Acoplamento Poço-Reservatório via Elementos Finitos.

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PRH-32

Finite elements for well-reservoir coupling

Acoplamento Poço-Reservatório via
Elementos Finitos

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Rio de Janeiro
April, 2004

To my fathers
To my brothers

Abstract

We analyze the coupling across an interface of fluid and porous media flows. Some applications are: *Coupling surface and groundwater water flow*, *well and oil reservoir* and *biofluid dynamics models (several organs can be viewed as a porous medium, organs like brain, heart, lung)*. We consider the Stokes equations in the fluid region and Darcy law for the filtration velocity in the porous medium. Beavers-Joseph conditions for the interface are considered. We use the porous pressure as a Lagrange multiplier to couple the model and develop inf-sup conditions at the continuous and discrete levels. Using the second order Taylor-Hood and the lowest Raviart-Thomas finite elements, optimal discrete approximations and inf-sup conditions based on constructing Fortin's interpolations are provided. Numerical experiments are presented.

Keywords: *Inf-sup* condition, Stokes-Darcy, Finite Elements.

Acknowledgment

I want to thank, first of all, to IMPA for providing the adequate atmosphere of study.

I am also thankful to my adviser Professor Marcus Sarkis for his help and collaboration during the accomplishment of the work and to the IMPA researchers who were my professors during this time, specially to Professor Carlos Isnard.

I thank Professor Felipe Linares and Professor Paulo Goldfeld for taking part in the examination committee.

I further express my gratefulness to my colleagues from IMPA.

I am grateful to my family in Colombia for their encouragement and support.

Finally I want to thank to Agência Nacional do Petróleo (ANP-PRH32) for financial help.

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April, 2004.

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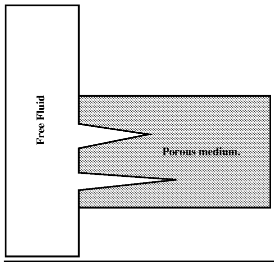
Introduction.

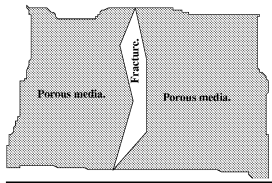
In this short chapter we present an overview of some applications of coupling fluid flow with porous media flow (and related models, i.e., models that consist of heterogeneous submodels.).



A simplified transversal view of a perforated well.

Application to oil industry. The main application to oil industry is the oil reservoir modeling. We can obtain more realistic models of coupling well-reservoir that include the geometry of the well completion[1]. Overall productivity of perforated wells is influenced by this geometry, for instance, the number and length of the perforations. We also can consider models of loss of recovering due to obstructions in perforated wells.

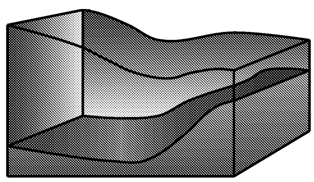
We have applications to modeling of disconnected fractures (the fractures are separated from each other with a gap).



A simplified transversal view of a disconnected fracture.

Similar models can be applied to situations involving interfaces between domains with different physical properties, for example, linear and nonlinear coupling.

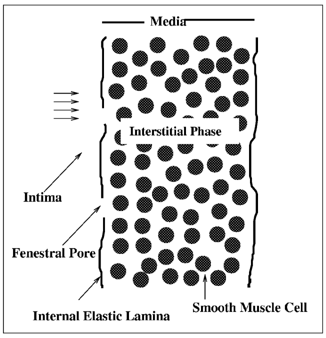
Coupling groundwater and surface water flow. Consider water, and sand as the porous media. In this case the model helps in the task to understand how beach are formed and how is the dynamic caused by the water. This model can be used to simulate the effect of flooding in dry areas and combined with transport-diffusion models can be used to study the propagation and diffusion of pollutants dispersed in water. We remark that in the model presented here, the interface between the fluid and the porous media is a rigid hypersurface. Models with not rigid and/or moving interface have a different analysis.



Coupling groundwater and surface water flow.

On the other hand, groundwater is an important water source for both domestic and agricultural usage. The ability to manage groundwater resources requires an understanding of groundwater/surface water interactions and if we (somehow) know the geometry of water reservoir we can apply models similar to the one presented here in order to improve the rep-

[1]Open-Hole completion or a Through-Casing completion using "perforating guns".



Tanken from Am J Physiol Heart Circ Physiol 278:H1589–H1597, 2000

A simplified transversal view of a blood vessel wall.

resentations of these interactions.

Applications to Biomathematics. In the human body we can find several organs that we can view as a porous medium (organs like brain, heart, lung, and kidney) and several (bio)fluids (as blood) that interact with them[2]. For instance, the numerical modeling of solutes absorption processes by the arterial wall helps to understand the relationships between the local features of blood flow and the arterial wall (in this case the arterial wall must be considered as rigid). The dynamics of different blood solutes (as oxygen and low density lipids) as well as their absorption through the arterial walls, could play a relevant role in the genesis of some pathologies (such as astherosclerosis). In the case of arterial walls we can model interstitial flow in media. In a simplified transversal view of a blood vessel wall the main components are the *media* which is considered as the porous medium. The *internal elastic lamina* which is an impermeable barrier to water flow, except for *fenestral pores*, that separates *intima* (the most inner part of the vessel wall) from media.

[2]Although blood is not a Newtonian fluid (it is a suspension of red blood cells, white blood cells and platelets in plasma), the Newtonian assumption is considered acceptable as a first approximation for the flow in medium-to-large vessels.

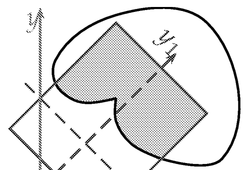
Chapter 1

Preliminaries and Notations.

The aim of this chapter is to present a survey of background results needed in this work. Basic notions and notations are presented.

Section 1.1 presents some results from Sobolev spaces. Most of the material consists of standard definitions and basic results. The main results for our purpose are Lemmas 1.6, and 1.7. They characterize the Sobolev space $H^{1/2}(\Gamma)$ and its dual. The space $H^{1/2}(\Gamma)$ is chosen for one of the variables (the Lagrange multiplier) in the weak formulation of the problem studied in Chapter 2. Lemma 1.14 is used to establish the *inf-sup* condition, in order to get existence and uniqueness of the solution. Section 1.3 discusses the general abstract saddle point theory which is the framework to study numerical solutions of Stokes equation and related models. Basic results about finite element approximations are presented. Fortin's criterion (Lemma 1.21) is an important tool to achieve our objective.

1.1 Sobolev spaces.



A Lipschitz domain.

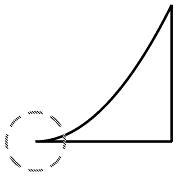
Given $\Omega \subset \mathbb{R}^n$, a Lipschitz domain[1], let $L^2(\Omega)$ be the space of square Lebesgue integrable functions, that is:

$$L^2(\Omega) := \left\{ \psi : \Omega \to \mathbb{R} / \int_\Omega |\psi|^2 < \infty \right\}$$

[1]This means that $\partial\Omega$ is locally the graph of a Lipschitz continuous function and that Ω lies on one side of this graph. Usually, n is two or three and Ω is a bounded and open set.

with the usual norm given by $\|\psi\|_{L^2(\Omega)} := \left(\int_\Omega |\psi|^2\right)^{\frac{1}{2}}$ which is obtained from the inner product:

$$(\phi,\psi)_{L^2(\Omega)} := \int_\Omega \phi\psi.$$


A Non Lipschitz domain.

Denote by $L^2_0(\Omega)$ the subspace of $L^2(\Omega)$ involving the functions of zero average.

The space $\boldsymbol{L}^2(\Omega) := L^2(\Omega)^n$ is the cartesian product of $L^2(\Omega)$ n times with the norm

$$\|f\|^2_{L^2(\Omega)} := \sum_1^n \|f_i\|^2_{L^2(\Omega)}.$$

Let $C_0^\infty(\Omega)$ be the space of infinitely differentiable functions having compact support in Ω. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be an n-tuple of nonnegative integers. α is called a *multiindex*. Define:

$$|\alpha| := \sum_1^n \alpha_j, \quad \alpha! := \alpha_1!\alpha_2!\ldots\alpha_n!.$$

It is used the shorthand $\partial_j = \frac{\partial}{\partial x_j}$ for partial derivatives on $\mathbb{R}^n$. Higher-order derivatives are expressed by:

$$\partial^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1}\left(\frac{\partial}{\partial x_2}\right)^{\alpha_2}\cdots\left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}.$$

Denote by ∂u, ∇u or $\operatorname{grad} u$ the n-tuple of functions $(\partial_1 u, \ldots, \partial_n u)$.

Let $\mathcal{D}(\Omega)$ be the space $C_0^\infty(\Omega)$ with the following sense of convergence: $\{f_n\}$ converges if there exist a function $f \in C_0^\infty(\Omega)$ such that the supports of $\{f_n\}$ are all contained in a compact subset of Ω and their derivatives $\{\partial^\alpha f_n\}$ converge uniformly to $\partial^\alpha f$ for all multiindex α.

Let the dual of $\mathcal{D}(\Omega)$ be the space of distributions, i.e., the space of linear functionals on $\mathcal{D}(\Omega)$ that are continuous with respect to the notion of convergence defined above. It is denoted by $\mathcal{D}'(\Omega)$.

The notation $\langle\cdot,\cdot\rangle$ is used for the duality pair between $\mathcal{D}(\Omega)$ and its dual, that is, we write:

$$\langle f,\psi\rangle, \quad \psi \in \mathcal{D}(\Omega),\ f \in \mathcal{D}'(\Omega).$$

If f is a distribution and α is a multiindex it is possible to define its derivative (in the sense of distributions) $\partial^\alpha f$ by:

$$\langle\partial^\alpha f,\psi\rangle = (-1)^{|\alpha|}\langle f,\partial^\alpha\psi\rangle, \quad \psi \in \mathcal{D}(\Omega).$$

Now we define a very important family of subspaces of $L^2(\Omega)$, the *Sobolev Spaces*. By definition, a function $f \in L^2(\Omega)$ belongs to $H^k(\Omega)$, $k \in \mathbb{N}$, if $|\alpha| \leq k$ implies $\partial^\alpha f \in L^2(\Omega)$, or more precisely:

$$H^k(\Omega) := \Big\{ f \in L^2(\Omega) \ : \ \forall \alpha, |\alpha| \leq k, \quad \text{exists} \ \ f_\alpha \in L^2(\Omega), \\ \text{s. t. } \langle \partial^\alpha f, \psi \rangle = \int_\Omega f_\alpha \psi, \ \ \forall \psi \in \mathcal{D}(\Omega) \Big\}.$$

In $H^k(\Omega)$ it is considered the following inner product:

$$(f, g)_{H^k(\Omega)} := \sum_{|\alpha| \leq k} (\partial^\alpha f, \partial^\alpha g)_{L^2(\Omega)}.$$

which gives the norm:

$$\|f\|^2_{H^k(\Omega)} := (f, f)_{H^k(\Omega)} = \sum_{|\alpha| \leq k} \|\partial^\alpha f\|^2_{L^2(\Omega)} = \sum_{|\alpha| \leq k} \int_\Omega |\partial^\alpha f|^2.$$

A very important functional on $H^k(\Omega)$ is the seminorm given by:

$$|f|^2_{H^k(\Omega)} := \sum_{|\alpha| = k} \|\partial^\alpha f\|^2_{L^2(\Omega)} = \sum_{|\alpha| = k} \int_\Omega |\partial^\alpha f|^2.$$

For other definitions of Sobolev (and related) spaces see [11].

There are different ways to define $H^s(\Omega)$, $s \in \mathbb{R}$, (see [5],[7],[11]). Here we use the one based on the following norm:

$$\|u\|^2_{H^s(\Omega)} := \|u\|^2_{H^{[s]}(\Omega)} + |u|^2_{H^s(\Omega)}$$

with $[s]$ the integer part of s and the seminorm:

$$|u|^2_{H^s(\Omega)} := \sum_{|\alpha| = [s]} \int_\Omega \int_\Omega \frac{\big(\partial^\alpha u(x) - \partial^\alpha u(y)\big)^2}{\|x - y\|^{n+2\sigma}} dx dy.$$

where $\sigma = s - [s]$.

Define $H^s_0(\Omega)$ as the closure of $\mathcal{D}(\Omega)$ in $H^s(\Omega)$. For $s < 0$, $H^s(\Omega)$ is by definition the dual space of $H_0^{-s}(\Omega)$ with the usual norm.

With Ω open and Lipschitz continuous, define the space $H^s(\partial\Omega)$ by the norm:

$$\|u\|^2_{H^s(\partial\Omega)} := \|u\|^2_{H^{[s]}(\partial\Omega)} + |u|^2_{H^s(\partial\Omega)}$$

with the seminorm:

$$|u|^2_{H^s(\partial\Omega)} := \sum_{|\alpha| = [s]} \int_{\partial\Omega} \int_{\partial\Omega} \frac{(\partial^\alpha u(x) - \partial^\alpha u(y))^2}{\|x - y\|^{2\sigma + n - 1}} dS_x dS_y, \quad \sigma = s - [s].$$

For $\Gamma \subsetneq \partial\Omega$, with non-vanishing $(n-1)$-dimensional measure and relatively open with respect to $\partial\Omega$, it is possible to define $H^s(\Gamma)$ in the same way as $H^s(\partial\Omega)$, that is, using the norm:

$$\|u\|^2_{H^s(\Gamma)} := \|u\|^2_{H^{[s]}(\Gamma)} + |u|^2_{H^s(\Gamma)}$$

with the seminorm ($\sigma = s - [s]$):

$$|u|^2_{H^s(\Gamma)} := \sum_{|\alpha|=[s]} \int_\Gamma \int_\Gamma \frac{(\partial^\alpha u(x) - \partial^\alpha u(y))^2}{\|x-y\|^{2\sigma+n-1}} dS_x dS_y.$$

Under suitable conditions on Ω this is equivalent to the following definition:

$$H^s(\Gamma) := \{u = v|_\Gamma \quad : \quad v \in H^s(\partial\Omega)\}$$

with the norm suggested by it.

Define:

$$H^s_{00}(\Gamma) := \{u \in H^s(\Gamma) \quad : \quad E_0(u) \in H^s(\partial\Omega)\}$$

where $E_0(u)$ is the extension by zero outside Γ. Put $N = \partial\Omega \setminus \Gamma$. Let Γ and N be connected subsets of $\partial\Omega$ and S_1, S_2 the end points of Γ and N. When $n = 2$ if $u \in H^{1/2}(\Gamma)$ we have from the definition of the $H^{1/2}(\partial\Omega)$ seminorm

$$\begin{aligned}
|E_0(u)|_{H^{1/2}(\partial\Gamma)} &= \int_{\partial\Omega}\int_{\partial\Omega} \frac{(E_0u(x) - E_0u(y))^2}{\|x-y\|^2} dS_x dS_y \\
&= \int_\Gamma\int_\Gamma \frac{(u(x)-u(y))^2}{\|x-y\|^2} dS_x dS_y + \\
&\qquad + 2\int_\Gamma\int_N \frac{(0-u(y))^2}{\|x-y\|^2} dS_x dS_y \\
&= |u|_{H^{1/2}(\Gamma)} + 2\int_\Gamma (u(y))^2 \int_N \frac{1}{\|x-y\|^2} dS_x dS_y \\
&= |u|_{H^{1/2}(\Gamma)} + 2\int_\Gamma (u(y))^2 r(y) dS_y
\end{aligned}$$

where for $y \in \Gamma$, $r(y) := \int_N \|x-y\|^{-2} dS_x$. Now let s_j be the distance along $\partial\Omega$ with S_j as starting point and let $x(s_j)$ be the point on $\partial\Omega$ whose distance to S_j is s_j. There exists $\epsilon > 0$ such that

$$0 < s_j < \epsilon \text{ implies } x(s_j) \in \Gamma$$
$$0 > s_j > -\epsilon \text{ implies } x(s_j) \in N.$$

Then $|E_0(u)|_{H^{1/2}(\partial\Omega)} < \infty$ is equivalent to the conditions:

$$\int_0^\epsilon (u(y(s_j)))^2 \int_0^\epsilon \frac{1}{(s_j - t)^2} dt ds_j < \infty, \qquad j = 1, 2. \tag{1.1}$$

This follows from the fact that the function $r(y(t))$ is bounded in the region given by $s_j \geq \epsilon$ and that $u \in L^2(\Gamma)$. We also use the fact that the function $\|x(s_j) - y(t)\|$ is equivalent to the function $|s_j - t|$ when $s_j < \epsilon$. To ensure this we have to suppose that the "angles" between Γ and N at S_j are not zero.

From (1.1) we get the conditions

$$\int_0^{\epsilon} (u(y(s_j)))^2 \frac{1}{s_j} ds_j < \infty, \qquad j = 1, 2.$$

These two conditions can be expressed as

$$\int_0^{L} (u(y(s_1)))^2 \left[\frac{1}{s_1} + \frac{1}{L - s_1}\right] ds_1 < \infty, \qquad j = 1, 2.$$

where L is the length of Γ. Then an equivalent norm for $H_{00}^{1/2}(\Gamma)$ is obtained by :

$$\|u\|_{H_{00}^{1/2}(\Gamma)} := |u|_{H^{1/2}(\Gamma)} + \int_0^{L} (u(y(s_1)))^2 \left[\frac{1}{s_1} + \frac{1}{L - s_1}\right] ds_1.$$

The following well known result will be used (see [11], [7], [14]):

Lemma 1.1 (Trace Theorem). *Let Ω be Lipschitz continuous. Then the operator $\gamma_0 : C^\infty(\overline{\Omega}) \to C^\infty(\overline{\partial\Omega})$, mapping a function into its restriction on the boundary, can be extended continuously to an operator $\gamma_0 : H^1(\Omega) \to H^{1/2}(\partial\Omega)$. Also, there is a continuous* lifting *operator $\mathbf{R}_0 : H^{1/2}(\partial\Omega) \to H^1(\Omega)$, such that $\gamma_0 \mathbf{R}_0 u = u$, for all $u \in H^{1/2}(\partial\Omega)$.*

The operator γ_0 is known as trace operator on $\partial\Omega$. Similarly, given $\Gamma \subset \partial\Omega$ with non-vanishing $(n-1)$-dimensional measure and relatively open with respect to $\partial\Omega$, a trace on Γ can be defined. We denote by $H_0^1(\Omega, \Gamma)$ the subspace of $H^1(\Omega)$ of functions that vanish on Γ.

Let P_k be the space of polynomials of degree k or less. Next lemma is very useful in order to look for equivalent norms in Sobolev spaces.

Lemma 1.2. *Let Ω be a (Lipschitz) continuous domain. $k, l \in \mathbb{Z}^+$, f_i, $i = 1, \ldots, l$ continuous functionals (not necessary linear) on $H^k(\Omega)$ such that $\phi \in P_{k-1}$ implies:*

$$\sum_{i=1}^{l} |f_i(\phi)|^2 = 0 \iff \phi = 0 \text{ on } \Omega. \tag{1.2}$$

Then there exists a constant, depending only on Ω and the functionals f_i, such that,

$$\|u\|_{H^k(\Omega)} \leq C_1 \left[|u|^2_{H^k(\Omega)} + \sum_{i=1}^{l} |f_i(u)|^2\right]^{1/2}.$$

Proof. Assume that for all $n \in \mathbb{Z}^+$ there exists a function $u_n \in H^k(\Omega)$ with $\|u_n\|_{H^k(\Omega)} = 1$ such that

$$\left[|u_n|^2_{H^k(\Omega)} + \sum_{i=1}^{l} |f_i(u_n)|^2\right]^{1/2} < \frac{1}{n}.$$

Then, if $|\alpha| = k$ we have

$$D^\alpha u_n \to 0 \tag{1.3}$$

and $f_i(u_n) \to 0$ for all i. Using the well known fact that the inclusion of $H^k(\Omega)$ into $H^{k-1}(\Omega)$ is completely continuous when Ω is continuous[2] (i.e., $\partial\Omega$ is continuous and Ω is locally at one side of $\partial\Omega$, see [11], pag. 108 theorem 6) we can pass to a convergent subsequence in $H^{k-1}(\Omega)$. Let

$$u = \lim_{H^{k-1}(\Omega)} u_n.$$

From (1.3) we see that $u_n \to u$ in $H^k(\Omega)$ and $D^\alpha u = 0$ when $|\alpha| = k$.

