k+2}\frac{\partial}{\partial x} + t_{k+2}\frac{\partial}{\partial y}) - (s_{k+2}\frac{\partial f}{\partial x} + t_{k+2}\frac{\partial f}{\partial y})\vec{R},$$

therefore

$$(k+1)(fs_{k+2} - gp_{k+2}) = x(s_{k+2}\frac{\partial f}{\partial x} + t_{k+2}\frac{\partial f}{\partial y} - p_{k+2}\frac{\partial g}{\partial x} - q_{k+2}\frac{\partial g}{\partial y})$$
$$(k+1)(ft_{k+2} - gq_{k+2}) = y(s_{k+2}\frac{\partial f}{\partial x} + t_{k+2}\frac{\partial f}{\partial y} - p_{k+2}\frac{\partial g}{\partial x} - q_{k+2}\frac{\partial g}{\partial y})$$

In particular,

$$\frac{fs_{k+2} - gp_{k+2}}{x} = \frac{ft_{k+2} - gq_{k+2}}{y},$$

or equivalently, $f(xs_{k+2} - yt_{k+2}) = g(xq_{k+2} - yp_{k+2})$, but $gcd(f, xq_{k+2} - yp_{k+2}) = 1$, it follows that $f|g$, and since f and g have the same degree then $g = rf$ where $r \in \mathbb{C}$. Substituting g we obtain the system of equations

$$\begin{pmatrix} (k+1)f - x\frac{\partial f}{\partial x} & -x\frac{\partial f}{\partial y} \\ -y\frac{\partial f}{\partial x} & (k+1)f - y\frac{\partial f}{\partial y} \end{pmatrix} \begin{pmatrix} s_{k+2} - rp_{k+2} \\ t_{k+2} - rq_{k+2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

multiply by the adjoint matrix $(k+1)f^2 \begin{pmatrix} s_{k+2} - rp_{k+2} \\ t_{k+2} - rq_{k+2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ then $s_{k+2} = rp_{k+2}$ and $t_{k+2} = rq_{k+2}$. Notice that this last calculation is true in arbitrary dimension. Finally, suppose that $s_{k+j} = rp_{k+j}$ and $t_{k+j} = rq_{k+j}$ for $j = 2, \ldots, i$. The $(2k+i+1)$-jet of the bracket is $[f\vec{R}, s_{k+i+1}\frac{\partial}{\partial x} + t_{k+i+1}\frac{\partial}{\partial y}] + [p_{k+i+1}\frac{\partial}{\partial x} + q_{k+i+1}\frac{\partial}{\partial y}, g\vec{R}] = 0$ because the symmetrical terms of the summation

$$\sum_{j=2}^{i}[p_{k+j}\frac{\partial}{\partial x} + q_{k+j}\frac{\partial}{\partial y}, s_{k+i+2-j}\frac{\partial}{\partial x} + t_{k+i+2-j}\frac{\partial}{\partial y}]$$

is zero. Then the equalities $s_{k+i+1} = rp_{k+i+1}$ and $t_{k+i+1} = rq_{k+i+1}$ follows in the same way to the case $k+2$. □

Observation: The condition over F is generic and means that $(0,0) \in \mathbb{C}^2$ is an isolated singularity of $\mathfrak{f}_0 = (f(x,y)x + p_{k+2}(x,y))\frac{\partial}{\partial x} + (f(x,y)y + q_{k+2}(x,y))\frac{\partial}{\partial y}$. With a similar condition the theorem is true in arbitrary dimension.

6 Milnor number of a diffeomorphism tangent to the identity

Let $F : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^n, 0)$ be a germ of holomorphic diffeomorphism, i.e., F is given by a convergent power series in a neighborhood of 0 as

$$F(X) = X + P_{k+1}(X) + P_{k+2}(X) + \cdots$$

where P_m is a homogeneous polynomial vector field of degree $m \geq k+1 \geq 2$. We say that G is analytic (resp. topological, C^i) conjugate to F if there exists a analytic diffeomorphism (resp. homeomorphism, C^i-homeomorphism) ϕ such that $G = \phi^{-1} \circ F \circ \phi$.
Let $\phi : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^n, 0)$ be a C^1 change of coordinates, i.e, ϕ a germ of homeomorphism C^1 fixing 0, such that $\phi(X) = AX + R(X)$ where $A \in Gl(n, \mathbb{C})$ and $\frac{|R(X)|}{|X|} \to 0$ when $X \to 0$, then $\phi^{-1}(X) = A^{-1}X + S(X)$ where $\frac{|S(X)|}{|X|} \to 0$ when $X \to 0$. Conjugating F by ϕ we have

$$\begin{aligned}
\phi^{-1} \circ F \circ \phi &= A^{-1}(F \circ \phi) + S(F \circ \phi) \\
&= A^{-1}(F \circ \phi) + S(\phi + P_{k+1}(\phi) + \cdots) \\
&= A^{-1}(\phi) + A^{-1}P_{k+1}(\phi) + o(|X|^{k+1}) + S(\phi) \\
&\qquad + \left(\int_0^1 \nabla S(\phi + t(P_{k+1}(\phi) + \cdots))dt \right) \cdot (P_{k+1}(\phi) + \cdots) \\
&= (A^{-1} + S)(\phi(X)) + A^{-1}P_{k+1}(AX + R(X)) + o(|X|^{k+1}) \\
&= X + A^{-1}P_{k+1}(AX) + o(|X|^{k+1})
\end{aligned}.$$

Notice that if $[X_0] \in \mathbb{C}P(n)$ is a fixed point of $P_{k+1} : \mathbb{C}P(n) \to \mathbb{C}P(n)$, then $[A^{-1}X_0]$ is a fixed point of $A^{-1}P_{k+1} \circ A : \mathbb{C}P(n) \to \mathbb{C}P(n)$. In addition, the class of similarity of $\nabla P_{k+1}[X_0]$ is a C^1 invariant, in particular, the eigenvalues of $\nabla P_{k+1}[X_0]$ are invariant.

It is clear that the constant k is a C^1-invariant. In fact, we are going to prove that k is a topological invariant when $P_{k+1}(X)$ does not have zeros different from $X = 0$.
Suppose that there exists $\epsilon > 0$ such that $F(X) \neq X$ for all $X \in B_\epsilon \setminus \{0\}$. For $0 < r \leq \epsilon$ define

$$\begin{array}{cccc} f_r : & S^{2n-1} & \longrightarrow & S^{2n-1} \\ & X & \longrightarrow & \frac{F(rX)-rX}{||F(rX)-rX||} \end{array}$$

where $S^{2n-1} = \{X \in \mathbb{C}^n | \, ||X|| = 1\}$. For $0 < r_1 < r_2 \leq \epsilon$ the functions f_{r_1} and f_{r_2} are homotopic, in particular we have that degree f_{r_1} = degree f_{r_2}. In fact, there exists a homotopy

$$\begin{array}{cccc} H : & (0,\epsilon) \times S^{2n-1} & \longrightarrow & S^{2n-1} \\ & (t,X) & \longrightarrow & \frac{F(tX)-tX}{||F(tX)-tX||} \end{array}$$

such that $H(r_1, X) = f_{r_1}(X)$ and $H(r_2, X) = f_{r_2}(X)$.

