Computing general equilibrium with incomplete markets and default

Susan Schommer

Adviser: Aloísio Araújo

Instituto Nacional de Matemática Pura e Aplicada

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Abstract

General economic equilibrium with incomplete markets and default represents an important tool in competitive economic theory, and has led to fundamental insights into the behavior of real and financial markets. The numerical computation of these models is important for understanding the behavior of the real world economy, and may lead to insights on the effects of regulation and welfare. However, the current approach to the computation of general equilibrium, based on homotopy methods, does not scale well to complex economic models such as general equilibrium with default. The computation is technically difficult because the utility function and constraints may not be smooth on the entire domain when considering default penalties and collateral models.

We consider a two-period exchange economy with default for two classes of models. In the first (collateral) model, promises have to be backed by a durable good, held by the borrower as collateral. In the second (default penalties) model, an agent incurs a loss in utility when she defaults, the loss increasing proportionately with the amount of default.

In Chapter 1 we compute general equilibrium with incomplete markets, collateral and default penalties. The computation of general equilibrium is treated as a nonlinear programming problem and solved by an optimization procedure for large computation - ALGENCAN - an Augmented Lagrangian Method for general nonlinear programming problems. We illustrate the proposed method by computing equilibria for some examples, showing its robustness.

In Chapter 2 (based on collaboration with Aloísio Araújo and Felix Kubler) we examine the effects of default and scarcity of collateralizable durable goods on risk-sharing. We assume that there is a large set of assets, but which distinguish themselves by the collateral requirement. There are at least as many assets available for trade as there are states of the world. In the example 1, if there is an abundance of commodities that can be used as collateral and if each agent owns a large fraction of these commodities, markets are complete and competitive equilibrium allocations Pareto optimal. If, on the other hand, the collateralizable durable good is scarce or if some agents do not own enough of the collateralizable durable good in the first period, markets can be endogenously incomplete, not all of the available assets are traded in the competitive equilibrium and allocations are not Pareto optimal. We give examples that show that welfare losses can be quantitatively large and

examine the scope for government intervention. We also show that if the borrower owns almost no durable goods, the only asset traded in equilibrium is the one with the lowest possible collateral - that can be interpreted as a subprime loan.

In Chapter 3 (based on collaboration with Aloísio Araújo) we examine, through numerical examples, when the equilibria allocations can approach the Pareto frontier by the use of a default mechanism. As in the Chapter 2 we assume that there are at least as many assets available for trade as there are states of the world. Our main focus is on the extent that the equilibria allocations approximate the Pareto frontier, and we exhibit how this quantitative problem is quite sensitive to qualitative features of the endowment distribution. In our examples, if the endowment distribution displays only heterogeneity between periods (e.g., one agent is the richest in the first period and another is the richest in all states of nature of the second period), some collateral equilibria are Pareto optimal. If the heterogeneity of the endowments is also manifest between states of nature, default penalties equilibria are often Pareto superior with respect to collateral equilibria.

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Chapter 1

Computing equilibrium with collateral and default penalties

1.1 Introduction

Most of the literature about the computation of general economic equilibrium has been done for complete markets (GE), the first efforts involving fixed-point algorithms in [Sca1], [Sca2], [ESca] and [HaSca], followed by a different approach, based on global Newton's methods in [Sm1] and [Sm2]. The optimization perspective in the form of variational inequality formulations, complementary formulations in particular, have been brought in [D],[Man],[Mat], [FDM] and [JoRW]. For others approaches see [Ja], [CoV] and [CoMPRV]. However, none of these approaches led to extensions for incomplete markets. According to [MS] a system GE is a market structure that is principally of theoretical interest, it can be viewed as an ideal system of markets.

General equilibrium with incomplete markets (GEI) is an extension of the GE with real and financial markets in which the structure of the markets is incomplete, where agents keep all their promises by assumption, i. e., GEI model without default. Up to now the computation of GEI has been approached through homotopy methods (see [BDE1], [BDE2], [DE], [ES], [S1], [S2], [KS1], [K], [KS2], [HeK], and [JKS]) and interior-point methods (see [Eb1] and [Eb2]). Homotopy methods possess goods theoretical properties, but these methods may be inappropriate to dealing with inequalities, see [W], and may fail to produce a solution even for relatively simple systems of nonlinear equations, see [NW]. Computing GE and GEI models is interesting as benchmark economies in which collateral and default penalties are absent, because there is no need to worry about default. It is easy to see the role collateral and default penalties play in the economy: in a world in which promises can exceed physical endowments, there is a substantial amount of default.

The extension of standard models of general equilibrium to allow for default represents an important tool in competitive economic theory, and has led to fundamental insights into the behavior of real and financial markets. Default can be either strategic or due to ill-fortune (bankruptcy). In order to allow for strategic default, but maintain some incentive for repayment, two classes of models has been studied, GEI with penalties and GEI with collateral. For the first class, default penalties are imposed on agents: the model considered in [DGS1] and [DGS2] an agent incurs a loss in utility when he defaults, the loss increasing proportionately with the value of default. The second class as considered in [GZ] agents have to put durable goods as collateral for their assets, but in case of default, the collateral is seized by the creditors. To obtain a better understanding of the impact of collateral requirements, penalties and default on the behavior of economies with incomplete markets, it is necessary to compute equilibrium in such models.

The GEI model with possibility of default is technically difficult because the utility function (for penalty model) and constraints (for collateral model) may not be smooth on the entire domain. In order to handle non-differentiable [KS3] approximate the equilibrium function for computing GEI model with collateral in the infinite-horizon exchange economy.

This chapter presents a general algorithm for the computation of general equilibrium in a range of models: complete markets, incomplete markets and incomplete markets with default (collateral and default penalties). We suggest to represent the general equilibrium model as a large system of nonlinear equations, and the optimization problem consists into solving this system. In all the cases the nonlinear models are solved using ALGENCAN (see [ABMS]) an Augmented Lagrangian Method for general nonlinear programming problems with bound-constraints (freely available at the TANGO Project web page [B]).

We consider examples that can simultaneously cover all cases, by reinterpreting in various ways the set of assets. Also, the algorithm has been implemented for examples of GEI models considered in the literature by [DE] and [S1] and also to a GEI model with collateral considered in [G1].

This chapter is organized as follows. In Section 1.2 we present the economic model as a nonlinear system and we discuss some of its essential features. In Section 1.3 we describe properties and usage of ALGENCAN. Numerical experiments are shown in Section 1.4.

1.2 The economy model

We consider a pure exchange economy over two time periods $t = 0, 1$ with uncertainty over the state of nature in second period denoted by the subscript $s \in \mathcal{S} = \{1, \ldots, S\}$. In the first period there isn't uncertainty. For convenience, the first period will sometimes be called state 0 so that in total there are $\mathcal{S}^* = \mathcal{S} + 1$ states (see [MS]).

Agents and Commodities: The economy consists of a finite number of H agents denoted by the superscript $h \in \mathcal{H} = \{1, \ldots, H\}$ and a finite number of L goods or commodities, denoted by the subscript $l \in \mathcal{L} = \{1, \ldots, L\}$. The commodities can be perishable or durable. We suppose that each commodity l is transformed into a vector $Y_{sl} = (Y_{sl1}, \ldots, Y_{slL}) \in \mathbb{R}^L_+$ in each state s. Each unit of the good l in the first period yields $Y_{sll'}$ units of good l' in state s.

Each agent has an initial endowment of the L goods in each state, $e^h \in \mathbb{R}^{S^*L}_+$. The preference ordering of agent h is represented by a utility function of the consumption $x^h = (x^h_0, x^h_1, \ldots, x^h_S) \in \mathbb{R}^{S^*L}_+$, denoted by $u^h : \mathbb{R}^{S^*L}_+ \to \mathbb{R}$. The spot prices of goods are represented by $p \in \mathbb{R}^{S^*L}_{++}$.

The characteristics of agent h are summarized by a utility function and endowment vector (u^h, e^h) satisfying:

A 1. *$u^h : \mathbb{R}^{S^*L}_+ \to \mathbb{R}$ is continuous on $\mathbb{R}^{S^*L}_+$ and C^∞ on $\mathbb{R}^{S^*L}_{++}$;*

A 2. *for each $x^h \in \mathbb{R}^{S^*L}_{++}$, $\nabla u^h(x) \in \mathbb{R}^{S^*L}_{++}$, and $f^T \nabla^2 u^h(x) f < 0$ for all $f \neq 0$ such that $\nabla u^h(x) f = 0$;*

A 3. $\sum_h e^h_{0l} > 0; \quad \forall l \in \mathcal{L}$*;*

A 4. $\sum_h e^h_{sl} + \sum_h \sum_{l'} Y_{sl'l} e^h_{0l'} > 0; \quad \forall s \in \mathcal{S}$ *and* $l \in \mathcal{L}$*;*

Assumptions A1 and A2 say that utility functions are continuous, strictly monotone, quasi-concave and smooth in the interior of the domain. Assumptions A3 and A4 says that the initial endowment is positive in the aggregate (see [GZ]). These are standard assumptions.

Asset, Collateral and Penalty: There are J real assets denoted by the subscript $j \in \mathcal{J} = \{1, \ldots, J\}$. Let $A_j \in \mathbb{R}^{SL}$ be the promise, per unit of asset j, of delivery of commodity l in each state s. Given commodity prices $p_{sl} \in \mathbb{R}_+$ the matrix of returns $V_{sj} = \sum_l p_{sl} A_{slj}$ completely describes the financial promise at the second period allowed by the real asset structure. The financial markets are said to be complete if $rank(V_{sj}) = S$. When $rank(V_{sj}) < S$, the financial markets are said to be incomplete (see [MS]). Let $q \in \mathbb{R}^J$ be the vector of the asset prices and $z^h \in \mathbb{R}^J$ be the portfolio of agent h, with $z^h = \theta^h - \varphi^h$ where $\theta^h \in \mathbb{R}^J_+$ are asset purchases of agent h and $\varphi^h \in \mathbb{R}^J_+$ are asset sales of agent h. Models may also incorporate a bound $Q^h_j \in \mathbb{R}_+$ on the sale of asset j by agent h.

In the two-period model of [GZ] to each promise j we must formally associate levels of collateral $C_j \in \mathbb{R}^L_+$, which are given exogenously and have the purpose of protecting the buyer when sellers do not honor their commitments. The collateral in this economy consists of shares in the physical assets. For simplicity, we assume that the collateral always has to be held by the borrower (i.e. in their notation this would be C^B_j).

Another class of model which allows for strategic default, but maintains some incentive for repayment, are default penalties. Following the model of [DGS1] and [DGS2] an agent incurs in a loss of utility when he defaults, the loss increasing proportionately with the value of default. We denote by $\lambda^h_{sj} \in \bar{\mathbb{R}}_+$ the real default penalty on agent h for asset j in state s. The effective payment is $D^h \in \mathbb{R}^{SLJ}_+$ and $K \in [0,1]^{SJ}$ is expected delivery rates on assets.

1.2.1 General equilibrium with complete markets (GE)

In this section we assume that markets are complete. As explained above, the financial markets are said to be complete if $rank(V_{sj}) = S$. In this case agents are unrestricted to transfer wealth back and forth between periods and states, all activity in an economy to be solved in a single period (first period), this is the classical Arrow-Debreu framework.

The economy with complete markets, E_{GE}, is characterized by the agents' utility functions $u = (u^h)_{h\in\mathcal{H}}$, the agents' endowment process $e = (e^h)_{h\in\mathcal{H}}$ and durability technologies $Y = (Y_{sl})_{s\in\mathcal{S},l\in\mathcal{L}}$. If markets are complete, agents can insure themselves against any type of contingency in period $t = 1$. Then, each agent h can sell his endowment $e^h = (e^h_0, e^h_1, \ldots, e^h_S)$ at the prices $p = (p_0, p_1, \ldots, p_S)$ to obtain the income pe^h and can purchase any consumption satisfying $px^h \leq pe^h + pYx^h_0$ (see [MS]) As the market are complete the buget set is thus defined by

$$B(p, e^h) = \{x^h \in \mathbb{R}^{S^*L}_+ | \sum_{l\in\mathcal{L}} p_{0l}(x^h_{0l} - e^h_{0l}) + \sum_{l\in\mathcal{L}} p_{sl}(x^h_{sl} - e^h_{sl} - \sum_{l'\in\mathcal{L}} Y_{sl'l} x^h_{0l'}) = 0\}$$

An Arrow-Debreu or competitive equilibrium (GE) consists of commodity prices $p \in \mathbb{R}^{S^*L}_{++}$ and plans $x^h \in \mathbb{R}^{S^*L}_{+}$ such that:

(i) Consumers optimize: $(\overline{x}^h) \in B(p, e^h) \Rightarrow u^h(\overline{x}^h) \leq u^h(x^h)$.

(ii) Commodity market clear in period 0: $\sum_{h\in\mathcal{H}}(\overline{x}^h_{0l} - e^h_{0l}) = 0, \quad \forall l \in \mathcal{L}$.

(iii) Commodity market clear in state s: $\sum_{h\in\mathcal{H}}(\overline{x}^h_{sl} - e^h_{sl} - \sum_{l'\in\mathcal{L}} Y_{sl'l}\overline{x}^h_{0l'}) = 0, \quad \forall s \in \mathcal{S}; \forall l \in \mathcal{L}$.

Therefore, under assumptions A1-A4, an equilibrium GE is characterized by the first-order conditions of all agents' utility maximization problems in GE model that are necessary and sufficient for optimality (item(i)) the market clearing equations (item(ii) and (iii)), boundary condition of consumption and a price normalization to ensure that $p \in \mathbb{R}^{S^*L}_{++}$ (due the homogeneity of budgetary constrains in the prices, we may and impose a normalization condition, by requiring prices to lie on the unit simplex, $\sum_{s\in\mathcal{S}^*}\sum_{l\in\mathcal{L}} p_{sl} = 1$). In this way an equilibrium is characterized as a simultaneous solution of a nonlinear system (see [J], page 189-190).

Consider the following set of equations:

First-order conditions in x^h at date $t = 0$:

$$\partial_{0l} u^h(\overline{x}^h) - \overline{\delta}^h \overline{p}_{0l} + \overline{\delta}^h \sum_{s\in\mathcal{S}}\sum_{l'\in\mathcal{L}} Y_{sll'}\overline{p}_{sl'} + {}_x\overline{\delta}^h_{0l} = 0; \quad \forall h \in \mathcal{H}; \forall l \in \mathcal{L}. \tag{1.1}$$

First-order conditions in x^h at date $t = 1$:

$$\partial_{sl} u^h(\overline{x}^h) - \overline{\delta}^h \overline{p}_{sl} + {}_x\overline{\delta}^h_{sl} = 0; \quad \forall h \in \mathcal{H}; \forall s \in \mathcal{S}; \forall l \in \mathcal{L}. \tag{1.2}$$

The budget constraint:

$$\sum_{l\in\mathcal{L}} \overline{p}_{0l}(\overline{x}^h_{0l} - e^h_{0l}) + \sum_{s\in S}\sum_{l\in\mathcal{L}} \overline{p}_{sl}(\overline{x}^h_{sl} - e^h_{sl} - \sum_{l'\in\mathcal{L}} Y_{sl'l}\overline{x}^h_{0l'}) = 0; \quad \forall h \in \mathcal{H}. \tag{1.3}$$

Equilibrium commodity markets clear at date $t = 0$:

$$\sum_{h\in\mathcal{H}}(\overline{x}^h_{0l} - e^h_{0l}) = 0; \quad \forall l \in \mathcal{L}. \tag{1.4}$$

Equilibrium commodity markets clear at date $t = 1$:

$$\sum_{h\in\mathcal{H}}(\overline{x}^h_{sl} - e^h_{sl} - \sum_{l'\in\mathcal{L}} Y_{sl'l}\overline{x}^h_{0l'}) = 0; \quad \forall s \in \mathcal{S}; \forall l \in \mathcal{L}. \tag{1.5}$$

The boundary conditions:

$${}_x\overline{\delta}^h_{sl}\overline{x}^h_{sl} = 0; \quad \forall h \in \mathcal{H}; \forall s \in \mathcal{S}^*; \forall l \in \mathcal{L}. \tag{1.6}$$

The simplex condition for prices:

$$\sum_{s\in\mathcal{S}^*}\sum_{l\in\mathcal{L}}\overline{p}_{sl}=1. \tag{1.7}$$

where $\delta \in \mathbb{R}_+^H$ are multipliers associated with the budget constraint and ${}_x\delta \in \mathbb{R}_+^{HS^*L}$ are multipliers associated with boundary conditions of x (consumption).

Computing equilibrium means solving the system 1.1-1.7 for given $e = (e^h)_{h\in\mathcal{H}}$ and $Y = (Y_{sl})_{s\in\mathcal{S},l\in\mathcal{L}}$. We added simple bounds on the variables in order to force the non-negativity denoted by

$$\Omega = \{y \in \mathbb{R}^n, y \geq 0\}. \tag{1.8}$$

where $y = (\delta, {}_x\delta, x, p)$ and $n = H(2S^*L+1)+S^*L$, and the system has $m = H(2S^*L+1)+S^*L+1$ equality equations.

1.2.2 General equilibrium with incomplete markets (GEI)

In this section we assume that markets are incomplete. As explained above, the financial markets are said to be incomplete if $rank(V_{sj}) < S$.

The economy with incomplete markets, E_{GEI}, is characterized by the agents' utility functions $u = (u^h)_{h\in\mathcal{H}}$, the agents' endowment process $e = (e^h)_{h\in\mathcal{H}}$, the asset structure $A = (A_j)_{j\in\mathcal{J}}$ and durability technologies $Y = (Y_{sl})_{s\in\mathcal{S},l\in\mathcal{L}}$. The financial markets provide instruments that enable each agent to redistribute income across states. We assume that there is a system of J financial assets, where asset j can be purchased for price q_j at date $t=0$ and delivers a random return $V_{sj} = \sum_l p_{sl}A_{slj}$ across the states s at date $t=1$. Let $z^h = (z_1^h, \ldots, z_J^h) \in \mathbb{R}^J$ denote the number of units of each of the J assets purchased by agent h, where $z_j^h < 0$ means short-selling asset j (see [MS]).

Given $p \in \mathbb{R}_{++}^{S^*L}$, and $q \in \mathbb{R}^J$ the agent h choose an allocation (x^h, z^h), subject to the budgetary restrictions. Then, the constrained problem of each agent is:

$$\begin{aligned}
&\max_{x^h\in\mathbb{R}_+^{S^*L}} \; u^h(x^h) \\
\text{s.t.}\quad & \text{there exists } z^h \in \mathbb{R}^J \text{ with} \\
&\sum_{l\in\mathcal{L}} p_{0l}(x_{0l}^h - e_{0l}^h) + \sum_{j\in\mathcal{J}} q_j z_j^h \leq 0; \\
&\sum_{l\in\mathcal{L}} p_{sl}(x_{sl}^h - e_{sl}^h - \sum_{l'\in\mathcal{L}} Y_{sl'l}x_{0l'}^h) - \sum_{l\in\mathcal{L}}\sum_{j\in\mathcal{J}} p_{sl}A_{slj}z_j^h \leq 0; \quad \forall s \in \mathcal{S}.
\end{aligned} \tag{1.9}$$

The homogeneity of budgetary constrains equation 1.9 allows us to impose a normalization condition on the prices, which we require to lie on the unit simplex for each $s \in \mathcal{S}^*$, $\sum_l p_{sl} = 1$. By strict monotonicity, the budget constraints can be written as equalities.

A general equilibrium with incomplete markets (GEI) for the economy E_{GEI} is a vector
$[(\overline{x}, \overline{z}); (\overline{p}, \overline{q})]$ with $(\overline{x}, \overline{z}) = (\overline{x}^h, \overline{z}^h)_{h\in\mathcal{H}}$ such that:

(i) Consumers optimize: $(\overline{x}^h, \overline{z}^h)$ solves problem 1.9.

(ii) Commodity market clear in period 0: $\sum_{h\in\mathcal{H}}(\overline{x}^h_{0l} - e^h_{0l}) = 0; \quad \forall l \in \mathcal{L}$.

(iii) Commodity market clear in state s: $\sum_{h\in\mathcal{H}}(\overline{x}^h_{sl} - e^h_{sl} - \sum_{l'\in\mathcal{L}} Y_{sl'l}\overline{x}^h_{0l'}) = 0$; $\forall s \in \mathcal{S}; \forall l \in \mathcal{L}$.

(iv) Asset market clear: $\sum_{h\in\mathcal{H}} \overline{z}^h_j = 0; \quad \forall j \in \mathcal{J}$.

Note that problem 1.9 is a convex programming problem. Moreover, the constraints are linear and linearly independent, thereby satisfying the basic Lagrangian constraint qualification. This imply that first-order conditions of all agents' utility maximization problems in 1.9 are necessary and sufficient. Therefore, under assumptions, an equilibrium GEI is characterized by the first-order conditions of all agents' utility maximization problems in GEI model (item(i)) the market clearing for commodities and asset equations (item(ii) to item(iv)), boundary condition of consumption and a price normalization. In this way an equilibrium is characterized as a simultaneous solution of a nonlinear system (see [S1] and [Eb2]).