Since $D^\alpha u = 0$ when $|\alpha| = k$ it is possible to construct u_h, $h \in \mathbb{R}$ such that $u_h \in P_{k-1}$ and $u_h \to u$ when $h \to 0$ (see [10], pag. 72). Then since P_{k-1} is closed in $H^k(\Omega)$ we get $u \in P_{k-1}$. In fact we can use:

$$u_h(x) := \frac{1}{h^n} \int_\Omega \omega_h(x-y)u(y)dy$$

where

$$r = \int_{\|x\|<h} e^{\frac{\|x\|^2}{\|x\|^2-h^2}} dx \quad \text{and} \quad \omega_h(x) := \begin{cases} \frac{1}{r} e^{\frac{\|x\|^2}{\|x\|^2-h^2}}, & \text{if } \|x\| < h \\ 0, & \text{if } \|x\| \geq h \end{cases}.$$

Now, $u \in P_{k-1}$, $f_i(u) = 0$ for $i = 1, \ldots, l$ and (1.2) imply $u = 0$.

On the other hand:

$$\|u\|_{H^k(\Omega)} = \lim \|u_n\|_{H^k(\Omega)} = 1.$$

which gives a contradiction. This ends the proof. ✠

The following two lemmas are important applications of Lemma 1.2 when $k = 1$ and they turn out to be particular cases of the functionals f_i.

Lemma 1.3 (Poincaré Inequality). *Let $u \in H^1(\Omega)$. Then there exist constants, depending only on Ω, such that*

$$\|u\|^2_{L^2(\Omega)} \leq C_1 |u|^2_{H^1(\Omega)} + C_2 \left(\int_\Omega u\right)^2.$$

[2]When $k = 1$ this fact is known as the Rellich Theorem.

In particular the seminorm $|\cdot|_{1,\Omega}$ is equivalent to the norm $\|\cdot\|_{H^1(\Omega)}$ in $H^1(\Omega)\cap L^2_0(\Omega)$. We can obtain similar results using the integral over a subset of Ω or even in part of the boundary using the Trace Theorem (Lemma 1.1).

Lemma 1.4 (Friedrichs Inequality). *Let $\Gamma\subset\partial\Omega$ with non-vanishing $(n-1)$-dimensional measure and relatively open with respect to $\partial\Omega$. Then there exist constants, depending only on Ω and Γ, such that for $u\in H^1(\Omega)$,*

$$\|u\|^2_{L^2(\Omega)}\le C_1|u|^2_{H^1(\Omega)}+C_2\|u\|^2_{L^2(\Gamma)}.$$

In particular, if $u\in H^1_0(\Omega,\Gamma)$ the $H^1(\Omega)$-seminorm is a norm equivalent to the $H^1(\Omega)$-norm.

Next lemmas are going to be useful, they characterize the space $H^{1/2}(\Gamma)$.

Lemma 1.5. *Given $\mu\in H^{1/2}(\Gamma)$, define:*

$$E_\eta(\mu):=\gamma_0(\phi)\ \textit{where } \phi \textit{ is the solution of }\ \begin{cases}-\Delta\phi=0 & \textit{in } \Omega\\ \phi=\mu & \textit{on } \Gamma\\ \partial_\eta\phi=0 & \textit{on } \partial\Omega-\Gamma\end{cases}\tag{1.4}$$

then $E_\eta(\mu)\in H^{1/2}(\partial\Omega)$ and $\|E_\eta(\mu)\|_{H^{1/2}(\partial\Omega)}\le C\|\mu\|_{H^{1/2}(\Gamma)}$.

This follows from classical regularity results (see [11], [13]) and the Trace Theorem (Lemma 1.1).

Remark. *In Lemma 1.5 we also have $\|\mu\|_{H^{1/2}(\Gamma)}\le\|E_\eta(\mu)\|_{H^{1/2}(\partial\Omega)}$.*

Using Lemma 1.5 we can consider $H^{1/2}(\Gamma)$ as a subset of $H^{1/2}(\partial\Omega)$. Then we have the following result:

Lemma 1.6. *Let Ω be Lipschitz continuous and $\Gamma\subset\partial\Omega$ with non-vanishing $(n-1)$-dimensional measure and relatively open with respect to $\partial\Omega$. Then for all $u\in H^{1/2}(\partial\Omega)$ there exist $u_1\in H^{1/2}(\Gamma)$ and $u_0\in H^{1/2}_{00}(\partial\Omega\setminus\Gamma)$ such that $u=E_\eta(u_1)+E_0(u_0)$. This decomposition is unique.*

Proof. Let $u\in H^{1/2}(\partial\Omega)$ and put $N=\partial\Omega\setminus\Gamma$. Take

$$u_1=u|_\Gamma \quad\text{and}\quad u_0=\phi|_N \text{ where } \phi=u-E_\eta(u_1)$$

and observe that $u_1\in H^{1/2}(\Gamma)$ and

$$\|E_\eta(u_1)\|_{H^{1/2}(\partial\Omega)}\le C\|u|_\Gamma\|_{H^{1/2}(\Gamma)}\le C\|u\|_{H^{1/2}(\partial\Omega)}$$

so $\phi\in H^{1/2}(\partial\Omega)$, and $E_0(u_0)=\phi$ because u and $E_\eta(u_1)$ coincide on Γ. For the uniqueness, if $0=E_\eta(u_1)+E_0(u_0)$ then $E_\eta(u_1)$ is a weak solution of the problem:

$$\begin{cases} -\Delta\phi = 0 & \text{in } \Omega \\ \phi = 0 & \text{on } \Gamma \\ \partial_\eta \phi = 0 & \text{on } \partial\Omega - \Gamma \end{cases} \tag{1.5}$$

so $u_1 = 0$. ☩

The space $H^{-1/2}(\partial\Omega)$ is by definition the dual of $H^{1/2}(\partial\Omega)$ with the usual norm. For $\Gamma \subset \partial\Omega$ with non-vanishing $(n-1)$-dimensional measure and relatively open with respect to $\partial\Omega$ we have two dual spaces, the dual of $H_{00}^{1/2}(\Gamma)$, denoted by $H_{00}^{-1/2}(\Gamma)$, and $H^{-1/2}(\Gamma)$ the dual space of $H^{1/2}(\Gamma)$. The first one is larger than the second.

If $f \in H^{-1/2}(\partial\Omega)$, then $f|_\Gamma = 0$ means by definition that:

$$\langle f, E_0(\phi)\rangle_{\partial\Omega} = 0 \quad \text{for all } \phi \in H_{00}^{1/2}(\Gamma). \tag{1.6}$$

We have the following result:

Lemma 1.7. *For all $f \in H^{-1/2}(\partial\Omega)$, there are $f_1 \in H^{-1/2}(\Gamma)$ and $f_0 \in H_{00}^{-1/2}(\partial\Omega \setminus \Gamma)$ such that, for all $u \in H^{1/2}(\partial\Omega)$, $u = E_\eta(u_1) + E_0(u_0)$ as in Lemma 1.6, and we have:*

$$\langle f, u\rangle_{\partial\Omega} = \langle f_1, u_1\rangle_\Gamma + \langle f_0, u_0\rangle_N \tag{1.7}$$

Proof. For $u_1 \in H^{1/2}(\Gamma)$ define:

$$\langle f_1, u_1\rangle_\Gamma = \langle f, E_\eta(u_1)\rangle_{\partial\Omega}$$

and for $u_0 \in H_{00}^{1/2}(N)$

$$\langle f_0, u_0\rangle_N = \langle f, E_0(u_0)\rangle_{\partial\Omega}$$

we have:

$$\langle f_1, u_1\rangle_\Gamma \le \|f\|_{H^{-1/2}(\partial\Omega)}\|E_\eta(u_1)\|_{H^{1/2}(\partial\Omega)} \le C\|f\|_{H^{-1/2}(\partial\Omega)}\|u_1\|_{H^{1/2}(\Gamma)}$$

so $f_1 \in H^{-1/2}(\Gamma)$, analogously $f_0 \in H_{00}^{-1/2}(N)$. Moreover:

$$\langle f_1, u_1\rangle_\Gamma + \langle f_0, u_0\rangle_N = \langle f, E_\eta(u_1) + E_0(u_0)\rangle_{\partial\Omega} = \langle f, u\rangle_{\partial\Omega}.$$

☩

Remark. *In particular, if $f \in H^{-1/2}(\partial\Omega)$ and $f|_N = 0$, i.e., if (1.6) holds in $H_{00}^{1/2}(N)$, we have from (1.7) that:*

$$\langle f, u\rangle_{\partial\Omega} = \langle f_1, u_1\rangle_\Gamma. \tag{1.8}$$

Then functionals in $H^{-1/2}(\partial\Omega)$ which are zero restricted to $\partial\Omega \setminus \Gamma$ can be identified with functionals in $H^{-1/2}(\Gamma)$.

Lemma 1.8. *Given $f_1 \in H^{-1/2}(\Gamma)$ we can define $f \in H^{-1/2}(\partial\Omega)$ by $\langle f, u\rangle_{\partial\Omega} := \langle f_1, u_1\rangle_\Gamma$, where $u = E_\eta(u_1) + E_0(u_0)$ as in Lemma 1.6.*

We have a similar result for $f_0 \in H_{00}^{-1/2}(\partial\Omega \setminus \Gamma)$.

In Lemma 1.2, when $k = 1$, the principal element of the proof is that the inclusion $H^1(\Omega) \subset H^0(\Omega)$ is completely continuous. Since the inclusion $H^{1/2}(\partial\Omega) \subset L^2(\partial\Omega)$ is compact when Ω is an open and bounded Lipschitz domain, we can prove the following in the same way :

Lemma 1.9. *Let Ω be a continuous bounded domain and f_i, $i = 1, \dots, l$, $l \in \mathbb{Z}^+$, continuous functionals in $H^{1/2}(\partial\Omega)$, such that, if ϕ is constant on $\partial\Omega$*

$$\sum_{i=1}^{l} |f_i(\phi)|^2 = 0 \iff \phi = 0 \text{ on } \partial\Omega.$$

then, there is a constant, depending only on Ω and the functionals f_i, $i = 1, \dots, k$ such that:

$$\|u\|_{L^2(\partial\Omega)} \leq C_1 \left[|u|^2_{H^{1/2}(\partial\Omega)} + \sum_{i=1}^{l} |f_i(u)|^2 \right]^{1/2}.$$

With the help of this lemma we can find equivalent norms in subspaces of $H^{1/2}(\partial\Omega)$. In particular, we can obtain similar results as the Poincaré Inequality (Lemma 1.3). This is still true if we consider $\Gamma \subset \partial\Omega$ with non-vanishing $(n-1)$-dimensional measure and relatively open with respect to $\partial\Omega$.

Other important function space is defined with the *div* operator, the H-div space:

$$\boldsymbol{H}(\text{div}, \Omega) := \left\{ \boldsymbol{u} := (u_i)_1^n \in \boldsymbol{L}^2(\Omega) \;\middle|\; \nabla\cdot\boldsymbol{u} := \sum_1^n \partial^i u_i \in L^2(\Omega) \right\}$$

with the norm: $\|\boldsymbol{u}\|^2_{\boldsymbol{H}(\text{div},\Omega)} := \|\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\nabla\cdot\boldsymbol{u}\|^2_{L^2(\Omega)}$.

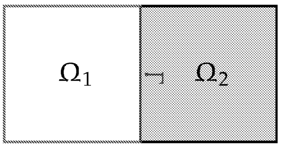
The following lemmas are going to be used:

Lemma 1.10 (Normal Trace Theorem). *Let Ω be Lipschitz continuous. Then the operator $\gamma_\eta : C^\infty(\overline{\Omega}) \to C^\infty(\overline{\partial\Omega})$, mapping a vector function into its normal component on the boundary, can be extended continuously to an operator $\gamma_\eta : \boldsymbol{H}(div, \Omega) \to H^{-1/2}(\partial\Omega)$. Also, there is a continuous* lifting *operator $\mathbf{R}_\eta : H^{-1/2}(\partial\Omega) \to \boldsymbol{H}(div, \Omega)$, such that $\gamma_\eta \mathbf{R}_\eta u = u$, for all $u \in H^{-1/2}(\partial\Omega)$.*

Lemma 1.11 (Green's formula). *If $\boldsymbol{u} \in \boldsymbol{H}(div, \Omega)$ and $\phi \in H^1(\Omega)$, then the following holds:*

$$\int_\Omega \boldsymbol{u}\cdot\nabla\phi + \int_\Omega \nabla\cdot\boldsymbol{u}\phi = \int_{\partial\Omega} (\boldsymbol{u}\cdot\boldsymbol{\eta})\phi. \tag{1.9}$$

The space $\boldsymbol{H}_0(\text{div},\Omega)$ consists of functions in H-div with vanishing normal component on $\partial\Omega$. $\boldsymbol{H}(\text{div}^0,\Omega)$, consists of functions in H-div with vanishing divergence. $\boldsymbol{H}_0(\text{div}_0,\Omega)$ is their intersection. Using (1.6) we can define the space $\boldsymbol{H}_0(\text{div},\Omega,\Gamma)$, $\Gamma\subset\partial\Omega$, as the subspace of $\boldsymbol{H}(\text{div},\Omega)$ of functions that its normal trace restricted to Γ is zero.



An open domain Ω with two subdomains.

In the next chapter we are going to work on an open domain Ω with two disjoint open subdomains Ω_1 and Ω_2 such that $\overline{\Omega}_1\cup\overline{\Omega}_2=\overline{\Omega}$. Put $\Gamma:=\overline{\Omega}_1\cap\overline{\Omega}_2$ and $\Gamma_j:=\partial\Omega_j\setminus\Gamma$. Then we are interested in vector functions $\boldsymbol{u}$ defined on Ω such that $\boldsymbol{u}|_{\Omega_1}$ belongs to $(H_0^1(\Omega,\Gamma_1))^2$ and $\boldsymbol{u}|_{\Omega_2}$ belongs to $\boldsymbol{H}_0(\text{div},\Omega,\Gamma_2)$. See Section 2.2.

Lemma 1.12. *There is a constant $C>0$, which depends only on Ω, such that for each $\boldsymbol{u}\in\boldsymbol{H}(div,\Omega)$ with $\int_{\partial\Omega}\boldsymbol{u}\cdot\boldsymbol{\eta}=\langle\boldsymbol{u}\cdot\boldsymbol{\eta},1\rangle_{\partial\Omega}=0$*

$$C\sup_{\substack{\phi\in H^{1/2}(\partial\Omega)\\ \phi\neq\, constant}}\frac{\langle\boldsymbol{u}\cdot\boldsymbol{\eta},\phi\rangle}{|\phi|_{H^{1/2}(\partial\Omega)}}\leq\|\boldsymbol{u}\cdot\boldsymbol{\eta}\|_{H^{-1/2}(\partial\Omega)}\leq\sup_{\substack{\phi\in H^{1/2}(\partial\Omega)\\ \phi\neq\, constant}}\frac{\langle\boldsymbol{u}\cdot\boldsymbol{\eta},\phi\rangle}{|\phi|_{H^{1/2}(\partial\Omega)}}.$$

Proof. Observe that if α is a constant $\langle\boldsymbol{u}\cdot\boldsymbol{\eta},\alpha\rangle=\alpha\langle\boldsymbol{u}\cdot\boldsymbol{\eta},1\rangle=0$ and that for $\phi\in H^{1/2}(\partial\Omega)$, ϕ nonconstant:

$$\frac{\langle\boldsymbol{u}\cdot\boldsymbol{\eta},\phi\rangle}{\|\phi\|_{H^{1/2}(\partial\Omega)}}\leq\frac{\langle\boldsymbol{u}\cdot\boldsymbol{\eta},\phi\rangle}{|\phi|_{H^{1/2}(\partial\Omega)}},$$

then

$$\|\boldsymbol{u}\cdot\boldsymbol{\eta}\|_{H^{-1/2}(\partial\Omega)}=\sup_{\substack{\phi\in H^{1/2}(\partial\Omega)\\ \phi\neq\text{ constant}}}\frac{\langle\boldsymbol{u}\cdot\boldsymbol{\eta},\phi\rangle}{\|\phi\|_{H^{1/2}(\partial\Omega)}}\leq\sup_{\substack{\phi\in H^{1/2}(\partial\Omega)\\ \phi\neq\text{ constant}}}\frac{\langle\boldsymbol{u}\cdot\boldsymbol{\eta},\phi\rangle}{|\phi|_{H^{1/2}(\partial\Omega)}}$$

which gives the right inequality.
By using Poincaré Inequality (Lemma 1.3), there exists a positive constant which depends only on Ω, such that

$$\|\psi\|^2_{H^{1/2}(\partial\Omega)}\leq c|\psi|^2_{H^{1/2}(\partial\Omega)}$$

holds for all $\psi\in H^{1/2}(\partial\Omega)$, with $\int_{\partial\Omega}\psi=0$. For $\phi\in H^{1/2}(\partial\Omega)$ nonconstant we have that

$$\psi:=\phi-\int_{\partial\Omega}\phi\neq 0$$

and

$$\frac{\langle\boldsymbol{u}\cdot\boldsymbol{\eta},\psi\rangle}{\|\psi\|_{H^{1/2}(\partial\Omega)}}=\frac{\langle\boldsymbol{u}\cdot\boldsymbol{\eta},\phi\rangle}{\|\psi\|_{H^{1/2}(\partial\Omega)}}\geq\frac{1}{c}\frac{\langle\boldsymbol{u}\cdot\boldsymbol{\eta},\phi\rangle}{|\psi|_{H^{1/2}(\partial\Omega)}}=\frac{1}{c}\frac{\langle\boldsymbol{u}\cdot\boldsymbol{\eta},\phi\rangle}{|\phi|_{H^{1/2}(\partial\Omega)}}.$$

✠

Using the argument in the last proof we can show:

Lemma 1.13. *Let $\Gamma \subset \partial\Omega$ with non-vanishing $(n-1)$-dimensional measure and relatively open with respect to $\partial\Omega$. There is a constant $C > 0$, which depends only on Ω, such that for each $\mu \in H^{-1/2}(\Gamma)$ with $\int_\Gamma \mu = \langle \mu, 1\rangle_\Gamma = 0$*

$$C \sup_{\substack{\phi \in H^{1/2}(\Gamma) \\ \phi \neq constant}} \frac{\langle \mu, \phi\rangle_\Gamma}{|\phi|_{H^{1/2}(\Gamma)}} \leq \|\mu\|_{H^{-1/2}(\Gamma)} \leq \sup_{\substack{\phi \in H^{1/2}(\Gamma) \\ \phi \neq constant}} \frac{\langle \mu, \phi\rangle_\Gamma}{|\phi|_{H^{1/2}(\Gamma)}}.$$

This gives an equivalent norm in the subspace of $H^{-1/2}(\Gamma)$ of zero average functions. Define, for $\mu \in H^{-1/2}(\Gamma)$, μ with zero average ($\langle \mu, 1\rangle_\Gamma = 0$):

$$|\mu|_{H^{-1/2}(\Gamma)} := \sup_{\substack{\phi \in H^{1/2}(\Gamma) \\ \phi \neq \text{constant}}} \frac{\langle \mu, \phi\rangle_\Gamma}{|\phi|_{H^{1/2}(\Gamma)}} = \sup_{\substack{\phi \in H^{1/2}(\Gamma) \\ \int_\Gamma \phi = 0}} \frac{\langle \mu, \phi\rangle_\Gamma}{|\phi|_{H^{1/2}(\Gamma)}}.$$

