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Volatility Calibration in Equity and Commodity Markets by Convex Regularization

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To Roseane, my wife.

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Abstract

We propose a theoretical approach to solve the inverse problem of local volatility calibration from quoted European option prices in Equity and Commodity markets. Based on the assumption that local volatility surface evolves with the initial state, and varies with time, we achieve a term structure calibration procedure for such surface. Thus, combining some Bochner and Sobolev space properties with some techniques of parabolic PDE analysis we characterize the direct problem and its related forward operator, as well.

We apply non-quadratic, convex regularization tools to the inverse problem leading to existence, stability and convergence rates results. Such convergence results are with respect to the noise level and are based on the concept of Bregman distances. Using the same techniques we establish a Morozov discrepancy principle for this problem.

We consider also a discretized set up, whose goal is to present convergence rates results accounting separately to the contribution of different sources of uncertainties.

We apply all the above results to commodity and equity markets.

Finally, we implement algorithms and develop numerical tests with synthetic as well as with real data from equity (SPX, MCD and PBR) and commodity (WTI Brent oil and HH natural gas) markets. As an example of synthetic experiments, we calibrate the local volatility surface from Heston model data.

Key words: Volatility Surface, Local Volatility Calibration, Convex Tikhonov Regularization, Convergence Rates, Commodity and Equity Markets.

Contents

Introduction

The valuation of financial instruments has been the subject of many practical and scientific works, since stocks and its derivatives are at the core of risk management. The attention paid to such problems increased substantially especially after the equity market crash of 1987. A crucial topic in pricing theory is volatility calibration. Volatility has different definitions according to the background of the researcher and the context. Intuitively it is understood as a synonym to the nervousness of the market, or to its agitation. See for example [28]. It is associated to the variance of the process along a moving window [28, 41, 45]. However, this is not the perspective here, since we are concerned with risk-neutral pricing and hedging. Thus, we follow the so-called sell-side approach and consider it as the aggregate diffusion coefficient given by traded prices of derivatives.

There are at least three different meanings for the concept of volatility that permeate the literature. The first one, and perhaps the original one concerns the historical volatility. It is connected to time series analysis and econometrics. This one is used to postulate the classical Black-Scholes model for stock prices. The second one is the so-called implied volatility. It is the value of the volatility that should be used in Black-Scholes formula to give a quoted price. This is a useful change of variable in one to one correspondence with prices, but in principle offers little difficulty to apply. Finally, we have the volatility surface which is the subject of interest of this thesis. Mathematically it can be seen as the diffusion coefficient of a parabolic PDE.

If we are able to accurately determine such volatility, we would be able to price different financial instruments, including exotic derivatives in accordance with the market. Such instruments are used in general to minimize risk in many different situations. As an example, we could foresee the possibility of an oil company using future and other derivatives as hedging devices in its strategic cash-flow management.

The classical reference presenting a way to price European vanilla options derived from a given stock is the Black-Scholes-Merton model [8, 40]. It is based in no-arbitrage conditions leading to a partial differential equation by assuming that volatility is constant in time and different contracts. However, in practice such approach proved to be unrealistic. An evidence of it is the implied volatility, which can be obtained from European vanilla options prices under the Black-Scholes-Merton model. When time or the stock price vary, the implied volatility changes a lot, it shows that Black-Scholes volatility is not constant.

Under certain hypotheses, following Derman and Kani [20], Dupire [21] and Gatheral [28], we can describe local volatility as the unique state dependent diffusion parameter consistent with European option prices. Such quantity can also be thought as the conditional expectation of all instantaneous volatilities, from stochastic volatility models [28]. In other words, local volatility models are not separate from the the class of stochastic volatility models. Another interesting aspect of such quantity is that it can depend heavily on the initial state price. See [28].

Therefore, following the ideas presented in [21, 28], we assume a initial state dependent local volatility framework. Under these hypotheses, in order to characterize the direct problem, we propose a theoretical approach based on Bochner and Sobolev spaces techniques [29, 52], using also the usual parabolic PDE approach [23, 38]. More specifically, we define a parameter to solution map

$$\mathcal{U} : D(\mathcal{U}) \subset U \longrightarrow V.$$

relating families of volatility surfaces with families of European vanilla option prices surfaces, both indexed by the initial state prices. The generality of such characterization of the direct problem allows us to use many different identification methods, in special convex regularization.

Having established the direct problem, we pass to the solution of the calibration problem. In other words, we take a family of prices surfaces $\widetilde{\mathcal{U}}$ and search for a family of local volatility surfaces $\mathcal{A}^\dagger \in D(\mathcal{U})$ satisfying

$$\widetilde{\mathcal{U}} = \mathcal{U}(\mathcal{A}^\dagger).$$

However, we shall see that this inverse problem is ill-posed in the sense of Hadamard [35]. By applying convex regularization tools [48] the original problem becomes the following minimization problem:

$$\text{argmin}\left\{\|\widetilde{\mathcal{U}} - \mathcal{U}(\mathcal{A})\|_V^2 + \alpha f(\mathcal{A})\right\} \qquad \text{subject to } \mathcal{A} \in D(\mathcal{U}).$$

It leads us to obtain results concerning existence and stability of solutions. Note that, the constant α is the regularization parameter and $f : D(f) \to [0, +\infty]$ is a convex functional, both have to be properly chosen. We obtain also a convergence analysis when we consider data corrupted by noise, by using Bregman distance techniques [10, 48]. We establish results concerning an optimal choice for the regularization parameter α in a very general setting [3, 42]. The optimality here means that, α is chosen in order to minimize the residual $\|\widetilde{\mathcal{U}} - \mathcal{U}(\mathcal{A})\|$. Furthermore, we consider all these methods under a discretized set up [44].

We observe that, considering indexed families of surfaces of prices and local volatilities can be helpful since we increase the amount of data used in the calibration process. It becomes better when we assume a smoother dependence on such index, like Hölder continuity with index greater than 1/2. It can be thought as a non-Markovian dependence on the history of such families.

Under a very general framework, we choose convex regularization in order to tackle this inverse problem. It provides much more general tools than the standard quadratic Tikhonov regularization. Such generality is useful, since it allows us to introduce desirable properties concerning the solution, through the choice of the regularization functional. This idea is largely used in statistical inverse problems. See [49]. For instance, this approach generalizes many approaches presented in literature, as [17, 19, 23, 36, 39].

Another aspect explored in this thesis is the application of this framework to the context of commodity markets. Indeed, commodity futures and their derivatives have become key players in the portfolios of many corporations, especially those in the energy sector. To the best of our knowledge, this is the first time local volatility calibration with convex regularization is considered to tackle this problem.

The main contribution of this thesis can be summarized as follows:

1. Extending regularization results for local volatility calibration to a more general framework that incorporates the dependence of the local volatility surface on the current stock price.

2. Developing a convergence analysis in a general context for the above problem.

3. Establishing a Morozov discrepancy principle in order to find the regularization parameter appropriately.

4. Considering a discretized setting, accounting separately for the uncertainties introduced in this problem by different sources.

5. Applying the calibration techniques to commodity markets.

We illustrate the results established in this thesis by numerical tests. We seek to highlight the robustness and reliability of the regularized solutions.

It is important to note that, we do not apply any filtering approach as Kalman filter [49] or an econometric approach [41, 45]. The method presented in this thesis can be classified as a non-parametric approach, since we are concerned with the search of a family of the functional diffusion parameters known as local volatility surfaces.

The idea behind the direct problem is the pricing of European call option prices by a generalization of Black-Scholes-Merton model. Namely, consider a risk-neutral filtered probability space $(\Omega, \mathscr{U}, \mathbb{F}, \widetilde{\mathbb{P}})$, where $\mathbb{F} = \{\mathbb{F}_t\}_{t\in\mathbb{R}}$. Assume that the prices of a given stock follow the Itô's dynamics

$$\begin{cases} dS(t) &= r(t)S(t) + \sqrt{\nu(t)}S(t)dW^{\widetilde{\mathbb{P}}}(t, S(t)) \quad \text{for } t \in [t_0, T_{\max}] \\ S(t_0) &= S_0 \end{cases} \tag{1}$$

with $r(t)$ the risk-free interest rate, $\nu(t)$ the stochastic volatility, $W^{\widetilde{\mathbb{P}}}$ a $\widetilde{\mathbb{P}}$-Brownian motion and S_0 the deterministic initial stock price. In general, we do not observe ν. Strongly related to stochastic volatility model is the well-known local volatility, which can be defined by the following expected value [21, 28]:

$$\sigma(S_0, T, K) = \sqrt{\mathbb{E}^{\widetilde{\mathbb{P}}}[\nu(T)|S(T) = K]}. \tag{2}$$

Note that, since $r(t)$ is known, it follows that the distribution of the stochastic process $\{S(t)\}_{t\in[t_0,T]}$ with respect to $\widetilde{\mathbb{P}}$ is uniquely determined by ν, which remains unknown. During this thesis, instead of trying to describe the stochastic volatility ν, we shall work with the local volatility function $\sigma(S_0, t, S(t))$.

The simplest example of a derivative for such stock is an European call option with maturity T and strike K. Such contract gives to its owner the right but not the obligation to buy at time T a share of the underlying stock for the price K. Thus, if $S(T) > K$ the option has the positive value $S(T) - K$. However, if $S(t) \leq K$, it follows that the option is worthless. In other words, the value of the option at maturity T with strike K is given by the payoff

$$C(T, K) = \max(0, S(T) - K).$$

The present value of the option is assumed to be the mean in the risk-neutral measure of the discounted payoff given the actual stock price, i.e.,

$$C(T, K, s, t) = \mathrm{e}^{-r(T-t)}\mathbb{E}^{\widetilde{\mathbb{P}}}[\max(0, S(T) - K)|S(t) = s]. \tag{3}$$

See Chapters 2 and 3 of [37]. It follows that, when $\nu(t) = \sigma(S_0, t, S(t))$ in (1) is a deterministic function of the time and the stock price, fixing maturity and strike, $C(t, s)$ satisfies the generalized Black-Scholes problem

$$\begin{cases} \dfrac{\partial C}{\partial t} + \dfrac{1}{2}\sigma^2(t, s)S^2\dfrac{\partial^2 C}{\partial s^2} + b\dfrac{\partial C}{\partial s} & = & 0, & t \in [0, T], s > 0 \\ C(T, s) & = & \max(0, s - K) & , s > 0. \end{cases} \tag{4}$$

The model presented in [8] can be found by choosing σ constant.

The model developed by Dupire [21], which is assumed in this thesis, is based on the same assumption of a functional deterministic volatility. As we mentioned above, he has shown that $\sigma(t, S)$ is the unique state dependent diffusion parameter consistent with European option prices which can be obtained directly from such prices. However the formula stating this result is highly unstable, since it is based on the differentiation of observed data. We shall see it in more detail at the beginning of Chapter 1.

The plan of this thesis is the following:

In Chapter 1 we make a survey on the theory of local volatility calibration, presenting some known results from literature, in special [17, 19, 23]. We first analyze the direct problem and present some background properties of the forward operator. Then, we present results extracted from [17, 19, 23], concerning Tikhonov regularization of such calibration problem. At the end of the chapter we make a short literature review concerning local volatility calibration.

Chapter 2 is concerned with the direct problem. We first introduce the concept of Fourier transform in $L^2(0, T, H^{1+\varepsilon}(D))$ and define the Bochner-Sobolev space $H^l(0, S, H^{1+\varepsilon}(D))$, where D is the set where the direct problem is stated. Under such framework we prove some background properties concerning the forward operator. The most important are continuity, compactness and Frechét differentiability. At the end of this chapter we establish that our analysis can be simplified by assuming that volatility is independent of the initial state of the stock price. It implies that normalized prices for different initial states are obtained by the same diffusion parameter.

Chapter 3 concerns the analysis of the inverse problem by applying Tikhonov regularization.

Following the techniques of [48], we first present results concerning the existence and stability of regularized solutions. After, making use of tools from convex regularization, we establish a convergence analysis. Using the same techniques we also present a Morozov discrepancy principle. We consider also a discretized set up whose goal was to account separately some different sources of uncertainties in the convergence analysis.

In Chapter 4 we apply all the results of the previous chapters in the context of commodity markets, considering European vanilla options on commodity futures. We base our analysis on the assumption that risk-neutral future prices are martingales. In addition, we characterize the direct problem in two different ways. In the first one, we consider the forward operator related to a generalization of Black's model. In the second one, we consider a Dupire's equation for options on futures. After we establish that such forward operators are particular cases of the one studied in the previous chapters, we apply the previous results concerning existence, stability and convergence rates to the related inverse problem.

The numerical results are presented in Chapter 5. There, we implement algorithms and develop numerical tests with synthetic as well as real data from equity and commodity markets. We first make a small description of the numerical solution to the inverse problem, by presenting a solution of the forward problem and a characterization of the conjugate gradient of the Tikhonov functional. Then, we mention that the minimization problem can be solved by available optimization methods, as the conjugate gradient method with Wolfe's conditions. In the first set of experiments, we illustrate the robustness of the method by comparing reconstructed volatility solutions with the original ones for different noise levels. In the second and third parts of the numerical tests we present reconstructed volatility surfaces with MCD, PBR, SPX, HH and WTI data.

Chapter 1

A Short Survey of Local Volatility Calibration

In this chapter, we first present the direct problem, the forward operator and its domain, which in turn is defined by Dupire's equation [21]. See for example [17, 19, 23]. After, we present a small collection of results extracted from literature about properties of this operator and its Frechét derivative, which are in the core of the theoretical analysis of inverse problems presented in [17, 19, 23].

Then, we present some results about Tikhonov regularization as existence, stability, convergence and convergence rates.

At the end of this chapter we present a small literature review about volatility calibration.

1.1 Dupire's Equation: The Direct Problem

Let $(\Omega, \mathscr{U}, \mathbb{F}, \widetilde{\mathbb{P}})$ be a filtered probability space , with $\mathbb{F} = \{\mathbb{F}_t\}_{t\in\mathbb{R}}$ a suitable filtration and $\widetilde{\mathbb{P}}$ the risk-neutral measure. Thus, we consider a stock which prices are governed by the Itô's dynamics

$$dS(t) = rS(t)dt + \sigma(t, S(t))S(t)dW(t), \text{ for } t \in [0, T] \text{ and } S(0) = S_0 > 0 \text{ non-random.} \quad (1.1)$$

The process $\{W(t)\}_{t\in\mathbb{R}}$ is a Brownian motion under $\widetilde{\mathbb{P}}$, r is the interest rate, which is assumed constant for simplicity and σ is the volatility. Note that, following Dupire [21], $\sigma = \sigma(t, s)$ is a deterministic function of $(t, s) \in [0, \infty) \times (0, \infty)$.

Let us define the starting point of our analysis: Dupire's equation (see [21]). Firstly, consider an European vanilla option price $C(T, K, t, S(t))$ at time t written on the stock $S(t)$ from (1.1), with maturity T and strike K. Thus, when we fix t and $S(t)$, writing $t = 0$ and $S(0) = S_0$, it follows that $C(S_0, T, K)$ satisfies Dupire's equation:

$$\begin{cases} -\dfrac{\partial C}{\partial T} + \dfrac{1}{2}\sigma^2 K^2 \dfrac{\partial^2 C}{\partial K^2} - rK\dfrac{\partial C}{\partial K} = 0 & T > 0,\ K \geq 0 \\ C(T = t, K) = (S_0 - K)^+, & \text{for } K > 0. \end{cases} \quad (1.2)$$

By the no-arbitrage condition implied by the risk-neutral framework, it follows that

$$\frac{\partial^2 C}{\partial K^2} > 0$$

and then, as $K > 0$ we have that $\sigma(T, K)$ is defined by $C(S_0, T, K)$ through the formula

$$\sigma(T, K) = \sqrt{2\frac{\dfrac{\partial C}{\partial T} + rK\dfrac{\partial C}{\partial K}}{K^2\dfrac{\partial^2 C}{\partial K^2}}}. \tag{1.3}$$

In other words, the formula (1.3) first presented in [21] shows a way to find directly σ as a function of (t, S) when $t = T$ and $S = K$, from quoted European vanilla option prices.

Unfortunately, such formula is not robust since it involves differentiation of data, which in general is sparse and corrupted by noise, implying that, the expression in the square root signal could be negative or even unbounded.

Therefore, we have to find an alternative way to reconstruct σ as it is of great importance in the pricing of many financial instruments in financial and commodity markets.

In order to apply classical techniques of parabolic PDE theory (see [38]), we perform the change of variables $y := \log(K/S_0)$ and $\tau := T - t$. Thus, we define

$$u(\tau, y) := C(\tau + t, S_0 \mathrm{e}^y) \text{ and } a(\tau, y) := \frac{1}{2}\sigma^2(\tau + t, S_0 \mathrm{e}^y),$$

it follows that $u(\tau, y)$ satisfies

$$\begin{cases} -\dfrac{\partial u}{\partial \tau} + a\left(\dfrac{\partial^2 u}{\partial y^2} - \dfrac{\partial u}{\partial y}\right) + r\dfrac{\partial u}{\partial y} = 0 & \tau > 0,\ y \in \mathbb{R} \\ u(0, y) = S_0(1 - \mathrm{e}^y)^+, & y \in \mathbb{R}, \end{cases} \tag{1.4}$$

Note that, σ and a are assumed strictly positive and are related by a smooth bijection (under suitable conditions). Thus, in what follows we shall work only with variance a instead of volatility σ since it simplifies the direct and inverse problems analysis.

Let $I \subset \mathbb{R}$ be a probably unbounded open interval, denote by $D = (0, T) \times I$ the set where problem (1.4) is defined. From [23] we know that (1.4) has a unique solution in $W^{1,2}_{2,loc}(D)$, the space of functions $u : (\tau, y) \in D \mapsto u(\tau, y) \in \mathbb{R}$ such that, it has locally squared integrable weak derivatives up to order one in τ and up to order two in y.

For latter purposes, we define now the set where a lives. Take scalar constants $a_1, a_2 \in \mathbb{R}$ such that $0 < a_1 \leq a_2 < +\infty$ and a fixed function $a_0 \in H^{1+\varepsilon}(D)$, with $\varepsilon > 0$ and $a_1 \leq a_0 \leq a_2$. Define

$$Q := \{a \in a_0 + H^{1+\varepsilon}(D) : a_1 \leq a \leq a_2\} \tag{1.5}$$

Now, define the forward operator

$$\begin{array}{rcl} G : Q \subset H^{1+\varepsilon}(D) & \longrightarrow & W_2^{1,2}(D) \\ a \in Q & \longmapsto & u(a) - u(a_0) \in W_2^{1,2}(D) \end{array} \tag{1.6}$$

where $u(a)$ and $u(a_0)$ are the unique solutions of (1.4) with diffusion parameter a and a_0, respectively. Note that a_0 is fixed and given. It is necessary to introduce $u(a_0)$ in the definition of operator $G(\cdot)$ because every solution $u(a)$ of (1.4) for a given diffusion parameter a is in $W^{1,2}_{loc,2}(D)$. See [23, 38]. If we subtract it by $u(a_0)$, i.e, take $u(a) - u(a_0)$, then $u(a) - u(a_0)$ is in $W_2^{1,2}(D)$. Such property follows by the linearity of equation (1.4), which implies that $u(a) - u(a_0)$ satisfy a parabolic problem with homogeneous boundary conditions.

Therefore we can state now the **Direct Problem**:
Given a function $a \in Q$, construct $G(a) = u(a) - u(a_0)$, where $u(a)$ is the unique solution of (1.4) with diffusion parameter a.

1.2 Background Properties of the Forward Operator

Following [48], we present below some properties of the operator $G(\cdot)$ as weak-continuity, weak-closedness and Frechét differentiability. They are required to implement Tikhonov regularization. Such properties are extracted specially from [17, 19, 23].

Observe that, we have chosen the space $H^{1+\varepsilon}(D)$ with $\varepsilon \in (0, 1]$ so that Q has nonempty interior. See [19]. Furthermore, we have that Q is convex and weakly closed in $H^{1+\varepsilon}(D)$.

The following proposition states that calibrating local volatility from quoted European call option prices is an ill-posed problem, since the forward operator is compact.

Proposition 1 (Proposition A2 and A3 of [23]). *The operator $G(\cdot)$ is continuous and compact. Moreover it is weakly (sequentially) continuous and weakly closed.*

Now we state the differentiability of $G(\cdot)$.

Proposition 2 (Proposition 4.1 of [23]). *The operator $G(\cdot)$ admits a one sided derivative at $a \in Q$, in the direction h such that $a + h \in Q$. Its Frechét derivative $G'(a)$ can be extended as a bounded linear operator on $H^{1+\varepsilon}(D)$, i.e., it satisfies*

$$\|G'(a)h\|_{W_2^{1,2}(D)} \leq c\|h\|_{H^{1+\varepsilon}(D)}.$$

Moreover, $G'(a)$ satisfies the Lipschitz condition

$$\|G'(a) - G'(a+h)\|_{\mathcal{L}\left(H^{1+\varepsilon}(D), W_2^{1,2}(D)\right)} \leq \gamma\|h\|_{H^{1+\varepsilon}(D)}$$

for all $a, h \in Q$ such that $a + h \in Q$.

We observe that in [23] all the results were obtained with $\varepsilon = 0$ in $H^{1+\varepsilon}(D)$. However, as it is stated in Section 1.3 of [19], Proposition 1 and 2 hold true with $\varepsilon \in (0, 1]$.

Since the embedding $W_2^{1,2}(D) \subset L^2(D)$ is linear and bounded, we have that Proposition 1 and 2 still hold true if we change from $W_2^{1,2}(D)$ to $L^2(D)$ in the definition of $G(\cdot)$. This is an important change in the topology of the range of $G(\cdot)$ as it is easier to work with $\langle\cdot,\cdot\cdot\rangle_{L^2(D)}$ instead of $\langle\cdot,\cdot\cdot\rangle_{W_2^{1,2}(D)}$.

The next two results present a characterization of the range of $G'(a)$ as a subset of $L^2(D)$ and $G'(a)^*$ as a subset of $H^{1+\varepsilon}(D)$. They are important players in the proof of convergence rates results for Tikhonov regularization.

Proposition 3 (Lemma 1.4.1 of [19]). *The Frechét derivative of $G(\cdot)$ is injective and compact.*

Proposition 4 (Lemma 1.4.2 of [19]). *The adjoint operator $G'(a^\dagger)^*$ of $G'(a^\dagger)$, has a trivial kernel. Thus, the range of $G'(a^\dagger)^*$ is dense in $H^{1+\varepsilon}(D)$.*

Now we present a very important result, namely the tangential cone condition. It is a sufficient condition to state results that guarantee the convergence of iterative regularization methods. See Chapter 3 of [19].

Proposition 5 (Theorem 1.4.2 of [19]). *The operator $G(\cdot)$ satisfies the local tangential cone condition*

$$\|G(a) - G(\tilde{a}) - G'(\tilde{a})(a-\tilde{a})\|_{W_2^{1,2}(D)} \leq \gamma \|G(a) - G(\tilde{a})\|_{W_2^{1,2}(D)}, \quad \textit{with } \gamma < 1/2,$$

for all $a, \tilde{a}$ in an open ball $B(a^*, \rho) \subset Q$ for some $\rho > 0$.

1.3 Volatility Calibration by Tikhonov Regularization

During the last section we have provided some remarkable properties about the operator G. Continuing this survey, we present below some results extracted from literature about Tikhonov regularization applied to the local volatility calibration of European call options.

As it was mentioned above, we are interested in solving the inverse problem of volatility calibration, i.e., given a set of European call option prices traded at a given day t,

$$C(t) = \{C(t, T_m, K_n) : m = 1, ..., M \text{ and } n = 1, ..., N\},$$

find its related local volatility surface σ, assuming the framework developed in Sections 1.1 and 1.2. In other words, we want to find $a^\dagger \in Q$ satisfying

$$G(a^\dagger) = C(t) - u(a_0). \tag{1.7}$$

In addition we let the data be corrupted by a noise of level $\delta > 0$, i.e.,

$$\|u(a^\dagger) - C(t)\|_{L^2(D)} \leq \delta.$$

We denote by u^δ the noisy data $C(t) - u(a_0)$.

Roughly speaking, we consider a noisy data since for a given model there are many sources of uncertainty. For example, there is a finite amount of option prices which are presented in a sparse grid. In addition, the present model is continuous, which leads to uncertainties introduced interpolation.

Since there is no easy way to solving such problem directly we present below how Tikhonov regularization helps us to find reliable approximations of $a^\dagger$ and what happens when $\delta \to 0$.

Let $f_{a_0} : Q \to [0, \infty]$ be a convex and weakly lower semi-continuous functional. Define the Tikhonov regularization functional

$$\mathcal{F}_{\alpha,u^\delta}(a) = \|G(a) - u^\delta\|^2_{L^2(D)} + \alpha f_{a_0}(a). \tag{1.8}$$

Instead of looking for $\tilde{a}$, an element of Q satisfying (1.7), we shall find a minimizer for (1.8) in Q. Assuming further that $f_{a_0}(\cdot)$ is coercive we have that for each $c > 0$, the set

$$L_\alpha(c) = \{a \in Q : \mathcal{F}_{\alpha,u^\delta}(a) \leq c\} \subset H^{1+\varepsilon}(D)$$

is weakly pre-compact. Thus, from [48] we have the next three results.

Theorem 1 (Existence - Theorem 2.1.1 of [19])**.** *If $\alpha > 0$ and $u^\delta \in L^2(D)$, then there exists a minimizer in Q for the functional $\mathcal{F}_{\alpha,u^\delta}(\cdot)$ defined in (1.8).*

Theorem 2 (Stability - Theorem 2.1.1 of [19])**.** *If $\alpha > 0$ and $\{u_k\}_{k\in\mathbb{N}} \subset L^2(D)$ is a sequence converging strongly to u in $L^2(D)$, then every sequence of minimizers $\{a_k\}_{k\in\mathbb{N}} \subset Q$ of $\mathcal{F}_{\alpha,u_k}(\cdot)$, has a weakly convergent subsequence $\{a_{k_l}\}_{l\in\mathbb{N}}$. The limit of such subsequence $\tilde{a}$ is a minimizer of $\mathcal{F}_{\alpha,u}(\cdot)$. Furthermore $f_{a_0}(a_{k_l}) \to f_{a_0}(\tilde{a})$.*

Theorem 3 (Convergence - Theorem 2.1.1 of [19])**.** *If (1.7) has a solution in Q, then it has an f_{a_0}-minimizing solution. Thus, assume that such solution for the noiseless problem with data $\overline{u}$ exists and $\alpha = \alpha(\delta) > 0$ satisfies*

$$\lim_{\delta\to 0} \alpha(\delta) = 0 \quad \text{and} \quad \lim_{\delta\to 0} \frac{\delta^2}{\alpha(\delta)} = 0.$$

Take a sequence $\{\delta_k\}_{k\in\mathbb{N}}$ converging to zero and assume further that $u_k \in L^2(D)$ satisfies $\|u_k - \overline{u}\|_{L^2(D)} \leq \delta_k$. Denote $\alpha_k := \alpha(\delta_k)$. Then, every sequence $\{a_k\}_{k\in\mathbb{N}} \subset Q$ with a_k minimizing $\mathcal{F}_{\alpha_k,u_k}(\cdot)$ has a weakly convergent subsequence $\{a_{k_l}\}_{l\in\mathbb{N}}$ with limit $a^\dagger \in Q$. Such limit is an f_{a_0}-minimizing solution (1.7) and $f_{a_0}(a_{k_l}) \to f(a^\dagger)$. Furthermore, if there is a unique f_{a_0}-minimizing solution $a^\dagger$ of (1.7), then the entire sequence $\{a_k\}_{k\in\mathbb{N}}$ converges weakly to $a^\dagger$.

1.4 A Convergence Analysis

Theorem 3 states the convergence of the regularized solutions when the noise level δ goes to zero, given a suitable choice for the regularizing parameter α. However, it does not state its convergence rate, hence this is the purpose of the present section.

Before starting the convergence analysis below, we have to introduce some fundamental concepts in convex regularization as the Bregman distance (divergence) related to f_{a_0}, q-coerciveness and the source condition. Bregman distance is not a distance, in general, however it is a useful tool to define a weaker sense of convergence than the norm one. q-coerciveness states a strong relation between Bregman distances and the space norm, under certain hypotheses. The source condition is in the core of the characterization of the range of $G'(a)^*$ which is used to prove convergence rates. A good reference on this topic is [48].

Now, we have the following definitions.

Definition 1. *Let U be a Banach space and $f : D(f) \subset U \to [0, \infty]$ be a convex functional with sub-differential $\partial f(a)$ in $a \in D(f)$. The Bregman distance (or divergence) of f at $a \in D(f)$ and $\xi \in \partial f(a) \subset U^*$ is defined by*

$$D_\xi(b, a) = f(b) - f(a) - \langle \xi, b - a \rangle,$$

for every $b \in D(f)$, with

$$\langle \cdot, \cdot\cdot \rangle : U^* \times U \to \mathbb{R},$$

the dual product of U^* and U. Moreover, the set

$$\mathcal{D}_B(f) = \{a \in D(f) \ : \ \partial f(a) \neq \emptyset\}$$

is called the Bregman domain of f.

We remark that the Bregman domain $D_B(f)$ is dense in $D(f)$ and the interior of $D(f)$ is a subset of $D_B(f)$. Furthermore, the map $b \in D(f) \mapsto D_\xi(b, a)$ is convex, non-negative and satisfies $D_\xi(a, a) = 0$. In addition, if f is strictly convex, then $D_\xi(b, a) = 0$ if and only if $b = a$. For a survey on Bregman distances see Chapter I of [10].

Definition 2. *For $1 \leq q < \infty$, the Bregman distance $D_\xi(\cdot, a)$ is said to be q-coercive with constant $\zeta > 0$ if*

$$D_\xi(b, a) \geq \zeta \|b - a\|_U^q,$$

for every $b \in D(f)$.

The canonical example

$$f(a) = \|a - a_0\|_U^2,$$

is 2-coercive with constant 1 since $\partial f(a) = \{2(a - a_0)\}$ for every $a \in D(f)$ and

$$D_{2(a-a_0)}(b, a) = \|b - a\|_U^2.$$

As in Section 2.1.2 of [19], in order to state Theorem 4 the following assertions are sufficient:

(i) The inverse problem (1.7) has at least one f_{a_0}-minimizing solution denoted here by $a^\dagger$, which is in the Bregman domain associated to f_{a_0}.

(ii) There exist constants $\beta_1 \in [0, 1)$, $\beta_2 \geq 0$ and $\xi^\dagger$, an element of the sub-differential of f_{a_0} at $a^\dagger$, denoted by $\partial f_{a_0}(a^\dagger)$, such that

$$\langle \xi^\dagger, a^\dagger - a \rangle \leq \beta_1 D_{\xi^\dagger}(a, a^\dagger) + \beta_2 \|G(a) - G(a^\dagger)\| \tag{1.9}$$

for $a \in L_{\beta_{\max}}(\rho)$, with $\rho > 0$ and $\beta_{\max}$ satisfying $\rho > \beta_{\max} f_{a_0}(a^\dagger)$.

Note that the first assumption holds immediately for this specific problem (see Section 2.1 of [19]) and the second one follows by Lemma 2.

The following two Lemmas are concerned with the so-called source condition and q-coerciveness. As it is mentioned above, such concepts are in the core of convergence analysis and rates. See [48].

Lemma 1 (Lemma 2.1.1 of [19]). *For every $\xi \in \partial f_{a_0}(a^\dagger)$, there exists an $\omega^\dagger \in L^2(D)$ and an $r \in H^{1+\varepsilon}(D)$ satisfying the source condition*

$$\xi = G'(a^\dagger) * \omega^\dagger + r \tag{1.10}$$

with $\|r\|$ arbitrarily small.

Lemma 2 (Lemma 2.1.2 of [19]). *For every $\xi \in \partial f_{a_0}(a^\dagger)$ take $\omega^\dagger$ and r as in Lemma 1, satisfying*

$$\underline{c}(C\|\omega^\dagger\| + \|r\|) < 1$$

and denote such estimate by β_1. Assume further that f_{a_0} is 1-coercive with constant $\underline{c}$. Then (1.9) in assertion (ii) above holds.

As mentioned above, these lemmas are sufficient to ensure the convergence rates below.

Theorem 4 (Theorem 2.1.2 of [19]). *If $\alpha = \alpha(\delta) \approx \delta$ then, for $\xi \in \partial f_{a_0}(a^\dagger)$ and a_α^δ minimizer of (1.8) satisfy*

$$\|G(a_\alpha^\delta) - u^\delta\| = O(\delta) \quad \text{and} \quad D_\xi(a_\alpha^\delta, a^\dagger) = O(\delta).$$

Moreover, there exists $c > 0$ such that $f_{a_0}(a_\alpha^\delta) \leq f_{a_0}(a^\dagger) + \delta/c$ for every $\delta > 0$ satisfying the estimate $\beta(\delta) \leq \beta_{\max}$.

1.5 Local Volatility Calibration: A Micro Review

Due to its great importance in financial markets, volatility calibration has been the subject of a huge amount of academic and practical works. Thus, in order to make a brief review in this topic, we have chosen only a few works due to its importance and its strong connection with this thesis.

Dupire in [21] introduced the local volatility framework and presented the closed formula (1.3) reconstructing the local volatility surface from quoted European option prices. Such quantity defines the unique risk neutral diffusion process compatible with these option prices. However, this formula can not be used in practice since the available set of quoted prices are in general too much sparse and noisy.

In [4], Avellaneda *et al* have proposed, under a risk-neutral framework, the following problem: *find a control function σ^2 minimizing Kullback-Leibler entropy such that, the risk neutral call option prices match the quoted prices.* Thus, under certain hypotheses they found a Hamilton-Jacobi-Bellman equation and hence a closed formula for such σ^2 as a deterministic function of maturity T and strike K. The drawback of such approach is that the resulting local volatility surface, is in general non-smooth, which may vary abruptly.

In [36], Jackson *et al* have proposed local volatility calibration from quoted European vanilla option prices under the Black-Scholes framework. They have assumed that, for each time t $\sigma(t, S)$ is a spline whose weights are determined by the penalized (regularized) minimization of the difference between quoted and Black-Scholes prices. By numerical examples, the method proved to be robust despite not having any theoretical justification.

In [39], Lagnado and Osher have proposed a regularization approach to solve the inverse problem of smile calibration from quoted option prices under a generalized Black-Scholes framework. They have assumed that index volatility is a deterministic function of index level and time.

In [17] and [23], following [39], local volatility framework was considered and the related calibration problem was tackled by Tikhonov regularization. Moreover, in both papers results concerning convergence rates are discussed.

Following the same ideas of Tikhonov regularization, but assuming that volatility depends only on time, are [31, 34].

In [22], a Lagrangian optimal control strategy was proposed to solve the identification problem of Dupire's local volatility. They presented a globalized sequential quadratic programming (SQP) algorithm combined with a primal-dual active set strategy. They established results concerning existence of local optimal solutions and of Lagrange multipliers. Furthermore, a sufficient second-order optimality condition was proved. They also presented some numerical results.

The thesis [19] presents two different regularization techniques in order to tackle such local volatility calibration problem. The first technique is based on convex regularization tools, generalizing results present in [17, 23]. In this context, making use of Bregman distance techniques a convergence analysis with rates was established. The second technique was an iterative method, which was based on a tangential cone condition. In the latter case, a discrepancy principle was also stated.

Chapter 2

Forward Operators

Following Dupire's framework [21, 28], we fix the current time and define the local volatility surface as a function of time to maturity, strike price and the current stock price. Thus, when time evolves, such local volatility surface changes with respect to the stock price. In this chapter we define and present some basic properties of the forward operator which associates families of local volatility surfaces to families of European call option price surfaces, both parameterized by the underlying stock price. In other words, we shall define the map

$$\mathcal{U} : \{\sigma(S_0) \ : \ S_0 \in [S_{\min}, S_{\max}]\} \longmapsto \{u(S_0, \sigma(S_0)) \ : \ S_0 \in [S_{\min}, S_{\max}]\},$$

where $u(S_0, \sigma(S_0)) = u[S_0, \sigma(S_0)](t, S_t)$ is the call price given by the model at time t and underlying price S_t. Note that, such operator is a generalization of the operator $G(\cdot)$ of Chapter 1.

Another important feature of this framework is the possibility of such dependence on the stock price be non-Markovian, in the sense that, the (local) history of the process accounts for understanding how the present state is.

In order to define rigorously such framework, we make use of Bochner space techniques. A small survey of this topic can be found in Appendix A. The main reference for the first two sections of the present chapter is [29].

2.1 Definitions

We define below the forward operators and its domains. Thus, it is necessary to define first the spaces containing its domains and images, as well.

Given a time interval, say $[0, \overline{T}]$, the realized prices $S(t)$ vary within the interval $[S_{\min}, S_{\max}]$ with $t \in [0, \overline{T}]$. For technical reasons we perform the change of variables $s = S(t) - S_{\min}$ such that $s \in [0, S_{\max} - S_{\min}]$. Hence, for each $s \in [0, S_{\max} - S_{\min}]$ we denote $a(s) := a(s, \tau, y)$ the squared variance divided by 2 for the stock price $S(t) - S_{\min} = s$. In what follows, we shall represent s by t and $S_{\max} - S_{\min}$ by S.

Since we need some regularity of the volatility surface with respect to the stock price in the analysis below, the following definition introduces the concept of the Fourier transform in

Bochner spaces which is necessary in order to define some Sobolev-type Bochner spaces.

Definition 3. *Given $\mathcal{A} \in L^2(0, S, H^{1+\varepsilon}(D))$, with $\mathcal{A} : s \mapsto a(s)$ (see Appendix), we define its Fourier series $\hat{\mathcal{A}} = \{\hat{a}(k)\}_{k\in\mathbb{Z}}$ by*

$$\hat{a}(k) := \frac{1}{2S}\int_0^S a(s)\exp(-iks\pi/S)ds + \frac{1}{2S}\int_{-S}^0 a(-s)\exp(-iks\pi/S)ds.$$

It is well defined, since $\{s \mapsto a(s)\exp(-iks2\pi/S)\}$ is weakly measurable and

$$L^2(0, S, H^{1+\varepsilon}(D)) \subset L^1(0, S, H^{1+\varepsilon}(D))$$

by the Cauchy-Schwartz inequality.

Now we present the definition of the space containing the domain of the forward operator.

Definition 4. *We define $H^l(0, S, H^{1+\varepsilon}(D))$ as the space of $\mathcal{A} \in L^2(0, S, H^{1+\varepsilon}(D))$, such that*

$$\|\mathcal{A}\|_l := \sum_{k\in\mathbb{Z}}(1 + |k|^l)^2\|\hat{a}(k)\|^2_{H^{1+\varepsilon}(D)_{\mathbb{C}}} < \infty,$$

where $H^{1+\varepsilon}(D)_{\mathbb{C}} = H^{1+\varepsilon}(D)\oplus iH^{1+\varepsilon}(D)$ is the complexification of $H^{1+\varepsilon}(D)$. $H^l(0, S, H^{1+\varepsilon}(D))$ is a Hilbert space with the inner product

$$\langle\mathcal{A}, \widetilde{\mathcal{A}}\rangle_l := \sum_{k\in\mathbb{Z}}(1 + |k|^l)^2\langle a(k), \tilde{a}(k)\rangle_{H^{1+\varepsilon}(D)_{\mathbb{C}}}.$$

The following proposition is extracted from Lemma 3.2 of [29], it states the continuous inclusion of $H^l(0, S, H^{1+\varepsilon}(D))$ into $C(0, S, H^{1+\varepsilon}(D))$ when $l > 1/2$. For the proof see [29].

Proposition 6. *For $l > 1/2$, each $\mathcal{A} \in H^l(0, S, H^{1+\varepsilon}(D))$ has a continuous representative and the mapping $i_l : H^l(0, S, H^{1+\varepsilon}(D)) \hookrightarrow C(0, S, H^{1+\varepsilon}(D))$ is continuous (bounded). Defining $\langle\mathcal{A}, x\rangle_{H^{1+\varepsilon}(D)} := \{s \mapsto \langle a(s), x\rangle\}$ for $x \in H^{1+\varepsilon}(D)$ and $\mathcal{A} \in H^l(0, S, H^{1+\varepsilon}(D))$, we have that $\langle\mathcal{A}, x\rangle_{H^{1+\varepsilon}(D)} \in H^l([0, S])$ and $\|\langle\mathcal{A}, x\rangle_{H^{1+\varepsilon}(D)}\|_{H^l([0,S])} \leq \|\mathcal{A}\|_l\|x\|_{H^{1+\varepsilon}(D)}$. Moreover, for every $\mathcal{A}, \mathcal{B} \in L^2(0, S, H^{1+\varepsilon}(D))$,*

$$\langle\mathcal{A}, \mathcal{B}\rangle_{L^2(0,S,H^{1+\varepsilon}(D))} = \sum_{k\in\mathbb{Z}}\langle\hat{a}(k), \hat{b}(k)\rangle_{H^{1+\varepsilon}(D)_{\mathbb{C}}}.$$

Now we define the set

$$\mathfrak{Q} := \{\mathcal{A} \in H^l(0, S, H^{1+\varepsilon}(D)) : a(s) \in Q,\ \forall s \in [0, S]\}, \tag{2.1}$$

i.e., each $\mathcal{A}$ in $\mathfrak{Q}$ is the map $\mathcal{A} : s \in [0, S] \mapsto a(s) \in Q$. Note that $\mathfrak{Q}$ is a generalization of Q from Section 1, as it shall be the domain of the direct operator $\mathcal{U}(\cdot)$.