Consider the following set of equations:

First-order conditions in x^h at date $t = 0$:

$$\partial_{0l}u^h(\overline{x}^h) - \overline{\delta}^h_0\overline{p}_{0l} + \sum_{s\in\mathcal{S}} \overline{\delta}^h_s \sum_{l'\in\mathcal{L}} Y_{sll'}\overline{p}_{sl'} + {}_x\overline{\delta}^h_{0l} = 0; \quad \forall h \in \mathcal{H}; \forall l \in \mathcal{L}. \tag{1.10}$$

First-order conditions in x^h at date $t = 1$:

$$\partial_{sl}u^h(\overline{x}^h) - \overline{\delta}^h_s\overline{p}_{sl} + {}_x\overline{\delta}^h_{sl} = 0; \quad \forall h \in \mathcal{H}; \forall s \in \mathcal{S}; \forall l \in \mathcal{L}. \tag{1.11}$$

The budget constraint at date $t = 0$:

$$\sum_{l\in\mathcal{L}} \overline{p}_{0l}(\overline{x}^h_{0l} - e^h_{0l}) + \sum_{j\in\mathcal{J}} \overline{q}_j\overline{z}^h_j = 0; \quad \forall h \in \mathcal{H}. \tag{1.12}$$

The budget constraint at date $t = 1$:

$$\sum_{l\in\mathcal{L}} \overline{p}_{sl}(\overline{x}^h_{sl} - e^h_{sl} - \sum_{l'\in\mathcal{L}} Y_{sl'l}\overline{x}^h_{0l'}) - \sum_{l\in\mathcal{L}}\sum_{j\in\mathcal{J}} \overline{p}_{sl}A_{slj}\overline{z}^h_j = 0; \quad \forall h \in \mathcal{H}; \forall s \in \mathcal{S}. \tag{1.13}$$

First-order conditions in z:

$$\sum_{s\in\mathcal{S}}\sum_{l\in\mathcal{L}}\overline{\delta}_s^h\overline{p}_{sl}A_{slj}-\overline{\delta}_0^h\overline{q}_j=0;\quad \forall h\in\mathcal{H};\forall j\in\mathcal{J}. \tag{1.14}$$

Equilibrium commodity markets to clear at date $t=0$:

$$\sum_{h\in\mathcal{H}}(\overline{x}_{0l}^h-e_{0l}^h)=0;\quad \forall l\in\mathcal{L}. \tag{1.15}$$

Equilibrium commodity markets to clear at date $t=1$:

$$\sum_{h\in\mathcal{H}}(\overline{x}_{sl}^h-e_{sl}^h-\sum_{l'\in\mathcal{L}}Y_{sl'l}\overline{x}_{0l'}^h)=0;\quad \forall s\in\mathcal{S};\forall l\in\mathcal{L}. \tag{1.16}$$

Equilibrium asset markets to clear:

$$\sum_{h\in\mathcal{H}}\overline{z}_j^h=0;\quad \forall j\in\mathcal{J}. \tag{1.17}$$

The boundary conditions:

$${}_x\overline{\delta}_{sl}^h\overline{x}_{sl}^h=0;\quad \forall h\in\mathcal{H};\forall s\in\mathcal{S}^*;\forall l\in\mathcal{L}. \tag{1.18}$$

The simplex condition for prices:

$$\sum_{l\in\mathcal{L}}\overline{p}_{sl}=1.\quad \forall s\in\mathcal{S}^* \tag{1.19}$$

where $\delta\in\mathbb{R}_+^{HS^*}$ are multipliers associated with the budget constraints and and ${}_x\delta\in\mathbb{R}_+^{HS^*L}$ are multipliers associated with boundary conditions of x.

Computing equilibrium means solving the system 1.10-1.19 for given $e=(e^h)_{h\in\mathcal{H}}$, $A=(A_j)_{j\in\mathcal{J}}$ and $Y=(Y_{sl})_{s\in\mathcal{S},l\in\mathcal{L}}$. We added simple bounds on the variables denoted by

$$\Omega=\{y\in\mathbb{R}^n\quad \text{s.t.}\quad (\delta,{}_x\delta,x,p)\geq 0,\quad \text{and}\quad \iota\leq(z,q)\leq\upsilon\}. \tag{1.20}$$

where $y=(\delta,{}_x\delta,x,z,p,q)$ and $n=H(S^*(2L+1)+J)+S^*L+J$, and the system has $m=H(S^*(2L+1)+J)+S^*L+J+S^*$ equality equations. The vectors $\iota\in(\mathbb{R}\cup-\infty)^n$ and $\upsilon\in(\mathbb{R}\cup\infty)^n$ are specified lower and upper bounds on the variables.

1.2.3 General equilibrium with incomplete markets and collateral (GEIC)

The economy with exogenous collateral, E_{GEIC}, is characterized by the agents' utility functions $u = (u^h)_{h\in\mathcal{H}}$, the agents' endowment process $e = (e^h)_{h\in\mathcal{H}}$, the asset structure $A = (A_j, C_j)_{j\in\mathcal{J}}$ and durability technologies $Y = (Y_{sl})_{s\in\mathcal{S},l\in\mathcal{L}}$. The GEIC with collateral is a extending of the general equilibrium with incomplete markets (GEI) models with default and exogenously specified collateral. As in the two-period model of [GZ], agents can default on your promises without any utility penalties, but to each promise j we must formally associate levels of collateral. Agents default on their promises whenever the market value of the shares they hold as collateral is lower than the face value of their promise. Let $\theta^h = (\theta^h_1, \ldots, \theta^h_J) \in \mathbb{R}^J_+$ denote the number of units of each of the J assets bought by agent h, and $\varphi^h = (\varphi^h_1, \ldots, \varphi^h_J) \in \mathbb{R}^J_+$ means short-selling asset j.
Given $p \in \mathbb{R}^{S^*L}_{++}$, and $q \in \mathbb{R}^J_+$ the agent h chooses an consumption and portfolios $(x^h, \theta^h, \varphi^h)$, subject to the budget constraints.

$$
\begin{aligned}
&\max_{x^h\in\mathbb{R}^{S^*L}_+} u^h(x^h) \\
\text{s.t.}\quad & \text{there exists } \theta^h \in \mathbb{R}^J_+ \text{ and } \varphi^h \in \mathbb{R}^J_+ \text{ with} \\
& \sum_{l\in\mathcal{L}} p_{0l}(x^h_{0l} - e^h_{0l}) + \sum_{j\in\mathcal{J}} q_j(\theta^h_j - \varphi^h_j) \le 0; \\
& \sum_{l\in\mathcal{L}} p_{sl}(x^h_{sl} - e^h_{sl} - \sum_{l'\in\mathcal{L}} Y_{sl'l}x^h_{0l'}) \\
& - \sum_{j\in\mathcal{J}} (\theta^h_j - \varphi^h_j) \min\left\{\sum_{l\in\mathcal{L}} p_{sl}A_{slj}, \sum_{l\in\mathcal{L}} p_{sl}\sum_{l'\in\mathcal{L}} Y_{sl'l}C_{l'j}\right\} \le 0; \quad \forall s \in \mathcal{S}. \\
& x^h_{0l} - \sum_{j\in\mathcal{J}} \varphi^h_j C_{lj} \ge 0; \quad \forall l \in \mathcal{L}.
\end{aligned}
\tag{1.21}
$$

In state s, an asset j pays $\min\left\{\sum_{l\in\mathcal{L}} p_{sl}A_{slj}, \sum_{l\in\mathcal{L}} p_{sl}\sum_{l'\in\mathcal{L}} Y_{sl'l}C_{l'j}\right\}$. We refer to the last inequality constraint in the agent's problem as the collateral constraint. We replace $\min\{a, b\}$ by r and add the inequalities $2r - a - b \le 0$ and the equality $(r - a)(r - b) = 0$. As in the GEI model the homogeneity of budgetary constrains equation 1.21 in the prices implies that can be normalized. So we consider prices to lie on the unit simplex, $\sum_l p_l = 1$ for each $s \in \mathcal{S}^*$.

A competitive equilibrium is defined as usual by agents' optimality and market clearing.

Definition 1. *An equilibrium for the economy E_{GEIC} is a vector $[(\overline{x},\overline{\theta},\overline{\varphi});(\overline{p},\overline{q})]$ with $(\overline{x},\overline{\theta},\overline{\varphi}) = (\overline{x}^h,\overline{\theta}^h,\overline{\varphi}^h)_{h\in\mathcal{H}}$ such that:*

(i) $(\overline{x}^h,\overline{\theta}^h,\overline{\varphi}^h)$ *solves problem 1.21.*

(ii) $\sum_{h\in\mathcal{H}}(\overline{x}^h_{0l} - e^h_{0l}) = 0; \quad \forall l \in \mathcal{L}.$

(iii) $\sum_{h\in\mathcal{H}}(\overline{x}^h_{sl} - e^h_{sl} - \sum_{l'\in\mathcal{L}} Y_{sl'l}\overline{x}^h_{0l'}) = 0; \quad \forall s \in \mathcal{S}; \forall l \in \mathcal{L}.$

(iv) $\sum_{h\in\mathcal{H}}(\overline{\theta}^h_j - \overline{\varphi}^h_j) = 0; \quad \forall j \in \mathcal{J}.$

The following theorem follows from [GZ].

Theorem 1. *For an economy E_{GEIC}, under assumptions A1-A4 there exists a GEIC equilibrium.*

As in the GEI model, equilibrium is defined through a set of equations describing first-order conditions, market clearing, price normalization and boundary conditions. We add additional inequalities and equalities describing: $r_{sj} = \min\left\{\sum_{l\in\mathcal{L}} p_{sl}A_{slj}, \sum_{l\in\mathcal{L}} p_{sl}\sum_{l'\in\mathcal{L}} Y_{sl'l}C_{l'j}\right\}$ and portfolio conditions, requiring that agents are forbidden to buy and sell the same asset. The portfolio condition collapses what may appear as redundant equilibria in the numerical computation as in [MRT].

Consider the following set of equalities and inequalities:

First-order conditions in x^h at date $t = 0$:

$$\partial_{0l}u^h(\overline{x}^h) - \overline{\delta}^h_0\overline{p}_{0l} + \sum_{s\in S}\overline{\delta}^h_s\sum_{l'\in\mathcal{L}} Y_{sll'}\overline{p}_{sl'} + {}_{col}\overline{\delta}^h_l = 0; \quad \forall h \in \mathcal{H}; \forall l \in \mathcal{L}. \tag{1.22}$$

First-order conditions in x^h at date $t = 1$:

$$\partial_{sl}u^h(\overline{x}^h) - \overline{\delta}^h_s\overline{p}_{sl} + {}_{x}\overline{\delta}^h_{sl} = 0; \quad \forall h \in \mathcal{H}; \forall s \in \mathcal{S}; \forall l \in \mathcal{L}. \tag{1.23}$$

The budget constraint at date $t = 0$:

$$\sum_{l\in\mathcal{L}}\overline{p}_{0l}(\overline{x}^h_{0l} - e^h_{0l}) + \sum_{j\in\mathcal{J}}\overline{q}_j(\overline{\theta}^h_j - \overline{\varphi}^h_j) = 0; \quad \forall h \in \mathcal{H}. \tag{1.24}$$

The budget constraint at date $t = 1$:

$$\sum_{l\in\mathcal{L}} \overline{p}_{sl}(\overline{x}_{sl}^h - e_{sl}^h - \sum_{l'\in\mathcal{L}} Y_{sl'l}\overline{x}_{0l'}^h) - \sum_{j\in J}(\overline{\theta}_j^h - \overline{\varphi}_j^h)\overline{r}_{sj} = 0; \quad \forall h \in \mathcal{H}; \forall s \in \mathcal{S}. \tag{1.25}$$

The first-order conditions with respect to θ:

$$_{\theta}\overline{\delta}_j^h + \sum_{s\in S}\overline{\delta}_s^h r_{sj} - \overline{\delta}_0^h \overline{q}_j = 0; \quad \forall h \in \mathcal{H}; \forall j \in \mathcal{J}. \tag{1.26}$$

And the first-order conditions with respect to φ:

$$_{\varphi}\overline{\delta}_j^h - \sum_{s\in\mathcal{S}}\overline{\delta}_s^h r_{sj} + \overline{\delta}_0^h \overline{q}_j - \sum_{l\in\mathcal{L}} {}_{col}\overline{\delta}_l^h C_{lj} = 0; \quad \forall h \in \mathcal{H}; \forall j \in \mathcal{J}. \tag{1.27}$$

Equilibrium commodity markets to clear at date $t = 0$:

$$\sum_{h\in\mathcal{H}}(\overline{x}_{0l}^h - e_{0l}^h) = 0; \quad \forall l \in \mathcal{L}. \tag{1.28}$$

Equilibrium commodity markets to clear at date $t = 1$:

$$\sum_{h\in\mathcal{H}}(\overline{x}_{sl}^h - e_{sl}^h - \sum_{l'\in\mathcal{L}} Y_{sl'l}\overline{x}_{0l'}^h) = 0; \quad \forall s \in S; \forall l \in \mathcal{L}. \tag{1.29}$$

Equilibrium asset markets to clear:

$$\sum_{h\in\mathcal{H}}(\overline{\theta}_j^h - \overline{\varphi}_j^h) = 0; \quad \forall j \in \mathcal{J}. \tag{1.30}$$

The simplex condition for prices:

$$\sum_{l\in\mathcal{L}}\overline{p}_{sl} = 1; \quad \forall s \in \mathcal{S}^*. \tag{1.31}$$

The boundary conditions:

$$_{x}\overline{\delta}_{sl}^h \overline{x}_{sl}^h = 0; \quad \forall h \in \mathcal{H}; \forall s \in \mathcal{S}; \forall l \in \mathcal{L}. \tag{1.32}$$

$$_{\theta}\overline{\delta}_j^h \overline{\theta}_j^h = 0; \quad \forall h \in \mathcal{H}; \forall j \in \mathcal{J}. \tag{1.33}$$

$$_{\varphi}\overline{\delta}_j^h \overline{\varphi}_j^h = 0; \quad \forall h \in \mathcal{H}; \forall j \in \mathcal{J}. \tag{1.34}$$

$$_{col}\overline{\delta}_l^h(-\overline{x}_{0l}^h + \sum_{j\in\mathcal{J}}\varphi_j^h C_{lj}) = 0; \quad \forall h \in \mathcal{H}; \forall l \in \mathcal{L}. \tag{1.35}$$

Inequality to r_{sj}

$$2\overline{r}_{sj} - \sum_{l\in\mathcal{L}} \overline{p}_{sl} A_{slj} - \sum_{l\in\mathcal{L}} \overline{p}_{sl} \sum_{l'\in\mathcal{L}} Y_{sl'l} C_{l'j} \leq 0; \quad \forall s \in \mathcal{S}; \forall j \in \mathcal{J}. \tag{1.36}$$

Equality to r_{sj}

$$(\overline{r}_{sj} - \sum_{l\in\mathcal{L}} \overline{p}_{sl} A_{slj})(\overline{r}_{sj} - \sum_{l\in\mathcal{L}} \overline{p}_{sl} \sum_{l'\in\mathcal{L}} Y_{sl'l} C_{l'j}) = 0; \quad \forall s \in \mathcal{S}; \forall j \in \mathcal{J}. \tag{1.37}$$

Inequality to x at date $t = 0$:

$$-\overline{x}^h_{0l} + \sum_{j\in\mathcal{J}} \varphi^h_j C_{lj} \leq 0; \quad \forall h \in \mathcal{H}; \forall l \in \mathcal{L}. \tag{1.38}$$

Portfolio condition:

$$\overline{\varphi}^h_j \overline{\theta}^h_j = 0; \quad \forall h \in \mathcal{H}; \forall j \in \mathcal{J}. \tag{1.39}$$

where $\delta \in \mathbb{R}^{HS^*}_+$ are multipliers associated with the budget constraints, ${}_x\delta \in \mathbb{R}^{HSL}_+$, ${}_\theta\delta \in \mathbb{R}^{HJ}_+$ and ${}_\varphi\delta \in \mathbb{R}^{HJ}_+$ are multipliers associated with boundary conditions of x, θ and φ, and ${}_{col}\delta \in \mathbb{R}^{HL}_+$ are multipliers associated with the collateral constrains.

Computing equilibrium means solving the system 1.22-1.39 for given $e = (e^h)_{h\in\mathcal{H}}$, $A = (A_j, C_j)_{j\in\mathcal{J}}$ and $Y = (Y_{sl})_{s\in\mathcal{S},l\in\mathcal{L}}$. We added simple bounds on the variables in order to force the non-negativity denoted by

$$\Omega = \{y \in \mathbb{R}^n \text{s.t.} y \geq 0\}. \tag{1.40}$$

where $y = (\delta, x, \theta, \varphi, p, q, {}_x\delta, {}_\theta\delta, {}_\varphi\delta, r, {}_{col}\delta)$ and $n = H(S^*(L+1) + 4J + L(S+1)) + S^*L + J(S+1)$ and the system has $mequal = H(S^*(L+1) + 5J + L(S+1)) + S^*(L+1) + J(S+1)$ equality equations and $mineq = HL + SJ$ inequality equations.

1.2.4 General equilibrium with incomplete markets and default penalties (GEI$_\lambda$)

The economy with default penalties, E_{GEI_λ}, is characterized by the agents' utility functions $u = (u^h)_{h\in\mathcal{H}}$, the agents' endowment process $e = (e^h)_{h\in\mathcal{H}}$, durability technologies $Y = (Y_{sl})_{s\in\mathcal{S},l\in\mathcal{L}}$, the asset structure $A = (A_j)_{j\in\mathcal{J}}$, real default penalties $\lambda = (\lambda^h_{sj})_{h\in\mathcal{H},s\in\mathcal{S},j\in\mathcal{J}}$, and bound on sale of asset $Q = (Q^h_j)_{h\in\mathcal{H},j\in\mathcal{J}}$. The GEI$_\lambda$ is a extending of the GEI models with default and

exogenously specified penalties $\lambda^h_{sj} > 0$, where λ^h_{sj} is the utility loss of the agent h for defaulting an unit of the value of asset j in state s. The fraction of promise payments $K \in [0,1]^{SJ}$ is endogenous in the model. Agents are permitted to deliver whatever they want of their own promises, represented by $D^h \in \mathbb{R}^{SLJ}_+$, but they are penalized $\lambda^h_{sj} p_{ls}$ for every unit of good l they fail to deliver in state s from their engagement through asset j.
Given $p \in \mathbb{R}^{S^*L}_{++}$, $q \in \mathbb{R}^J_+$ and $K \in [0,1]^{SJ}$ the agent h can choose an allocation $(x^h, \theta^h, \varphi^h, D^h)$, to maximize utility subject to the budget constraints.

Then, the constrained problem of each agent is:

$$
\begin{aligned}
&\max_{x^h\geq 0, \varphi^h \geq 0, D^h \geq 0} u^h(x) - \sum_{j\in\mathcal{J}} \sum_{s\in S} \frac{\lambda^h_{sj}[\sum_{l\in\mathcal{L}} p_{sl} A_{slj} \varphi^h_j - \sum_{l\in\mathcal{L}} p_{sl} D^h_{slj}]}{\sum_{l\in\mathcal{L}} p_{sl} b_{sl}} \\
\text{s.t.} \quad & \text{there exists } \theta^h \in \mathbb{R}^J_+ \text{ with} \\
& \sum_{l\in\mathcal{L}} p_{0l}(x^h_{0l} - e^h_{0l}) + \sum_{j\in\mathcal{J}} q_j(\theta^h_j - \varphi^h_j) \leq 0; \\
& \sum_{l\in\mathcal{L}} p_{sl}(x^h_{sl} - e^h_{sl} - \sum_{l'\in\mathcal{L}} Y_{sl'l} x^h_{0l'}) \\
& + \sum_{l\in\mathcal{L}} \sum_{j\in\mathcal{J}} p_{sl} D^h_{slj} - \sum_{l\in\mathcal{L}} \sum_{j\in\mathcal{J}} \theta^h_j K_{sj} p_{sl} A_{slj} \leq 0; \quad \forall s \in \mathcal{S}; \\
& \varphi^h_j - Q^h_j \leq 0; \quad \forall j \in \mathcal{J}; \\
& \sum_{l\in\mathcal{L}} p_{sl} D^h_{slj} - \sum_{l\in\mathcal{L}} \varphi^h_j p_{sl} A_{slj} \leq 0; \quad \forall s \in \mathcal{S} \quad \forall j \in \mathcal{J}.
\end{aligned}
\tag{1.41}
$$

where $b_{sl} \in \mathbb{R}_+$ is exogenously specified with $b_{sl} \neq 0$. We have that $[\sum_{l\in\mathcal{L}} p_{sl} A_{slj} \varphi^h_j - \sum_{l\in\mathcal{L}} p_{sl} D^h_{slj}]$ is the money value of the default of h on his promise to deliver on asset j in state s. The default in real terms is: to divide the money value of the default by $\sum_{l\in\mathcal{L}} p_{sl} b_{sl}$. We refer to the last inequality constraint in the agent's problem as the non-negativity of default.

As in the GEIC model the homogeneity of budgetary constrains equation 1.41 in the prices implies that can be normalized. So we consider prices to lie on the unit simplex.

Definition 2. *An equilibrium for the economy E_{GEI_λ} is a vector $[(\overline{x}, \overline{\theta}, \overline{\varphi}, \overline{D}); (\overline{p}, \overline{q}, \overline{K})]$ with $(\overline{x}, \overline{\theta}, \overline{\varphi}, \overline{D}) = (\overline{x}^h, \overline{\theta}^h, \overline{\varphi}^h, \overline{D}^h)_{h\in\mathcal{H}}$ such that*

(i) $(\overline{x}^h, \overline{\theta}^h, \overline{\varphi}^h, \overline{D}^h)$ solves problem 1.41.