We have the following result:

Lemma 1.14. *For $\phi \in H^{1/2}(\Gamma)$ with $\int_\Gamma \phi = 0$ we have:*

$$|\phi|_{H^{1/2}(\Gamma)} = \sup_{\substack{\mu \in H^{-1/2}(\Gamma) \\ \langle \mu, 1\rangle = 0}} \frac{\langle \mu, \phi\rangle_\Gamma}{|\mu|_{H^{-1/2}(\Gamma)}}.$$

Proof. Denote $|\cdot|_{H^{1/2}(\Gamma)}$ by $|\cdot|_{\frac{1}{2}}$ and $\|\cdot\|_{H^{1/2}(\Gamma)}$ by $\|\cdot\|_{\frac{1}{2}}$.
Let $G := \{\alpha\phi \ : \ \alpha \in \mathbb{R}\}$. Define the linear functional $g : G \to \mathbb{R}$ by

$$g(\alpha\phi) := \alpha|\phi|^2_{\frac{1}{2}}.$$

Let $p : H^{1/2}(\Gamma) \to \mathbb{R}$ defined by

$$p(\lambda) := |\phi|_{\frac{1}{2}}|\lambda|_{\frac{1}{2}}$$

we have $g(\alpha\phi) \leq p(\alpha\phi)$ for all $\alpha \in \mathbb{R}$. Then from Hahn-Banach theorem (see [3]), g can be extended to a linear functional μ_0 defined on $H^{1/2}(\Gamma)$ with:

$$\langle \mu_0, \lambda\rangle_\Gamma \leq p(\lambda) = |\phi|_{\frac{1}{2}}|\lambda|_{\frac{1}{2}} \qquad \forall \lambda \in H^{1/2}(\Gamma),$$

then, using that $|\cdot|_{\frac{1}{2}} \leq \|\cdot\|_{\frac{1}{2}}$ we get $\mu_0 \in H^{-1/2}(\Gamma)$ and moreover, if we put $\mu_1 = \mu_0 - \int_\Gamma \mu_0$ we have:

$$\langle \mu_1, \phi\rangle_\Gamma = \langle \mu_0, \phi\rangle_\Gamma = g(\phi) = |\phi|^2_{\frac{1}{2}}$$

and

$$|\mu_1|_{H^{-1/2}(\Gamma)} := \sup_{\substack{\lambda \in H^{1/2}(\Gamma) \\ \int_\Gamma \lambda = 0}} \frac{\langle \mu_1, \lambda\rangle_\Gamma}{|\lambda|_{\frac{1}{2}}} = \sup_{\substack{\lambda \in H^{1/2}(\Gamma) \\ \int_\Gamma \lambda = 0}} \frac{\langle \mu_0, \lambda\rangle_\Gamma}{|\lambda|_{\frac{1}{2}}} = |\phi|_{\frac{1}{2}}$$

Then

$$\sup_{\substack{\mu\in H^{-1/2}(\Gamma)\\ \int_\Gamma \mu=0}} \frac{\langle \mu,\phi\rangle_\Gamma}{|\mu|_{H^{-1/2}(\Gamma)}} \geq \frac{\langle \mu_1,\phi\rangle_\Gamma}{|\mu_1|_{H^{-1/2}(\Gamma)}} = \frac{|\phi|^2_{\frac{1}{2}}}{|\phi|_{\frac{1}{2}}} = |\phi|_{\frac{1}{2}}$$

On the other hand, it is clear that for all μ with zero average we have:

$$|\mu|_{H^{-1/2}(\Gamma)} \geq \frac{\langle \mu,\phi\rangle_\Gamma}{|\phi|_{\frac{1}{2}}}$$

which gives the other inequality. This ends the proof. ∎

An alternative proof is obtained by considering $(H^{1/2}(\Gamma)\cap L^2_0(\Gamma))'$, the dual space of $H^{1/2}(\Gamma)\cap L^2_0(\Gamma)$, and by observing that a functional μ_0 defined on $H^{1/2}(\Gamma)\cap L^2_0(\Gamma)$ can be extended to one defined on $H^{1/2}(\Gamma)$, say μ, by the following formula: $\langle \mu,\lambda\rangle := \langle \mu_0,\lambda_0\rangle$ where $\lambda\in H^{1/2}(\Gamma)$ and $\lambda_0 := \lambda - \int_\Gamma \lambda$.

Sometimes it is useful to work without worry about the domain "size", only the shape matters. Because of this fact, it have been used scaled norms instead of the norms defined above.

We can obtain a scaled norm of $H^s(\Omega)$, Ω with diameter H, by taking the standard definition of $\|\cdot\|_{H^s(\widehat{\Omega})}$, where $\widehat{\Omega}$ is a region of diameter one with the same shape of Ω, and doing a dilatation.

Using this procedure we obtain:

$$|||u|||^2_{H^1(\Omega)} = \frac{1}{H^2}\|u\|^2_{L^2(\Omega)} + |u|^2_{H^1(\Omega)}.$$

We can, of course, consider the dual spaces with the norm subordinated to the scaled norm. As an example of the use of scaled norm we present the next result which is the analogous of the trace theorem in $H-$div.

Lemma 1.15. *Given $u\in \mathbf{H}(div,\Omega)$, H the diameter of Ω, then there is a constant $C>0$, which is independent of the diameter of Ω, such that:*

$$|||\boldsymbol{u}\cdot\boldsymbol{\eta}|||^2_{H^{-1/2}(\partial\Omega)} \leq C\Big[\|\boldsymbol{u}\|^2_{L^2(\Omega)} + H^2\|\nabla\cdot\boldsymbol{u}\|^2_{L^2(\Omega)}\Big]. \tag{1.10}$$

Proof. We first state the result for a reference region $\widehat{\Omega}$ of diameter one and then the general result follows after applying a scaling argument.

Considerer $\phi\in H^{1/2}(\partial\widehat{\Omega})$. Denote by ϕ its extension $\mathbf{R}_0(\phi)$. Then using Green's formula we have:

$$\int_{\partial\widehat{\Omega}} \phi\boldsymbol{u}\cdot\boldsymbol{\eta} = \int_{\widehat{\Omega}} \boldsymbol{u}\cdot\nabla\phi + \int_{\widehat{\Omega}} \nabla\cdot\boldsymbol{u}\phi$$

then

$$\langle \boldsymbol{u}\cdot\boldsymbol{\eta}, \phi\rangle = \int_{\partial\widehat{\Omega}} \phi\boldsymbol{u}\cdot\boldsymbol{\eta} \le \left[\|\boldsymbol{u}\|_{L^2(\hat{\Omega})} + \|\nabla\cdot\boldsymbol{u}\|_{L^2(\hat{\Omega})}\right]\|\phi\|_{H^1(\hat{\Omega})}$$

$$\le C\left[\|\boldsymbol{u}\|_{L^2(\hat{\Omega})} + \|\nabla\cdot\boldsymbol{u}\|_{L^2(\hat{\Omega})}\right]\|\phi\|_{H^{1/2}(\partial\hat{\Omega})},$$

this ends the proof. ✠

Note that the right hand side of (1.10) is a multiple of a scaled norm in $\boldsymbol{H}(\text{div}, \Omega)$.

If we work with scaled norms, then the constant that appears in Lemmas 1.5, 1.7, 1.12, and 1.13 are independent of de diameter of Ω. (See [15]).

1.2 A note on Green's formula.

In Section 1.1 all the definitions of Sobolev spaces were based in the space $L^2(\Omega)$. There are analogous definitions for spaces based on $L^p(\Omega)$ where

$$L^p(\Omega) := \left\{\psi : \left(\int_\Omega |\psi|^p\right)^{\frac{1}{p}} < \infty\right\}$$

with the norm suggested by its definition. We denote by $W^s_p(\Omega)$ the analogous to $H^s(\Omega)$ based on $L^p(\Omega)$ instead of $L^2(\Omega)$. That is, $W^s_p(\Omega)$ is defined by the norm:

$$\|u\|^p_{W^s_p(\Omega)} := \|u\|^p_{W^{[s]}_p} + |u|^p_{W^s_p}$$

where for m integer:

$$\|u\|^p_{W^m_p} := \sum_{|\alpha|\le m} \|\partial^\alpha u\|^p_{L^p(\Omega)}$$

and the seminorm $|\ \ |_{W^{[s]}_p}$ is defined by:

$$|u|^p_{W^s_p(\Omega)} := \sum_{|\alpha|=[s]} \int_\Omega\int_\Omega \frac{(\partial^\alpha u(x) - \partial^\alpha u(y))^p}{\|x-y\|^{n+p\sigma}} dxdy, \qquad \sigma = s - [s].$$

We have the following result (see [7]):

Lemma 1.16. *Let Ω be a bounded open subset of $\mathbb{R}^n$ with Lipschitz boundary. Then*

$$W^s_l(\Omega) \subset W^t_q(\Omega) \tag{1.11}$$

for $0 \le t \le s$ and $q \ge l$ such that $s - n/l = t - n/q$.

Let $n \geq 2$. Given $p' < 2$, we have $W^1_{p'}(\Omega) \subset W^0_{q'}(\Omega)$ where $1 - \frac{n}{p'} = -\frac{n}{q'}$ or $q' = \frac{np'}{n-p'}$. Then if $\psi \in L^q(\Omega)$ where $\frac{1}{q} + \frac{1}{q'} = 1$, i.e.,

$$q = \frac{np'}{np' + p' - n} \tag{1.12}$$

the integral $\int_\Omega \psi\phi$ is well defined for $\phi \in W^1_{p'}(\Omega)$. Note that $\frac{2n}{n+2} \leq p' < 2$ implies $1 < q \leq 2$.

In the Green's formula, Lemma 1.11, we note that if $\boldsymbol{u} \in (L^p(\Omega))^n$ with $\nabla \cdot u \in L^q(\Omega)$, q in (1.12), and $\phi \in W^1_{p'}(\Omega)$, $\frac{1}{p} + \frac{1}{p'} = 1$, then the left side of equation (1.9) still makes sense, that is, we can compute the expression

$$\int_\Omega \boldsymbol{u} \cdot \nabla \phi + \int_\Omega \nabla \cdot \boldsymbol{u} \phi$$

for all $\phi \in W^1_{p'}(\Omega)$. In particular this makes sense for $\boldsymbol{u} \in (L^p(\Omega))^n$ with $\nabla \cdot u \in L^2(\Omega)$ when $q \leq 2$ (because $L^2(\Omega) \subset L^q(\Omega)$).

Then the Green's formula remains valid in these spaces (Ω is a bounded open subset of $\mathbb{R}^n$ with Lipschitz boundary, see [7]). That is:

$$\int_\Omega \boldsymbol{u} \cdot \nabla \phi + \int_\Omega \nabla \cdot \boldsymbol{u} \phi = \int_{\partial\Omega} (\boldsymbol{u} \cdot \boldsymbol{\eta}) \phi. \tag{1.13}$$

$\phi \in W^1_{p'}(\Omega)$ and $u \in L^p(\Omega)$ with $\nabla \cdot u \in L^q(\Omega)$, q in (1.12).

We have the following general result about trace of functions in $W^s_p(\Omega)$ (see [7], pag 38, theorem 1.5.1.3.).

Lemma 1.17. *Let Ω be a bounded open subset of $\mathbb{R}^n$ with a Lipschitz boundary $\partial\Omega$. Then the mapping $u \to \gamma_0 u$ which is defined for $u \in C^{0,1}(\overline{\Omega})$, has a unique extension as an operator from $W^1_{p'}(\Omega)$ onto $W^{1-1/p'}_{p'}(\partial\Omega)$. This operator has a right continuous inverse independent of p'.*

Now if $p > 2$ we have $p' < 2$ and in order to (1.13) makes sense we only need $\nabla \cdot u \in L^q(\Omega)$, q in (1.12).

Observe that if $\Gamma \subset \partial\Omega$ and f is defined by $f = 1$ on Γ and $f = 0$ on $\partial\Omega \setminus \Gamma$ then f belong to $W^{1-1/p'}_2(\partial\Omega) = H^{1-1/p'}(\partial\Omega)$ because $1 - \frac{1}{p'} < \frac{1}{2}$. Then (since f is bounded on $\partial\Omega$) we have that $f \in W^{1-1/p'}_{p'}(\partial\Omega)$ so we can use Lemma 1.17 to find $\phi \in W^1_{p'}(\Omega)$ such that $\phi|_{\partial\Omega} = f$. Applying Green's formula we see that we can calculate:

$$\int_{\Gamma_0} u \cdot \eta$$

with

$$\left| \int_{\Gamma_0} u \cdot \eta \right| \leq C\big[\|u\|_{L^p(\Omega)} + \|\nabla \cdot u\|_{L^2(\Omega)}\big].$$

when $\nabla \cdot u \in L^2(\Omega)$. Summarizing, given a function in $\boldsymbol{H}(\text{div}, \Omega) \cap L^p(\Omega)$, $p > 2$, we can compute the mean of $u \cdot \eta$ in part of the boundary $\partial\Omega$ of Ω.

1.3 Approximation of saddle point problems.

1.3.1 Abstract variational problems.

In this section it is presented a quick overview of basic results of abstract variational problems (see [1] chapter 4, or [6] chapter 2).

Let X and M be two Hilbert spaces with norms $\|\ \|_X$ and $\|\ \|_M$ and dual spaces X' and M'.
Suppose that $a : X \times X \to \mathbb{R}$ and $b : X \times M \to \mathbb{R}$ are two continuous bilinear forms, then there is associated the following two linear operator $A : X \to X'$ and $B : X \to M'$ defined by:

$$\begin{aligned} \langle Au, v\rangle &= a(u,v) \quad \forall\ (u,v) \in X \times X. \\ \langle Bv, \mu\rangle &= b(v,\mu) \quad \forall\ (v,\mu) \in X \times M. \end{aligned}$$

Consider the following problem:

$$\left\{ \begin{array}{l} \textit{For } \ell \in X' \textit{ and } \chi \in M', \textit{ find } (u,\lambda) \in X \times M \textit{ such that:} \\ \qquad a(u,v) + b(v,\lambda) = \langle \ell, v\rangle \quad \forall\ v \in X \\ \qquad\qquad\qquad b(u,\mu) = \langle \chi, \mu\rangle \quad \forall\ \mu \in M. \end{array} \right. \tag{1.14}$$

which is equivalent to the problem:

$$\left\{ \begin{array}{l} \textit{For } \ell \in X' \textit{ and } \chi \in M', \textit{ find } (u,\lambda) \in X \times M \textit{ such that:} \\ \qquad\qquad Au + B'\lambda = \ell \quad \text{in } X' \\ \qquad\qquad\qquad Bu = \chi \quad \text{in } M'. \end{array} \right. \tag{1.15}$$

If $\Phi : X \times M \to X' \times M'$, defined by $\Phi(u,v) := (Au + B'\mu, Bv)$, is an isomorphism from $X \times M$ onto $X' \times M'$, then the problem (1.15) is said *well-posed*.

Set $V(\chi) := \left\{ \upsilon \in X \big/ B\upsilon = \chi \right\}$ and consider the problem :

$$\left\{ \begin{array}{l} \textit{Find } u \in V(\chi) \textit{ such that:} \\ \qquad\qquad a(u,\upsilon) = \langle \ell, \upsilon\rangle \quad \forall \upsilon \in V = V(0). \end{array} \right. \tag{1.16}$$

Consider the set $V^0 = \left\{ g \in X' \ :\ \langle g, \upsilon\rangle = 0 \text{ in } V(0) \right\}$.

Lemma 1.18. *Under above considerations the following assertions are equivalent:*

(*i*) *There is a constant $\beta > 0$ such that:*

$$\inf_{\mu \in M} \sup_{\upsilon \in X} \frac{b(\upsilon,\mu)}{\|\upsilon\|_X \|\mu\|_M} \geq \beta > 0$$

(ii) B' *is an isomorphism from* M *onto* V^0 *and* $\|B'\mu\|_{X'} \geq \beta\|\mu\|_M$

(iii) B *is an isomorphism from* $V^\perp$ *onto* M' *and* $\|Bv\|_{M'} \geq \beta\|v\|_X$, $v \in V^\perp$.

And if $\pi : X' \to V'$ *is defined by:*

$$\langle \pi f, v\rangle = \langle f, v\rangle \ \ \forall f \in X' \quad \forall v \in V$$

Then the problem in (1.14) is well-posed if and only if πA *is an isomorphism from* V *onto* V' *and* $b(\cdot,\cdot)$ *satisfies the* inf-sup *condition, i.e., condition* (i) *above.*

The following result is an important corollary of Lemma 1.18.

Lemma 1.19 (Brezzi's splitting theorem). *If* $a(\cdot,\cdot)$ *is* $V-$*elliptic (or coercive), i.e., if there exists a constant* $\alpha > 0$ *such that:*

$$a(v,v) \geq \alpha\|v\|_X^2 \ \ \textit{for all } v \in X$$

then the problem (1.14) is well-posed if and only if the bilinear form $b(\cdot,\cdot)$ *satisfies the* inf-sup *condition.*

For the purpose of this work, Brezzi's splitting theorem will be more useful since we work with continuous bilinear forms that are elliptic. For instance, the bilinear form defined by $(u,v) \mapsto (\nabla u, \nabla v)_\Omega$ where $u, v \in H_0^1(\Omega,\Gamma)$ (see [1], [13]).

1.3.2 Approximation.

Now, we are interested in solving problem (1.14) numerically. The Ritz-Galerkin approach is used.
Let $X^h \subset X$ and $M^h \subset M$ be finite dimensional spaces. The parameter h is the discretization parameter. Consider the problem:

$$\left\{\begin{array}{lr} \textit{For } \ell \in X' \textit{ find } (u^h, p^h) \in X^h \times M^h \textit{ such that:} & \\ \qquad a(u^h, v) + b(v, p^h) = \langle \ell, v\rangle & \forall\, v \in V^h \\ \qquad\qquad\qquad b(u^h, q) = 0 & \forall\, q \in M^h. \end{array}\right. \quad (1.17)$$

As before define $V^h := \{v \in X^h : b(v,q) = 0 \ \forall\, q \in M^h\}$. Then the problem (1.17) is equivalent to:

$$\left\{\begin{array}{l} \textit{Find } u^h \in V^h \textit{ such that:} \\ \qquad a(u^h, v) = \langle \ell, v\rangle \quad \forall v \in V^h. \end{array}\right. \quad (1.18)$$

and find $p^h \in M^h$ with $b(v, p^h) = -a(u^h, v) + \langle \ell, v\rangle$ in X^h. We have the following lemma which is easy to prove (see [2], chapter 8 and 10).

Lemma 1.20. *Let V and V^h be subspaces of a Hilbert space X. Assume that the bilinear form $a : X \times X \to \mathbb{R}$ is C-continuous and α-elliptic on V^h for all h. Given $\ell \in X'$ let $u \in V$ solve:*

$$a(u,v) = \langle \ell, v \rangle \quad \text{for all } v \in V. \tag{1.19}$$

and $u^h \in V^h$ solve

$$a(u^h,v) = \langle \ell, v \rangle \quad \text{for all } v \in V^h. \tag{1.20}$$

Then

$$\|u - u^h\|_X \leq \left(1 + \frac{C}{\alpha}\right) \inf_{v \in V^h} \|u - v\|_X + \frac{1}{\alpha} \sup_{v \in V^h \setminus \{0\}} \frac{|a(u - u^h, v)|}{\|v\|_X}.$$

Let Ω be an open and bounded domain such that $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$, where Ω_1 and Ω_2 are disjoint open sets. Let $\Gamma := \overline{\Omega}_1 \cap \overline{\Omega}_2$ be the interface between Ω_1 and Ω_2 and $\Gamma_j := \partial\Omega \setminus \Gamma$, $j = 1, 2$. The model presented in Chapter 2, coupling fluid flow with porous media flow is going to be reduced to problem (1.14). To get an approximated solution we use problem (1.17). The solution $\boldsymbol{u}$ is a function defined on Ω such that $\boldsymbol{u}_1 := \boldsymbol{u}|_{\Omega_1} \in (H_0^1(\Omega_1, \Gamma_1))^2$ and $\boldsymbol{u}_2 := \boldsymbol{u}|_{\Omega_2} \in \boldsymbol{H}_0(\text{div}, \Omega_2, \Gamma_2)$ and $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ satisfy a continuity condition on Γ (see Section 2.1.3). In order to obtain an approximated $\boldsymbol{u}$ we first use the above result to show that we can approximate $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ separately. Finally we use this result to show that we can approximate the set of functions $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ satisfying the required condition on (the interface) Γ.