Definition 6.1 *Let $F \in \mathrm{Diff}_1(\mathbb{C}^n, 0)$. Define the Milnor number of F, $\mu(F)$ as the topological degree of f_r where r is sufficient small such that 0 is a isolated fixed point of F, and ∞ otherwise.*

Observation: If $F(X) = X + P_{k+1}(X) + \cdots$ where $P_{k+1}(X) = 0$ only if $X = 0$, then $\mu(F) = (k+1)^n$. In fact, in the homotopy above we have

$$\begin{aligned} \lim_{t\to 0} H(t,X) &= \lim_{t\to 0} \frac{F(tX) - tX}{||F(tX) - tX||} = \lim_{t\to 0} \frac{P_{k+1}(tX) + p_{k+2}(X) + \cdots}{||P_{k+1}(tX) + p_{k+2}(X) + \cdots||} \\ &= \lim_{t\to 0} \frac{P_{k+1}(X) + t(p_{k+2}(X) + \cdots)}{||P_{k+1}(X) + t(p_{k+2}(X) + \cdots)||} = \frac{P_{k+1}(X)}{||P_{k+1}(X)||} \end{aligned}.$$

Since $P_{k+1}(X) \neq 0$ for $X \neq 0$ it is possible to define F_0 like this limit and f_0 is homotopic to f_r. Moreover, it is easy to proof that degree $f_0 = (k+1)^n$.
In general, if we suppose that $R(X) = P_{k+1}(X) + \cdots + P_m(X)$ has 0 as an isolated zero, then

$$\begin{array}{cccc} H : & [0,1] \times S^{2n-1} & \longrightarrow & S^{2n-1} \\ & (t,X) & \longrightarrow & \frac{R(rX)+t(P_{m+1}(rX)+P_{m+2}(rX)+\cdots)}{||R(rX)+t(P_{m+1}(rX)+P_{m+2}(rX)+\cdots)||} \end{array}$$

is a homotopy between $H(0,X) = \frac{R(rX)}{||R(rX)||}$ and $H(1,X) = f_r$ where r is sufficient small. To estimate r, we define the function $\rho : S^{2n-1} \longrightarrow (0,\epsilon]$ where

$$\rho(Y) = \max \left\{ t \in (0,\epsilon] \left| \frac{||P_{m+1}(tY) + \cdots ||}{||P_{k+1}(tY) + \cdots + P_m(tY)||} \leq \frac{1}{2} \quad \forall t \in (0,\epsilon] \right. \right\}.$$

This function is well defined, in fact, we know that exists l with $k+1 \leq l \leq m$ such that $P_l(Y) \neq 0$ and $P_s(Y) = 0$ for all $s < l$, because 0 is an isolated zero. Then

$$\lim_{t\to 0} \frac{||P_{m+1}(tY) + \cdots ||}{||P_{k+1}(tY) + \cdots + P_m(tY)||} = \lim_{t\to 0} \frac{t^{m+1-l}||P_{m+1}(Y) + \cdots ||}{||P_l(Y) + \cdots + t^{m-l}P_m(tY)||} = 0.$$

By the continuity in the variable t, we conclude that ρ is well defined and $0 < \rho(Y) \leq \epsilon$ for all $Y \in S^{2n-1}$, moreover as ρ is upper semi-continuous, we define $r = \min_{Y\in S^{2n-1}} \rho(Y) > 0$.
Now, it is easy to prove that H is a homotopy because

$$\begin{aligned} ||R(rX) + t(P_{m+1}(rX) + P_{m+2}(rX) + \cdots)|| &\geq ||R(rX)|| - t||P_{m+1}(rX) + \cdots || \\ &\geq \left(1 - \frac{t}{2}\right) ||R(rX)|| > 0 \end{aligned}$$

For example, in the case $F(x,y) = \begin{pmatrix} x + f(x,y)p_l(x,y) + p_{k+2}(x,y) \\ y + f(x,y)q_l(x,y) + q_{k+2}(x,y) \end{pmatrix}$ where $(0,0)$ is an isolated fixed point and $g.c.d.(p_l,q_l) = 1$, calculation the local degree of F in $(0,0)$ is equivalent to calculating the multiplicity of the solution $(0,0)$ in the system

$$\begin{array}{c} f(x,y)p_l(x,y) + p_{k+2}(x,y) = 0 \\ f(x,y)q_l(x,y) + q_{k+2}(x,y) = 0 \end{array}.$$

Multiplying the first equation by q_l and the second by p_l and subtracting we obtain

$$p_{k+2}(x,y)q_l(x,y) - p_l(x,y)q_{k+2}(x,y) = 0,$$

this is a homogeneous polynomial of degree $(l+k+2)$, therefore the solutions of this equation are of the form $x = \lambda_j y$ where $j = 1,\ldots,l+k+2$. Substituting in the first equation we have

$$x^{k+1}(f(1,\lambda_j)p_l(1,\lambda_j) - p_{k+2}(1,\lambda_j)x) = 0.$$

Using the Bezout theorem, the number of solutions of the initial system is $(k+2)^2$, it follows that the multiplicity of $(0,0)$ is

$$(k+2)^2 - (k+l+2) + \#\{\lambda_j | f(1,\lambda_j) = 0\}.$$

Theorem 6.1 (Camacho, Lins, Sad) *The Milnor number is topologically invariant, i.e. if $F, G \in \mathrm{Diff}_1(\mathbb{C}^n, 0)$ are topologically conjugate then $\mu(F) = \mu(G)$.*

Proof: It follows from the proof of the theorem A in [4]. □

7 Convergent Orbits

Definition 7.1 *Let $F \in \mathrm{Diff}_{k+1}(\mathbb{C}^n, 0)$, where $F(X) = X + F_{k+1}(X) + \cdots$. We say that $[V] \in \mathbb{C}P(n-1)$ is a characteristic direction of F, if there exists $\lambda \in \mathbb{C}$ such that*

$$F_{k+1}(V) = \lambda V$$

Moreover the direction $[V]$ is called non degenerate if $\lambda \neq 0$.

Definition 7.2 *Let $F \in \mathrm{Diff}_2(\mathbb{C}^n, 0)$. A parabolic curve for F at the origin is an injective map $\varphi : D_1 \to \mathbb{C}^n$, where $D_1 = \{x |\ |x-1| \leq 1\}$, holomorphic in $int(D_1)$, such that $\varphi(0) = 0$, $\varphi(D_1)$ is invariant under F and $(F|_{D_1})^n \to 0$ when $n \to \infty$*

Theorem 7.1 (Hakim [13]) *Let $F \in \mathrm{Diff}_{k+1}(\mathbb{C}^n, 0)$, then for every non-degenerate characteristic direction $[V]$ there are a parabolic curves tangent to $[V]$ at the origin.*

Theorem 7.2 (Abate [1]) *Let $F \in \mathrm{Diff}_{k+1}(\mathbb{C}^2, 0)$ such that the origin is a isolated fixed point. Then there exist k parabolic curves for F at the origin.*

Denote $r(v) = q_{k+1}(1, v) - vp_{k+1}(1, v)$ and $p(v) = p_{k+1}(1, v)$.