(ii) $\sum_{h\in\mathcal{H}}(\overline{x}^h_{0l} - e^h_{0l}) = 0; \quad \forall l \in \mathcal{L}.$

(iii) $\sum_{h\in\mathcal{H}}(\overline{x}^h_{sl} - e^h_{sl} - \sum_{l'\in\mathcal{L}} Y_{sl'l}\overline{x}^h_{0l'}) = 0; \quad \forall s \in \mathcal{S}; \forall l \in \mathcal{L}.$

(iv) $\sum_{h\in\mathcal{H}}(\overline{\theta}^h_j - \overline{\varphi}^h_j) = 0; \quad \forall j \in \mathcal{J}.$

(v) $\overline{K}_{sj} = \frac{\sum_{h\in\mathcal{H}}\sum_{l\in\mathcal{L}}\overline{p}_{sl}\overline{D}^h_{slj}}{\sum_{h\in\mathcal{H}}\sum_{l\in\mathcal{L}}\overline{p}_{sl}A_{slj}\overline{\varphi}^h_j}; \quad \forall s \in \mathcal{S}; \forall j \in \mathcal{J}.$

As in the GEIC model conditions (i-iv) says that all agents optimize and markets are clear. Condition (v) says that each lender of an asset is correct in his expectation about the fraction of promises that do in fact get delivered (see [DGS2]). The economy satisfying standard assumptions, for any positive default penalties and bound on sale of asset admits a default penalties equilibrium.

A 5. *For any* $\lambda^h_{sj} \in \overline{\mathbb{R}}_+$ *and* $Q^h_j \in \mathbb{R}_+$*;*

The following theorem follows from [DGS2]

Theorem 2. *For an economy E_{GEI_λ}, under assumptions A1-A4, and A5 there exists an equilibrium.*

As in the GEIC model, equilibrium is defined through a set of equations describing first-order conditions, market clearing, price normalization, boundary conditions, inequalities for non-negativity of the default, bound on sale of asset and portfolio conditions.

Consider the following set of equalities and inequalities:

First-order conditions in x^h at date $t = 0$:

$$\partial_{0l}u^h(\overline{x}^h) - \overline{\delta}^h_0\overline{p}_{0l} + \sum_{s\in\mathcal{S}}\overline{\delta}^h_s\sum_{l'\in\mathcal{L}}Y_{sll'}\overline{p}_{sl'} + {}_x\overline{\delta}^h_{0l} = 0; \quad \forall h \in \mathcal{H}; \forall l \in \mathcal{L}. \tag{1.42}$$

First-order conditions in x^h at date $t = 1$:

$$\partial_{sl}u^h(\overline{x}^h) - \overline{\delta}^h_s\overline{p}_{sl} + {}_x\overline{\delta}^h_{sl} = 0; \quad \forall h \in \mathcal{H}; \forall s \in \mathcal{S}; \forall l \in \mathcal{L}. \tag{1.43}$$

The budget constraint at date $t = 0$:

$$\sum_{l\in\mathcal{L}}\overline{p}_{0l}(\overline{x}^h_{0l} - e^h_{0l}) + \sum_{j\in\mathcal{J}}\overline{q}_j(\overline{\theta}^h_j - \overline{\varphi}^h_j) = 0; \quad \forall h \in \mathcal{H}. \tag{1.44}$$

The budget constraint at date $t = 1$:

$$\sum_{l\in\mathcal{L}} \overline{p}_{sl}(\overline{x}^h_{sl} - e^h_{sl} - \sum_{l'\in\mathcal{L}} Y_{sl'l}\overline{x}^h_{0l'}) + \sum_{l\in\mathcal{L}}\sum_{j\in\mathcal{J}} \left(p_{sl}\overline{D}^h_{slj} - \overline{K}_{sj}p_{sl}A_{slj}\overline{\theta}^h_j \right) = 0; \quad \forall h \in \mathcal{H}; \forall s \in \mathcal{S}. \tag{1.45}$$

The first-order conditions with respect to θ:

$$_\theta\overline{\delta}^h_j + \sum_{s\in\mathcal{S}}\sum_{l\in\mathcal{L}} \overline{\delta}^h_s \overline{K}_{sj}p_{sl}A_{slj} - \overline{\delta}^h_0 \overline{q}_j = 0; \quad \forall h \in \mathcal{H}; \forall j \in \mathcal{J}. \tag{1.46}$$

And the first-order conditions with respect to φ:

$$_\varphi\overline{\delta}^h_j - {}_B\overline{\delta}^h_j + \overline{\delta}^h_0\overline{q}_j - \sum_{s\in\mathcal{S}} \frac{\lambda^h_{sj}\sum_{l\in\mathcal{L}}\overline{p}_{sl}A_{slj}}{\sum_{l\in\mathcal{L}} p_{sl}b_{sl}} + \sum_{l\in\mathcal{L}} {}_d\overline{\delta}^h_{sj} + \sum_{l\in\mathcal{L}} \overline{p}_{sl}A_{slj} = 0; \quad \forall h \in \mathcal{H}; \forall j \in \mathcal{J}. \tag{1.47}$$

First-order conditions with respect to D:

$$\frac{\lambda^h_{sj}\overline{p}_{sl}}{\sum_{l\in\mathcal{L}} p_{sl}b_{sl}} - \overline{\delta}^h_s\overline{p}_{sl} - {}_d\overline{\delta}^h_{sj}\overline{p}_{sl} + {}_D\overline{\delta} = 0 \quad \forall h \in \mathcal{H}; \forall s \in \mathcal{S}; \forall l \in \mathcal{L}; \forall j \in \mathcal{J}. \tag{1.48}$$

Equilibrium commodity markets to clear at date $t = 0$:

$$\sum_{h\in\mathcal{H}} (\overline{x}^h_{0l} - e^h_{0l}) = 0; \quad \forall l \in \mathcal{L}. \tag{1.49}$$

Equilibrium commodity markets to clear at date $t = 1$:

$$\sum_{h\in\mathcal{H}} (\overline{x}^h_{sl} - e^h_{sl} - \sum_{l'\in\mathcal{L}} Y_{sl'l}\overline{x}^h_{0l'}) = 0; \quad \forall s \in \mathcal{S}; \forall l \in \mathcal{L}. \tag{1.50}$$

Equilibrium asset markets to clear:

$$\sum_{h\in\mathcal{H}} (\overline{\theta}^h_j - \overline{\varphi}^h_j) = 0; \quad \forall j \in \mathcal{J}. \tag{1.51}$$

Equilibrium aggregate proportion of delivers:

$$\sum_{h\in\mathcal{H}}\sum_{l\in\mathcal{L}} \overline{p}_{sl}A_{slj}\overline{\varphi}^h_j\overline{K}_{sj} - \sum_{h\in\mathcal{H}}\sum_{l\in\mathcal{L}} \overline{p}_{sl}\overline{D}^h_{slj} = 0; \quad \forall s \in \mathcal{S}; \forall j \in \mathcal{J}. \tag{1.52}$$

The simplex condition for prices:

$$\sum_{l\in\mathcal{L}} \overline{p}_{sl} = 1; \quad \forall s \in \mathcal{S}^*. \tag{1.53}$$

Inequality for non-negativity of the default:

$$\sum_{l\in\mathcal{L}} \overline{p}_{sl}\overline{D}^h_{slj} - \sum_{l\in\mathcal{L}} \overline{p}_{sl}A_{slj}\overline{\varphi}^h_j \leq 0; \quad \forall h \in \mathcal{H}; \forall s \in \mathcal{S}; \forall j \in \mathcal{J}. \tag{1.54}$$

Inequality for bound on sale of asset:

$$\overline{\varphi}^h_j - Q^h_j \leq 0; \quad \forall h \in \mathcal{H}; \forall j \in \mathcal{J}. \tag{1.55}$$

The boundary conditions:

$${}_x\overline{\delta}^h_{sl}\overline{x}^h_{sl} = 0; \quad \forall h \in \mathcal{H}; \forall s \in \mathcal{S}^*; \forall l \in \mathcal{L}. \tag{1.56}$$

$${}_\theta\overline{\delta}^h_j\overline{\theta}^h_j = 0; \quad \forall h \in \mathcal{H}; \forall j \in \mathcal{J}. \tag{1.57}$$

$${}_\varphi\overline{\delta}^h_j\overline{\varphi}^h_j = 0; \quad \forall h \in \mathcal{H}; \forall j \in \mathcal{J}. \tag{1.58}$$

$${}_D\overline{\delta}^h_{slj}\overline{D}^h_{slj} = 0; \quad \forall h \in \mathcal{H}; \forall s \in \mathcal{S}; \forall l \in \mathcal{L}; \forall j \in \mathcal{J}. \tag{1.59}$$

$${}_d\overline{\delta}^h_{sj}\left(\sum_{l\in\mathcal{L}} \overline{p}_{sl}\overline{D}^h_{slj} - \sum_{l\in L} \overline{p}_{sl}A_{slj}\overline{\varphi}^h_j\right) = 0, \quad \forall h \in \mathcal{H}; \forall s \in \mathcal{S}; \forall j \in \mathcal{J}. \tag{1.60}$$

$${}_B\overline{\delta}^h_j(\overline{\varphi}^h_j - Q^h_j) = 0; \quad \forall h \in \mathcal{H}; \forall j \in \mathcal{J}. \tag{1.61}$$

Portfolio condition:

$$\overline{\varphi}^h_j\overline{\theta}^h_j = 0; \quad \forall h \in \mathcal{H}; \forall j \in \mathcal{J}. \tag{1.62}$$

where $\delta \in \mathbb{R}^{HS^*}_+$ are multipliers associated with the budget constraints, ${}_x\delta \in \mathbb{R}^{HS^*L}_+$, ${}_\theta\delta \in \mathbb{R}^{HJ}_+$, ${}_\varphi\delta \in \mathbb{R}^{HJ}_+$ and ${}_D\delta \in \mathbb{R}^{HSLJ}_+$ are multipliers associated with boundary condition of the x, θ, φ and D, ${}_d\delta \in \mathbb{R}^{HSJ}_+$ are multipliers associated with the non-negativity of the default and ${}_B\overline{\delta} \in \mathbb{R}^{HJ}_+$ are multipliers associated with the bound on sale of asset.

Computing equilibrium means solving the system 1.42-1.62 for given $e = (e^h)_{h\in\mathcal{H}}$, $Y = (Y_{sl})_{s\in\mathcal{S},l\in\mathcal{L}}$, $A = (A_j)_{j\in\mathcal{J}}$, $\lambda = (\lambda^h_{sj})_{h\in\mathcal{H},s\in\mathcal{S},j\in\mathcal{J}}$, $Q = (Q^h_j)_{h\in\mathcal{H},j\in\mathcal{J}}$ and $b = (b_s)_{s\in\mathcal{S}}$, and we added simple bounds on the variables in order to force the non-negativity denoted by

$$\Omega = \{y \in \mathbb{R}^n \text{s.t.} y \geq 0\}. \tag{1.63}$$

where $y = (\delta, x, \theta, \varphi, D, p, q, K, {}_x\delta, {}_\theta\delta, {}_\varphi\delta, {}_D\delta, {}_d\delta, {}_B\delta)$ and $n = H(S^*(2L+1) + 5J + SJ(2L+1)) + S^*L + J + SJ$. The system has $mequal = H(S^*(2L+1) + 6J + SJ(2L+1)) + S^*(L+1) + J(S+1)$ equality equations and $mineq = HS^*J$ inequality equations.

1.3 The Optimization Solver

A code for each model described in Section 1.2 computes the nonlinear system and also computes analytical derivatives. In all cases the nonlinear models are solved using ALGENCAN (see [ABMS]). ALGENCAN is a recently introduced Augmented Lagrangian method for smooth general-constrained minimization. The method considers the following nonlinear programming problem:

$$\begin{aligned} &\min f(x), x \in \Omega \\ \text{s.t.} \quad & h_i(x) = 0, i = 1, \dots, m; g_i(x) \leq 0, i = 1, \dots, p \end{aligned} \tag{1.64}$$

where $\Omega = \{x \in \mathbb{R}^n | \iota \leq x \leq \upsilon\}$, and the vectors $\iota \in (\mathbb{R} \cup -\infty)^n$ and $\upsilon \in (\mathbb{R} \cup \infty)^n$ are specified lower and upper bounds on the variables.

For given $\rho > 0$, we define the Augmented Lagrangian as:

$$L_\rho(x, \eta, \mu) = f(x) + \frac{\rho}{2} \left\{ \sum_{i=1}^{m} \left[h_i(x) + \frac{\eta_i}{\rho} \right]^2 + \sum_{i=1}^{p} \left[\max \left(0, g_i(x) + \frac{\mu_i}{\rho} \right) \right]^2 \right\} \tag{1.65}$$

where $\eta_i \in \mathbb{R}$ and $\mu_i \geq 0$ are multipliers.

In our case $f(x) = 0$ and Ω is defined in the Section 1.2.

The main algorithm defined by [ABMS] consist of a sequence of (approximate) minimization of $L_\rho(x, \eta, \mu)$ subject to $x \in \Omega$, followed by the updating of η, μ and ρ. Each approximate minimization of L will be called an Outer Iteration.

Algorithm 1. *Let* $x_0^k \in \mathbb{R}^n$, *be an arbitrary initial point. The given parameters for the execution of the algorithm are:* $\tau \in [0, 1), \gamma > 1, \rho_1 > 0, -\infty < \eta_{min} < \eta_{max} < \infty, \mu_{max} > 0, \eta_i^1 \in [\eta_{min}, \eta_{max}] \forall i = 1, \dots, m, \mu_i^1 \in [0, \mu_{max}] \forall i = 1, \dots, p$. *Finally,* $\{\varepsilon_k\} \subset \mathbb{R}_+$ *is a sequence of tolerance parameters such that* $\lim_{k \to \infty} \varepsilon_k = 0$.

Step 1. *Initialization: Set* $k \leftarrow 1$. *For* $i = 1, \dots, p$ *compute* $V^0 = \max\{0, g_i(x^0)\}$.

Step 2. *Solving the subproblem: Compute* $x^k \in \mathbb{R}^n$*, such that there exist* $v^k \in \mathbb{R}^m$*,* $u^k \in \mathbb{R}^p$ *satisfying*

$$\| \nabla L_{\rho^k}(x^k, \eta^k, \mu^k) + \sum_{i=1}^{m} v_i^k \nabla h_i(x^k) + \sum_{i=1}^{p} u^i \nabla g_i(x^k) \| \leq \varepsilon_k \tag{1.66}$$

$$u_i^k \geq 0, g_i(x^k) \leq \varepsilon_k \forall i = 1, \ldots, p \tag{1.67}$$

$$g_i(x^k) < -\varepsilon_k \Rightarrow u_i^k = 0, \forall i = 1, \ldots, p \tag{1.68}$$

$$\| h(x^k) \| \leq \varepsilon_k \tag{1.69}$$

If it is not possible to find x^k *satisfying 1.66-1.69, stop the execution of the algorithm.*

Step 3. *Estimate multipliers: For all* $i = 1, \ldots, m$ *compute* $\eta_i^{k+1} = \eta_i^k + \rho^k h_i(x^k)$ *For all* $i = 1, \ldots, p$ *compute* $\mu_i^{k+1} = \max\{0, \mu_i^k + \rho^k g_i(x^k)\}$

Step 4. *Update the penalty parameter: If* $\max\{\| h(x^k) \|_\infty, \| V^k \|_\infty\} \leq \tau \max\{\| h(x^{k-1}) \|_\infty, \| V^{k-1} \|_\infty\}$*,then define* $\rho^{k+1} = \rho^k$*. Else, define* $\rho^{k+1} = \gamma\rho^k$

Step 5. *Begin a new outer iteration: Set* $k \leftarrow k + 1$*. Go to Step 2.*

Codes for each model and ALGENCAN are in Fortran77 and the compiler gfortran.

The random generation of inputs, including initial points, collateral, default penalties, endowments, promises and parameters of the utility function, was made through shell manipulation of the output of the random number generator /dev/urandom (provided by Linux), so to obtain uniform distributions in specified ranges.

1.4 Numerical results

This section shows the results of the numerical experimentation we carried out in order to verify the performance of algorithm for the computation of general economic equilibrium for small, medium and large scale. The first class of examples cover all models in the section 1.2, by reinterpreting in various ways the set of assets, these examples we call test examples. The second class of examples is taken of the literature for collateral model (according to [G1]) and for GEI model (according to [DE] and [S1]).

We implemented the examples using the ALGENCAN with bound defined by Section 1.2. The results reported are obtained with the stopping tolerance $\varepsilon^k = 10^{-08}$ and setting $maxout = 50$. All the examples were run on a Intel® Pentium® dual-core processor T2330, 2 GB of RAM memory and Linux operating system.

1.4.1 Test examples

The set of examples for testing the numerical performance of our code consider a two period model with logarithmic utility functions:

$$u^h = \sum_{l \in \mathcal{L}} \log(x^h_{0l}) + \sum_{s \in \mathcal{S}} \varepsilon_s \sum_{l \in \mathcal{L}} \log(x^h_{sl}) \tag{1.70}$$

where ε is the probability of states of nature. We assume $\varepsilon_s = 1/S$ for all $s \in \mathcal{S}$. In the economy there are perishable and durable goods, the consumption-durability technology is:

$$Y_s = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \forall s \in \mathcal{S} \tag{1.71}$$

meaning that 1 unit of perishable good becomes 0 units of both goods, and 1 unit of durable good becomes 1 unit of durable good in the second period for each state. We assume that $A_{sj} = [1 \quad 0]^T$ for all $s \in \mathcal{S}$ and $j \in \mathcal{J}$ which promises 1 unit of perishable good and no durable good in the second period. In the GEI$_\lambda$ the bound on sale of asset $Q^h_j = 20$ for all $h \in \mathcal{H}$ and $j \in \mathcal{J}$ and the $b_{sl} = 1$ for all $s \in \mathcal{S}$ and $l \in \mathcal{L}$.

We generated several problems in this class with number of variables (dimension) ranging from $n = 20$ to $n = 452$, varying the number of H agents, S states, L goods and J assets for several collateral requirements and penalties. The problems are divided in three sets of small, medium and large problems. We first solved the small problems in order to show the features of equilibrium economic and performance of the algorithm. In order to verify capabilities of algorithm we solved for large problems. To demonstrate its performance we taking the initial guess random $y^n \approx U[0, 10]$ for 1000 samples and to show its robustness we applied it to several random choice of collateral and penalties for 10000 samples for medium scale problems.

Remark 1. *As we consider the logarithm function for utility, equilibria always have positive consumption, so we do not need to consider the boundary conditions of the consumption. Thus the effective dimension for GE, GEI and GEI_λ models is smaller by HS^*L and GEIC model is smaller by HSL.*

Example 1: small scale

We consider an example with two agents $\mathcal{H} = \{1, 2\}$, two states in the second period $\mathcal{S}^* = \{0, 1, 2\}$, two goods $\mathcal{L} = \{1_p, 2_d\}$ one perishable and one durable and one asset $\mathcal{J} = \{1\}$. Each agent with the same utility function defined in equation 1.70 and the consumption-durability technology defined in equation 1.71. We suppose that endowments are:
$e^1 = (e^1_{01_p}, e^1_{02_d}, e^1_{11_p}, e^1_{12_d}, e^1_{21_p}, e^1_{22_d}) = (4, 4, 4, 0, 4, 0)$;
$e^2 = (e^2_{01_p}, e^2_{02_d}, e^2_{11_p}, e^2_{12_d}, e^2_{21_p}, e^2_{22_d}) = (2, 1, 6, 0, 2, 0)$.
We explain the results and features of equilibrium for each model and report the performance of the algorithm in the Table 1.3.

- **GE equilibrium**: The GE model described in the section 1.2.1 is a system with infinite default penalties and no collateral. Taking the initial guess random for 1000 samples the algorithm converged for 884 samples (see performance Table 1.3). Equilibrium for consumption, prices and the utilities are:
  $p = (p_{01_p}, p_{02_d}, p_{11_p}, p_{12_d}, p_{21_p}, p_{22_d}) = (0.19, 0.44, 0.06, 0.11, 0.09, 0.11)$;
  $x^1 = (x^1_{01_p}, x^1_{02_d}, x^1_{11_p}, x^1_{12_d}, x^1_{21_p}, x^1_{22_d}) = (4.2, 3.5, 7, 3.5, 4.2, 3.5)$;
  $x^2 = (x^2_{01_p}, x^2_{02_d}, x^2_{11_p}, x^2_{12_d}, x^2_{21_p}, x^2_{22_d}) = (1.8, 1.5, 3, 1.5, 1.8, 1.5)$.
  $u^1 = 5.6311$ and $u^2 = 2.2419$

- **GEI equilibrium**: The GEI model described in the section 1.2.2 is a system with infinite default penalties for the asset J, so that full delivery is assured even with no collateral. Taking the initial guess random for 1000 samples the algorithm converged for 960 samples (see performance Table 1.3). Equilibrium for consumption, portfolio, prices and the utilities are:
  $p = (0.29, 0.71, 0.33, 0.67, 0.45, 0.55)$;
  $x^1 = (4.26, 3.64, 6.01, 3.01, 4.56, 3.80)$;
  $x^2 = (1.74, 1.36, 3.98, 1.99, 1.44, 1.2)$;
  $z^1_1 = 0.7491, z^2_1 = -0.7491, q_1 = 0.2388$

$u^1 = 5.6153$ and $u^2 = 2.1703$

- **Collateral equilibrium** $GEIC$: We now introduce the possibility of collateral in a world where there are no penalties. We suppose that agents have to hold C_j units of durable good in order to sell one unit of asset j and the borrower holds the collateral. The GEIC model is a system described in the section 1.2.3. Taking the initial guess random algorithm converges for positive collateral requirements (see performance Table 1.3). In the Table 1.1 we report the collateral, portfolio ($z_1^1 = -z_1^2$ in this example), price of the assets and utilities.