One of the necessary conditions to state the existence of Tikonov regularization in what follows is that $\mathcal{U}(\cdot)$ has to be weakly closed. This requires that $\mathfrak{Q}$ must be weakly closed as well. Thus, we have the following proposition:

Proposition 7. *For $l > 1/2$, the set $\mathfrak{Q}$ is weakly closed and it has a nonempty interior in $H^l(0,S,H^{1+\varepsilon}(D))$.*

Proof: Take a sequence $\{\mathcal{A}_n\}_{n\in\mathbb{N}} \subset \mathfrak{Q}$ weakly convergent to $\widetilde{\mathcal{A}} \in H^l(0,S,H^{1+\varepsilon}(D))$. We want to show that, given a weak zero neighborhood U of $H^{1+\varepsilon}(D)$, then for a sufficient large n, $a_n(s) - a(s) \in U$ for every $s \in [0,S]$. A weak zero neighborhood U of $H^{1+\varepsilon}(D)$ is defined by a set of $\alpha_1, ..., \alpha_K \in H^{1+\varepsilon}(D)$ and an $\epsilon > 0$ such that $g \in H^{1+\varepsilon}(D)$ is an element of U if $\max_{k=1,...,K} |\langle g, \alpha_n\rangle| < \epsilon$.

Since the imersion $H^l([0,S]) \hookrightarrow C([0,S])$ is compact and $H^l([0,S])$ is reflexive, it follows that each weak zero neighborhood of $H^l([0,S])$ is a zero neighborhood of $C([0,S])$. Furthermore, from Proposition 6 we know that $\langle \mathcal{A}, \alpha\rangle_{H^{1+\varepsilon}(D)} \in H^l([0,S])$ with its norm bounded by $\|\mathcal{A}\|_l\|\alpha\|_{H^{1+\varepsilon}(D)}$, for every $n \in \mathbb{N}$ and $\alpha \in H^{1+\varepsilon}(D)$. Thus, we take the smallest closed ball centered at zero, B, which contains $\langle \widetilde{\mathcal{A}}, \alpha_k\rangle_{H^{1+\varepsilon}(D)}$ with $k = 1, ..., K$ and every $\langle \mathcal{A}_n, \alpha_k\rangle_{H^{1+\varepsilon}(D)}$ with $n \in \mathbb{N}$ and $k = 1, ..., K$. Therefore, choosing $\epsilon > 0$ as above, it is true that for each $k = 1, ..., K$, there are $f_{k,1}, ..., f_{k,M(k)} \in H^l([0,S])$ and $\eta_k > 0$, such that $\|f\|_{C([0,S])} < \epsilon$ for every $f \in B$ with $\max_{m=1,...,M(k)} |\langle f, f_{k,m}\rangle| < \eta_k$. Hence, we define $\mathcal{C}_{k,m} := \alpha_k \otimes f_{k,m} \in H^l(0,S,H^{1+\varepsilon}(D))^*$ and the weak zero neighborhood $A = \cap_{k=1}^K A_k$ of $H^l(0,S,H^{1+\varepsilon}(D))$ with

$$A_k := \{\mathcal{A} \in H^l(0,S,H^{1+\varepsilon}(D)) \ : \ |\langle \mathcal{A}, \mathcal{C}_{k,m}\rangle| \leq \eta_k, \ m = 1, ..., M(k)\}.$$

As A is a weak zero neighborhood of $H^l(0,S,H^{1+\varepsilon}(D))$, it is true that for sufficiently large n, $\mathcal{A}_n - \widetilde{\mathcal{A}} \in A$, which implies that $a_n(s) - \tilde{a}(s) \in U$ for every $s \in [0,S]$, i.e., $\{a_n(s)\}_{n\in\mathbb{N}}$ weakly converges to $\tilde{a}(s)$ for every $s \in [0,S]$. As Q is weakly closed, we have that $\tilde{a}(s) \in Q$ for every $s \in [0,S]$, which shows that $\widetilde{\mathcal{A}} \in \mathfrak{Q}$. The assertion that the interior of $\mathfrak{Q}$ is nonempty follows from the fact that $H^l(0,S,H^{1+\varepsilon}(D)) \hookrightarrow C(0,S,H^{1+\varepsilon}(D))$ is continuous and bounded (note that, given $\epsilon > 0$, $\widetilde{\mathcal{A}} = \{s \mapsto \tilde{a}(s)\}$ with $\underline{a} + \epsilon \leq \tilde{a}(s) \leq \overline{a} + \epsilon$ for every $s \in [0,S]$ is in the interior of $\mathfrak{Q}$). ■

We introduce now the forward operator $\mathcal{U}(\cdot)$, which associates each path of the volatility surface to a path of option prices surface. For this, it is necessary to define the operator $F(\cdot,\cdot\cdot)$ which takes the pair (s,a) of a given stock price level and a volatility surface and associates to $u(s,a)$ a surface of option prices. We stress that, in what follows, we always assume that $l > 1/2$. Thus, for a given $a_0 \in Q$ we define the following operators:

$$\begin{array}{rcl} F : [0,S] \times Q & \longrightarrow & W_2^{1,2}(D) \\ (s,a) & \longmapsto & u(s,a) - u(s,a_0) \end{array}$$

and

$$\begin{array}{rcl} \mathcal{U} : \mathfrak{Q} & \longrightarrow & L^2(0,S,W_2^{1,2}(D)), \\ \mathcal{A} & \longmapsto & \{s \in [0,S] \mapsto F(s,a(s)) \in W_2^{1,2}(D)\}. \end{array}$$

Note that, for each $s \in [0,S]$, $F(s,\cdot) = G(\cdot)$.

2.2 Properties

The next result is the first step towards a rigorous use of regularization techniques in inverse problems analysis as it states, among other things, compactness and weak closedness of operator $F(\cdot,\cdot\cdot)$. Such properties are at the core of a similar result for the main forward operator $\mathcal{U}(\cdot)$. Note that, compactness implies ill-posedness of the inverse problem and weak closedness is a necessary condition to prove existence of a solution to the regularized inverse problem.

Proposition 8. *The operator $F : [0,S]\times Q \longrightarrow W_2^{1,2}(D)$ is continuous and compact. Moreover, it is sequentially weakly continuous and weakly closed.*

Proof: Let $\{(s_n,a_n)\}_{n=1}^{\infty} \subset [0,S]\times Q$ and $\{u^n\}_{n=1}^{\infty}$ be such that $u^n = u(s_n,a_n)$ and $(s_n,a_n)\to(s,a)$ in the product topology of $[0,S]\times H^{1+\varepsilon}(D)$. We know that $u^n - u(s_n,a_0) \in W_2^{1,2}(D)$. Let us introduce some notations, $w^n := u^n - u(s_n,a_0)$, $\tilde{w} := u(s,a)-u(s,a_0)$, $h^n = u(s_n,a_0)-u(s,a_0)$ and $v^n := w^n - \tilde{w}^n$. By linearity, w^n, $\tilde{w}^n$ and v^n solve

$$-w^n_\tau + a(w^n_{yy} - w^n_y) + bw^n_y = -(a_n - a_0)(u_{yy}(s_n,a_0) - u_y(s_n,a_0)),$$

$$-\tilde{w}_\tau + a(\tilde{w}_{yy} - \tilde{w}_y) + b\tilde{w}_y = -(a-a_0)(u_{yy}(s,a_0) - u_y(s,a_0)) \quad \text{and}$$

$$-v^n_\tau + a(v^n_{yy} - v^n_y) + bv^n_y = -(a_n - a)(u^n_{yy} - u^n_y) - (a-a_0)(h^n_{yy} - h^n_y),$$

respectively, with homogeneous boundary conditions. Thus, v^n satisfies the estimate

$$\|v^n\|_{W_2^{1,2}(D)} \leq C\left(\|a_n - a\|_{L^2(D)}\|u^n\|_{H^1(D)} + \|a-a_0\|_{L^2(D)}\|h^n_y\|_{W_2^{0,1}D}\right).$$

It remains to prove that $\|h^n_y\|_{W_2^{0,1}(D)} \to 0$. As above, by linearity, h^n solves

$$\begin{cases} -h^n_\tau + a_0(h^n_{yy} - h^n_y) + bh^n_y &= 0 \\ h^n(0,y) &= (S_0(s_n) - S_0(t))(1-e^y)^+ \end{cases}$$

and satisfies the estimates

$$|h^n| \leq |s_n - s| \quad \textit{and} \quad \|h^n_y\|_{W_2^{0,1}(D)} \leq C,$$

with C depending only on the parameters. Thus $h^n \to 0$ almost everywhere in D as $n\to 0$. Thus, $h^n \to 0$ in $W^{1,2}_{2,loc}(D)$, which implies that $\|h^n_y\|_{W_2^{0,1}(D)} \to 0$, as this sequence is uniformly bounded. Hence $\|v^n\|_{W_2^{1,2}(D)} \to 0$ which shows that $F : [0,S]\times Q \longrightarrow W_2^{1,2}(D)$ is continuous.

Let, $\{(s_n,a_n)\}_{n=1}^{\infty} \subset [0,S]\times Q$ be such that $(s_n,a_n) \rightharpoonup (s,a)$, i.e., in the product topology defined by the standard topology of $[0,S]$ and the weak topology of $H^{1+\varepsilon}(D)$. Moreover, let u^n be such that, $u^n = u(s_n,a_n)$. As Q is weakly closed, it follows that $a\in Q$ and we set $u := u(s,a)$. We shall prove that $F(s_n,a_n) \to F(s,a)$ in $W_2^{1,2}(D)$, where $F(s_n,a_n) = u^n - u(s_n,a_0)$ and $F(s,a) = u(s,a) - u(s,a_0)$. If $D_c \subset D$ is compact, then, by Sobolev embedding theorems (see, e.g., [53]) $a_n \to a$ in $L^2(D_c)$. Take v^n and h^n as above. Set $f^n := (a_n - a)(u_{yy} - u_y) + (a-a_0)(h^n_{yy} - h^n y)$. Thus, we can split $f^n = f^n_1 + f^n_2$ with $f^n_1 = f^n\mathcal{X}_{D_c}$, $f^n_2 = f^n - f^n_1$ and

$D_c := [0,S] \times [-M,M]$. Hence

$$\|v^n\|_{W_2^{1,2}(D)} \leq C \left(\|a_n - a\|_{L^2(D_c)} \|u_y^n\|_{H^1(D_c)} + \|a_n - a\|_{L^2(D\backslash D_c)} \|u_y^n\|_{H^1(D\backslash D_c)} + \right.$$
$$\left. \|a - a_0\|_{L^2(D_c)} \|h_y^n\|_{H^1(D_c)} + \|a - a_0\|_{L^2(D\backslash D_c)} \|h_y^n\|_{H^1(D\backslash D_c)} \right).$$

By Sobolev embedding, it follows that $\|a_n - a\|_{L^2(D_c)} \to 0$ and $\|h_y^n\|_{H^1(D_c)} \to 0$. $\|u_y\|_{H^1(D\backslash D_c)} \to 0$ and $\|h_y^n\|_{H^1(D\backslash D_c)} \to 0$ as $M \to \infty$ due to its uniform boundedness. Therefore, F is compact, and weakly continuous. The assertion of F being weakly closed follows from Q being weakly closed and the weak continuity of F. ■

We define now the concept of Frechét equi-differentiability for a family of operators, since it is a fundamental concept in order to establish the Frechét differentiability of operator $\mathcal{U}(\cdot)$. We shall prove in the next proposition that $\{F(s,\cdot) : \ s \in [0,S]\}$ is Frechét equi-differentiable.

Definition 5. *We call a family of operators $\{\mathcal{F}_s : Q \longrightarrow W_2^{1,2}(D) \mid s \in [0,S]\}$ as Frechét equi-differentiable, if for all $\tilde{a} \in Q$ and $\epsilon > 0$, there is a $\delta > 0$, such that*

$$\sup_{s\in[0,S]} \|\mathcal{F}_t(\tilde{a}+h) - \mathcal{F}_s(\tilde{a}) - \mathcal{F}_s'(\tilde{a})h\| \leq \epsilon \|h\|,$$

for $\|h\|_{H^{1+\varepsilon}(D)} < \delta$ and $\mathcal{F}_s'(\tilde{a})$ the Frechét derivative of $\mathcal{F}_s(\cdot)$ at $\tilde{a}$.

Proposition 9. *The family of operators $\{F(s,\cdot) : Q \longrightarrow W_2^{1,2}(D) \mid s \in [0,S]\}$ is Frechét equi-differentiable.*

Proof: Given $\tilde{a} \in Q$ and $\epsilon > 0$, define $w = F(s,\tilde{a}+h) - F(s,\tilde{a}) - \partial_a F(s,\tilde{a})h$, it is equivalent to $w = u(s,\tilde{a}+h) - u(s,\tilde{a}) - \partial_a u(s,\tilde{a})h$. We denote $v := u(s,\tilde{a}+h) - u(s,\tilde{a})$. Thus, by linearity w satisfies

$$-w_\tau + \tilde{a}(w_{yy} - w_y) + bw_y = h(v_{yy} - v_y),$$

with homogeneous boundary condition. Such problem does not depend on s, as $\tilde{a}$ is independent of s. From the proof of Proposition 8, we have

$$\|w\|_{W_2^{1,2}(D)} \leq C\|h\|_{L^2(D)} \|v\|_{W_2^{1,2}(D)}$$

By the continuity of F, given an $\epsilon > 0$ we can chose $h \in H^{1+\varepsilon}(D)$ with $\|h\|_{H^{1+\varepsilon}(D)} \leq \delta$, such that $\|v\|_{W_2^{1,2}(D)} \leq \epsilon/C$ and thus the assertion follows. ■

Finally, we present one of the principal results of this section, which states compactness and weak closedness of the forward operator $\mathcal{U}(\cdot)$. These properties are at the core of inverse problems analysis [26, 48].

Proposition 10. *The forward operator $\mathcal{U} : \mathfrak{Q} \longrightarrow L^2(0,S,W_2^{1,2}(D))$ is well-posed, continuous and compact. Moreover, it is sequentially weakly continuous and weakly closed.*

Proof: Well Posedness: Take an arbitrary $\widetilde{\mathcal{A}} \in \mathfrak{Q}$, by the continuity of $\widetilde{\mathcal{A}}$ (see Proposition 6) and F, it follows that $t \mapsto F(s,\tilde{a}(s))$ is continuous and then weakly measurable. Therefore, $s \mapsto \|F(s,a(s))\|_{W_2^{1,2}(D)}$ is bounded, then $\mathcal{U}(\widetilde{\mathcal{A}}) \in L^2(0,S,W_2^{1,2}(D))$, which asserts the well-posedness of $\mathcal{U}(\cdot)$.

Continuity: As $F : [0, S] \times Q \longrightarrow W_2^{1,2}(D)$ is continuous, it follows by Proposition 6 that the set $\{F(s,\cdot) \mid s \in [0,S]\} \subset C(Q, W_2^{1,2}(D))$ is uniformly equi-continuous, i.e., given $\epsilon > 0$, there is a $\delta > 0$ such that, for all $a, \tilde{a} \in Q$ satisfying $\|a - \tilde{a}\| < \delta$, we have that

$$\sup_{s\in[0,S]} \|F(s,a) - F(s,\tilde{a})\| < \epsilon.$$

Thus, given $\epsilon > 0$ and $\mathcal{A}, \widetilde{\mathcal{A}} \in \mathfrak{Q}$ such that $\sup_{s\in[0,S]} \|a(s) - \tilde{a}(s)\|_{H^{1+\varepsilon}(D)} < \delta$, then, by the uniform equi-continuity of $\{F(s,\cdot), s \in [0,S]\}$, it follows that

$$\|\mathcal{U}(\mathcal{A}) - \mathcal{U}(\widetilde{\mathcal{A}})\|^2_{L^2(0,S,W_2^{1,2}(D))} = \int_0^S \|F(s,a(s)) - F(s,\tilde{a}(s))\|^2_{W_2^{1,2}(D)} ds < \epsilon^2 \cdot S,$$

which asserts the continuity of $\mathcal{U}(\cdot)$.

Compactness: Assume that the set $\mathbb{B} \subset \mathfrak{Q}$ is bounded, we shall prove that

$$\mathcal{U}(\mathbb{B}) \subset L^2(0, S, W_2^{1,2}(D))$$

is pre-compact. As $\mathbb{B}$ is weakly pre-compact, given an $\epsilon > 0$, it is sufficient to find a weak zero neighborhood U of $H^l(0, S, H^{1+\varepsilon}(D))$ such that, if $\{\mathcal{A}_n\}_{n\in\mathbb{N}} \subset \mathbb{B}$ converges weakly to $\widetilde{\mathcal{A}}$, then for n sufficiently large $\mathcal{A}_n - \widetilde{\mathcal{A}} \in U$ and $\|\mathcal{U}(\mathcal{A}_n) - \mathcal{U}(\widetilde{\mathcal{A}})\|_{L^2(0,S,W_2^{1,2}(D))} < \epsilon$. Thus, we shall find a set of functionals $\mathcal{C}_{n,m} \in H^l(0, S, H^{1+\varepsilon}(D))^*$, which defines such zero neighborhood.

By the compactness of F, we have that, given $\mathcal{A} \in \mathbb{B}$, take $\rho \geq \sup_{s\in[0,S]} \|a(s)\|$ and define $B = B(0,\rho) \cap Q$, where $B(0,\rho) \subset H^{1+\varepsilon}(D)$ is the open ball with center 0 and radius ρ. Thus, by the weakly continuity of $F(\cdot,\cdot\cdot)$ we have that given an $\epsilon > 0$, there are $\alpha_1, ..., \alpha_N \in H^{1+\varepsilon}(D)$ and $\delta > 0$, such that

$$\sup_{s\in[0,S]} \|F(s,a) - F(s,\tilde{a})\| < \epsilon$$

for all $a, \tilde{a} \in B$ with

$$\max\{|\langle a - \tilde{a}, \alpha_n\rangle_{H^{1+\varepsilon}(D)}| \,|n = 1, ..., N\} < \delta. \tag{2.2}$$

By Proposition 6 it follows that $\langle \mathcal{A}, \alpha_n\rangle_{H^{1+\varepsilon}(D)} \in H^l([0,S])$ with its norm bounded by $\|\mathcal{A}\|_l \|\alpha_n\|_{H^{1+\varepsilon}(D)}$. Then, there is a closed and bounded ball

$$A \subset H^l([0,S]) \text{ containing } \langle \mathcal{A}, \alpha_n\rangle_{H^{1+\varepsilon}(D)}, \text{ for all } n = 1, ..., N,$$

and $\mathcal{A} \in \mathbb{B}$. By the fact that $H^l([0,S])$ is compactly immersed into $C([0,S])$, it is true that, each weak zero neighborhood of $H^l([0,S])$ is a zero neighborhood of $C([0,S])$. Thus, for $n = 1, ..., N$ and the same $\delta > 0$ of (2.2), there are $f_{n,1}, ..., f_{n,M(n)} \in H^l([0,S])$ and $\xi_n > 0$ such that, $\|f\|_{C([0,S])} < \delta$ for every $f \in A$ satisfying

$$\max\{|\langle f, \alpha_n\rangle_{H^{1+\varepsilon}(D)} \mid m = 1, ..., M(n)\} < \xi_n.$$

Now, we can define $\mathcal{C}_{n,m} := \alpha_n \otimes f_{n,m}$, with $n = 1, ..., N$ and $m = 1, ..., M(n)$ as an element of

$H^l(0,S,H^{1+\varepsilon}(D))^*$, where, for each $\mathcal{A} \in H^l(0,S,H^{1+\varepsilon}(D))$, we have that

$$\langle \mathcal{A}, \mathcal{C}_{n,m} \rangle_l = \langle \langle \mathcal{A}, \alpha_n \rangle_{H^{1+\varepsilon}(D)}, f_{n,m} \rangle_{H^l([0,S])} = \sum_{k\in\mathbb{Z}} (1+|k|^l)^2 \langle \hat{a}(k), \alpha_n \rangle_{H^{1+\varepsilon}(D)} \hat{f}_{n,m}(k).$$

These functionals define the weak zero neighborhood $U := \cap_{n=1}^N U_n$ with

$$U_n := \{\mathcal{A} \in H^l(0,S,H^{1+\varepsilon}(D)) : |\langle \mathcal{A}, \mathcal{C}_{n,m} \rangle_l| < \xi_n, \ m = 1, ..., M(n)\}.$$

Therefore, if $\{\mathcal{A}_k\}_{k\in\mathbb{N}} \subset \mathbb{B}$ converges weakly to $\widetilde{\mathcal{A}} \in \mathbb{B}$, then for a sufficient large k, $\mathcal{A}_k - \widetilde{\mathcal{A}} \in U$ and by the definition of U, we have that for each $n = 1, ..., N$,

$$\xi_n > |\langle \mathcal{A} - \widetilde{\mathcal{A}}, \mathcal{C}_{n,m} \rangle_l| = |\langle \langle \mathcal{A} - \widetilde{\mathcal{A}}, \alpha_n \rangle_{H^{1+\varepsilon}(D)}, f_{n,m} \rangle_{H^l([0,S])}|$$

for all $m = 1, ..., M(n)$. By the choice of the $f_{n,m} \in H^l([0,S])$, it follows that

$$\|\langle \mathcal{A}_k - \widetilde{\mathcal{A}}, \alpha_n \rangle_{H^{1+\varepsilon}(D)}\|_{H^l([0,S])} < \delta \text{ for all } n = 1, ..., N.$$

Thus,

$$\sup_{s\in[0,S]} \|F(s,a(s)) - F(s,\tilde{a}(s))\| < \epsilon$$

which implies that $\|\mathcal{U}(\mathcal{A}_k) - \mathcal{U}(\widetilde{\mathcal{A}})\|_{L^2(0,S,W_2^{1,2}(D))} \leq \epsilon \cdot T$.

Weak Continuity: The weak continuity follows directly from the proof of compactness, as we use the same framework, only changing the compactness of F, by the weakly equi-continuity of $\{F(s,\cdot) : \ s \in [0,S]\}$ on bounded subsets of Q.

Weak Closedness: It is true as $\mathfrak{Q}$ is weakly closed and $\mathcal{U}(\cdot)$ is weakly continuous. ■

Continuing the analysis of the forward operator $\mathcal{U}(\cdot)$, we now turn to results concerning the regularity of the operator $\mathcal{U}(\cdot)$ and properties of its Frechét derivative $\mathcal{U}'(\cdot)$, including Lipschitz continuity. These are important conditions in order to establish a convergence analysis.

Proposition 11. *The operator $\mathcal{U}(\cdot)$ admits a one sided derivative at $\widetilde{\mathcal{A}} \in \mathfrak{Q}$ in the direction $\mathcal{H}$, such that $\widetilde{\mathcal{A}} + \mathcal{H} \in \mathfrak{Q}$. The derivative $\mathcal{U}'(\widetilde{\mathcal{A}})$ satisfies*

$$\left\|\mathcal{U}'(\widetilde{\mathcal{A}})\mathcal{H}\right\|_{L^2(0,S,W_2^{1,2}(D))} \leq c\|\mathcal{H}\|_{H^l(0,S,H^{1+\varepsilon}(D))}.$$

Moreover, $\mathcal{U}'(\widetilde{\mathcal{A}})$ satisfies the Lipschitz condition

$$\left\|\mathcal{U}'(\widetilde{\mathcal{A}}) - \mathcal{U}'(\widetilde{\mathcal{A}} + \mathcal{H})\right\|_{\mathcal{L}\left(H^l(0,S,H^{1+\varepsilon}(D)), L^2(0,S,W_2^{1,2}(D))\right)} \leq \gamma\|\mathcal{H}\|_{H^l(0,S,H^{1+\varepsilon}(D))}$$

for all $\widetilde{\mathcal{A}}, \mathcal{H} \in \mathfrak{Q}$ such that $\widetilde{\mathcal{A}}, \widetilde{\mathcal{A}} + \mathcal{H} \in \mathfrak{Q}$.

Proof: By Proposition 9, the family of operators $\{F(s,\cdot) \ : \ s \in [0,S]\}$ is Frechét equi-differentiable. Take $\widetilde{\mathcal{A}}, \mathcal{H} \in H^l(0,S,H^{1+\varepsilon}(D))$, such that $\widetilde{\mathcal{A}}, \widetilde{\mathcal{A}} + \mathcal{H} \in \mathfrak{Q}$. Thus, define the one sided derivative of $\mathcal{U}(\cdot)$ at $\widetilde{\mathcal{A}}$ in the direction $\mathcal{H}$ as

$$\mathcal{U}'(\widetilde{\mathcal{A}})\mathcal{H} := \{s \mapsto \partial_a F(s, \tilde{a}(s))h(s)\},$$

where for each $s \in [0, S]$, dropping t to easy the notation, $\partial_a F(s, \tilde{a})h$ is the solution of

$$-v_\tau + a(v_{yy} - v_y) + bv_y = h(u_{yy} - u_y)$$

with homogeneous boundary conditions and $u = u(s, a(s))$. From Proposition 8 we have the estimate

$$\|\partial_a F(s, \tilde{a}(s))h(t)\|_{W_2^{1,2}(D)} \leq C\|h(t)\|_{L^2(D)}\|u_{yy}(s, \tilde{a}(s)) - u_y(s, \tilde{a}(s))\|_{L^2(D)}.$$

Where $\|u_{yy}(s, a) - u_y(s, a)\|_{L^2(D)}$ is uniformly bounded in $[0, S] \times Q$. Thus, $\mathcal{U}'(\widetilde{\mathcal{A}})\mathcal{H}$ is well defined and

$$\begin{aligned}
\left\|\mathcal{U}'(\widetilde{\mathcal{A}})\mathcal{H}\right\|^2_{L^2(0,S,W_2^{1,2}(D))} &= \int_0^S \|\partial_a F(s, \tilde{a}(s))h(t)\|^2_{W_2^{1,2}(D)} ds \\
&\leq C\int_0^S \|h(t)\|_{L^2(D)}\|u_{yy}(s, \tilde{a}(s)) - u_y(s, \tilde{a}(s))\|_{L^2(D)} ds \\
&\leq c\int_0^S \|h(t)\|^2_{L^2(D)} ds = c\|\mathcal{H}\|^2_{H^l(0,S,H^{1+\varepsilon}(D))}
\end{aligned}$$

Therefore, $\mathcal{U}'(\widetilde{\mathcal{A}})$ can be extended to a bounded linear operator from $H^l(0, S, H^{1+\varepsilon}(D))$ to $L^2(0, S, W_2^{1,2}(D))$.

Now, let $\widetilde{\mathcal{A}}, \mathcal{H}, \mathcal{G} \in H^l(0, S, H^{1+\varepsilon}(D))$ be such that,

$$\widetilde{\mathcal{A}}, \widetilde{\mathcal{A}} + \mathcal{H}, \widetilde{\mathcal{A}} + \mathcal{G}, \widetilde{\mathcal{A}} + \mathcal{H} + \mathcal{G} \in Q.$$

Denote, $v := u(s, a(s) + h(t)) - u(s, a(s))$. Thus,

$$w := (\partial_a u(s, a(s) + h(t))g(s) - \partial_a u(s, a(s))g(s))$$

satisfies

$$-w_\tau + a(w_{yy} - w_y) = -q[v_{yy} - v_y] - h[(\partial_a u(s, a + h)g)_{yy} - (\partial_a u(s, a + h)g)_y],$$

with homogeneous boundary conditions (dropping the dependence on s). As above, we have

$$\begin{aligned}
\left\|\mathcal{U}'(\widetilde{\mathcal{A}} + \mathcal{H})\mathcal{G} - \mathcal{U}'(\widetilde{\mathcal{A}})\mathcal{G}\right\|^2_{L^2(0,S,W_2^{1,2}(D))} &= \int_0^S \|w\|^2_{W_2^{1,2}(D)} ds \\
&\leq c_1\int_0^S \|q(t)\|^2_{L^2(D)}\|v_{yy}(s, \tilde{a}(s)) - v_y(s, \tilde{a}(s))\|^2_{L^2(D)} ds \\
&+c_2\int_0^S \|h(t)\|^2_{L^2(D)}\|\partial_a u(s, a(s) + h(s))g(s)\|^2_{W_2^{1,2}(D)} ds \\
&\leq C\|\mathcal{H}\|^2_{H^l(0,S,H^{1+\varepsilon}(D))}\|\mathcal{G}\|^2_{H^l(0,S,H^{1+\varepsilon}(D))},
\end{aligned}$$

which yields the Lipschitz condition. ■

The following result is a consequence of the compactness of $\mathcal{U}(\cdot)$.

Proposition 12. *The Frechét derivative of the operator $\mathcal{U}(\cdot)$ is injective and compact.*

Proof: Take $\mathcal{H} \in \ker\left(\mathcal{U}'(\widetilde{\mathcal{A}})\right)$. Thus, from the proof of Proposition 11, we have

$$h(s) \cdot (u_{yy} - u_y) = 0.$$

However, for each t, $G = u_{yy} - u_y$ is the solution of

$$\begin{cases} \partial_\tau G = \dfrac{1}{2}\left(\partial^2_{yy} - \partial_y\right)(a(s)G + bG) \\ G\,|_{\tau=0} = \delta(y)\ , \end{cases}$$

i.e. G is the Green's function of the Cauchy problem above. Thus, $G > 0$ for every y,$\tau > 0$ and $s \in [0, S]$. Therefore $h(t) = 0$. As it is true for every $s \in [0, S]$, then it yields the result. ■

As in Section 1.2 we make use of the bounded embedding

$$L^2(0, S, W^{1,2}_2(D)) \subset L^2(0, S, L^2(D)).$$

Since it implies that $\mathcal{U}$ satisfies the same results presented above with $L^2(0, S, L^2(D))$ in the place of $L^2(0, S, W^{1,2}_2(D))$. Thus, we characterize the range of $\mathcal{U}'(\mathcal{A})$ as a subset of $L^2(0, S, L^2(D))$ and the range of $\mathcal{U}'(\mathcal{A})^*$ as a subset of $H^l(0, S, H^{1+\varepsilon}(D))$ in order to proceed in Chapter 3 the convergence analysis.

Proposition 13. *The operator $\mathcal{U}'(\mathcal{A}^\dagger)^*$ has a trivial kernel.*

Proof: For simplicity take $b = 0$. Denote

$$\mathcal{L} := -\partial_\tau + a(\partial yy - \partial_y)$$

the parabolic operator of equation 1.4 with homogeneous boundary condition and $\mathcal{G}_{u_{yy}-u_y}$ the multiplication operator by $u_{yy}-u_y$. Thus, for each $s \in [0, S]$, we have $\partial_a u(s, \tilde{a}(s)) = \mathcal{L}^{-1}\mathcal{G}_{u_{yy}-u_y}$, where $\mathcal{L}^{-1}$ is the left inverse of $\mathcal{L}$ with null boundary conditions. By definition of

$$\mathcal{U}'(\widetilde{\mathcal{A}})^* : L^2(0, S, L^2(D)) \longrightarrow H^l(0, S, H^{1+\varepsilon}(D)),$$

we have,

$$\left\langle \mathcal{U}'(\widetilde{\mathcal{A}})\mathcal{H}, \mathcal{Z} \right\rangle_{L^2(0,S,L^2(D))} = \langle \mathcal{H}, \Phi \rangle_{H^l(0,S,H^{1+\varepsilon}(D))},$$

$\forall\ \mathcal{H} \in H^l(0, S, H^{1+\varepsilon}(D))$ and $\forall\ \mathcal{Z} \in L^2(0, S, L^2(D))$, with $\Phi = \mathcal{U}'(\widetilde{\mathcal{A}})^*\mathcal{Z}$. Thus, given any $\mathcal{Z} \in \ker\left(\mathcal{U}'(\widetilde{\mathcal{A}})^*\right)$, it follows that

$$\begin{aligned} 0 &= \left\langle \mathcal{U}'(\widetilde{\mathcal{A}})\mathcal{H}, \mathcal{Z} \right\rangle_{L^2(0,S,L^2(D))} = \int_0^S \left\langle \mathcal{L}^{-1}\mathcal{G}_{u_{yy}-u_y} h(s), z(s) \right\rangle_{L^2(D)} ds \\ &= \int_0^S \left\langle \mathcal{G}_{u_{yy}-u_y} h(s), [\mathcal{L}^{-1}]^* z(s) \right\rangle_{L^2(D)} ds = \int_0^S \left\langle \mathcal{G}_{u_{yy}-u_y} h(s), g(s) \right\rangle_{L^2(D)} ds \end{aligned}$$

where g is a solution of the adjoint equation

$$g_\tau + (ag)_{yy} + (ag)_y = z$$

for each $s \in [0, S]$, with homogeneous boundary conditions. Since $z(t) \in L^2(D)$, we have that $g(s) \in H^{1+\varepsilon}(D)$ (see [38]) and $g \in L^2\left(0, S, H^{1+\varepsilon}(D)\right)$. Since $\mathcal{G} > 0$, from the proof of Proposition 12 and $h \in H^l(0, S, H^{1+\varepsilon}(D))$ is arbitrary, it follows that $g = 0$. Therefore $\mathcal{Z} = 0$ almost everywhere in $s \in [0, S]$. It yields that $\ker\left(\mathcal{U}'(a)^*\right) = \{0\}$. ■

Remark 1. *We remark that, from the last proposition we have*

$$\ker\{\mathcal{U}'(\widetilde{\mathcal{A}})\} = \{0\} \Rightarrow \overline{\mathcal{R}\left\{\left(\mathcal{U}'(\widetilde{\mathcal{A}})\right)^*\right\}} = H^l(0, S, H^{1+\varepsilon}(D)).$$

In other words, the range of the adjoint operator of the Frechét derivative of the forward operator $\mathcal{U}$ *at* $\widetilde{\mathcal{A}}$ *is dense in* $H^l(0, S, H^{1+\varepsilon}(D))$.

To finish this section we shall present below the tangential cone condition for $\mathcal{U}$ which is a sufficient condition to state iterative regularization techniques as Landweber iteration. Such result shall be used latter in Chapter 3.6. The present Theorem is based on a similar result from Section 1.2, Proposition 5.

Theorem 5. *The map* $\mathcal{U}(\cdot)$ *satisfies the local tangential cone condition*

$$\left\|\mathcal{U}(\mathcal{A}) - \mathcal{U}(\widetilde{\mathcal{A}}) - \mathcal{U}'(\widetilde{\mathcal{A}})(\mathcal{A} - \widetilde{\mathcal{A}})\right\|_{L^2(0,S,W_2^{1,2}(D))} \leq \gamma \left\|\mathcal{U}(\mathcal{A}) - \mathcal{U}(\widetilde{\mathcal{A}})\right\|_{L^2(0,S,W_2^{1,2}(D))} \tag{2.3}$$

for all $\mathcal{A}, \widetilde{\mathcal{A}}$ *in a ball* $B(\mathcal{A}^*, \rho) \subset \mathfrak{Q}$ *with some* $\rho > 0$ *and* $\gamma < 1/2$.

Proof: Take $\mathcal{A}, \widetilde{\mathcal{A}} \in \mathfrak{Q}$, then for each $s \in [0, S]$ we have (see Theorem 1.4.2 of [19])

$$\|F(s, a(s)) - F(s, \tilde{a}(s)) - \partial_a F(s, \tilde{a}(s))(a(s) - \tilde{a}(s))\|_{W_2^{1,2}(D)} \leq \gamma \|F(s, a(s)) - F(s, \tilde{a}(s))\|_{W_2^{1,2}(D)}$$

thus, by the continuity of $\mathcal{A}$, $\widetilde{\mathcal{A}}$ and $F(\cdot, \cdot\cdot)$,

$$\{t \mapsto \|F(s, a(s)) - F(s, \tilde{a}(s)) - \partial_a F(s, \tilde{a}(s))(a(s) - \tilde{a}(s))\|_{W_2^{1,2}(D)}\}$$

and

$$\{t \mapsto \|F(s, a(s)) - F(s, \tilde{a}(s))\|_{W_2^{1,2}(D)}\}$$

are bounded. Thus,

$$\{t \mapsto F(s, a(s)) - F(s, \tilde{a}(s)) - \partial_a F(s, \tilde{a}(s))(a(s) - \tilde{a}(s))\}$$

and

$$\{t \mapsto F(s, a(s)) - F(s, \tilde{a}(s))\}$$

are weakly measurable and then they are in $L^2(0, S, W_2^{1,2}(D))$.

Therefore, we have

$$
\begin{aligned}
&\left\| \mathcal{U}(\mathcal{A}) - \mathcal{U}(\widetilde{\mathcal{A}}) - \mathcal{U}'(\widetilde{\mathcal{A}})(\mathcal{A} - \widetilde{\mathcal{A}}) \right\|^2_{L^2(0,S,W_2^{1,2}(D))} = \\
&\int_0^S \|F(s,a(s)) - F(s,\tilde{a}(s)) - \partial_a F(s,\tilde{a}(s))(a(s) - \tilde{a}(s))\|^2_{W_2^{1,2}(D)}\, ds \\
&\leq \gamma^2 \int_0^S \|F(s,a(s)) - F(s,\tilde{a}(s))\|^2_{W_2^{1,2}(D)}\, ds = \gamma^2 \left\| \mathcal{U}(\mathcal{A}) - \mathcal{U}(\widetilde{\mathcal{A}}) \right\|^2_{L^2(0,S,W_2^{1,2}(D))}
\end{aligned}
$$

■

Corollary 1. *The operator $\mathcal{U}$ is injective.*

Proof: Suppose that $\mathcal{A}^\dagger, \widetilde{\mathcal{A}} \in H^l(0,S,H^{1+\varepsilon}(D))$ satisfy

$$\mathcal{U}(\mathcal{A}^\dagger) = \mathcal{U}(\widetilde{\mathcal{A}}) = \widetilde{\mathcal{U}}.$$

Then, by the Tangential Cone Condition it follows that

$$\mathcal{U}'(\mathcal{A}^\dagger)(\mathcal{A}^\dagger - \widetilde{\mathcal{A}}) = 0,$$

i.e.,

$$(\mathcal{A}^\dagger - \widetilde{\mathcal{A}}) \in \ker\{\mathcal{U}'(\mathcal{A}^\dagger)\}.$$

By Proposition 12 $\ker\{\mathcal{U}'(\mathcal{A}^\dagger)\} = \{0\}$, implying that $\mathcal{A}^\dagger = \widetilde{\mathcal{A}}$. ■

2.3 The Case $\sigma(S_0, T, K) = \sigma(T, K)$

Now we show what happens when we assume that local volatility is (locally) independent of the initial state of the underlying asset price, i.e., $\sigma(S_0, T, K) = \sigma(T, K)$ assuming further that r is constant in equation (1.4). Such hypothesis simplifies a lot our analysis, as the differential operator is linear.

For a given set of European call prices surfaces parameterized by the underlying asset price

$$\{u\,(S, a(S)) \;:\; S \in [S_{\min}, S_{\max}]\}$$

normalize them by the asset prices, i.e., define $v(S) := u\,(S, a(S))\,/S$. Then for each $S \in [S_{\min}, S_{\max}]$, v satisfies

$$
\begin{cases}
-\dfrac{\partial v}{\partial \tau} + a\left(\dfrac{\partial^2 v}{\partial y^2} - \dfrac{\partial v}{\partial y}\right) - r\dfrac{\partial v}{\partial y} = 0, \text{ for } (\tau, y) \in D \\[2ex]
v(0,y) = (1 - \mathrm{e}^y)^+, \text{ for } y \in \mathbb{R},
\end{cases}
\tag{2.4}
$$

with $\lim_{y\to-\infty} v(s,y) = 1$ and $\lim_{y\to+\infty} v(s,y) = 0$.

Since b is constant and $a(S) = a$ is independent of S, it follows that v is also independent of S, i.e., $v(S, a(S)) = v(a)$. In other words, the normalized prices v satisfy the same equation (2.4) for every $S \in [S_{\min}, S_{\max}]$.

Therefore, under such hypothesis, we can take quoted European option prices traded on different values of $S(t)$ and normalize them by the correspondent value of $S(t)$ and then they satisfy (2.4).

Such remark is justified since it is a way to increase the amount of input data in inverse problems analysis presented, for example in Chapter 1 and in [17, 19, 23]. Hence we avoid to use the framework of the last two sections. However,we must be carefull when doing things this way since we might introduce more noise into the input data. Thus, the result would have as much uncertainty as we were using the data for a single day.

Chapter 3

The Inverse Problem

Now, we tie up the inverse problem of extracting the local volatility surface from quoted European call option prices. Thus, following the notation of Chapter 2, we want to associate to each set of European option prices surfaces the corresponding set of local volatility surfaces, both parameterized by the underlying initial state of stock prices. In other words, we want to define appropriately the inverse map

$$\{C(S)\ :\ S \in [S_{\min}, S_{\max}]\} \longmapsto \{\sigma(S)\ :\ S \in [S_{\min}, S_{\max}]\}.$$

However, defining it is challenging due to its intrinsic ill-posedness. Note that, working with past prices and local volatility surfaces allows us to incorporate more information into the evaluation of the present value of σ.

In this chapter we shall see that the framework introduced in Chapter 2 is sufficient to state results concerning existence, stability of solutions for the regularized problem and a convergence analysis. Such results extend the latter ones presented in Chapter 1.