Lemma 7.1 *Let $F \in \mathrm{Diff}_{k+1}(\mathbb{C}^2, 0)$ and suppose that $(x_n, y_n) = F(x_{n-1}, y_{n-1})$ is a sequence converging to 0 such that $\frac{y_n}{x_n} \to v$ when $n \to \infty$. Then*

$$\lim_{n\to\infty} \frac{1}{nx_n^k} = -kp(v)$$

Proof: After a blow up at $0 \in \mathbb{C}^2$ in the chart $y = xv$, we obtain the diffeomorphism

$$\begin{pmatrix} x_1 \\ v_1 \end{pmatrix} = \tilde{F}(x, v) = \begin{pmatrix} x + x^{k+1}p(v) + x^{k+2}p_{k+2}(1, v) + \cdots \\ v + x^k r(v) + \cdots \end{pmatrix}$$

Rewriting the first equation

$$\begin{aligned} \frac{1}{x_1^k} &= \frac{1}{(x + x^{k+1}p(v) + x^{k+2}p_{k+2}(1, v) + \cdots)^k} \\ &= \frac{1}{x^k}(1 + x^k p(v) + x^{k+1}p_{k+2}(1, v) + \cdots)^{-k} \\ &= \frac{1}{x^k} - kp(v) + o(x). \end{aligned}$$

Let's define $(x_j, v_j) = F(x_{j-1}, v_{j-1})$. From the equation above we get the telescopic sum

$$\frac{1}{x_n^k} - \frac{1}{x^k} = \sum_{j=1}^{n} \frac{1}{x_j^k} - \frac{1}{x_{j-1}^k} = -\sum_{j=1}^{n}(kp(v_{j-1}) + o(x_{j-1}))$$

Divide by n and make n tends to ∞, we deduce

$$\lim_{n\to\infty} \frac{1}{nx_n^k} = -k \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} kP(v_{j-1}) + \lim_{n\to\infty} \sum_{j=1}^{n} \frac{1}{n} o(x_{j-1}) = -kp(v). \qquad \square$$

Proposition 7.1 *Let $(x_0, y_0) \in U$ open neighborhood of 0 such that the sequence $(x_j, y_j) = F(x_{j-1}, y_{j-1})$ converge to 0, and $\frac{y_j}{x_j}$ converge to $v \in \mathbb{C}P(1) = \overline{\mathbb{C}}$, then $r(v) = 0$.*

Proof: From the lemma 7.1. we have $\lim_{n\to\infty} \frac{1}{nx_n^k} = -kp_{k+1}(1, v)$ and the same way $\lim_{n\to\infty} \frac{1}{ny_n^k} = -kq_{k+1}(\frac{1}{v}, 1)$. Dividing these relations we get

$$\frac{1}{v^k} = \lim_{n\to\infty} \frac{x_n^k}{y_n^k} = \frac{\lim_{n\to\infty} \frac{1}{ny_n^k}}{\lim_{n\to\infty} \frac{1}{nx_n^k}} = \frac{q_{k+1}(\frac{1}{v}, 1)}{p_{k+1}(1, v)},$$

therefore $v^{k+1}q_{k+1}(\frac{1}{v}, 1) - vp_{k+1}(1, v) = r(v) = 0$. $\square$

Theorem 7.3 *Let $F : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ be a dicritic diffeomorphism fixing zero, i.e. F can be represented by a convergent series*

$$F(x,y) = \begin{pmatrix} x + xp_k(x,y) + p_{k+2}(x,y) + \cdots \\ y + yp_k(x,y) + q_{k+2}(x,y) + \cdots \end{pmatrix},$$

and $\tilde{F} = \Pi^* F : (\tilde{\mathbb{C}}^2, D) \to (\tilde{\mathbb{C}}^2, D)$ the continuous extension of the diffeomorphism after making the blow-up in $(0,0)$. Then there exist open sets $U^+, U^- \subset \tilde{\mathbb{C}}^2$ such that

1. *$\overline{U^+ \cup U^-}$ is a neighborhood of $D \setminus \{(1 : v) \in D | p_k(1,v) = 0\}$.*
2. *For all $P \in U^+$, the sequence $\{\tilde{F}^n(P)\}_{n\in\mathbb{N}}$ converge and $\lim_{n\to\infty} \tilde{F}^n(P) \in D$.*
3. *For all $P \in U^-$, the sequence $\{\tilde{F}^{-n}(P)\}_{n\in\mathbb{N}}$ converge and $\lim_{n\to\infty} \tilde{F}^{-n}(P) \in D$.*

Proof: Making a blow up at $(0,0)$, and regarding the diffeomorphism in the chart (x,v), $v = \frac{y}{x}$ we obtain

$$\begin{pmatrix} x_1 \\ v_1 \end{pmatrix} = \tilde{F}(x,v) = \begin{pmatrix} x + x^{k+1}p(v) + \sum_{j=2}^{\infty} a_{k+j}(v)x^{k+j} \\ v + \sum_{j=1}^{\infty} b_{k+j}(v)x^{k+j} \end{pmatrix},$$

where a_j and b_j are polynomial of degree less than $j+2$. Let q be a arbitrary point in $D \setminus \{(1 : v) \in D | p(v) = 0\}$. We can suppose making a linear change of coordinates that $q = (0,0) \in \tilde{\mathbb{C}}^2$. Since $\tilde{F}$ is holomorphic in some neighborhood of $(0,0)$ there exist $r_1, r_2 > 0$ and constants $C_1, C_2 > 0$ such that

$$|| \sup_{|v|\le r_1} a_j(v)|| \le C_1 r_2^j, \text{ and } || \sup_{|v|\le r_1} b_j(v)|| \le C_2 r_2^j \text{ for all } j.$$

Now making a ramificated change of coordinates $w = \frac{1}{x^k}$, it follows that

$$\begin{pmatrix} w_1 \\ v_1 \end{pmatrix} = \overline{F}(w,v) = \begin{pmatrix} w - kp(v) + \sum_{j=1}^{\infty} c_j(v)\frac{1}{w^{\frac{j}{k}}} \\ v + \sum_{j=1}^{\infty} b_{k+j}(v)\frac{1}{w^{1+\frac{j}{k}}} \end{pmatrix},$$

is holomorphic in $\{(w,v) \in \mathbb{C}_k \times \mathbb{C} | \ |v| < r_1, \ |w| > r_2^k\}$, where $\mathbb{C}_k$ is a k-fold covering of $\mathbb{C}^*$. In particular there exist $C_3, C_4 > 0$ such that

$$|| \sup_{|v|\le r_1} c_j(v)|| \le C_3 r_2^j \text{ for all } j \text{ and } \left| \sum_{j=1}^{\infty} b_{k+j}(v)\frac{1}{w^{1+\frac{j}{k}}} \right| < \frac{C_4}{|w|^{1+\frac{1}{k}}}.$$