Table 1.1: Equilibrium for portfolio, asset prices and utilities of the collateral model

Collateral	z_1^1	q_1	u^1	u^2
0.3	2.5631	0.1073	5.6200	2.1965
0.8333	0.7491	0.2388	5.6153	2.1703
5	0.2424	0.2475	5.6126	2.1596

The default in the both states occurs with $C = 0.3$ and with $C = 0.5$ the default occurs in the second state, the portfolio and price of assets change but the consumption is the same. For $C = 0.8333$ the value of the collateral exceeds his debt in the first state and the collateral is equal his debt in the second state, there is no default and the equilibrium is the same GEI. For $C > 1.8$ the marginal utility for consumption is negative for agent 2 in the good 2, in this case the trade decreases when the collateral requirement increase.

- **Default penalties equilibrium** GEI_λ: We now introduce the possibility penalties without collateral. The GEI_λ model is a system described in the section 1.2.4. Taking the initial guess random the algorithm converges for positive penalties. For penalty defined by $\lambda_s^h = \frac{\delta_s^h \sum_l p_{sl}}{\varepsilon_s}$ the agents' deliveries is equal his debt. In this example the penalties for the agent 1 are $\lambda_s^2 = [0.50, 0.48]$, so that his is the lender and the agent 2 borrower, so the penalties are showed only agent 2 in Table 1.2.

Table 1.2: Equilibrium for portfolio, asset prices and utilities of the default penalties model

λ_1^2	λ_2^2	z_1^1	q_1	u^1	u^2
0.74	1.40	0.3883	0.2182	5.6127	2.1468
0.75	1.53	0.7491	0.2388	5.6153	2.1703
0.94	0.09	2.3389	0.0916	5.6239	2.2029

The default in both states occurs for $\lambda_s^2 = [0.74, 1.40]$, the delivery rates on asset are $K_s = [0.90, 0.88]$. For penalty $\lambda_s^2 = [0.75, 1.53]$ default not occurs and the delivery rates on asset is $K = 1$ in both states, the equilibrium is the same GEI. For penalty $\lambda_s^2 = [0.94, 0.09]$ default occurs in the state 2 and no default in state 1.

Table 1.3 shows the number of variables (dimension), number of convergence for 1000 samples (CPU time) and average number of iterations (average time of convergence) for each model.

Table 1.3: Performance for small equilibrium problems

Model	dimension	N. samples (CPU time)	Aver. Iter. (time)
GE	20	884 (33.29s)	6.48 (0.01s)
GEI	27	996 (2m4.14s)	8 (0.03s)
GEIC $C = 0.3$	39	331 (3m17.79s)	10.14 (0.10s)
GEIC $C = 0.8333$	39	348 (3m0.01s)	9.53 (0.11s)
GEIC $C = 5$	39	327 (2m22.12s)	8.5 (0.10s)
GEI$_\lambda$ $\lambda_s^2 = [0.74, 1.40]$	57	140 (5m18.79s)	9.19 (0.12s)
GEI$_\lambda$ $\lambda_s^2 = [0.75, 1.53]$	57	66 (6m55.806s)	11.26 (0.11s)
GEI$_\lambda$ $\lambda_s^2 = [0.94, 0.09]$	57	126 (5m41.47s)	9.60 (0.08s)

GE and GEI models converged for the majority of initial points. The GEI_λ model converged less frequently than the GEIC model, but in case of convergence the average number of iterations (and time) was quite similar (around 10 iterations and 0.10 seconds).

Example 2: medium scale

We consider similar economy as Example 1 with two goods $L = 2$ one perishable and one durable and the same consumption-durability technology for each state in the second period. But now we suppose that there are three agents $H = 3$, three states in the second period $S = 3$ and two assets $J = 2$. We suppose that endowments are:
$e^1_{sl} = (4, 4, 1, 1, 2, 0, 4, 0); \quad \forall s \in S^*; \forall l \in L$
$e^2_{sl} = (1, 2, 1, 0, 2, 0, 4, 0); \quad \forall s \in S^*; \forall l \in L$
$e^3_{sl} = (2, 3, 3, 0, 1, 0, 2, 0); \quad \forall s \in S^*; \forall l \in L$
A code for random generation of initial points and collateral was provided. If a solution was found the code uses this solution as initial point for solving the following equilibrium. We solved 3996 samples (with CPU time 229m20.51s) GEIC with random choice of the collateral between 0 and 3 (see Figure 1.1).

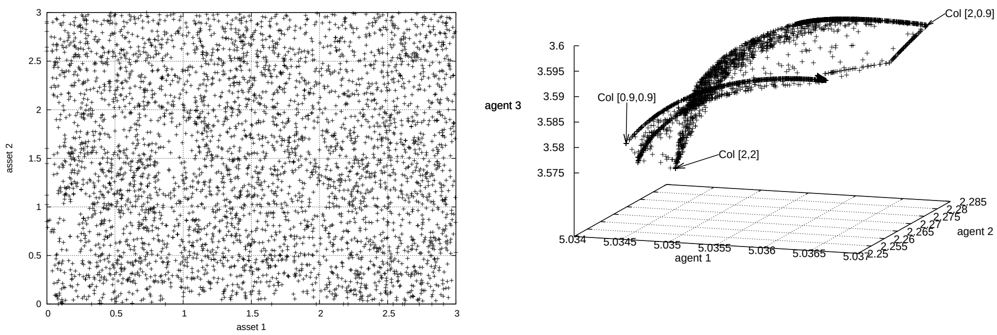

Figure 1.1: Random choice of the collateral (left) and corresponding utilities at equilibrium (right)

In GEIC model the default in the first and second state occurs with $C_{2j} = [0.9, 0.9]$ in both assets and utilities are $u^1 = 5.0339$, $u^2 = 2.2664$ and $u^3 = 3.5759$. For collateral $C_{2j} = [2, 2]$ there is no default whatever the states of nature and assets, the equilibrium is the same GEI that are $u^1 = 5.0348$, $u^2 = 2.2499$ and $u^3 = 3.5769$. For collateral $C_{2j} = [2, 0.9]$ the default in the first and second state occurs in asset 2 and utilities are $u^1 = 5.0366$, $u^2 = 2.2818$ and $u^3 = 3.5976$. In all cases both assets are traded (see Figure 1.1).

In GEI$_\lambda$ model a code for random generation of initial points and penalties was provided. If a solution was found the code uses this solution as initial point and a random walk for penalties for solving the following equilibrium. We solved 3296 samples (with CPU time 1277m33.61s) GEI$_\lambda$ with random choice of the penalties between 0 and 3 (see Figure 1.2).

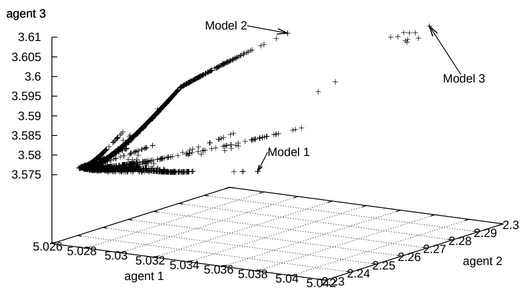

Figure 1.2: Utilities at equilibrium for random choice of the penalties

In Figure 1.2 we highlighted three extremal equilibria. In the Model 1 (see Figure 1.2 arrow Model 1) the penalties for agent 2 are $\lambda^2_{s1} = [1.91, 0.78, 2.35]$ and $\lambda^2_{s2} = [2.04, 1.53, 1.62]$ and for agent 3 are $\lambda^3_{s1} = [2.40, 1.32, 1.23]$ and $\lambda^3_{s2} = [1.68, 2.01, 1.22]$ for s=1,2,3 the agent 2 and 3 are borrowers and the agent 1 is lender with penalties $\lambda^1_{s1} = [1.27, 2.80, 2.23]$ and $\lambda^1_{s2} = [0.94, 2.91, 1.18]$, only asset 2 is traded and the deliveries are in states 1, 2 and 3 with no default in all states of nature, the equilibrium is the same GEI . In the Model 2 (see Figure 1.2 arrow Model 2) the penalties $\lambda^2_{s1} = [0.01, 1.82, 2.31]$ and $\lambda^2_{s2} = [0.31, 2.93, 1.15]$ for s=1,2,3 the agent 2 is borrower and agent 1 and 3 are lenders[1], only asset 1 is traded and the deliveries are in states

[1]The penalties for agent 1 are $\lambda^1_{s1} = [1.52, 2.52, 1.39]$ and $\lambda^1_{s2} = [2.20, 2.06, 0.52]$ and

2 and 3 with no default in both states and default in the state 1. The utilities are $u^1 = 5.0301$, $u^2 = 2.2952$ and $u^3 = 3.6001$. In the Model 3 (see Figure 1.2 arrow Model 3) the penalties are $\lambda^2_{s1} = [2.12, 0.92, 0.86]$ and $\lambda^2_{s2} = [0.03, 1.86, 2.84]$ and $\lambda^3_{s1} = [0.94, 0.15, 2.8]$ and $\lambda^3_{s2} = [1.34, 0.48, 0.52]$ for s=1,2,3 the agent 2 and 3 are borrowers (only trade in the asset 2 and 1 respective) and agent 1 is lender with penalties $\lambda^1_{s1} = [1.37, 0.14, 0.75]$ and $\lambda^1_{s2} = [1.00, 2.54, 0.97]$, both assets are traded with default in the state 1 (asset 2) and state 2 (asset 1) and no default in the state 2 and 3 (asset 2) and state 1 and 3 (asset 1). The utilities are $u^1 = 5.0388$, $u^2 = 2.2925$ and $u^3 = 3.6075$. The utilities for GE equilibrium are $u^1 = 5.0489$, $u^2 = 2.3077$ and $u^3 = 3.6413$. Notice that the collateral and default penalty equilibria are Pareto superior respect to GEI equilibrium, but not on the Pareto frontier.

Taking the initial guess random $y^n \approx U[0, 10]$ for 1000 samples, the Table 1.4 show the number of variables (dimension), number of convergence for 1000 samples (CPU time) and average number of iterations (average time of convergence) for each model.

Table 1.4: Performance for medium equilibrium problems

Model	dimension	N. samples (CPU time)	Aver. Iter. (time)
GE	35	875 (42.58s)	6.51 (0.02s)
GEI	52	939 (5m58.93s)	10.3 (0.19s)
GEIC $C_{2j} = [0.9, 0.9]$	82	124 (28m59.52s)	14.59 (0.92s)
GEIC $C_{2j} = [2, 2]$	82	109 (24m24.12s)	16.04 (0.88s)
GEIC $C_{2j} = [2, 0.9]$	82	176 (34m29.15s)	11.95 (1.29s)
GEI$_\lambda$ Model 1	172	36 (67m28.32s)	14.66 (0.83s)
GEI$_\lambda$ Model 2	172	55 (62m24.47s)	13.05 (0.74s)
GEI$_\lambda$ Model 3	172	108 (35m8.908s)	13.05 (0.75s)

As in Example 1, the performance for medium scale equilibrium problems

for agent 3 are $\lambda^3_{s1} = [0.56, 0.59, 2.24]$ and $\lambda^3_{s2} = [1.89, 2.91, 2.80]$

shows that the average number of iterations for GEIC and GEI$_\lambda$ is quite similar, around 14 iterations.

Example 3: large scale

In the following we report results from the computation of equilibria for much larger GEIC and GEI$_\lambda$ models. To demonstrate its performance we taking the initial guess random $y^n \approx U[0, 10]$ for 1000 samples for a variety of random choice of the collateral, penalties and endowments. The utility function and probability of the states as same of the Examples 1 and 2.

- **Collateral equilibrium GEIC**: In this item we consider a GEIC economy with two goods $\mathcal{L} = \{1_p, 2_d\}$ and consumption-durability technology as same of the Examples 1 and 2. The assets promises 1 in each perishable good in the second period and no durable good.

  The Model 1 considers four states in the second period $S = 4$, four assets $J = 4$ and three agents $H = 3$. Model 2 considers four agents $H = 4$ with the same number of states and assets of the Model 1. Model 3 considers five states in the second period $S = 5$, five assets $J = 5$ and four agents $H = 4$. Model 4 considers five agents $H = 5$ with the same number of states and assets of the Model 3.

  A code for random generation of initial points, endowments and collateral was provided. For a variety of random choice of the collateral between 0 and 3 and endowments between 0 and 6 we created samples taking the initial guess random for 1000 samples. Table 1.5 show number of variables (dimension), number of convergence for 1000 samples (CPU time) and average number of iterations (average time of convergence).

- **Default penalties equilibrium** GEI_λ: In this item we consider a GEI$_\lambda$ economy with two assets $J = 2$ with only perishable goods, in this case the consumption-durability technology $Y_{sl} = 0$ for all $s \in S$ and $l \in L$ and the assets promises 1 in each perishable good.

  The Model 1 considers three states in the second period $S = 3$, three goods $L = 3$ and three agents $H = 3$. Model 2 considers four agents $H = 4$ with the same number of states and goods of the Model 1. Model 3 considers four goods $L = 4$ with the same number of states and agents of the Model 2. Model 4 considers four states $S = 4$ with the same number of agents and goods of the model 3.

Table 1.5: Performance for large equilibrium problems of the collateral model

Model	dimension	N. samples (CPU time)	Aver. Iter. (time)
GEIC: Model 1	129	435 (71m5.347s)	11.69 (1.72s)
GEIC: Model 2	162	263 (190m23.514s)	12.27 (3.32s)
GEIC: Model 3	202	231 (491m56.085s)	12.78 (8.77s)
GEIC: Model 4	242	150 (841m50.701s)	11.85 (10.99s)

A code similar for GEIC economy was provided, but now we have penalties. For a variety of random choice of the penalties between 0 and 3 and endowments between 0 and 6 we created samples taking the initial guess random for 1000 samples. Table 1.6 show number of variables (dimension), number of convergence for 1000 samples (CPU time) and average number of iterations (average time of convergence).

Table 1.6: Performance for large equilibrium problems of the default penalties model

Model	dimension	N. samples (CPU time)	Aver. Iter. (time)
GEI_λ: Model 1	224	206 (236m36.355s)	6.79 (0.64s)
GEI_λ: Model 2	292	197 (673m39.490s)	7.05 (2.02s)
GEI_λ: Model 3	360	198 (1055m45.751s)	7.12 (2.7s)
GEI_λ: Model 4	452	200 (2002m31.717s)	7.14 (5.47s)

1.4.2 Literature examples

The purpose of the examples in this section is show the performance of the algorithm for small models usually considered by economic theorists as well as larger models considered by economic applications.

The first example is taken from [G1] and [GZ] for collateral model, this example is interesting because each agent have different utility function and the constraint of collateral are active. In the literature for default penalties models [DGS2] consider simplest examples with logarithmic utility functions similar the section 1.4.1. The second and third examples we solved for GEI models according to [DE] and [S1].

Example 4: collateral model

In the literature for two-period models [G1] and [GZ] consider the GEIC economy with two agents $H = 2$, two states in the second period $S = 2$, two goods $L = 2$ one perishable (food) and one durable (house) and one asset $J = 1$. The consumption-durability technology and promise is the same in Section 1.4.1.

The utility function for each agent according to [G1] is:

$$u^1 = x^1_{01_p} + x^1_{02_d} + (1-\varepsilon)(x^1_{11_p} + x^1_{12_d}) + \varepsilon(x^1_{21_p} + x^1_{22_d})$$

$$u^2 = 9x^2_{01_p} - 2(x^2_{01_p})^2 + 15x^2_{02_d} + (1-\varepsilon)(x^2_{11_p} + 15x^2_{12_d}) + \varepsilon(x^2_{21_p} + 15x^2_{22_d})$$

Furthermore, he supposes that endowments are:

$$e^1 = (e^1_{01_p}, e^1_{02_d}, e^1_{11_p}, e^1_{12_d}, e^1_{21_p}, e^1_{22_d}) = (20, 1, 20, 0, 20, 0);$$

$$e^2 = (e^2_{01_p}, e^2_{02_d}, e^2_{11_p}, e^2_{12_d}, e^2_{21_p}, e^2_{22_d}) = (4, 0, 50, 0, 3, 0).$$

The collateral requirement is $C = 1/15$.

The GEIC model is a system with 47 variables. Taking the initial guess random for 1000 samples the total CPU time was 15m20.12s. The algorithm converged for 712 samples. The average number of iterations for convergence was 10.82, the average of time for convergence was 0.29 seconds. With $\varepsilon = 0.5$ the equilibrium for consumption, portfolio, prices and utilities is:
$p = (0.0769, 0.9231, 0.0625, 0.9375, 0.25, 0.75)$;
$x^1 = (23, 0, 35, 0, 23, 0); x^2 = (1, 1, 35, 1, 0, 1)$;
$z^1_1 = 15; z^2_1 = -15; q_1 = 0.0462$;
$u^1 = 52$ and $u^2 = 54.5$

Example 5: GEI model Demarzo and Eaves

For the GEI economy in [DE] with two-period $t = 0, 1$, three agents $H = 3$, three states in the second period $S = 3$, two goods $L = 2$ both perishables and two assets $J = 2$. Agents 1 and 2 are identical, have identical endowments $e^1 = e^2$ and utilities $u^1 = u^2$. The endowments of the agents are:

$$e^1_{sl} = e^2_{sl} = (10, 10, 25, 20, 20, 20, 15, 20); \quad \forall s \in S^*; \forall l \in L$$

$e^3_{sl} = (20, 20, 5, 10, 10, 10, 15, 10); \quad \forall s \in S^*; \forall l \in L$

The utilities are given by:

$$u^h(x^h) = -\sum_{s \in S^*} \varepsilon_s \left(B - (x^h_{s1})^{\beta^h} (x^h_{s2})^{1-\beta^h} \right)^2 \tag{1.72}$$

where $B = 57$, $\varepsilon_s = (1, 1/3, 1/3, 1/3)$, $\beta^1 = \beta^2 = 3/4$ and $\beta^3 = 1/4$. The asset matrix A is given by: $A^T = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & -1 & 1 & 0 & 2 & -1 \end{bmatrix}$

For this example the system has 52 variables. Taking the initial guess random for 1000 samples the total CPU time was 28m56.27s. The algorithm converged for 793 samples for the same point of equilibrium in [DE], but considerably faster. The average number of iterations for convergence in our algorithm was 11.31, and the average of time for convergence was 1.33 seconds. In [DE] with homotopy method the convergence was in 80 iterations. For the same example, but with interior-point method in [Eb2] the convergence was in 20 iterations, with a specific initial point.

According to [S1] the main difficulty in finding an equilibrium is that for prices p such that $rank(\sum_l p_{sl} A_{slj}) < J$ the demand functions x^h are usually discontinuous. He called a price p with $rank(\sum_l p_{sl} A_{slj}) = J$ a good price and if $rank(\sum_l p_{sl} A_{slj}) < J$ then p is bad price. These discontinuities make the computation of equilibrium for economies that do have equilibrium very difficult. Well-known solution approaches such as Newtons methods or conventional homotopy methods that fundamentally rely on continuity are not applicable, because the Jacobian is singular. Then, he introduces the homotopy algorithm with penalties for transactions on the asset markets. A penalty function on the portfolio holdings of the constraint agents depending on a homotopy parameter $t \in [0, 1]$ results in the continuity of the agents demand function for $t < 1$. Starting at $t = 0$ he plan to increase t to $t = 1$ thereby eliminating the penalty function and returning to the original model. Then we choose others starting point for the prices called for Schmedders bad prices, such that: $p = 0.5$ for all periods and all goods and the remaining variables at the starting point according to [S1] are $\sigma^3 = 170.45$, $\sigma^1 = \sigma^2 = 0$ for all periods, $x^h = e^h$ for all periods and goods, $z^h = 0$, and $q = 0$ for all assets. At these prices the asset return matrix is:

$$\sum_l p_{sl} A_{slj} = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$$

Note that $rank(\sum_l p_{sl} A_{slj}) = 1$, so the starting price vector is bad. A traditional Newton's method can run into difficulties. If Jacobian is singular the Newton step d^k is not even defined. The our algorithm converges for the same point of equilibrium in 12 iterations (1.16 seconds).

Example 6: GEI model Schmedders

To show the quality and robustness of the algorithm as [S1] we applied it to thousands of randomly generated economies and examined its behavior. The GEI economies of Experiment II in [S1] have $H = 2$ agents, $S = 3$ states in second period, $L = 2$ goods and $J = 2$ assets. Agent's utility function were given by

$$u^h = \sum_{s \in S^*} \varepsilon_s(\alpha^h \log(x^h_{s1}) + (1 - \alpha^h) \log(x^h_{s2})) \tag{1.73}$$

where is $\varepsilon_s = 1/S$ for all $s \in \mathcal{S}$ and $\alpha^1 = \frac{1}{3}$ and $\alpha^2 = \frac{2}{3}$. The asset matrix and endowments are random, such that:

$$A_{slj} \approx U[0, 5]$$
$$e^h_{sl} \approx U[10, 30]$$

.

For this example the system have 38 variables. Taking the initial guess random for 1000 samples the total CPU time was 4m30.137s. The algorithm converged for 990 samples. The average number of iterations for convergence was 11.4, the average of time for convergence was 0.17 seconds.

In order to demonstrate how the algorithm behaves in large-scale, we consider the GEI economy of Example 6 with $S = 5$ states, $L = 5$ goods, $J = 2$ assets where the Model 1 considers 3 agents, Model 2 has 15 agents and Model 3 has 30 agents. The asset matrix and endowments are random as Example 6 and the $\alpha^h_l \approx U[0, 1]$. The Table 1.7 displays the number of variables (dimension), number of convergence for 1000 samples (CPU time) and average number of iterations (average time of convergence). The algorithm performed well in terms of convergence and number of iterations. The scale of the problem only affects the cost of computation, mainly because of the cost of function evaluations.