It is important to note that, the main contribution of this chapter are the following:

1. We establish a Morozov discrepancy principle, which is proved in details under a very general setting.

2. We also consider a discretized setting, in order to quantify separately the uncertainties related to different sources.

3.1 The Regularized Problem

As mentioned above, we are concerned with defining properly the inverse problem of local volatility calibration under the framework of Chapter 2. It can be summarized as

Given a trajectory of European call option prices surfaces $\widetilde{\mathcal{U}} = \{t \mapsto \tilde{u}(t)\} \in L^2(0, S, L^2(D))$, find the associated trajectory of local variances surfaces $\widetilde{\mathcal{A}} = \{t \mapsto \tilde{a}(t)\} \in \mathfrak{Q}$, satisfying

$$\widetilde{\mathcal{U}} = \mathcal{U}(\widetilde{\mathcal{A}}). \tag{3.1}$$

We call $\widetilde{\mathcal{U}}$ the observable variable and $\widetilde{\mathcal{A}}$ the unknown. This is an idealized situation as the model (3.1) has no uncertainties associated. Thus, to be more realistic, we assume that we can only observe corrupted data $\mathcal{U}^\delta$, satisfying a perturbed version of (3.1),

$$\mathcal{U}^\delta = \widetilde{\mathcal{U}} + \mathcal{E} = \mathcal{U}(\widetilde{\mathcal{A}}) + \mathcal{E} \tag{3.2}$$

where $\mathcal{E} = \{t \mapsto E(t)\}$ compiles all the uncertainties associated to this problem and $\widetilde{\mathcal{U}}$ is the unobservable noiseless data. We assume further that, the norm of $\mathcal{E}$ is bounded by the noise level $\delta > 0$. Moreover, for each $t \in [0, T]$, we assume that $\|E(t)\| \leq \delta/T$. These hypotheses imply that

$$\|\mathcal{U}^\delta - \widetilde{\mathcal{U}}\|_{L^2(0,S,L^2(D))} \leq \delta \quad \text{and} \quad \|u^\delta(t) - \tilde{u}(t)\|_{L^2(D)} \leq \delta/T \text{ for every } t \in [0, T]. \tag{3.3}$$

Proposition 10 says that $\mathcal{U}(\cdot)$ is compact, implying that the associated inverse problem is ill-posed. It means that such inverse problem cannot be solved directly in a stable way. Hence, we must apply regularization techniques. This, roughly speaking, relies on stating the original problem under a more robust setting. More specifically, instead of looking for an $\mathcal{A}^\delta \in \mathfrak{Q}$ satisfying (3.2), we shall search an $\mathcal{A}^\delta \in \mathfrak{Q}$ minimizing the following Tikhonov functional

$$\mathcal{F}^{\mathcal{U}^\delta}_{\mathcal{A}_0,\alpha}(\mathcal{A}) = \|\mathcal{U}^\delta - \mathcal{U}(\mathcal{A})\|^2_{L^2(0,S,L^2(D))} + \alpha f_{\mathcal{A}_0}(\mathcal{A}). \tag{3.4}$$

We shall see later that, in this setting, such minimizers are in an appropriate sense good approximations for solutions of (3.1). Note that, we allow the existence of more than one solution.

We observe further that, in this framework, in order to regularize (3.2), we had to introduce the term $\alpha f_{\mathcal{A}_0}$ where α and $f_{\mathcal{A}_0} : \mathfrak{Q} \to [0, \infty]$ are known as regularization parameter and regularization functional, respectively.

Roughly speaking, α controls how much regularization enters into the new framework and thus, it has to be properly chosen according to the noise level δ. We present a way to find α in Section 3.5.

The functional $f_{\mathcal{A}_0}$ introduces some prior information concerning the solution of (3.2), as convexity or smoothness. We also note that the value $\mathcal{A}_0$ in the function $f_{\mathcal{A}_0}$ is an element of $\mathfrak{Q}$, which can be thought of as an *a priori* approximation for a proper solution of (3.1).

The technical aspects concerned with the numerical solution of such minimization problem and some numerical examples shall be presented in Chapter 5.

3.2 Some Properties of Minimizers

In this section we shall see that, under certain hypotheses on the regularizing functional $f_{\mathcal{A}_0}$, it turns out that the functional (3.4) has stable minimizers. In other words, Tikhonov regularization gives us solutions for (3.2) approximating solutions for (3.1), in a proper sense. Such result is obtained by an appropriate choice of α as a function of the noise level δ.

We assume that the regularizing functional $f_{\mathcal{A}_0} : \mathfrak{Q} \to [0, \infty]$ is convex, coercive and weakly lower semi-continuous. A good reference on this topic is [25]. Note that, these assumptions are not too strong as they are fulfilled by a huge class of functionals on $H^l(0, S, H^{1+\varepsilon}(D))$. A canonical example is the squared distance between $\mathcal{A}$ and $\mathcal{A}_0$, $f_{\mathcal{A}_0}(\mathcal{A}) = \|\mathcal{A} - \mathcal{A}_0\|^2_{H^l(0,S,H^{1+\varepsilon}(D))}$, which is used in classical Tikhonov regularization.

Thus, because of such properties of $\mathcal{U}$ and $f_{\mathcal{A}_0}$ and since $\mathfrak{Q}$ is weakly closed, it follows that the sets

$$\mu_\alpha(M) = \left\{ \mathcal{A} \in \mathfrak{Q} \,\middle|\, \mathcal{F}^{\delta}_{\mathcal{A}_0,\alpha}(\mathcal{A}) \leq M \right\}$$

are weakly pre-compact and the restriction of $\mathcal{U}$ to $\mu_\alpha(M)$ is weakly continuous. From [48], we have the following three theorems concerned with existence, stability and convergence of minimizers for (3.4).

Theorem 6 (Existence)**.** *For every $\mathcal{U}^\delta \in L^2(0, S, L^2(D))$, there exists at least one element of $\mathfrak{Q}$ minimizing $\mathcal{F}^{\mathcal{U}^\delta}_{\mathcal{A}_0,\alpha}(\cdot)$, the functional defined in (3.4).*

Before stating the following theorem, we first introduce the concept of stability of minimizers in the definition below.

Definition 6 (Stability)**.** *We say that the minimizers of (3.4) are stable if for small perturbations on the data $\mathcal{U}$, say $\mathcal{U}_\epsilon$, some minimizer of (3.4) with data $\mathcal{U}_\epsilon$ is an approximation of a minimizer of the same functional with data $\mathcal{U}$. In other words, for every sequence $\{\mathcal{U}_k\}_{k\in\mathbb{N}} \subset L^2(0, S, W^{1,2}_2(D))$ converging strongly to $\mathcal{U}$, the sequence $\{\mathcal{A}_k\}_{k\in\mathbb{N}} \subset \mathfrak{Q}$ of minimizers of $\mathcal{F}^{\mathcal{U}^k}_{\mathcal{A}_0,\alpha}(\cdot)$ has a subsequence converging weakly to $\widetilde{\mathcal{A}}$, a minimizer of $\mathcal{F}^{\mathcal{U}}_{\mathcal{A}_0,\alpha}(\cdot)$.*

Hence, we can state the following theorem.

Theorem 7 (Stability)**.** *The minimizers of (3.4) are stable in the sense of Definition 6. Furthermore, if there exists a solution to (3.1), then there is at least one $f_{\mathcal{A}_0}$-minimizing solution for such problem.*

The next result states that, when we have a proper choice for the regularization parameter α in (3.4) as a function of the noise level δ, we can prove that it is possible to find a sequence of minimizers of (3.4) converging weakly to an $f_{\mathcal{A}_0}$-minimizing solution for (3.1), given that it has a solution.

Theorem 8 (Convergence)**.** *We assume that (3.1) has a solution and the map $\alpha : (0, \infty) \to (0, \infty)$, satisfies*

$$\lim_{\delta\to 0} \alpha(\delta) = 0 \quad \textit{and} \quad \lim_{\delta\to 0} \frac{\delta^2}{\alpha(\delta)} = 0. \tag{3.5}$$

Moreover, we assume further that the sequence $\{\delta_k\}_{k\in\mathbb{N}}$ converges to 0 and the elements of $\{\mathcal{U}_k\}_{k\in\mathbb{N}}$, with $\mathcal{U}_k = \mathcal{U}^{\delta_k}$, satisfy $\|\widetilde{\mathcal{U}} - \mathcal{U}_k\| \leq \delta_k$, with $\widetilde{\mathcal{U}}$ the noiseless data of (3.1). Denote $\alpha(\delta_k)$ by α_k for every $k \in \mathbb{N}$. Then every sequence $\{\mathcal{A}_k\}_{k\in\mathbb{N}}$ of minimizers of $\mathcal{F}^{\mathcal{U}_k}_{\mathcal{A}_0,\alpha_k}(\cdot)$, converges weakly to $\mathcal{A}^\dagger$, the unique $f_{\mathcal{A}_0}$-minimizing solution of (3.1), with $f_{\mathcal{A}_0}(\mathcal{A}_k) \to f_{\mathcal{A}_0}(\mathcal{A}^\dagger)$.

By the three results above we conclude that Tikhonov regularization gives us, reliable approximate solutions for (3.1). Note that, all of them depend strongly on the magnitude of δ and then on the choice of α and $f_{\mathcal{A}_0}$. Thus, Theorem 8 says the smaller δ is, the less dependent on $f_{\mathcal{A}_0}$ the solutions are, which makes them more reliable.

3.3 A Convergence Analysis

Making use of Convex Analysis techniques, since we have assumed that the regularization functional $f_{\mathcal{A}_0}$ is convex, weakly lower semi-continuous and coercive, we shall provide a convergence analysis, aiming to state convergence rates. In order to proceed with such analysis, as in Chapter 1, we need the concepts of Bregman distance related to $f_{\mathcal{A}_0}$, q-coerciveness and the source condition.

For the definitions of Bregman Distance with some of its properties as q-coerciveness, the reader is referred to Definitions 1 and 2 in Chapter 1.

In what follows we always assume that (3.1) has an $f_{\mathcal{A}_0}$-minimizing solution which is an element of the Bregman domain $\mathcal{D}_B(f_{\mathcal{A}_0})$ (see Definition 1).

Before stating the result about convergence rates, we need the following two auxiliary lemmas. The first one introduces a so-called source condition, which is a necessary condition to state convergence rates. See [48]. Note that, these results are generalizations of Lemma 1 and Lemma 2 from Chapter 1.

Lemma 3. *For every $\xi^\dagger \in \partial f_{\mathcal{A}_0}(\mathcal{A}^\dagger)$, there exist*

$$\omega^\dagger \in L^2(0,S,L^2(D)) \text{ and } \mathscr{E} \in H^l(0,S,H^{1+\varepsilon}(D))$$

such that

$$\xi^\dagger = \left[\mathcal{U}'(\mathcal{A}^\dagger)\right]^* \omega^\dagger + \mathscr{E} \tag{3.6}$$

holds. Moreover, $\mathscr{E}$ can be taken such that $\|\mathscr{E}\|_{H^l(0,S,H^{1+\varepsilon}(D))}$ is arbitrarily small.

Lemma 3 follows from the fact that $\mathcal{R}(\mathcal{U}'(\mathcal{A}^\dagger)^*)$ is dense in $H^l(0,S,H^{1+\varepsilon}(D))$, see Proposition 13 in Chapter 2. Observe also that, here we are identifying $L^2(0,S,L^2(D))^*$ with $L^2(0,S,L^2(D))$ and $H^l(0,S,H^{1+\varepsilon}(D))^*$ with $H^l(0,S,H^{1+\varepsilon}(D))$ due to the reflexivity of Hilbert spaces.

Lemma 4. *Let $\xi^\dagger$ be in $\partial f_{\mathcal{A}_0}(\mathcal{A}^\dagger)$, by Lemma 3 it satisfies (3.6). We assume that there exist constants $C, \underline{c} > 0$ such that with $\omega^\dagger$ and $\mathscr{E}$ as in Lemma 3, we have that*

$$\beta_1 := \frac{1}{\underline{c}}\left(C\|\omega^\dagger\| + \|\mathscr{E}\|\right) \text{ is in } [0,1).$$

We assume further that the Bregman distance with respect to $f_{\mathcal{A}_0}$ is 1-coercive with constant $\underline{c}$. Then it follows that the inequality

$$\langle \xi^\dagger, \mathcal{A}^\dagger - \mathcal{A}\rangle \leq \beta_1 D_{\xi^\dagger}(\mathcal{A},\mathcal{A}^\dagger) + \beta_2\|\mathcal{U}(\mathcal{A}) - \mathcal{U}(\mathcal{A}^\dagger)\|_{L^2(0,S,L^2(D))} \tag{3.7}$$

holds with $\beta_2 \geq 0$.

Proof: By (3.6), we have that

$$\xi^\dagger = \left[\mathcal{U}'(\mathcal{A}^\dagger)\right]^* \omega^\dagger + \mathscr{E},$$

thus,

$$\langle \xi^\dagger, \mathcal{A} - \mathcal{A}^\dagger \rangle = \langle \xi^\dagger - \mathscr{E}, \mathcal{A} - \mathcal{A}^\dagger \rangle + \langle \mathscr{E}, \mathcal{A} - \mathcal{A}^\dagger \rangle = \langle \omega^\dagger, \mathcal{U}'(\mathcal{A}^\dagger)(\mathcal{A} - \mathcal{A}^\dagger) \rangle + \langle \mathscr{E}, \mathcal{A} - \mathcal{A}^\dagger \rangle,$$

which implies that, denoting $U := H^l(0, S, H^{1+\varepsilon}(D))$ and $V := L^2(0, S, W_2^{1,2}(D))$,

$$\begin{aligned} |\langle \xi^\dagger, \mathcal{A} - \mathcal{A}^\dagger \rangle| &\leq \left(\|\omega^\dagger\|_{V^*} \|\mathcal{U}'(\mathcal{A}^\dagger)\|_{\mathcal{L}(U,V)} + \|\mathscr{E}\|_{U^*} \right) \|\mathcal{A} - \mathcal{A}^\dagger\|_U \\ &\leq \left(C\|\omega^\dagger\|_{V^*} + \|\mathscr{E}\|_{U^*} \right) \|\mathcal{A} - \mathcal{A}^\dagger\|_U . \end{aligned}$$

Since the Bregman distance with respect to $f_{\mathcal{A}_0}$ is 1-coercive and by definition of β_1, we have from the last inequality

$$|\langle \xi^\dagger, \mathcal{A} - \mathcal{A}^\dagger \rangle| \leq \beta_1 D_{\xi^\dagger}(\mathcal{A}, \mathcal{A}^\dagger) \leq \beta_1 D_{\xi^\dagger}(\mathcal{A}, \mathcal{A}^\dagger) + \beta_2 \|\mathcal{U}(\mathcal{A}) - \mathcal{U}(\mathcal{A}^\dagger)\|_V, \tag{3.8}$$

which shows (3.7). ■

By [48] the theorem below, which states the convergence rates, is a consequence of Lemma 4, under the hypotheses presented during this Chapter.

Theorem 9 (Convergence Rates)**.** *Under the hypothesys of this section, let the map $\alpha : (0, \infty) \to (0, \infty)$ be such that $\alpha(\delta) \approx \delta$. Then*

$$D_{\xi^\dagger}(\mathcal{A}^\delta, \mathcal{A}^\dagger) = \mathcal{O}(\delta) \quad \text{and} \quad \|\mathcal{U}(\mathcal{A}^\delta) - \mathcal{U}^\delta\| = \mathcal{O}(\delta).$$

Moreover, there exists a constant $c > 0$, such that $f_{\mathcal{A}_0}(\mathcal{A}^\delta) \leq f_{\mathcal{A}_0}(\mathcal{A}^\dagger) + \delta/c$ for every δ with $\alpha(\delta) \leq \beta_{\max}$.

Theorem 9 states how fast is the convergence of solutions obtained by Tikhonov regularization to an $f_{\mathcal{A}_0}$-minimizing solution of (3.1) when the noise level goes to zero. It can be seen as a measure of reliability of solutions, since it quantifies how better solutions are when the noise level decreases.

Remark 2. *It is important to note that convergence rates in terms of Bregman distances imply in rates in terms of norm convergence when we have q-coerciveness. See Definition 2 in Chapter 1. One example is the quadratic functional*

$$f_{\mathcal{A}_0}(\mathcal{A}) = \|\mathcal{A} - \mathcal{A}_0\|^2_{H^l(0,S,H^{1+\varepsilon}(D))},$$

which is 2-coercive with constant 1, since its Bregman distance is

$$D_{2(\widetilde{\mathcal{A}} - \mathcal{A}_0)}(\mathcal{A}, \widetilde{\mathcal{A}}) = \|\mathcal{A} - \widetilde{\mathcal{A}}\|^2_{H^l(0,S,H^{1+\varepsilon}(D))}.$$

Thus, by Theorem 9 it follows that

$$\mathcal{O}(\delta) = D_{2(\mathcal{A}^\dagger - \mathcal{A}_0)}(\mathcal{A}^\delta, \mathcal{A}^\dagger) = \|\mathcal{A}^\delta - \mathcal{A}^\dagger\|^2_{H^l(0,S,H^{1+\varepsilon}(D))}.$$

3.4 Kullback-Leibler: a Regularization Functional

Relative Entropy is a quantity extracted from Communication Theory which generalizes Shanon entropy. See [16]. Roughly speaking, it can be thought as a measure of distance between two probability distributions, thus, under many different contexts it has been applied in the place of Euclidean measures. An important example can be found in Inverse Problems regularization. In [18] there is a deep analysis comparing both measures when solving linear inverse problems. Examples of applications of relative entropy to solve/regularize some inverse problems can be found in [4, 19, 24, 47].

From our knowledge, the first work to use such distance as in the solution of local volatility calibration problem is [4]. However, only in [19] such distance is applied as the regularization functional in Tikhonov analysis to solve the same inverse problem.

Thus, following [19] we shall use Kullback-Leibler entropy in Tikhonov regularization. The novelty here is that we are under the framework of Chapter 2 and thus we shall use an integrated form of Kullback-Leibler entropy. As it was said earlier, such approach has the advantage of incorporating past information to obtain the present one, as in on-line estimation (see [5]).

First of all, observe that $Q \subset H^{1+\varepsilon}(D) \cap L^\infty_{>0}(D) \subset L^1(D)$, where $L^\infty_{>0}(D)$ is the class of strictly positive $L^\infty(D)$ functions. Thus, we can define the Kullback-Leibler (KL) relative entropy as

$$\begin{array}{rcl} KL : Q \times Q & \longrightarrow & \mathbb{R}^+ \cup \{+\infty\} \\ (a,b) & \longmapsto & KL(a,b) \end{array} \tag{3.9}$$

where

$$KL(a,b) := \begin{cases} \displaystyle\int_D a(\tau,y)\log\left(\frac{a(\tau,y)}{b(\tau,y)}\right) - a(\tau,y) + b(\tau,y)d\tau dy, & \text{if it exists} \\ +\infty, & \text{otherwise.} \end{cases} \tag{3.10}$$

Under the context of last section, this functional can be interpreted as the Bregman distance of Shanon entropy

$$\mathcal{S}(a) := \begin{cases} \displaystyle\int_D a(\tau,y)\log\left(a(\tau,y)\right) d\tau dy, & \text{if it exists} \\ +\infty, & \text{otherwise.} \end{cases} \tag{3.11}$$

See [16]. We remark that the domains of $\mathcal{S}$ and of its sub-gradient, are $\mathcal{D}(\mathcal{S}) = L^\infty_{\geq 0}(D)$, the class of essentially bounded and non-negative functions, and $\mathcal{D}(\partial\mathcal{S}) = L^\infty_{>0}(D)$, respectively.

We present below some properties of Kullback-Leibler relative entropy and Shanon entropy. For its proofs, the reader is referred to [24] and [47].

Lemma 5. *The following assertions hold:*

1. *$\mathcal{D}(\partial\mathcal{S})$ is strictly included in $L^1_{\geq 0}(D) := \{a \in L^1(D) \mid a \geq 0 \text{ a.e.}\}$ and has empty interior under the norm topology of $L^1(D)$;*

2. *the map $(a,b) \mapsto KL(a,b)$ is convex;*

3. *the maps* $a \mapsto KL(a, \overline{b})$ *and* $b \mapsto KL(\overline{a}, b)$ *are weakly lower semi-continuous in* $L^1(D)$*;*

4. *for any* $C > 0$ *and* $b \in L^{\infty}_{\leq 0}$ *the sets.*

$$\mathcal{E}_C := \{a \in L^1(D)\,|KL(a,b) \leq C\}$$

are weakly closed and weakly compact in $L^1(D)$*;*

5. *if* $D \subset \mathbb{R}^2$ *is bounded with Lipschitz boundary then*

$$\|a-b\|^2_{L^1(D)} \leq \left(\frac{2}{3}\|a\|_{L^1(D)} + \frac{4}{3}\|b\|_{L^1(D)}\right) KL(a,b) \tag{3.12}$$

note that, we assume here the convention $0 \cdot (+\infty) = 0$*; furthermore, for the sequences* $\{a_k\}_{k\in\mathbb{N}}$ *and* $\{b_k\}_{k\in\mathbb{N}}$ *in* $L^1(D)$*, with one of them bounded, we have*

$$\lim_{k\to\infty} KL(a_k, b_k) = 0 \quad \Rightarrow \quad \lim_{k\to\infty} \|a_k - b_k\|_{L^1(D)} = 0 \tag{3.13}$$

As we are dealing with trajectories of elements of Q, we define below the integrated form of Kullback-Leibler.

$$\mathscr{KL}(\mathcal{A}, \widetilde{\mathcal{A}}) := \int_0^T KL(a(t), \tilde{a}(t))dt, \tag{3.14}$$

with $(\mathcal{A}, \widetilde{\mathcal{A}}) \in \mathfrak{Q} \times \mathfrak{Q}$. This is the integrated version of Kullback-Leibler distance and it is the Bregman distance of

$$\mathscr{S}(\mathcal{A}) := \int_0^T \mathcal{S}(a(t))dt, \tag{3.15}$$

Lemma 6. *The following assertions about* $\mathscr{KL}(\cdot, \cdot\cdot)$ *hold for* $D \subset \mathbb{R}^2$ *bounded:*

1. *the map* $(\mathcal{A}, \widetilde{\mathcal{A}}) \mapsto \mathscr{KL}(\mathcal{A}, \widetilde{\mathcal{A}})$ *is convex;*

2. *the maps* $\mathcal{A} \mapsto \mathscr{KL}(\mathcal{A}, \widetilde{\mathcal{A}})$ *and* $\mathcal{A} \mapsto \mathscr{KL}(\widetilde{\mathcal{A}}, \mathcal{A})$ *are weakly lower semi-continuous in* $L^1(0,T,L^1(D))$*;*

3. *for sequences* $\{\mathcal{A}_k\}_{k\in\mathbb{N}}$ *and* $\{\mathcal{B}_k\}_{k\in\mathbb{N}}$ *in* $L^1(0,T,L^1(D))$*, with one of them bounded, we have*

$$\lim_{k\to\infty} \mathscr{KL}(\mathcal{A}_k, \mathcal{B}_k) = 0 \quad \Rightarrow \quad \lim_{k\to\infty} \|\mathcal{A}_k - \mathcal{B}_k\|_{L^1(0,T,L^1(D))} = 0 \tag{3.16}$$

Proof: The first item follows directly from item one of Lemma 5. For the second item, we note that, any weak zero-neighborhood of $L^1(0,T,L^1(D))$ is generated by $f \otimes \gamma \in L^1(0,T,L^1(D))^*$ with $f \in L^{\infty}[0,T]$ and $\gamma \in L^{\infty}(D)$. Moreover, for each $\gamma \in L^{\infty}[0,T]$, and $\mathcal{A} \in \mathfrak{Q}$, $\langle \gamma, \mathcal{A}\rangle = \{t \mapsto \langle \gamma, a(t)\rangle\} \in C[0,T]$. Now, given a sequence $\{\mathcal{A}_k\}_{k\in\mathbb{N}} \subset \mathfrak{Q}$ converging weakly to $\mathcal{A}_0 \in \mathfrak{Q}$ in

$L^1(0,T,L^1(D))$. It is equivalent to $\langle f\otimes\gamma, \mathcal{A}_k-\mathcal{A}_0\rangle \to 0$, where

$$\begin{aligned}\langle f\otimes\gamma, \mathcal{A}_k-\mathcal{A}_0\rangle &= \int_0^T f(t)\langle\gamma, \mathcal{A}_k-\mathcal{A}_0\rangle(t)dt\\ &= \int_0^T f(t)\int_D \gamma(x)(a_k(t,x)-a_0(t,x))dxdt \to 0.\end{aligned}$$

As f and γ are arbitrary, we have that $\{a_k(t)\}_{k\in\mathbb{N}}$ converges weakly to $a_0(t)$ t-almost surely. Therefore, the weak lower semi-continuity of $\mathcal{A}\mapsto\mathscr{KL}(\mathcal{A},\widetilde{\mathcal{A}})$ and $\mathcal{A}\mapsto\mathscr{KL}(\widetilde{\mathcal{A}},\mathcal{A})$ follows from the item 3 of Lemma 5. Finally, the third item follows from item 4 of Lemma 5. ■

Lemma 7. *Assume that $\mathcal{U}(\cdot)$ is continuous in the weak topologies of*

$$L^1(0,T,L^1(D)) \text{ and } L^2\left(0,T,L^2(D)\right).$$

Moreover, the set $D\subset\mathbb{R}^2$ is bounded with Lipschitz boundary. Fix $a^*\in\mathcal{D}_B(S)$ with $a^*\neq 0$, define $\mathcal{A}^*:=\{t\mapsto a^*\}$. Then, for any $M>0$, the sets

$$\mu_{\alpha,\mathcal{U}^\delta}(M) := \left\{\mathcal{A}\in\mathcal{D}_B(\mathscr{S})\,\middle|\,\mathcal{F}^{\mathcal{U}^\delta}_{\mathcal{A}^*,\alpha}(\mathcal{A})\le M\right\}$$

are weakly pre-compact.

Proof: For $M>0$ fixed, take a sequence $\{\mathcal{A}_k\}_{k\in\mathbb{N}}\subset\mu_{\alpha,\mathcal{U}^\delta}(M)$. By (3.13) it follows that $\{\|\mathcal{A}_k\|\}_{k\in\mathbb{N}}$ is uniformly bounded in $L^1(D)$. Since

$$\|a_k(t)\|^2_{L^2(D)} = \int_D |a_k(t,x)|^2dx \le a_2\int_D |a_k(t,x)|dx = a_2\cdot\|a_k(t)\|_{L^1(D)}$$

where the constant $a_2>0$, from the definition of Q, is a superior bound for its elements. It implies that

$$\|\mathcal{A}_k\|^2_{L^2(0,T,L^2(D))} \le a_2\|\mathcal{A}_k\|_{L^1(0,T,L^1(D))}.$$

Thus, $\{\mathcal{A}_k\}_{k\in\mathbb{N}}$ has a weak cluster point $\widetilde{\mathcal{A}}$ in $L^2(0,T,L^2(D))$. As $\mathfrak{Q}$ is weakly closed and the weak topology of $L^2(0,T,L^2(D))$ contains the weak topology of $H^l(0,S,H^{1+\varepsilon}(D))$, we have that $\widetilde{\mathcal{A}}\in\mathfrak{Q}$.

In other direction, given $\gamma\in L^1(D)^*$ and $\mathcal{A}\in L^1(0,T,L^1(D))$, define

$$\langle\gamma,\mathcal{A}\rangle := \{t\mapsto\langle\gamma,a(t)\rangle\}.$$

It is in $L^1([0,T])$. As $\gamma\in L^\infty(D)$ and D has bounded Lebesgue's measure, it is in $L^2(D)$. Given $f\in L^\infty([0,T])$, it is in $L^2([0,T])$, and thus $f\otimes\gamma\in L^1(0,T,L^1(D))^*$ and $L^2(0,T,L^2(D))$, where

$$\langle f\otimes\gamma,\mathcal{A}_k-\widetilde{\mathcal{A}}\rangle := \langle f,\langle\gamma,\mathcal{A}_k-\widetilde{\mathcal{A}}\rangle\rangle = \int_0^T f(t)\langle\gamma,a_k(t)-\tilde{a}(t)\rangle dt.$$

This shows that, a weak zero-neighborhood of $L^1(0,T,L^1(D))$ is a weak zero-neighborhood of $L^2(0,T,L^2(D))$. Hence, $\widetilde{\mathcal{A}}$ is a weak cluster point of $\{\mathcal{A}_k\}_{k\in\mathbb{N}}$ in $L^1(0,T,L^1(D))$. Let $\{\mathcal{A}_{k_l}\}_{l\in\mathbb{N}}$ be a subsequence of $\{\mathcal{A}_k\}_{k\in\mathbb{N}}$ converging to $\widetilde{\mathcal{A}}$. Thus, by the weak lower semi-continuity of the integrated Kullback-Leibler functional we have

$$\mathscr{K}\mathscr{L}(\widetilde{\mathcal{A}}, \mathcal{A}^*) \leq \liminf \mathscr{K}\mathscr{L}(\mathcal{A}_k, \mathcal{A}^*).$$

By the continuity of $\mathcal{U}(\cdot)$ in the weak topology of $L^1(0, T, L^1(D))$, we achieve that

$$\mathcal{F}^{\mathcal{U}^\delta}_{\mathcal{A}^*,\alpha}(\widetilde{\mathcal{A}}) = \|\mathcal{U}(\widetilde{\mathcal{A}}) - \mathcal{U}^\delta\|^2_{L^2(0,T,L^2(D))} + \alpha\mathscr{K}\mathscr{L}(\widetilde{\mathcal{A}}, \mathcal{A}^*) \leq M,$$

that finishes the proof. ■

The following result states the principal aspects of the Kullback-Leibler-Tikhonov regularization technique.

Theorem 10. *Let the set $D \subset \mathbb{R}^2$ be bounded. Then the functional*

$$\mathcal{F}^{\mathcal{U}^\delta}_{\mathcal{A}^*,\alpha}(\mathcal{A}) = \|\mathcal{U}(\mathcal{A}) - \mathcal{U}^\delta\|^2_{L^2(0,T,L^2(D))} + \alpha\mathscr{K}\mathscr{L}(\mathcal{A}, \mathcal{A}^*)$$

has a minimizer in $\mathfrak{Q}$. Moreover, the minimizers are stable and convergent in the sense of Theorems 7 and 8.

The theorem above follows from standard Tikhonov regularization results (see [26, 48]) and the Remark below.

Remark 3. *By the boundedness of D, we have, $Q \subset L^2(D) \subset L^1(D)$, $Q \subset \mathcal{D}_B S$. Moreover, $W^{1,2}_2(D)$ is continuously embedded into $L^2(D)$, which implies in the continuous embedding of $L^2(0, T, W^{1,2}_2(D))$ into $L^2(0, T, L^2(D))$. Thus, by the weak continuity of $\mathcal{U} : \mathfrak{Q} \to L^2(0, T, W^{1,2}_2(D))$ it follows that $\mathcal{U} : \mathfrak{Q} \to L^2(0, T, L^2(D))$ is weakly continuous. Hence, Lemma 7 holds. Furthermore, for $C > 0$ sufficiently large, the convex set $\mathfrak{Q}$ is a subset of $\mathcal{E}_C$, which implies it is weakly closed under the topology of $L^1(0, T, L^1(D))$.*

3.5 Morozov's Principle

In Tikhonov regularization theory the choice of $\alpha(\delta)$ is crucial, since it asserts how much the prior information has to be taken into account in the presence of noise level δ. On one side, for α too large, the influence of the prior could be larger than necessary, leading to biased solutions. On the other side, if α is too small, the instability of the problem and the uncertainty caused by the presence of noise will decrease the reliability of solutions.

Since this choice is fundamental to regularize the inverse problem appropriately, we can find in literature many ways to find a proper α. See for example [26]. One of the most robust techniques is the Morozov's discrepancy principle (see [42]), which is the way we have chosen to find α.

Following [42] and the notation of the present work, remember that $\delta > 0$ is the noise level, $\widetilde{\mathcal{U}}$ and $\mathcal{U}^\delta$ are the noiseless and noisy data, respectively. In addition, they satisfy

$$\|\widetilde{\mathcal{U}} - \mathcal{U}^\delta\| \leq \delta. \tag{3.17}$$

From such relation we conclude that any $\mathcal{A} \in \mathfrak{Q}$ satisfying

$$\|\mathcal{U}(\mathcal{A}) - \mathcal{U}^\delta\| \leq \delta \tag{3.18}$$

could be an approximate solution for (3.1). If $\mathcal{A}_\alpha^\delta$ is a minimizer of (3.4), then we define the map

$$\begin{array}{rcl} g : \mathbb{R}_+ & \longrightarrow & \mathbb{R}_+ \\ \alpha \in \mathbb{R}_+ & \longmapsto & g(\alpha) := \|\mathcal{U}(\mathcal{A}_\alpha^\delta) - \mathcal{U}^\delta\| \end{array} \tag{3.19}$$

Morozov's discrepancy principle says that the regularization parameter α should be chosen from the condition

$$g(\alpha) = \|\mathcal{U}(a_\alpha^\delta) - \mathcal{U}^\delta\| = \delta \tag{3.20}$$

when it is possible. In other words, the regularized solution should not satisfy the data more accurately than up to the noise level.

Thus, let the noise level δ and the data $\mathcal{U}^\delta$ be fixed. We have now the following definition.

Definition 7. *Define the functionals*

$$\begin{array}{rcl} L : \mathfrak{Q} & \longrightarrow & \mathbb{R}_+ \cup \{+\infty\} \\ \mathcal{A} \in \mathfrak{Q} & \longmapsto & L(\mathcal{A}) = \|\mathcal{U}(\mathcal{A}) - \mathcal{U}^\delta\|, \end{array}$$

$$\begin{array}{rcl} H : \mathfrak{Q} & \longrightarrow & \mathbb{R}_+ \cup \{+\infty\} \\ \mathcal{A} \in \mathfrak{Q} & \longmapsto & H(\mathcal{A}) = f_{\mathcal{A}_0}(\mathcal{A}), \end{array}$$

and

$$\begin{array}{rcl} I : \mathbb{R}_+ & \longrightarrow & \mathbb{R}_+ \cup \{+\infty\} \\ \alpha \in \mathbb{R}_+ & \longmapsto & I(\alpha) = \mathcal{F}_{\mathcal{A}_0,\alpha}^{\mathcal{U}^\delta}(\mathcal{A}_\alpha^\delta). \end{array}$$

We also define:

1. *The set of all minimizers of the functional (3.4) for each $\alpha \in (0, \infty)$*

$$M_\alpha := \left\{ \mathcal{A}_\alpha^\delta \in \mathfrak{Q} : L(a_\alpha^\delta) \leq L(\mathcal{A}),\ \forall \mathcal{A} \in H^l(0, S, H^{1+\varepsilon}(D)) \right\}.$$

   Note that we have extended $L(\mathcal{A})$ to be equal to $\|\mathcal{U}(\mathcal{A}) - \mathcal{U}^\delta\|$ when $\mathcal{A} \in \mathfrak{Q}$ and be equal to $+\infty$ otherwise.

2. *The set $\mathcal{L}$ of all $f_{\mathcal{A}_0}$-minimizing solutions of (3.2).*

As in [3], we define more accurately the Morozov's discrepancy principle.

Definition 8. *For $1 < \tau_1 \leq \tau_2$ we choose $\alpha = \alpha(\delta, \mathcal{U}^\delta) > 0$ such that*

$$\tau_1 \delta \leq \|\mathcal{U}(\mathcal{A}_\alpha^\delta) - \mathcal{U}^\delta\| \leq \tau_2 \delta \tag{3.21}$$

holds for some $\mathcal{A}_\alpha^\delta$ in M_α.

The following lemma states some properties of $L(\cdot)$, $H(\cdot)$ and $I(\cdot)$., its proof is simple and it can be found in Section 2.6 of [50], Lemma 1.

Lemma 8. *The functional $H(\cdot)$ is non-increasing and the functionals $L(\cdot)$ and $I(\cdot)$ are non-decreasing with respect to $\alpha \in (0,\infty)$. In other words, for $0 < \alpha < \beta$ we have*

$$\sup_{\mathcal{A}_\alpha^\delta \in M_\alpha} L(\mathcal{A}_\alpha^\delta) \leq \inf_{\mathcal{A}_\beta^\delta \in M_\beta} L(\mathcal{A}_\beta^\delta),$$

$$\inf_{\mathcal{A}_\alpha^\delta \in M_\alpha} H(\mathcal{A}_\alpha^\delta) \geq \inf_{\mathcal{A}_\beta^\delta \in M_\beta} H(\mathcal{A}_\beta^\delta),$$

$$I(\alpha) \leq I(\beta)$$

Lemma 9. *The functional $I : (0,\infty) \to [0,\infty]$ is continuous. The sets*

$$M := \left\{ \alpha > 0 \,\middle|\, \inf_{\mathcal{A}_\alpha^\delta \in M_\alpha} L(\mathcal{A}_\alpha^\delta) < \sup_{\mathcal{A}_\alpha^\delta \in M_\alpha} L(\mathcal{A}_\alpha^\delta) \right\}$$

and

$$N := \left\{ \alpha > 0 \,\middle|\, \inf_{\mathcal{A}_\alpha^\delta \in M_\alpha} H(\mathcal{A}_\alpha^\delta) < \sup_{\mathcal{A}_\alpha^\delta \in M_\alpha} H(\mathcal{A}_\alpha^\delta) \right\}$$

are at most countable and coincide. Moreover, the maps $L(\cdot)$ and $H(\cdot)$ are continuous in $(0,\infty)\backslash M$.

Proof: Let $\{\alpha_n\}_{n\in\mathbb{N}} \subset (0,\infty)$ be a convergent sequence converging to α^*. Thus, we can choose a sequence $\{\mathcal{A}_n\}_{n\in\mathbb{N}}$ where, for each $n \in \mathbb{N}$, $\mathcal{A}_n := \mathcal{A}_{\alpha_n}^\delta$, a minimizer of 3.4 with α replaced by α_n. By the coerciveness of $f_{\mathcal{A}_0}$, $\{\mathcal{A}_n\}_{n\in\mathbb{N}}$ is bounded and thus, it has a weakly convergent subsequence denoted by $\{\mathcal{A}_k\}_{k\in\mathbb{N}}$ with limit $\mathcal{A}^*$.
As $\mathcal{U}(\cdot)$ is weakly continuous, the norm of $L^2(0,S,W_2^{1,2}(D))$ is lower semi-continuous and $f_{a_0}(\cdot)$ is weakly lower semi-continuous, we have

$$\begin{aligned} \mathcal{F}^{\mathcal{U}^\delta}_{\mathcal{A}_0,\alpha^*}(\mathcal{A}^*) &\leq \liminf_{k\to\infty} \mathcal{F}^{\mathcal{U}^\delta}_{\mathcal{A}_0,\alpha_k}(\mathcal{A}_k) &&\leq \limsup_{k\to\infty} \mathcal{F}^{\mathcal{U}^\delta}_{\mathcal{A}_0,\alpha_k}(\mathcal{A}_k) \\ &\leq \lim_{k\to\infty} \mathcal{F}^{\mathcal{U}^\delta}_{\mathcal{A}_0,\alpha_k}(\mathcal{A}) &&= \mathcal{F}^{\mathcal{U}^\delta}_{\mathcal{A}_0,\alpha^*}(\mathcal{A}), \forall\, \mathcal{A} \in \mathfrak{Q}, \end{aligned}$$

which shows that $\mathcal{A}^* \in M_{\alpha^*}$. Thus, by the monotonicity of $I(\cdot)$ we have that $I(\cdot)$ is continuous at α^*.
If $\alpha \in M$ there are $\mathcal{A}, \widetilde{\mathcal{A}} \in M_\alpha$ such that $L(\mathcal{A}) < L(\widetilde{\mathcal{A}})$, as $I(\mathcal{A}) = I(\widetilde{\mathcal{A}})$, we have

$$L(\mathcal{A}) \pm \alpha H(\mathcal{A}) < G(\widetilde{\mathcal{A}}) \pm \alpha H(\widetilde{\mathcal{A}}) \Leftrightarrow I(\mathcal{A}) - \alpha H(\mathcal{A}) < I(\widetilde{\mathcal{A}}) - \alpha H(\widetilde{\mathcal{A}}) \Leftrightarrow H(\mathcal{A}) > H(\widetilde{\mathcal{A}})$$

and $M \subset N$. The other inclusion is analogous. The countability of M follows by the fact that for each $\alpha \in M$ we can associate the interval $\mathcal{I}_\alpha = (\inf_{\mathcal{A}_\alpha^\delta \in M_\alpha} L(\mathcal{A}_\alpha^\delta), \sup_{\mathcal{A}_\alpha^\delta \in M_\alpha} L(\mathcal{A}_\alpha^\delta))$. By the monotonicity of L we have that for each $\alpha, \beta \in M$, $\mathcal{I}_\alpha \cap \mathcal{I}_\beta = \emptyset$. Therefore, we can define an injective map that associates each $\alpha \in M$ to an element of $\mathcal{I}_\alpha \cap \mathbb{Q}$.
The continuity of L and H with respect to α out of M follows by the same argument above about the continuity of I, with $\{\alpha_n\}_{n\in\mathbb{N}}$ and α^* in $(0,\infty)\backslash M$. ■

The following lemma follows from the fact that the sets M_α are weakly closed.