In order to apply Lemma 1.20 we need to show that there exists a $u \in X$ such that (1.19) holds. As before, in order to obtain existence and uniqueness in the discrete problem, i.e., problem (1.17), we have to verify the *inf-sup* condition w.r.t the spaces X^h and M^h. A very useful result in this direction is the following (see [1] and [13]):

Lemma 1.21 (Fortin's criterion). *Suppose that $b : X \times M \to \mathbb{R}$ satisfies the inf-sup condition (i.e., the continuous inf-sup condition holds). Are equivalents:*

i) There exists a bounded linear operator $\Pi^h : X \to X^h$ such that

$$b\big(v - \Pi^h(v), p^h\big) = 0 \quad \text{for all } p^h \in M^h.$$

and $\|\Pi^h\| = \sup_{v \in X \setminus 0} \frac{\|\Pi^h(v)\|_X}{\|v\|_X} < c$, c is a constant independent of h.

ii) The finite element spaces X^h and M^h satisfy the inf-sup *condition.*

Proof. Let $p^h \in M^h$ then

$$\begin{aligned}
\beta \|p^h\|_M &\leq \sup_{v \in X \setminus \{0\}} \frac{b(v, p^h)}{\|v\|_X} && \text{continuous } inf\text{-}sup \text{ condition.} \\
&= \sup_{v \in X \setminus \{0\}} \frac{b\big(\Pi^h(v), p^h\big)}{\|v\|_X} \\
&\leq \sup_{v \in X \setminus \{0\}} \frac{b\big(\Pi^h(v), p^h\big)}{\frac{1}{c}\|\Pi^h(v)\|_X} \\
&= c \sup_{v \in X^h \setminus \{0\}} \frac{b(v^h, p^h)}{\|v^h\|_X}
\end{aligned}$$

and conversely, given $v \in V$ define $\Pi^h(v)$ by:

$$\begin{aligned}
(\Pi^h(v), w)_X + b(w, p^h) &= (v, w)_X \quad \forall w \in V^h. \\
b(\Pi^h(v), q^h) &= b(v, q^h) \quad \forall q^h \in M^h.
\end{aligned} \tag{1.21}$$

the bilinear form $(\cdot, \cdot)$ is X-elliptic then we use the previous section results to obtain unique solution $\Pi^h(v)$ and:

$$\|\Pi^h(v)\|_X \leq \tilde{c}\|v\|_X$$

which gives the bound required for Π^h because the constant $\tilde{c}$ depends only on the coercivity constant.

✠

1.3.3 A note on triangulations.

Let $\Omega \subset \mathbb{R}^n$ be a bounded polygonal or polyhedral domain with Lipschitz continuous boundary. A *triangulation* or *mesh* is a non-overlapping partition of Ω into *elements*. In the next chapter we are going to use triangulations consisting of triangles in two dimensions. In general a triangulations consists of triangles or affinely mapped rectangles in two dimensions, and of tetrahedra or affinely mapped parallelepipeds in three dimensions[3]. Let $h > 0$, a family of triangulations of Ω is a partition of Ω $\mathcal{T}^h$ such that:

$\mathcal{T}^h = \{K : K = F_K(\hat{K})\}$

$\hat{K}$ is a reference element e F_K is an affine mapping.

$\cup_{K \in \mathcal{T}^h} \overline{K} = \overline{\Omega}$; $K \cap K' =$ empty if $K \neq K'$;

$h = \max_{K \in \mathcal{T}^h} h_K$ where $h_K := \sup_{K \times K} \|x - y\|_{\mathbb{R}^n}$ is the diameter of K

[3]Affinely mapped from a reference element. In the case of two dimensional triangulations made of triangles the reference element has vertices (0,0),(0,1) and (1,0).

A family triangulations $\mathcal{T}^h$ is called *geometrically conforming* if the intersection between the closure of two different elements is either empty, a vertex, and edge, or a face that is common to both elements.

The family $\mathcal{T}^h$ is called *shape regular* if there exists a constant independent of h_K, such that

$$h_K \leq C\rho_K, \quad \text{for all } K \in \mathcal{T}^h.$$

where ρ_K is the radius of the largest circle or sphere contained in K.

If $\mathcal{T}^h$ is shape regular and there exists a constant independent of h such that

$$h_K \geq Ch \quad \text{for all } K \in \mathcal{T}^h.$$

we say that $\mathcal{T}^h$ is *quasi-uniform*.

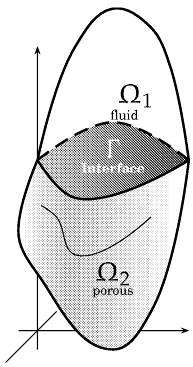
Chapter 2

Coupling fluid flow with porous media flow.

In this chapter is presented the problem of interest of these notes. First the PDE framework is introduced and later the weak formulation is analyzed.

In Section 2.1 is presented the problem to be analyzed together with the systems of differential equations that constitute the model, including the interface matching conditions which are the central part of the model. Section 2.2 is the heart of this chapter (and of the whole work). It is derived the weak formulation of the model presented in Section 2.1 and the inf-sup condition is established. This condition (according to the abstract saddle point theory) guarantees the well-posedness of the problem.

2.1 The problem.



The Problem: Coupling fluid flows with porous media flow

Consider the following problem: An incompressible fluid in a region Ω_1 can flow both ways across an interface Γ into a saturated porous medium domain Ω_2. The model that is going to be used for this problem is the *Stokes equations* for the fluid region and *Darcy equations* for the porous region. These two systems are going to be coupled at the interface with adequate conditions.

In general, $\Omega_1,\ \Omega_2 \subset \mathbb{R}^n$, $\Omega = \text{int}(\overline{\Omega}_1 \cup \overline{\Omega}_2)$, Ω_1 and Ω_2 are Lipschitz, so it is possible to define outward unit normal vectors, denoted by $\boldsymbol{\eta}_j$, $j = 1, 2$. The tangent vectors to Γ are denoted by $\boldsymbol{\tau}_1$ ($n = 2$), or $\boldsymbol{\tau}_j$, $j = 1, 2$ ($n = 3$).

Define $\Gamma_j := \partial\Omega_j \setminus \Gamma$. The fluid velocities are denoted by $\boldsymbol{u}_j : \Omega_j \to \mathbb{R}^n$, $j = 1, 2$. The fluid pressure are $p_j : \Omega_j \to \mathbb{R}$, $j = 1, 2$.

The velocities and pressures have different (physical) roles in each region, and for this reason they are studied separately and later we study the interface conditions.

2.1.1 The fluid region.

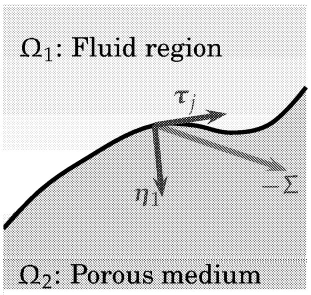
As it was mentioned previously, the model for the fluid region is the Stokes equations. The equations basically consist of conservation of mass and conservation of momentum and we have:

$$\begin{cases} -\mu\Delta u_1 + \nabla p_1 = f_1 & \text{in } \Omega_1 \quad \text{conservation of momentum,} \\ \nabla\cdot u_1 = 0 & \text{in } \Omega_1 \quad \text{conservation of mass,} \\ u_1 = \mathbf{0} & \text{on } \Gamma_1 \quad \text{no slip,} \end{cases} \tag{2.1}$$

where μ is the viscosity.

It is possible to express this formulation in an equivalent way that is going to be more useful. Let $\boldsymbol{D}v = (d_{ij}(v))_{n\times n}$ where

$$d_{ij}(v) := \frac{1}{2}\left(\partial^i v_j + \partial^j v_i\right).$$



$\Sigma = -p\eta_1 + 2\mu D(u_1)\eta_1$ is the Cauchy stress (or traction) vector

$\boldsymbol{D}$ is the linearized strain tensor. Note that $2\boldsymbol{D}v = \nabla v + [\nabla v]^T$. Define the operator $T(v,p) = (T_{ij}(v,p))$ where:

$$T_{ij}(v,p) := -p\delta_{ij} + 2\mu d_{ij}(v).$$

Then $T(v,p) = -pI + \mu(\nabla v + \nabla v^T)$ and:

$$\nabla\cdot T(v,p) = -\nabla\cdot(pI) + \mu\left(\nabla\cdot\nabla v + \nabla\cdot[\nabla v]^T\right).$$

Condition $\nabla\cdot v = 0$ implies $\nabla\cdot[\nabla v]^T = 0$. Then (2.1) can be written as:

$$\begin{cases} -\nabla\cdot T(u_1,p_1) = f_1 & \text{in } \Omega_1 \quad \text{conservation of momentum,} \\ \nabla\cdot u_1 = 0 & \text{in } \Omega_1 \quad \text{conservation of mass,} \\ u_1 = \mathbf{0} & \text{on } \Gamma_1 \quad \text{no slip,} \end{cases} \tag{2.2}$$

Observe that from Cauchy formula we get that

$$\Sigma(u_1,p_1) := T(u_1,p_1)\eta_1$$

is the force on $\partial\Omega_1$ acting on the fluid volume inside Ω_1, i.e., Σ is the Cauchy stress (or traction) vector. The force on Γ from Ω_1 is then $-\Sigma(u_1,p_1)$.

2.1.2 The porous region.

For Ω_2 it is used Darcy's law, i.e., (u_2,p_2) satisfies on Ω_2:

$$\begin{cases} u_2 = -\kappa\nabla p & \text{in } \Omega_2 \quad \text{Darcy's law} \\ \nabla\cdot u_2 = g_2 & \text{in } \Omega_2 \quad \text{conservation of mass,} \\ u_2\cdot\eta_2 = 0 & \text{on } \Gamma_2 \quad \text{no flow through } \Gamma_2, \\ \int_{\Omega_2} g_2 = 0 & \qquad\quad \text{solvability condition,} \end{cases} \tag{2.3}$$

Here, κ represent the rock permeability divided by the fluid viscosity.

The (only) force acting on the interface from Ω_2 is the one given by p_2 in the direction of η_2 and must be equal to the component of Σ in this direction. The other component of Σ, i.e., $\Sigma\cdot\tau_1$ is more difficult to analyze and we consider it separately in Section 2.1.3.

Note that it is assumed that Ω_2 is saturated with the same fluid and only absolute permeability of the porous region is considered. More general cases can be analyzed in a similar way, for example, the case with two fluids in which it has to be considered relative permeability instead of absolute permeability, etc.

2.1.3 Interface matching conditions.

The systems presented above must be coupled across Γ. The following conditions are imposed:

Conservation of mass across Γ: It is expressed by:

$$u_1\cdot\eta_1 + u_2\cdot\eta_2 = 0 \text{ on } \Gamma. \tag{2.4}$$

This means that the fluid that is leaving a region enters in the other one.

Balance of normal forces across Γ:

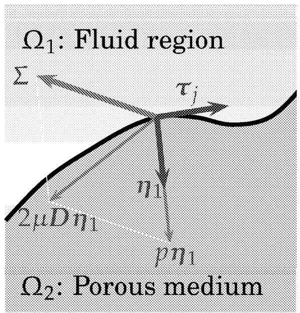
$$p_1 - 2\mu\eta_1^T D(u_1)\eta_1 = p_2 \quad \text{on } \Gamma. \tag{2.5}$$

Note that this was already mentioned in the previous section and it results from applying the Cauchy stress (or traction) vector to η_1.
Observe that:

$$2\eta_1^T D(u)\eta_1 = \eta_1^T(\nabla u + \nabla u^T)\eta_1 = 2\eta_1^T\nabla u\eta_1.$$

Then (2.5) can be written, more familiar but less intuitive, as:

$$p_1 - 2\mu\eta_1^T\nabla u\eta_1 = p_2 \quad \text{on } \Gamma. \tag{2.6}$$



$\Sigma = -p\eta_1 + 2\mu D(u_1)\eta_1$

Beavers-Joseph-Saffman condition: This condition is a kind of empirical law that gives an expression for the component of Σ in the direction of τ. It is expressed by:

$$u_1\cdot\tau_j = -\frac{\sqrt{\tilde\kappa}}{\alpha_1}2\eta_1^T D(u_1)\tau_j \quad j = 1, d-1; \text{ on } \Gamma. \tag{2.7}$$

Here[1] $\tilde\kappa = \mu\kappa$.
A related conditions is

$$(u_1 - u_2)\cdot\tau_j = -\frac{\sqrt{\tilde\kappa}}{\alpha_1}2\eta_1^T D(u_1)\tau_j \quad j = 1, d-1; \text{ on } \Gamma. \tag{2.8}$$

[1]In general k is a symetric and uniformly positive definite tensor, in this case $\tilde\kappa = \tau_j\cdot\mu\kappa\cdot\tau_j$.

which is known as the Beavers-Joseph condition. But it turns out in practice that the component of u_2 in τ_j is small compared with that of u_1. According to the previous subsection, when more general cases are considered, suitable interface conditions have to be imposed. An analytical way to find the right interface conditions is via homogenization (see [8]).

2.2 Weak formulation and analysis.

Let:

$$X_1 := \left\{ v_1 \in \boldsymbol{H}^1(\Omega_1) \;:\; v_1\Big|_{\Gamma_1} = \boldsymbol{0} \text{ and } \int_{\partial\Omega_1} v_1 \cdot \eta_1 = 0 \right\}, \qquad M_1 := L_0^2(\Omega_1),$$

note that if Γ_1 has non-vanishing $(n-1)$-dimensional measure and is relatively open with respect to $\partial\Omega_1$ then the $H^1(\Omega_1)$-seminorm is a norm equivalent to$\|\;\|_{H^1(\Omega)}$.
Observe that velocities in X_1 have zero mean flux across $\partial\Omega_1$ and that $X_1 \subset (H_0^1(\Omega, \Gamma_1))^n$.

For Ω_2 we have the following spaces:

$$X_2 := \left\{ v_2 \in \boldsymbol{H}(\mathrm{div}, \Omega_2) \;:\; v_2 \cdot \eta_2\Big|_{\Gamma_2} = \boldsymbol{0} \text{ and } \int_{\partial\Omega_2} v_2 \cdot \eta_2 = 0 \right\}, \qquad M_2 := L_0^2(\Omega_2).$$

The restriction of $v_2 \cdot \eta_2$ to Γ_2 is taken in the sense given in (1.6). Using Lemma 1.13 we have an equivalent norm to $\|\cdot\|_{H^{-1/2}(\partial\Omega)}$ involving only the seminorm of $H^{1/2}(\Omega)$. Note that $X_2 \subset H_0(\mathrm{div}, \Omega, \Gamma_2)$.

Define $X := X_1 \times X_2$ with the usual norm, i.e., given $v = (v_1, v_2) \in X$, then

$$\|v\|_X^2 := \|v_1\|_{\boldsymbol{H}^1(\Omega_1)}^2 + \|v_2\|_{\boldsymbol{H}(\mathrm{div},\Omega_2)}^2.$$

Given $q = (q_1, q_2) \in M_1 \times M_2$, define $\tilde{q} : \Omega \to \mathbb{R}$ by:

$$\tilde{q} := \begin{cases} q_1 & \text{in } \Omega_1 \\ q_2 & \text{in } \Omega_2 \end{cases}$$

and $M := M_1 \times M_2$ with $\|q\|_M^2 := \|q_1\|_{L^2(\Omega_1)}^2 + \|q_2\|_{L^2(\Omega_2)}^2 = \|\tilde{q}\|_{L^2(\Omega)}^2$.

We start with the Stokes equation (2.1). For all $v_1 \in X_1$ we have:

$$(-\mu\Delta u_1 + \nabla p_1, v_1)_{\Omega_1} = (-\mu\Delta u_1, v_1)_{\Omega_1} + (\nabla p_1, v_1)_{\Omega_1} = (f_1, v_1)_{\Omega_1} \quad (2.9)$$

From the Green formula we get:

$$\begin{aligned}
-(\Delta \boldsymbol{u}_1, \boldsymbol{v}_1)_{\Omega_1} =& -(\nabla\cdot\nabla \boldsymbol{u}_1, \boldsymbol{v}_1)_{\Omega_1} \\
=&(\nabla \boldsymbol{u}_1, \nabla \boldsymbol{v}_1)_{\Omega_1} - (\nabla \boldsymbol{u}_1 \boldsymbol{\eta}, \boldsymbol{v}_1)_{\partial\Omega_1} \\
=&(\nabla \boldsymbol{u}_1, \nabla \boldsymbol{v}_1)_{\Omega_1} - (\nabla \boldsymbol{u}_1 \boldsymbol{\eta}, \boldsymbol{v}_1)_{\Gamma} \\
=&(\nabla \boldsymbol{u}_1, \nabla \boldsymbol{v}_1)_{\Omega_1} - (\boldsymbol{\eta}_1 \boldsymbol{\eta}_1^T \nabla \boldsymbol{u}_1 \boldsymbol{\eta}_1 + \sum_{j=1}^{n-1} \boldsymbol{\tau}_j \boldsymbol{\tau}_j^T \nabla \boldsymbol{u}_1 \boldsymbol{\eta}_1, \boldsymbol{v}_1)_{\Gamma} \\
=&(\nabla \boldsymbol{u}_1, \nabla \boldsymbol{v}_1)_{\Omega_1} - \langle \boldsymbol{\eta}_1^T \nabla \boldsymbol{u}_1 \boldsymbol{\eta}_1, \boldsymbol{v}_1\cdot \boldsymbol{\eta}_1 \rangle_{\Gamma} - \sum_{j=1}^{n-1} \langle \boldsymbol{\tau}_j^T \nabla \boldsymbol{u}_1 \boldsymbol{\eta}_1, \boldsymbol{v}_1\cdot \boldsymbol{\tau}_j \rangle_{\Gamma},
\end{aligned}$$

Using the fact that $\nabla\cdot \boldsymbol{u}_1 = 0$ implies

$$\int_{\Omega_1} \nabla \boldsymbol{u}_1^T : \nabla \boldsymbol{v}_1 = \int_{\partial\Omega_1} \nabla \boldsymbol{u}_1^T \boldsymbol{\eta}_1\cdot \boldsymbol{v}_1,$$

we get

$$-(\Delta \boldsymbol{u}_1, \boldsymbol{v}_1)_{\Omega_1} = 2(\boldsymbol{D}(\boldsymbol{u}_1), \nabla \boldsymbol{v}_1)_{\Omega_1} - 2\langle \boldsymbol{\eta}_1^T \boldsymbol{D}(\boldsymbol{u}_1) \boldsymbol{\eta}_1, \boldsymbol{v}_1 \boldsymbol{\eta}_1 \rangle_{\Gamma} - 2 \sum_{j=1}^{n-1} \langle \boldsymbol{\tau}_j^T \boldsymbol{D}(\boldsymbol{u}_1) \boldsymbol{\eta}_1, \boldsymbol{v}_1 \boldsymbol{\tau}_j \rangle_{\Gamma}.$$

For the second integral in (2.9) we have:

$$(\nabla p_1, \boldsymbol{v}_1)_{\Omega_1} = \langle p_1, \boldsymbol{v}_1 \boldsymbol{\eta}_1 \rangle_{\Gamma} - (p_1, \nabla\cdot \boldsymbol{v}_1)_{\Omega_1}.$$