Choose $r_1 > r > 0$ such that $\left|\frac{p(v)}{p(0)} - 1\right| < \frac{1}{4}$ for all $|v| < r$, and

$$R > \max\left\{ r_2^k \left(\frac{4C_3}{k|p(0)|} + 1\right)^k, \frac{(2C_4)^k}{r^k}\left(1 + \frac{4C(k)}{k|p(0)|}\right)^k, 1 \right\},$$

where $C(k) = \int_0^\infty \frac{1}{(1+x^2)^{\frac{k+1}{2k}}} dx < k+1$. Let

$$V_q^+(a) = \left\{ (w,v) \in \mathbb{C}_k \times \mathbb{C} \middle| \ |v| < a, \quad \left|\arg\left(-\frac{w}{p(0)} - \frac{2R}{|p(0)|}\right)\right| < \frac{2\pi}{3} \right\},$$

where arg is defined from $\mathbb{C}_k^*$ to $(-\pi, \pi]$. It is easy to prove that $V_q^+(r) \subset \{(w,v) \in \mathbb{C}_k \times \mathbb{C} | \ |v| < r, \ |w| > R\}$.

Claim: For all $p = (w_0, v_0) \in V_q^+(\frac{r}{2})$ we have

1. $\overline{F}^n(p) = (w_n, v_n) \in V_q^+(r)$ for all $n \in \mathbb{N}$.
2. The sequence $\{v_n\}_{n\in\mathbb{N}}$ converge.
3. $|w_n| \to \infty$ when $n \to +\infty$.

Proof of the claim: For a arbitrary point (w,v) in $V_q^+(r)$ we have

$$\begin{aligned}
|w_1 - (w - kp(0))| &\le \left| k(p(0) - p(v)) + \sum_{j=1}^{\infty} c_j(v) \frac{1}{w^{\frac{j}{k}}} \right| \\
&\le k|p(0) - p(v)| + \sum_{j=1}^{\infty} C_3 \frac{r_2^j}{|w|^{\frac{j}{k}}} < k\frac{|p(0)|}{4} + C_3 \frac{r_2}{|w|^{\frac{1}{k}} - r_2} \\
&< k\frac{|p(0)|}{4} + C_3 \frac{r_2}{r_2\left(\frac{4C_3}{k|p(0)|} + 1\right) - r_2} = k\frac{|p(0)|}{2}.
\end{aligned}$$

Let $p = (w_0, v_0) \in V_q^+(\frac{r}{2})$ and suppose that $(w_i, v_i) \in V_q^+(r)$ for $i = 0, \ldots, j-1$. Then

$$|w_j - (w_0 - jkp(0))| = \left|\sum_{i=1}^{j} w_i - (w_{i-1} - kp(0))\right| \le \sum_{i=1}^{j} |w_i - (w_{i-1} - kp(0))| < jk\frac{|p(0)|}{2}$$

therefore $w_j = w_0 - jkp(0)(1+\delta_j)$ where $|\delta_j| < \frac{1}{2}$, and

$$-\frac{w_j}{p(0)} - \frac{2R}{|p(0)|} = -\frac{w_0}{p(0)} - \frac{2R}{|p(0)|} + jk(1+\delta_j),$$

but since $|\arg(1+\delta_j)| < \frac{\pi}{3}$ it follows that $|\arg(-\frac{w_j}{p(0)} - \frac{2R}{|p(0)|})| < \frac{2\pi}{3}$. Now notice that $|v_i| < |v_{i-1}| + \frac{C_4}{|w_{i-1}|^{1+\frac{1}{k}}}$ for all $i = 1, \ldots, j$, therefore

$$|v_j| < |v_0| + \sum_{i=1}^{j} \frac{C_4}{|w_{i-1}|^{1+\frac{1}{k}}},$$

but

$$\begin{aligned}\sum_{i=1}^{j} \frac{1}{|w_{i-1}|^{1+\frac{1}{k}}} &< \sum_{i=-\infty}^{\infty} \frac{1}{(R^2 + (\frac{ik|p(0)|}{2})^2)^{\frac{k+1}{2k}}} \\ &< \frac{1}{R^{1+\frac{1}{k}}} + 2\int_0^\infty \frac{1}{(R^2 + (\frac{k|p(0)|}{2})^2 x^2)^{\frac{k+1}{2k}}} dx \\ &< \frac{1}{R^{\frac{1}{k}}} + \frac{4}{k|p(0)|R^{\frac{1}{k}}} \int_0^\infty \frac{1}{(1+x^2)^{\frac{k+1}{2k}}} dx = \frac{1}{R^{\frac{1}{k}}}\left(1 + \frac{4C(k)}{k|p(0)|}\right) \\ &< \frac{r}{2C_4},\end{aligned} \tag{5}$$

it follows that $|v_j| < r$, and therefore $(w_j, v_j) \in V_q^+(r)$. Now we have trivially that $|w_j| = |w_0 - jkp(0)(1+\delta_j)| \to \infty$ when $n \to \infty$ and $\{v_j\}_{j\in\mathbb{N}}$ is a Cauchy sequence because $\sum_{i=0}^{\infty} \frac{C_4}{|w_i|^{1+\frac{1}{k}}}$ is a Cauchy series.

In the same way we can obtain the open set V_q^- changing in the proof the diffeomorphism germ F by F^{-1}. Finally we conclude the proof making $U^+ = \bigcup V_q^+$ and $U^- = \bigcup V_q^-$ where $q \in D \setminus \{(1:v) \in D | p_k(1,v) = 0\}$. □

Theorem 7.4 *Let $F \in \mathrm{Diff}_{k+1}(\mathbb{C}^2, 0)$, where $F(x,y) = \begin{pmatrix} x + p_{k+1}(x,y) + \cdots \\ y + q_{k+1}(x,y) + \cdots \end{pmatrix}$ and $r(v) = vp_{k+1}(1,v) - q_{k+1}(1,v) \not\equiv 0$. Suppose that $v_0 \in \mathbb{C}$ satisfies $r(v_0) = 0$ and $\Re(\frac{r'(v_0)}{p(v_0)}) > 0$. Then there exist open sets U^+ and U^- such that $(0,0) \in \partial U^\pm$ and for each point $(a^+, b^+) \in U^+$ and $(a^-, b^-) \in U^-$ the sequences $(a_n^\pm, b_n^\pm) = F^{\pm n}((a^\pm, b^\pm))$ converge and*

$$\lim_{n\to\infty} F^{\pm n}(a^\pm, b^\pm) = 0 \text{ and } \lim_{n\to\infty} \frac{b_n^+}{a_n^+} = \lim_{n\to\infty} \frac{b_n^-}{a_n^-} = v_0.$$

Proof: For a linear change of coordinates we can suppose that $v_0 = 0$. Let $r(v) = vs(v)$, $\alpha = s(0)$, $\beta = p(0)$ and $\tilde{F} : (\tilde{\mathbb{C}}^2, D) \to (\tilde{\mathbb{C}}^2, D)$ the continuous extension of the diffeomorphism after making the blow-up Π at $(0,0)$, i.e. in the chart (x, v) where $y = vx$

$$(x_1, v_1) = \tilde{F}(x, v) = \begin{pmatrix} x + x^{k+1}p(v) + x^{k+2}(\cdots) \\ v + x^k r(v) + x^{k+1}(\cdots) \end{pmatrix}.$$