Lemma 10. *To each $\overline{\alpha} > 0$ there exist $\mathcal{A}_1, \mathcal{A}_2 \in M_{\overline{\alpha}}$ such that*

$$L(\mathcal{A}_1) = \inf_{\mathcal{A} \in M_{\overline{\alpha}}} L(\mathcal{A}) \quad \textit{and} \quad L(\mathcal{A}_2) = \sup_{\mathcal{A} \in M_{\overline{\alpha}}} L(\mathcal{A})$$

For the remaining part of this Section, we assume that $f_{\mathcal{A}_0}(\mathcal{A}_0) = 0$.

Proposition 14. *Let $1 < \tau_1 \leq \tau_2$ be fixed. Suppose that $\|\mathcal{U}(\mathcal{A}_0) - \mathcal{U}^\delta\| > \tau_2 \delta$. Then we can find $\alpha_1, \alpha_2 > 0$, denoting $\mathcal{A}_1 := \mathcal{A}^\delta_{\alpha_1}$ and $\mathcal{A}_2 := \mathcal{A}^\delta_{\alpha_2}$, such that*

$$L(\mathcal{A}_1) < \tau_1 \delta \leq \tau_2 \delta < L(\mathcal{A}_2).$$

Proof: First let the sequence $\{\alpha_n\}_{n \in \mathbb{N}}$ converge to 0. Thus, we can find a sequence $\{\mathcal{A}_n\}_{n \in \mathbb{N}}$ with $\mathcal{A}_n \in M_{\alpha_n}$ for each $n \in \mathbb{N}$. Now, let $\mathcal{A}^\dagger$ be an $f_{\mathcal{A}_0}$-minimizing solution of (3.2). Hence, we have

$$L(\mathcal{A}_n)^2 \leq I(\alpha_n) \leq \mathcal{F}^{\mathcal{U}^\delta}_{\mathcal{A}_0, \alpha_n}(\mathcal{A}^\dagger) \leq \delta^2 + \alpha_n f_{\mathcal{A}_0}(\mathcal{A}^\dagger)$$

thus, for n sufficiently large we have $L(\mathcal{A}_n)^2 < (\tau_1 \delta)^2$, as $\alpha_n f_{\mathcal{A}_0}(a^\dagger) \to 0$ and for this n we can set $\alpha_1 := \alpha_n$.

Now, we assume that $\alpha_n \to \infty$ and we take $\mathcal{A}_n$ as before, the

$$H(\mathcal{A}_n) \leq \frac{1}{\alpha_n} I(\alpha_n) \leq \frac{1}{\alpha_n} \mathcal{F}^{\mathcal{U}^\delta}_{\mathcal{A}_0, \alpha_n}(\mathcal{A}_0) = \frac{1}{\alpha_n} \|\mathcal{U}(\mathcal{A}_0) - \mathcal{U}^\delta\| \to 0 \ \ \textit{as } n \to \infty.$$

Thus, $\lim_{n \to \infty} f_{\mathcal{A}_0}(\mathcal{A}_n) = 0$, which implies that $\{\mathcal{A}_n\}_{n \in \mathbb{N}}$ converges weakly to $\mathcal{A}_0$. Then, by the weak continuity of $\mathcal{U}(\cdot)$ and the lower semi-continuity of the norm, we have

$$\|\mathcal{U}(\mathcal{A}_0) - \mathcal{U}^\delta\| \leq \liminf_{n \to \infty} \|\mathcal{U}(\mathcal{A}_n) - \mathcal{U}^\delta\|,$$

which shows the existence of α_2 such that $L(\mathcal{A}^\delta_{\alpha_2}) > \tau_2 \delta$. ■

Now we require that, there is no $\alpha > 0$ with $\mathcal{A}_1, \mathcal{A}_2 \in M_\alpha$ such that

$$\|\mathcal{U}(\mathcal{A}_1) - \mathcal{U}^\delta\| < \tau_1 \delta \leq \tau_2 \delta < \|\mathcal{U}(\mathcal{A}_2) - \mathcal{U}^\delta\|.$$

If such α exists, $\mathcal{A}_1$ would be a sufficiently good approximation for the solution of (3.2) give that $\|\mathcal{U}(\mathcal{A}^\dagger) - \mathcal{U}^\delta\| \leq \delta$. Thus, we can state the theorem below.

The proof of the Theorem below is the same of Theorem 3.10 of [3].

Theorem 11. *Under the assumption above and the assumptions of Proposition 14, we have the existence of an $\alpha := \alpha(\delta) > 0$ and $\mathcal{A}^\delta_\alpha \in M_\alpha$ such that*

$$\tau_1 \delta \leq \|\mathcal{U}(\mathcal{A}^\delta_\alpha) - \mathcal{U}^\delta\| \leq \tau_2 \delta. \tag{3.22}$$

We present now the first main result of this section. It states that, if we use (3.21) to choose $\alpha = \alpha(\delta, \mathcal{U}^\delta)$, we have that such parameter satisfies the hypothesis (3.5) in Theorem 8, which states the weak convergence of the regularized solutions $\mathcal{A}_\alpha^\delta$ to an $f_{\mathcal{A}_0}$-minimizing solution of the inverse problem of local volatility calibration.

Theorem 12. *Let $\delta > 0$ and $\mathcal{U}^\delta$ satisfy the hypothesis above. Then the regularizing parameter $\alpha = \alpha(\delta, \mathcal{U}^\delta)$ obtained through Morozov's discrepancy principle (3.21) satisfies (3.5), i.e., the limits below*

$$\lim_{\delta\to 0+} \alpha(\delta,\mathcal{U}^\delta) = 0 \quad \text{and} \quad \lim_{\delta\to 0+} \frac{\delta^2}{\alpha(\delta,\mathcal{U}^\delta)} = 0.$$

hold.

Proof: Take a sequence $\{\delta_n\}_{n\in\mathbb{N}}$ in $(0,+\infty)$ converging to zero, fix the noiseless data $\widetilde{\mathcal{U}}$ and denote $\alpha_n := \alpha(\delta_n, \mathcal{U}^{\delta_n})$ the regularizing parameter chosen from (3.21). Let $\mathcal{A}_n := \mathcal{A}_{\alpha_n}^{\delta_n}$ be a minimizer of (3.4) with δ_n, α_n and $\mathcal{U}^{\delta_n}$. Thus we can define a sequence $\{\mathcal{A}_n\}_{n\in\mathbb{N}}$. As $f_{\mathcal{A}_0}(\cdot)$ is coercive, such sequence has a weakly convergent subsequence, denoted by $\{\mathcal{A}_k\}_{k\in\mathbb{N}}$ with limit $\widetilde{\mathcal{A}}$.

By the weak lower semi-continuity of $\|\mathcal{U}(\cdot) - \widetilde{\mathcal{U}}\|$ and $f_{\mathcal{A}_0}$ we have the following estimates

$$\|\mathcal{U}(\widetilde{\mathcal{A}}) - \widetilde{\mathcal{U}}\| \leq \liminf_{k\to\infty} \|\mathcal{U}(\mathcal{A}_k) - \widetilde{\mathcal{U}}\| \leq \liminf_{k\to\infty} \|\mathcal{U}(\mathcal{A}_k) - \mathcal{U}^{\delta_k}\| + \delta_k \leq \liminf_{k\to\infty} (\tau_2 + 1)\delta_k = 0, \tag{3.23}$$

i.e., $\widetilde{\mathcal{A}}$ is a solution for the inverse problem. From (3.21), we have that

$$\lim_{k\to\infty} \|\mathcal{U}(\mathcal{A}_k) - \mathcal{U}^{\delta_k}\| \leq \lim_{k\to\infty} \tau_2 \delta_k = 0 \tag{3.24}$$

Therefore, for every $\mathcal{A}^\dagger \in \mathcal{L}$ we have the estimate

$$\tau_1^2\delta_k^2 + \alpha_k f_{\mathcal{A}_0}(\mathcal{A}_k) \leq \|\mathcal{U}(\mathcal{A}_k) - \mathcal{U}^{\delta_k}\|^2 + \alpha_k f_{\mathcal{A}_0}(\mathcal{A}_k) \leq \delta_k + \alpha_k f_{\mathcal{A}_0}(\mathcal{A}^\dagger), \tag{3.25}$$

which shows that $\limsup_{k\to\infty} f_{\mathcal{A}_0}(\mathcal{A}_k) \leq f_{\mathcal{A}_0}(\mathcal{A}^\dagger)$. It follows that

$$f_{\mathcal{A}_0}(\widetilde{\mathcal{A}}) \leq \liminf_{k\to\infty} f_{\mathcal{A}_0}(\mathcal{A}_k) \leq \limsup_{k\to\infty} f_{\mathcal{A}_0}(\mathcal{A}_k) \leq f_{\mathcal{A}_0}(\mathcal{A}^\dagger) \tag{3.26}$$

for every $\mathcal{A}^\dagger \in \mathcal{L}$. As $\mathcal{A}^\dagger$ is an $f_{\mathcal{A}_0}$-minimizing solution of the inverse problem, it follows that $\widetilde{\mathcal{A}} \in \mathcal{L}$ and $f_{\mathcal{A}_0}(\widetilde{\mathcal{A}}) = f_{\mathcal{A}_0}(\mathcal{A}^\dagger)$. Furthermore, it implies that $f_{\mathcal{A}_0}(\mathcal{A}_k) \to f_{\mathcal{A}_0}(\widetilde{\mathcal{A}})$.

We prove now the first limit in the theorem. Assume that there exists an $\overline{\alpha} > 0$ and a subsequence $\{\alpha_k\}_{k\in\mathbb{N}}$ such that $\alpha_k \geq \tilde{\alpha}$ for every $k \in \mathbb{N}$. As above select a sequence $\{\mathcal{A}_k\}_{k\in\mathbb{N}}$ of minimizers of (3.4) with δ_k, α_k and $\mathcal{U}^{\delta_k}$. Define further the sequence $\{\overline{\mathcal{A}}_k\}_{k\in\mathbb{N}}$ of minimizers of (3.4) with δ_k, $\overline{\alpha}$ and $\mathcal{U}^{\delta_k}$.

As L in non-decreasing with respect to α and by (3.21), it follows that

$$\|\mathcal{U}(\overline{\mathcal{A}}_k) - \mathcal{U}^{\delta_k}\| \leq \|\mathcal{U}(\mathcal{A}_k) - \mathcal{U}^{\delta_k}\| \leq \tau_2\delta_k \to 0 \tag{3.27}$$

On the other hand,

$$\begin{aligned} \limsup_{k\to\infty} \overline{\alpha} f_{\mathcal{A}_0}(\overline{\mathcal{A}}_k) &\leq \limsup_{k\to\infty} \left(\|\mathcal{U}(\overline{\mathcal{A}}_k) - \mathcal{U}^{\delta_k}\|^2 + \overline{\alpha} f_{\mathcal{A}_0}(\overline{\mathcal{A}}_k) \right) \\ &\leq \limsup_{k\to\infty} \left(\|\mathcal{U}(\mathcal{A}^\dagger) - \mathcal{U}^{\delta_k}\|^2 + \overline{\alpha} f_{\mathcal{A}_0}(\mathcal{A}^\dagger) \right) = \overline{\alpha} f_{\mathcal{A}_0}(\mathcal{A}^\dagger), \end{aligned} \tag{3.28}$$

for every $\mathcal{A}^\dagger \in \mathcal{L}$, as $\mathcal{U}(\mathcal{A}^\dagger) = \widetilde{\mathcal{U}}$, the noiseless data. By the coerciveness of $f_{\mathcal{A}_0}$, it follows that it has a convergent subsequence, denoted by $\{\overline{\mathcal{A}}_k\}_{k\in\mathbb{N}}$, with limit $\widetilde{\mathcal{A}} \in \mathfrak{Q}$. Thus, by (3.26), (3.27), the weak lower semi-continuity of $\|\mathcal{U}(\cdot) - \widetilde{\mathcal{U}}\|$ and $f_{\mathcal{A}_0}$, follows that

$$\|\mathcal{U}(\widetilde{\mathcal{A}}) - \widetilde{\mathcal{U}}\| \le \liminf_{k\to\infty} \|\mathcal{U}(\overline{\mathcal{A}}_k) - \widetilde{\mathcal{U}}\| \le \liminf_{k\to\infty} \left(\|\mathcal{U}(\overline{\mathcal{A}}_k) - \mathcal{U}^{\delta_k}\| + \delta_k\right) = 0 \tag{3.29}$$

$$f_{\mathcal{A}_0}(\widetilde{\mathcal{A}}) \le \liminf_{k\to\infty} f_{\mathcal{A}_0}(\overline{\mathcal{A}}_k) \le \limsup_{k\to\infty} f_{\mathcal{A}_0}(\overline{\mathcal{A}}_k) \le f_{\mathcal{A}_0}(\mathcal{A}^\dagger), \tag{3.30}$$

for every $\mathcal{A}^\dagger \in \mathcal{L}$. As above, $f_{\mathcal{A}_0}(\widetilde{\mathcal{A}}) = f_{\mathcal{A}_0}(\mathcal{A}^\dagger)$ and thus $f_{\mathcal{A}_0}(\overline{\mathcal{A}}_k) \to f_{\mathcal{A}_0}(\mathcal{A}^\dagger)$. From the above estimates, we have

$$\begin{aligned}
\|\mathcal{U}(\widetilde{\mathcal{A}}) - \widetilde{\mathcal{U}}\|^2 + \overline{\alpha} f_{\mathcal{A}_0}(\widetilde{\mathcal{A}}) &\le \liminf_{k\to\infty} \left(\|\mathcal{U}(\overline{\mathcal{A}}_k) - \mathcal{U}^{\delta_k}\|^2 + \overline{\alpha} f_{\mathcal{A}_0}(\overline{\mathcal{A}}_k)\right) \\
&\le \liminf_{k\to\infty} \left(\|\mathcal{U}(\mathcal{A}) - \mathcal{U}^{\delta_k}\|^2 + \overline{\alpha} f_{\mathcal{A}_0}(\mathcal{A})\right) \\
&= \|\mathcal{U}(\mathcal{A}) - \mathcal{U}^{\delta_k}\|^2 + \overline{\alpha} f_{\mathcal{A}_0}(\mathcal{A}),
\end{aligned}$$

for every $\mathcal{A} \in \mathfrak{Q}$, i.e., $\widetilde{\mathcal{A}}$ is a minimizer for (3.4) with $\overline{\alpha}$ and the noiseless data $\widetilde{\mathcal{U}}$. Thus, by the convexity of $f_{\mathcal{A}_0}$ we have, for every $t \in [0,1)$,

$$f_{\mathcal{A}_0}((1-t)\widetilde{\mathcal{A}} + t\mathcal{A}_0) \le (1-t) f_{\mathcal{A}_0}(\widetilde{\mathcal{A}}) + t f_{\mathcal{A}_0}(\mathcal{A}_0) = (1-t) f_{\mathcal{A}_0}(\widetilde{\mathcal{A}}).$$

On the other hand, from the above estimates it follows that

$$\begin{aligned}
\overline{\alpha} f_{\mathcal{A}_0}(\widetilde{\mathcal{A}}) &= \|\mathcal{U}(\widetilde{\mathcal{A}}) - \widetilde{\mathcal{U}}\|^2 + \overline{\alpha} f_{\mathcal{A}_0}(\widetilde{\mathcal{A}}) \\
&\le \|\mathcal{U}((1-t)\widetilde{\mathcal{A}} + t\mathcal{A}_0) - \widetilde{\mathcal{U}}\|^2 + \overline{\alpha} f_{\mathcal{A}_0}((1-t)\widetilde{\mathcal{A}} + t\mathcal{A}_0) \\
&\le \|\mathcal{U}((1-t)\widetilde{\mathcal{A}} + t\mathcal{A}_0) - \widetilde{\mathcal{U}}\|^2 + \overline{\alpha}(1-t) f_{\mathcal{A}_0}(\widetilde{\mathcal{A}})
\end{aligned}$$

This implies that $\overline{\alpha} t f_{\mathcal{A}_0}(\widetilde{\mathcal{A}}) \le \|\mathcal{U}((1-t)\widetilde{\mathcal{A}} + t\mathcal{A}_0) - \widetilde{\mathcal{U}}\|^2$. As $\widetilde{\mathcal{U}} = \mathcal{U}(\widetilde{\mathcal{A}})$, by Proposition 11 with $\mathcal{H} = \mathcal{A}_0 - \mathcal{A}$, we have

$$\overline{\alpha} f_{\mathcal{A}_0}(\widetilde{\mathcal{A}}) \le \lim_{t\to 0^+} \frac{1}{t} \|\mathcal{U}((1-t)\widetilde{\mathcal{A}} + t\mathcal{A}_0) - \widetilde{\mathcal{U}}\|^2 = 0$$

Therefore, $f_{\mathcal{A}_0}(\widetilde{\mathcal{A}}) = 0$ which, by hypothesis, can only hold if $\widetilde{\mathcal{A}} = \mathcal{A}_0$. However, by hypothesis, for every $\delta > 0$, $\mathcal{A}_0$ is chosen such that $\|\mathcal{U}(\mathcal{A}_0) - \mathcal{U}^\delta\| \ge \tau_1 \delta > \delta$. Thus,

$$\|\mathcal{U}(\mathcal{A}_0) - \widetilde{\mathcal{U}}\| \ge \|\mathcal{U}(\mathcal{A}_0) - \mathcal{U}^\delta\| - \|\mathcal{U}^\delta - \widetilde{\mathcal{U}}\| \ge (\tau_1 - 1)\delta > 0.$$

Therefore, we have achieved a contradiction with the fact that $\widetilde{\mathcal{U}} = \mathcal{U}(\widetilde{\mathcal{A}}) = \mathcal{U}(\mathcal{A}_0)$. We conclude that $\alpha(\delta, \mathcal{U}^\delta) \to 0$ when $\delta \to 0$.

Now for the second limit, take the subsequence $\{\mathcal{A}_k\}_{k\in\mathbb{N}}$ of the sequence of the beginning of the proof, with $\delta_k \to 0$. We know that this sequence converges weakly to an $f_{\mathcal{A}_0}$-minimizing solution of the inverse problem $\mathcal{A}^\dagger$, with $f_{\mathcal{A}_0}(\mathcal{A}_k) \to f_{\mathcal{A}_0}(\mathcal{A}^\dagger)$. Thus, from (3.21), it follows that

$$\begin{aligned} \tau_1^2\delta_k^2 + \alpha_k f_{\mathcal{A}_0}(\mathcal{A}_k) &\leq \|\mathcal{U}(\mathcal{A}_k) - \mathcal{A}^{\delta_k}\|^2 + \alpha_k f_{\mathcal{A}_0}(\mathcal{A}_k) \\ &\leq \|\mathcal{U}(\mathcal{A}^\dagger) - \mathcal{A}^{\delta_k}\|^2 + \alpha_k f_{\mathcal{A}_0}(\mathcal{A}^\dagger) \\ &= \delta_k^2 + \alpha_k f_{\mathcal{A}_0}(\mathcal{A}^\dagger). \end{aligned}$$

This implies that

$$(\tau_1^2 - 1)\frac{\delta^2}{\alpha_n} \leq f_{\mathcal{A}_0}(\mathcal{A}^\dagger) - f_{\mathcal{A}_0}(\mathcal{A}_k) \to 0.$$

■

From the above theorem we conclude that choosing the regularizing parameter by Morozov's discrepancy principle, defined in (3.21) implies that the regularized solutions weakly converge to a $f_{\mathcal{A}_0}$-minimizing solution of the inverse problem.

Now we shall see in the theorem below that such choice of α as a function of δ implies the convergence rates result of Theorem 9. This is the second main result of this section.

Theorem 13. *Assume that $\mathcal{A}_\alpha^\delta$ is a minimizer of (3.4)and $\alpha = \alpha(\delta, \mathcal{U}^\delta)$ is chosen by Morozov's discrepancy principle (3.21). Then, Theorem 9 holds, i.e., we have the following estimates*

$$\|\mathcal{U}(\mathcal{A}_\alpha^\delta) - \mathcal{U}(\mathcal{A}^\dagger)\| = \mathcal{O}(\delta) \quad \text{and} \quad D_{\xi^\dagger}(\mathcal{A}_\alpha^\delta, \mathcal{A}^\dagger) = \mathcal{O}(\delta), \tag{3.31}$$

with $\xi^\dagger \in \partial f_{\mathcal{A}_0}(\mathcal{A}^\dagger)$.

Proof: Assume that $\mathcal{A}^\dagger$ is an $f_{\mathcal{A}_0}$-minimizing solution of (3.1) and $\mathcal{A}_\alpha^\delta \in M_\alpha$. The first estimate is trivial as

$$\|\mathcal{U}(\mathcal{A}_\alpha^\delta) - \mathcal{U}(\mathcal{A}^\dagger)\| \leq \|\mathcal{U}(\mathcal{A}_\alpha^\delta) - \mathcal{U}^\delta\| + \|\mathcal{U}^\delta - \mathcal{U}(\mathcal{A}^\dagger)\| \leq (\tau_2 + 1)\delta.$$

By (3.21) it follows that

$$\begin{aligned} \tau_1\delta^2 + \alpha f_{\mathcal{A}_0}(\mathcal{A}_\alpha^\delta) &\leq \|\mathcal{U}(\mathcal{A}_\alpha^\delta) - \mathcal{U}^\delta\|^2 + \alpha f_{\mathcal{A}_0}(\mathcal{A}_\alpha^\delta) \\ &\leq \|\mathcal{U}(\mathcal{A}^\dagger) - \mathcal{U}^\delta\|^2 + \alpha f_{\mathcal{A}_0}(\mathcal{A}^\dagger) \quad \leq \quad \delta^2 + \alpha f_{\mathcal{A}_0}(\mathcal{A}^\dagger), \end{aligned}$$

implying that $f_{\mathcal{A}_0}(\mathcal{A}_\alpha^\delta) \leq f_{\mathcal{A}_0}(\mathcal{A}^\dagger)$ as $\tau_1 - 1 > 0$. Hence, by the definition of Bregman distance we have, for every $\xi^\dagger \in \partial f_{\mathcal{A}_0}(\mathcal{A}^\dagger)$ satisfying the source condition,

$$\begin{aligned} D_{\xi^\dagger}(\mathcal{A}_\alpha^\delta, \mathcal{A}^\dagger) &= f_{\mathcal{A}_0}(\mathcal{A}_\alpha^\delta) - f_{\mathcal{A}_0}(\mathcal{A}^\dagger) - \langle \xi^\dagger, \mathcal{A}_\alpha^\delta - \mathcal{A}^\dagger \rangle \\ &\leq |\langle \xi^\dagger, \mathcal{A}_\alpha^\delta - \mathcal{A}^\dagger \rangle| = |\langle \mathcal{U}'(\mathcal{A}^\dagger)^*\omega^\dagger + \mathcal{E}, \mathcal{A}_\alpha^\delta - \mathcal{A}^\dagger \rangle| \\ &\leq \|\omega^\dagger\| \|\mathcal{U}'(\mathcal{A}^\dagger)(\mathcal{A}_\alpha^\delta - \mathcal{A}^\dagger)\| + \|\mathcal{E}\| \|\mathcal{A}_\alpha^\delta - \mathcal{A}^\dagger\| \\ &\leq (1+\gamma)\|\omega^\dagger\| \|\mathcal{U}(\mathcal{A}_\alpha^\delta) - \mathcal{U}(\mathcal{A}^\dagger)\| + \frac{1}{\zeta}\|\mathcal{E}\| D_{\xi^\dagger}(\mathcal{A}_\alpha^\delta, \mathcal{A}^\dagger) \end{aligned} \tag{3.32}$$

The last inequality follows by the tangential cone condition and the 1-coerciveness with constant ζ of $f_{\mathcal{A}_0}$. As $\xi^\dagger$ can be chosen with $\|\mathcal{E}\|$ arbitrarily small, it follows that

$$1 - \frac{1}{\zeta}\|\mathcal{E}\| > 0$$

and then, by (3.21)

$$D_{\xi^\dagger}(\mathcal{A}_\alpha^\delta, \mathcal{A}^\dagger) \leq \frac{\zeta}{\zeta - \|\mathcal{E}\|}(1+\gamma)\|\omega^\dagger\|\|\mathcal{U}(\mathcal{A}_\alpha^\delta) - \mathcal{U}(\mathcal{A}^\dagger)\| \leq \tau_2 \frac{\zeta}{\zeta - \|\mathcal{E}\|}(1+\gamma)\|\omega^\dagger\| \cdot \delta.$$

■

Remark 4. *For $f_{\mathcal{A}_0}$ q-coercive with $q > 1$, a reasoning as the one used in (3.32), gives that*

$$\begin{aligned} D_{\xi^\dagger}(\mathcal{A}_\alpha^\delta, \mathcal{A}^\dagger) &\leq \beta_1 (D_{\xi^\dagger}(\mathcal{A}_\alpha^\delta, \mathcal{A}^\dagger))^{1/q} + \beta_2 \|\mathcal{U}(\mathcal{A}_\alpha^\delta) - \mathcal{U}(\mathcal{A}^\dagger)\| \\ &\leq \frac{\beta_1^q}{q} + \frac{1}{q} D_{\xi^\dagger}(\mathcal{A}_\alpha^\delta, \alpha) + \beta_2 \|\mathcal{U}(\mathcal{A}_\alpha^\delta) - \mathcal{U}(\mathcal{A}^\dagger)\|. \end{aligned}$$

Assume further that $\beta_1 = \mathcal{O}(\delta^{\frac{1}{q}})$. As

$$\|\mathcal{U}(\mathcal{A}_\alpha^\delta) - \mathcal{U}(\mathcal{A}^\dagger)\| = \mathcal{O}(\delta)$$

it follows that

$$\|\mathcal{A}_\alpha^\delta - \mathcal{A}^\dagger\|^q \leq \frac{1}{\zeta} D_\xi(\mathcal{A}_\alpha^\delta, \mathcal{A}^\dagger) = \mathcal{O}(\delta).$$

Therefore we can say that Morozov's discrepancy principle is a very robust way to finding α as a function of δ since such procedure fulfills all the hypotheses that concern the existence, stability, convergence and convergence rates results. Furthermore, it can be easily implemented as a numerical code.

3.6 A Discretized Approach

In this section we shall present a way to quantify uncertainties on data and model through a discretized approach for Tikhonov regularization. Such framework is of interest since it allows us to separate how each source of uncertainty corrupt solutions.

The main reference for this analysis is [44].

3.6.1 The Discrete Forward Operator

In what follows, to simplify notation, we denote $H^l(0, S, H^{1+\varepsilon}(D))$ by U and $L^2(0, S, L^2(D))$ by V. Thus, we take a nested sequence $\{U_n\}_{n \in \mathbb{N}}$ of finite dimensional subspaces of U such that its union is dense in U, i.e.,

$$\overline{\cup_{n \in \mathbb{N}} U_n} = U,$$

Define the finite dimensional sub-domains

$$Q_n := Q \cap U_n \neq \emptyset, \text{ for every } n \in \mathbb{N}.$$

An example of orthonormal basis generating a sequence of finite dimensional subspaces U_n is a finite element basis of bilinear functions in $H^1(D)$. Such example is of interest as the number of elements is set equal to the number of observed points in the price surface, for each day.

Now we present the discretized versions of the direct operators of Chapter 2. For $m, N \in \mathbb{N}$, take a partition $0 = t_0 < t_1 < \ldots < t_N = T$ of the interval $[0, T]$. Thus we define the approximate operators

$$\begin{array}{rcl} F_m : [0,T] \times Q & \longrightarrow & H^{1,2} \\ (t,a) & \longmapsto & F_m(t,a) \end{array} \tag{3.33}$$

and

$$\begin{array}{rcl} \mathcal{U}_N^m : \mathfrak{Q} & \longrightarrow & L^2(0,S,W_2^{1,2}(D)) \\ \mathcal{A} & \longmapsto & \left\{ t \mapsto \sum_{j=1}^{N} F_m(t_j, a(t_j)) \mathcal{X}_{(t_{j-1},t_j]}(t) \right\}. \end{array} \tag{3.34}$$

Note that, $F_m(\cdot, \cdot\cdot)$ and $\mathcal{U}_N^m(\cdot)$ approximate $F(\cdot, \cdot\cdot)$ and $\mathcal{U}(\cdot)$ from Chapter 2, respectively. Moreover, we assume that F_m is continuous.

In addition, we assume the estimates

$$\|F_m(t,a) - F(t,a)\| \leq \varepsilon_m \quad \text{for every } (t,a) \in [0,T] \times Q \text{ and} \tag{3.35}$$

$$\|\mathcal{U}_N^m(\mathcal{A}) - \mathcal{U}(\mathcal{A})\| \leq \rho_N^m \quad \text{for every } \mathcal{A} \in \mathfrak{Q}. \tag{3.36}$$

where the constants ρ_N^m and ε_m depend only on (m, N) and m, respectively, with $\lim_{m,N\to\infty} \rho_N^m = 0$ and $\lim_{m\to\infty} \varepsilon_m = 0$. In Chapter 5 we shall choose a Crank-Nicholson scheme to approximate the operator $F(\cdot, \cdot\cdot)$.

3.6.2 Discrete Tikhonov Regularization

Now we are concerned with finding a minimizer for the Tikhonov functional

$$\begin{aligned} \mathcal{F}^{m,\alpha}_{\mathcal{U}^\delta, a_0}(\mathcal{A}) &= \|\mathcal{U}_N^m(\mathcal{A}) - \mathcal{U}^\delta\|^2_{L^2(0,S,L^2(D))} + \alpha f_{\mathcal{A}_0}(\mathcal{A}) \\ &= \sum_{j=1}^{N} \|F_m(t_j, a(t_j)) - u^\delta(t_j)\|^2_{L^2(D)} \cdot (t_j - t_{j-1}) + \alpha f_{\mathcal{A}_0}(\mathcal{A}) \end{aligned} \tag{3.37}$$

subject to $\mathcal{A} \in \mathfrak{Q}_n$. In order to establish the analysis of this inverse problem, we first make the following assumptions.

Assumption 1. *(a) For every $\mathcal{A} \in U$ there exists a sequence $\{\mathcal{A}_n\}_{n\in\mathbb{N}} \subset U$, with $\mathcal{A}_n \in U_n$ for every $n \in \mathbb{N}$ and $\mathcal{A}_n \to \mathcal{A}$.*

(b) The functional $f_{\mathcal{A}_0}(\cdot)$ is weakly lower semi-continuous and coercive.

Lemma 11. *1. For every $m, N \in \mathbb{N}$ the operators $\mathcal{U}_N^m(\cdot)$ and $F_m(\cdot, \cdot\cdot)$ are weakly closed.*

2. For every $M > 0$, $\alpha > 0$ and every $m, n \in \mathbb{N}$, the level sets

$$\mu^m_{\alpha,N}(M) = \left\{ \mathcal{A} \in \mathfrak{Q} \,\middle|\, \sum_{j=1}^{N} \|F(t_j, \mathcal{A}(t_j))\|^2 + \alpha f_{a_0}(\mathcal{A}) \leq M \right\}$$

and

$$\nu^{m,n}_{\alpha,N}(M) = \left\{ \mathcal{A} \in \mathfrak{Q}_n \,\middle|\, \sum_{j=1}^{N} \|F_m(t_j, \mathcal{A}(t_j))\|^2 + \alpha f_{a_0}(\mathcal{A}) \leq M \right\}$$

are weakly pre-compact.

Proof: As the operators $\mathcal{U}(\cdot)$ and $F(\cdot, \cdot\cdot)$ are weakly continuous and closed (see Section 2), the first item follows by the reflexiveness of the spaces $H^{1+\varepsilon}(D)$, $L^2(D)$ and $L^2(0, S, L^2(D))$. As $\mathfrak{Q}$ and $\mathfrak{Q}_n$ are weakly closed, by the coerciveness and weakly lower semi-continuity of f_{a_0} the second item follows by the first one. ■

Now we are ready to state the discretized versions of the theorems of Chapter 3.

Theorem 14 (Existence). *For $m, n, N \in \mathbb{N}$ and $\alpha, \delta > 0$ fixed, assume valid the Assumption 1 and the estimates (3.35) and (3.36). Then there exists a minimizer for the functional (3.37).*

Theorem 15 (Stability). *Under the same hypothesis of Theorem 14, the minimizers of (3.37) are stable, i.e., if the sequence $\{\mathcal{U}_k\}_{k\in\mathbb{N}}$ converges strongly to $\mathcal{U}^\delta$, then a sequence $\{\mathcal{A}_k\}_{k\in\mathbb{N}}$ of minimizers of (3.37) with $\mathcal{U}^\delta$ replaced by $\mathcal{U}_k$, has a subsequence $\{\mathcal{A}_{k_l}\}_{l\in\mathbb{N}}$ converging weakly to $\mathcal{A}^*$, a minimizer of (3.37) with $\lim\limits_{l\to\infty} f_{a_0}(\mathcal{A}_{k_l}) = f_{a_0}(\mathcal{A}^*)$.*

The proofs of the theorems above are the same of Proposition 2.3 of [44] and follows almost directly from the Assumption 1 and the estimates (3.36), and (3.35). The following result is more delicate, as it asserts convergence of regularized solutions to a $f_{\mathcal{A}_0}$-minimizing solution of the inverse problem (3.1) when $\delta \to 0$ and $N, m, n \to \infty$. Thus, we need an auxiliary lemma.

Lemma 12. *Assume that Problem (3.1) has an $f_{\mathcal{A}_0}$-minimizing solution $\mathcal{A}^\dagger$ in the interior of $\mathfrak{Q}$. Furthermore, there is $r > 0$ small enough, such that, the open ball $B(\mathcal{A}^\dagger, r) \subset \mathfrak{Q}$. Then, there exists a sequence $\{\mathcal{A}_n\}_{n\in\mathbb{N}} \subset \mathfrak{Q}$ where, for n sufficiently large, $\mathcal{A}_n \in \mathfrak{Q}_n \cap B(\mathcal{A}^\dagger, r)$, $\|\mathcal{A}_n - \mathcal{A}^\dagger\| \to 0$ and $f_{\mathcal{A}_0}(\mathcal{A}_n) \to f_{\mathcal{A}_0}(\mathcal{A}^\dagger)$ as $n \to \infty$.*

Proof: The functional $f_{a_0}(\cdot)$ is convex and weakly lower semi-continuous. Furthermore, $\mathfrak{Q}$ is convex and closed. Thus, $f_{\mathcal{A}_0}(\cdot)$ is in fact lower semi-continuous (see [25], Corollary 2.2). By Corollary 2.5 of [25], $f_{a_0}(\cdot)$ is in fact continuous in the interior of Q. Thus, take the sequence $\{\mathcal{A}_n\}_{n\in\mathbb{N}}$ of Assumption 1. By definition we have $\mathcal{A}_n \to \mathcal{A}^\dagger$. Hence, for n sufficiently large, $\mathcal{A}_n \in U_n \cap B(\mathcal{A}^\dagger, r) \subset \mathfrak{Q}_n$. As $f_{\mathcal{A}_0}(\cdot)$ is continuous in the interior of $\mathfrak{Q}$, we have that $f_{\mathcal{A}_0}(\mathcal{A}_n) \to f_{\mathcal{A}_0}(\mathcal{A}^\dagger)$. ■

Theorem 16 (Convergence). *Let Assumption 1 be satisfied. Furthermore, assume that:*

1. The parameter $\alpha = \alpha(\delta, m, N, n) > 0$ is such that

$$\lim_{\substack{\delta\to 0\\ m,N,n\to\infty}} \alpha(\delta, m, N, n) = 0 \quad \textit{and} \quad \lim_{\substack{\delta\to 0\\ m,N,n\to\infty}} \frac{(\max\{\delta, \rho^m_N, \lambda_n\})^2}{\alpha(\delta, m, N, n)} = 0, \tag{3.38}$$

where ρ_N^m *is defined in (3.36) and* $\lambda_n = \|\mathcal{U}'(\mathcal{A}^\dagger)(\mathcal{A}_n - \mathcal{A}^\dagger)\|$, *with* $\mathcal{A}_n$ *defined above and* $\lim_{n\to\infty} \gamma_n = 0$.

If (3.35) and (3.36) hold, then every sequence of minimizers of (3.37), $\{\mathcal{A}_k\}_{k\in\mathbb{N}}$ *with* $\mathcal{A}_k = \mathcal{A}_{m_k,N_k,n_k}^{\alpha_k,\delta_k}$ *and* $\alpha_k := \alpha(m_k, N_k, n_k)$, *where* $\delta_k \to 0$, $m_k, N_k, n_k \to \infty$, *has a subsequence* $\{\mathcal{A}_{k_l}\}_{l\in\mathbb{N}}$ *converging weakly to* $\widetilde{\mathcal{A}}$, *an* $f_{\mathcal{A}_0}(\cdot)$*-minimizing solution of (3.2) with* $f_{\mathcal{A}_0}(\mathcal{A}_{k_l}) \to f_{\mathcal{A}_0}(\widetilde{\mathcal{A}})$ *as* $l \to \infty$. *Furthermore, if* $\mathcal{A}^\dagger$ *is the unique solution of (3.1), then the entire sequence* $\{\mathcal{A}_k\}_{k\in\mathbb{N}}$ *converges weakly to* $\mathcal{A}^\dagger$ *with* $f_{\mathcal{A}_0}(\mathcal{A}_k) \to f_{\mathcal{A}_0}(\mathcal{A}^\dagger)$ *as* $k \to \infty$.

Proof: Let $\mathcal{A}_{N,n}^{\delta,m}$ be a minimizer of (3.37), which existence is guaranteed by Theorem 14. Moreover, take $\mathcal{A}^\dagger$ and $\{\mathcal{A}_n\}_{n\in\mathbb{N}}$ as in Lemma 12. Thus, we have

$$\begin{gathered}\|\mathcal{U}_N^m(\mathcal{A}_{N,n}^{\delta,m}) - \mathcal{U}^\delta\|^2 + \alpha f_{\mathcal{A}_0}(\mathcal{A}_{N,n}^{\delta,m}) \le \|\mathcal{U}_N^m(\mathcal{A}_n) - \mathcal{U}^\delta\|^2 + \alpha f_{\mathcal{A}_0}(\mathcal{A}_n) \le \\ \Big(\|\mathcal{U}_N^m(\mathcal{A}_n) - \mathcal{U}(\mathcal{A}_n)\| + \|\mathcal{U}(\mathcal{A}_n) - \mathcal{U}(\mathcal{A}^\dagger)\| + \|\mathcal{U}(\mathcal{A}^\dagger) - \mathcal{U}^\delta\|\Big)^2 + \alpha f_{\mathcal{A}_0}(\mathcal{A}_n) \le \\ \big(\rho_N^m + \|\mathcal{U}(\mathcal{A}_n) - \mathcal{U}(\mathcal{A}^\dagger)\| + \delta\big)^2 + \alpha f_{\mathcal{A}_0}(\mathcal{A}_n).\end{gathered}$$

By the continuity of $\mathcal{U}(\cdot)$, the definition of $\mathcal{A}_n$ and the Tangential Cone Condition of $\mathcal{U}(\cdot)$, we have

$$\begin{aligned}\|\mathcal{U}_N^m(\mathcal{A}_{N,n}^{\delta,m}) - \mathcal{U}^\delta\|^2 + \alpha f_{\mathcal{A}_0}(\mathcal{A}_{N,n}^{\delta,m}) &\le (\rho_N^m + C\lambda_n + \delta)^2 + \alpha f_{\mathcal{A}_0}(\mathcal{A}_n) \\ &\le \mathcal{C}\left(\max\{\rho_N^m, \lambda_n, \delta\}\right)^2 + \alpha f_{\mathcal{A}_0}(\mathcal{A}_n),\end{aligned}$$

where $C := \dfrac{1}{1-\gamma}$ and $\mathcal{C}$ is a suitable constant, independent of m, n, N and α. Thus, we achieve the following inequality,

$$f_{\mathcal{A}_0}(\mathcal{A}_{N,n}^{\delta,m}) \le \mathcal{C}\frac{(\max\{\rho_N^m, \lambda_n, \delta\})^2}{\alpha} + f_{\mathcal{A}_0}(\mathcal{A}_n),$$

which combined with Lemma 12 implies that

$$\limsup_{\substack{\delta\to 0\\ m,N,n\to\infty}} f_{\mathcal{A}_0}(\mathcal{A}_{N,n}^{\delta,m}) \le f_{\mathcal{A}_0}(\mathcal{A}^\dagger) \quad \text{and} \quad \lim_{\substack{\delta\to 0\\ m,N,n\to\infty}} \|\mathcal{U}(\mathcal{A}_{N,n}^{\delta,m}) - \mathcal{U}^\delta\| = 0.$$

Denote $\alpha_k := \alpha(\delta_k, m_k, N_k, n_k)$ and $\mathcal{A}_k := \mathcal{A}_{N_k,n_k}^{\delta_k,m_k}$. Since the sequence $\{\mathcal{F}_{\mathcal{U}^{\delta_k},a_0}^{m_k,\alpha_k}(\mathcal{A}_k)\}_{k\in\mathbb{N}}$ is bounded, by the coerciveness of $f_{\mathcal{A}_0}(\cdot)$, $\{\mathcal{A}_k\}_{k\in\mathbb{N}}$ is bounded. Thus, by Lemma 11 it has a weakly convergent subsequence $\{\mathcal{A}_{k_l}\}_{l\in\mathbb{N}}$, converging to some $\widetilde{\mathcal{A}} \in \mathfrak{Q}$. Therefore, from the weak lower-semi-continuity of $f_{\mathcal{A}_0}(\cdot)$ and the estimate above, we have

$$f_{\mathcal{A}_0}(\widetilde{\mathcal{A}}) \le \liminf_{l\to\infty} f_{\mathcal{A}_0}(\mathcal{A}_{k_l}) \limsup_{l\to\infty} f_{a_0}(\mathcal{A}_{k_l}) \le f_{\mathcal{A}_0}(\mathcal{A}^\dagger).$$

Thus, by the weak continuity of $\mathcal{U}(\cdot)$, and the lower semi-continuity of the norm, $\widetilde{\mathcal{A}}$ is an $f_{\mathcal{A}_0}$-minimizing solution of (3.2). ■

3.6.3 A Convergence Analysis

We are now concerned with the problem of stating how fast the solution obtained using Tikhonov regularization converges to a solution of (3.2), when $\delta \to 0$ and $m, N, n \to \infty$. Another important feature of the following result is that it quantifies how much information is necessary to get a reliable solution.