By replacing in (2.9), using conditions (2.6) and (2.7) and defining:

$$a_1(\boldsymbol{u}_1, \boldsymbol{v}_1) := 2\mu(\boldsymbol{D}\boldsymbol{u}_1, \nabla \boldsymbol{v}_1)_{\Omega_1} + \sum_{j=1}^{n-1} \frac{\mu\alpha_1}{\sqrt{\tilde{\kappa}}} \langle \boldsymbol{u}_1 \cdot \boldsymbol{\tau}_j, \boldsymbol{v}_1 \cdot \boldsymbol{\tau}_j \rangle_{\Gamma} \quad \text{for all } \boldsymbol{u}_1, \boldsymbol{v}_1 \in \boldsymbol{X}_1, \tag{2.10}$$

$$b_1(\boldsymbol{v}_1, q_1) := -(q_1, \nabla\cdot \boldsymbol{v}_1)_{\Omega_1} \quad \text{for all } \boldsymbol{v}_1 \in \boldsymbol{X}_1 \text{ and } q_1 \in M_1, \tag{2.11}$$

we get *for all* $\boldsymbol{v}_1 \in \boldsymbol{X}_1$ *and* $q_1 \in M_1$

$$\left\{ \begin{array}{ll} a_1(\boldsymbol{u}_1, \boldsymbol{v}_1) + b_1(\boldsymbol{v}_1, p_1) + \langle p_2, \boldsymbol{v}_1\cdot \boldsymbol{\eta}_1 \rangle_{\Gamma} & = (\boldsymbol{f}_1, \boldsymbol{v}_1)_{\Omega_1} \\ b_1(\boldsymbol{u}_1, q_1) & = 0. \end{array} \right. \tag{2.12}$$

For all $\boldsymbol{v}_2 \in X_2$ we have

$$(k^{-1}\boldsymbol{u}_2 + \nabla p_2, \boldsymbol{v}_2)_{\Omega_2} = (k^{-1}\boldsymbol{u}_2, \boldsymbol{v}_2)_{\Omega_2} + (\nabla p_2, \boldsymbol{v}_2)_{\Omega_2} = 0. \tag{2.13}$$

By using Green's formula we get

$$(\nabla p_2, \boldsymbol{v}_2)_{\Omega_2} = \langle p_2, \boldsymbol{v}_2\cdot \boldsymbol{\eta}_2 \rangle_{\Gamma} - (p_2, \nabla\cdot \boldsymbol{v}_2)_{\Omega_2},$$

and defining

$$a_2(\boldsymbol{u}_2, \boldsymbol{v}_2) := (k^{-1}\boldsymbol{u}_2, \boldsymbol{v}_2)_{\Omega_2} \quad \text{for all } \boldsymbol{u}_2, \boldsymbol{v}_2 \in \boldsymbol{X}_2,$$

$$b_2(\boldsymbol{v}_2, p_2) := -(p_2, \nabla\cdot \boldsymbol{v}_2)_{\Omega_2} \quad \text{for all } \boldsymbol{u}_2 \in \boldsymbol{X}_2 \text{ and } p_2 \in M_2,$$

we have *for all* $\boldsymbol{v}_2 \in \boldsymbol{X}_2$ *and* $q_2 \in M_2$

$$\begin{cases} a_2(\boldsymbol{u}_2, \boldsymbol{v}_2) + b_2(\boldsymbol{v}_2, p_2) + \langle p_2, \boldsymbol{v}_2 \cdot \boldsymbol{\eta}_2 \rangle_\Gamma & = 0 \\ b_2(\boldsymbol{u}_2, q_2) & = -(g_2, q_2)_{\Omega_2}. \end{cases} \tag{2.14}$$

Define $a : X \times X \to \mathbb{R}$ and $b : X \times M \to \mathbb{R}$ by:

$$a(\boldsymbol{u}, \boldsymbol{v}) := a_1(\boldsymbol{u}_1, \boldsymbol{v}_1) + a_2(\boldsymbol{u}_2, \boldsymbol{v}_2), \qquad b(\boldsymbol{v}, p) := b_1(\boldsymbol{v}_1, p_1) + b_2(\boldsymbol{v}_2, p_2), \tag{2.15}$$

then we obtain, using (2.4):

$$\begin{cases} a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) + \langle \boldsymbol{v}_1 \cdot \boldsymbol{\eta}_1 + \boldsymbol{v}_2 \cdot \boldsymbol{\eta}_2, p_2 \rangle_\Gamma & = (f_1, \boldsymbol{v}_1)_{\Omega_1} \\ b(\boldsymbol{u}, q) & = -(g_2, q_2)_{\Omega_2} \\ \langle \boldsymbol{u}_1 \cdot \boldsymbol{\eta}_1 + \boldsymbol{u}_2 \cdot \boldsymbol{\eta}_2, q_2 \rangle_\Gamma & = 0. \end{cases} \tag{2.16}$$

To uncouple the two subproblems we use the Darcy pressure, this could also be done using Stokes problem. Introduce the Lagrange multiplier λ :

$$\lambda = p_2 = p_1 - 2\mu \boldsymbol{\eta}_1^T \nabla \boldsymbol{u} \boldsymbol{\eta}_1. \tag{2.17}$$

Then we get :

$$\begin{cases} a_1(\boldsymbol{u}_1, \boldsymbol{v}_1) + b_1(\boldsymbol{v}_1, p_1) + \langle \boldsymbol{v}_1 \cdot \boldsymbol{\eta}_1, \lambda \rangle_\Gamma & = (f_1, \boldsymbol{v}_1)_{\Omega_1} & \forall \boldsymbol{v}_1 \in \boldsymbol{X}_1 \\ a_2(\boldsymbol{u}_2, \boldsymbol{v}_2) + b_2(\boldsymbol{v}_2, p_2) + \langle \boldsymbol{v}_2 \cdot \boldsymbol{\eta}_2, \lambda \rangle_\Gamma & = 0 & \forall \boldsymbol{v}_2 \in \boldsymbol{X}_2 \\ b_1(\boldsymbol{u}_1, q_1) & = 0 & \forall q_1 \in M_1 \\ b_2(\boldsymbol{u}_2, q_2) & = -(g_2, q_2)_{\Omega_2} & \forall q_2 \in M_2 \\ \langle \boldsymbol{u}_1 \cdot \boldsymbol{\eta}_1 + \boldsymbol{u}_2 \cdot \boldsymbol{\eta}_2, \mu \rangle_\Gamma & = 0 & \forall \mu \in \Lambda \end{cases} \tag{2.18}$$

where Λ is not yet defined.

We have to choose a suitable function space Λ for λ. Observe that λ has to be applied to functions of the form $\boldsymbol{v}_j \cdot \boldsymbol{\eta}_j$, where $\boldsymbol{v}_1 \in (H_0^1(\Omega, \Gamma_1))^n$ and $\boldsymbol{v}_2 \in \boldsymbol{H}_0(\text{div}, \Omega_2, \Gamma_2)$. This implies $\boldsymbol{v}_1 \cdot \boldsymbol{\eta}_1 \in H_{00}^{1/2}(\Gamma)$ and then $E_0(\boldsymbol{v}_1 \cdot \boldsymbol{\eta}_1) \in H^{1/2}(\partial\Omega_2) \subset H^{-1/2}(\partial\Omega_2)$. Here we use the fact that $H_{00}^{1/2}(\Gamma)$, which is the set of restrictions of functions in $H_0^1(\Omega_1, \Gamma)$, is equivalent to the trace of $H_0^1(\Omega_2, \Gamma)$ on Γ if the shape and the measure of Ω_1 are of the same order as those of Ω_2 (see [7], [11]). The Normal Trace Theorem, Lemma 1.10, implies that $\boldsymbol{v}_2 \cdot \boldsymbol{\eta}_2 \in H^{-1/2}(\partial\Omega_2)$. Then $\boldsymbol{v}_1 \cdot \boldsymbol{\eta}_1 + \boldsymbol{v}_2 \cdot \boldsymbol{\eta}_2 \in H^{-1/2}(\partial\Omega_2)$.

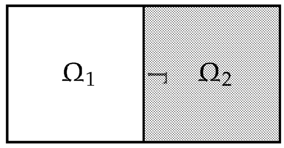
Remember that $\boldsymbol{v}_2 \cdot \boldsymbol{\eta}_2 \Big|_{\Gamma_2} = 0$ means that:

$$\langle \boldsymbol{v}_2 \cdot \boldsymbol{\eta}_2, E_0(\phi) \rangle = 0 \quad \forall \phi \in H_{00}^{1/2}(\Gamma_2)$$

and that Lemma 1.6 says that

$$H^{1/2}(\partial\Omega_2) = H^{1/2}_{00}(\Gamma_2) \oplus H^{1/2}(\Gamma).$$

Then we choose for λ the subspace $\Lambda := H^{1/2}(\Gamma)$. Other reason to choose $H^{1/2}(\Gamma)$ instead of $H^{1/2}_{00}(\Gamma)$ as in [9] is that the Lagrange multiplier in this case is pressure, and there is no physical reason for the pressure p_2 to vanish on the interface relative boundary points (with respect to $\partial\Omega_j$).

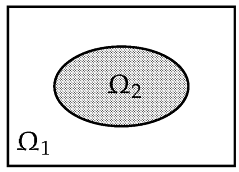


$\Gamma \neq \partial\Omega_2$.

Observe that when $\Gamma = \partial\Omega_2$, i.e., when the porous region is totally surrounded by the fluid region, $H^{1/2}_{00}(\Gamma) = H^{1/2}(\Gamma) = H^{1/2}(\partial\Omega_2)$.

Define $b_\Gamma : \boldsymbol{X} \times \Lambda \to \mathbb{R}$ by:

$$b_\Gamma(\boldsymbol{v}, \mu) := \langle v_1 \cdot \boldsymbol{\eta}_1, \mu\rangle_\Gamma + \langle v_2 \cdot \boldsymbol{\eta}_2, \mu\rangle_\Gamma, \quad \boldsymbol{v} = (v_1, v_2) \in X, \mu \in \Lambda, \tag{2.19}$$



$\Gamma = \partial\Omega_2$.

here $\langle\, ,\, \rangle_\Gamma$ for $v_j \in H^{-1/2}(\Omega_j)$ and $\mu \in H^{1/2}(\Gamma)$ is given (as in Lemma 1.7) by:

$$\langle v_j \cdot \boldsymbol{\eta}_j, \mu\rangle_\Gamma = \int_\Gamma v_j \cdot \boldsymbol{\eta}_j \mu := \langle v_j \cdot \boldsymbol{\eta}_j, E_{\eta_j}(\mu)\rangle_{\partial\Omega_j} \tag{2.20}$$

where $E_\eta(\mu)$ is defined in Lemma 1.5. See the Remark after Lemma 1.7.

Lemma 2.1. *$b_\Gamma : \boldsymbol{X} \times \Lambda \to \mathbb{R}$ defined in (2.19) and (2.20) is continuous.*

Proof. Let $\mu \in H^{1/2}(\Gamma)$. Then

$$\begin{aligned}
\int_\Gamma v_2 \cdot \boldsymbol{\eta}_2 \mu &= \langle v_2 \cdot \boldsymbol{\eta}_2, E_\eta(\mu)\rangle_{\partial\Omega_2} \\
&= \int_{\partial\Omega_2} v_2 \cdot \boldsymbol{\eta}_2 E_\eta(\mu) \\
&\leq \|v_2\|_{H^{-1/2}(\Omega_2)} \|E_\eta(\mu)\|_{H^{1/2}(\partial\Omega_2)} \\
&\leq C\|\boldsymbol{v}\|_X \|\mu\|_\Lambda.
\end{aligned}$$

The same holds for $j = 1$. ✠

If λ is a constant function, we already have $b_\Gamma(\boldsymbol{v}, \lambda) = 0$ for $\boldsymbol{v} \in \boldsymbol{X}$. Then it is convenient to modify Λ to

$$\Lambda := H^{1/2}(\Gamma) \cap L^2_0(\Gamma) \quad \text{with norm } |\cdot|_{H^{1/2}(\Gamma)}. \tag{2.21}$$

We finally arrive to the weak formulation of the problem: *find* $(\boldsymbol{u}, p, \lambda) \in \boldsymbol{X} \times M \times \Lambda$ *satisfying, for all* $(\boldsymbol{v}, q, \mu) \in \boldsymbol{X} \times M \times \Lambda$:

$$\begin{cases}
a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) + b_\Gamma(\boldsymbol{v}, \lambda) &= \ell(\boldsymbol{v}) \\
b(\boldsymbol{u}, q) &= g(q) \\
b_\Gamma(\boldsymbol{u}, \mu) &= 0.
\end{cases} \tag{2.22}$$

where $\ell(v) := (f_1, v_1)_{\Omega_1}$ for all $v \in \boldsymbol{X}$ and $g(q) := -(g_2, q_2)_{\Omega_2}$ for all $q \in M$.

Remark. *We note however that the space we choose for Λ is richer than $H_{00}^{1/2}(\Gamma)$ (which is the one used in [9]) and therefor closer will be $v_1 \cdot \eta_1$ and $v_2 \cdot \eta_2$ near the interface end points.*

Remark. *The Korn inequality implies that the bilinear form a_1 defined in (2.10) is X_1-elliptic (See [1], [11]). Then the bilinear form a defined in (2.15) is X-elliptic.*

Define

$$\boldsymbol{V} := \{v \in \boldsymbol{X} \,:\, b_\Gamma(v, \mu) = 0 \;\; \forall \mu \in \Lambda\}\,. \tag{2.23}$$

The set $\boldsymbol{V}$ is closed because the linear map $B_\Gamma : \boldsymbol{X} \to \Lambda'$ defined by $B_\Gamma(v)\mu := b_\Gamma(v, \mu)$ is continuous and $V = \ker(B_\Gamma)$.

Then we can formulate problem (2.22) as:

$$\left\{ \begin{array}{lll} a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) & = \ell(\boldsymbol{v}) & \forall\, \boldsymbol{v} \in \boldsymbol{V} \\ b(\boldsymbol{u}, q) & = g(q) & \forall\, q \in M \end{array} \right. \tag{2.24}$$

Now, define

$$\boldsymbol{Z} := \{v \in \boldsymbol{X} \,:\, b(v, q) = 0 \;\; \forall q \in M\}\,. \tag{2.25}$$

Then we can also formulate problem (2.22) as:

$$\left\{ \begin{array}{lll} a(\boldsymbol{u}, \boldsymbol{v}) + b_\Gamma(\boldsymbol{v}, \lambda) & = \ell(\boldsymbol{v}) & \forall\, \boldsymbol{v} \in \boldsymbol{Z} \\ b_\Gamma(\boldsymbol{u}, \mu) & = 0 & \forall\, \mu \in \Lambda \end{array} \right. \tag{2.26}$$

Remember that we are working with pressure of zero mean in each subdomain, then from the usual *inf-sup* condition for Stokes we can easily derive the *inf-sup* condition needed in formulation (2.24).

Lemma 2.2. *There is a constant $\alpha > 0$ s.t:*

$$\inf_{\substack{p \in M \\ p \neq 0}} \sup_{\substack{v \in V \\ v \neq 0}} \frac{b(v, p)}{\|v\|_X \|p\|_M} \geq \alpha > 0.$$

To show that the weak formulation (2.26) is stable next lemma shows that the inf-sup condition between spaces Z and Λ holds (see [13]).

Lemma 2.3. *There is a constant $\gamma > 0$ s.t:*

$$\inf_{\substack{\lambda \in \Lambda \\ \lambda \neq 0}} \sup_{\substack{v \in Z \\ v \neq 0}} \frac{b_\Gamma(v, \lambda)}{\|v\|_X \|\lambda\|_\Lambda} \geq \gamma > 0.$$

Proof. Fix $\lambda \in \Lambda$ then $\lambda \in H^{1/2}(\Gamma)$ and $\int_\Gamma \lambda = 0$, in particular if $\lambda \neq 0$ then is nonconstant.

From Lemma 1.14 we have that there exists $\mu_1 \in H^{-1/2}(\Gamma)$ such that $\langle \mu_1, 1\rangle_\Gamma = 0$ and:

$$\frac{\langle \mu_1, \lambda\rangle_\Gamma}{|\mu_1|_{H^{-1/2}(\Gamma)}} \geq \frac{1}{2}|\lambda|_{H^{1/2}(\Gamma)}. \tag{2.27}$$

From Lemma 1.8 and (1.13) we get $\mu \in H^{-1/2}(\partial\Omega_2)$, given by:

$$\langle \mu, \phi\rangle_{\partial\Omega_2} := \langle \mu_1, \phi|_\Gamma\rangle_\Gamma. \tag{2.28}$$

with

$$|\mu|_{H^{-1/2}(\partial\Omega_2)} \leq C_1|\mu_1|_{H^{-1/2}(\Gamma)} \tag{2.29}$$

and zero mean on $\partial\Omega_2$, $\langle \mu, 1\rangle_{\partial\Omega_2} = \langle \mu_1, 1\rangle_\Gamma = 0$.
By using the normal trace theorem, Lemma 1.10, and a continuous Stokes problem (μ has zero mean on $\partial\Omega_2$) we can find $v_2 \in \boldsymbol{H}(\text{div}, \Omega_2)$ with $\nabla\cdot v = 0$ in Ω_2 and such that:

$$\|v_2\|_{\boldsymbol{H}(\text{div},\Omega_2)} \leq C_2|\mu|_{H^{-1/2}(\partial\Omega_2)} \tag{2.30}$$

$$v\cdot\boldsymbol{\eta}_2 = \mu. \tag{2.31}$$

Observe that $v_2 \in \boldsymbol{X}_2$. In fact if $\phi \in H^{1/2}_{00}(\Gamma_2)$ then:

$$\langle v_2\cdot\boldsymbol{\eta}_2, \phi\rangle_{\partial\Omega_2} = \langle \mu, \phi\rangle_{\partial\Omega_2} = \langle \mu_1, \phi|_\Gamma\rangle_\Gamma = \langle \mu_1, 0\rangle_\Gamma = 0$$

and $\langle v_2\cdot\boldsymbol{\eta}_2, 1\rangle_{\partial\Omega_2} = \langle \mu_1, 1\rangle_\Gamma = 0$.