1.5 Conclusions

In this Chapter we implemented a optimization algorithm ALGENCAN for several classes of general equilibrium models. The algorithm was shown to be robust for the computation of equilibria in GEI with default for large-scale

Table 1.7: Performance for GEI equilibrium problems

Model	dimension	N. samples (CPU time)	Aver. Iter. (time)
GEI: 3 agents	146	993 (50m25.489s)	12.82 (2.56s)
GEI: 15 agents	602	985 (695m33.140s)	12.67 (31.6s)
GEI: 30 agents	1172	984 (1971m2.655s)	12.54 (99.31s)

computations. Given its robustness, this algorithm seems to be a promising alternative for computing equilibria.

Chapter 2

Regulated collateral with incomplete markets

2.1 Introduction

We examine how scarcity and an unequal distribution of collateralizable good affects risk-sharing in an economy with default and durable goods. Individuals have to put up durable goods as collateral when they want to take short positions in financial markets. They are allowed to default on their promises without any punishment, but in the case of default, the collateral associated with the promise is seized and distributed among the creditors. We show that if agents do not own enough durable goods that can be used as collateral, markets can be endogenously incomplete and competitive equilibrium allocations suboptimal. The amount of collateralizable goods needed to achieve an efficient allocation is often unrealistically large.

The vast majority of debt, especially if it extends over a long period of time, is guaranteed by tangible assets called collateral. For example, residential homes serve as collateral for short-term and long-term loans to households, equipment and plants are often used as collateral for corporate bonds, and investors can borrow money to establish a position in stocks, using these as collateral. [DGS2] and [GZ] incorporate default and collateral into the standard two-period GEI model. In a two period model, default can be prevented by collateral and by utility-penalties that can be thought of a reduced form representation of reputation effects present in more realistic dynamic models. While it is true that in developed economies it is also possible to take substantial positions in unsecured debt and default on collateralized obligations has consequences beyond the loss of the collateral, to simplify our analysis we follow [GZ] and abstract from these considerations by assuming that collateral constitutes the only enforcement mechanism.

In our model, there are two periods, with uncertainty over the states of the world in the second period. There are two commodities, one perishable and the second durable. The durable good serves as collateral. An asset in this model is characterized by its collateral- (or margin-) requirement which specifies how much of the durable good needs to be used to back a short-position in this asset and its state-contingent promises in the second period. The actual payoff of the asset will be the minimum of this promise and the value of the associated collateral. The margin requirement that dictates how much collateral one has to hold in order to borrow one dollar and via the possibility of default also determines the payoff of an asset.

In the simplest version of the model, the only assets available for trade are promising one unit of the perishable good across all states. Throughout the work, we consider the generic case where the price of collateral will differ across states. If there are S states, S assets with safe promises but with appropriately chosen collateral requirements will span the marketed subspace, i.e. it appears as if they will 'complete markets'. However, scarce collateralizable durable good will generally imply that agents cannot hold arbitrary portfolios but will instead lead to allocations which can be very far away from the complete markets competitive equilibrium.

While it appears somewhat arbitrary to limit the set of available securities to assets that promise a safe payout, this assumption can be motivated by the observation that few individuals hold short-position in assets other than debt.

If everybody owns enough collateralizable durable good, the resulting equilibrium allocation does not depend on the exact promises of the assets. However, with scarce collateralizable durable good, this assumption turns out to be crucial for our analysis – although the payoffs of the S assets are linearly independent, large positions are necessary to derive the desired payoff of a portfolio in the second period.

The first observation of the model is that even if there are many more securities than states of the world available for trade, if they all promise a safe payoff, it is without loss of generality to consider S different securities. While in the standard model this is a trivial observation, in our model with collateral, it is crucial that these margin requirements to these securities are chosen to ensure that for each security there is exactly one state where the price of the underlying collateral is exactly equal to the promise of the asset.

If there is enough of the collateralizable durable good in the sense that each agent is endowed with so much of the durable good that he can hold a portfolio of all assets without the collateral constraint being binding, the model is equivalent to a standard Arrow-Debreu model and competitive equilibrium allocations are Pareto optimal.

If, on the other hand, the collateralizable durable good is scarce, most assets are not traded in equilibrium and markets 'appear' incomplete. As [GZ] and [G2] point out, 'scarce collateral' rations the volume of trade since there will always be a gap between utility of buying and dis-utility of selling an asset. The rationing does not reduce volume of trade proportionally but chokes off all trade in most contracts. We show that with several states and several agents still there are generally more than one contract being actively traded in equilibrium. However, it is also true that often not all contracts are traded and that many agents trade only in one of the available assets.

It is easy to see that the resulting equilibrium allocation is not Pareto optimal. This simply follows from the fact that not every agent trades every security, but would also be true in a model with certainty, if collateral requirements are binding. This in itself is not very surprising. Given the simple 'contracting' technology that does not allow for agents to deliver on promises without collateral, it is clear that with too little collateralizable goods in the economy, there cannot be complex trade in securities and full risk-sharing is impossible.

In such situation, an interesting question is whether the subprime loans can lead to a Pareto-improvement.

In the literature, what is understood as subprime loans are mortgage loans for borrowers who do not qualify for prime loans, due to weak credit history or income level. To compensate for these credit risks subprime loans carry higher interest rates compared to prime loans. In our collateral model, assets with low collateral margins can be interpreted as subprime. In particular they do carry higher interest rates, and are bought by agents who lack collateralizable durable goods in the present. However, it is particularly remarkable in our examples that in the presence of enough collateralizable durable goods, agents avoid subprime loans, even when there is little certainty about future income.

It is clear that understanding the effect of subprime loans on the economy is of fundamental importance in today's highly sophisticated financial markets. While this work does not intend to explain the subprime mortgage crisis, but only to analyze in which situations of equilibrium, low requirement of the collateral is optimal for the economy, we hope that the computation of equilibrium will eventually lead to a good understanding of the role subprime should play in a healthy market.

A more interesting question is whether the allocation is constrained sub-optimal in the sense that a government intervention can lead to a Pareto-improvement. There are several interventions one can think of (see e.g. [GP]). The most interesting one in this model is clearly to consider the effects of a regulation of margin-requirements. In other words, can it be Pareto-improving to force agents to trade only a subset of the available assets. [GZ] are the first to address this question. They show that without price effects, competitive equilibrium allocations are constrained efficient and a regulation can never be Pareto-improving. However, even without price effects, it is likely that the majority of agents benefits from a regulation, although this is not Pareto-improving.

It is a quantitative question how much collateralizable goods needs to be in the economy to achieve Pareto optimality and how the collateralizable goods needs to be distributed. Our examples show how for realistic levels of collateralizable goods welfare losses due to endogenously incomplete markets can be large. Furthermore, the examples illustrate that regulation of margin requirements generally does not lead to Pareto-improvements. However, often a majority of agents would favor such a regulation since it is welfare improving for them. Also, the examples elucidate that subprime loans are optimal when the borrower owns almost no collateralizable goods.

In our model, collateral levels are exogenously given, but since all possible collateral levels are in principal available for trade, one can think of the market picking out the collateral levels. [AOP] and [AFP] develop a model where collateral levels are determined endogenously and set by the lender. It is subject to further research to compare the welfare consequences and assets traded in our model to their analysis.

Clearly our focus on a two period model has important implications for our welfare analysis. In a dynamic model (see e.g. [APT] or [KS3]), where agents can accumulate the durable good over time, the distribution of collateral is endogenous. It is an important open question to evaluate the welfare consequences in such a model.

The Chapter is organized as follows. In Section 2.2 we present the GEI model with collateral. In Section 2.3, we discuss small theoretical results. In Section 2.4 we give examples of the GEI for two-period model with collateral.

2.2 The economy model

As in the Chapter 1 we consider a pure exchange economy over two time periods $t = 0, 1$ with uncertainty over the state of nature in period 1 denoted by the subscript $s \in \mathcal{S} = \{1, \ldots, S\}$. For convenience, the first period will sometimes be called state 0 so that in total there are $\mathcal{S}^* = S + 1$ states.

The economy consists of a finite number of H agents denoted by the superscript $h \in \mathcal{H} = \{1, \ldots H\}$ and $L = 2$ goods or commodities, denoted by the subscript $l \in \mathcal{L} = \{1, 2\}$. Throughout the analysis we assume that good 1 is perishable and good 2 is durable and that the consumption-durability technology is $Y_s = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ for each s, meaning that 1 unit of perishable good becomes 0 units of both goods, and 1 unit of durable good becomes 1 unit of durable good in period 1.

Each agent has an initial endowment of the L goods in each state, $e^h \in \mathbb{R}_+^{S^*L}$. The preference ordering of agent h is represented by a utility function $u^h : \mathbb{R}_+^{S^*L} \rightarrow \mathbb{R}$, defined over consumption $x^h = (x_0^h, x_1^h, \ldots, x_S^h) \in \mathbb{R}_+^{S^*L}$.

The characteristics of agent h are summarized by a utility function and endowment vector (u^h, e^h) satisfying standard assumptions A1-A4 listed in the Chapter 1.

In each state $s = 0, \ldots, S$ there are complete spot markets - the spot prices of the commodities across states are denoted by $p \in \mathbb{R}_{++}^{S^*L}$.

There are J real assets denoted by the subscript $j \in \mathcal{J} = \{1, \ldots, J\}$. We will assume that:

A 6. *Each asset only promises payments in commodity 1.*

We will also assume that the promises are independent of the states of nature. Hence we can normalize so that each asset promises one unit of commodity 1 in each state of nature.

We associate with each asset $j \in \mathcal{J}$ a collateral requirement $C_{2j} \geq 0$. We assume that the collateral always has to be held by the borrower, in order to simplify our analysis. Agents have to hold C_{2j} units of good 2 in order to sell one unit of asset j. We assume that collateral being held to secure some asset j cannot be used for any other asset and long position of other assets cannot be used to secure a short-position. This will turn out to be a strong assumption with important welfare implications.

Agents default on their promises whenever the market value of the durable good they hold as collateral is lower than the face value of their promise. Given equilibrium prices p, the actual payoff of asset j in states s is therefore $\min(p_{s1}, p_{s2}C_{2j})$. Let $q \in \mathbb{R}_+^J$ denote the prices of assets in period zero. It is useful to define the margin requirement on asset j as

$$\mu_j = \frac{C_{2j}p_{02} - q_j}{q_j}.$$

Finally, let $\theta^h = (\theta^h_1, \ldots, \theta^h_J) \in \mathbb{R}^J_+$ denote the number of units of each of the J assets bought by agent h, and $\varphi^h = (\varphi^h_1, \ldots, \varphi^h_J) \in \mathbb{R}^J_+$ the short-positions in the assets.

The economy with collateral, E_{GEIC}, is characterized by the agents' utility functions $u = (u^h)_{h\in\mathcal{H}}$, the agents' endowment process $e = (e^h)_{h\in\mathcal{H}}$ and the asset structure $(C_{2j})_{j\in\mathcal{J}}$.

Given $p \in \mathbb{R}^{S^*L}_{++}$, and $q \in \mathbb{R}^J_+$ the agent h chooses consumption and portfolios $(x^h, \theta^h, \varphi^h)$, to maximize utility subject to the budget constraints.

$$\begin{aligned}
&\max_{x^h \in \mathbb{R}^{S^*L}_+} u^h(x^h) \\
\text{s.t.} \quad & \text{there exists } \theta^h \in \mathbb{R}^J_+ \text{ and } \varphi^h \in \mathbb{R}^J_+ \text{ with} \\
& \sum_{l\in\mathcal{L}} p_{0l}(x^h_{0l} - e^h_{0l}) + \sum_{j\in\mathcal{J}} q_j(\theta^h_j - \varphi^h_j) \le 0; \\
& \sum_{l\in\mathcal{L}} p_{sl}(x^h_{sl} - e^h_{sl} - x^h_{02}) \\
& - \sum_{j\in\mathcal{J}} (\theta^h_j - \varphi^h_j) \min\{p_{s1}, p_{s2}C_{2j}\} \le 0; \quad \forall s \in \mathcal{S}. \\
& x^h_{02} - \sum_{j\in\mathcal{J}} \varphi^h_j C_{2j} \ge 0; \quad \forall l \in \mathcal{L}.
\end{aligned} \tag{2.1}$$

In state s, an asset j pays $\min\left\{p_{s1}, \sum_{l\in\mathcal{L}} p_{s2}C_{2j}\right\}$, an agent has endowments and receives x_{02} units from his 'investment' in the first period. As in the Chapter 1 we refer to the last inequality constraint in the agent's problem as the 'collateral' constraint.

As in the Chapter 1 a GEIC equilibrium is defined as usual by agents' optimality and market clearing. The existence follows from [GZ] as cited in the Chapter 1.

As a benchmark for welfare, we also consider the Arrow-Debreu or general equilibrium (GE) in the present durable goods, as shown in the Chapter 1.

2.3 Some theoretical observations

Throughout the analysis we want to think of the set of assets $\mathcal{J}$ as a finite but very large set. It is easy to see that 'generically' (e.g. for an open and full-measure set of individual endowments) for each GEI equilibrium we have $p_{s2}/p_{s1} \neq p_{s'2}/p_{s'1}$ for each $s, s' \in \mathcal{S}$. For the remainder of this Chapter, we will only consider this generic case.

The assets in $\mathcal{J}$ could distinguish themselves both with respect to promises and collateral requirements. As explained in the introduction, we focus on the case where all assets promise a safe pay-off.

2.3.1 Complete set of collateral requirements

We assume throughout that each asset j promises one unit of good 1 in each state $s = 1, ..., S$. The assets therefore distinguish themselves only by their collateral requirement C_{2j} and not by their promises. Of course, given the assumptions on default, this will also imply that the assets have different payoffs. We write $(C_{2j})_{j \in \mathcal{J}}$ to characterize all assets.

The first insight is that if the set $\mathcal{J}$ is very large many assets are collinear and not all assets are being traded in equilibrium, or differently put, there exists an equivalent equilibrium with trade in only a few assets (see also [GZ] for an explanation). In fact, it is clear that if for two assets j and j', $C_{2j} < C_{2j'} \leq \min_s p_{s1}/p_{s2}$, assets j and j' are collinear and it is without loss of generality to only consider equilibria with $\varphi_j = 0$.

More interestingly, if there are S states and there is a set of asset $\mathcal{J}^{CC}$ (CC standing for complete set of collateral requirements) that consists of S assets $j \in \mathcal{J}^{CC} \subset \mathcal{J}$ such that for each state $s = 1, ..., S$ there is a $j \in \mathcal{J}^{CC}$ with $C_{2j} p_{s2} = p_{s1}$, then it is without loss of generality to assume that only these S assets are traded. The set $\mathcal{J}^{CC}$ denotes the set of 'active' assets and contains S securities such that for each one, there is a different state in which the holder is indifferent between defaulting and paying the full promise. If $\mathcal{J}^{CC} \subset \mathcal{J}$, we say there is a complete set of collateral requirements.

The following proposition formalizes this issue.

Proposition 1. *Given an economy* $\left((u^h)_{h \in \mathcal{H}}, (e^h)_{h \in \mathcal{H}}, (C_{2j})_{j \in \mathcal{J}}\right)$ *and a GEI equilibrium* $[(\overline{x}, \overline{\theta}, \overline{\varphi}); (\overline{p}, \overline{q})]$*, suppose that for each* s *there is a* $j \in \mathcal{J}$ *with* $C_{2j} p_{s2} = p_{s1}$*, then* $\overline{x}$ *and* $\overline{p}, \overline{q}$ *are GEIC equilibrium consumptions and prices for any economy* $\left((u^h)_{h \in \mathcal{H}}, (e^h)_{h \in \mathcal{H}}, (C_{2j})_{j \in \tilde{\mathcal{J}}},\right)$ *if* $\mathcal{J} \subset \tilde{\mathcal{J}}$.

Proof. Let $\delta \in \mathbb{R}_+^{HS^*}$ denote the vector of multipliers associated with the $S+1$ spot budget constraints and let ${}_{col}\delta \in \mathbb{R}_+^H$ denote the multiplier associated with the collateral constraints. Agent h chooses positive θ_j^h if

$$-\delta_0^h q_j + \sum_{s \in \mathcal{S}} \delta_s^h \min\{p_{s1}, p_{s2}C_{2j}\} = 0; \tag{2.2}$$

Agent h chooses positive φ_j^h if

$$\delta_0^h q_j - \sum_{s \in \mathcal{S}} \delta_s^h \min\{p_{s1}, p_{s2}C_{2j}\} - {}_{col}\delta C_{2j} = 0; \tag{2.3}$$

Assume now that $i, j \in \mathcal{J}$, $C_{2i} < C_{2j}$ and that there is no $k \in \mathcal{J}$ with $C_{2i} < C_{2k} < C_{2j}$. If an individual holds an asset $\tilde{j} \in \tilde{\mathcal{J}}$ with $C_{2i} < C_{2\tilde{j}} < C_{2j}$, there obviously exists a $\mu > 0$ such that $\mu C_{2i} + (1-\mu)C_{2j} = C_{2\tilde{j}}$. The individual can then simply hold μ units of asset i and $(1-\mu)$ units of asset j, obtaining the same payoff in the second period and holding the exact same collateral. By Equation (2.2), the first order condition of the lender, if the individual is indifferent between holding asset i at price q_i, asset j at price q_j or holding asset j' at price $\mu q_i + (1-\mu)q_j$. Therefore the cost to the borrower of holding asset $\tilde{j}$ is the same as holding the portfolio of i and j that gives the same payoff. $\square$

Having established that it is without loss of generality[1] to limit ourselves to S assets $\mathcal{J}^{CC}$, the question is then if scarce durable good held as collateral implies that not all of these S assets are traded.

We denote a GEIC equilibrium in which the set of assets available for trade contains $\mathcal{J}^{CC}$ by GEICC, a GEI equilibrium with complete collateral. On the other, we might consider a situation where the set of assets available for trade is regulated exogenously and might not contain all potential assets would like to trade. We refer to a GEI equilibrium with an exogenously fixed set of collateral requirements as a GEIRC equilibrium – GEI with regulated collateral. For a given economy, there might a unique GEICC equilibrium, there are obviously always infinitely many GEIRC equilibria depending on different available assets. The GEICC equilibrium can be viewed as that specific GEIRC equilibrium for which adding assets (differing only by the collateral requirement) does not change the equilibrium allocation.

[1]Of course, there is a subtle issue concerning the possibility of multiple equilibria. Throughout the work, all statements are made about one of the equilibria.

2.3.2 Complete versus incomplete markets

We first consider the situation where there is an abundance of collateralizable goods in the economy and each individual owns enough of it to back his promises. Since the payoffs of the S securities in $\mathcal{J}$ are linearly independent, markets are complete. The first welfare theorem applies. It is a quantitative question, how much collateralizable goods is needed for this. In the numerical section below we address this question. For now it is useful to note that this is a joint conditions on endowments and preferences. What is important is that each agent owns collateral and that the value of this collateral is sufficiently large to secure all the short positions the agent has to take in order to span his desired consumption. Theorem 2 in [GZ] illustrates this point for the trivial case of no uncertainty. As we will see in the next section, to assume that collateralizable durable good is 'sufficiently large' in a world where all assets promise a safe payoff and variations in payoffs are only due to variations in prices is generally unrealistic. Agents have to take on huge short-position in some assets and would need to own a lot of collateralizable goods.

If, on the other hand, the amount of collateralizable goods is small or some agent own no collateralizable goods at all, not all S contracts in $\mathcal{J}$ will be traded in equilibrium. Instead, the examples below show that only very few assets are traded. Just like in the GEI model with incomplete markets, allocations will not be Pareto-efficient.

In the situation of scarce collateralizable goods the collateral constraints will be binding and ${}_{col}\delta^h > 0$ in Equation 2.3. Whether a particular contract will now be traded depends on the multipliers δ^h across agents. Generally they will not be collinear and some assets are more attractive to both agents than others. These will be the only asset traded in equilibrium.

In particular, it is easy to see that if an agent is poor in the first period and owns no durable good, he wants to finance his first period consumption in the durable good by selling an asset which promises to hand the durable good over to the lender in the second period. As [GZ] observe, this is essentially a rental contract. The agent buys the durable good and borrows as much money as possible on an asset that defaults for sure, i.e. hands the durable good to the lender in all states of the world tomorrow.

If the agent is very poor and his marginal utility for the durable good is large, this will be the only asset he will sell. He will not trade in any asset with a large collateral requirement since this cannot be used to finance extra consumption in the first period.

More interestingly, we give an example below (Example 2), where two agents trade in a unique asset that does not default in all states. We will argue that this asset has the best risk-sharing characteristics for the two agents.

2.3.3 Welfare when markets are incomplete

Following [GZ] it is a natural question whether it is efficient that precisely the assets in $\mathcal{J}$ are traded or if it is possible to make everybody in the economy better off by restricting trade to take place in other (possible fewer) assets. As [GZ] put it, 'Given that the markets choose the asset structure, we are compelled to ask whether the market chooses the asset structure efficiently'.

Neither [GZ] nor we can provide a complete answer to the problem. If all agents have identical homothetic utility (an assumption often made in applied work), the answer is simple. The market chooses the asset structure efficiently. The following result is a (trivial) special case of Theorem 3 in [GZ], since identical homothetic utility is sufficient for prices to be independent of the wealth distribution.