Since $\tilde{F}|_D = id|_D$ we can forget the dynamic in $D = \{x = 0\}$ and make a ramificate change of coordinates $w = \frac{1}{x^k}$. Then F is representing in the (w, v) coordinates system as

$$(w_1, v_1) = \overline{F}(x, v) = \begin{pmatrix} w - kp(v) + o(\frac{1}{w^{1/k}}) \\ v + \frac{s(v)}{w} v + o(\frac{1}{w^{1+1/k}}) \end{pmatrix}.$$

In particular, there exists a neighborhood $A = \{(w, v) \in \mathbb{C}_k^* \times \mathbb{C} |\ |w| > R_1,\ |v| < r_1\}$ of $(\infty, 0)$ and constants $C_1, C_2 > 0$ such that

$$|w_1 - (w - kp(v))| < C_1 \frac{1}{|w|^{1/k}} \quad \text{and} \quad \left| v_1 - v\Big(1 + \frac{s(v)}{w}\Big) \right| < C_2 \frac{1}{|w|^{1+1/k}}.$$

Let denote $\theta = \frac{\pi}{2} - |\arg(\frac{\alpha}{\beta})| > 0$. Choose $0 < r < r_1$ such that

$$\left| \frac{p(v)}{\beta} - 1 \right| < \frac{1}{2} \sin \frac{\theta}{2}, \quad \left| \arg \frac{s(v)}{\alpha} \right| < \frac{\theta}{2}$$

and $R > \max\left\{ R_1 + 1, \left(\frac{2C_1}{|\beta| k \sin \frac{\theta}{2}} \right)^k, \frac{(2C_2)^k}{r^k} \left(1 + \frac{2C(k)}{k(1 - \sin \frac{\theta}{2})|\beta|} \right)^k \right\}$. Let define

$$V^+(a) = \{(w, v) \in A |\ |v| < a,\ \Re(\frac{w}{\alpha} e^{\pm \frac{\theta}{2} i}) < -R\}.$$

See that for all $(w, v) \in V^+(r)$ we have $\Re(\frac{w}{s(v)}) < -R$, i.e. $\frac{s(v)}{w}$ is in the disc of center $-\frac{1}{R}$ and radius $\frac{1}{R}$, it follows that

$$\left| 1 + \frac{s(v)}{w} \right| < 1.$$

In addition

$$\begin{aligned}|w_1-(w-k\beta)| &\le |k(\beta-p(v))|+C_1\frac{1}{|w|^{1/k}}\\ &\le \frac{k|\beta|}{2}\sin\frac{\theta}{2}+\frac{k|\beta|}{2}\sin\frac{\theta}{2}\\ &= k|\beta|\sin\frac{\theta}{2}.\end{aligned}$$

Let $p=(w_0,v_0)\in V^+(\frac{r}{2})$ and suppose that $(w_i,v_i)\in V^+(r)$ for $i=0,\dots,j-1$. Then

$$|w_j-(w_0-jk\beta)|\le\sum_{i=1}^{j}|w_i-(w_{i-1}-k\beta)|<jk|\beta|\sin\frac{\theta}{2},$$

therefore $w_j=w_0-jk\beta(1+\delta_j)$ where $|\delta_j|<\sin\frac{\theta}{2}$. Now since $\frac{w_j}{\alpha}=\frac{w_0}{\alpha}-jk\frac{\beta}{\alpha}(1+\delta_j)$, and

$$\left|\arg(\frac{\beta}{\alpha}(1+\delta_j))\right|\le\left|\arg\frac{\beta}{\alpha}\right|+|\arg(1+\delta_j)|\le\frac{\pi}{2}-\theta+\frac{\theta}{2}=\frac{\pi}{2}-\frac{\theta}{2},$$

it follows that $\Re(\frac{\beta}{\alpha}(1+\delta_j)e^{\pm\frac{\theta}{2}i})\ge 0$ and therefore $\Re(\frac{w_j}{\alpha}e^{\pm\frac{\theta}{2}i})<-R$.
In order to bound v_j, see that

$$|v_i|<\left|\left(1+\frac{s(v_{i-1})}{w_{i-1}}\right)v_{i-1}\right|+\frac{C_2}{|w_{i-1}|^{1+\frac{1}{k}}}\le|v_{i-1}|+\frac{C_2}{|w_{i-1}|^{1+\frac{1}{k}}}$$

for all $i=1,\dots,j$, using a similar calculation as the inequality in (5) we have

$$|v_j|<|v_0|+\sum_{i=1}^{j}\frac{C_2}{|w_{i-1}|^{1+\frac{1}{k}}}<r.$$

It follows that $(w_j,v_j)\in V^+(r)$, $|w_j|=|w_0-jkp(0)(1+\delta_j)|\to\infty$ when $n\to\infty$ and $\{v_j\}_{j\in\mathbb{N}}$ is a Cauchy sequence because $\sum_{i=0}^{\infty}\frac{C_2}{|w_i|^{1+\frac{1}{k}}}$ is a Cauchy series. Moreover from the proposition 7.1 we know that $\{v_j\}_{j\in\mathbb{N}}$ converge to 0. □

8 Normal forms

Let $\tilde{F}, \tilde{G} \in \mathrm{Diff}_{k+1}(\mathbb{C}^2, 0)$ such that their $(k+1)$-jet are equal and suppose that they are formally conjugate. Let $F(x,v) = \begin{pmatrix} x + x^{k+1}p(v) + \cdots \\ v + x^k r(v) + \cdots \end{pmatrix}$ and $G(x,v) = \begin{pmatrix} x + x^{k+1}p(v) + \cdots \\ v + x^k r(v) + \cdots \end{pmatrix}$ be the diffeomorphisms obtained after the blow-up at 0 where $r(v)$ and $p(v)$ are polynomials of degree $(k+2)$ and $(k+1)$ respectively. Let H be a formal diffeomorphism that conjugates F and G. Notice that $H(x,v) = \begin{pmatrix} x + \sum_{l=1}^{\infty} x^{l+1} h_{1,l}(v) \\ v + \sum_{l=1}^{\infty} x^l h_{2,l}(v) \end{pmatrix}$ where $h_{1,l}$ and $h_{2,l}$ are polynomial of degree $l+1$ and $l+2$ respectively, in particular $h_{1,l}$ and $h_{2,l}$ are holomorphic functions in $\mathbb{C}$. The following definition is borrowed from [26].

Definition 8.1 *A formal Taylor series in x is called semiformal in U if its coefficients holomorphically depend on v in the same domain U. A formal change of coordinates is called semiformal in U if its components are semiformal series in U.*

It is clear that if H is a semiformal change of coordinates in $\overline{\mathbb{C}}$, then F and G are semiformally conjugates in $\overline{\mathbb{C}}$. We are interested in finding some semiformal invariant, for that we need the following lemma.