Choosing $v_1 = 0$, we have $v := (v_1, v_2) \in \boldsymbol{Z}$ and:

$$\begin{aligned}
\frac{b_\Gamma(v,\lambda)}{\|v\|_X} &= \frac{\langle v_1\cdot\boldsymbol{\eta}_1, \lambda\rangle_\Gamma + \langle v_2\cdot\boldsymbol{\eta}_2, \lambda\rangle_\Gamma}{\|v_2\|_{\boldsymbol{H}(\text{div},\Omega_2)}} && \text{by (2.19)}\\
&= \frac{0 + \langle v_2\cdot\boldsymbol{\eta}_2, E_{\eta_2}(\lambda)\rangle_{\partial\Omega_2}}{\|v_2\|_{\boldsymbol{H}(\text{div},\Omega_2)}} && \text{by (2.20)}\\
&= \frac{\langle v_2\cdot\boldsymbol{\eta}_2, E_{\eta_2}(\lambda)\rangle_{\partial\Omega_2}}{\|v_2\|_{\boldsymbol{H}(\text{div},\Omega_2)}}\\
&\geq \frac{1}{C_2}\frac{\langle v_2\cdot\boldsymbol{\eta}_2, E_{\eta_2}(\lambda)\rangle_{\partial\Omega_2}}{|\mu|_{H^{-1/2}(\partial\Omega_2)}} && \text{by (2.30)}\\
&= \frac{1}{C_2}\frac{\langle \mu, E_{\eta_2}(\lambda)\rangle_{\partial\Omega_2}}{|\mu|_{H^{-1/2}(\partial\Omega_2)}} && \text{by (2.31)}\\
&= \frac{1}{C_2}\frac{\langle \mu_1, E_{\eta_2}(\lambda)|_\Gamma\rangle_\Gamma}{|\mu|_{H^{-1/2}(\partial\Omega_2)}} && \text{by (2.28)}\\
&= \frac{1}{C_2}\frac{\langle \mu_1, \lambda\rangle_\Gamma}{|\mu|_{H^{-1/2}(\partial\Omega_2)}}\\
&\geq \frac{1}{C_2}\frac{1}{2}|\lambda|_{H^{1/2}(\Gamma)} && \text{by (2.27)}
\end{aligned}$$

With Lemma 2.2 and Lemma 2.3 we can show:

Lemma 2.4. *There is a constant* $\beta > 0$ *s.t:*

$$\inf_{\substack{(p,\lambda)\in M\times\Lambda \\ p\neq 0,\lambda\neq 0}} \sup_{\substack{v\in X \\ v\neq 0}} \frac{b(v,p)+b_\Gamma(v,\lambda)}{\|v\|_X[\|p\|_M+\|\lambda\|_\Lambda]} \geq \beta > 0.$$

Proof. Given $(p,\lambda) \in M \times \Lambda$, $p \neq 0$, $\lambda \neq 0$, from Lemma 2.2 there exists $v \in V$ such that

$$\frac{b(v,p)}{\|v\|_X} \geq \alpha\|p\|_M > 0,$$

where α independent of p. From Lemma 2.3 there exists $z \in Z$ such that

$$\frac{b_\Gamma(z,\lambda)}{\|z\|_X} \geq \gamma\|\lambda\|_\Lambda > 0,$$

where γ independent of λ. Observe that $v + z \neq 0$. Then from definitions of V (2.23) and Z (2.25) we get $b(z,p) = 0$ and $b_\Gamma(v,\lambda) = 0$ and:

$$\begin{aligned}
\frac{b(v+z,p)+b_\Gamma(v+z,\lambda)}{\|v+z\|_X} &= \frac{b(v,p)+b(z,p)+b_\Gamma(v,\lambda)+b_\Gamma(z,\lambda)}{\|v+z\|_X} \\
&= \frac{b(v,p)+b_\Gamma(z,\lambda)}{\|v+z\|_X} \\
&\geq \frac{\alpha\|p\|_M\|v\|_X+\gamma\|\lambda\|_\Lambda\|z\|_X}{\|v+z\|_X} \\
&\geq \min\{\alpha,\gamma\}\frac{\|v\|_X+\|z\|_X}{\|v+z\|_X}\left[\|p\|_M+\|\lambda\|_\Lambda\right] \\
&\geq \min\{\alpha,\gamma\}\left[\|p\|_M+\|\lambda\|_\Lambda\right] \\
&= \frac{\beta}{2}\left[\|p\|_M+\|\lambda\|_\Lambda\right]
\end{aligned}$$

2.3 Remarks about the weak formulation.

Remark 1. In the Stokes equation (2.1) we have used zero divergence and zero boundary conditions. It is possible to start with nonhomogeneous boundary conditions and nonzero divergence, i.e.,

$$\begin{cases} -\mu\Delta \boldsymbol{u}_1 + \nabla p &= \boldsymbol{f}_1 \quad \text{in } \Omega_1 \\ \nabla\cdot \boldsymbol{u}_1 &= g_1 \quad \text{in } \Omega_1 \\ \boldsymbol{u}_1 &= \boldsymbol{h}_1 \quad \text{on } \Gamma_1. \end{cases} \tag{2.32}$$

The nonhomogeneous boundary condition can be reduced to the homogeneous case when $\boldsymbol{h}_1 \in (H^{1/2}(\Gamma_1))^n$. In fact, since $\boldsymbol{h}_1 \in (H^{1/2}(\Gamma_1))^n$ we can consider $\boldsymbol{\mu} := E_\eta(\boldsymbol{h}_1)$ and define:

$$\hat{\boldsymbol{h}}_1 := \boldsymbol{\mu} + c\boldsymbol{\theta}, \tag{2.33}$$

where $\theta \in H_{00}^{1/2}(\Gamma)$ with $\int_\Gamma \boldsymbol{\theta}\cdot\eta_1 = 1$ (so $\hat{\boldsymbol{h}}_1|_{\Gamma_1} = \boldsymbol{h}_1$) and the constant c is chosen such that:

$$\int_{\Omega_1} g_1 = \int_{\partial\Omega_1} \hat{\boldsymbol{h}}_1\cdot\boldsymbol{\eta}_1. \tag{2.34}$$

It is possible to construct $\boldsymbol{\omega}_1$ such that[2]:

$$\boldsymbol{\omega}_1|_{\partial\Omega_1} = \hat{\boldsymbol{h}}_1 \quad\text{and}\quad \nabla\cdot\boldsymbol{\omega}_1 = g_1 \quad\text{in}\quad \Omega_1.$$

Now put $\boldsymbol{u}_1 = \boldsymbol{\omega}_1 + \boldsymbol{\zeta}_1$ where $\boldsymbol{u}_1$ satisfies (2.32). So we are looking for $\boldsymbol{\zeta}_1$ that satisfy:

$$\begin{cases} -\mu\Delta\boldsymbol{\zeta}_1 + \nabla p &= \boldsymbol{f}_1 + \mu\Delta\boldsymbol{\omega}_1 \quad \text{in } \Omega_1 \\ \nabla\cdot\boldsymbol{\zeta}_1 &= 0 \quad \text{in } \Omega_1 \\ \boldsymbol{\zeta}_1 &= 0 \quad \text{on } \Gamma_1 \end{cases} \tag{2.35}$$

Analogously, we use homogeneous boundary conditions in the porous region. The nonhomogeneous case, i.e.,

$$\begin{cases} \boldsymbol{u}_2 &= -\kappa\nabla p \quad \text{in } \Omega_2 \quad \text{Darcy's law.} \\ \nabla\cdot\boldsymbol{u}_2 &= g_2 \quad \text{in } \Omega_2 \\ \boldsymbol{u}_2\cdot\boldsymbol{\eta}_2 &= h_2 \quad \text{on } \Gamma_2 \end{cases} \tag{2.36}$$

can be considered. In this case we need $h_2 \in H^{-1/2}(\Gamma_2)$. We can define $\hat{h}_2 \in H^{-1/2}(\partial\Omega_2)$ by:

$$\langle \hat{h}_2, \lambda\rangle_{\partial\Omega_2} := \langle \hat{\boldsymbol{h}}_1\cdot\eta_2, \lambda|_\Gamma\rangle_\Gamma + \langle h_2, \lambda|_{\Gamma_2}\rangle_{\Gamma_2} \tag{2.37}$$

and then use the (normal) trace theorem to get $\boldsymbol{\omega}_2 \in \boldsymbol{H}(\text{div}, \Omega_2)$ such that:

$$\boldsymbol{\omega}_2\cdot\eta = \hat{h}_2.$$

Put $\boldsymbol{u}_2 = \boldsymbol{\omega}_2 + \boldsymbol{\zeta}_2$. Then we look for $\boldsymbol{\zeta}_2$ such that:

$$\begin{cases} \boldsymbol{\zeta}_2 &= -\kappa\nabla p - \boldsymbol{\omega}_2 \quad \text{in } \Omega_2 \\ \nabla\cdot\boldsymbol{\zeta}_2 &= g_2 - \nabla\cdot\boldsymbol{\omega}_2 \quad \text{in } \Omega_2 \\ \boldsymbol{\zeta}_2\cdot\boldsymbol{\eta}_2 &= 0 \quad \Gamma_2, \end{cases} \tag{2.38}$$

where the last equation follows from the definition of $\hat{h}_2$. The compatibility condition is now:

$$\int_{\Omega_2} (g_2 - \nabla\cdot\boldsymbol{\omega}_2) = 0,$$

[2]Take w solution of $\begin{cases} -\Delta w + \nabla p &= 0 \quad \text{in } \Omega_1 \\ \nabla\cdot w &= g_1 \quad \text{in } \Omega_1 \\ w &= \hat{h}_1 \quad \text{on } \partial\Omega_1 \end{cases}$, note that $\int_{\Omega_1} g_1 = \int_{\partial\Omega_1} \hat{\boldsymbol{h}}_1\cdot\boldsymbol{\eta}_1$.

but

$$
\begin{aligned}
\int_{\Omega_2} \nabla\cdot \boldsymbol{\omega}_2 &= \langle \hat{h}_2, 1\rangle \\
&= \langle \hat{\boldsymbol{h}}_1 \cdot \eta_2, 1\rangle_\Gamma + \langle h_2, 1\rangle_{\Gamma_2} && \text{by definition of } \hat{h}_2 \text{ (2.37)} \\
&= -\int_\Gamma \hat{\boldsymbol{h}}_1 \cdot \boldsymbol{\eta}_1 + \int_{\Gamma_2} h_2 \\
&= \int_{\Gamma_1} \hat{\boldsymbol{h}}_1 \cdot \boldsymbol{\eta}_1 - \int_{\Omega_1} g_1 + \int_{\Gamma_2} h_2 && \text{by (2.34)} \\
&= \int_{\Gamma_1} \boldsymbol{h}_1 \cdot \boldsymbol{\eta}_1 + \int_{\Gamma_2} h_2 - \int_{\Omega_1} g_1
\end{aligned}
$$

then the compatibility condition becomes:

$$
\int_{\Omega_1} g_1 + \int_{\Omega_2} g_2 - \int_{\Gamma_1} \boldsymbol{h}_1 \cdot \boldsymbol{\eta}_1 - \int_{\Gamma_2} h_2 = 0 \tag{2.39}
$$

which is quite intuitive. This become more clear if we look at the weak formulation. With $\boldsymbol{\omega}_j$ constructed as before define $\boldsymbol{\omega} := (\boldsymbol{\omega}_1, \boldsymbol{\omega}_2)$. We have to: *find* $(\zeta, p, \lambda) \in \boldsymbol{X} \times M \times \Lambda$ *satisfying, for all* $(\boldsymbol{v}, q, \mu) \in \boldsymbol{X} \times M \times \Lambda$:

$$
\left\{
\begin{array}{ll}
a(\zeta, \boldsymbol{v}) + b(\boldsymbol{v}, p) + b_\Gamma(\boldsymbol{v}, \lambda) & = \ell(\boldsymbol{v}) - a(\boldsymbol{\omega}, \boldsymbol{v}) \\
b(\boldsymbol{u}, q) & = g(q) + b(\boldsymbol{u}, q) \\
b_\Gamma(\zeta, \mu) & = 0,
\end{array}
\right. \tag{2.40}
$$

which is the same problem (2.22) with a different right hand side.

Remark 2. Instead of the spaces $\boldsymbol{X}$ and M used to obtain the weak formulation (2.22) we can use:

$$
\boldsymbol{Y}_1 := \left\{ \boldsymbol{v}_1 \in \boldsymbol{H}^1(\Omega_1) \ : \ \boldsymbol{v}_1\Big|_{\Gamma_1} = \boldsymbol{0} \right\}, \qquad N_1 := L^2(\Omega_1).
$$

and

$$
\boldsymbol{Y}_2 := \left\{ \boldsymbol{v}_2 \in \boldsymbol{H}(\mathrm{div}, \Omega) \ : \ \boldsymbol{v}_2 \cdot \eta_2\Big|_{\Gamma_2} = \boldsymbol{0} \right\}, \qquad N_2 := L^2(\Omega_2).
$$

with $\boldsymbol{Y} := \boldsymbol{Y}_1 \times \boldsymbol{Y}_2$ and

$$
N := \left\{ (q_1, q_2) \in N_1 \times N_2 \ : \ \int_{\Omega_1} q_1 + \int_{\Omega_2} q_2 = 0 \right\},
$$

and obtain an equivalent weak formulation. This is the weak formulation presented in [9]. The advantage of the weak formulation (2.22) is that permit us to work with seminorms instead of norms and that the space Λ for the Lagrange multipliers is bigger and is not restricted to have zero value at the end points of the interface. The equivalence between the two formulations follows from the solvability condition and the divergence theorem.

Chapter 3

Finite Element Approximation

In Section 2.1 was presented the problem of coupling fluid with porous media flow in its continuous form, now a finite element approximation is discussed. The discrete spaces are defined in Section 3.1 for both regions and for the Lagrange multipliers. The properties of these spaces are analyzed in Sections 3.2 and 3.3. Finally in Section 3.4 we establish the discrete inf-sup condition related to the weak formulation (2.24).

3.1 Discretization.

From now on we assume that Ω is two dimensional and it has polygonal boundary. Let $\mathcal{T}_j^h$ be a (geometrically conforming, shape regular and quasi-uniform) triangulation of Ω_j. We assume that they match at the interface Γ which is a polyhedral. We choose the following spaces for the fluid region:

$$\boldsymbol{X}_1^h := \left\{ \boldsymbol{u} \in \boldsymbol{X}_1 \ : \ \boldsymbol{u}_K = \tilde{\boldsymbol{u}}_K \circ F_K^{-1} \text{ on } K \text{ and } \tilde{\boldsymbol{u}}_K \in P_2(\tilde{K})^2 \right\} \cap C^0(\overline{\Omega}_1)^2, \quad (3.1)$$

where $\boldsymbol{u}_K := \boldsymbol{u}|_K$ and

$$M_1^h := \left\{ p \in M_1 \ : \ p_K = \tilde{p}_K \circ F_K^{-1} \text{ on } K \text{ and } \tilde{p}_K \in P_1(\tilde{K})^2 \right\} \cap C^0(\overline{\Omega}_1), \quad (3.2)$$

that is, we use the triangular Taylor-Hood finite elements of order two (see [2], [4], [13]).

For the porous region we are going to use the lowest order Raviart-Thomas finite elements based on triangles. In general the Raviart-Thomas elements in a cell are defined by (see [1], [4], [6]):

$$RT_k(K) := (P_k(K))^n + P_k(K)x,$$

and if $u \in RT_k(K)$ then $\nabla\cdot u \in P_k(K)$ and $u\cdot\eta|_{e_i} \in P_k(e_i)$, for all edge e_i. Then we choose:

$$X_2^h := \{u \in X_2 \ : \ u|_K = u_K \in RT_0(K)\}, \tag{3.3}$$

and

$$M_2^h := \left\{p \in M_2 \ : \ p|_K = p_K = \tilde{p}_K \circ F_K^{-1} \text{ on } K \text{ and } \tilde{p}_K \in P_0(\tilde{K})^2\right\}. \tag{3.4}$$

Observe that in the previous definitions the boundary conditions are included.

For the Lagrange multiplier space we choose:

$$\Lambda^h := \{\lambda \in \lambda \ : \ \lambda|_e = \lambda_e \text{ is constant for all edge } e \text{ in } \Gamma\}. \tag{3.5}$$

We note that we use nonconforming finite elements associated to Λ since picewise constant functions do not belongs to $H^{1/2}(\Gamma)$.

Define $X^h := X_1^h \times X_2^h$ and of course:

$$V^h := \left\{v \in X^h \ : \ b_\Gamma(v,\mu) = 0 \ \ \forall \mu \in \Lambda^h\right\}.$$

3.2 Approximation properties of Taylor-Hood finite elements.

The domain of reference is Ω_1. In order to simplify the notation we omit the subscript that refers to the domain. In particular, in this section X^h denotes X_1^h .

Let $\pi_1 : X \to X^h$ be the L_2-projector[1] defined from X into the space X^h in (3.1), i.e., π_1 is defined by:

$$\pi_1(v) \in X^h, \ \ \int_K \pi_1(v)\cdot w = \int_K v\cdot w \quad \text{for all } w \in X^h.$$

It is know (see [1]) that π_1 is bounded, i.e.,

$$\|\pi_1(v)\|_{H^1(\Omega)^2} \leq C\|v\|_{H^1(\Omega)^2} \tag{3.6}$$

and

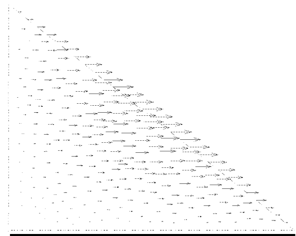
$$\|v - \pi_1(v)\|_{L^2(\Omega)^2} \leq \bar{C}h\|v\|_{H^1(\Omega)^2} \tag{3.7}$$

Given $K \in \mathcal{T}^h$ and e edge of K let $\eta_e^{(K)} = (\eta_e^1, \eta_e^2)$ denotes the normal to e exterior to K, $\tau_e = (\tau_e^1, \tau_e^2)$ the tangential vector to e and x_e the midpoint of the edge e. Each interior edge belongs to two triangles K_1 and K_2. Let η_e denote one of the directions $\eta_e^{(K_1)}$ or $\eta_e^{(K_2)}$. For boundary edges η_e denote $\eta_e^{(K)}$.

[1] We can use Clement interpolation instead of the L_2-projector. See [1].

Let $\phi_i^{(K)}$, $i = 1,2,3$, be the Taylor-Hood basis functions based on the midpoints of the edges of K. Put $\boldsymbol{\psi}_i^{(K)} := \phi_i^{(K)}\boldsymbol{\eta}_{e_i}$, $i = 1,2,3$, and $w_i^{(K)} := \phi_i^{(K)}\boldsymbol{\tau}_{e_i}$, $i = 1,2,3$. Observe that:

$$\begin{aligned} &\textstyle\int_K \boldsymbol{\psi}_i^{(K)}\cdot\boldsymbol{\eta}_{e_i} \neq 0, \qquad \boldsymbol{\psi}_i^{(K)}\cdot\boldsymbol{\tau}_{e_i} = 0 \quad i = 1,2,3. \\ &w_i^{(K)}(x_{e_i})\cdot\boldsymbol{\tau}_{e_i} \neq 0, \quad w_i^{(K)}\cdot\boldsymbol{\eta}_{e_i} = 0 \quad i = 1,2,3. \end{aligned} \tag{3.8}$$



Taylor-Hood basis function based on the midpoints of an edge of K.

Consider the following subspaces of $\boldsymbol{X}^h$:

$$\boldsymbol{\Psi}^h := \{v \in \boldsymbol{X}^h : v|_K \in \text{Span}\{\boldsymbol{\psi}_1^{(K)}, \boldsymbol{\psi}_2^{(K)}, \boldsymbol{\psi}_3^{(K)}\}\} \cap \boldsymbol{X}^h \tag{3.9}$$

and

$$\boldsymbol{W}^h := \{v \in \boldsymbol{X}^h : v|_K \in \text{Span}\{w_1^{(K)}, w_2^{(K)}, w_3^{(K)}\}\} \cap \boldsymbol{X}^h. \tag{3.10}$$

Note that if $v \in \boldsymbol{\Psi}^h$, then $v\cdot\boldsymbol{\eta}|_\Gamma \in H_{00}^{1/2}(\Gamma)$ and $v\cdot\boldsymbol{\tau}|_{\partial\Omega} = 0$. Also note that if $v \in \boldsymbol{W}^h$ then $v\cdot\boldsymbol{\tau}|_\Gamma \in H_{00}^{1/2}(\Gamma)$ and $v\cdot\boldsymbol{\eta}|_{\partial\Omega} = 0$.