Theorem 3. *If all agents have identical homothetic utility, given a GEICC equilibrium with actively traded assets $\mathcal{J}^{CC}$, there is no other set of assets $\mathcal{J}'$ such that in the resulting GEIRC equilibrium all agents are better off.*

Below, we give an example where agents do not have identical utility but the GEICC equilibrium allocation is still constrained efficient. The example suggests, that there is no generic sense in which GEICC allocations are always inefficient when preferences are heterogeneous. However, this is as much as we know.

2.4 Risk-sharing with scarce collateralizable goods

In this section, we describe three numerical examples. The first example illustrates the point that the Arrow-Debreu allocation can be achieved if there is sufficient collateralizable goods in the economy.

We then give two examples that illustrate how scarce collateralizable goods leads to a situation where only very few assets are traded and welfare losses due to imperfect risk-sharing are large. In these examples, allocations are always far from Arrow-Debreu allocation. We use the algorithm described in Chapter 1 to approximate GEIC equilibrium numerically.

2.4.1 Example 1: two agents, two states

The first example illustrates how with a complete set margin requirements, plentiful collateralizable goods can lead to the Arrow-Debreu allocation, while an unequal distribution of collateralizable goods might imply a situation where markets are endogenously incomplete and welfare losses compared to the Arrow-Debreu allocation are large.

We first consider the simplest two period model with two states in period 1 $\mathcal{S}^* = \{0, 1, 2\}$ and two agents, $\mathcal{H} = \{1, 2\}$. Each individual $h = 1, 2$ has a utility function of the form:

$$u^h = 0.2\log(x^h_{01}) + 0.8\log(x^h_{02}) + \frac{1}{2}\sum_{s=1}^{2}(0.2\log(x^h_{s1}) + 0.8\log(x^h_{s2}))$$

Suppose first that collateralizable durable goods is plentiful and endowments are:

$$e^1_{sl} = (4, 2, 4, 0, 4, 0); \quad \forall s \in S^*; \forall l \in L$$
$$e^2_{sl} = (2, 2, 6, 0, 2, 0); \quad \forall s \in S^*; \forall l \in L$$

The GEICC equilibrium allocation is identical to the (unique because of Cobb-Douglas utility) Arrow-Debreu allocation for this economy. The collateral requirements for the two assets traded are $C_{21} = 0.1$ and $C_{22} = 0.16667$. With two states, these two assets are sufficient to complete the markets. The crucial point of this example is that each agent has so much collateralizable goods (and its price is so high) that collateral constraints are not binding and agents can trade to the complete markets allocation. Agent 1's portfolio $z^1_j = \theta^1_j - \phi^1_j$ is given by $z^1_1 = 5.2$ and $z^1_2 = -4$.

The situation is very different if instead of taking the durable good to be evenly distributed among agents, we assume that agent 1 initially owns the entire amount of the durable good, i.e.

$$e^1_{02} = 4, \quad e^2_{02} = 0.$$

Since we assumed identical homothetic utility, collateral requirements on the two assets are as above, $C_{21} = 0.1$ and $C_{22} = 0.16667$. However, in this case, agent 2 needs to borrow just to be able to consume the durable good in period 0. As this is relatively more important to him than any risk-sharing considerations, the agent will not establish a long position in asset 2 and the only asset traded in the GEICC equilibrium is asset 1. Agent 1's portfolio is now $z_1^1 = 2.66667$ and $z_2^1 = 0$ and second period risk is not shared at all. It is clear that now there should be substantial welfare losses compared to the Arrow-Debreu allocation.

Throughout the work, we report welfare numbers in terms of wealth equivalence compared to the Arrow-Debreu allocation. That is, for log-utility and the case of no discounting (these are the preferences considered throughout the work), if u^{hGEIC} denotes an agent's utility in the GEIC equilibrium and u^{hAD} denotes his utility in the Arrow-Debreu equilibrium, we compute $WR^h = \exp(\frac{u^{hGEIC} - u^{hAD}}{2})$ and we call welfare rate. If we multiply consumption in the Arrow-Debreu equilibrium by WR in all states, we obtain an allocation that gives the agent the same utility as in the GEIC equilibrium. That is a number of say 0.95 means that an agent would be willing to decrease his consumption in the Arrow-Debreu allocation by 5 percent in each state to avoid the incomplete markets consumption.

The welfare rate in this example are

$$(WR^1, WR^2) = (0.9998, 0.9567).$$

Naturally, the collateral requirement hurts agent 2 much more than agent 1. But also agent 1's welfare is clearly below its Arrow-Debreu level. Both the borrower and the lender are hurt by the fact that the borrower faces a binding collateral constraint and cannot take simultaneous long and short-positions to share risk in the second period.

As mentioned in the previous section, 'plentiful' collateralizable durable goods can lead to complete markets, while in a situation where an agent owns very little or no collateralizable goods, often only one asset is traded and welfare losses can be substantial. The example illustrates this point, but leaves open the question of what happens when there is little collateralizable goods in the economy as a whole and of what assets are traded when $\mathcal{J}^{CC}$ consists of more than only two assets. The next example provides some insights to this question.

2.4.2 Example 2: two agents, four states

We now consider an example with four states in period 1 $\mathcal{S}^* = \{0, 1, \ldots, 4\}$ and two agents, $\mathcal{H} = \{1, 2\}$, each with identical utility,

$$u^h = \log(x_{01}^h) + \log(x_{02}^h) + \frac{1}{4}\sum_{s=1}^{4}(\log(x_{s1}^h) + \log(x_{s2}^h))$$

We consider a variety of profiles of endowments, differing by the distribution of the durable (collateralizable) good in the first period.

$$e^1_{0l} = (4, \eta), e^1_{1l} = e^1_{2l} = (1, 0), e^1_{3l} = e^1_{4l} = (2, 0);$$
$$e^2_{0l} = (1, (1 - \eta)), e^2_{1l} = e^2_{3l} = (1, 0), e^2_{2l} = e^2_{4l} = (2, 0.2).$$

In the first period agent 1 is rich (the natural lender in the example) and agent 2 is poor. In the second period both agents face identically distributed shocks to endowments of good 1 that are independent across agents. In addition, agent 2 has random endowments in the durable good. The parameter η determines how the collateralizable durable good is distributed between the two agent. We consider $\eta \geq 1/2$.

Since we assume identical homothetic utility, spot-prices do not depend on η. The set $\mathcal{J}^{CC}$ consists of the four assets with collateral requirements $C_{21} = 0.5$, $C_{22} = 0.4$, $C_{23} = 0.333$ and $C_{24} = 0.3$. The margin requirements depend on the interest rate and therefore on the distribution of first period endowments, η. The assets' payment in the states defined by $\frac{\min\{p_{s1}, p_{s2}C_{2j}\}}{p_{s1}}$ follows in Table 2.1:

Table 2.1: Assets' payment in the states

Assets	state 1	state 2	state 3	state 4
j=1	1	1	1	1
j=2	0.8	1	1	1
j=3	0.667	0.833	1	1
j=4	0.6	0.75	0.9	1

Obviously, if agents would not face a collateral constraint in period 0, markets would be complete and the Arrow-Debreu allocation (which is unique since we assume Cobb-Douglas utility) would be achieved. However, since agents do face collateral constraints, the Arrow-Debreu allocation is not achieved for any value of η.

Assets traded

We first examine, how portfolios depend on the distribution of the durable good and how with an unequal distribution, the fact that collateralizable goods is scarce implies that only one of the four assets is traded and the collateral requirement is uniquely determined endogenously.

The following Table 2.2 denotes the portfolio-holding $\theta - \varphi$ of agent 1 for different values of η.

Table 2.2: Portfolio agent 1

η	asset 1	asset 2	asset 3	asset 4
0.5	-0.48	1.08	-0.81	0
0.75	0	0.77	-0.15	0
0.8	0	0.70	-0.01	0
0.81	0	0.68	0	0
0.85	0	0.15	0	0.62
0.9	0	0	0	0.67
0.95	0	0	0	0.51

If the borrower owns almost nothing of the durable good in period zero and has to buy it to consume it, the only asset traded in equilibrium is the one with the lowest possible margin requirement. The borrower buys the durable good and borrows as much as possible for it while he is for sure going to default in all states in the second period. Any asset with a higher margin requirement is not optimal since the additional price the lender would be willing to pay to get a payoff above one in states 1,2 or 3 is not sufficiently high for the borrower to forgo extra consumption in period zero. The little collateralizable goods he owns he needs to use to finance the margin on the loan he takes out just to buy more of the durable good.

For $\eta = 0.85$, two assets are traded, the full default asset, but also asset 2, that pays back in full in states 2, 3 and 4. This is traded for risk-sharing in the second period. In state 2, the borrower is rich (has endowments of 2) while the lender is poor (has endowments of 1). So the lender values this asset relatively more than the borrower, making its price high enough so that the borrower is willing to take on the extra collateral (compared to assets 3 and 4). Around $\eta = 0.81$ there is a robust region of endowments in good 2, for which in fact asset 2 is the only asset traded. For these distributions of endowments, remarkably there is a unique asset determined collateral-requirement that is not equal to the lowest collateral available. Since the two agents want to share the risk in the second period, if agent 2 has sufficient collateralizable goods to sell only asset 2 to finance first period consumption, he does so because this asset fetches the relatively best price among the 4 available assets.

For $\eta = 0.8$, the borrower, agent 2, himself starts buying assets in the first period. He holds a long-position in asset 3 and borrows money in asset 2. Asset 2 pays in full in states 2, 3 and 4 but defaults in state 1. In state 3, the borrower is poor (endowments of one) while the lender is rich (endowments of 2), so buying asset 3 is a way for the borrower to insure himself against the second period risk. As the table shows this is true for all η between 0.75 and 0.8.

Finally, for $\eta = 0.5$ both agents have sufficient collateralizable goods to establish large short positions in some assets, however, the collateral requirement is still binding for both agents. Asset 4 (which obviously would be traded in a situation with collateral and default) is still not traded in equilibrium. Instead, agent 1 has relatively large short positions in assets 1 and 3. Agent 2 only uses asset 3 to borrow and finance first period consumption. He takes long-positions in assets 1 and 3 to insure against the second period risk. Even with both agents owning substantial amounts of the durable good in the first period, trade only takes place in 3 of the four available assets.

This obviously raises the question how large the welfare losses due to this trade in a restricted set of assets are and if welfare can be improved by 'forcing' agents to trade in other assets through a regulation of the margin requirements.

Welfare

We first consider the case of a complete set of collateral requirements and ask how large the welfare losses are that are implied by the fact that agents cannot commit to pay their promises, i.e. we compare the GEIC equilibrium welfare to the welfare's agents would obtain in an Arrow-Debreu equilibrium. The following Table 2.3 shows this welfare rate that due to default and collateral for different values of η.

As one would expect, the welfare losses due to default and collateral are large when the borrower has little collateralizable goods. In particular for $\eta > 0.9$, the possible welfare gains from better enforcement of intertemporal contracts are very large. Note that for each value of η, we compare the GEIC welfare to the Arrow-Debreu equilibrium welfare for that given economy. So

Table 2.3: Welfare rate for distribution of durable good: agent 1 and 2

η	Lender (agent 1)	Borrower (agent 2)
0.5	0.993	0.996
0.75	0.992	0.989
0.8	0.992	0.982
0.85	0.991	0.978
0.9	0.989	0.969
0.95	0.988	0.933

for $\eta = 0.95$, agent 2 would gain more than 7 percent if he could commit to pay back all promises and trade in all assets without holding any collateral. This would allow the agent to buy more of the durable good in the first period and to ensure against his endowment risk in the second period. Since with log-utility risk aversion is relatively low, the endowment risk is actually not the main source of the welfare losses. They are mostly due to the fact that the agent cannot afford to consume very much of the durable good. In fact, agent 2's GEIC equilibrium consumption in period zero is $x_1^2(0) = 0.76$ and $x_2^2(0) = 0.15$ while in the Arrow-Debreu equilibrium it is $x_1^2(0) = 1.10$ and $x_2^2(0) = 0.22$. Of course, his second period consumption is higher in all states, but this shows how collateral skews consumption away from the efficient Arrow-Debreu allocation.

Even for the case of an equal distribution of collateralizable durable goods, the welfare losses are still substantial. As we argued in the previous section, not all assets are traded and collateralizable goods is still scarce. Moreover, it is surprising that he fact that the natural borrower (agent 2) has more collateralizable goods available to finance large short-positions, does not necessarily bring the other agent closer to the complete markets welfare. Between $\eta = 0.8$ and $\eta = 0.5$ the welfare losses remain more or less constant for agent 1, while there are still substantial improvements for agent 2.

The fact that even for $\eta = 0.5$, the welfare losses due to default and collateral are still substantial (certainly significantly positive) makes highlight that it is not clear ex ante what 'plentiful collateralizable goods' means. In this example, agents spend 50 percent of their income on the consumption of the durable good. That seems to large fraction but not suffices to create so much collateralizable durable goods that markets are complete.

Regulating the margin requirement

What happens if, instead of allowing the agents to trade in assets with arbitrary margin requirements, we consider a situation where there is a fixed set of S assets with 'optimally' set margin requirements?

As explained above, under the assumption made in this example that all agents have identical homothetic utility, equilibrium allocations must be constrained efficient, and it is impossible to make both agents better off by exogenously selecting margin requirements. This obviously does not imply, however, that all possible margin-requirements are Pareto-ranked.

To illustrate these points, we consider two cases. First we assume $\eta = 0.95$. In this case, it seems likely that GEIC equilibrium allocations are in fact Pareto-ranked, with the GEICC allocation yielding the highest utility for both agents: Forcing both agents to trade in an asset that does not default in all states will reduce trade in the single asset traded and most likely make most agents worse off. This intuition turns out to be correct. In Figure 2.1, we show a few different points in utility (measured in terms of welfare rate) space for a sample 895 values of collateral between 0 and 1.

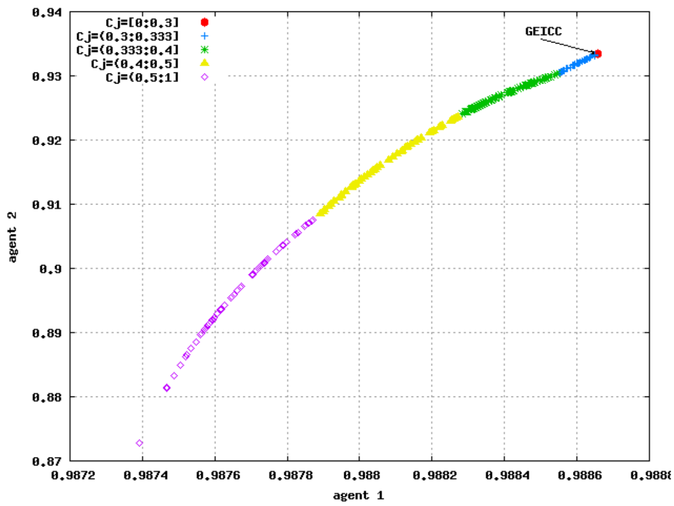

Figure 2.1: Regulated collateral for $\eta = 0.95$

The figure clearly shows that all equilibria are Pareto-ranked with the largest utility arising from the GEICC allocation

We also consider the case $\eta = 0.85$. In this case, the GEIC equilibria are not Pareto-ranked as Figure 2.2 shows for a sample 350 values of collateral between 0 and 1.

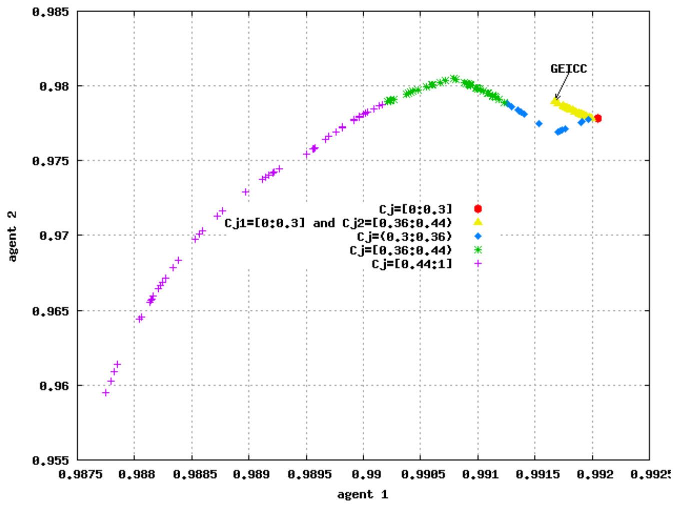

Figure 2.2: Regulated collateral for $\eta = 0.85$

While it is true that there is no GEIC equilibrium that is Pareto-better than the complete collateral equilibrium, there are GEIC equilibria that make agent 1 better and there are other GEIC equilibria that make agent 2 better off. In particular, somewhat counter-intuitively, the lender, agent 1, would be better off if is trade only takes place in the asset that defaults in all states. The point in the graph that gives him the highest utility corresponds to a situation where only the full default asset is available for trade. In this case, all agents of type 2 borrow heavily, since the collateral requirement is not an issue. The equilibrium interest rate is so high that agent 1 is compensated by the high interest rate for the fact that the only available asset has bad risk-sharing properties.

On the other hand, the point that gives agent 2 the highest utility corresponds to the case where only one asset with collateral requirement 0.4 is traded. This asset only defaults in one state. If it is the only asset traded in equilibrium, its interest rate is so favorable that agent 2 is well off. Agent 1 naturally is hurt by the low interest rate.

Subprime loans

As explained above, if the borrower owns almost no durable goods in the first period (case when $(1-\eta) = 0.05$), the GEIC equilibrium allocations are Pareto-ranked and the only asset traded is the one with lower collateral requirement. Figure 2.1 illustrates this. The highest utility[2] for both agents is with collateral between 0 and 0.3 (red point), this is true for all $\eta \geq 0.88$. In these cases, subprime is good for agent 1 (he benefits with higher interest rate) and agent 2 (poor borrower buys the durable good), because the only asset traded in equilibrium is the one with the lowest collateral. When the borrower has more of the durable good (case when $(1-\eta) = 0.15$), the GEIC equilibria are not Pareto-ranked and now subprime is good only for agent 1 (rich). Figure 2.2 shows the largest utility for agent 1 with collateral between 0 and 0.3 (red point) and the largest utility for agent 2 with collateral between 0.36 and 0.44 (green points), this behavior holding for all $0.82 \leq \eta \leq 0.87$.

2.4.3 Example 3: three agents

Finally, we consider an example with 3 agents. It seems clear that enough heterogeneity among agents should lead to trade in several assets even when collateralizable goods is scarce and unequally distributed among agents. As in Example 2, we first discuss portfolios, then report welfare rate due to collateral and finally show how the GEICC equilibrium welfare compares to welfare achieved in regulated economies.

To keep the example relatively simple, we assume that there are $S = 3$ states in the second period. The three agents' endowments are given by

$$
\begin{aligned}
e^1_{0l} = (4,\eta), e^1_{1l} = (1,0), e^1_{2l} = (4,0), e^1_{3l} = (2,0);\\
e^2_{0l} = (1,\gamma), e^2_{1l} = (1,0), e^2_{2l} = (2,0), e^2_{3l} = (4,0);\\
e^3_{0l} = (2,1-\eta-\gamma), e^3_{1l} = (2,0.2), e^3_{2l} = (2,0), e^3_{3l} = (2,0.2);
\end{aligned}
$$

We assume that agents have heterogeneous utility. Under this assumption, Theorem 3 from above does not imply and there could be GEIRC allocations that are Pareto-better than the GEICC allocation. However, in this example this turns out not to be the case. Utility functions are

$$
u^h = \alpha^h \log(x^h_{01}) + (1-\alpha^h)\log(x^h_{02}) + \frac{1}{3}\sum_{s=1}^{3}(\alpha^h \log(x^h_{s1}) + (1-\alpha^h)\log(x^h_{s2})),
$$

[2]Measure by Welfare Rate

with

$$\alpha^1 = 0.7, \quad \alpha^2 = 0.77, \alpha^3 = 0.625.$$

With these preferences, the collateral requirements of the traded assets (i.e. of the assets in $\mathcal{J}^{CC}$) obviously vary with η and γ since spot prices will vary. However, since preferences are quite similar, the variation is relatively small. In most cases, the collateral requirements for the three assets traded are around $C_{21} = 0.644, C_{22} = 0.288$ and $C_{23} = 0.362$. In all the cases we consider, asset 1 pays back in full in all states, asset 2 pays one unit in state 2, but defaults in states 1 and 3 and asset 3 defaults only in state 1, but pays one unit in both states 2 and 3.

Portfolios

As before, we first examine which assets are actively traded in the GEICC equilibrium, depending on the distribution of collateralizable goods which we parametrize by η, γ. We report portfolios of agent 1 and of agent 2 in Table 2.4.

Table 2.4: Portfolio agent 1 and 2

η	γ	Portfolio agent 1	Portfolio agent 2
0.9	0.05	(0.00,0.34,0.76)	(0,-0.34,0)
0.8	0.1	(0.06,0.42,0.74)	(0,-0.42,0)
0.8	0.2	(0.00,0.00,1.15)	(0,0,-0.48)
0.6	0.3	(0.18,-0.15,0.98)	(0.05,0.15,-0.54)
0.6	0.2	(0.55,0.00,0.46)	(0.00,0.00,-0.46)
0.6	0.1	(0.64,0.04,0.24)	(0.00,-0.1,-0.24)
0.4	0.3	(0.49,-0.32,0.52)	(0.13,0.13,-0.52)

Although now, there are almost always more than just one asset being traded in equilibrium, the same logic as in Example 2 above can be applied to understand which assets are traded. In terms of risk-sharing, agent 1 would ideally want to hold an asset that pays a lot in state 1, little in state

2 and substantial in state 3. Among the available assets, asset 1 comes the closest to this pattern in the sense that it is the only asset that pays in full in state 1. Asset 2 seems the worst since it only pays in full in state 2. Agent 2, on the other hand, would like to borrow since he is poor today and pay back more in state 3 than in state 2. He is also poor in state 1, so asset 1 is not a good asset to borrow in. Agent 3 is relatively rich in states 1 and 3, but wants to pay less in state 2. So asset 1 seems to be a good asset to be traded between agent 1 and 3, while asset 3 seems to be a good asset for trade between agent 1 and 2. But asset 3 is also attractive for agent 3 since he pays in full in state 3, where he is relatively rich. Asset 2 is the worst for agent 3.