This leads to the following:

Theorem 17 (Convergence Rates)**.** *Let the hypothesis of Theorem 16, Lemma 12 and Lemma 4 be satisfied. Define* $\gamma_n := \|\mathcal{A}_n - \mathcal{A}^\dagger\|$*, with* $\{\mathcal{A}_n\}_{n\in\mathbb{N}}$ *and* $\mathcal{A}^\dagger$ *as in Lemma 12. If* $\alpha \sim \max\{\delta, \lambda_n, \rho_N^m\}$ *and* $\alpha \cdot \beta_1 < 1$*, with* β_1 *from Lemma 4, then*

$$D_{\xi^\dagger}(\mathcal{A}_{n,N}^{\delta,m}, \mathcal{A}^\dagger) = \mathcal{O}(\delta + \lambda_n + \rho_N^m + \zeta_n).$$

Proof: Take $\{\mathcal{A}_n\}_{n\in\mathbb{N}}$ and $\mathcal{A}^\dagger$ as in Lemma 12, then by the proof of Theorem 16, we have

$$\|\mathcal{U}_N^m(\mathcal{A}_{n,N}^{\delta,m}) - \mathcal{U}^\delta\| + \alpha f_{\mathcal{A}_0}(\mathcal{A}_{n,N}^{\delta,m}) \leq (\rho_N^m + C\lambda_n + \delta)^2 + \alpha f_{\mathcal{A}_0}(\mathcal{A}_n).$$

We denote $\beta_n := (\rho_N^m + C\lambda_n + \delta)^2$. Thus, by the definition of Bregman distance and the 1-coerciveness of $f_{\mathcal{A}_0}(\cdot)$, we have

$$\begin{aligned}
&\|\mathcal{U}_N^m(\mathcal{A}_{n,N}^{\delta,m}) - \mathcal{U}^\delta\|^2 + \alpha D_{\xi^\dagger}(\mathcal{A}_{n,N}^{\delta,m}, \mathcal{A}^\dagger) \leq \beta_n + \alpha\left(f_{\mathcal{A}_0}(\mathcal{A}_n) - f_{\mathcal{A}_0}(\mathcal{A}^\dagger) - \langle \xi^\dagger, \mathcal{A}_{n,N}^{\delta,m}\rangle\right)\\
&= \beta_n + \alpha D_{\xi^\dagger}(\mathcal{A}_n, \mathcal{A}^\dagger) - \alpha\langle \xi^\dagger, \mathcal{A}_{n,N}^{\delta,m} - \mathcal{A}_n\rangle = \beta_n + \alpha\left[\zeta_n - \langle \xi^\dagger, \mathcal{A}_{n,N}^{\delta,m} - \mathcal{A}^\dagger\rangle - \langle \xi^\dagger, \mathcal{A}^\dagger - \mathcal{A}_n\rangle\right]\\
&\leq \beta_n + \alpha\left[\zeta_n + \beta_1 D_{\xi^\dagger}(\mathcal{A}_{n,N}^{\delta,m}, \mathcal{A}^\dagger) + \beta_1 D_{\xi^\dagger}(\mathcal{A}_n, \mathcal{A}^\dagger)\right] = \beta_n + \alpha(1+\beta_1)\zeta_n + \alpha\beta_1 D_{\xi^\dagger}(\mathcal{A}_{n,N}^{\delta,m}, \mathcal{A}^\dagger).
\end{aligned}$$

The last inequality follows from the first inequality of (3.8) in the proof of Lemma 4. Thus,

$$\|\mathcal{U}(\mathcal{A}_{n,N}^{\delta,m}) - \mathcal{U}^\delta\|^2 + \alpha(1-\beta_1) D_{\xi^\dagger}(\mathcal{A}_{n,N}^{\delta,m}, \mathcal{A}^\dagger) \leq \beta_n + \alpha(1+\beta_1)\zeta_n.$$

Therefore, if $\alpha \sim \max\{\rho_N^m, \lambda_n, \delta\}$, then

$$D_{\xi^\dagger}(\mathcal{A}_{n,N}^{\delta,m}, \mathcal{A}^\dagger) = \mathcal{O}(\rho_N^m + \lambda_n + \delta + \zeta_n)$$

■

Thus, we have established Theorem 17 which states convergence rates accounting separately different sources of uncertainties. It is important since we can now measure separately how much a given discretization method affects the resulting volatility surfaces.

Chapter 4

Applications to Commodity Markets

Commodity futures and their derivatives have become key players in the portfolios of many corporations, especially those in the energy sector. In order to apply the regularization techniques presented in the past chapters we have to characterize the direct problem by making assumptions in the dynamics of commodity future contract prices and its derivatives. We shall see that such assumptions lead us to a Dupire's equation [21] and a generalization of Black's model [7] for European Vanilla options on futures.

4.1 Risk-Neutral Pricing of Commodity Future Contracts and Another Derivatives

In this section we characterize the so called *Direct Problem (DP)*. It defines the price of European vanilla options on a future contract as a function of the local volatility surface of such futures. Thus, we first define how the risk-neutral prices of a future contract evolves with time. Then, since volatilities are assumed deterministic functions, we construct a generalization of Black's equation (see [7]) and an application of Dupire's equation for futures. Such models are used to define the forward operators. We further present some of background properties of such operators that are necessary to state Tikhonov analysis.

4.1.1 Pricing Commodity Futures

Under the classical Black-Scholes framework, F. Black in [7] has shown that the risk-neutral prices of futures contracts, with a fixed maturity, are martingales. Another related approach is presented in [33], where the authors start from the hypothesis that future prices are martingales. The authors use a change of numeraire, which implies that under this new currency, the price of a futures contract is modeled as a zero-coupon bond following the Heath-Jarrow-Morton (HJM) model [30]. In other words, the *commodity market* defined by future prices (which are described by a martingale) is equivalent to a *money market* defined by a numeraire and a zero-coupon bond. Note that, these results were achieved for electricity markets, however they can be easily extended to oil and gas markets.

Here we assume that volatilities are deterministic functions of the current time t, maturity

T and current future prices $F_{t,T}$. In what follows, we assume further that the risk-neutral prices of future contracts are positive-valued martingales, i.e., for each fixed maturity T we shall have that

$$\frac{dF_{t,T}}{F_{t,T}} = \sigma(t,T,F_{t,T})dW(t) \text{ for } 0 < t \leq T < \infty, \text{ with } F_{t,t} = S(t) \forall\ t. \tag{4.1}$$

with $\sigma(t,T,f)$ in a admissible class of deterministic functions. Note that, Equation (4.1) implies that $F_{t,T}$ is a martingale. Moreover, (4.1) is an infinite family of stochastic differential equations parameterized by maturity T. Existence results for such families have been established by Pilipovic [43].

4.1.2 Pricing European Vanilla Options on Commodity Futures

We assume that the risk-neutral prices of a future contract $F(t,T)$ follows (4.1) with

$$0 \leq t \leq T \leq \tilde{T}$$

and $\tilde{T}$ fixed. We shall present a PDE approach to price an European vanilla option

$$P = P(t, T', F_{t,T}, K)$$

at time t written on $F_{t,T}$, with maturity $T' \leq T$, strike K and payoff $P(F_{T',T}, K)$.

Following the classical portfolio method used to find Black-Scholes equation (see [8]), assume that, for instance, $P = P(t,f)$ is of class $C^{1,2}$. Thus, by Itô's formula [37], it follows that

$$\begin{aligned} dP_t &= \frac{\partial P}{\partial t}dt + \frac{\partial P}{\partial f}dF_{t,T} + \frac{1}{2}\frac{\partial^2 P}{\partial f^2}\sigma^2(t,T,F_{t,T})F_{t,T}^2 dt \\ &= \left(\frac{\partial P}{\partial t} + \frac{1}{2}\frac{\partial^2 P}{\partial f^2}\sigma^2(t,T,F_{t,T})F_{t,T}^2\right)dt + \sigma(t,T,F_{t,T})F_{t,T}\frac{\partial P}{\partial f}dW. \end{aligned}$$

Hence, defining a portfolio Π with a long position in Δ shares of $F_{t,T}$ and selling an option $P(t, F_{t,T})$, we have

$$\Pi_t = \Delta F_{t,T} - P(t, F_{t,T}). \tag{4.2}$$

By the self financing assumption we have,

$$\begin{aligned} d\Pi_t &= \Delta dF_{t,T} - dP(t,F_{t,T}) \\ &= \Delta dF_{t,T} - \left(\frac{\partial P}{\partial t} + \frac{1}{2}\frac{\partial^2 P}{\partial f^2}\sigma^2(t,T,F_{t,T})F_{t,T}^2\right)dt - \frac{\partial P}{\partial f}dF_{t,T} \qquad (4.3) \\ &= r\Pi_t dt, \end{aligned}$$

where the last equality holds only by the delta-hedging assumption $\Delta = \partial P/\partial f$.

Since the long position at the Futures contract has no cash flow, it follows that $d\Pi_t =$

$-rP(t, F_{t,T})dt$. Thus, we achieve the following PDE, known as Black's equation (see [7]),

$$\frac{\partial P}{\partial t}(t,f) + \frac{1}{2}\sigma^2(t,T,f)f^2\frac{\partial^2 P}{\partial f^2} = rP(t,f) \text{ with } 0 \leq t \leq T' \leq T,\ f > 0, \tag{4.4}$$

with terminal condition

$$P(T', f) = g(K, f). \tag{4.5}$$

Since σ depends explicitly on the maturity T of the future contract, it follows that (4.4)-(4.5) form an infinite family of Terminal Value Problems parameterized by $T \in [0, \tilde{T}]$.

Making the change of variables $\tau = T' - t$ and $y = \log(f/K)$. We define further that $a(\tau, T, y) := \frac{1}{2}\sigma^2(T' - \tau, T, K\mathrm{e}^y)$ and $v(\tau, y) := P(T' - \tau, K\mathrm{e}^y)$, with $\tau > 0$ and $y \in I \subset \mathbb{R}$, an open interval, not necessarily bounded. Hence, by (4.4)-(4.5) it follows that $v(\tau, y)$ satisfies

$$\begin{cases} \dfrac{\partial v}{\partial \tau}(\tau, y) &= a(\tau, T, y)\left(\dfrac{\partial^2 v}{\partial y^2}(\tau, y) - \dfrac{\partial v}{\partial y}(\tau, y)\right), \text{ for } \tau \in (0, +\infty) \text{ and } y \in I \\ v(\tau = 0, y) &= g(K, K\mathrm{e}^y), \text{ for } y \in I. \end{cases} \tag{4.6}$$

For each $T \in [0, \tilde{T}]$ and a in an admissible class, we shall see that (4.6), with initial condition $g(K, K\mathrm{e}^y) = K(1 - \mathrm{e}^y)^+$, has a unique solution denoted by $v(T, a)$ in $W^{1,2}_{2,loc}(D)$, with $D := (0, \infty) \times I$. For a given a_0 in the same class of a, we shall see further that $u(T, a) - u(T, a_0) \in W^{1,2}_2(D)$.

Dupire's Equation for Futures

We present now another important approach to price European vanilla options. It is an application of Dupire's Equation to European call options on Futures.

Firstly, we fix the current time t, the maturity of the future contract $T \leq \tilde{T}$ and the current price of the Futures $F_{t,T}$. Then, we let the price of an European call option $C(t, T', F_{t,T}, K)$ vary with strike $K > 0$ and option's maturity $0 \leq T' \leq \tilde{T}$. We now follow the same ideas presented in [21] to find Dupire's equation for European call options on spot prices. Recall that call prices are "discounted" expected values of the payoff $(F_{0,T} - K)^+$. Hence, assuming further that the distribution of $F_{T',T}$ is absolutely continuous with respect to Lebesgue's measure, it follows that

$$\begin{aligned} C(T, F_{0,T}, T', K) &= \mathbb{E}[(F_{T',T} - K)^+ | F_{0,T}] \\ &= \int_K^\infty (F_{T',T} - K)\pi(T, F_{T',T}, T', F_{0,T})dF_{T',T} \end{aligned} \tag{4.7}$$

where $\pi(T, F_{T',T}, T', F_{0,T})$ is the density of $F_{T',T}$ given that, at $t = 0$, the price of the future contract is $F_{0,T}$.

For each fixed T, if $F_{t,T}$ follows (4.1), then the density π satisfy the Fokker-Planck equation

$$\frac{\partial \pi}{\partial T'} = \frac{1}{2}\frac{\partial^2}{\partial F^2_{T',T}}\left(\sigma^2 F^2_{T',T}\pi\right). \tag{4.8}$$

Differentiating C twice with respect to K, from (4.7) we have

$$\frac{\partial^2 C}{\partial K^2}(T', K) = \pi(T, K, T', F_{0,T}).$$

Now, differentiating C with respect to T' in (4.7), integrating by parts and using the equality above we find

$$\frac{\partial C}{\partial T'} = \frac{1}{2}\sigma^2 K^2 \frac{\partial^2 C}{\partial K^2}.$$

Note that we have omitted the dependence of C and σ on T, T' and K.

Therefore, we have the following Initial Value Problem,

$$\begin{cases} \dfrac{\partial C}{\partial T'} & = & \dfrac{1}{2}\sigma^2(T', T, K)K^2\dfrac{\partial^2 C}{\partial K^2}, \text{ for } 0 < T' \leq T \text{ and } K > 0, \\ C(T' = 0, K) & = & (F_{0,T} - K)^+, \text{ for } K > 0. \end{cases} \tag{4.9}$$

Note that, the term of first derivative disappeared as it is originated by the drift term which from (4.1) is zero. The same equation was obtained for a forward contract in the first chapter of [28].

Since future contracts are derivative contracts, we could assume that instead of σ being dependent on the maturity of the future contract, it could depend on the current commodity's spot price $S(0)$, even though it is unknown, in general.

Hence, making the change of variables $y = \log(K/F_{0,T})$, with $y \in I \subset \mathbb{R}$ an open interval, we define $V(T', y) := C(T', F_{0,T}\mathrm{e}^y)/F_{0,T}$ and $a(S(0), T', y) := \frac{1}{2}\sigma^2(S(0), T', F_{0,T}\mathrm{e}^y)$. Thus, $V(T', y)$ satisfies

$$\begin{cases} \dfrac{\partial V}{\partial T'}(T', y) & = & a(S(0), T', y)\left(\dfrac{\partial^2 V}{\partial y^2}(T', y) - \dfrac{\partial V}{\partial y}(T', y)\right), \text{ for } 0 < T' \leq T \text{ and } y \in I, \\ V(T' = 0, y) & = & (1 - \mathrm{e}^y)^+, \text{ for } y \in I, \end{cases} \tag{4.10}$$

which is the normalized Dupire's equation for European call options on spot prices. Note that (4.10) does not depend on $F_{0,T}$. Therefore, this is an appealing approach as it permits us to assume that options on future contracts with different maturities satisfy the same equation (4.10), up to a normalization and a change of variables.

Figure 4.1 presents the resulting European call option prices surface on Brent oil (WTI) futures. Note that such call prices are divided by the current commodity future price. Note also that it is very similar to a surface of European call prices on spot prices. Figure 4.2 presents the implied volatility surface obtained from this renormalized European call option prices of Figure 4.1.

Such result permits us to directly apply the techniques developed for options on spot prices, in particular the theory of local volatility calibration presented in [17, 19, 23] and Chapter 3. In Section 5.3.4 we shall present some numerical tests with local volatility calibration under this framework.

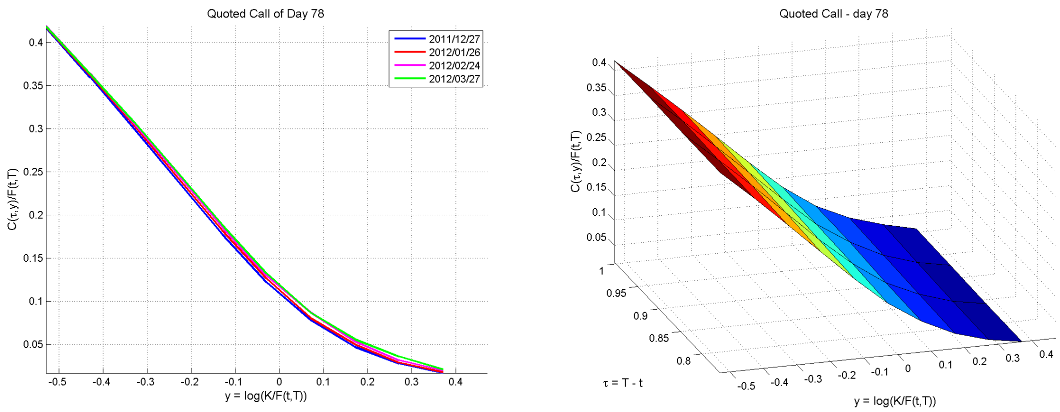

Figure 4.1: Normalized European call option prices on Futures of Brent oil (WTI) traded on 2011/03/16, where for each option's maturity T', the call prices $C(t,T,T')$ are divided by the current futures price $F(t,T)$

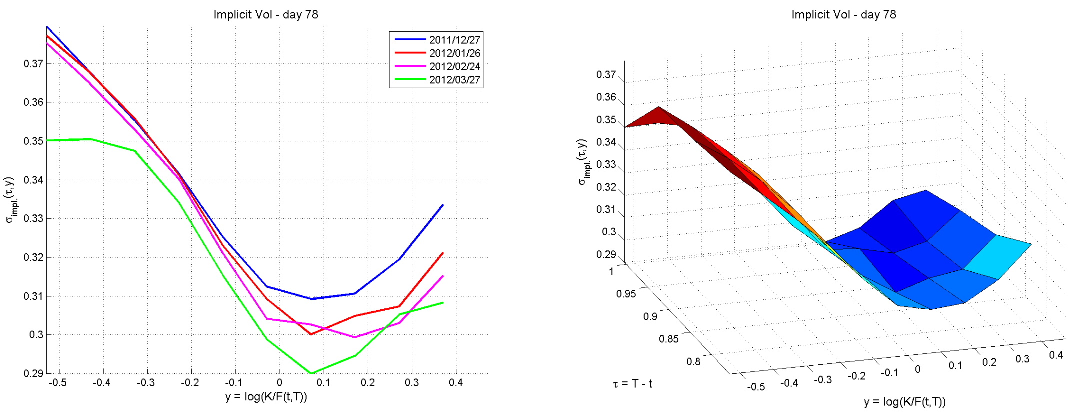

Figure 4.2: Implied volatility obtained from European call prices on Futures of Brent oil (WTI) traded on 2011/03/16 and presented in Figure 4.1.

4.1.3 Forward Operators

In what follows we shall apply part of the theory developed in [17, 19, 23] and in Chapter 3 on the inverse problem of volatility calibration of European vanilla options on Future prices.

Such analysis can be performed by using both problems, (4.6) or (4.10). Each one has intrinsic advantages. The first one can be used to evaluate the dependence of volatility on Futures' maturity whereas the second one can show us how volatility's shape evolve as the underlying commodity's spot price change.

Thus, we shall precisely determine the forward operators related to each problem and then some of its relevant properties necessary to make the subsequent analysis. Some of these properties are weak continuity, compactness and Frechét differentiability.

Black's Forward Operator for Futures

In this section we are first concerned with stating a precise direct problem defined by (4.6). Thus, given $\tilde{T} \geq 0$, an upper bound for maturities of future contracts, take $T \in [0, \tilde{T}]$ as the variable representing the expiration of a future contract. The option's maturity T' is always in $[0, T]$ and the strike price is denoted by $K > 0$. We define $D^1 := [0, \tilde{T}] \times I$ the set where problem (4.6) is stated. Thus, take fixed positive constants $\underline{a}$ and $\overline{a}$ such that $0 < \underline{a} \leq \overline{a} < \infty$. Furthermore, chose $a_0 \in H^{1+\varepsilon}(D^1)$ such that $\underline{a} \leq a_0 \leq \overline{a}$. Now define the set

$$Q^1 := \left\{a \in a_0 + H^{1+\varepsilon}(D^1)\ :\ \underline{a} \leq a \leq \overline{a}\right\}. \tag{4.11}$$

From Chapter 1 we have that Q^1 is convex and it has a nonempty interior when $0 < \varepsilon \leq 1$.

For $T' \leq T$ and $K > 0$ fixed, we denote $t = T - T'$ and $\hat{T} = \tilde{T} - T'$, hence $t \in [0, \hat{T}]$. Therefore, taking $\mathcal{A} \in H^s(0, \hat{T}, H^{1+\varepsilon}(D))$ with $s > 1/2$, we define the set

$$\mathfrak{Q}^1 := \left\{\mathcal{A} \in H^s(0, \hat{T}, H^{1+\varepsilon}(D^1))\ :\ a(t) \in\ Q^1\ \forall\, t\ \in [0, \hat{T}]\right\}, \tag{4.12}$$

which shall be the domain of the forward operator related to (4.6).

The following result shows the existence of a unique solution for problem (4.6) for an European Put option. This result follows almost directly from Proposition A1 of [23], which is based on Theorem IV 9.2 of [38].

Theorem 18. *If $a \in Q^1$ then there exists an unique solution $v \in W^{1,2}_{2,loc}(D^1)$ for problem (4.6) with initial condition $P(K, e^y) = K(1 - e^y)^+$. Furthermore, it satisfies*

$$|v| \leq K \ \text{and}\ \left\|\frac{\partial v}{\partial y}\right\|_{W^{0,1}_2(D^1)} \leq C, \tag{4.13}$$

with C depending on the bounds of the norm of a.

Proof: Observe that, when O defines one European put option we have that $P(K, \mathrm{e}^y) = (K - e^y)^+$ in (4.6). Thus, if we define the function

$$\overline{v}(\tau, y) = \frac{1}{\sqrt{4\pi\tau}} \int_{-\infty}^{\log K} \mathrm{e}^{-\frac{(y-x)^2}{4\tau}} K(1 - \mathrm{e}^x)dx,$$

which, for each (τ, y), is bounded in modulus by K and we have that $\overline{v}$ is an element of $W^{1,2}_{2,loc}(D^1)$. Now, assume formally that, v is a solution of (4.6), thus $w := v - \overline{v}$ satisfies

$$-\frac{\partial w}{\partial \tau} + a\left(\frac{\partial^2 w}{\partial y^2} - \frac{\partial w}{\partial y}\right) = \frac{\partial \overline{v}}{\partial \tau} - a\left(\frac{\partial^2 \overline{v}}{\partial y^2} - \frac{\partial \overline{v}}{\partial y}\right) \tag{4.14}$$

with homogeneous boundary conditions. Note that, the right side of (4.14) defines an element of $L^2(D^1)$. Thus, Proposition A1 of [23] says that there exists a unique function w satisfying (4.14) with homogeneous boundary conditions and the estimate

$$\|w\|_{W^{1,2}_2(D^1)} \leq \left\|\frac{\partial \overline{v}}{\partial \tau} - a\left(\frac{\partial^2 \overline{v}}{\partial y^2} - \frac{\partial \overline{v}}{\partial y}\right)\right\|_{L^2(D^1)}. \tag{4.15}$$

From (4.15) we have the estimate (4.13). ■

Note that, from Put-Call parity and Theorem 18, we have automatically the existence of Call prices. Having stated the existence of solution, proceeding as in Chapter 2, we define now that forward operators related to (4.6): Thus, we define the forward operators related to Equation (4.6).

$$\begin{array}{rcl} G_1 : Q^1 \subset H^{1+\varepsilon}(D^1) & \longrightarrow & W_2^{1,2}(D^1) \\ a \in Q^1 & \longmapsto & v(a) - v(a_0), \end{array} \tag{4.16}$$

$$\begin{array}{rcl} F_1 : [0, \hat{T}] \times Q^1 \subset [0, \hat{T}] \times H^{1+\varepsilon}(D) & \longrightarrow & W_2^{1,2}(D^1) \\ (t, a) \in [0, \hat{T}] \times Q^1 & \longmapsto & v(t, a) - v(t, a_0), \end{array} \tag{4.17}$$

with $v(t, a)$ solution of (4.6) with futures maturing at $T = t + T'$ and

$$\begin{array}{rcl} \mathcal{U}_1 : \mathfrak{Q}^1 \subset H^s(0, \hat{T}, H^{1+\varepsilon}(D^1)) & \longrightarrow & L^2(0, \hat{T}, W_2^{1,2}(D^1)) \\ \mathcal{A} \in \mathfrak{Q}^1 & \longmapsto & \mathcal{U} = \{t \mapsto F_1(t, a(t)) = v(t, a(t)) - v(t, a_0)\}. \end{array} \tag{4.18}$$

Note that, as in (4.11) $a_0 \in Q^1$ is fixed. Furthermore, for each $t \in [0, \hat{T}]$, $F_1(t, \cdot) = G_1(\cdot)$.

It follows that for each futures' maturity T, the problem (4.6) for an European put is a particular case of (1.4). Just take $r = 0$ and $S_0 = K$ in (1.4) to obtain (4.6). It permits us to directly apply all the results obtained for the forward operator of Chapter 1 and [17, 19, 23] to the operator $G_1(\cdot)$ from (4.16). Such results are summarized in the following theorem:

Theorem 19. *The operator G_1 defined in (4.16), has the following properties:*

(i) G_1 is compact, weakly (sequentially) continuous and weakly closed (Proposition A.2 and A.3 of [23]).

(ii) G_1 is continuous and admits a one sided derivative at each $a \in Q^1$, in direction h such that $a + h \in Q^1$ (Proposition 4.1 of [23]).

(iii) $G_1'(a)$ can be extended to a bounded linear operator on $H^{1+\varepsilon}(D^1)$ (Proposition 4.1 of [23]).

(iv) $G_1'(\cdot)$ is Lipschitz continuous in Q^1, injective and compact (Proposition 4.1 of [23] and Lemma 1.4.1 of [19]).

(v) The operator $G_1'(a^\dagger)^*$ has a trivial kernel (Lemma 1.4.2 of [19]).

(vi) G_1 satisfies the local tangential cone condition

$$\|G_1(a) - G_1(\tilde{a}) - G_1'(\tilde{a})(a - \tilde{a})\|_{W_2^{1,2}(D^1)} \leq \gamma \|G_1(a) - G_1(\tilde{a})\|_{W_2^{1,2} D^1}, \text{ with } \gamma < \frac{1}{2}, \tag{4.19}$$

for all $a, \tilde{a}$ in an open ball $B(a^*, \rho) \subset Q^1$ with some $\rho > 0$ (Theorem 1.4.2 of [19]).

Now, by proceeding as above, we apply some results from Chapter 2 to the operators $F_1(\cdot, \cdot\cdot)$ and $\mathcal{U}_1(\cdot)$. We summarize such results in the next theorem. Before stating the theorem, we note that from Proposition 7 the set $\mathfrak{Q}^1$ is weakly closed and has a nonempty interior in $H^l(0, \hat{S}, H^{1+\varepsilon}(D^1))$ for $s > 1/2$. We observe also that the operators F_1 and $\mathcal{U}_1$ are particular cases of the operators F and $\mathcal{U}$ from Chapter 2, respectively.

Theorem 20. *The operators F_1 and $\mathcal{U}_1$ defined in (4.17) and (4.18), respectively, have the following properties,*

(i) $F_1(\cdot,\cdot\cdot)$ is continuous, compact, weakly continuous and weakly closed (Proposition 8 of Section 2).

(ii) The family $\{F_1(t,\cdot) : Q \to W_2^{1,2}(D) \;:\; t \in [0,\tilde{T}]\}$ is Frechét equi-differentiable (Proposition 9 of Section 2).

(iii) $\mathcal{U}_1$ is continuous, compact, weakly (sequentially) continuous and weakly closed (Proposition 10 of Section 2).

(iv) $\mathcal{U}_1$ has an one sided derivative at $\widetilde{\mathcal{A}} \in \mathfrak{Q}^1$ in direction $\mathcal{H}$, such that $\mathcal{A}+\mathcal{H} \in \mathfrak{Q}^1$ (Proposition 11 of Section 2).

(v) $\mathcal{U}_1'(\mathcal{A})$ can be extended to a bounded linear operator on $H^l(0,\hat{S},H^{1+\varepsilon}(D^1))$ (Proposition 11 of Section 2).

(vi) $\mathcal{U}_1'(\cdot)$ is Lipschitz continuous in $\mathfrak{Q}^1$, injective and compact (Proposition 11 and Proposition 12 of Section 2).

(vii) $\mathcal{U}_1'(\mathcal{A}^\dagger)^*$ has a trivial kernel (Proposition 13 of Section 2).

(viii) $\mathcal{U}_1(\cdot)$ satisfies the local tangential cone condition

$$\left\|\mathcal{U}_1(\mathcal{A}) - \mathcal{U}_1(\widetilde{\mathcal{A}}) - \mathcal{U}_1'(\mathcal{A} - \widetilde{\mathcal{A}})\right\|_{L^2(0,\hat{S},W_2^{1,2}(D^1))} \leq \gamma \left\|\mathcal{U}_1(\mathcal{A}) - \mathcal{U}_1(\widetilde{\mathcal{A}})\right\|_{L^2(0,\hat{S},W_2^{1,2}(D^1))} \tag{4.20}$$

for all $\mathcal{A}, \widetilde{\mathcal{A}}$ in an open ball $B(\mathcal{A}^*,\rho) \subset \mathfrak{Q}^1$ with some $\rho > 0$ and $\gamma < 1/2$ (Theorem 5 of Section 2).

The item (i) of the last two theorems above states the compactness of operators G_1 and $\mathcal{U}_1$. In other words, the inverse problem of calibrating local volatility from quoted European put option prices on Futures contracts is ill-posed. Thus, we have to apply some technique of regularization to find a reliable approximation to the local volatility surface related to such prices. Note that, many of the properties presented in Theorems 19 and 20 are useful tools to state Tikhonov analysis and we shall use them in sections 4.2 and 4.3.

Dupire's Forward Operators for Futures

Under the framework of Section 4.1.2 we assume that normalized European call option prices on futures satisfy (4.10). Thus, following the analysis presented in Chapter 2, we assume that local variance a depends on the current commodity spot price, instead of its Futures price. Thus, at a fixed time t, options depending on Futures with different maturities satisfy the same equation. Under such hypotheses, the present problem is a particular case of the one that motivated Chapter 2. Thus, we shall use the operators G, F and $\mathcal{U}$ of the chapters 1 and 2 as well as its properties within the present context. Therefore, all the results obtained in Chapters 1 and Chapter 2 follow for the present direct problem. It is summarized in the theorem below.

Theorem 21. *Under the hypothesis above we have that Theorem 19 follows for $G(\cdot)$ in the place of $G_1(\cdot)$ and Theorem 20 follows for $F(\cdot,\cdot\cdot)$ and $\mathcal{U}(\cdot)$ in the place of $F_1(\cdot,\cdot\cdot)$ and $\mathcal{U}_1(\cdot)$, respectively.*

As we mentioned above, such analysis is important to identify the way local volatility surfaces evolve with commodities spot prices. Moreover, if such dependence is non Markovian, i.e., if the past measures account for the present or even for the future ones, then, when we incorporate such information into the analysis, it could improve our results.

Note that, the commodity spot price does not appear explicitly in our analysis, thus, we shall not present any model in order to obtain it from futures prices, since they are, in general, unknown.

4.2 Volatility Calibration by Tikhonov Regularization

In this section we shall apply the same techniques developed in Chapter 3. It can be performed as we shall assume the framework presented in Section 4.1 concerned with its forward operators. It was stated that these operators are particular cases of the operators from Chapter 1 and Chapter 2, among other things.

4.2.1 Regularized Inverse Problems

We first assume that the hypothesis of Sections 4.1.2 and 4.1.3 are fulfilled. It follows that local volatility surfaces depend on the maturities of the underlying future contracts. As it is the case of Chapter 3, we shall present a way to reconstruct the whole trajectory

$$\{\mathcal{A} : t \in [0,\hat{T}] \mapsto a(t) \in H^{1+\varepsilon}(D)\}.$$

Proceeding as in Chapter 3, the inverse problem can be summarized as, *given a set of European put option prices $\widetilde{\mathcal{U}}$, we have to find an $\mathcal{A}^\dagger \in \mathfrak{Q}_1$ such that*

$$\widetilde{\mathcal{U}} = \mathcal{U}_1(\mathcal{A}^\dagger). \tag{4.21}$$

Furthermore, we assume that the data is corrupted by (unknown) noise $\mathcal{E} = \{t \mapsto E(t)\}$ of level $\delta > 0$, i.e., instead of $\mathcal{U}_1(\widetilde{\mathcal{A}})$ we consider

$$\mathcal{U}^\delta := \widetilde{\mathcal{U}} + \mathcal{E}, \tag{4.22}$$

with $\sup_{t\in[0,\hat{T}]} \|E(t)\| \leq \delta$. Under such assumptions, following Tikhonov analysis [48], instead of solving (4.21), we are now concerned with finding a minimizer $\mathcal{A} \in \mathfrak{Q}_1$ for the Tikhonov functional

$$\begin{aligned} \mathcal{F}^\delta_{\mathcal{A}_0,\alpha}(\mathcal{A}) &= \left\|\mathcal{U}^\delta - \mathcal{U}_1(\mathcal{A})\right\|_{L^2(0,\hat{T},L^2(D))} + \alpha f_{\mathcal{A}_0}(\mathcal{A}) \\ &= \int_0^{\hat{T}} \|u^\delta(t) - u(t,a(t))\|_{L^2(D)} dt + \alpha f_{\mathcal{A}_0}(\mathcal{A}). \end{aligned} \tag{4.23}$$

Observe that, finding a minimizer for (4.23) is a more stable way to presenting an approximate solution for (4.21).

Although the conceptual difference between the operator $\mathcal{U}$ related to Dupire's equation (4.10) and the operator $\mathcal{U}_1$ related to Black's model (4.6), it follows that the Tikhonov functional associated to $\mathcal{U}$ is (4.23) with $\mathcal{U}$ in the place of $\mathcal{U}_1$.

Note that, from Section 4.1.3 we can directly apply the Tikhonov analysis of Chapter 3 following [48]. Thus, the following results are immediate consequences of the equivalent results stated in Chapter 3. We list them here for the sake of completeness.

4.2.2 Existence of a Minimizer

In this section we show that the functional (4.23) has minimizers. Moreover, we shall see that when the noise level δ goes to zero, such minimizers converges to a solution for (4.21). Hence, in order to starting such analysis, we have to make some hypotheses on the regularization functional $f_{\mathcal{A}_0}(\cdot)$.

As in Section 3, we assume that $f_{\mathcal{A}_0}(\cdot) : \mathfrak{D} \to [0, \infty]$ is convex, coercive and sequentially lower semi-continuous. Such hypotheses implies that the level sets

$$\mu_\alpha(M) := \left\{ \mathcal{A} \in \mathfrak{D} : \mathcal{F}^{\delta}_{\mathcal{A}_0,\alpha}(\mathcal{A}) \leq M \right\} \tag{4.24}$$

are weakly pre-compact. Theorem 20 implies that the restriction of $\mathcal{U}_1$ (and $\mathcal{U}$) to $\mu_\alpha(M)$ is weakly continuous. Now from Section 3.2 of [48], we have the following results:

Theorem 22 (Existence and Stability). *For every $\mathcal{U}_1 \in L^2(0, \hat{T}, L^2(D))$, there exists a minimizer of the functional (4.23) within the set $\mathfrak{D}_1$. Such minimizers are stable in the sense of Definition 6 from Chapter 3. Furthermore, when the inverse problem has a solution, then it has an $f_{\mathcal{A}_0}(\cdot)$-minimizing one.*

For a proof, see Theorems 3.22, 3.23 and 3.25 of [48].

The next theorem shows that when the noise level δ goes to zero, it follows that the regularized solutions obtained by minimizing (4.23) are (at least in a weak sense) good approximations for an $f_{\mathcal{A}_0}(\cdot)$-minimizing solution for the inverse problem (4.21). A proof for such result can be found at [48], Theorem 3.26.

Theorem 23 (Convergence). *Assume that there exists an $\widetilde{\mathcal{A}} \in \mathfrak{D}$ solving the inverse problem with $\mathcal{U}(\widetilde{\mathcal{A}}) = \widetilde{\mathcal{U}}$ and the map $\alpha : (0, \infty) \to (0, \infty)$, satisfies*

$$\lim_{\delta \to 0} \alpha(\delta) = 0 \quad \textit{and} \quad \lim_{\delta \to 0} \frac{\delta^2}{\alpha(\delta)} = 0. \tag{4.25}$$

Moreover, assume that the sequence $\{\delta_k\}_{k\in\mathbb{N}}$ converges to 0 and the elements of $\{\mathcal{U}_k\}_{k\in\mathbb{N}} \subset L^2(0, \hat{T}, W^{1,2}_2(D))$, with $\mathcal{U}_k = \mathcal{U}^{\delta_k}$, satisfy $\|\widetilde{\mathcal{U}} - \mathcal{U}_k\| \leq \delta_k$. Denote $\alpha(\delta_k)$ by α_k for every $k \in \mathbb{N}$. The minimizers of $\mathcal{F}^{\delta_k}_{\mathcal{A}_0,\alpha_k}(\cdot)$ with $k \to \infty$ form a sequence $\{\mathcal{A}_k\}_{k\in\mathbb{N}}$, with a subsequence converging weakly to $\mathcal{A}^\dagger$, an $f_{\mathcal{A}_0}(\cdot)$-minimizing elment of $\mathfrak{D}$, satisfying $\mathcal{U}(\mathcal{A}^\dagger) = \widetilde{\mathcal{U}}$, with $f_{\mathcal{A}_0}(\mathcal{A}_k) \to f_{\mathcal{A}_0}(\mathcal{A}^\dagger)$. In addition, if $\mathcal{A}^\dagger$ is the unique $f_{\mathcal{A}_0}(\cdot)$-minimizing solution for the Inverse problem, then the entire sequence $\{\mathcal{A}_k\}_{k\in\mathbb{N}}$ converges weakly to $\mathcal{A}^\dagger$.

By Theorem 21 and Chapter 3 we have the following theorem.

Theorem 24. *Under the hypothesis of Theorem 21 it follows that Theorems 22 and 23 remain valid with $\mathcal{U}(\cdot)$ in the place of $\mathcal{U}_1(\cdot)$.*

4.2.3 A Convergence Analysis

As above, we apply directly results from the Chapter 3 to the present inverse problem. Thus, making the same hypotheses of Section 3.3, we shall present an estimate of how fast the minimizers $\mathcal{A}^\delta$ of (4.23) converges to $\mathcal{A}^\dagger$, an $f_(\mathcal{A}_0, \cdot)$-minimizing solution for the inverse problem (4.21), when the noise level δ goes to zero.

Note that, in order to apply such results, some fundamental concepts in convex regularization theory are necessary, thus, the reader is referred to Chapter 1 (e.g. Definitions 1 and 2 presents the concept of Bregman distance related to $f_{\mathcal{A}_0}(\cdot)$ and q-coerciveness with constant $\zeta > 0$).

With such concepts in mind, we can state the following theorem.

Theorem 25 (Convergence Rates, from [48])**.** *Assume that $\mathcal{U}$, $f_{\mathcal{A}_0}(\cdot)$, $\mathfrak{Q}$, $H^s(0,\hat{T},H^{1+\varepsilon}(D))$ and $L^2(0,\hat{T},W^{1,2}_2(D))$ satisfy the conditions of Theorems 22 and 23. Furthermore, let the map $\alpha:(0,\infty)\to(0,\infty)$ be such that $\alpha(\delta)\approx\delta$. Then*

$$D_{\xi^\dagger}(\mathcal{A}^\delta,\mathcal{A}^\dagger)=\mathcal{O}(\delta) \quad \textit{and} \quad \|\mathcal{U}(\mathcal{A}^\delta)-\mathcal{U}^\delta\|=\mathcal{O}(\delta).$$

Moreover, there exists a constant $c>0$, such that $f_{\mathcal{A}_0}(\mathcal{A}^\delta)\le f_{\mathcal{A}_0}(\mathcal{A}^\dagger)+\delta/c$ for every δ with $\alpha(\delta)\le\beta_{\max}$.