Let $\pi_0 : X \to \boldsymbol{\Psi}^h$ be (locally) defined by :

$$\pi_0(v) \in \text{Span}\{\boldsymbol{\psi}_1^K, \boldsymbol{\psi}_2^K, \boldsymbol{\psi}_3^K\}, \quad \text{such that} \int_{e_i} \pi_0(v)\cdot\boldsymbol{\eta} = \int_{e_i} v\cdot\boldsymbol{\eta}, \ \ i = 1,2,3.$$

for all $K \in \mathcal{T}^h$. In other words, $\pi_0(v) = \alpha_1\boldsymbol{\psi}_1 + \alpha_2\boldsymbol{\psi}_2 + \alpha_3\boldsymbol{\psi}_3$ where

$$\alpha_i := \frac{\int_{e_i} v\cdot\boldsymbol{\eta}}{\int_{e_i} \boldsymbol{\psi}_i\cdot\boldsymbol{\eta}}.$$

We can interpret π_0 as a normal trace on each edge e_i, $i = 1,2,3$, (which is continuous in the norms $\|\cdot\|_{H^1(\Omega)}$ and $\|\cdot\|_{H^{1/2}(\Gamma)}$) followed by an L_2 projection in the piecewise constant functions en each e_i and then each constant is replaced by some multiple of $\boldsymbol{\psi}_i$. Then it is continuous. From the Trace Theorem and a scaling argument, or by using the the scaled $H^1(K)$ norm presented in Section 1.1, we have that:

$$|\alpha_i|^2 \leq c\left(\frac{1}{h^2}\|v\|_{L^2(K)^2}^2 + |v|_{H^1(K)^2}^2\right).$$

Then

$$|\pi_0(v)|_{H^1(\Omega)} \leq c_1 \max_{1\leq i\leq 3}|\alpha_i|^2 \leq c_3\left(\frac{1}{h^2}\|u\|_{L^2(\Omega)^2}^2 + |u|_{H^1(\Omega)^2}^2\right) \tag{3.11}$$

and

$$\|\pi_0(v)\|_{L^2(\Omega)^2}^2 \leq c_2 h^2 \max_{1\leq i\leq 3}|\alpha_i|^2 \leq c_4\left(\|v\|_{L^2(\Omega)^2}^2 + h^2|v|_{H^1(\Omega)^2}^2\right). \tag{3.12}$$

Observe that

$$\int_K \nabla\cdot\pi_0(v) = \int_{\partial K} \pi_0(v)\cdot\boldsymbol{\eta} = \int_{\partial K} v\cdot\boldsymbol{\eta} = \int_K \nabla\cdot v. \tag{3.13}$$

Define $\rho_0 : X \to X^h$ by:

$$\rho_0(v) := \pi_1(v) + \pi_0(v - \pi_1(v)), \tag{3.14}$$

then we have the following result,

Lemma 3.1. *The operator ρ_0 defined in (3.14) is bounded*

$$\|\rho_0(v)\|_{H^1(\Omega)^2} \leq C_0\|v\|_{H^1(\Omega)^2} \tag{3.15}$$

and

$$\|v - \rho_0(v)\|_{L^2(\Omega)^2} \leq \bar{C}_0 h\|v\|_{H^1(\Omega)^2}. \tag{3.16}$$

We also have

$$\int_e \rho_0(v)\cdot\boldsymbol{\eta}_e = \int_e v\cdot\boldsymbol{\eta}_e \quad \textit{for all edge } e. \tag{3.17}$$

Proof. From (3.12) we have

$$\begin{aligned}
\sum_{K\in\mathcal{T}^h} \|\pi_0(v - \pi_1(v))\|^2_{L^2(K)^2} &\leq c_4 \sum_{K\in\mathcal{T}^h} \left(\|v - \pi_1(v)\|^2_{L^2(K)^2} + h^2|v - \pi_1(v)|^2_{H^1(K)^2}\right) \\
&\leq c_4\left(\bar{C}^2h^2\|v\|^2_{H^1(\Omega)^2} + (1+C)^2h^2\|v\|^2_{H^1(\Omega)^2}\right) \qquad \text{by (3.7)} \\
&\leq h^2\tilde{C}^2\|v\|^2_{H^1(\Omega)^2} \qquad (3.18)
\end{aligned}$$

then, using an inverse estimate (see [1]) and (3.18) we get

$$\|\pi_0(v - \pi_1(v))\|_{H^1(K)^2} \leq \hat{C}\frac{1}{h}\|\pi_0(v - \pi_1(v))\|_{L^2(K)^2} \leq \hat{C}\tilde{C}\|v\|_{H^1(\Omega)^2}.$$

Then

$$\begin{aligned}
\|\rho_0(v)\|_{H^1(\Omega)^2} &\leq \|\pi_1(v)\|_{H^1(\Omega)^2} + \|\pi_0(v - \pi_1(v))\|_{H^1(\Omega)^2} \quad \text{by definition of } \rho_0 \\
&\leq C\|v\|_{H^1(\Omega)^2} + \tilde{C}\hat{C}\|v\|_{H^1(\Omega)^2}. \\
&\leq (C + \tilde{C}\hat{C})\|v\|_{H^1(\Omega)^2}.
\end{aligned}$$

To show (3.16) we have that

$$\begin{aligned}
\|v - \rho_0(v)\|_{L^2(\Omega)^2} &= \|v - \pi(v) - \pi_0(v - \pi_1(v))\|_{L^2(\Omega)^2} \quad \text{by definition of } \rho_0. \\
&\leq \|v - \pi_1(v)\|_{L^2(\Omega)^2} + \|\pi_0(v - \pi_1(v))\|_{L^2(\Omega)^2} \\
&\leq \bar{C}h\|v\|_{H^1(\Omega)^2} + \tilde{C}h\|v\|_{H^1(\Omega)^2} \qquad \text{by (3.7) and (3.18)} \\
&\leq \bar{C}_0 h\|v\|_{H^1(\Omega)^2}.
\end{aligned}$$

The other assertion is straightforward. ∎

Remark. *This is a situation similar to that of the MINI elements (see [1]) where Taylor-Hood elements are enriched with a bubble function in each triangle (this bubble is a function of degree three that is null on the boundary of the element). In our case we have one "bubble" of degree two for each edge (with support in the interior of at most two triangles) instead of one bouble for each triangle. In this case the "bubbles" are already elements of the finite element spaces under consideration.*

Given $q \in M^h$, define (locally) $\boldsymbol{\rho}_1(p) \in \boldsymbol{W}^h$ by

$$\boldsymbol{\rho}_1(q)|_K \in \text{Span}\{w_1^{(K)}, w_2^{(K)}, w_3^{(K)}\}$$

with

$$\boldsymbol{\rho}_1(q)(x_{e_i}) \cdot \boldsymbol{\eta} = 0 \text{ and } \boldsymbol{\rho}_1(q)(x_{e_i}) \cdot \boldsymbol{\tau} = \nabla q(x_{e_i}) \cdot \boldsymbol{\tau} \tag{3.19}$$

for all interior edge e_i and $\boldsymbol{\rho}_1(q)|_e = 0$ for all boundary edge e. Note that $\boldsymbol{\rho}_1(q)$ is zero at the vertices of all elements of $\mathcal{T}^h$ and observe that $\boldsymbol{\rho}_1(q) \in H^1(\Omega)$ because the above equation are consistent in neighbor triangles which gives $\boldsymbol{\rho}_1(q)$ continuous (see [1], Chapter II, theorem 5.2).

Lemma 3.2. *Suppose that $\mathcal{T}^h$ is non-degenerate and has no triangle with two edges on $\partial\Omega$ and consider the operator $\boldsymbol{\rho}_1$ defined in (3.19). Then*

$$\|\boldsymbol{\rho}_1(q^h)\|_{L^2(\Omega)^2} \le C_1 |q^h|_{H^1(\Omega)} \quad \textit{for all } q^h \in M^h \tag{3.20}$$

and there exists a positive constant such that:

$$\int_\Omega \boldsymbol{\rho}_1(q^h)\cdot\nabla q^h \ge \bar{C}_1 |q^h|^2_{H^1(\Omega)} \ge C_1 \|q^h\|^2_{L^2(\Omega)} \quad \textit{for all } q^h \in M^h. \tag{3.21}$$

Proof. It is possible to calculate the integral of a quadratic function over a triangle using the value of the function in the three edge mid points, this is a quadrature formula (that integrate exact any affine quadratic function). Applying this formula we get:

$$\begin{aligned}
\int_K \boldsymbol{\rho}_1(q^h)\nabla q^h &= \frac{|K|}{3}\sum_{e\subset\Omega} \boldsymbol{\rho}_1(q^h)(x_e)\cdot\nabla q^h(x_e) \\
&= \frac{|K|}{3}\sum_{e\subset\Omega} |\boldsymbol{\tau}_e\cdot\nabla q^h(x_e)|^2 && \text{by (3.19)} \\
&= \frac{|K|}{3}\sum_{e\subset\Omega} |\boldsymbol{\tau}_e\cdot\nabla q^h_K|^2 && \nabla q^h_K := (\nabla q^h)|_K \text{is constant on K.} \\
&\ge |K| C |\nabla q^h_K|^2 && \boldsymbol{\tau}_e \text{ with } e\subset\Omega \text{ span } \mathbb{R}^2.\ K \text{ is a non-degenerated triangle with at least two edges inside } \Omega \\
&= C\int_K |\nabla q^h_K|^2.
\end{aligned}$$

Then

$$\int_\Omega \boldsymbol{\rho}_1(q^h)\cdot\nabla q^h \ge C|q^h|^2_{H^1(\Omega)} \ge \bar{C}_1 \|q^h\|^2_{L^2(\Omega)} \tag{3.22}$$

here we used a Poincaré Inequality.

Using scaling argument on the functions $w_i^{(K)}$ and the fact that ∇q^h is constant in each triangle we obtain (3.20). ✠

From (3.21) and the boundedness of $\boldsymbol{\rho}_1$ we get that the spaces $\boldsymbol{W}^h$ (with the $\|\ \|_{L^2(\Omega)}$-norm) and M^h (with the $|\ |_{H^1(\Omega)}$-norm) satisfy the inf-sup condition independent of h with respect to the bilinear form defined (in (2.11)) by:

$$b_1(v,q) := -(q, \nabla\cdot v)_\Omega \quad \text{for all } v \in \boldsymbol{X} \text{ and } q \in M,$$

and observe that if $v \in \boldsymbol{W}^h$ then $v\cdot\boldsymbol{\eta} = 0$ on $\partial\Omega$ and then $b_1(v,q) = \int_\Gamma v \cdot \nabla p$ by the Green formula.

Then, according to the Brezzi's splitting theorem (Lemma 1.19), if $\hat{a}$ is any continuous coercive bilinear form defined on $\boldsymbol{W}^h$ we can always get a stable solution $w \in \boldsymbol{W}^h$ of:

$$\begin{cases} \hat{a}(w,v) + b_1(w,p) &= \hat{a}(f,v)_\Omega \quad \forall\, v \in \boldsymbol{W}^h \\ b_1(w,q) &= b_1(f,q)_\Omega \quad \forall\, q \in M^h \end{cases} \tag{3.23}$$

where $f \in L^2(\Omega)$. We use $\hat{a}(w,v) = \int_\Omega vw$, i.e., the L^2-inner product.

Given f, denote by $\rho_2(f)$ the solution of (3.23), then

$$\|\rho_2(f)\|_{L^2(\Omega)} \le C_2 \|f\|_{L^2(\Omega)} \tag{3.24}$$

and $b_1(\rho_2(f), q^h) = b_1(f, q^h)$ for $q^h \in M^h$. We have the following result:

Lemma 3.3. *Suppose that $\mathcal{T}^h$ is non-degenerate and has no triangle with two edges on $\partial\Omega$. Then (X^h, M^h) satisfy the* inf-sup *condition.*

This is a direct consequence of Fortin's criterion (Lemma 1.21) and the following:

Lemma 3.4. *Suppose that $\mathcal{T}^h$ is non-degenerate and has no triangle with two edges on $\partial\Omega$. There exists a bounded linear operator $\mathcal{I}^{TH} : X \to X^h$ such that*

$$b_1(v - \mathcal{I}^{TH}(v), p^h) = 0 \quad \textit{for all } p^h \in M^h$$

and $\|\mathcal{I}^{TH}\| < c$, c constant independent of h.

Proof. Define

$$\mathcal{I}^{TH}(v) := \rho_0(v) + \boldsymbol{\rho}_2(v - \rho_0(v)). \tag{3.25}$$

Observe that:

$$\begin{aligned} \|\mathcal{I}^{TH}(v)\|_{H^1(\Omega)} &\le \|\rho_0(v)\|_{H^1(\Omega)} + \|\boldsymbol{\rho}_2(v - \rho_0(v))\|_{H^1(\Omega)} \\ &\le C_0\|v\|_{H^1(\Omega)} + \tilde{C}\frac{1}{h}\|\rho_2(v - \rho_0(v))\|_{L^2(\Omega)} && \text{by (3.15) and inverse estimate.} \\ &\le C_0\|v\|_{H^1(\Omega)} + \tilde{C}C_2\frac{1}{h}\|v - \rho_0(v)\|_{L^2(\Omega)} && \text{by (3.24)} \\ &\le C_0\|v\|_{H^1(\Omega)} + \tilde{C}\bar{C}_0C_2\|v\|_{H^1(\Omega)}. && \text{by (3.16)} \end{aligned}$$

Then the operator $\mathcal{I}^{TH}$ is bounded (with constant independent of h). More over for $p^h \in M^h$ we get:

$$\begin{aligned} b_1(\mathcal{I}^{TH}(v), p^h) &= b_1(\rho_0(v), p^h) + b_1(\rho_2(v - \rho_0(v)), p^h) \\ &= b_1(\rho_0(v), p^h) + b_1(v - \rho_0(v), p^h) \qquad \text{by definition of } \rho_2. \\ &= b_1(v, p^h). \end{aligned}$$

✠

Lemma 3.5. *Consider the operator $\mathcal{I}^{TH}$ defined in (3.25). We have:*

$$\|v - \mathcal{I}^{TH}(v)\|_{L^2(\Omega)} \leq Ch\|v\|_{H^1(\Omega)}. \tag{3.26}$$

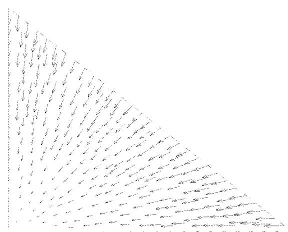
Proof. From the definition of $\mathcal{I}^{TH}$ we have:

$$\begin{aligned} \|v - \mathcal{I}^{TH}(v)\|_{L^2(\Omega)} &\leq \|v - \rho_0(v)\|_{L^2(\Omega)} + \|\rho_2(v - \rho_0(v))\|_{L^2(\Omega)} \\ &\leq \|v - \rho_0(v)\|_{L^2(\Omega)} + C_2\|v - \rho_0(v)\|_{L^2(\Omega)} \qquad \text{by (3.24)} \\ &\leq (1 + C_2)\bar{C}_0 h\|v\|_{H^1(\Omega)}. \qquad \text{by (3.16)} \end{aligned}$$

✠

3.3 Approximation properties of Raviar-Thomas finite elements.

It is considered the case of interest $k = 0$. In this section the domain of reference is Ω_2. As in the previous subsection, we omit the subscript that refers to the domain.



Raviart-Thomas basis function.

Note that the velocities in $RT_0(K)$, $K \in \mathcal{T}$, are of the form:

$$v(x_1, x_2) = \begin{pmatrix} a \\ b \end{pmatrix} + c \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

By using the results presented in Section 1.2 we can work with function in $H(\text{div}, \Omega)$ less regular than funtions in $H^1(\Omega)$ (see [1], [4]).

Lemma 3.6. *For $K \in \mathcal{T}^h$ and $p > 2$, p fixed, define $\mathcal{I}_K^{RT} : \boldsymbol{H}(div, K) \cap L^p(K)^n \to RT_0(K)$ by*

$$\mathcal{I}_K^{RT}(v) \cdot \boldsymbol{\eta}|_e = \frac{1}{|e|} \int_e v \cdot \boldsymbol{\eta}$$

and define $\mathcal{I}^{RT} : \boldsymbol{H}(div, \Omega) \cap L^p(\Omega) \to RT_0$ locally by:

$$\mathcal{I}^{RT}(v)|_K = \mathcal{I}_K^{RT}(v).$$

Then

$$\int_\Omega \nabla\cdot\big(v - \mathcal{I}^{RT}(v)\big)q^h = 0 \quad \textit{for all } q^h \in M^h$$

and there is a constant $C > 0$ *such that* $\|\mathcal{I}^{RT}(v)\|_{H(div,\Omega)} \le C\|v\|_{H(div,\Omega)\cap L^p(\Omega)}$.

Proof. By using the divergence theorem and the definition of $\mathcal{I}^{RT}$ we have:

$$\begin{aligned}
\int_\Omega \nabla\cdot\big(v - \mathcal{I}^{RT}(v)\big)\cdot\eta &= \sum_{K\in\mathcal{T}} q_K^h \int_K \nabla\cdot\big(v - \mathcal{I}^{RT}(v)\big)\cdot\eta \\
&= \sum_{K\in\mathcal{T}} q_K^h \sum_{\substack{e \text{ edge}\\ \text{of } K}} \int_e \big(v - \mathcal{I}^{RT}(v)\big)\cdot\eta \\
&= \sum_{K\in\mathcal{T}} q_K^h \sum_{\substack{e \text{ edge}\\ \text{of } K}} 0 \\
&= 0.
\end{aligned}$$

To prove the second statement, observe that $\mathcal{I}^{RT}(v)$ has constant divergence in each element K and that the operator $\nabla\cdot : RT_0 \to M^h$ is surjective. In fact, given $f \in M^h$ we can find[2] a unique $u \in H^2(\Omega)\cap H^1_0(\Omega)$ such that $\Delta u = f$ in Ω. Taking $v = \nabla u$ it is easy to verify that

$$\int_K \nabla\cdot\mathcal{I}^{RT}(v) = \int_{\partial K} \nabla\cdot v = \int_K f$$

and note that this construction from M^h into RT_0 is continuous. Observe also that from Lemma 1.16 we obtain $\nabla\cdot u \in L^p(\Omega)$. Finally observe that the following diagram conmute:

$$\begin{array}{ccc}
\boldsymbol{H}(\mathrm{div},\Omega)\cap L^p(\Omega)^n & \xrightarrow{\mathcal{I}^{RT}} & RT_0(\Omega) \\
\nabla\cdot\big\downarrow & & \nabla\cdot\big\downarrow \\
L^2(\Omega) & \xrightarrow{\pi} & M^h(\Omega)
\end{array}$$

where π is the L^2-projection into M^h. This follows from the fact that

$$\int_K \nabla\cdot\mathcal{I}^{RT}(v) = \int_K \nabla\cdot v = \int_K \pi(\nabla\cdot v)$$

and that the first and last functions in this equality are constant in K. ∎

By using Poincaré Inequality (Lemma 1.3) and a scaling argument we can show (see [1])

Lemma 3.7. *Consider* $\mathcal{I}^{RT}$ *defined in (3.6). Then if* $v \in H^1(\Omega)$

$$\|v - \mathcal{I}^{RT}(v)\|_{L^2(\Omega)^2} \le h|v|_{H^1(\Omega)}.$$

By using the Fortin's idea we can establish the inf-sup condition for the spaces (X^h, M^h) defined in (3.3) and (3.4) respectively.

[2]Here we need Ω to be convex, but we can, if necessary, enlarge the domain by finitely many triangles

3.4 Discrete inf-sup condition.

We have the following

Lemma 3.8. *Suppose that $\mathcal{T}_1^h$ is non-degenerate and has no triangle with two edges on $\partial\Omega_1$. For p fixed, $p > 2$, there exists a linear continuous operator*

$$\Pi^h : \boldsymbol{V} \cap [\boldsymbol{X}_1^h \times L^p(\Omega_2)^n] \to \boldsymbol{V}^h$$

such that

$$b(\Pi^h v - v, q^h) = 0 \text{ for all } q^h \in M^h \tag{3.27}$$

and

$$\|\Pi^h v\|_X \leq C\|v\|_X. \tag{3.28}$$

Proof. We can put $\Pi^h(v) = (\Pi_1^h(v), \Pi_2^h(v))$ and use for Π_1^h the operator given by Lemma 3.4, i.e., $\Pi^h(v) := \mathcal{I}^{TH}(v_1)$.

We are going to construct Π_2^h.

Let $\lambda = \Pi_1^h(v)|_\Gamma = \mathcal{I}^{TH}(v_1)|_\Gamma$. Construct[3] ω_2 such that:

$$\begin{cases} \nabla\cdot\omega_2 &= \nabla\cdot v_2 \quad \text{in } \Omega_2 \\ \omega_2 &= E_0(\lambda) \quad \text{on } \partial\Omega_2 \end{cases} \tag{3.29}$$

and $\|\omega_2\|_{H^1(\Omega_2)} \leq C(\|\nabla\cdot v_2\|_{L^2(\Omega_2)} + \|\lambda\|_{H^{1/2}(\Gamma)}) \leq C\|v\|_X$.