Unfortunately, in the first case agent 2 is so poor that he only borrows to finance consumption in the durable good, therefore he trades in asset 2 which defaults in all states and, as we pointed out, can be interpreted as a rental contract. In this case, agent 3, although poor, still trades in asset 3 which gives him a relatively better price although it requires him to hold more collateralizable good.

In the second case, both agents become more wealthy, but still agent 2 is stuck with asset 2 (he is still too poor to trade in any other asset). Instead agent 3 starts borrowing both in asset 1 and in asset 3, which are both good assets for him to share risk with agent 1.

In the third case, agent 3 has no durable goods. Agent 2 is not yet rich enough to trade in anyting but asset 3. There is a unique asset being traded in equilibrium by all three agents.

Let us now consider $\eta = 0.6$ – there is a lot of collateralizable goods both for agents 2 and 3. In the first case, agent 2 owns a lot of collateralizable good, now instead of using asset 2 to borrow, he actually goes long in asset 2 and borrows only in asset 3. Agent 3 borrows in assets 1 and 3. Agent 1 goes short in asset 2, to share risk with agent 2.

For $\gamma = 0.2$ agent 2 has not enough collateralizable good anymore to borrow enough so that he can take long-positions in some assets. He exclusively borrows in asset 3, which with some collateralizable good is the best way to at the same time borrow and share risk with agent 1. For $\gamma = 0.1$, agent 2 is quite poor again and has to do some of the borrowing in the full default asset.

Finally, for the case $\eta = 0.4, \gamma = 0.3$, we have a situation where the three available assets go a long way to share second period risk for the three agents. We now ask, how much of their Arrow-Debreu welfare the agents can achieve for the different distributions of collateralizable durable goods.

Welfare

As in Examples 1 and 2, we now want to examine how scarce collateralizable goods leads to welfare loss. In this example, we in addition want to point out how one agent's welfare-losses can depend on the distribution of collateralizable durable good between the to other agents. The following Table 2.5 shows this welfare rate that due to default and collateral for different values of η and γ.

Table 2.5: Welfare rate for the distribution of durable good: agent 1, 2 and 3

η	γ	Agent 1	Agent 2	Agent 3
0.9	0.05	0.9756	0.9470	0.9773
0.8	0.1	0.9775	0.9718	0.9845
0.8	0.2	0.9783	0.9973	0.9612
0.6	0.3	0.9815	0.9926	0.9810
0.6	0.2	0.9831	0.9856	0.9906
0.6	0.1	0.9804	0.9662	0.9966
0.4	0.3	0.9836	0.9886	0.9928

For all distributions of collateralizable goods, welfare losses are substantial for all three agents. In the first case, despite the fact that agent 1 is the lender and his collateral constraint is not binding, his welfare losses due to collateral are substantial. The fact that with scarce collateralizable durable good agents 2 and 3 are not willing to trade in asset 1 leads to little risk sharing in the second period. The table also shows that even for $\eta = 0.4, \gamma = 0.3$, i.e. in a situation where every agent owns substantial collateralizable good, welfare losses compared to incomplete markets are fairly large, in particular for agents 1 and 2.

Agent 2 is actually the one who seems to be hurt relatively least by default and collateral requirements. This is surprising since he is relatively poor in the first period, having the greatest need to borrow. However, the payoffs of the available assets are best for him - in states where he is poor he has to pay back relatively less than in states where he is rich.

Regulating the collateral requirements

As before, we ask how regulating the collateral requirements and not allowing agents to trade in arbitrary assets can influence welfare. Note that since preferences are heterogeneous, Theorem 2 no longer applies and it could be possible that the GEICC allocation is constrained optimal. To investigate this question, we search for GEIRC equilibria that could be Pareto better.

We examine the case $\eta = 0.8, \gamma = 0.1$. The Figure 2.3 shows a three dimensional scatter plot of different utility (measured in terms of welfare rate) levels corresponding to different GEIRC allocations for a sample 813 values of collateral between 0 and 1.

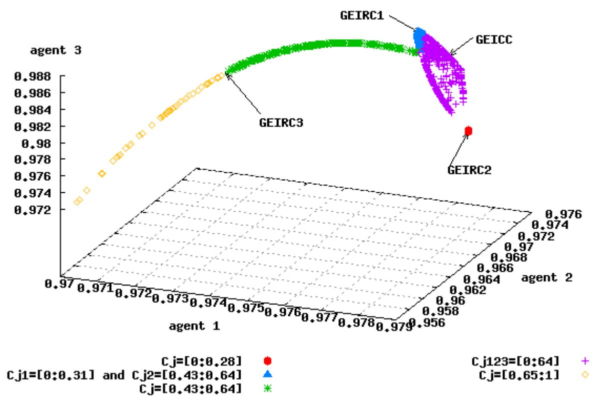

Figure 2.3: Regulated collateral for $\eta = 0.8, \gamma = 0.1$

While it is still impossible to Pareto-improve on the GEICC allocation, both agents 2 and 3 can obtain relatively gains through a regulation. It is clear that the allocations are not Pareto-ranked, but the Figure 2.3 also shows that the GEICC allocation cannot be Pareto-dominated. It is a bit difficult to see in a 3D scatter plot, but it turns out that both agent 2 and 3 prefer a regulated equilibrium. Figure 2.4 illustrates this.

While the GEICC allocation is not Pareto-dominated, the figure shows that agents 2 and 3 can do much better. At the GEICC equilibrium the collateral requirements are $C_{21} = 0.644, C_{22} = 0.288$ and $C_{23} = 0.362$. The

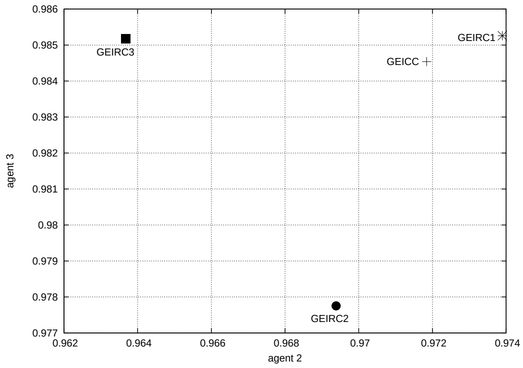

Figure 2.4: Regulated collateral for $\eta = 0.8, \gamma = 0.1$ - Agents 2 and 3

point denoted by GEIRC1 in the figure corresponds to the equilibrium where trade is restricted to take place only in two assets and collateral levels are set exogenously to $C_{22} = 0.286$ and $C_{23} = 0.45$.

The first asset has been eliminated and the collateral requirement on the third asset is higher than its endogenous level. While agents 2 and 3 benefit from this, the next figure shows that agent 1 loses.

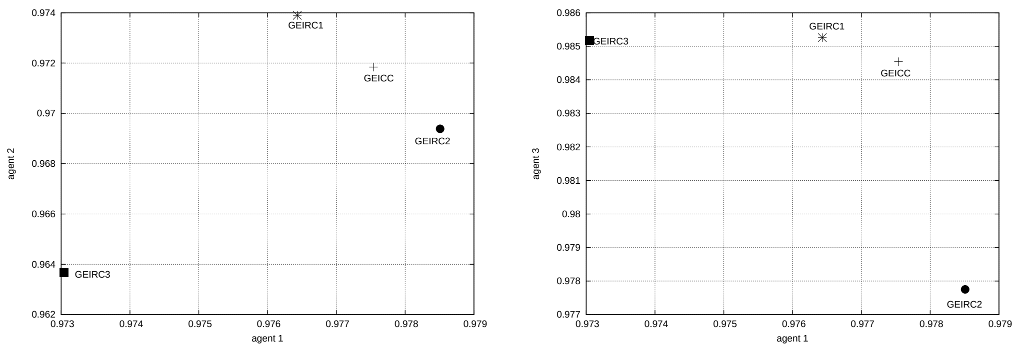

Figure 2.5: Regulated collateral for $\eta = 0.8, \gamma = 0.1$ - Agents 1 and 2 (left) and Agents 1 and 3 (right)

Figure 2.5 shows that agent 1 is best off in a GEIRC where all agents are only allowed to trade in a asset that fully defaults. The point GEIRC2 in the figure corresponds to this point. Since trade takes place in two assets, there is still significant risk sharing possible, however, the equilibrium interest rate decreases making it easier for the borrowers to finance first period consumption.

The situation here is similar to the previous example. The lender is better off in a situation where all trade takes place in the full default asset (i.e. essentially only rental contracts are traded). While trade in this asset does not allow for any risk-sharing, if all borrowers are forced to borrow only in this asset, the equilibrium interest rate rises so dramatically that in fact the lender is compensated for the lack of risk sharing by a high interest rate.

Finally, restricting all agents to trade in an asset that never defaults makes all agents worse off. The point GEIRC3 shows the equilibrium that arises if there is only one asset available for trade and its collateral requirement ensures (exactly) full delivery in all three states.

Subprime loans

As in the Example 2 we examine in which situations subprime is optimal for agents. In this example for all η and γ, GEIRC equilibria are not Pareto-ranked, then there not exist an economy where subprime is optimal for all agents. The case examined above with $\eta = 0.8, \gamma = 0.1$, subprime is good for agent 1 (rich) and subprime is good for agent 2 (poor) in one asset. The Figure 2.3 shows the largest utility for agent 1 with collateral between 0 and 0.28 (red point), the largest utility for agent 2 with collateral between 0 and 0.31 in one asset traded and another with collateral between 0.43 and 0.64 (blue points). The largest utility for agent 3 with collateral between 0.43 and 0.64 (green points).

2.5 Conclusion

In this Chapter we consider a model with default and collateral and demonstrate how scarcity of collateralizable goods can lead to large welfare losses in the absence of other mechanisms to enforce intertemporal contracts.

Following [GZ] we allow for a complete set of collateral requirements and show that in the GEIC equilibrium generally only few of the possible contracts are traded.

In our examples, equilibrium is constrained efficient in the sense that a regulation of collateral requirements never leads to a Pareto-improvement for all agents. However, we show that equilibria corresponding to different regulated collateral requirements are often not Pareto-ranked, and some agents can be benefited with regulation. We show, through examples, that when the borrower owns almost nothing of the durable good a subprime loan is optimal for both agents (lender and borrower). The lender always benefits from subprime loans, his risk being completely absorbed by the higher interest rate carried by these loans.

The welfare losses due to collateral can be large if some agents own small amounts of the collateralizable durable goods.

Chapter 3

Approaching the Pareto frontier with collateral and default penalties equilibria

3.1 Introduction

It is well known that incomplete markets equilibrium allocations are generically inefficient (see [MS]). When markets are incomplete, there do not exist claims that pay off for each possible state of world, thus, the optimal risk-sharing no longer holds. Moreover, in models that allow for default, even if there are many more securities than states of the world available for trade, markets can be endogenously incomplete leading to non Pareto optimal allocations. We have seen examples of this already in Chapter 2, where, the source of incompleteness was the scarcity of collateralizable durable goods.

Basic questions in general equilibrium analysis are concerned with the conditions under which an equilibrium will be efficient, and in such case, which efficient equilibria can be achieved. We examine, through examples, situations where collateral and default penalties equilibrium allocations can approach Pareto optimality. According to [G1], it is expected that an increase in available collateral (coming either by an improvement of the legal system, or by the increase of the number of durable goods) is welfare improving.

In the literature on general equilibrium with possibility of default, there is basically two classical ways of enforcing the honoring of financial commitments: collateral and default penalties, as explained in the Chapter 1.

In our model, there are two periods, with uncertainty over the states of the world in the second period. There are two commodities, the durable good serving as collateral. In the simplest version of the model, the only assets available for trade are promising one unit of the first good across all states. We suppose that there are S states and S assets for trade with the same promises, they can be distinguished either for their collateral requirements, or by the utility loss they trigger in case of default.

Our main focus is on the extent that the equilibria allocations approximate the Pareto optimal frontier, and we exhibit examples where this quantitative problem is quite sensitive to qualitative features of the endowment distribution. If the endowment distribution displays only heterogeneity between periods (e.g., one agent is the richest in the first period and another is the richest in all states of nature of the second period), the examples suggest that for some collateral equilibria can approach Pareto optimality. If the heterogeneity is also manifest between states of nature, the examples suggest that default penalties equilibria are often Pareto superior to collateral equilibria.

We also consider sets of examples shedding light on the following problem: Under which conditions do collateral equilibria coincide with the Arrow-Debreu (complete markets) equilibria, in the case when the endowments in the states of nature are heterogeneous? This is only achieved when the fraction of consumption expenditure on collateralizable durable goods is unrealistically large: this is shown in example 1 in the case where only one durable good is present.

In order to determine how equilibria do or do not approach Pareto optimality, we solved the Pareto frontier numerically. Our approach to this problem is to consider different income distribution in the Arrow-Debreu equilibrium.

This Chapter is organized as follows. In Section 3.2 we present the model with collateral (GEIC), the model with default penalties (GEI$_\lambda$) and the relations of the competitive equilibrium with the Pareto frontier. In Section 3.3 we compute numerically the GEI and the Pareto frontier for example economies of the two-period models with collateral and default penalties.

3.2 The economy model

As in Chapter 1 and 2 we consider a pure exchange economy over two time periods $t = 0, 1$ with uncertainty over the state of nature in period 1 denoted by the subscript $s \in \mathcal{S} = \{1, \ldots, S\}$. For convenience, the first period will sometimes be called state 0 so that in total there are $\mathcal{S}^* = S + 1$ states.

The economy consists of a finite number of H agents denoted by the superscript $h \in \mathcal{H} = \{1, \ldots H\}$ and $L = 2$ goods or commodities, denoted by the subscript $l \in \mathcal{L} = \{1, 2\}$. As in Chapter 2, we make the assumption that each one unit of the durable good in the first period yields one unit of the durable good in the second period.

Each agent has an initial endowment of the L goods in each state, $e^h \in \mathbb{R}_+^{S^*L}$. The preference ordering of agent h is represented by a utility function of the consumption $x^h = (x_0^h, x_1^h, \ldots, x_S^h) \in \mathbb{R}_+^{S^*L}$, denoted by $u^h : \mathbb{R}_+^{S^*L} \to \mathbb{R}$.

The characteristics of agent h are summarized by a utility function and endowment vector (u^h, e^h) satisfying standard assumptions A1-A4 listed in the Chapter 1.

In each state $s = 0, \ldots, S$ there are complete spot markets - the spot prices of the commodities across states are denoted by $p \in \mathbb{R}_{++}^{S^*L}$.

There are J real assets denoted by the subscript $j \in \mathcal{J} = \{1, \ldots, J\}$. We will assume that each asset only promises payments in commodity 1. We will also assume that the promises are independent of the states of nature. Hence we can normalize so that each asset promises one unit of commodity 1 in each state of nature. Such assumptions were also made through our analysis in Chapter 2.

In the two-period model of [GZ], to each promise j we must formally associate levels of collateral held by the borrower $C_j \in \mathbb{R}_+^L$, which is given exogenously and has the purpose of protecting the buyer when sellers do not honor their commitments.

Another model which allows for strategic default, but still maintains some incentive for repayment, are default penalties. According to [DGS2] an agent incurs a loss in utility when he defaults, the loss increasing proportionately with the value of the default and is denoted by $\lambda_{sj}^h \in \bar{\mathbb{R}}_+$ the real default penalty on agent h for asset j in state s. The effective payment is $D^h \in \mathbb{R}_+^{SLJ}$ and $K \in [0, 1]^{SJ}$ is expected delivery rates on assets. In general, one must also enforce an exogenous finite bound on the sale of assets to guarantee the existence of equilibria for the default penalties model. However, under assumption A6 (in the Chapter 2), the existence of equilibrium is guaranteed even without such a bound. Also, we assume that the $b_{sl} = 1$ for all $s \in S$ and $l \in L$.

Let $q \in \mathbb{R}_+^J$ denote the prices of assets in period zero and $\theta^h = (\theta_1^h, \ldots, \theta_J^h) \in$

$\mathbb{R}^J_+$ denote the number of units of each of the J assets bought by agent h, and $\varphi^h = (\varphi^h_1, \ldots, \varphi^h_J) \in \mathbb{R}^J_+$ the short-positions in the assets.

Collateral model

The economy with collateral, E_{GEIC}, is characterized by the agents' utility functions $u = (u^h)_{h\in\mathcal{H}}$, the agents' endowment process $e = (e^h)_{h\in\mathcal{H}}$ and the collateral requirements $(C_j)_{j\in J}$ as in the Chapter 2, but now we can be consider one or two durable goods. In this case, in state s, an asset j pays $\min\left\{p_{s1}, \sum_{l\in\mathcal{L}} p_{sl}C_{lj}\right\}$.

As in the Chapter 1 a GEIC equilibrium is defined as usual by agents' optimality and market clearing. The existence follows from [GZ].

Default penalties model

The economy with default penalties, E_{GEI_λ}, is characterized by the agents' utility functions $u = (u^h)_{h\in\mathcal{H}}$, the agents' endowment process $e = (e^h)_{h\in\mathcal{H}}$ and default penalties $\lambda = (\lambda^h_{sj})_{h\in\mathcal{H},s\in\mathcal{S},j\in\mathcal{J}}$.

In the GEI$_\lambda$, λ^h_{sj} is the utility loss of the agent h for defaulting an unit of the value of asset j in state s. The fraction of promise payments $K \in [0,1]^{SJ}$ is endogenous in the model. Agents are permitted to deliver whatever they want of their own promises, represented by $D^h \in \mathbb{R}^{SLJ}_+$, but they are penalized $\lambda^h_{sj}p_{ls}$ for every unit of good l they fail to deliver in state s from their engagement through asset j.

As explained in Chapter 1 the homogeneity of budgetary constraints allows us to impose a normalization condition on the prices, which we require to lie on the unit simplex for each $s \in \mathcal{S}^*$, $\sum_l p_{sl} = 1$. Given a normalized $p \in \mathbb{R}^{S^*L}_{++}$, $q \in \mathbb{R}^J_+$ and $K \in [0,1]^{SJ}$ the agent h can choose an allocation

$(x^h, \theta^h, \varphi^h, D^h)$, to maximize utility subject to the budget constraints.

$$
\begin{aligned}
&\max_{x^h \geq 0, \varphi^h \geq 0, D^h \geq} u^h(x) - \sum_{j \in \mathcal{J}} \sum_{s \in \mathcal{S}} \lambda^h_{sj} [\varphi^h_j p_{s1} - \sum_{l \in \mathcal{L}} p_{sl} D^h_{slj}] \\
\text{s.t.} \quad & \text{there exists } \theta^h \in \mathbb{R}^J_+ \text{ with} \\
& \sum_{l \in \mathcal{L}} p_{0l}(x^h_{0l} - e^h_{0l}) + \sum_{j \in \mathcal{J}} q_j (\theta^h_j - \varphi^h_j) \leq 0; \\
& \sum_{l \in \mathcal{L}} p_{sl}(x^h_{sl} - e^h_{sl} - x^h_{0l}) \\
& + \sum_{l \in \mathcal{L}} \sum_{j \in \mathcal{J}} p_{sl} D^h_{slj} - \sum_{j \in \mathcal{J}} \theta^h_j K_{sj} p_{s1} \leq 0; \quad \forall s \in S; \\
& \sum_{l \in \mathcal{L}} p_{sl} D^h_{slj} - \varphi^h_j p_{s1} \leq 0; \quad \forall s \in S \quad \forall j \in J.
\end{aligned}
\tag{3.1}
$$

As in the Chapter 1 a GEI_λ equilibrium is defined by agents' optimality, market clearing and rate of delivery.

The economy satisfying standard assumptions, default penalties equilibria exist for any positive default penalties, under the assumption of an exogenously enforced finite bound on the sale of assets, a result that follows from [DGS2]. In our restricted setting, however, equilibria exist irrespective of such a bound on sales, a result due to [GP].

Theorem 4. *For an economy E_{GEI_λ}, under assumptions A1-A4 and A6 there exists an equilibrium.*

As we are interested in solving the Pareto frontier and the impact of collateral requirements and default penalties, it will be useful to used the definition of Arrow-Debreu equilibrium or general equilibrium (GE) in the present durable goods, as shown in the Chapter 1.

3.2.1 Pareto frontier

Given a set of alternative allocations of goods for a set of individuals, we say that an allocation is in the Pareto frontier or is Pareto optimal if there is no other allocation where at least one individual is better off while no individual is worse off.

Two classical theorems give the correspondence between competitive equilibrium and efficient allocation by assuming

A1-A4:

- The first theorem of welfare states that any competitive equilibrium or Arrow-Debreu equilibrium leads to an efficient allocation of resources.
- The second theorem of welfare states the converse, that any efficient allocation can be sustainable by a competitive equilibrium.