Lemma 8.1 *Let F and G as above. Suppose that $G - F = \begin{pmatrix} x^{k+l+1}\phi_1(v) + \cdots \\ x^{k+l}\phi_2(v) + \cdots \end{pmatrix}$ and that $H(x,v) = \begin{pmatrix} x + x^{l+1}h_1(v) + \cdots \\ v + x^l h_2(v) + \cdots \end{pmatrix}$ conjugates F and G, i.e. $G \circ H = H \circ F$. Then*

$$r(v)\begin{pmatrix} h_1(v) \\ h_2(v) \end{pmatrix}' - \begin{pmatrix} (k-l)p(v) & p'(v) \\ kr(v) & r'(v) - lp(v) \end{pmatrix}\begin{pmatrix} h_1(v) \\ h_2(v) \end{pmatrix} = \begin{pmatrix} \phi_1(v) \\ \phi_2(v) \end{pmatrix}. \quad (6)$$

Proof: For the lemma 5.1 we have

$$0 = G\circ H - H\circ F = G\circ H - H\circ(G + (F - G))$$
$$= G\circ H - H\circ G - \nabla H(G)\cdot(F - G) + \begin{pmatrix} x^{2k+2l+2}(\cdots) \\ x^{2k+2l}(\cdots) \end{pmatrix}$$
$$= \begin{pmatrix} (k+1)x^k p(v) & x^{k+1}p'(v) \\ kx^{k-1}r(v) & x^k r'(v) \end{pmatrix} \begin{pmatrix} x^{l+1}h_1(v) \\ x^l h_2(v) \end{pmatrix} - \begin{pmatrix} (l+1)x^l h_1(v) & x^{l+1}h_1'(v) \\ lx^{l-1}h_2(v) & x^l h_2'(v) \end{pmatrix} \begin{pmatrix} x^{k+1}p(v) \\ x^k r(v) \end{pmatrix}$$
$$+ \begin{pmatrix} x^{k+l+1}\phi_1(v) \\ x^{k+l}\phi_2(v) \end{pmatrix} + \begin{pmatrix} x^{k+l+2}(\cdots) \\ x^{k+l+1}(\cdots) \end{pmatrix}$$
$$= \begin{pmatrix} x^{k+l+1}((k-l)p(v)h_1(v) + p'(v)h_2(v) - r(v)h_1'(v) + \phi_1(v)) \\ x^{k+l}(kr(v)h_1(v) + (r'(v) - lp(v))h_2(v) - r(v)h_2'(v) + \phi_2(v)) \end{pmatrix} + \begin{pmatrix} x^{k+l+2}(\cdots) \\ x^{k+l+1}(\cdots) \end{pmatrix}$$

In particular, we obtain the linear differential equation (6). □

Proposition 8.1 *Let F and G as in the lemma 8.1. Suppose that $r(v) \not\equiv 0$ and U is a simply connected open set of $\mathbb{C} \setminus \{r(v) = 0\}$. Then (6) has holomorphic solution in U.*

Proof: Substituting $g_i = e^{\int \frac{lp(v)}{r(v)}dv} h_i$ and $\psi_i = e^{\int \frac{lp(v)}{r(v)}dv}\phi_i$, $i = 1, 2$, in (6), we obtain the system of equations

$$r(v)g_1'(v) = kp(v)g_1(v) + p'(v)g_2(v) + \psi_1(v) \tag{7}$$
$$r(v)g_2'(v) = kr(v)g_1(v) + r'(v)g_2(v) + \psi_2(v) \tag{8}$$

Finding $g_1(v)$ in (8) and substituting in (7) we obtain

$$r\left(g_2' - \frac{r'}{r}g_2 - \frac{\psi_2}{r}\right)' = kp\left(g_2' - \frac{r'}{r}g_2 - \frac{\psi_2}{r}\right) + kp'g_2 + k\psi_1.$$

This equation is equivalent to

$$\left(g_2' - \frac{r' + kp}{r}g_2\right)' = \left(\frac{\psi_2}{r}\right)' + k\frac{r\psi_1 - p\psi_2}{r^2}.$$

Now integrating and multiplying by $\frac{1}{r}e^{-k\int \frac{p}{r}dv}$ it follows that

$$\left(\frac{1}{r}e^{-k\int \frac{p}{r}dv}g_2\right)' = \frac{\psi_2}{r^2}e^{-k\int \frac{p}{r}dv} + \frac{k}{r}e^{-k\int \frac{p}{r}dv}\int \frac{r\psi_1 - p\psi_2}{r^2}dv.$$

Integrating once more, and substituting g_1, ψ_1 and ψ_2 we conclude that

$$h_2 = r\left[e^{(k-l)\int \frac{p}{r}dv}\int\left(\frac{\phi_2}{r^2}e^{-(k-l)\int \frac{p}{r}dv}+\frac{k}{r}e^{-k\int \frac{p}{r}dv}\int\frac{r\phi_1-p\phi_2}{r^2}e^{l\int \frac{p}{r}dv}dv\right)dv\right]$$

and

$$\begin{aligned} h_1 =& e^{-l\int \frac{p}{r}dv}\int\frac{r\phi_1-p\phi_2}{r^2}e^{l\int \frac{p}{r}dv}dv+ \\ &+p\left[e^{(k-l)\int \frac{p}{r}dv}\int\left(\frac{\phi_2}{r^2}e^{-(k-l)\int \frac{p}{r}dv}+\frac{k}{r}e^{-k\int \frac{p}{r}dv}\int\frac{r\phi_1-p\phi_2}{r^2}e^{l\int \frac{p}{r}dv}dv\right)dv\right]. \end{aligned}$$

Observe that (h_1, h_2) is a holomorphic solution of (6) where the integrals are calculated from a fixed point $p_0 \in U$ and independent of the path taken because U is simply connected. □

Proposition 8.2 *Let v_0 be a root of $r(v)$. Suppose that $\frac{p(v_0)}{r'(v_0)} \notin \mathbb{Q}$. Then there exists a holomorphic solution (h_1, h_2) of (6) in some neighborhood of v_0 if and only if one of the following condition is true*

1. *$l \neq k$.*
2. *$l = k$ and $p'(v_0)\phi_2(v_0) = (r'(v_0) - kp(v_0))\phi_1(v_0)$.*

Moreover, that neighborhood is independent of the integer l.

Proof: Let $r(v) = (v - v_0)s(v)$ and $a = \frac{p(v_0)}{s(v_0)} \notin \mathbb{Q}$, we are interested in finding a solution of

$$(v-v_0)X' - \begin{pmatrix} (k-l)\frac{p(v)}{s(v)} & \frac{p'(v)}{s(v)} \\ k(v-v_0) & \frac{r'(v)-lp(v)}{s(v)} \end{pmatrix} X = \frac{1}{s(v)}\begin{pmatrix} \phi_1(v) \\ \phi_2(v) \end{pmatrix} \tag{9}$$

in some neighborhood of v_0.