We define $\Pi_2^h(v)$ as the finite element interpolant of $\boldsymbol{\omega_2} \in X_2$ that is:

$$\Pi_2^h(v) := \mathcal{I}^{RT}(\omega_2) \tag{3.30}$$

where $\mathcal{I}^{RT}$ is given in Lemma (3.6). Then (3.27) holds, let $q = (q_1, q_2) \in M^h$, for $K \in \mathcal{T}$ we have:

$$\begin{aligned} (\nabla\cdot\Pi_2^h v, q_2)_K &= (\nabla\cdot\mathcal{I}^{RT}(\omega_2), q_2)_K && \text{definition of } \Pi_2^h \\ &= (\nabla\cdot\omega_2, q_2)_K && \text{by Lemma 3.6} \\ &= (\nabla\cdot v_2, q_2)_K && \text{by (3.29).} \end{aligned}$$

Finally we prove that $\Pi^h(v) \in V^h$. Let e be an edge that meets the interface Γ (i.e., $e \in \Gamma$) and $\lambda \in \Lambda^h$. Then:

$$\begin{aligned} (\Pi_2^h v\cdot\eta_2, \lambda)_e &= (\mathcal{I}^{RT}(\omega_2)\cdot\eta_2, \lambda)_e && \text{definition of } \Pi_2^h \\ &= (\omega_2\cdot\eta_2, \lambda)_e && \text{definition of } \mathcal{I}^{RT} \\ &= (\Pi_1^h v\cdot\eta_2, \lambda)_e && \text{Lemma 3.4 and (3.29).} \end{aligned}$$

Summing over $e \subset \Gamma$ and using the definition of $\boldsymbol{V}^h$ (3.5) we get:

$$(\Pi_2^h v\cdot\eta_2 - \Pi_1^h v\cdot\eta_2, \lambda)_e = 0$$

which gives the assertion. ✠

[3] see the footnote on page 31.

Remark. *We can try to get ω_2 in (3.29) (or a correction $\omega_2 - v_2$) by using Raviat-Thomas basis functions but this is as hard as solving (3.29) because the normal degrees of freedom are linked in neighbor triangles.*

Remark. *If we assume that the solution is regular, say, it belongs to $\boldsymbol{V} \cap H^1(\Omega_1)^n \times H^1(\Omega_2)^n$. Then from the definition of b_Γ in (2.19) we see that $v_1 \cdot \boldsymbol{\eta} = v_2 \cdot \boldsymbol{\eta}$, in this case we do not have to solve (3.29) and then:*

$$\Pi^h(\boldsymbol{v}) = (\mathcal{I}^{TH}(v_1), \mathcal{I}^{RT}(v_2)). \tag{3.31}$$

In this case we have the approximations properties of both interpolations.

By using the Fortin's idea we can establish the

Lemma 3.9. *Suppose that $\mathcal{T}_1^h$ is non-degenerate and has no triangle with two edges on $\partial\Omega_1$. Then $(\boldsymbol{V}^h, M^h)$ satisfies the discrete* inf-sup *condition,i.e., there is a constant $\beta > 0$ s.t:*

$$\inf_{\substack{q^h \in M^h \\ \lambda \neq 0}} \sup_{\substack{v^h \in \boldsymbol{V}^h \\ v \neq 0}} \frac{b(\boldsymbol{v}^h, q^h)}{\|\boldsymbol{v}^h\|_X \|q^h\|_M} \geq \beta > 0.$$

Let $(\boldsymbol{u}, p)$ the solutions of the second weak formulation (2.24). Let $(\boldsymbol{u}^h, p^h)$ the discrete solution of the same formulation using the finite element spaces presented in the previous section. Then we know that the continuous and the discrete inf-sup condition is satisfied in the spaces $(\boldsymbol{V}, M)$ and $(\boldsymbol{V}^h, M_j^h)$. Then :

$$\|\boldsymbol{u} - \boldsymbol{u}^h\|_X + \|p - p^h\|_M \leq C\Big[\inf_{\boldsymbol{v}^h \in \boldsymbol{V}^h} \|\boldsymbol{u} - \boldsymbol{v}^h\|_X + \inf p \in M^h \|p - p^h\|_M\Big] + \\ + \sup_{\boldsymbol{v}^h \in \boldsymbol{V}^h \setminus \{0\}} \frac{|a(\boldsymbol{u}, \boldsymbol{v}^h) + b(\boldsymbol{v}^h, p) - \ell(\boldsymbol{v}^h)|}{\|\boldsymbol{v}^h\|_X}$$

To analyze the last term in this inequality define:

$$\Theta(\boldsymbol{v}^h) := -\langle p_2, v_1^h \cdot \boldsymbol{\eta}_1 + v_2^h \cdot \boldsymbol{\eta}_2 \rangle_\Gamma, \quad \boldsymbol{v}^h \in \boldsymbol{V}^h. \tag{3.32}$$

Remember that $\lambda := p_2$ and $\lambda \in \Lambda$. Define $\bar{\lambda} \in \Lambda^h$ by $\lambda_e := \int_e p_2$ for all edge e such that $e \subset \Gamma$. Observe that $\hat{\lambda}$ is the L^2-projection of p_2 into Λ^h. Then

$$\begin{aligned} \Theta(\boldsymbol{v}^h) &= -\langle p_2, v_1^h \cdot \boldsymbol{\eta}_1 \rangle_\Gamma - \langle p_2, v_2^h \cdot \boldsymbol{\eta}_2 \rangle_\Gamma \\ &= -\langle p_2, v_1^h \cdot \boldsymbol{\eta}_1 \rangle_\Gamma - \langle \bar{\lambda}, v_2^h \cdot \boldsymbol{\eta}_2 \rangle_\Gamma \qquad v_2^h \cdot \boldsymbol{\eta}_2 \text{ is constant in } e \\ &= \langle \bar{\lambda} - p_2, v_1^h \cdot \boldsymbol{\eta}_1 \rangle_\Gamma \end{aligned}$$

hence

$$
\begin{aligned}
|\Theta(\boldsymbol{v}^h)| &= |\sum_{e\subset\Gamma}\langle\bar{\lambda} - p_2, \boldsymbol{v}_1^h\cdot\boldsymbol{\eta}_1\rangle_e| \\
&\leq \sum_{\subset\Gamma}\|\boldsymbol{v}_1^h\|_{L^2(e)}\|\bar{\lambda} - p_2\|_{L^2(e)} \\
&\leq \|\boldsymbol{v}_1^h\|_{L^2(\Gamma)}\Big[\sum_{e\subset\Gamma}\|\bar{\lambda} - p_2\|^2_{L^2(e)}\Big]^{\frac{1}{2}}
\end{aligned}
$$

Observe that from Poincaré and Freidrichs type inequalities we get

$$\|\boldsymbol{v}_1^h\|_{L^2(\Gamma)} \leq C\|\boldsymbol{v}^h\|_X$$

and that $\bar{\lambda}$ is the L^2-projection of p_2, so:

$$\|\bar{\lambda} - p_2\|_{L^2(e)} \leq Ch_e|p_2|_{H^{1/2}(e)}$$

where h_e is the diameter of e; then

$$|\Theta(\boldsymbol{v}^h)| \leq C\Big[\sum_{e\subset\Gamma} h_e|p_2|_{H^{1/2}(e)})^2\Big]^{\frac{1}{2}}\|\boldsymbol{v}^h\|_X$$

which gives a bound for the consistency error.

Chapter 4

Numerical Examples and Final Remarks.

4.1 Numerical examples.

Consider the problem presented in figure (4.1).

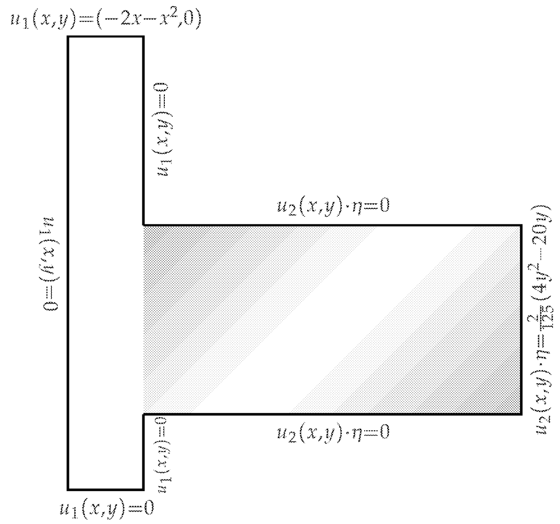


Figure 4.1: Test problem 1.

More precisely, set $\Omega_1 = [-2,0] \times [-2,10]$, $\Omega_2 = [0,10] \times [0,5]$, $\Gamma =$

$\{0\} \times [0,5]$, $\Gamma_j = \partial\Omega_j \setminus \Gamma$, $j = 1,2$. We have the problem:

$$\left\{\begin{array}{lll} -0.01\Delta \boldsymbol{u}_1 + \nabla p & = \mathbf{0} & \text{in } \Omega_1 \\ \nabla\cdot \boldsymbol{u}_1 & = 0 & \text{in } \Omega_1 \\ \boldsymbol{u}_1(x,y) & = (-2x - x^2, 0) & \text{on } \Gamma_{1D} = [-2,0] \times \{10\} \\ \boldsymbol{u}_1(x,y) & = (0,0) & \text{on } \Gamma_1 \setminus \Gamma_{1D} \\ \boldsymbol{u}_2 & = -10^{-8}\nabla p & \text{in } \Omega_2 \\ \nabla\cdot \boldsymbol{u}_2 & = 0 & \text{in } \Omega_2 \\ \boldsymbol{u}_2\cdot \eta_2 & = \frac{2}{125}(4y^2 - 20y) & \text{on } \Gamma_{2D} = \{10\} \times [0,5] \\ \boldsymbol{u}_2\cdot \eta_2 & = 0 & \text{on } \Gamma_2 \setminus \Gamma_{2D} \end{array}\right. \tag{4.1}$$

This problem satisfies condition (2.39). Using discretization presented above, i.e., Taylor Hood coupled with Raviart Thomas finite elements, we get the velocities in figure (4.2). Mesh information is in Table (4.1).

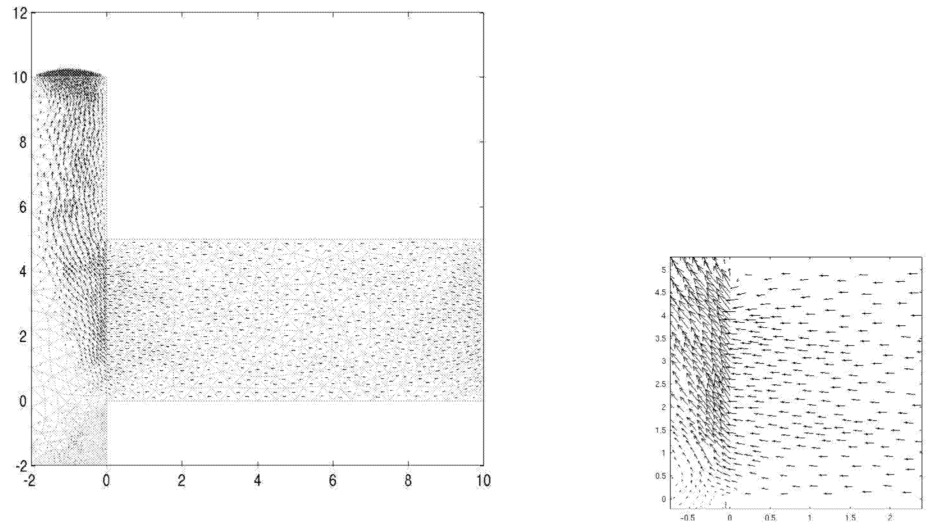


Figure 4.2: Numerical Solution.

	Free Fluid	Porous Medium	Interface
Vertices	428	407	
Edges	120	80	20
Triangles	834	732	

Table 4.1: Mesh information. Test problem 1.

If we look at the geometry of the interface a more interesting example is presented in figure (4.3). We used $\mu = 0.01$, $\kappa = 10^{-8}$ and $\alpha = 0,1$. We look

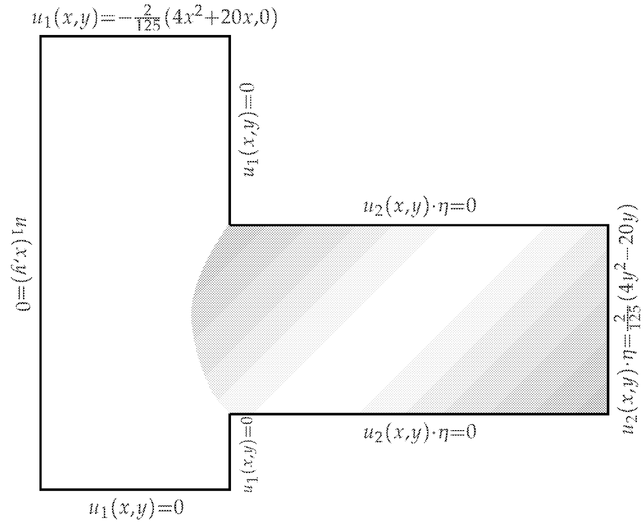


Figure 4.3: Test problem 2.

	Free Fluid	Porous Medium	Interface
Vertices	691	481	
Edges	112	95	25
Triangles	1268	865	

Table 4.2: Mesh information. Test problem 2.

for velocities such that $\nabla\cdot\boldsymbol{u}_1 = \nabla\cdot\boldsymbol{u}_2 = 0$, so this problem satisfies condition (2.39). Using the same finite elements we get the results in figure (4.4), (4.5) for velocities and (4.6) for pressures. Mesh information is in Table (4.2).

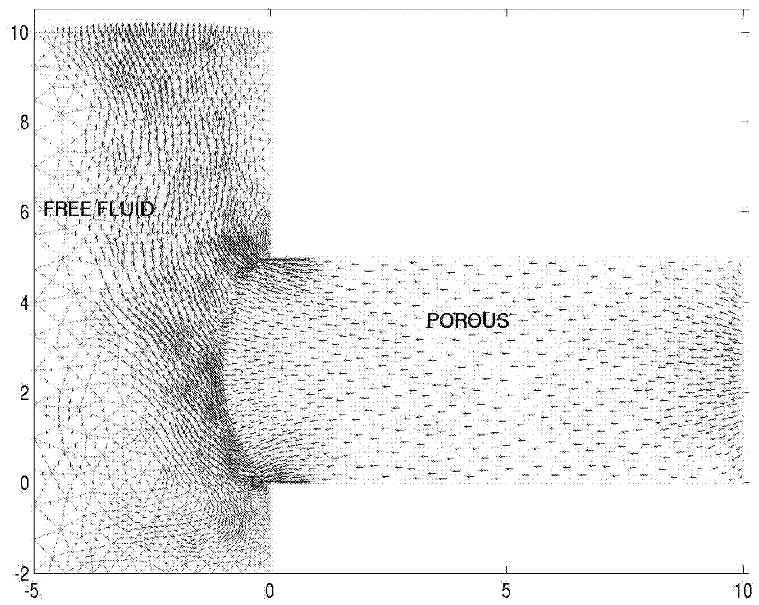


Figure 4.4: Numerical Solution. Test problem 2, $\alpha = 0$.

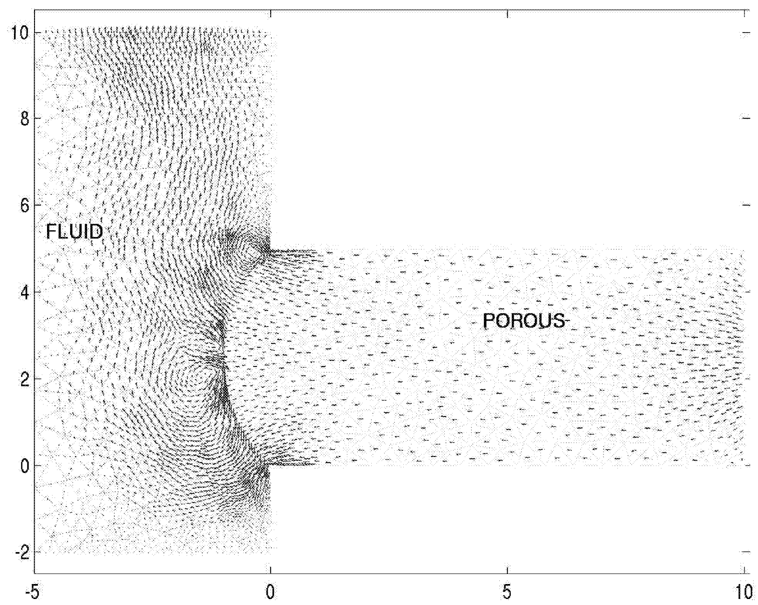


Figure 4.5: Numerical Solution. Test problem 2, $\alpha_1 = 1$.

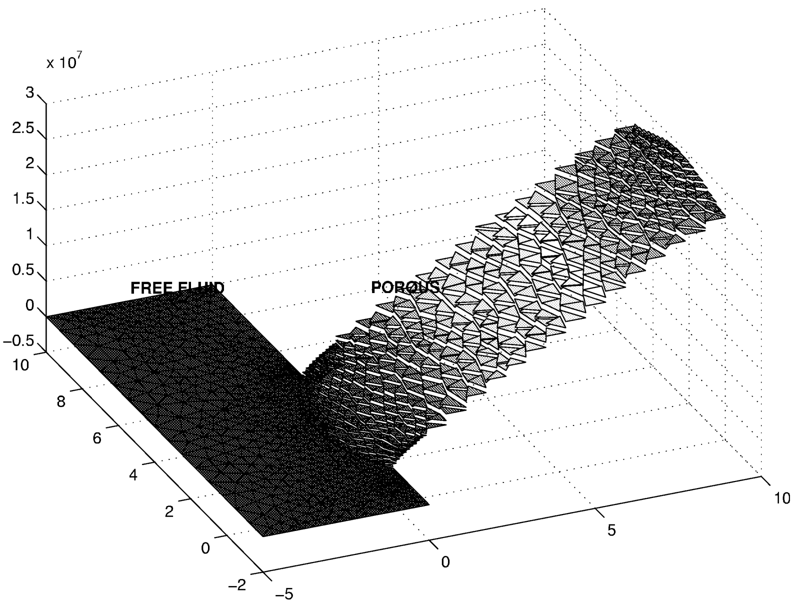


Figure 4.6: Numerical Solution. Test problem 2, $\alpha_1 = 0$. Remark that pressure is not necessary equal at the interface because $\lambda := p_2 = p_1 - 2\mu\boldsymbol{\eta}_1^T \boldsymbol{D}(\boldsymbol{u}_1)\boldsymbol{\eta}_1$.

4.2 Final remarks.

In Chapter 2 we presented the problem of coupling fluid and porous media flow. As we saw in the previous subsections this is a problem with many interesting applications. The model was presented in its partial differential equations form and its weak formulation was derived. The appropriate inf-sup condition was proved in order to get existence and uniqueness of the solution. Then a finite element model was proposed using Taylor-Hood finite elements for the fluid region coupled with Raviart-Thomas elements in the porous region. Some examples of solution calculated using this finite element scheme were presented.

More general models (with two submodels) can be considered. For example the filtration on a cigarette. These models can be studied in a similar way. The hardest part is to get the right interface conditions and to include them in the weak formulation of the problem thinking in a computational way. Other delicate part is to choose the right spaces of Lagrange multipliers to uncouple the submodels.

More general models with several subdomains and adequate conditions between neighbor subdomains can also be considered.

In applications as the examples above we have to be careful because one of our hypothesis is that the domains are of the same order of magnitude when they are compared with the size of the interface. By example in the well reservoir simulation the size of the well is very small compared with that of the reservoir, then it is convenient to use this model close to the well coupled with other adequate model for the rest of the reservoir.

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