Corollary 2. *The same rate of convergence of Theorem 25 is achieved for $\mathcal{U}$ in the place of $\mathcal{U}_1$.*

In other words, under the concept of Bregman distance, we can measure how far a minimizer of (4.23) is from an $f_{\mathcal{A}_0}(\cdot)$-minimizing solution of the inverse problem (4.21). Such distance is of order $\sqrt{\delta}$ with the noise level δ representing all the uncertainties related to the data ${\mathcal{U}^\delta}_1$.

Note further that, all the results concerned with Morozov's discrepancy principle and Kulback-Leibler regularization of Chapter 3 remain valid in the present context.

4.3 A Discretized Setting

We present now a discretized setting for Tikhonov analysis, applying results from Section 3.6. Such approach is interesting since it allows us quantifying how far (in an appropriate sense) a discretized solution is from a true one. It uses the same techniques of standard convex Tikhonov analysis from [48] combined with the property that general elements of separable Hilbert spaces can be approximated by finite dimensional ones. We put such analysis here as we have to work with discrete sets of data and discrete models as finite difference schemes to solve PDEs. We shall see in Chapter 5 that the discretized direct problem can be solved by a Crank-Nicholson scheme with a regular grid in time and space.

4.3.1 Discrete Forward Operators

First of all, the analysis presented here is based on [44] with a suitable change of notation. As in Section 3.6, denote $H^s(0,\hat{T},H^{1+\varepsilon}(D))$ by U and $L^2(0,\hat{T},W_2^{1,2}(D)$ by V. Take a nested sequence $\{U_n\}_{n\in\mathbb{N}}$ of finite dimensional subspaces of U such that

$$\overline{\cup_{n\in\mathbb{N}}U_n} = U.$$

Define the finite dimensional sub-domains

$$Q_n := Q\cap U_n \neq \emptyset, \text{ for every } n\in\mathbb{N}.$$

For $m,N\in\mathbb{N}$, take a partition $0=t_0<t_1<...<t_N=T$ of the interval $[0,T]$. The discrete case of operators (4.17) and (4.18) is presented below.

$$\begin{array}{rcl} F_m:[0,T]\times Q & \longrightarrow & W_2^{1,2}(D) \\ (t,a) & \longmapsto & F_m(t,a) \end{array} \tag{4.26}$$

and

$$\begin{array}{rcl} \mathcal{U}_N^m:\mathfrak{Q}\subset U & \longrightarrow & V \\ \mathcal{A} & \longmapsto & \left\{ t\mapsto \displaystyle\sum_{j=1}^N F_m(t_j,a(t_j))\mathcal{X}_{(t_{j-1},t_j]}(t)\right\}. \end{array} \tag{4.27}$$

Note that F_m is continuous since it is a composition of two continuous functions.

The following estimates are assumed for further application,

$$\|F_m(t,a)-F(t,a)\|\leq\varepsilon_m \quad \text{for every } (t,a)\in[0,T]\times Q \text{ and} \tag{4.28}$$

$$\|\mathcal{U}_N^m(\mathcal{A})-\mathcal{U}(\mathcal{A})\|\leq\rho_N^m \quad \text{for every } \mathcal{A}\in\mathfrak{Q}. \tag{4.29}$$

with ρ_N^m and ε_m depending only on (m,N) and m, respectively and $\lim\limits_{m,N\to\infty}\rho_N^m=0$. Furthermore, $\lim\limits_{m\to\infty}\varepsilon_m=0$.

4.3.2 Discrete Tikhonov Regularization

Instead of (4.23) we shall look for a minimizer in $\mathfrak{Q}$ for

$$\begin{array}{rcl} \mathcal{F}^{m,\alpha}_{\mathcal{U}^\delta,a_0}(\mathcal{A}) & = & \|\mathcal{U}_N^m(\mathcal{A})-\mathcal{U}^\delta\|^2_{L^2(0,S,W_2^{1,2}(D))}+\alpha f_{\mathcal{A}_0}(\mathcal{A}) \\ & = & \displaystyle\sum_{j=1}^N \|F_m(t_j,a(t_j))-u^\delta(t_j)\|^2_{H^{1,2}}\cdot(t_j-t_{j-1})+\alpha.f_{\mathcal{A}_0}(\mathcal{A}) \end{array} \tag{4.30}$$

Here we assume valid the hypotheses of Section 3.6. Note that Assumption 1 and thus, Lemma 11 are true. In order to establish the analysis of such inverse problem, we first assume that $f_{\mathcal{A}_0}(\cdot)$ is weakly lower semi-continuous, coercive and convex. Therefore, as in Section 3.6, we get discretized versions of Theorem 22 and Theorem 23 of Section 4.2.

Theorem 26 (Existence)**.** *For $m, n, N \in \mathbb{N}$ and $\alpha, \delta > 0$ fixed, assume valid Assumption 1 and the estimates (4.28) and (4.29). Then there exists a minimizer for the functional (4.30).*

Theorem 27 (Stability)**.** *Under the same hypothesis of Theorem 26, the minimizers of (4.30) are stable, i.e., if the sequence $\{\mathcal{U}_k\}_{k\in\mathbb{N}}$ converges strongly to $\mathcal{U}^\delta$, then a sequence $\{\mathcal{A}_k\}_{k\in\mathbb{N}}$ of minimizers of (4.30) with $\mathcal{U}^\delta$ replaced by $\mathcal{U}_k$, has a subsequence $\{\mathcal{A}_{k_l}\}_{l\in\mathbb{N}}$ converging weakly to $\mathcal{A}^*$, a minimizer of (4.30) with $\lim\limits_{l\to\infty} f_{\mathcal{A}_0}(\mathcal{A}_{k_l}) = f_{\mathcal{A}_0}(\mathcal{A}^*)$.*

The proofs of the theorems above are the same of Proposition 2.3 of [44] and follow almost directly from Assumption 1 and estimates (4.29) and (4.28). The following lemma and theorem were proved in Section 3.6. See Lemma 12 and Theorem 16.

Lemma 13. *Assume that problem (3.2) has an $f_{\mathcal{A}_0}$-minimizing solution $\mathcal{A}^\dagger$ in the interior of $\mathfrak{Q}$. Furthermore, there is $r > 0$ small enough, such that, the open ball $B(\mathcal{A}^\dagger, r) \subset \mathfrak{Q}$. Then, there exists a sequence $\{\mathcal{A}_n\}_{n\in\mathbb{N}} \subset \mathfrak{Q}$ where, for n sufficiently large, $\mathcal{A}_n \in \mathfrak{Q}_n \cap B(\mathcal{A}^\dagger, r)$ and $\|\mathcal{A}_n - \mathcal{A}^\dagger\| \to 0$ as $n \to \infty$.*

Theorem 28 (Convergence)**.** *Let Assumption 1 be satisfied. Furthermore, assume that*

1. *the parameter $\alpha = \alpha(\delta, m, N, n) > 0$ is such that*

$$\lim_{\substack{\delta\to 0\\ m,N,n\to\infty}} \alpha(\delta, m, N, n) = 0 \quad \text{and} \quad \lim_{\substack{\delta\to 0\\ m,N,n\to\infty}} \frac{(\max\{\delta, \rho_N^m, \lambda_n\})^2}{\alpha(\delta, m, N, n)} = 0, \tag{4.31}$$

where ρ_N^m is defined in (3.36) and $\lambda_n = \|\mathcal{U}'(\mathcal{A}^\dagger)(\mathcal{A}_n - \mathcal{A}^\dagger)\|$, with $\mathcal{A}_n$ defined above and $\lim\limits_{n\to\infty} \gamma_n = 0$.

If (3.35) and (3.36) hold, then every sequence of minimizers of (3.37), $\{\mathcal{A}_k\}_{k\in\mathbb{N}}$ with $\mathcal{A}_k = \mathcal{A}^{\alpha_k,\delta_k}_{m_k,N_k,n_k}$ and $\alpha_k := \alpha(m_k, N_k, n_k)$, where $\delta_k \to 0$, $m_k, N_k, n_k \to \infty$, has a subsequence $\{\mathcal{A}_{k_l}\}_{l\in\mathbb{N}}$ converging weakly to $\widetilde{\mathcal{A}}$, an $f_{\mathcal{A}_0}(\cdot)$-minimizing solution of (3.2) with $f_{\mathcal{A}_0}(\mathcal{A}_{k_l}) \to f_{\mathcal{A}_0}(\widetilde{\mathcal{A}})$ as $l \to \infty$. Furthermore, if $\mathcal{A}^\dagger$ is the unique solution of (3.2) then the entire sequence $\{\mathcal{A}_k\}_{k\in\mathbb{N}}$ converges weakly to $\mathcal{A}^\dagger$ with $f_{\mathcal{A}_0}(\mathcal{A}_k) \to f_{\mathcal{A}_0}(\mathcal{A}^\dagger)$ as $k \to \infty$.

Hence, we obtained a reliable discrete solution for the inverse problem by applying Tikhonov analysis.

4.3.3 A Convergence Analysis

As in Section 4.2 we present convergence rates, however, due to discretization effects, there are more uncertainty sources in addition to the noise level which have to be quantified. It can be interpreted as how much information is necessary to get a reliable solution with these techniques.

As it is mentioned above, we have no difference between the problem treated here and the one from Section 3.6, in a theoretical sense. Thus, using the framework of Bregman distances and convex analysis, we can state the following theorem:

Theorem 29 (Convergence Rates)**.** *Let the hypothesis of Theorem 28, Lemma 13 and Lemma 4 be satisfied. Define* $\gamma_n := \|\mathcal{A}_n - \mathcal{A}^\dagger\|$*, with* $\{\mathcal{A}_n\}_{n\in\mathbb{N}}$ *and* $\mathcal{A}^\dagger$ *as in Lemma 13. If* $\alpha \sim \max\{\delta, \lambda_n, \rho_N^m\}$ *and* $\alpha \cdot \beta_1 < 1$*, with* β_1 *from Lemma 4, then*

$$D_{\xi^\dagger}(\mathcal{A}_{n,N}^{\delta,m}, \mathcal{A}^\dagger) = \mathcal{O}(\delta + \lambda_n + \rho_N^m + \zeta_n).$$

Such result states that we can measure how far a discretized solution is from a $f_{\mathcal{A}_0}(\cdot)$-minimizing solution of (4.21), under the concept of Bregman distances. Such distance is of order $\sqrt{\delta + \lambda_n + \rho_N^m + \zeta_n}$, when δ is sufficiently small and m, n and N are sufficiently large. Note that, each quantity under the square-root signal represents a different source of uncertainty present in this model.

Chapter 5

Tikhonov Regularization in Practice: Numerical Results

In this chapter we present some numerical examples illustrating the theory developed in the past chapters. We first use Tikhonov regularization in order to reconstruct a known volatility surface from simulated data and make a comparison between the resulting surface and the original one, for different noise levels. After this with real data from equity and commodity markets we analyze through a graphical exposition the reconstructions of volatility surfaces.

5.1 A Numerical Solution for the Direct Problem

Recall that, for each $t \in [0, T]$, $F(t, a) = u(t, a) - u(t, a_0)$, where $u(t, a)$ is the solution of (1.4) and a_0 is given and fixed. In what follows we shall make use of the Crank-Nicholson scheme. See [51].

Now, we consider the same basis $\{\phi_k\}_{k=1}^{K}$ of bilinear finite elements presented in Example 4.3 from Chapter II of [9], where a regular triangular mesh on $[0, \mathcal{T}] \times [Y_{min}, Y_{max}]$ with K nodes is taken. Such basis is used here in order to find a discretization for $\mathcal{A}$ at each time $t \in [0, T]$.

Thus, for each $t \in [0, T]$, we define the discrete volatility surface

$$a_K(t, \tau, y) := \sum_{k=1}^{K} \langle a(t), \phi_k \rangle \phi_k(\tau, y)$$

and the discrete case of $\mathcal{A}$, $\mathcal{A}_K := \{t \mapsto a_K(t)\}$.

5.2 Solving the Minimization Problem

We shall use a gradient approach to solve numerically the minimization problem obtained by Tikhonov regularization. Thus, we present below a characterization of the conjugate of the gradient of the residual $\|\mathcal{U}(\cdot) - \mathcal{U}^\delta\|^2$. We shall see also that such characterization implies that a conjugate problem has to be solved at each iteration of this minimization method.

Defining the functional

$$J^\delta(\mathcal{A}) := \|\mathcal{U}(\mathcal{A}) - \mathcal{U}^\delta\|^2_{L^2(0,S,L^2(D))} = \int_0^T \|F(t,a(t)) - u^\delta(t)\|^2_{L^2(D)}dt \tag{5.1}$$

it follows that the Tikhonov functional (3.4) can be written as

$$\mathcal{F}^{\mathcal{U}^\delta}_{a_0,\alpha}(\mathcal{A}) = \|\mathcal{U}^\delta - \mathcal{U}(\mathcal{A})\|^2_{L^2(0,S,L^2(D))} + \alpha f_{a_0}(\mathcal{A}) = J^\delta(\mathcal{A}) + \alpha f_{a_0}(\mathcal{A}).$$

Hence, it is necessary to evaluate the gradient of $J(\cdot)$ to perform the minimization of (3.4). Then, we can represent the gradient $\nabla J^\delta(\mathcal{A})$ by

$$\begin{aligned}
\langle \nabla J^\delta(\mathcal{A}), \mathcal{H}\rangle_{H^l(0,S,H^{1+\varepsilon}(D))} &= 2\langle \mathcal{U}(\mathcal{A}) - \mathcal{U}^\delta, \mathcal{U}'(\mathcal{A})\mathcal{H}\rangle_{L^2(0,S,L^2(D))}\\
&= 2\int_0^T \langle F(t,a(t)) - u^\delta(t), \partial_a F(t,a(t))h(t)\rangle_{L^2(D)}dt\\
&= 2\int_0^T \langle F(t,a(t)) - u^\delta(t), \mathcal{L}^{-1}\mathcal{G}_{u_{yy}-u_y}h(t)\rangle_{L^2(D)}dt\\
&= 2\int_0^T \langle (\mathcal{L}^{-1})^*(F(t,a(t)) - u^\delta(t)), \mathcal{G}_{u_{yy}-u_y}h(t)\rangle_{L^2(D)}dt\\
&= 2\int_0^T \langle v(t,a(t)), \mathcal{G}_{u_{yy}-u_y}h(t)\rangle_{L^2(D)}dt\\
&= 2\int_0^T \int_D \{[v(u_{yy}-u_y)h(t)](t,a(t))\}(\tau,y)d\tau dy dt
\end{aligned} \tag{5.2}$$

Note that, in the last four equalities above we used the fact of $\partial_a F(t,a(t))h(t)$ is the solution of

$$-w_\tau + a(w_{yy} - w_y) + bw_y = h(t)(u_{yy}(t,a) - u_y(t,a))$$

with homogeneous boundary condition, where $\mathcal{L}$ represents the differential operator of the equation above and $\mathcal{G}_{u_{yy}-u_y}$ is the operator of multiplication by $u_{yy}-u_y$ (see the proof of Proposition 13). We further note that the term v of the two last equalities above is the adjoint state which, for each t, is the solution of

$$v_\tau + (av)_{yy} + (av)_y + bv_y = u(t,a) - u^\delta(t) \tag{5.3}$$

with homogeneous boundary condition. We also have that $v \in L^2(0,S,W^{1,2}_2(D))$. In order to solve Equation (5.3) numerically we shall apply a Crank Nicholson scheme.

Now, we consider a given initial set of volatility surfaces $\mathcal{A}^0$, and define $J^\delta(\mathcal{A}^j)$ and $\nabla J^\delta(\mathcal{A}^j)$ by (5.1) and (5.2), respectively. Thus, we perform iterations like

$$\mathcal{A}_{j+1} = \mathcal{A}_j - \beta_j\left[\nabla J^\delta(\mathcal{A}_j) + \alpha f'_{a_0}(\mathcal{A}_j)\right] \tag{5.4}$$

until a discrepancy principle

$$\tau_1\delta \le \|\mathcal{U}(\mathcal{A}_j) - \mathcal{U}^\delta\|^2_{L^2(0,S,W^{1,2}_2(D))} \le \tau_2\delta \tag{5.5}$$

is reached.

Note that the parameter β_j can be chosen at each step, for example, by a steepest decent+Armijo's rule.

We observe also that $f_{a_0}(\cdot)$ is not necessarily Gateaux differentiable.However, such calculations are made in a discretized setting, simplifying all the theoretical issues related to this problem. A good reference for such subject is [26].

5.3 Numerical Results

We present now a series of numerical results obtained with synthetic as well as real data. We first present the results with synthetic data illustrating some results from previous chapters. After we present some reconstructions of volatility surface obtained from real data in equity and commodity markets. We present reconstructed volatility surfaces for MCD, PBR, SPX, WTI Brent oil and HH natural gas.

5.3.1 Synthetic Data

We are now concerned with presenting reconstructions of volatility surfaces from simulated data. Thus, we compare the resulting surfaces with the original ones presenting the relative local error for different noise levels.

Such tests were performed as follows:

We first define the volatility surface:

$$\sigma(s,u,x) = \begin{cases} 0.4(1-0.4\mathrm{e}^{-0.5(u-s)})\cos(1.25\pi\log(x/s)), & \text{if } (u,x)\in(0,1.25]\times[-0.4,0.4], \\ 0.4, & \text{otherwise.} \end{cases} \tag{5.6}$$

After we use F_m with m large in order to generate the (almost) continuous noiseless data $\widetilde{\mathcal{U}}$. Then to be consistent with real situations and avoiding the so called inverse crime [49], we consider prices in a scarcer grid. Roughly speaking, we generate the continuous data $\widetilde{\mathcal{U}}$ but we chose only a small set of data denoted by $\mathcal{U}^\delta$, which is restricted to a smaller range of strikes and maturities. Thus, in order to test the sensitivity of the method with respect to the noise level, we increase the amount of price data.

As an example, for the first test, whose results are presented in Figure 5.1, we choose the following set of maturities (in years) and strikes (in log-moneyness variable) to collect the data $\mathcal{U}^\delta$:

$$\{(0.08, 0.16, 0.24, 0.40, 0.64, 1.2)\times(-0.4,-0.3,-0.2,-0.1,0,0.1,0.2,0.3,0.4)\}.$$

The minimization process is done with the same grid we have used to generate $\widetilde{\mathcal{U}}$.

Another possible way to generate $\widetilde{\mathcal{U}}$ is by using a Monte Carlo integration method with the risk-neural pricing formula

$$C(T,K) = \mathbb{E}^{\widetilde{\mathbb{P}}}\left[e^{-r(T-t)}(S_T-K)\,|S_t=s\right], \tag{5.7}$$

combined with the Euler method to simulate a large number of paths of (1.1). See [32]. Note that, time t is fixed, r is the constant interest rate, $\widetilde{\mathbb{P}}$ is the risk-neutral measure and s is the current value of the stock price.

In order to perform such tests, we have chosen:

- The interest rate is $r = 3\%$ per year.
- The stock price is $S_0 = 1$.
- The log-moneyness domain is $[-5, 5]$.
- The time to maturity domain is $[0, 1.2]$.
- The time and log-moneyness step sizes are $\Delta t = 0.004$ and $\Delta y = 0.1$, respectively. Such choice respects the CFL condition $\Delta t/(\Delta y)^2 < 1/2$.
- The first regularizing functional is

$$f_{\mathcal{A}_0}(\mathcal{A}) = \|\mathcal{A} - \mathcal{A}_0\|^2_{L^2(D)} + \|L_1(\mathcal{A} - \mathcal{A}_0)\|^2_{L^2(D)} + \|L_2(\mathcal{A} - \mathcal{A}_0)\|^2_{L^2(D)}$$

  where L_1 and L_2 are discrete versions of the operators ∇ of first derivatives and Δ of second derivatives, respectively. More specifically, L_1 is the centered first derivative in time and space and L_2 is the centered second derivative in space only.
- The regularization parameter is chosen to be approximately 0.5% of the noise level δ.

Some reconstructions can be found in Figure 5.1, Figure 5.2, Figure 5.3 and Figure 5.4. For each one a different noise level is considered. Here we only consider uncertainties related to discretization. For each case we take a different number of maturities, and consequently the noise level is inversely proportional to the number of maturities considered.

In each figure we have three different images, the first present the original volatility surface at the grid where the inverse problem is solved. The second shows the reconstructed surface at the grid where data is collected. The last one presents the relative local error of volatility for the first six maturities of the set taken for each case.

Comparing the reconstructions with the original surface at each figure, we can see that the solutions are getting better as the noise level decreases. It can be thought as an illustration of theorems concerning convergence. It is important to observe that, for every time we increase the number of maturities considered, we use the resulting surface of the previous case as the initial guess.

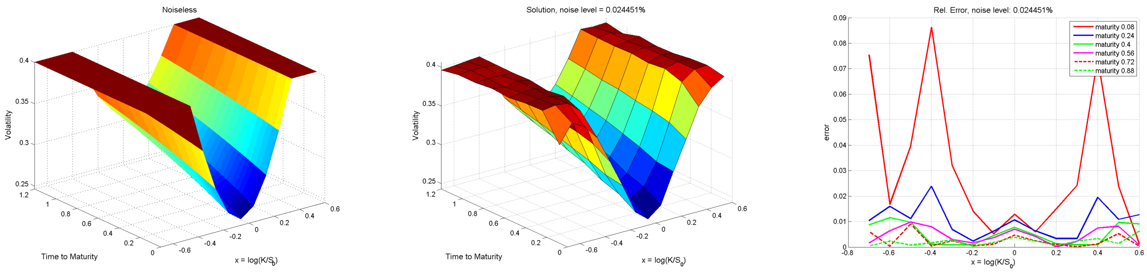

Figure 5.1: The first image shows the true volatility surface, the second is the reconstructed one and the third is the relative error.

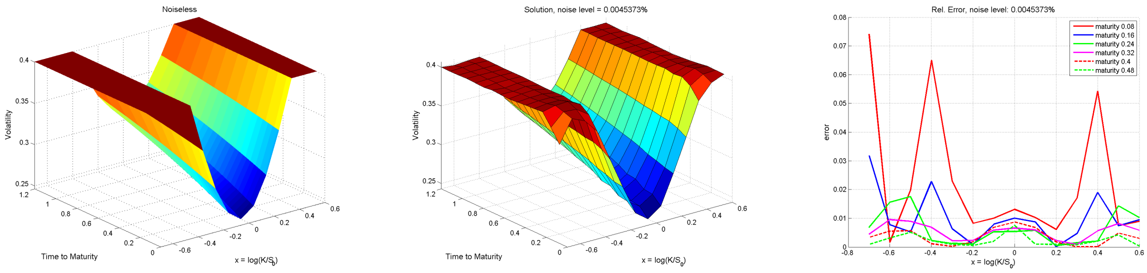

Figure 5.2: The first image shows the true volatility surface, the second is the reconstructed one and the third is the relative error.

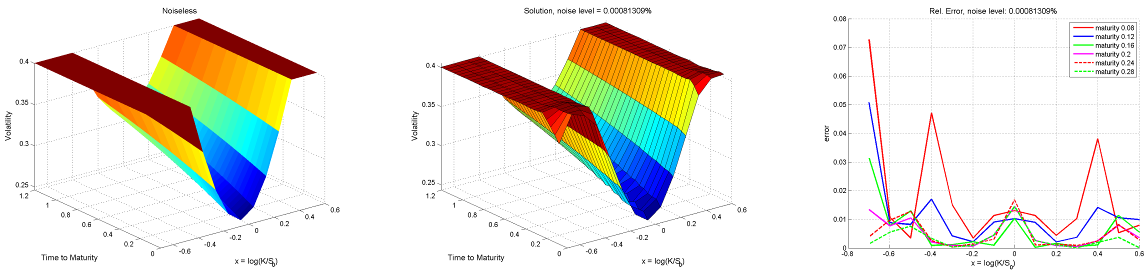

Figure 5.3: The first image shows the true volatility surface, the second is the reconstructed one and the third is the relative error.

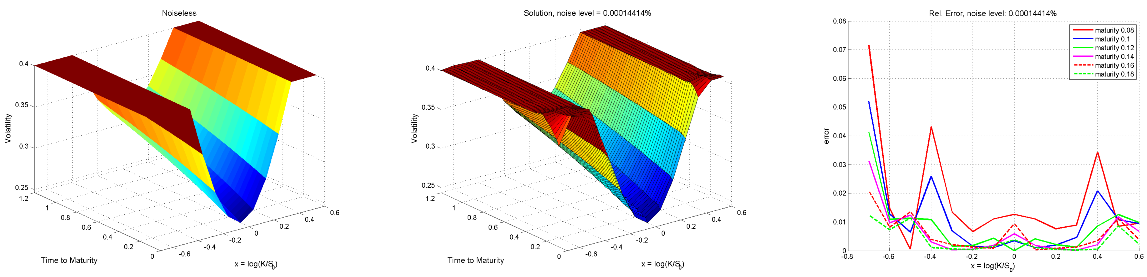

Figure 5.4: The first image shows the true volatility surface, the second is the reconstructed one and the third is the relative error.

5.3.2 Equity Market Data

Now we shall perform tests with real data. In this first part we consider only data from equity markets for the MCD, PBR and SPX prices traded at different days in the New York Stock Exchange during the years of 2009, 2010 and 2011. The reconstructed solutions respects many of the financial properties expected by practitioners of market. We would like to thank again to Bernardo Lima and Carlos Eduardo Moura who provided us the data used.

Mc.Donald's (MCD)

We consider here European call options on the MCD equity prices traded during the year of 2009. We first extracted the implied volatility surface from such prices and calculated the arithmetic mean of its values. This mean was used as the constant *a priori* and the initial guess for the volatility surface. We did not use directly such surface since it presented many outliers. We have chosen two different regularization functionals, in the first case, the regularization functional and parameter were chosen as above. In the second case, we used Kullback-Leibler entropy as the regularization functional, with the result obtained in the first case as the prior. In the latter case, we have performed a Morozov's discrepancy choice for the regularization parameter α.

The reconstructions can be found in Figure 5.5 and Figure 5.7. In each figure we have four pictures, the first shows quoted and experimental call prices. The second presents the relative local error for call prices. The third and fourth present the reconstructed local volatility. We depicted it in two different views, first we present it for each maturity and then we present the surface.

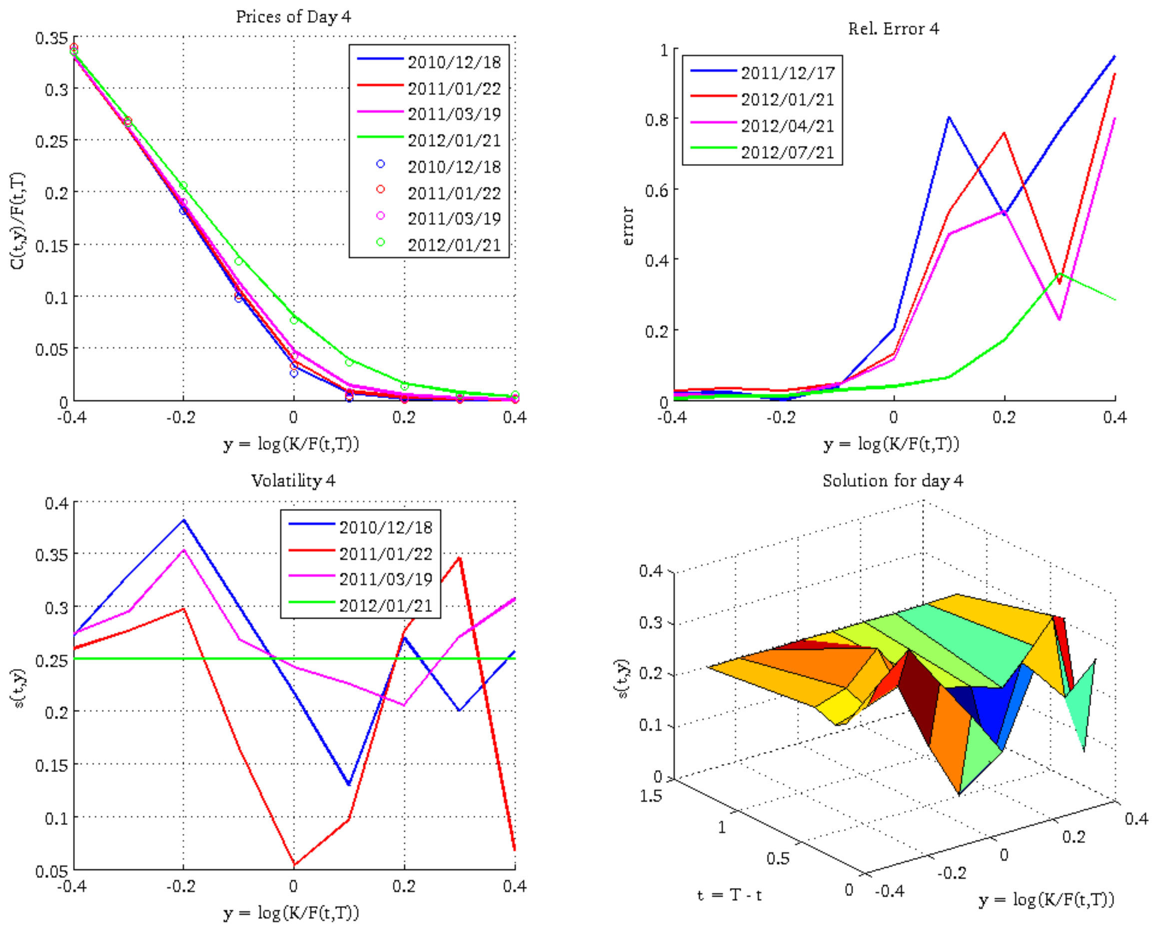

Figure 5.5: Day 4: The first image shows quoted and simulated MCD call option prices. The second one displays the relative error between such simulated and quoted prices. The third and fourth ones present the reconstructed local volatility.

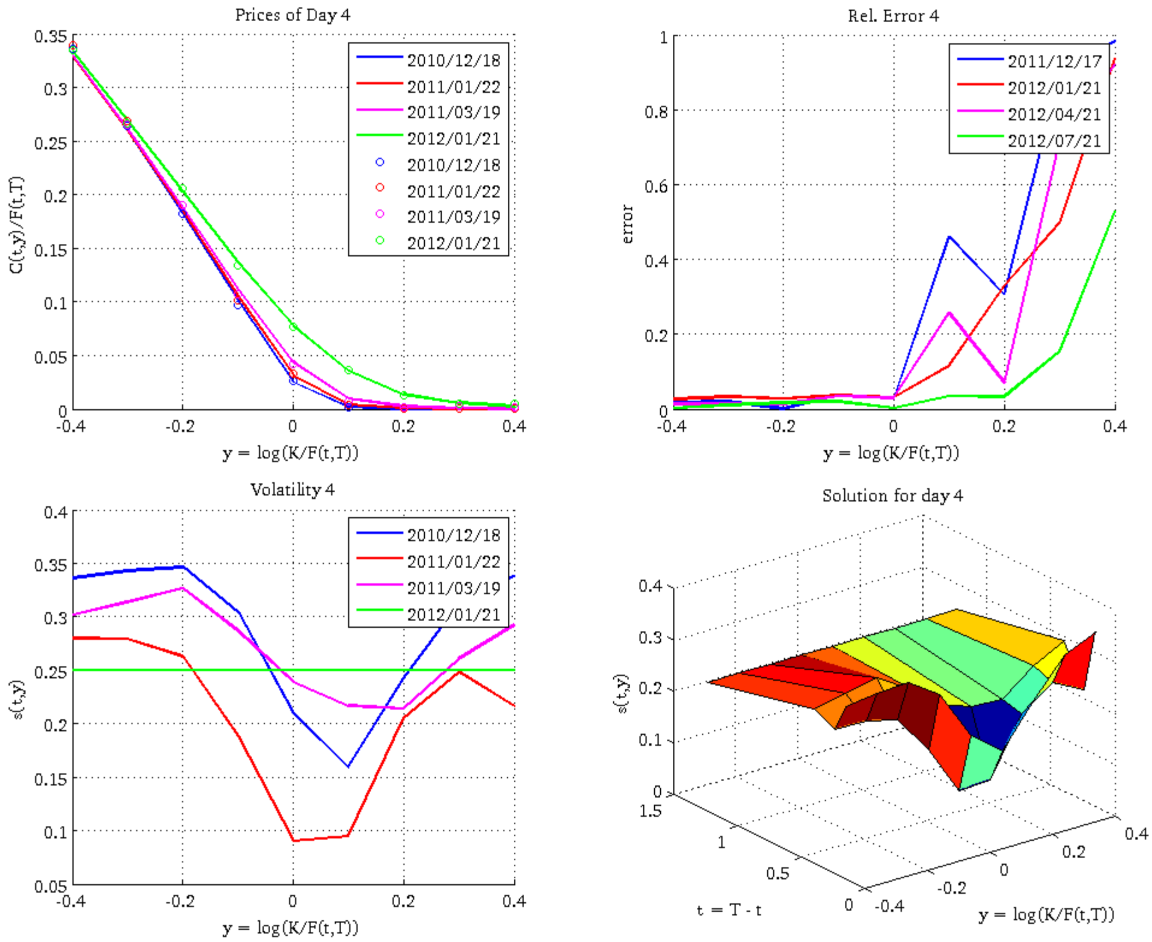

Figure 5.6: Day 4: The first image shows quoted and simulated MCD call option prices. The second one displays the relative error between such simulated and quoted prices. The third and fourth ones present the reconstructed local volatility. Solution obtained with Kullback-Leibler entropy as the regularization functional.

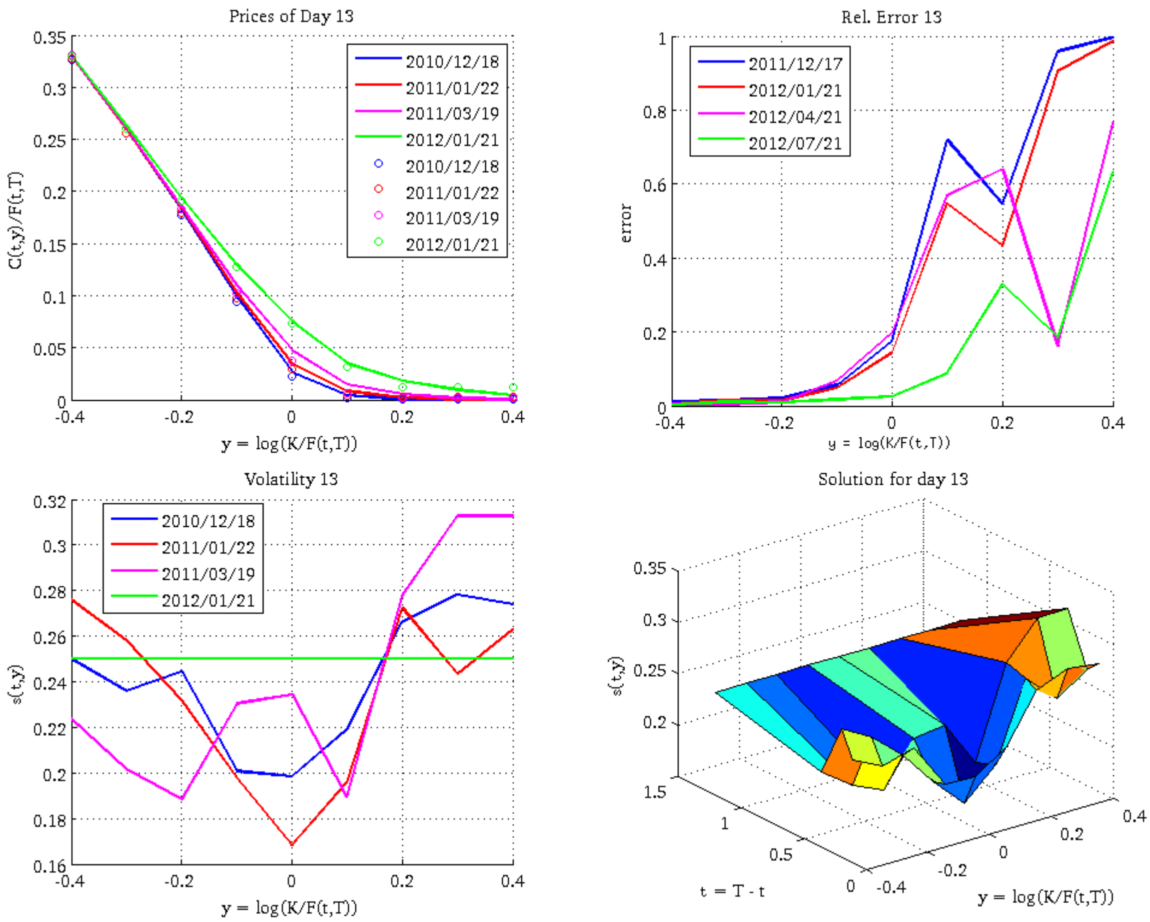

Figure 5.7: Day 13: The first image shows quoted and simulated MCD call option prices. The second one displays the relative error between such simulated and quoted prices.The third and fourth ones present the reconstructed local volatility.

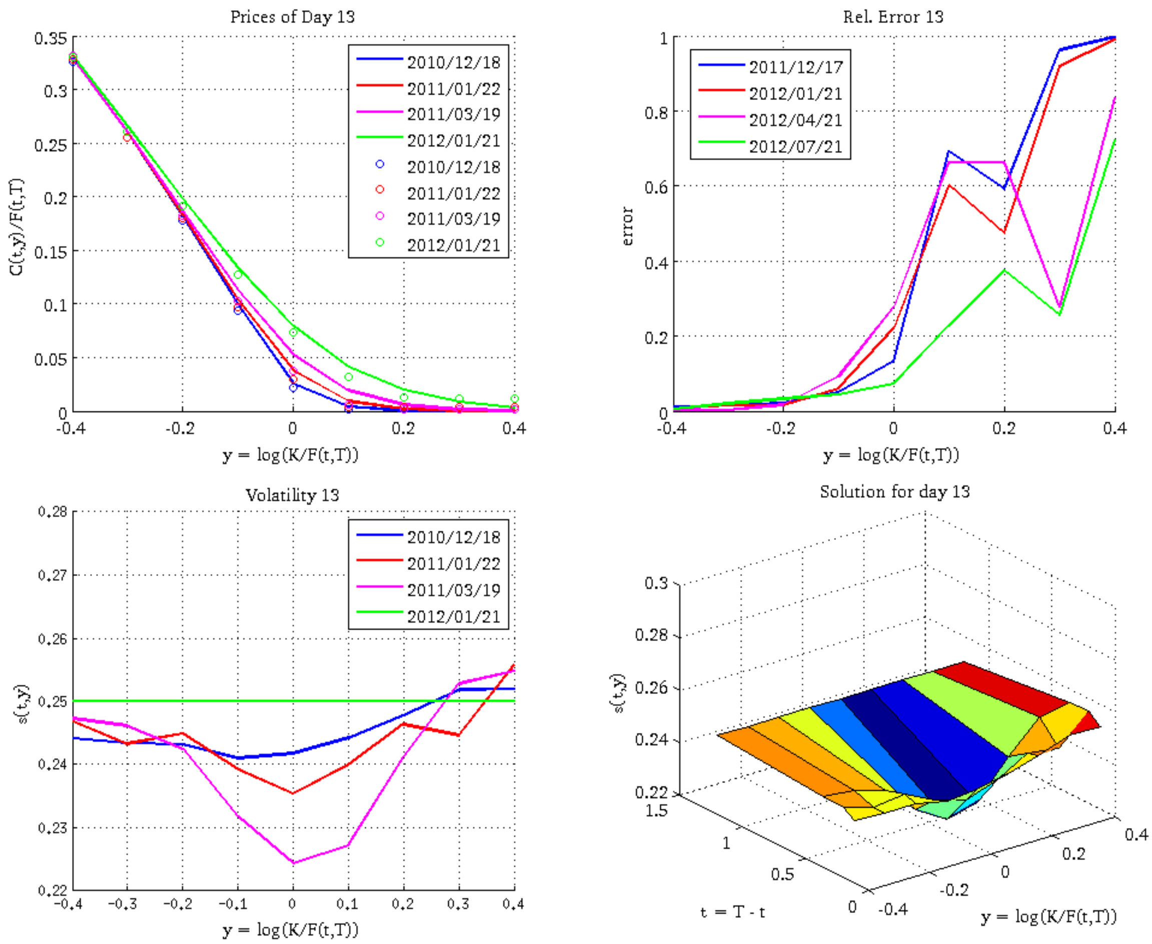

Figure 5.8: Day 13: The first image shows quoted and simulated MCD call option prices. The second one displays the relative error between such simulated and quoted prices.The third and fourth ones present the reconstructed local volatility. Solution obtained with Kullback-Leibler entropy as the regularization functional.

Petroleo Brasileiro S.A (PBR)

We present below some results obtained for quoted European call option prices on PBR equity prices traded during the year of 2009 at NYSE. We proceed in the same way as for MCD. Thus, the results are presented in the same configuration.