Since the difference between the eigenvalues of $\begin{pmatrix} (k-l)\frac{p(v_0)}{s(v_0)} & \frac{p'(v_0)}{s(v_0)} \\ 0 & \frac{r'(v_0)-lp(v_0)}{s(v_0)} \end{pmatrix}$ is not a integer, it is known (See [27]) that there exists a holomorphic change of coordinates $Z = P(v)X$ ($P(v_0) = Id$) in some neighborhood $V = B(v_0, R)$ of v_0 such that the system of differential equations (9) is equivalent to

$$(v - v_0)Z' - \begin{pmatrix} (k-l)a & a\frac{p'(v_0)}{p(v_0)} \\ 0 & 1 - la \end{pmatrix} Z = \begin{pmatrix} \psi_1(v) \\ \psi_2(v) \end{pmatrix} \tag{10}$$

In addition, R only depend on the radius of convergence of the series that defines locally each function in (9) around v_0, and then independent of l. Thus, we know that $\begin{pmatrix} \psi_1(v) \\ \psi_2(v) \end{pmatrix} = \sum_{j=0}^{\infty} B_j (v - v_0)^j$ for all $|v - v_0| < R$, and then if we write $Z = \sum_{j=0}^{\infty} A_j (v - v_0)^j$, we obtain for each j a linear equation

$$\begin{pmatrix} j - (k-l)a & -a\frac{p'(v_0)}{p(v_0)} \\ 0 & j - (1 - la) \end{pmatrix} A_j = B_j.$$

Notice that, in the case $l \neq k$ or $j \neq 0$ this equation has solution because $a \notin \mathbb{Q}$, and when $l = k$ and $j = 0$ we obtain the linear equation

$$\begin{pmatrix} 0 & -\frac{p'(v_0)}{r'(v_0)} \\ 0 & -1 + k\frac{p'(v_0)}{r'(v_0)} \end{pmatrix} A_j = \frac{1}{r'(v_0)} \begin{pmatrix} \phi_1(v_0) \\ \phi_2(v_0) \end{pmatrix},$$

which has solution if and only if $p'(v_0)\phi_2(v_0) = (r'(v_0) - kp(v_0))\phi_1(v_0)$. Moreover

$$\lim_{j\to\infty} ||A_j||^{\frac{1}{j}} \leq \lim_{j\to\infty} \left\| \begin{pmatrix} j - (k-l)a & -a\frac{p'(v_0)}{p(v_0)} \\ 0 & j - 1 - la \end{pmatrix}^{-1} \right\|^{\frac{1}{j}} \cdot ||B_j||^{\frac{1}{j}} = \frac{1}{R},$$

Therefore that the solution is holomorphic in $B(v_0, R)$. □

Theorem 8.1 *Let* $F(x, v) = \begin{pmatrix} x + x^{k+1}p(v) + \cdots \\ v + x^k r(v) + \cdots \end{pmatrix}$ *and* v_0 *be a root of* $r(v)$. *Suppose that* $0 \neq \frac{p(v_0)}{r'(v_0)} \notin \mathbb{Q}$. *Then there exists* $\lambda_{v_0} \in \mathbb{C}$ *such that* F *is semiformally conjugate with*

$$F_{\lambda_{v_0}} = \begin{pmatrix} x + x^{k+1}p(v) + \lambda_{v_0} x^{2k+1} \\ v + x^k r(v) \end{pmatrix}$$

in some neighborhood of v_0.

Proof: We will go to construct the semiformal conjugation H by successive approximations. Define by induction the sequences F_l and H_l from the initial condition $F_0 = F$ and $H_0 = id$.
Now for $l \geq 1$ and $l \neq k$, from the proposition 8.1 we know that the system of differential equations (6) in the lemma 8.1, where $G = F_{l-1}$ and

$$F_{l-1} - F_{\lambda_{v_0}} = \begin{pmatrix} x^{k+l+1}\phi_{1,l}(v) + \cdots \\ x^{k+l}\phi_{2,l}(v) + \cdots \end{pmatrix},$$

has holomorphic solution $(h_{1,l}, h_{2,l})$ in some neighborhood U that is independent of l.
In the case $l = k$ see that

$$F_{k-1} - F_{\lambda_{v_0}} = \begin{pmatrix} x^{2k+1}(\phi_{1,k}(v) - \lambda_{v_0}) + \cdots \\ x^{2k}\phi_{2,k}(v) + \cdots \end{pmatrix},$$

then if we define

$$\lambda_{v_0} = \phi_1(v_0) - \frac{p'(v_0)\phi_2(v_0)}{r'(v_0) - kp(v_0)},$$

it follows for the proposition 8.1, that there exists holomorphic solution $(h_{1,k}, h_{2,k})$ of (6).
In any case we could define

$$H_l(x, v) = \begin{pmatrix} x + x^{l+1}h_{1,l}(v) \\ v + x^l h_{2,l}(v) \end{pmatrix} \text{ and } F_l = H_l^{-1} \circ F_{l-1} \circ H_l.$$

See that in this case F_l and H_l are holomorphic diffeomorphisms in U.
Moreover, $H = \lim_{n\to\infty} H_n \circ \cdots \circ H_0$ is a semiformal diffeomorphism that conjugates F and $F_{\lambda_{v_0}}$ in U. □

Since this conjugation is defined locally around some root of $r(v)$, and we are interested in some global conjugation defined at some neighborhood of the divisor, first we need to construct some diffeomorphism that can be locally conjugate with F around every point of the divisor.
For that, let $L_F(v)$ denote the Lagrange interpolation Polynomial of the points

$$(v_1, \lambda_1), \ldots, (v_{k+2}, \lambda_{k+2}),$$

where v_j is root of $r(v) = 0$ and λ_j is the constant found in the theorem 8.1.
It is a simple consequence of the theorem 8.1 that for all U simply connected

open set that contains only one root of $r(v) = 0$ there exists a semiformal conjugation in U that conjugates F and

$$\begin{pmatrix} x \\ v \end{pmatrix} \mapsto \begin{pmatrix} x + x^{k+1}p(v) + L_F(v)x^{2k+1} \\ v + x^k r(v) \end{pmatrix}. \tag{11}$$

Notice that the diffeomorphism represented by (11) does not necessarily come from the blow up of some element of $\mathrm{Diff}(\mathbb{C}^2, 0)$.
But it is easy to see, using the theorem 8.1 that the blow up of

$$F_L(x,y) = \begin{pmatrix} x + P_{k+1}(x,y) + L_F(\frac{y}{x})x^{2k+1} \\ y + Q_{k+1}(x,y) - x^{2k-1}yL_F(\frac{y}{x}) \end{pmatrix}$$

is locally semiformally conjugate with (11), and then it is locally semiformally conjugate with F for $k \geq 2$. Moreover, in the chart (x, v) the diffeomorphism F_L has the following representation

$$\Pi^* F_L(x,v) = \begin{pmatrix} x + x^{k+1}p(v) + L_F(v)x^{2k+1} \\ v + x^k r(v) - x^{2k}r(v)p(v) + x^{3k}(\cdots) \end{pmatrix}.$$

Let $\mathcal{U} = \{U_i\}_{i=1,\ldots,k+2}$ a covering of $\mathbb{C}P(1) = \Pi^{-1}(0)$ such that U_i, $U_i \cap U_j$ are simply connected open sets, such that no four of them have a nonempty intersection and v_i is in U_j if and only if $i = j$. For each j there exists a semiformal diffeomorphism H_j such that $H_j \circ F = F_L \circ H_j$ in U_j.
Let $H_{ij} = H_j \circ H_i$ semiformal diffeomorphism defined in $U_j \cap U_i$. Observe that each $H_{i,j}$ commutes with F_L.