The first-order conditions that characterize Pareto efficient allocation are the same that characterize the market equilibrium. In this sense, the Pareto frontier can be explored by considering the Arrow-Debreu equilibria corresponding to alternative economies, with the same aggregate endowments but differing through the individual allocations (see [V], p. 331).

Even if agents are heterogeneous, but the allocations are Pareto optimal the marginal rate of substitution between each pair of goods is the same for every agent when the allocations belong to the interior of the domain. If two agents had different marginal rates of substitution between some pair of goods, they could arrange a small trade that would make them both better off, contradicting the assumption of Pareto efficiency ([V], p. 331). We refer to this feature of the complete contingent claims equilibrium as complete risk-sharing. When markets are incomplete, there do not exist claims that pay off for each possible state of the world. Generically as a result, marginal rates of substitution in consumption across different states are not equated across consumers and full insurance does not occur. This yields another way of verifying that the equilibrium is in the Pareto frontier.

3.3 Approaching the Pareto frontier by the use of default mechanism

In this section, we describe three numerical examples. The first example considers an economy with heterogeneity among states of nature in the second period. In this case, collateral is not enough for complete risk-sharing and default penalties equilibria can be Pareto superior. We also investigate conditions under which collateral equilibria approach or coincide with Arrow-Debreu (complete markets) equilibria.

The second example illustrates when collateral equilibrium is Pareto optimal in a situation where there is heterogeneity in the endowments between periods.

The third example considers an economy with heterogeneity in the endowments between states and periods. In this example, the default penalties equilibrium is Pareto superior to the collateral equilibria.

We use the algorithm described in the Chapter 1 to approximate GEIC, GEI$_\lambda$ and GE equilibria numerically.

3.3.1 Example 1: heterogeneity among states

We first consider the simplest two period model with two states in period 1 $\mathcal{S}^* = \{0, 1, 2\}$ and two agents, $\mathcal{H} = \{1, 2\}$.

Each individual $h = 1, 2$ has a utility function of the form:

$$u^h = \log(x_{01}^h) + \log(x_{02}^h) + \frac{1}{2}\sum_{s=1}^{2}(\log(x_{s1}^h) + \log(x_{s2}^h))$$

We choose one example with heterogeneous endowments among states of nature in the second period.

$$\begin{array}{l} e_{0l}^1 = (3.15, 2.04), e_{1l}^1 = (5.07, 0), e_{2l}^1 = (2.71, 0); \\ e_{0l}^2 = (3.15, 2.04), e_{1l}^2 = (2.15, 0), e_{2l}^2 = (5.92, 0). \end{array} \tag{3.2}$$

The agents have the same endowments in the first period, but are different between states of nature. Agent 1 is richer than agent 2 in state 1, and agent 2 is richer than agent 1 in state 2.

We created 9216 samples for a variety of random choice of collateral C_j between 0 and 3 and 7247 samples for a variety of random choice of penalties λ_{sj}^h between 0 and 3. In Figure 3.1, we show these different utilities of the agents.

The Figure 3.1 clearly shows that the GEI$_\lambda$ equilibria are Pareto superior with respect to GEIC for this economy, because agents can improve the risk-sharing between states in the second period.[1]

In achieving this, it seems important that penalties be devised in such a way to penalize more harshly default in states of nature where the agent is better-off. We can identify one GEI$_\lambda$ equilibriumwhich is Pareto superior to all others, yielding utilities $u^1 = 3.9593$ and $u^2 = 3.9250$. In this equilibrium

[1]Of course, bad choices of the penalty parameter are not going to be Pareto superior (e.g., with zero penalty, no assets are traded in the GEI$_\lambda$ equilibrium), but it turns out that for most reasonable choices (7201 of the 7247 samples) the GEI$_\lambda$ equilibrium is Pareto superior to all allocations arising as GEIC equilibiria.

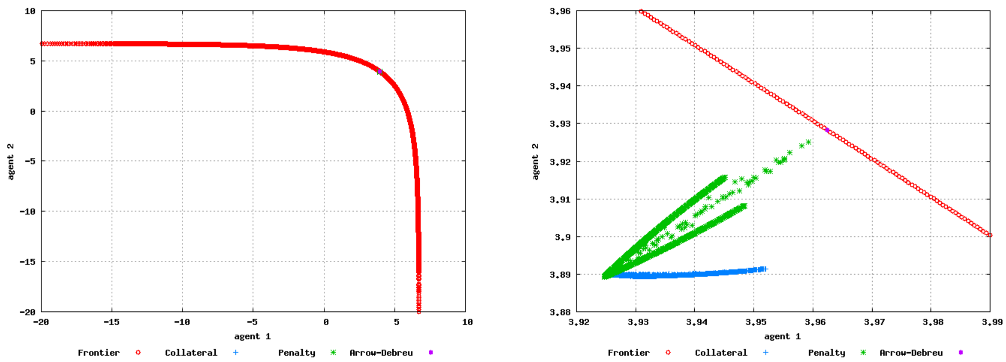

Figure 3.1: Pareto frontier (left) and Zoom Pareto frontier and Arrow-Debreu, Collateral and Penalty equilibria (right)

the first agent's portfolio $z_j^1 = \theta_j^1 - \phi_j^1$ is given by $z_1^1 = -1.30$ and $z_2^1 = 1.65$, i.e., the agent 1 sells the asset 1 and purchases the asset 2, because the penalties are large in the asset 2 in both states. The price of assets are: $q_1 = 0.1085$ and $q_2 = 0.0892$. The agents' penalties are given by $\lambda_{s1}^1 = [1.54, 0.03]$ and $\lambda_{s2}^2 = [0.01, 2.47]$ for $s = 1, 2$. The agent 1 pays in full in the first state and defaults in the second state (where his is poor). The agent 2 defaults in the state 1 (where his is poor) and pays in full in the state 2. We again identify one GEIC equilibriumas Pareto superior to all other GEIC equilibria, yielding utilities $u^1 = 3.9521$ and $u^2 = 3.8914$. As in the GEI$_\lambda$ equilibrium, agent 1 sells the asset 1 $z_1^1 = -3.67$ and purchases asset 2 $z_2^1 = 3.92$. The price of assets are: $q_1 = 0.2006$ and $q_2 = 0.1802$. The collateral requirements for the two assets traded are $C_1 = 0.57$ and $C_2 = 0.47$. In this case, asset 1 pays back in full in all states, and asset 2 defaults only in state 1. Agent 1 is benefited by the higher interest rate in the asset 2.

As in the Chapter 2 we compute the welfare rate $WR^h = \exp(\frac{u^{hGEI} - u^{hAD}}{2})$, where u^{hGEI} denotes an agent's utility in the GEI (collateral or default penalties) equilibrium and u^{hAD} denotes his utility in the Arrow-Debreu equilibrium. The maximal welfare rate in this example are attained, in each of GEI$_\lambda$ and GEIC, by the two equilibria highlighted above: for default penalties one has $(WR^1, WR^2) = (0.9992, 0.9992)$ and for collateral one has $(WR^1, WR^2) = (0.9974, 0.9908)$. The collateral requirement hurts both agents more than default penalties, especially for agent 2.

As explained above, an increase in available collateral, either through the increase of commodities that can used as collateral, or through the increase of the number of durable goods, will be welfare improving.

To illustrate these points, we examine two cases for GEIC equilibrium.

First we assume two durable goods (i.e. all goods are durable in this the economy), with the same endowments in the Equation 3.2. We created 12439 samples for a variety of random choice of collateral C_{lj} between 0 and 1. In this case, both agents are benefited, but unfortunately in distinct equilibria and the GEIC equilibria is not in the Pareto frontier, as Figure 3.2 shows.

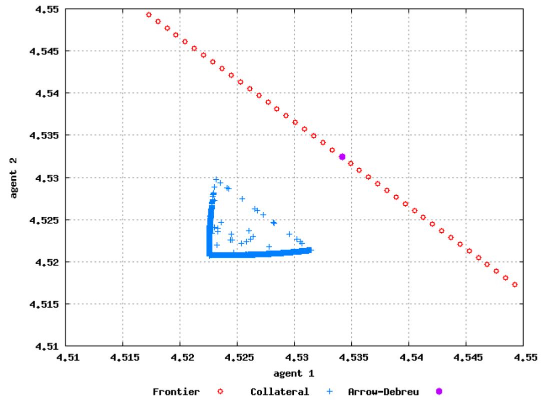

Figure 3.2: Pareto Frontier, and Arrow-Debreu and Collateral equilibria

The welfare rate for both agents when agent 1 has higher utility are $(WR^1, WR^2) = (0.9993, 0.9972)$ and when agent 2 has higher utility are $(WR^1, WR^2) = (0.9972, 0.9993)$

In the second case we examine how the increasing of commodities that can used as collateral allows one to reach the Arrow-Debreu equilibrium, as explained in the Chapter 2. We consider $L = 2$ with only the second good durable. We assume that each individual has a utility function of the form:

$$u^h = 0.1\log(x^h_{01}) + 0.9\log(x^h_{02}) + \frac{1}{2}\sum_{s=1}^{2}(0.1\log(x^h_{s1}) + 0.9\log(x^h_{s2}))$$

and we increased the durable good in the first period for each agent:

$$e^1_{02} = e^2_{02} = 10.$$

In this case for some GEIC equilibrium allocation coincides with Arrow-Debreu equilibrium allocationAs commented in the Chapter 2 the fraction of consumption expenditure on collateralizable durable goods has to be unrealistically large in order to allow agents to achieve the complete markets equilibrium.

We also, examine a variety of profiles of endowments, differing by the distribution of the perishable good in the second period. We assume the same sum of the endowments' agents in the Equation 3.2:

$$e^1_{0l} = (3.15, 2.04), e^1_{1l} = (\gamma 7.22, 0), e^1_{2l} = (\sigma 8.63, 0);$$
$$e^2_{0l} = (3.15, 2.04), e^2_{1l} = ((1-\gamma)7.22, 0), e^2_{2l} = ((1-\sigma)8.63, 0).$$

where γ and σ are values between 0 and 1.

We created 250 samples a variety of random choice of γ and σ and solved for collateral (set of asset $\mathcal{J}^{CC}$ defined in the Chapter 2) and Arrow-Debreu equilibrium. The Figure 3.3 shows the agents' utilities.

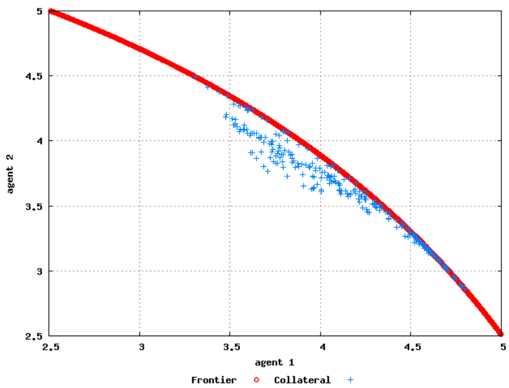

Figure 3.3: Pareto Frontier and Collateral equilibria

The Figure 3.3 shows that some collateral equilibria can lead Pareto frontier. The following is an example that the distribution of endowments leads to Pareto optimal with collateral equilibrium:

$$e^1_{0l} = (3.15, 2.04), e^1_{1l} = (6.93, 0), e^1_{2l} = (7.51, 0);$$
$$e^2_{0l} = (3.15, 2.04), e^2_{1l} = (0.29, 0), e^2_{2l} = (1.12, 0).$$

In this example, both agents are poor in the first state and rich in the second state, but agent 1 is richer than agent 2 through the entire second period. It makes sense for the agents to trade assets, so that agent 1 (respectively, agent 2) can consume more in the first period (respectively, second period). Still, in equilibrium, no agent is a "pure" seller or buyer: agent 1 sells asset 1 and agent 2 asset 2. In this equilibrium agent 1's portfolio is given by $z^1_1 = -3.97$ and $z^1_2 = 1.68$ and consumptions are:

$$x^1_{0l} = (3.80, 2.46), x^1_{1l} = (4.36, 2.46), x^1_{2l} = (5.21, 2.46);$$
$$x^2_{0l} = (2.49, 1.62), x^2_{1l} = (2.86, 1.62), x^2_{2l} = (3.42, 1.62).$$

It is interesting to investigate whether the characteristics behavior observed in this case can be also present in a situation where the endowments of the agents are different in the first period. The next example illustrates this.

3.3.2 Example 2: heterogeneity among periods

We now consider an example with three states in the second period $\mathcal{S}^* = \{0, 1, 2, 3\}$ and two agents, $\mathcal{H} = \{1, 2\}$, each with identical utility,

$$u^h = \log(x_{01}^h) + \log(x_{02}^h) + \frac{1}{3}\sum_{s=1}^{3}(\log(x_{s1}^h) + \log(x_{s2}^h))$$

We represent the economy with the following endowments:

$$e_{0l}^1 = (3, 2), e_{1l}^1 = (7, 0), e_{2l}^1 = (5, 0), e_{3l}^1 = (3, 0);$$
$$e_{0l}^2 = (6, 4), e_{1l}^2 = (5, 0), e_{2l}^2 = (3, 0), e_{3l}^2 = (1, 0).$$

In the first period, agent 2 is richer than agent 1 (natural borrower).

We created 5417 samples for a variety of random choices of collateral C_j between 0 and 2 and 7114 samples for a variety of random choice of penalties λ_{sj}^h between 0 and 2. In Figure 3.4, we show these different utilities of the agents.

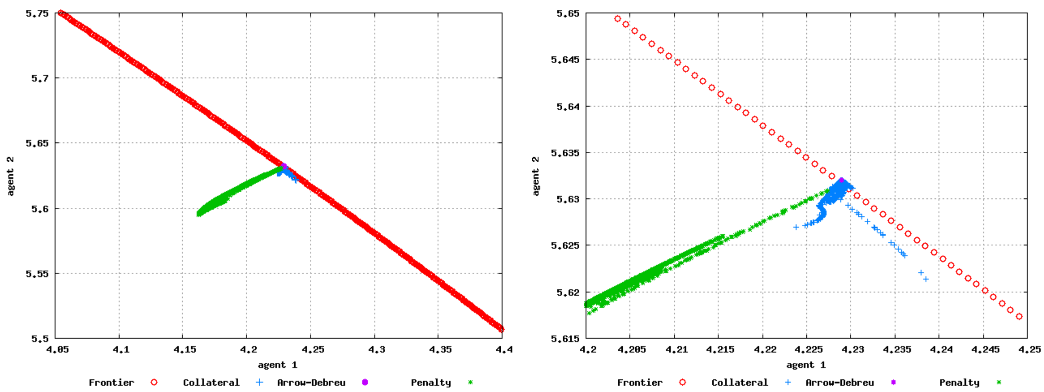

Figure 3.4: Pareto frontier, and Arrow-Debreu, Collateral and Penalty equilibria (left) and Zoom (right)

In this example it is clear that the for some collateral allocations are Pareto optimal. The collateral equilibrium allocation (set of asset $\mathcal{J}^{CC}$ defined in the Chapter 2) is identical to the Arrow-Debreu allocationFor this example, two assets (from the three available) are sufficient to reach Pareto frontier. The collateral requirements for the two assets traded are $C_1 = 0.5$ with no default in the state 1 and default in states 2 and 3, and $C_3 = 1.5$ with no default in all states, in equilibrium. The agent's portfolio is given by $z_1^1 = -1.04$, $z_2^1 = 0$ and $z_3^1 = -1$ and consumptions are:

$$x_{0l}^1 = (3.72, 2.48), x_{1l}^1 = (4.96, 2.48), x_{2l}^1 = (3.31, 2.48), x_{3l}^1 = (1.65, 2.48);$$
$$x_{0l}^2 = (5.28, 3.52), x_{1l}^2 = (7.04, 3.52), x_{2l}^2 = (4.69, 3.52), x_{3l}^2 = (2.35, 3.52).$$

We identify one GEI$_\lambda$ equilibriumas Pareto superior to all other GEI$_\lambda$ equilibria, yielding utilities $u^1 = 4.2274$ and $u^2 = 5.6309$. This equilibrium is not in the Pareto frontier because the agents have enough income to pay their debts in all states of nature. In equilibrium, the large penalties ensure no default in all states. This equilibrium is equivalent to the equilibrium with incomplete markets, with only one asset being traded (other assets are redundant). In fact, only one asset is traded for the default penalties equilibrium (asset 3), and the agent's portfolio is given by $z_3^1 = -1.44$.

3.3.3 Example 3: heterogeneity among states and periods

Now we consider an example with three agents $H = 3$ and we assume that there are $S = 3$ states in the second period. The utility function for each agent is the same of the Example 2.

The three agents' endowments are given by

$$e_{0l}^1 = (4, 4), e_{1l}^1 = (1, 0), e_{2l}^1 = (2, 0), e_{3l}^1 = (4, 0);$$
$$e_{0l}^2 = (1, 2), e_{1l}^2 = (1, 0), e_{2l}^2 = (5, 0), e_{3l}^2 = (3, 0);$$
$$e_{0l}^3 = (2, 3), e_{1l}^3 = (3, 0), e_{2l}^3 = (2, 0), e_{3l}^3 = (1, 0).$$

In the first period agent 1 is the richest (natural lender), agent 2 is the poorest and agent 3 is in between. In the second period for the perishable good the agents are the richest in different states: agent 1 in state 3, agent 2 in state 2 and agent 3 in state 1. In this way, we have heterogeneity between states and periods.

We created 5522 samples for a variety of random choice of collateral C_j between 0 and 3 and 9403 samples for a variety of random choice of penalties λ_{sj}^h between 0 and 3. As in Examples 1 and 2, we want to examine when the

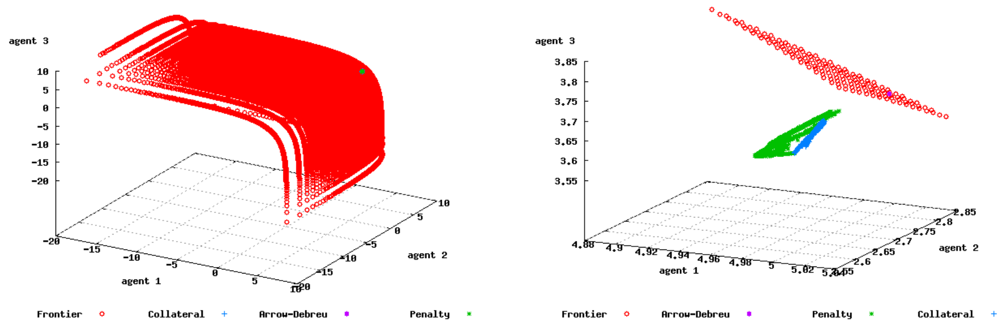

Figure 3.5: Pareto frontier (left) and Zoom Pareto frontier and Arrow-Debreu, Collateral and Penalty equilibria (right)

equilibria allocations (for collateral and default penalties) can approach the Pareto frontier.

Figure 3.5 shows a three dimensional scatter plot of different utility levels corresponding to different collateral and default penalties equilibria.

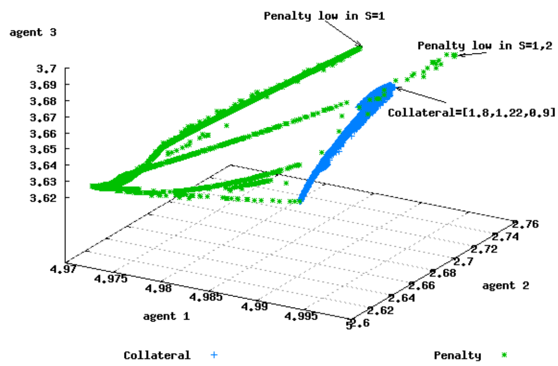

Figure 3.6: Collateral and Penalty equilibria

The Figure 3.5 shows that the equilibria with collateral and penalties are not in the Pareto frontier. However, the best GEI_λ equilibria are Pareto superior to all GEIC equilibria. As in the Example 1, borrowers can be benefited with low penalties in the states that they are poor. Figure 3.6

illustrates this.

Figure 3.6 shows that agent 2 is better off in a GEI$_\lambda$ when the penalty is low in the state 1 (poor state)The point denoted by Penalty low in S=1,2 in the Figure 3.6 corresponds to equilibrium where agents 1 and 3 are better off in a GEI$_\lambda$. In this equilibriumthe low penalty in the state 2 (poor state) benefits the agent 3. He holds a long-position in asset 1 and borrows money in asset 3. Agent 1 has relatively large long-positions in assets 1 and 3.

The following Table 3.1 shows the welfare rate for the three extremal equilibria highlighted in Figure 3.6.

Table 3.1: Welfare rate for extremal equilibria: agents 1, 2 and 3

Equilibrium	Agent 1	Agent 2	Agent 3
Low penalty in S=1,2	0.9956	0.9845	0.9950
Low penalty in S=1	0.9920	0.9950	0.9879
Collateral=[1.8,1.22,0.9]	0.9942	0.9829	0.9891

As explained above, GEI$_\lambda$ can be Pareto superior to GEIC equilibria. As one would expect, the collateral requirements of GEIC hurt agent 2 (he has little collateralizable goods) more than agents 1 and 3. Agent 2 trades in assets 2 and 3, with lower collateral requirements $C_2 = 1.22$ and $C_3 = 0.9$ and higher prices, which benefit the agent 1 (natural lender). The agent 3 still trades in asset 1 which gives him a relatively better price although it requires him to hold more collateral.

3.4 Conclusion

In this Chapter we consider a model with default penalties and collateral and examine when the equilibria allocations can be approach the Pareto frontier by the use of a default mechanism.

In our examples, collateral equilibria can be in the Pareto frontier if the endowment distribution displays only heterogeneity between periods. If the heterogeneity is also manifest between states of nature, default penalties equilibria are often Pareto superior with respect to collateral.

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