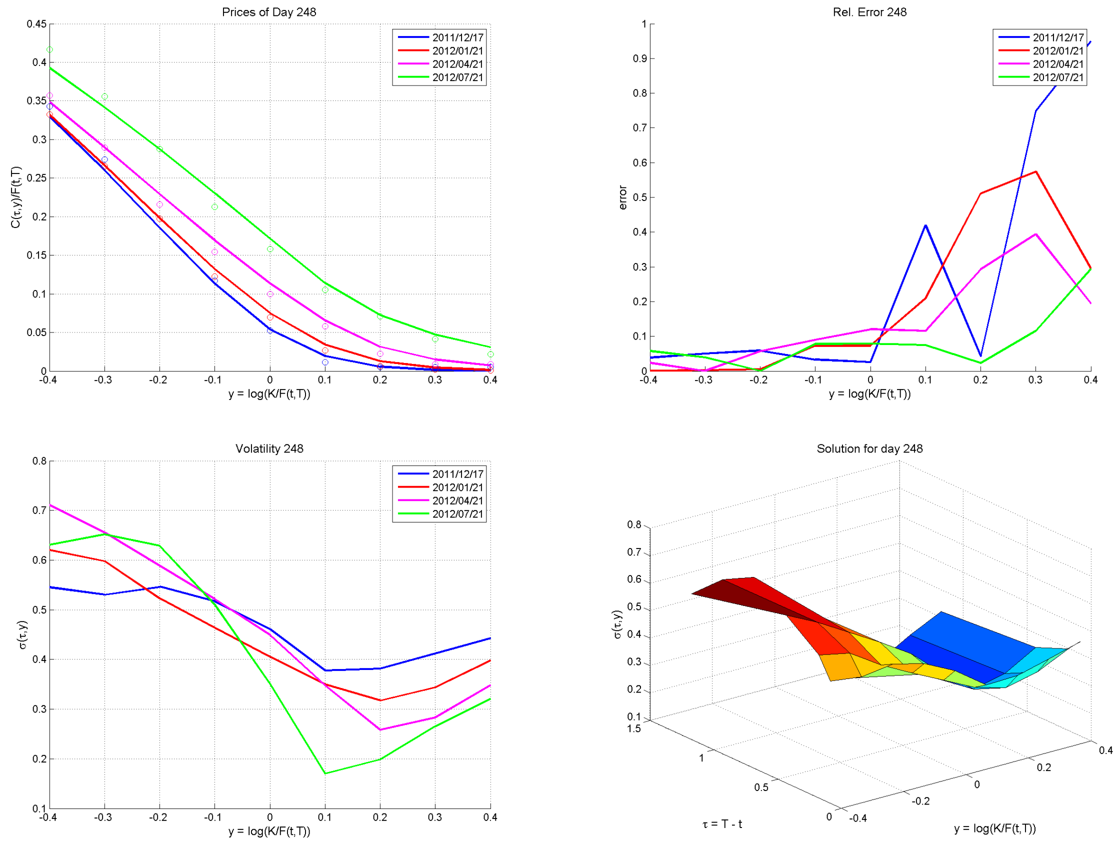

Figure 5.9: Day 248: The first image shows quoted and simulated PBR call option prices. The second one displays the relative error between such simulated and quoted prices.The third and fourth ones present the reconstructed local volatility.

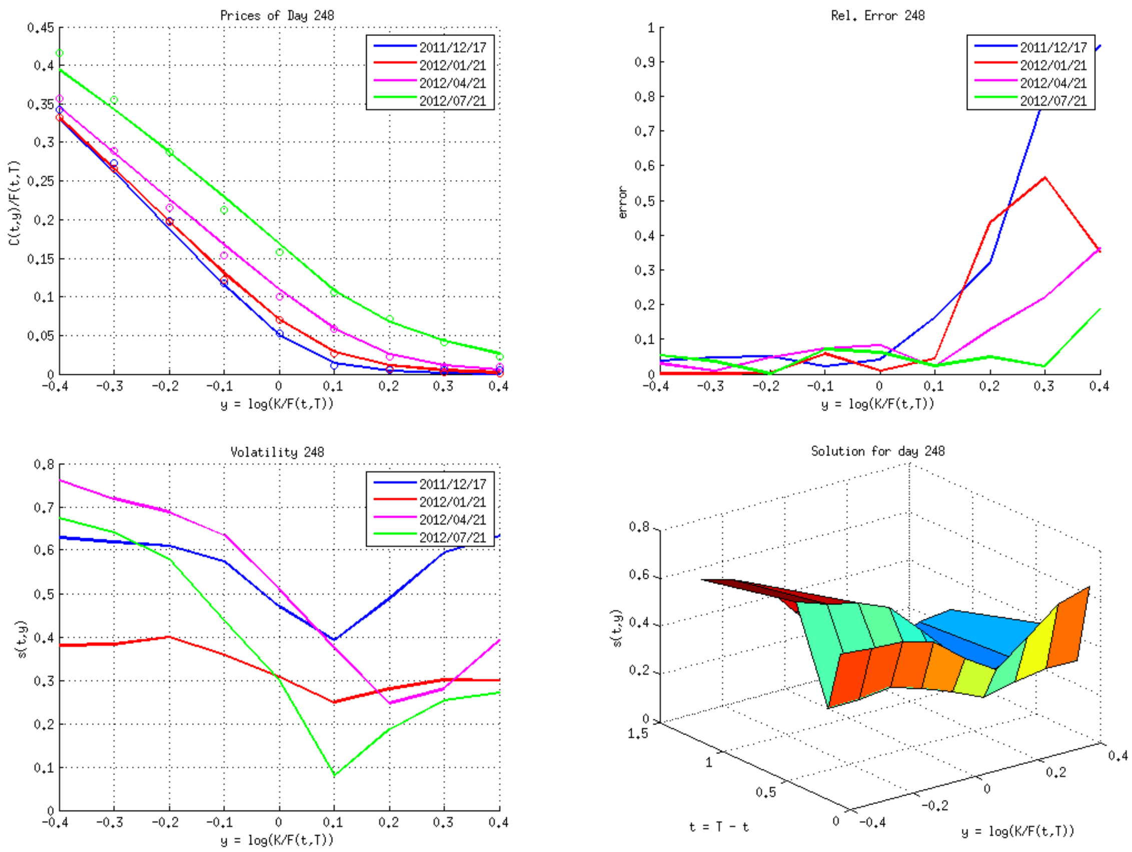

Figure 5.10: Day 248: The first image shows quoted and simulated PBR call option prices. The second one displays the relative error between such simulated and quoted prices.The third and fourth ones present the reconstructed local volatility. Solution obtained with Kullback-Leibler entropy as the regularization functional.

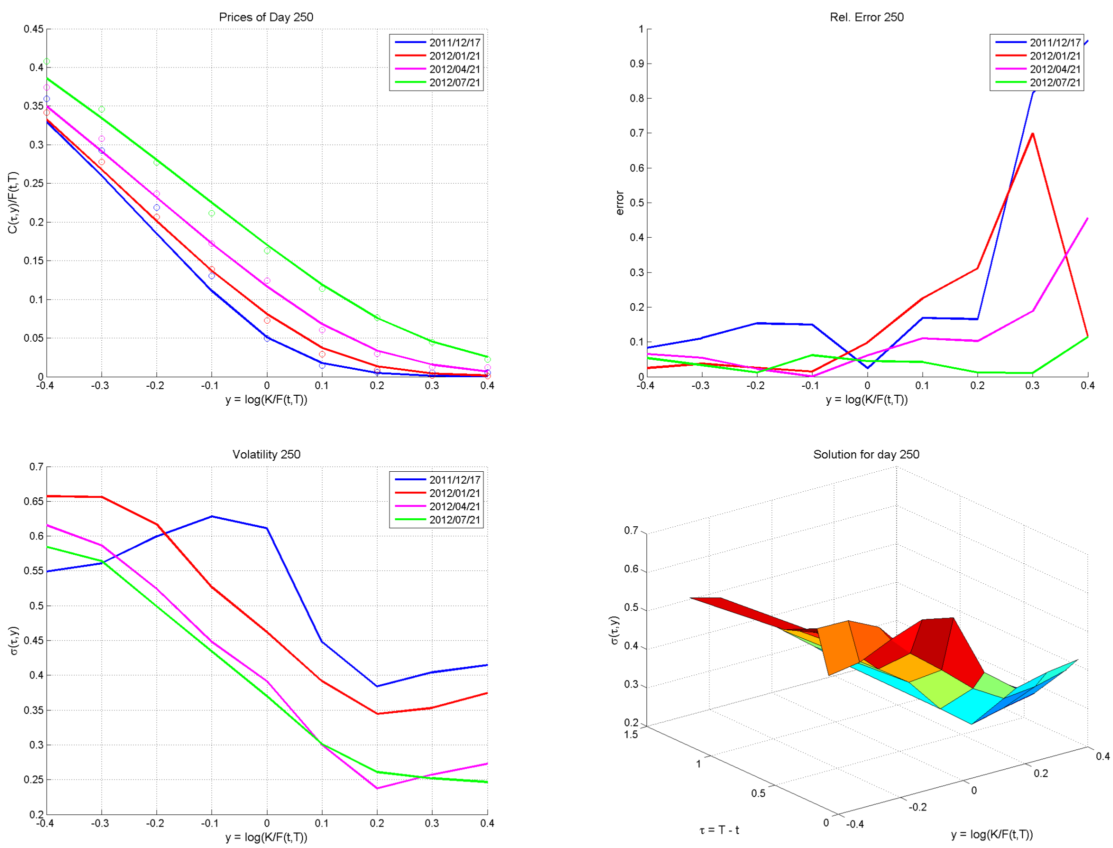

Figure 5.11: Day 250: The first image shows quoted and simulated PBR call option prices. The second one displays the relative error between such simulated and quoted prices.The third and fourth ones present the reconstructed local volatility.

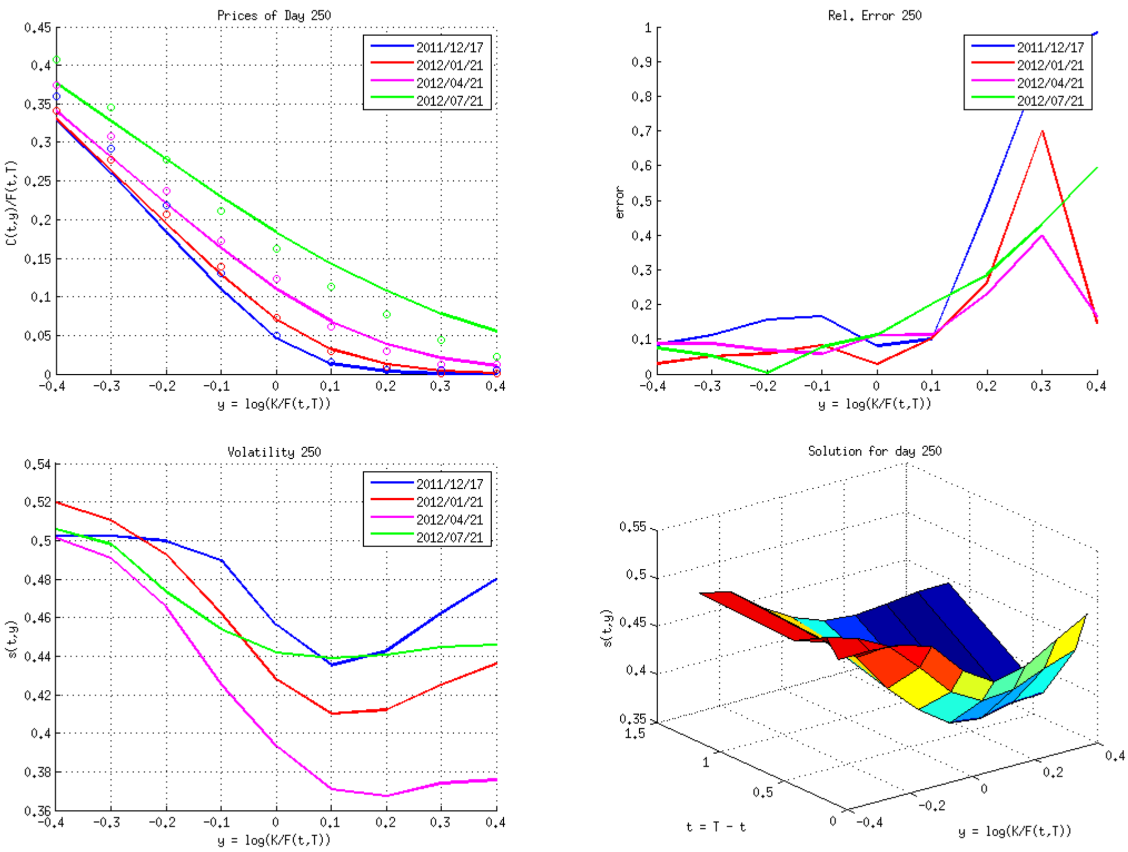

Figure 5.12: Day 250: The first image shows quoted and simulated PBR call option prices. The second one displays the relative error between such simulated and quoted prices.The third and fourth ones present the reconstructed local volatility. Solution obtained with Kullback-Leibler entropy as the regularization functional.

Standard and Poor's (SPX)

We present below some results obtained for quoted European call option prices on SPX index traded during the year of 2010 and 2011 at NYSE. We proceed in the same way as for MCD. Thus, the results are presented in the same configuration.

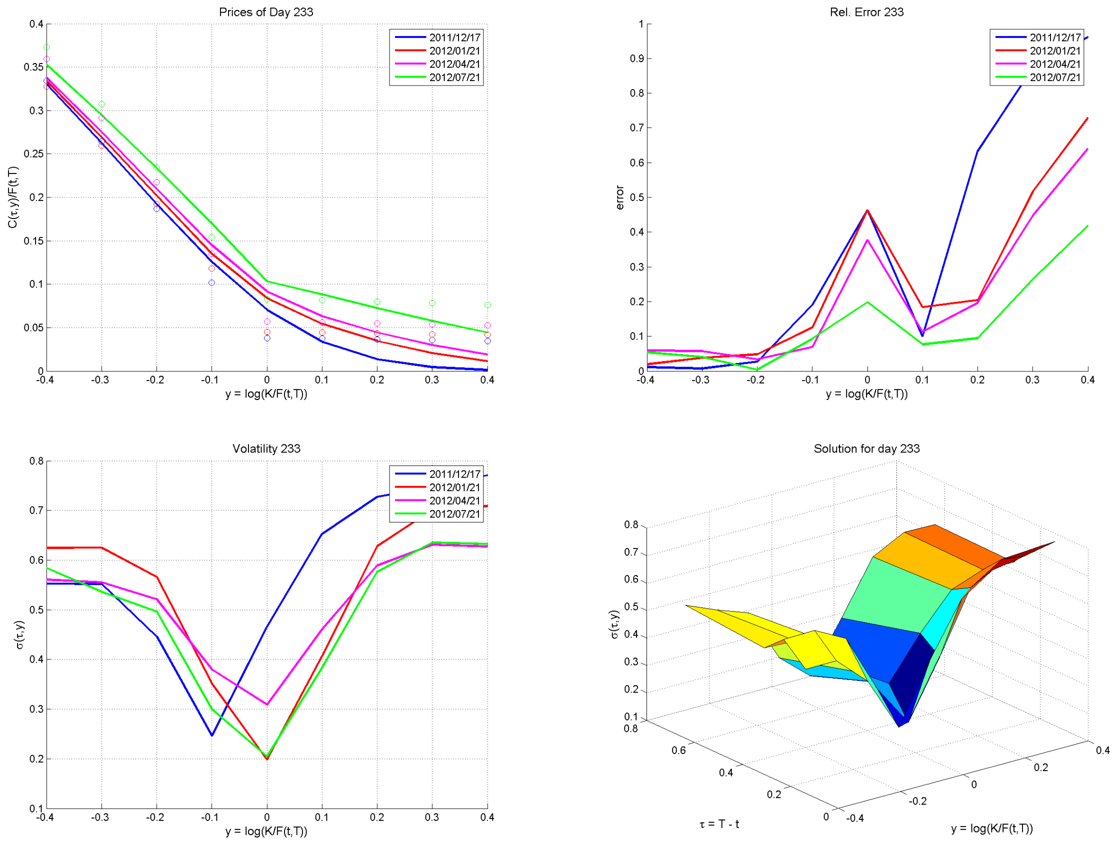

Figure 5.13: Day 233: The first image shows quoted and simulated SPX call option prices. The second one displays the relative error between such simulated and quoted prices.The third and fourth ones present the reconstructed local volatility.

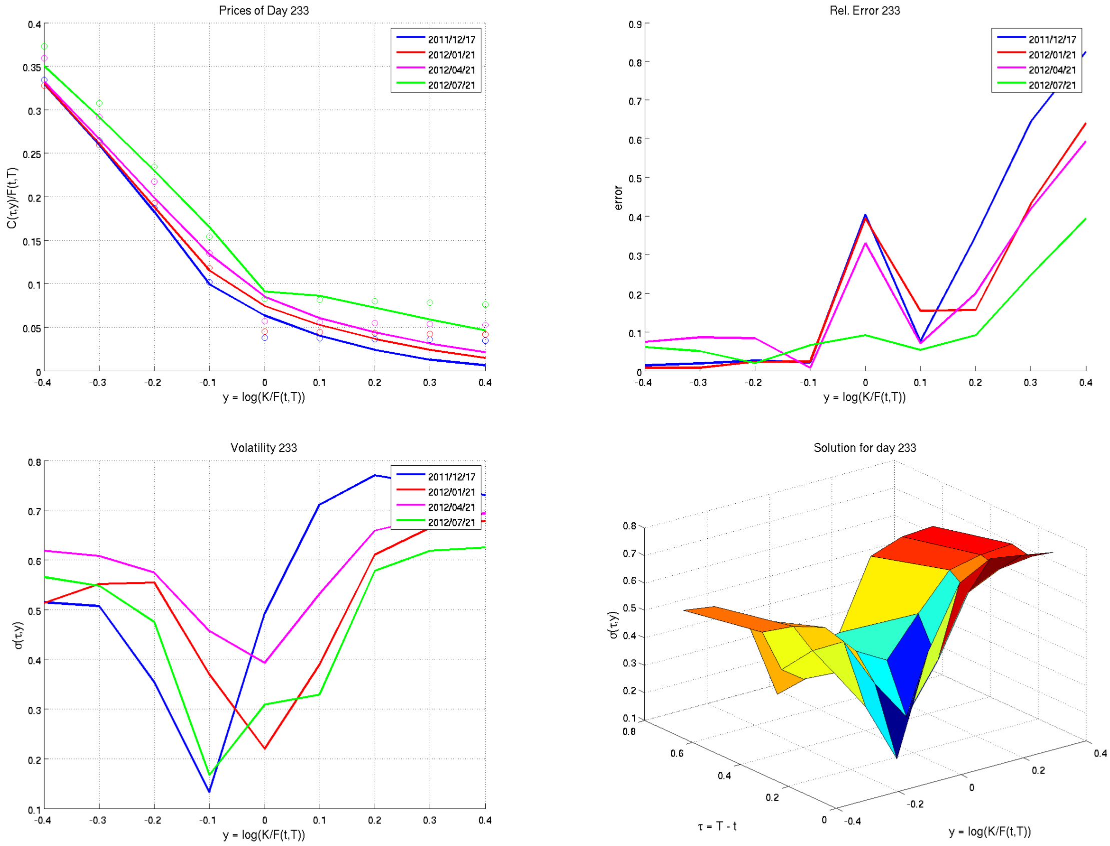

Figure 5.14: Day 233: The first image shows quoted and simulated SPX call option prices. The second one displays the relative error between such simulated and quoted prices.The third and fourth ones present the reconstructed local volatility. Solution obtained with Kullback-Leibler entropy as the regularization functional.

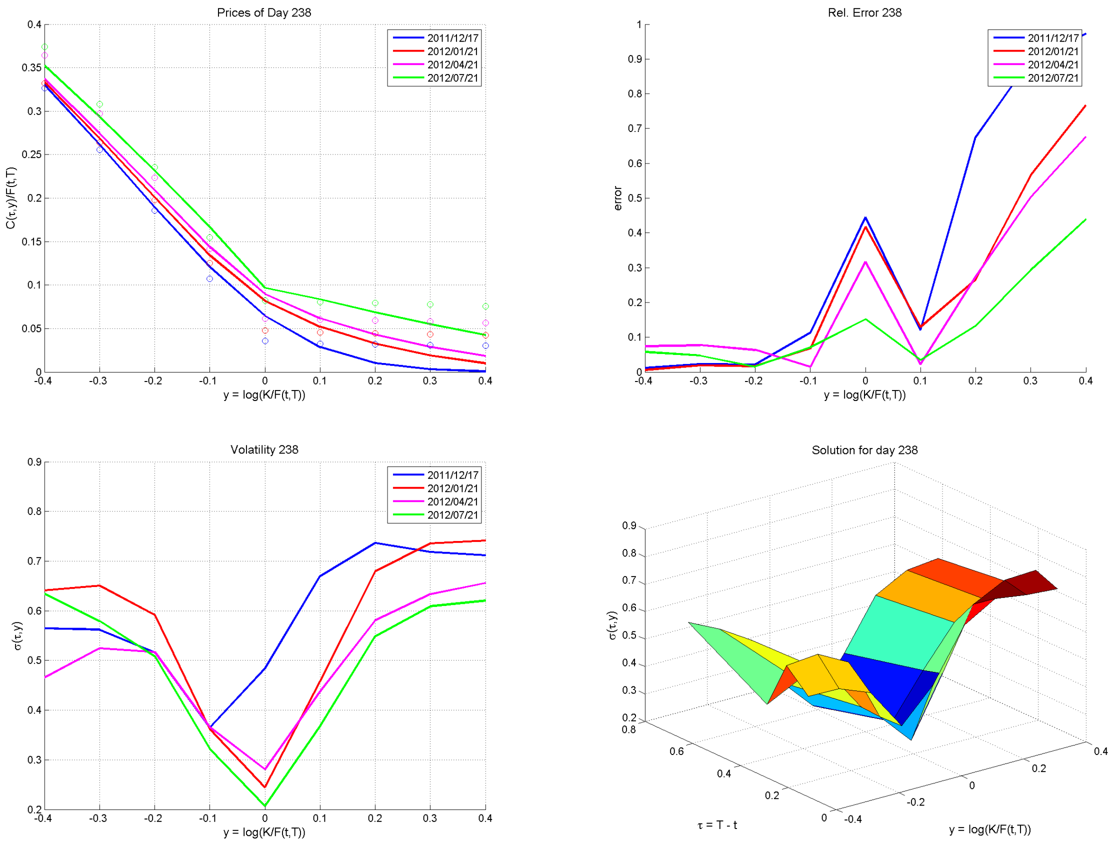

Figure 5.15: Day 238: The first image shows quoted and simulated SPX call option prices. The second one displays the relative error between such simulated and quoted prices.The third and fourth ones present the reconstructed local volatility.

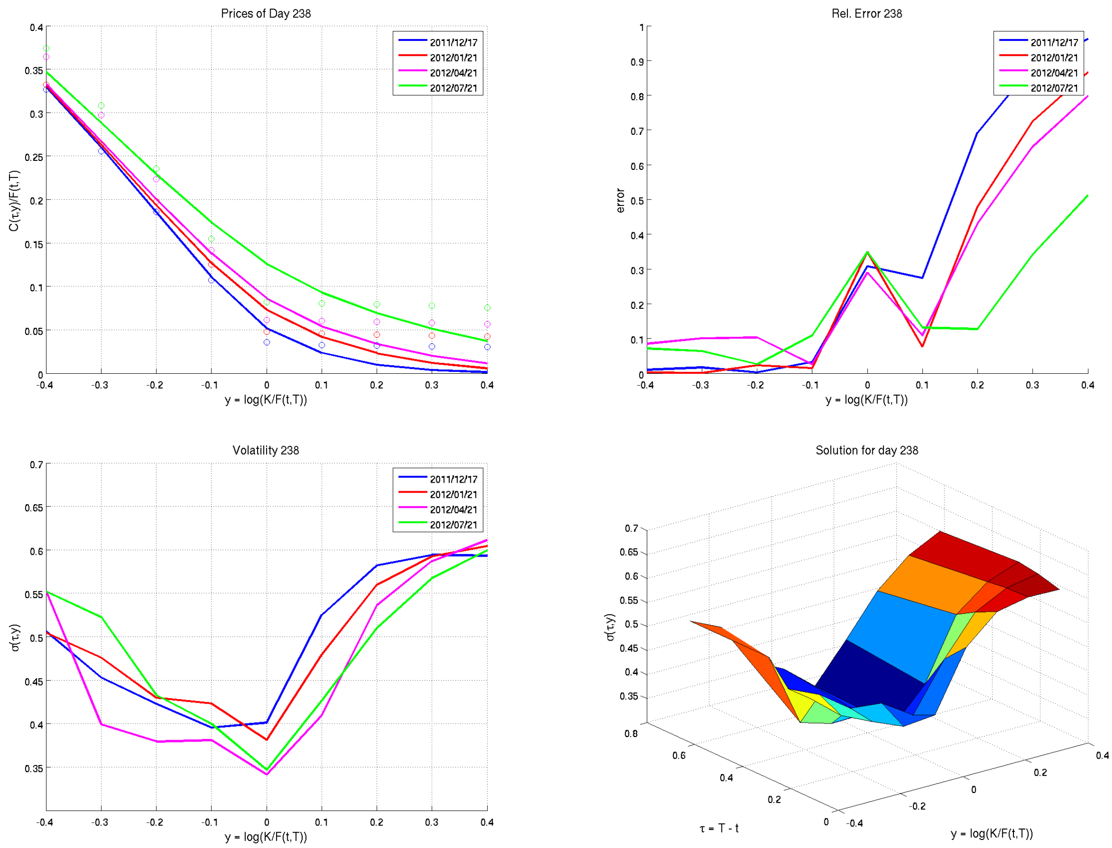

Figure 5.16: Day 238: The first image shows quoted and simulated SPX call option prices. The second one displays the relative error between such simulated and quoted prices.The third and fourth ones present the reconstructed local volatility. Solution obtained with Kullback-Leibler entropy as the regularization functional.

5.3.3 An Example with Heston Model

A very popular model in practice is the Heston model [28]:

$$\begin{aligned} &dS(t) = rS(t)dt + \sqrt{\nu(t)}S(t)dW_1(t) \text{ with } t \geq 0 \\ &d\nu(t) = \kappa(\theta - \nu(t))dt + \eta\sqrt{\nu(t)}dW_2(t) \\ &S(0) = S_0 \text{ and } \nu(0) = \nu_0, \end{aligned} \tag{5.8}$$

where W_1 and W_2 are correlated $\widetilde{\mathbb{P}}$-Brownian motions, with constant correlation parameterρ.

Note that the constant parameters

$$(\nu_0, \theta, \kappa, \eta, \rho)$$

are usually estimated from market data.

A natural question concerning the widespread use of Heston by practitioners, is *How would the local volatility surface look like if our prices were given by Heston model?* Thus, we try to answer this question by estimating the local volatility from price data generated in the following way:

1. Use the parameters
$$(\nu_0, \theta, \kappa, \eta, \rho)$$
to simulate (5.8).

2. Evaluate the formula
$$C(S_0, T, K) = \mathbb{E}^{\widetilde{\mathbb{P}}}[(S_0 - K)^+]$$
by a Monte Carlo integration in order to interpolate real data.

3. Use this interpolated data as $\mathcal{U}^\delta$ in the analysis presented above in order to find the family of local variance surfaces $\widetilde{\mathcal{A}}$.

Now we compare reconstructions obtained with interpolated and non-interpolated data.

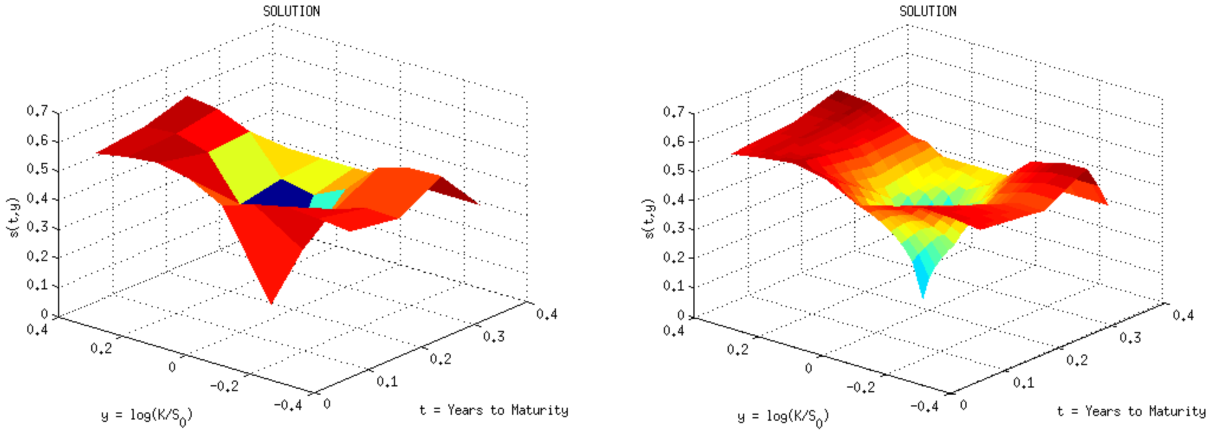

Figure 5.17: Local volatility reconstruction: The first picture was obtained by Tikhonov regularization directly from SPX data. For the second one we have used the same SPX data, but now interpolated with the help of Heston model.

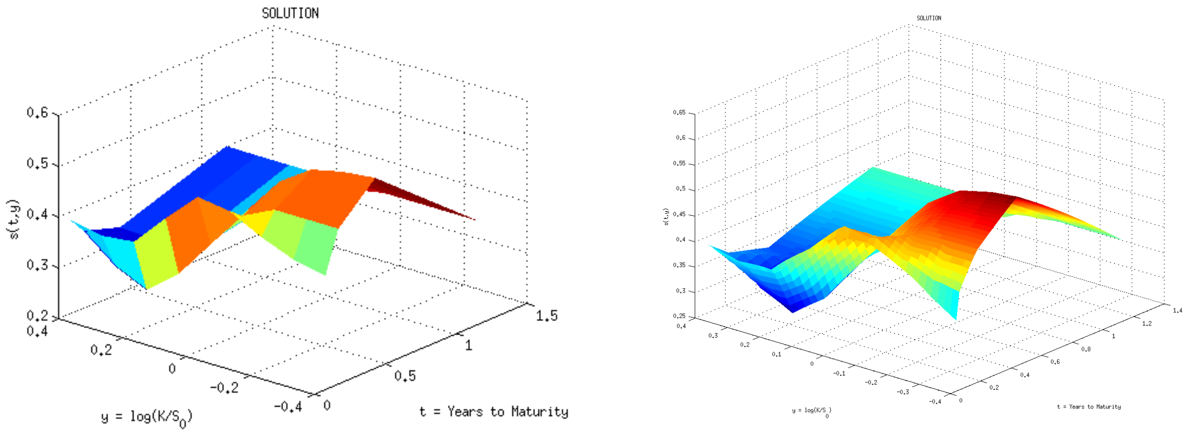

Figure 5.18: Local volatility reconstruction: The first picture was obtained by Tikhonov regularization directly from PBR data. For the second one we have used the same PBR data, but now interpolated with the help of Heston model.

5.3.4 Commodity Markets

We analyze the volatility surfaces calibrated from real data of commodity markets. We use European call option prices on futures data for Henry Hub natural gas and WTI Brent oil. Such data were traded during the year of 2011.

Since call prices with different maturities are functions of futures with different maturities, we have to perform some normalizations and make some assumptions. We first assume that such call prices satisfy the Dupire equation (4.9) and local volatility depends on the unknown spot price, which is the same for futures traded at the same day with different maturities. We assume also that volatility does not depend on futures' maturity. After, the change of variables $y = \log(K/F(t,T))$ we consider the normalized call prices

$$\widetilde{C}(t,T',T,y) = C(t,T',T,K = F(t,T)\mathrm{e}^{y})/F(t,T),$$

where T' is option's maturity, K is option's strike and C represent real call prices. It follows that these prices satisfy the same problem, namely (4.10).

After performing such normalizations, we proceed as in the case of equity markets. For each day we use such normalized data as $\mathcal{U}^{\delta}$ in the minimization procedure. We first extract the related implied volatility surface, calculate the mean of its values and use them. The implied volatility surface is used directly as the initial guess of $a = \sigma^2/2$ and its mean is used as the constant *a priori*. We choose the same regularization functional and parameter of the case of equity markets.

WTI Brent Oil

During this paragraph we use renormalized WTI Brent oil data. The numerical results are presented below. They are distributed as in the paragraph of equity market data, i.e., the first picture presents simulated and real call prices, the second is the local relative error for such prices. The third and fourth present local volatility.

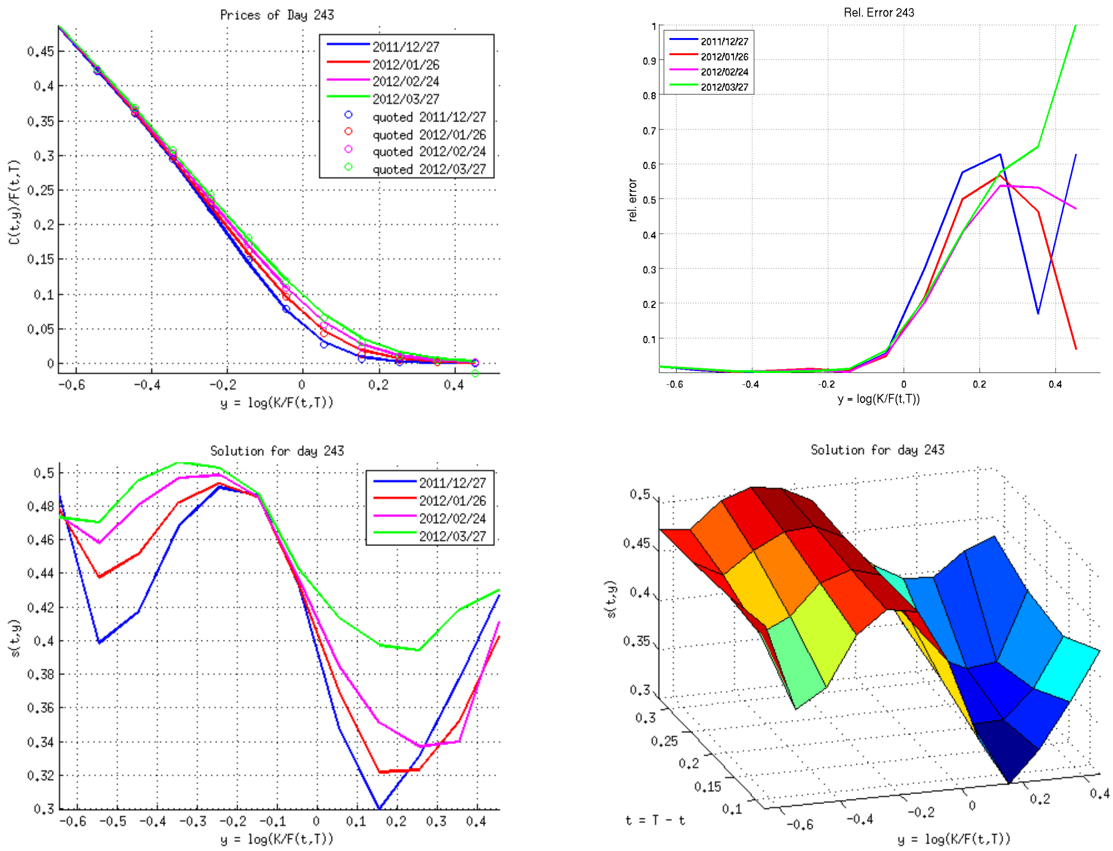

Figure 5.19: Day 242: The first image shows quoted and simulated WTI oil call option prices. The second one displays the relative error between such simulated and quoted prices.The third and fourth ones present the reconstructed local volatility.

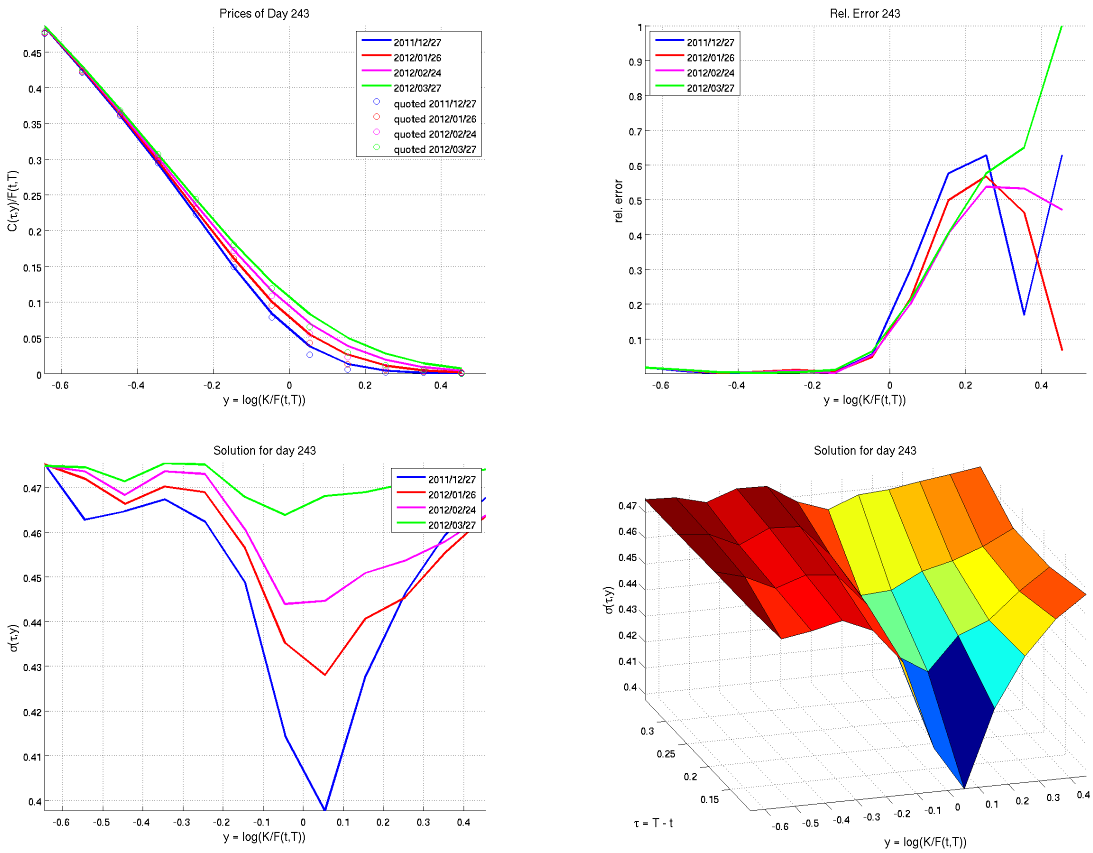

Figure 5.20: Day 242: The first image shows quoted and simulated WTI oil call option prices. The second one displays the relative error between such simulated and quoted prices.The third and fourth ones present the reconstructed local volatility. Solution obtained with Kullback-Leibler entropy as the regularization functional.

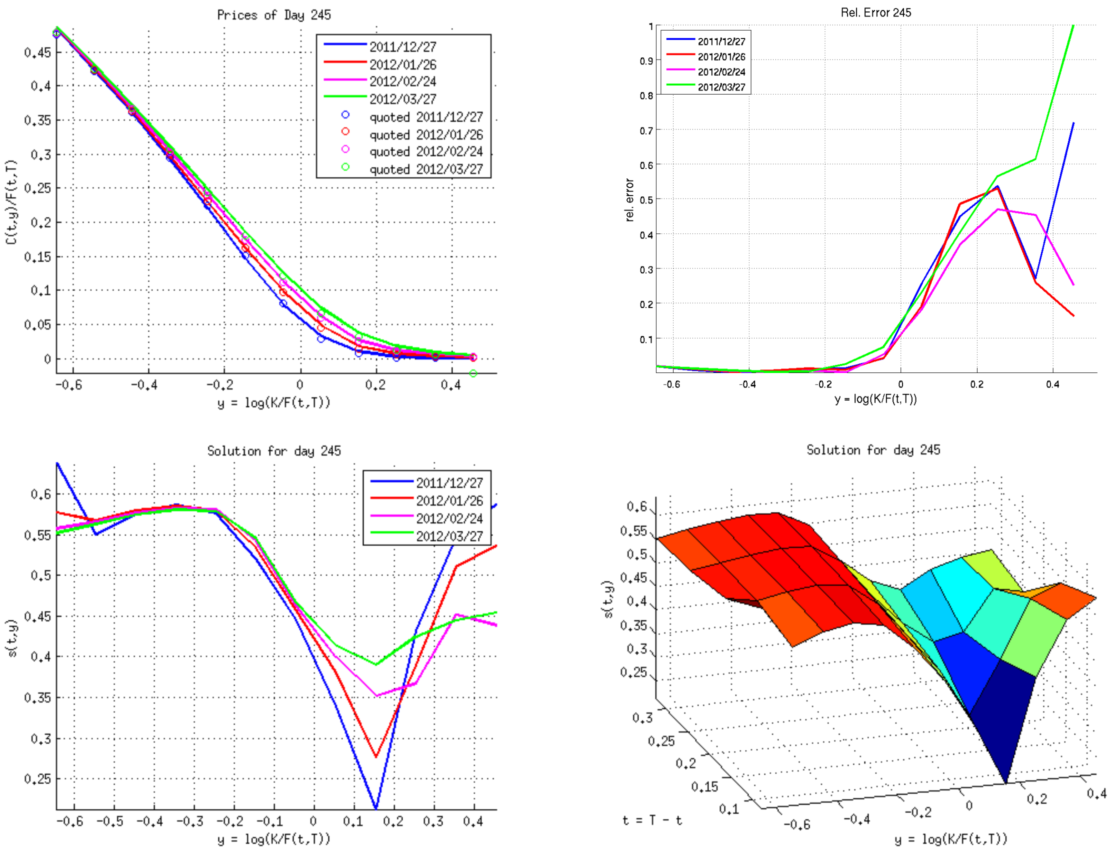

Figure 5.21: Day 244: The first image shows quoted and simulated WTI oil call option prices. The second one displays the relative error between such simulated and quoted prices.The third and fourth ones present the reconstructed local volatility.