Theorem 8.2 *The cocycle $\{H_{ij}\}$ determine the class of formal conjugation of F.*

First, we are going to prove that the cocycle $\{H_{ij}\}$ determine the class of semiformal conjugation of F. In fact suppose that $\{H_{ij} = H'_j \circ H'^{-1}_i\}$ is the cocycle associate to G. Then

$$H_j \circ H_i^{-1} = H'_j \circ H'^{-1}_i$$

in $U_i \cap U_j$, thus $H = H_j^{-1} \circ H'_j = H_i^{-1} \circ H'_i$ is a global semiformal diffeomorphism that conjugate F and G. Let

$$\begin{pmatrix} x \\ v \end{pmatrix} \mapsto \begin{pmatrix} x + \sum\limits_{j=k+1}^{\infty} a_{1j}(v)x^j \\ v + \sum\limits_{j=k}^{\infty} a_{2j}(v)x^j \end{pmatrix}$$

be the representation of H at the chart $(x,v) \in U_1$ and

$$\begin{pmatrix} y \\ s \end{pmatrix} \mapsto \begin{pmatrix} y + \sum_{j=k+1}^{\infty} b_{1j}(s) y^j \\ s + \sum_{j=k}^{\infty} b_{2j}(s) y^j \end{pmatrix}$$

be the representation of H at the chart $(y,s) \in U_2$ where $U_1 \simeq U_2 \simeq (\mathbb{C},0)\times\mathbb{C}$ and the change of coordinates between U_1 and U_2 is given by

$$\begin{array}{ccc} U_1 \cap U_2 & \to & U_1 \cap U_2 \\ (x,v) & \mapsto & (y,s) = (vx, \frac{1}{v}) \end{array}.$$

Therefore, we have that

$$\left(ys + \sum_{j=k+1}^{\infty} a_{1j}(\frac{1}{s})(ys)^j \right) \left(\frac{1}{s} + \sum_{j=k}^{\infty} a_{2j}(\frac{1}{s})(ys)^j \right) = y + \sum_{j=k+1}^{\infty} b_{1j}(s) y^j$$

and

$$\frac{1}{\frac{1}{s} + \sum_{j=k}^{\infty} a_{2j}(\frac{1}{s})(ys)^j} = s + \sum_{j=k}^{\infty} b_{2j}(s) y^j$$

for every $(y,s) \in (\mathbb{C},0) \times \mathbb{C}^*$.

Thus, it is easy to prove making the product and proceeding by induction that $a_{2j}(\frac{1}{s})s^{j-1}$ is holomorphic function in $\mathbb{C}$ and therefore $a_{2j}(v)$ has to be a polynomial of degree $\leq j-1$, the same way $a_{1j}(v)$ has to be a polynomial of degree $\leq j$. Then, we can conclude that H is a blow up of the formal diffeomorphism in the variables x, y that conjugate F and G. □

Observe that in the dicritic case, i.e. $r \equiv 0$, the system of equations (6) of the lemma 8.1 reduces to

$$\begin{pmatrix} (k-l)p(v) & p'(v) \\ 0 & -lp(v) \end{pmatrix} \begin{pmatrix} h_1(v) \\ h_2(v) \end{pmatrix} = \begin{pmatrix} \phi_1(v) \\ \phi_2(v) \end{pmatrix} \tag{12}$$

Theorem 8.3 *Let $\tilde{F} \in \widehat{\mathrm{Diff}}_{k+1}(\mathbb{C}^2,0)$ be dicritic diffeomorphism and $F(x,v) = \begin{pmatrix} x + x^{k+1}p(v) + x^{k+2}(\cdots) \\ v + x^{k+1}(\cdots) \end{pmatrix}$. Then there exists a unique rational function $q(v)$ such that F is semiformally conjugate to*

$$G_F = \begin{pmatrix} x + x^{k+1}p(v) + x^{2k+1}q(v) \\ v \end{pmatrix}$$

in $\overline{\mathbb{C}} \setminus \{p(v) = 0\}$*. In addition,* $q(v) = \frac{s(v)}{p(v)^{2k+1}}$ *where* $s(v)$ *is a polynomial of degree* $2k + 2 + 2k\partial(p(v))$*.*

Proof: Follows the same procedure to the proof of theorem 8.1. Define by induction the sequences F_n and H_n from the initial condition $F_0 = F$ and $H_0 = id$, and for $j > 0$ ($j \neq k$), if

$$F_{j-1} - G_F = \begin{pmatrix} x^{k+j+1}\phi_{1,j}(v) + \cdots \\ x^{k+j}\phi_{2,j}(v) + \cdots \end{pmatrix}$$

define

$$H_j = (x, v) = \begin{pmatrix} x + x^{j+1}h_{1,j}(v) \\ v + x^j h_{2,j}(v) \end{pmatrix} \text{ and } F_j = H_j^{-1} \circ F_{j-1} \circ H_j,$$

where $(h_{1,j}, h_{2,j})$ is the unique solution of (12), i.e. $h_{1,j} = \frac{\phi_{1,j}(v)}{(k-j)p(v)} + \frac{p'(v)\phi_{2,j}(v)}{jp(v)^2}$ and $h_{2,j}(v) = -\frac{\phi_{2,j}(v)}{jp(v)}$. See that the degree of $h_{1,j}$ and $h_{2,j}$ are $j+1$ and $j+2$. In the case $j = k$ we have that

$$F_{k-1} - G_F = \begin{pmatrix} x^{2k+1}(\phi_{1,k}(v) - q(v)) + \cdots \\ x^{2k}\phi_{2,k}(v) + \cdots \end{pmatrix},$$

where $q(v)$ is the unique function such that the system of equations (12) has solution, i.e. $h_{2,k}(v) = -\frac{\phi_{2,k}(v)}{kp(v)} = \frac{\phi_{1,k}(v) - q(v)}{p'(v)}$, therefore

$$q(v) = \phi_{1,k}(v) - \frac{p'(v)\phi_{2,k}(v)}{kp(v)}.$$

Finally, it is easy to prove inductively that $\phi_{i,j}$ ($i = 1, 2$) are of the form $\frac{\psi_{i,j}}{p(v)^{2j}}$ where $\psi_{1,j}$ and $\psi_{2,j}$ are polynomials of degree $k + j + 1 + 2j\partial(p(v))$ and $k + j + 2 + 2j\partial(p(v))$ respectively, and then we conclude that $q(v)$ is of form $q(v) = \frac{s(v)}{p(v)^{2k+1}}$ where $s(v)$ is a polynomial of degree $2k+2+2k\partial(p(v))$.□

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