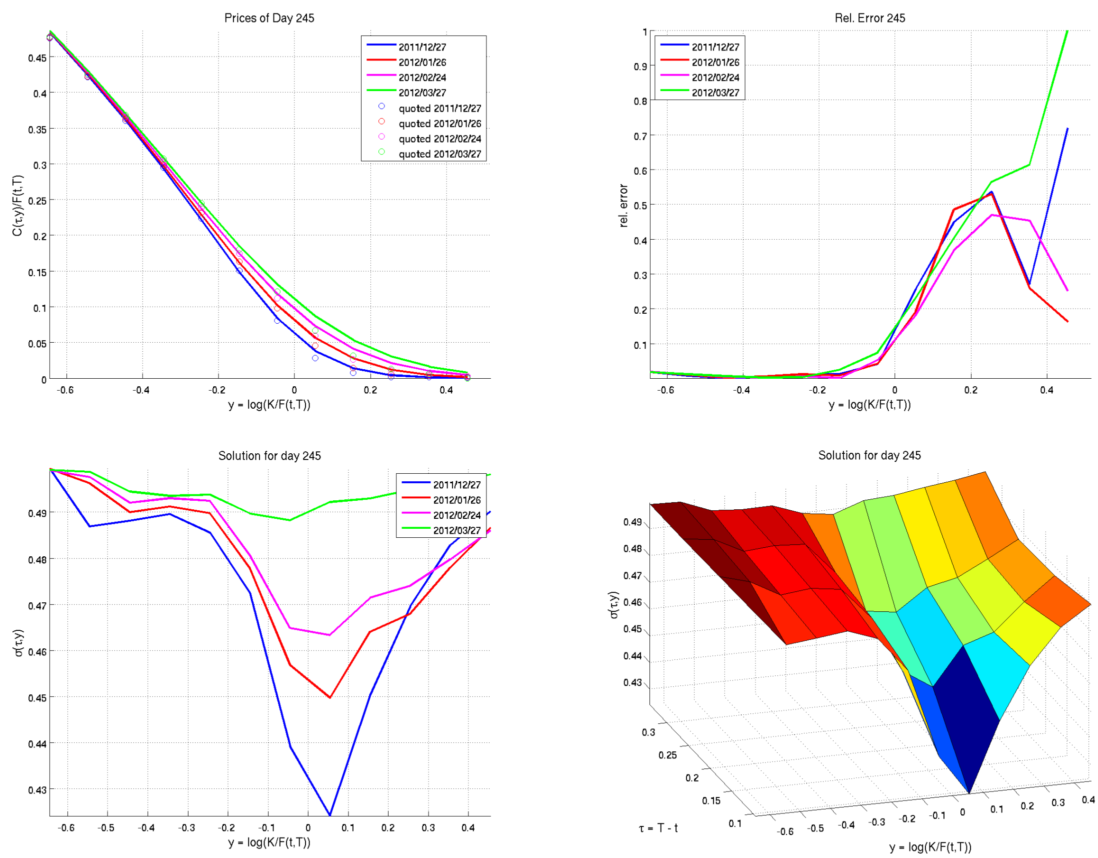

Figure 5.22: Day 244: The first image shows quoted and simulated WTI oil call option prices. The second one displays the relative error between such simulated and quoted prices.The third and fourth ones present the reconstructed local volatility. Solution obtained with Kullback-Leibler entropy as the regularization functional.

Henry Hub Natural Gas

Proceeding as in the previous paragraph we present some results for Henry Hub natural gas data.

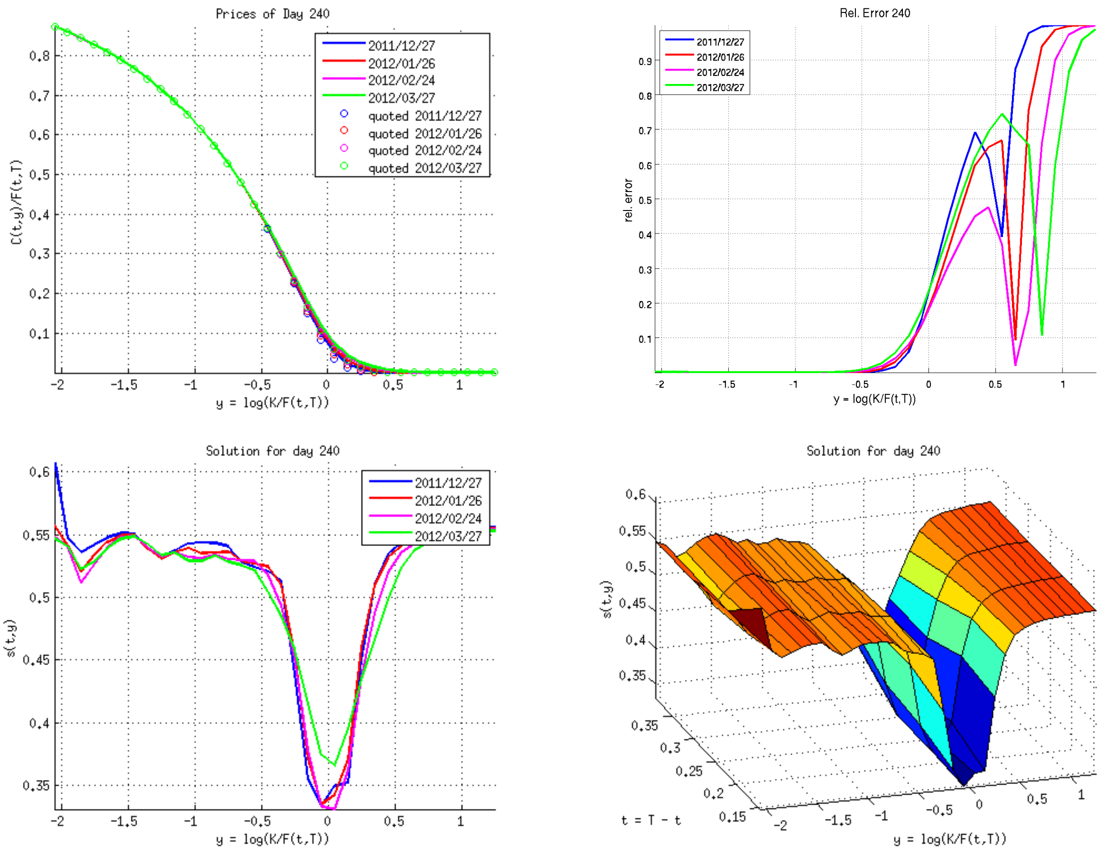

Figure 5.23: Day 239: The first image shows quoted and simulated HH natural gas call option prices. The second one displays the relative error between such simulated and quoted prices.The third and fourth ones present the reconstructed local volatility.

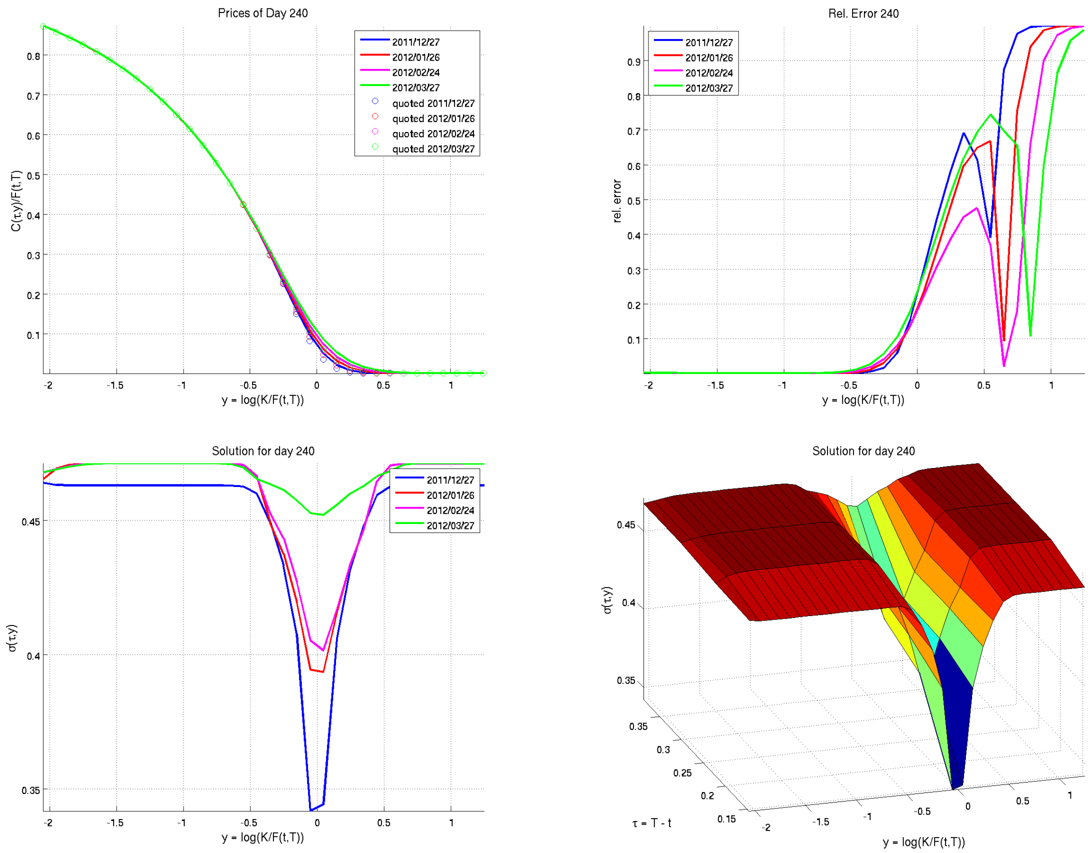

Figure 5.24: Day 239: The first image shows quoted and simulated HH natural gas call option prices. The second one displays the relative error between such simulated and quoted prices.The third and fourth ones present the reconstructed local volatility. Solution obtained with Kullback-Leibler entropy as the regularization functional.

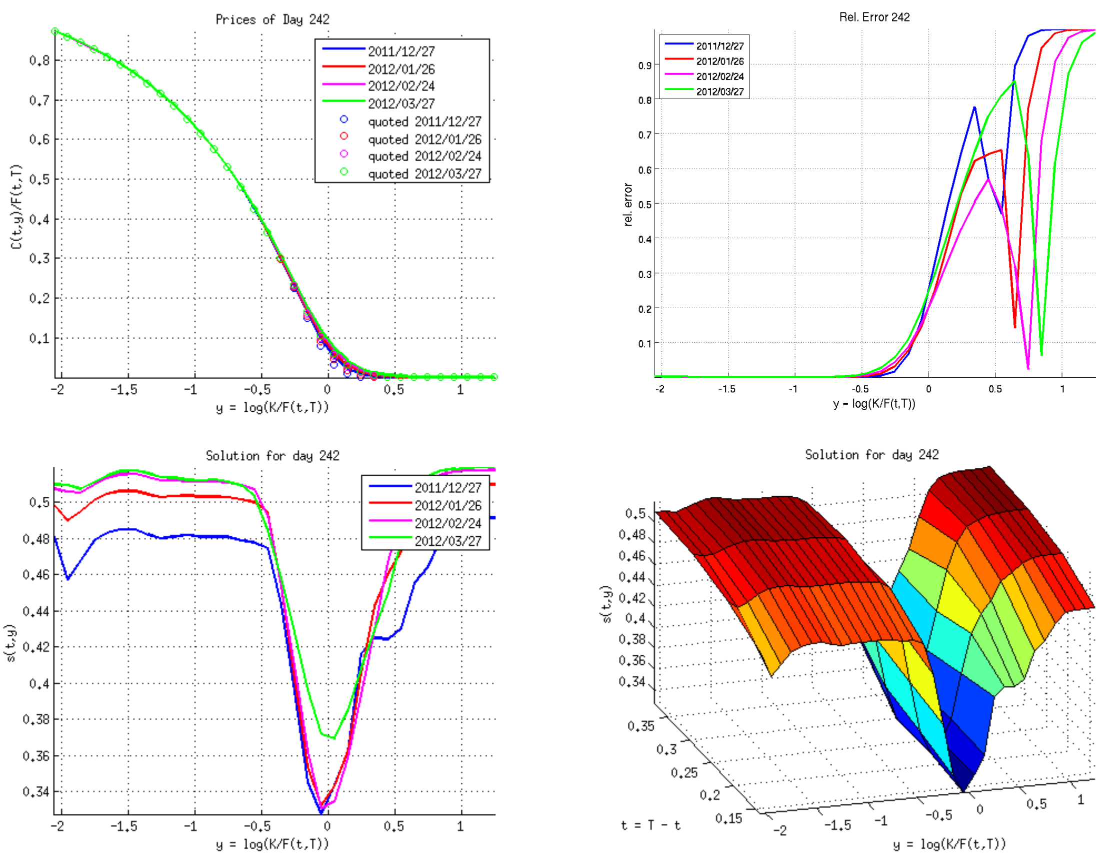

Figure 5.25: Day 241: The first image shows quoted and simulated HH natural gas call option prices. The second one displays the relative error between such simulated and quoted prices.The third and fourth ones present the reconstructed local volatility.

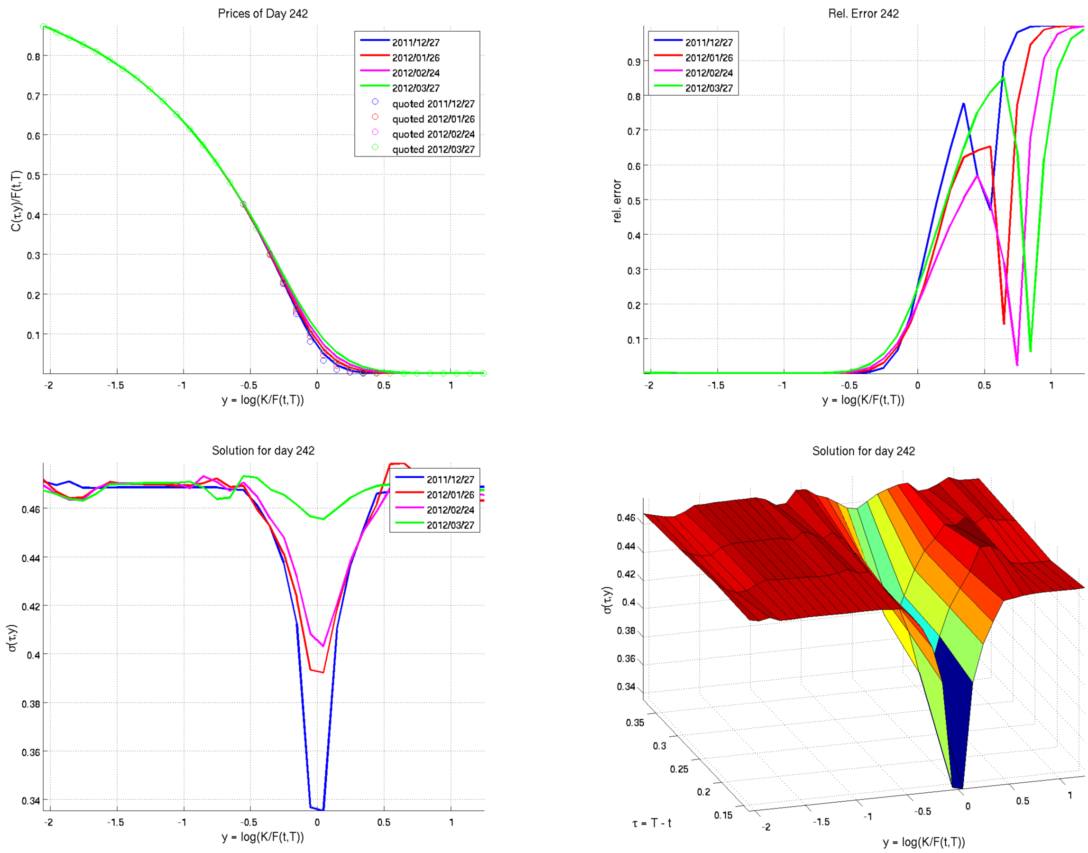

Figure 5.26: Day 241: The first image shows quoted and simulated HH natural gas call option prices. The second one displays the relative error between such simulated and quoted prices.The third and fourth ones present the reconstructed local volatility. Solution obtained with Kullback-Leibler entropy as the regularization functional.

Chapter 6

Conclusions and Future Directions

6.1 Conclusions

Making use of the Dupire framework [21], we considered models that allow the local volatility surface to depend on the initial state price [28]. In this context we defined the direct problem and deduced some background properties of the parameter to solution map. It relates families of volatility surfaces in $H^l(0, S, H^{1+\varepsilon}(D))$ with families of prices surfaces in $L^2(0, S, W_2^{1,2}(D))$, both indexed by the initial state price. In our context, the most relevant properties of the forward operator $\mathcal{U}$ are weak continuity, compactness and Frechét differentiability, since weak continuity is of fundamental importance in order to apply convex regularization. Compactness leads to the ill-posedness of the inverse problem. Frechét differentiability is important in order to develop iterative methods and to solve the regularized inverse problem by a gradient method.

We note that the direct problem was stated in a very general setting, that allows us to apply many different identification techniques. However, we have chosen to apply convex regularization tools, since it leads to robust identification techniques. Another motivation is that it generalizes many approaches found in literature as [17, 19, 23, 36, 39].

Thus, by applying regularization techniques, we established results concerning existence and stability of such families of volatilities solving the regularized inverse problem, even in the presence of noise. Concerning noise, we have stated a convergence analysis with rates based on the concept of Bregman distances for the case when the noise level goes to zero. Furthermore, we considered a discretized setup, presenting, under the same framework a convergence analysis with rates accounting separately the contribution of different sources of uncertainties.

We have stated also a very general Morozov discrepancy principle, which presents an optimal choice for the regularization parameter. It is a very useful tool since it states how much regularization is required for a given noisy data. This is the first time that this principle was stated for the volatility calibration problem.

In order to apply such techniques in the context of commodity markets, we made some assumptions on the evolution of risk-neutral prices of futures and its European vanilla options, obtaining two parabolic PDE approaches. The first one was a generalization of Black's model [7], where we have assumed that volatility is independent of the initial state but on the commodity futures' maturity. The second one was an application of Dupire's framework [21], based on the assumption that local volatility is dependent on the initial spot price instead of initial future price. In both cases, these assumptions were necessary to make some normalization and change of variables procedures. In the first case it allowed us to assume that options with different strikes derived from the same future to satisfy the same PDE. In the second case, it allowed us to assume that options derived from futures with different maturities satisfy the same PDE. Under these assumptions we stated for both cases that the identification procedures mentioned above apply within commodity futures context.

Note that all the results we have mentioned above were obtained for the identification of families of local volatilities surfaces indexed by the initial state price.

The numerical results with synthetic data illustrated the convergence to the solution when the noise level goes to zero. The resulting volatility surfaces reconstructed from real data were very satisfactory in both cases, with equity and commodity markets data. We observe that, with commodity data, we performed tests only with the framework related to Dupire's equation.

Therefore, by the theoretical results established in Chapter 3 and which were illustrated in Chapter 5 by numerical experiments, we can conclude that convex regularization is a good way of finding families of approximate local volatility surfaces. Another conclusion is that, such framework can be easily adapted to commodity markets, since we have found very interesting volatility surfaces in Chapter 5, proceeding "innocent" normalizations. However, such results have to be better understood under a financial viewpoint

6.2 Future Directions

We present below a short list of possible future directions:

- Implementing algorithms using Galerkin methods [2, 9] in order to solve the primal-dual parabolic PDE with an adaptative mesh.
- Making a comparison between quoted and simulated data for more general financial derivatives using the volatility surface obtained by these methods.
- Using the present framework with more identification techniques. In special Quasilinearization [6] and On-line estimation in a classical context [5] and in a statistical setup [49].
- Developing a similar analysis for a more general class of parabolic integro-differential operators related to jump-diffusions. See [13, 14, 15].
- Establishing a similar analysis to the case of American options, where we have to identify simultaneously the diffusion parameter and the related free boundary interface under a parabolic variational inequality problem. See [1, 2, 11, 12].

Appendix A

Technical Results and Notations

Here, we introduce some notation and present some technical results.

A.1 Banach Space Valued Maps

Let B be a Banach space, now we recall the notion of measurability for mappings with the form $f : [0,T] \longrightarrow B$ (see for example, [27, 46, 52]).

Definition 9. *(i) A function $s : [0,T] \longrightarrow B$ is called simple if it has the form*

$$s(t) = \sum_{j=1}^{m} \mathcal{X}_{A_j}(t) y_j, \;\; t \in [0,T]$$

where the sets $A_j \subset [0,T]$ are Lebesgue measurable and $y_j \in B$ with $j = 1, \ldots, m$.

(ii) A function $f : [0,T] \longrightarrow B$ is strongly measurable if there exists a sequence of simple functions as in (i) $\{s_k\}$, such that $\|s_k(t) - f(t)\|_B \to 0$ as $k \to \infty$, for almost every $t \in [0,T]$.

(iii) A function $f : [0,T] \longrightarrow B$ is weakly measurable if for each $\gamma \in B^*$, the map $t \mapsto \langle \gamma, f(t) \rangle$ is Lebesgue measurable.

(iv) A function $f : [0,T] \longrightarrow B$ is almost separably valued if there exists a subset $A \subset [0,T]$, with $|A| = 0$, such that $\{f(t) : t \in [0,T] - A\}$ is separable.

Theorem 30 (Pettis)**.** *$f : [0,T] \longrightarrow B$ is strongly measurable if f is weakly measurable and is almost separably valued.*

Definition 10. *(i) if $s(t) = \sum_{j=1}^{m} \mathcal{X}_{A_j}(t) y_j$ is a simple function, we define*

$$\int_0^T s(t)dt = \sum_{j=1}^{m} |A_j| y_j$$

(ii) We say that $f : [0,T] \longrightarrow B$ is summable if there exists a sequence of simple functions $\{s_k\}$, such that

$$\lim_{k\to\infty} \int_0^T \|s_k(t) - f(t)\|_B dt = 0.$$

(iii) If f is summable, we define

$$\int_0^T f(t)dt = \lim_{k\to\infty} \int_0^T s_k(t)dt.$$

Theorem 31 (Bochner)**.** *A strongly measurable function $f : [0,T] \longrightarrow B$ is summable if and only if $t \mapsto \|f(t)\|$ is summable. In this case*

$$\left\| \int_0^T f(t)dt \right\|_B \leq \int_0^T \|f(t)\|_B dt$$

and

$$\left\langle \gamma, \int_0^T f(t)dt \right\rangle = \int_0^T \langle \gamma, f(t) \rangle dt$$

for each $\gamma \in B^*$.

Definition 11. *Let H be a Hilbert space, we can denote by $L^2([0,T],H)$ the space of equivalence classes of strongly measurable functions $f : [0,T] \longrightarrow B$ which satisfy*

$$\|f\|^2_{L^2([0,T],H)} := \int_0^T \|f(t)\|^2_H dt < \infty,$$

and we can define the scalar product

$$\langle f, g \rangle_{L^2([0,T],H)} := \int_0^T \langle f(t), g(t) \rangle_H dt,$$

for every $f, g \in L^2([0,T],H)$.

If $B = H$, a separable Hilbert space, then we have the following theorem.

Proposition 15. *Let H be a separable Hilbert space. A map $f : [0,T] \longrightarrow H$ is weakly measurable if and only if it is strongly measurable. Furthermore, $L^2([0,T],H)$, with the above scalar product, is a Hilbert space.*

A.2 Equi-Continuity

Let X and Y be locally convex spaces. Fix the sets $B_X \subset X$ and $M \subset C(B_X, Y)$. M is called equi-continuous on B_X if for every $x_0 \in B_X$ and every zero neighborhood, $V \subset Y$ there is a zero neighborhood $U \subset X$ such that $G(x_0) - G(x) \in V$ for all $G \in M$ and all $x \in B_X$ with $x - x_0 \in U$. Furthermore, M is called uniformly equi-continuous if for every zero neighborhood $V \subset Y$ there exists a zero neighborhood $U \subset X$ such that $G(x) - G(x') \in V$ for all $G \in M$ and all $x, x' \in B_X$ with $x - x' \in U$.

From [29] we have the technical result:

Proposition 16. *Let* $F : [0,T] \times B_X \longrightarrow Y$ *be a function, and* B_X*,* X *and* Y *be as above. If* $M_1 := \{F(t,\cdot) : t \in [0,T]\} \subset C(B_X, Y)$*,* $M_2 := \{F(\cdot,x) : x \in B_X\} \subset C([0,T], Y)$ *and* M_1 *(respectively* M_2*) is equi-continuous, then* F *is continuous. Reciprocally, if* F *is continuous, then* M_1 *is equi-continuous and if additionally* B_X *is compact, then* M_2 *is equi-continuous, too.*

Bibliography

[1] Yves Achdou, *An inverse problem for parabolic variational inequalities in the calibration of american options*, SIAM journal of Control **43** (2005), no. 5, 1583–1615.

[2] Yves Achdou and Olivier Pironneau, *Computational Methods for Option Pricing*, Frontiers in Applied Mathematics, SIAM, 2005.

[3] Stephan Anzengruber and Ronny Ramlau, *Morozov's discrepancy principle for Tikhonov-type functionals with nonlinear operators*, Inverse Problems **26** (2010), no. 2.

[4] Marco Avellaneda, Craig Friedman, Richard Holmes, and Dominic Samperi, *Calibrating Volatility Surfaces Via Relative-Entropy Minimization*, Applied Mathematical Finance **4** (1997), 37–64.

[5] J. Baumeister, W. Scondo, M. Demetriou, and I. Rosen, *On-Line Parameter Estimation For Infinite-Dimensional Dynamical Systems*, SIAM Journal of Control and Optimization **35** (1997), no. 2, 678–713.

[6] Richard E. Bellman and Robert S. Roth, *Quasilinearization and the Identification Problem*, Series in Modern Applied Mathematics, vol. 2, World Scientific, 1983.

[7] Fischer Black, *The pricing of commodity contracts*, Journal of Financial Economics **3** (1976), 167–179.

[8] Fischer Black and Myron Scholes, *The pricing of options and corporate liabilities*, Journal of Political Economy **81** (1973), no. 3, 637–654.

[9] Dietrich Braess, *Finite Elements: Theory, Fast Solvers and Applications in Elasticity Theory*, third ed., Cambridge University Press, 2007.

[10] Dan Butnariu and Alfredo Iusem, *Totally Convex Functions for Fixed Points Computation and Infinite Dimensional Optimization*, Applied Optimization, vol. 40, Kluwer Academic, 2000.

[11] John Chadam and Xinfu Chen, *A Mathematical Analysis of the Optimal Exercise Boundary for American Put Options.*, SIAM Journal of Mathematical Analysis **38** (2006), 1613–1641.

[12] John Chadam, Lan Cheng, and David Saunders, *Analysis of an inverse First Passage Problem from Risk Management.*, SIAM Journal of Mathematical Analysis **38** (2006), 845–873.

[13] Rama Cont and Amel Bentata, *Forward equations for option prices in semimartingale models*, To appear in Finance & Stochastics in 2012. http://arxiv.org/abs/1001.1380.

[14] Rama Cont and Peter Tankov, *Nonparametric calibration of jump-diffusion processes*, Journal of Computational Finance **7** (2004), no. 3, 1–49.

[15] ———, *Retrieving Lévy processes from option prices: regularization of a nonlinear inverse problem*, SIAM Journal of Control and Optimization **45** (2006), no. 1, 1–25.

[16] Thomas Cover and Joy Thomas, *Elements of Information Theory*, second ed., John Wiley & Sons, 2006.

[17] Stephane Crepey, *Calibration of the local volatility in a generalized Black-Scholes model using Tikhonov regularization*, SIAM Journal of Mathematical Analysis **34** (2003), 1183–1206.

[18] Imre Csiszár, *Why Least Square and Maximum entropy? An Axiomatic Approach to Inference for Linear Inverse Problems*, The Annals of Statistics **19** (1991), no. 4, 2032–2066.

[19] Adriano De Cezaro, *On a Parabolic Inverse Problem Arising in Quantitative Finance: Convex and Iterative Regularization*, Ph.D. thesis, IMPA, Rio de Janeiro, 2010.

[20] Emanuel Derman and Iraj Kani, *Riding on a smile*, Risk **7** (1994), 32–39.

[21] Bruno Dupire, *Pricing with a smile*, Risk Magazine **7** (1994), 18–20.

[22] B. Düring, A. , Jüngel, and S. Volkwein, *Sequential Quadratic Programming Method for Volatility Estimation in Option Pricing*, Journal of Optimization Theory and Applications **139** (2008), no. 3, 515–540.

[23] Herbert Egger and Heinz Engl, *Tikhonov Regularization Applied to the Inverse Problem of Option Pricing: Convergence analysis and Rates*, Inverse Problems **21** (2005), 1027–1045.

[24] Paul Eggermont, *Maximum Entropy Regularization for Fredholm integral equations of the first kind*, SIAM Journal of Mathematical Analysis **24** (1993), 1557–1576.

[25] Ivar Ekland and Roger Teman, *Convex Analysis and Variational Problems*, North Holland, Amsterdan, 1976.

[26] Heinz Engl, Martin Hanke, and Andreas Neubauer, *Regularization of Inverse Problems*, Mathematics and its Applications, vol. 375, Kluwer Academic Publishers Group, Dordrecht, 1996.

[27] Lawrence Craig Evans, *Partial Differential Equations*, Graduate Studies in Mathematics, vol. 19, AMS, 1998.

[28] Jim Gatheral, *The Volatility Surface: A Practitioner's Guide*, Wiley Finance, John Wiley & Sons, 2006.

[29] Markus Haltmeier, Otmar Scherzer, and Antonio Leitão, *Tikhonov and iterative regularization methods for embedded inverse problems*, www.industrial-geometry.at/uploads/emb_preprint.pdf.

[30] D. Heath, R. Jarrow, and A. Morton, *Bond Pricing and the Term Structure of Interest Rates: A New Methodology.*, Econometrica **60** (1992), no. 1, 77–105.

[31] T. Hein and B. Hoffman, *On the nature of ill-posedness of an inverse problem arising in option pricing*, Inverse Problems **19** (2003), no. 6, 1319–1338.

[32] Desmond Higham, *An Algorithmic Introduction to Numerical Simulation of Stochastic Differential Equations*, SIAM Review **43** (2001), no. 3, 525–546.

[33] Juri Hinz and Martina Wilhelm, *Pricing flow commodities derivatives using fixed incoming market techniques*, International Journal of Theoretical and Applied Finance (IJTAF) **9** (2006), 1299–1321.

[34] B. Hoffman and R. Krämer, *On maximun entropy regularization for a specific incverse problem of option pricing*, Journal of Inverse and Ill-Posed Problems **13** (2005), no. 1, 41–63.

[35] Rafael Iorio and Valéria Iorio, *Fourier Analysis and Partial Differential Equations*, Cambridge Studies in Advanced Mathematics, vol. 70, Cambridge University Press, 2001.

[36] Nicholas Jackson, Endre Süli, and Sam Howinson, *Computation of Deterministic Volatility Surfaces*, Journal of Computational Finance **2** (1998), 5–32.

[37] Ralph Korn and Elke Korn, *Option Price and Portfolio Optimization: Modern Methods of Mathematical Finance*, Graduate Studies in Mathematics, vol. 31, AMS, 2001.

[38] Olga Ladyzenskaja, Vselvolod Solonnikov, and Nina Ural'ceva, *Linear and quasi-linear equations of parabolic type*, Translations of Mathematical Monographs, AMS, 1968.

[39] Ronald Lagnado and Stanley Osher, *A Techinque for Calibrating Derivative Security Pricing Models: Numerical Solution of an Inverse Problem*, Journal of Computational Finance **1** (1997), 13–25.

[40] Robert C. Merton, *Theory of Rational Option Pricing*, Bell Journal of Economics and Management Science **4** (1973), 141–183.

[41] Attilio Meucci, *Risk and Asset Allocation*, Springer Finance, Springer, 2005.

[42] Vladimir A. Morozov, *On the solution of functional equations by the method of regularization*, Soviet Mathematics Doklady **7** (1966), 414–417.

[43] Dragana Pilipovic, *Energy Risk: Valuing and Managing Energy Derivatives*, McGraw-Hill, 1997.

[44] Christiane Pöschl, Elena Resmerita, and Otmar Scherzer, *Discretization of variational regularization in Banach spaces*, Inverse Problems **26** (2010).

[45] Ricardo Rebonato, *Volatility and Correlation: The Perfect Hedger and the Fox*, second ed., John Wiley & Sons, 2004.

[46] Michael Reed and Barry Simon, *Methods of Modern Mathematical Physics I: Functional Analysis*, Academic Press, 1980.

[47] Elena Resmerita and Robert Anderssen, *Joint additive Kullback-Leibler residual minimization and regularization for linear inverse problems*, Mathematical Methods in The Applied Sciences **30** (2007), 1527–1544.

[48] Otmar Scherzer, Markus Grasmair, Harald Grossauer, Markus Haltmeier, and Frank Lenzen, *Variational Methods in Imaging*, Applied Mathematical Sciences, vol. 167, Springer, New York, 2008.

[49] Erkki Somersalo and Jari Kapio, *Statistical and Computational Inverse Problems*, Applied Mathematical Sciences, vol. 160, Springer, 2004.

[50] Andrey Tikhonov and Vasily Arsenin, *Nonlinear Ill-posed Problems*, Chapman and Hall, 1998.

[51] Paul Wilmott, Sam Howinson, and Jeff Dewynne, *The Mathematics of Financial Derivatives: A Student Introduction*, Cambridge University Press, 1995.

[52] Kosaku Yosida, *Functional Analysis*, Springer-Verlag, Heidelberg, 1995.

[53] William Ziemer, *Weakly Differentiable Functions*, Graduate Texts in Mathematics, vol. 120, Springer, 1989.

List of Symbols

$C(0, S, H^{1+\varepsilon}(D))$ the space of continuous $H^{1+\varepsilon}(D)$-valued maps, page 16

$C(T, K, s, t)$ option price at time t, when stock price is s, with maturity T and strike K, page 4

$C, C(T, K)$ option price, with strike K and maturity T, page 4

D the domain $(0, T) \times I$, page 8

$D_\xi(b, a)$ Bregman distance between a and b with respect to $\xi \in \partial f$, page 12

$E(t)$ noise variable at time t, page 28

$F, F(s, a)$ parameter to solution map at time s and parameter a, page 17

$F_m(t, a)$ finite dimensional version of $F(t, a)$, page 43

$F_{t,T}$ future contract price at time t eith maturity T, page 48

G forward operator for a single time, page 9

I open interval probably unbounded, page 8

$J^\delta(\mathcal{A})$ squared residual between the data $\mathcal{U}^\delta$ and $\mathcal{U}$ at $\mathcal{A}$ relative to the noise level $\delta > 0$, page 62

K strike price of a option, page 4

$KL(\cdot, \cdot\cdot)$ Kullback-Leibler relative entropy, page 32

$L^1(0, S, H^{1+\varepsilon}(D))$ Bochner integrable $H^{1+\varepsilon}(D)$-valued maps, page 16

Q domain of definition of the forward operator G for a single time, page 8

Q_n finite dimensional version of Q, page 42

$S, S(t)$ stock price, page 3

S_0 stock price at time $t = 0$, page 3

T maturity time of a option, page 4

$W_2^{1,2}(D)$ Sobolev space of type L^2 of functions on the domain D, with one derivative in time and up two derivatives in space, all them in $L^2(D)$, page 9

$W^{\widetilde{\mathbb{P}}}, W^{\widetilde{\mathbb{P}}}(t)$ standard Brownian motion in the risk-neutral probability measure $\widetilde{\mathbb{P}}$, page 3

$\mathcal{F}_{\alpha,u^\delta}(a)$ a Tikhonov functional, page 11

Ω sample space, page 3

$\mathfrak{Q}$ the domain where the forward operator $\mathcal{U}$ is defined in $H^l(0,S,H^{1+\varepsilon}(D))$, page 16

$H^l(0,S,H^{1+\varepsilon}(D))$ a Sobolev-Bochner space, page 16

$\mathcal{A}$ an element of $L^2(0,S,H^{1+\varepsilon}(D))$, page 16

$\alpha, \alpha(\delta)$ regularization parameter depending on the noise level δ, page 11

δ a positive noise level, page 10

$\hat{\mathcal{A}}$ the Fourier series of $\mathcal{A} \in L^2(0,S,H^{1+\varepsilon}(D))$, page 16

$H^{1+\varepsilon}(D)$ Sobolev space of type L^2 with order $1+\varepsilon$ on the domain D, page 8

$H^{1+\varepsilon}(D)_{\mathbb{C}}$ the complexification $H^{1+\varepsilon}(D) \oplus iH^{1+\varepsilon}(D)$ of $H^{1+\varepsilon}(D)$, page 16

$\langle \cdot, \cdot\cdot \rangle$ dual product, page 12

$\langle \cdot, \cdot\cdot \rangle_U$ the inner product of the space U., page 16

$\langle \cdot, \cdot\cdot \rangle_l$ the inner product of $H^l(0,S,H^{1+\varepsilon}(D))$, page 16

$L^2(0,S,H^{1+\varepsilon}(D))$ space of Bochner squared integrable $H^{1+\varepsilon}(D)$-valued maps, page 16

$\mathbb{E}$ expected value in the historical measure, page 4

$\mathbb{E}^{\widetilde{\mathbb{P}}}$ expected value in the risk-neutral probability measure $\widetilde{\mathbb{P}}$, page 4

$\mathbb{F}$ is a filtration, page 3

$\mathbb{R}$ real line, page 8

$\mathcal{D}_B(f)$ Bregman domain of the functional f, page 12

$\mathcal{E}$ $W_2^{1,2}(D)$-valued map representing the noise variable, page 28

$\mathcal{F}_{\mathcal{A}_0,\alpha}^{\mathcal{U}^\delta}$ Tikhonov functional, page 28

$\mathcal{F}_{\mathcal{U}^\delta,a_0}^{m,\alpha}$ finite dimensional Tikhonov functional, page 43

$\mathcal{S}(a)$ Shanon Entropy of a, page 32

$\mathscr{K}\mathscr{L}(\cdot,\cdot\cdot)$ Integrated Kullback-Leibler relative entropy, page 33

$\mathscr{S}(\mathcal{A})$ Integrated Shanon Entropy of $\mathcal{A}$, page 33

$\mathscr{U}$ a σ-algebra, page 3

$\nu, \nu(t)$ stochastic variance, page 3

∂f sub-differential of the functional f, page 12

$\partial_a F(s,a)h$ the Gateaux derivative of operator F at the variable a in the direction h, page 19

ρ_N^m distance between the operators $\mathcal{U}_N^m$ and $\mathcal{U}$, page 43

$\sigma(S_0, K, T)$ local volatility surface, page 3

$\sigma(t,s)$ local volatility function, page 7

$\widetilde{\mathcal{A}}$ a solution for the inverse problem, page 28

τ time to maturity, i.e., $\tau = T - t$, page 8

$\widetilde{\mathcal{U}}$ noiseless option price data, page 27

$\mathcal{U}^\delta$ observable option price data corrupted by noise of level $\delta > 0$, page 28

$\mathcal{U}$,$\mathcal{U}(\mathcal{A})$ the forward operator with parameter $\mathcal{A}$, page 17

$\mathcal{U}'(\mathcal{A})$ Frechét derivative of the operator $\mathcal{U}$ at $\mathcal{A}$, page 21

$\mathcal{U}'(\mathcal{A})\mathcal{H}$ Gateaux derivative of the operator $\mathcal{U}$ at $\mathcal{A}$ in the direction $\mathcal{H}$, page 21

$\mathcal{U}_N^m$ finite dimensional version of the operator $\mathcal{U}$, page 43

ε positive real constant in the interval $(0,1)$, page 8

ε_m distance between the operators F_m and F, page 43

$\widetilde{\mathbb{P}}$ risk-neutral probability measure, page 3

$a, a(\tau, y)$ local variance, squared volatility divided by 2 in the time to maturity versus log-moneyness variables, page 8

$a^\dagger$ solution of the inverse problem, page 11

a_0 a given element of $H^{1+\varepsilon}(D)$ bounded above and below by a_1 and a_2, respectively, page 8

a_1, a_2 positive real constant bounding the elements of Q, page 8

$f \otimes g$ Tensor product, page 21

f_{a_0} regularization functional with *a priori* a_0, page 11

$r, r(t)$ risk-free interest rate, page 3

t current time variable, page 3

$u(a)$ option prices surface satisfying equation (1.4) with diffusion parameter a, page 9

$u, u(\tau, y)$ option price at time to maturity versus log-moneyness variables, page 8

u^δ observable price data corrupted by noise of level $\delta > 0$, page 10

y log-moneyness variable $y = \log(K/S_0)$, page 8