ON DEFORMATIONS OF HOLOMORPHIC FOLIATIONS AND COMPLEX HYPERSURFACES

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Contents

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Part 1. Density and topological rigidity for holomorphic foliations

1. TOPOLOGICAL RIGIDITY

Let $\mathrm{Fol}(M)$ denote the set of holomorphic foliations on a complex manifold M. An *analytic deformation* of $\mathcal{F} \in \mathrm{Fol}(M)$ is an analytic family $\{\mathcal{F}_t\}_{t\in Y}$ of foliations on M, with parameters on an analytic space Y, such that there exists a point "0" $\in Y$ with $\mathcal{F}_0 = \mathcal{F}$. Here we will only consider deformations where $Y = \mathbb{D} \subset \mathbb{C}$ is a unitary disk. A *topological equivalence* (resp. *analytical equivalence*) between two foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ is a homeomorphism (resp. biholomorphism) $\phi : M \to M$, which takes leaves of $\mathcal{F}_1$ onto leaves of $\mathcal{F}_2$, and such that $\phi(\mathrm{Sing}(\mathcal{F}_1)) = \mathrm{Sing}(\mathcal{F}_2)$. The deformation $\{\mathcal{F}_t\}_{t\in\mathbb{D}}$ is *topologically trivial* (resp. *analytically trivial*) if there exists a continuous map (resp. holomorphic map) $\phi : M \times \mathbb{D} \to M$, such that each map $\phi_t = \phi(.,t) : M \to M$ is a topological equivalence (resp. analytical equivalence) between $\mathcal{F}_t$ and $\mathcal{F}_0$.

Let $\mathcal{C} \subset Fol(M)$ be a class of foliations. A foliation $\mathcal{F}_0 \in \mathcal{C}$ is topologically rigid in the class if any topologically trivial deformation $\{\mathcal{F}_t\}_{t\in\mathbb{D}}$ of $\mathcal{F}_0$ with $\mathcal{F}_t \in \mathcal{C}$ is analytically trivial.

We also say that $\mathcal{F}_0 \in \mathcal{C}$ is $\mathcal{U}$-topological rigid in the class $\mathcal{C}$, where $\mathcal{U} \subset M$ is an open subset, if any analytic deformation $\{\mathcal{F}_t\}_{t\in\mathbb{D}}$ of $\mathcal{F}_0$ with $\mathcal{F}_t \in \mathcal{C}$, $\forall t$; which is topologically trivial in $\mathcal{U}$, is in fact analytically trivial in M.

In this part we will be concerned with holomorphic foliations in $\mathbb{C}P(2)$. These foliations are motivated by Hilbert's sixteen problem on the number and location of limit cycles of polynomial differential equations

$$(\star)\begin{cases} \frac{dx}{dt} = P(x,y) \\ \frac{dy}{dt} = Q(x,y) \end{cases}$$

in the real plane $(x,y) \in \mathbb{R}^2$ where P and Q are relatively prime polynomials. A major attempt in this line was started in 1956 by a seminal work of I. Petrovski and E. Landis [18]. They consider $(\star)$ as a differential equations in the complex plane $(x,y) \in \mathbb{C}^2$, with t now being a complex time parameter. The integral curves of the vector field are now either singular points which correspond to the common zeros of P and Q, or complex curves tangent to the vector field which are holomorphically immersed in $\mathbb{C}^2$. This gives rise to a holomorphic foliation by complex curves with a finite number of singular points. One can easily see that this foliation extends to the complex projective plane $\mathbb{C}P(2)$, which is obtained by adding a line at infinity to the plane $\mathbb{C}^2$. Conversely any holomorphic foliation by curves on $\mathbb{C}P(2)$ is given in an affine space $\mathbb{C}^2 \hookrightarrow \mathbb{C}P(2)$ by a polynomial vector field $X = (P,Q) \in \mathfrak{X}(\mathbb{C}^2)$ with $\gcd(P,Q) = 1$.

Although they didn't solve this problem, however they introduced a truly novel method in geometric theory of ordinary differential equations. In 1978, Il'yashenko made a fundamental contribution to the problem. Following the general idea of Petrovski and Landis, he studied equations $(\star)$ with complex polynomials P and Q from a topological standpoint without particular attention to Hilbert's question.

We fix the line at infinity $L_\infty = \mathbb{C}P(2) \setminus \mathbb{C}^2$ and denote by $\mathcal{X}(n)$ the space of foliations of degree $n \in \mathbb{N}$ which leave invariant L_∞. Let us denote by $\mathcal{F}(n)$ the space of degree n foliations on $\mathbb{C}P(2)$ as introduced in [11]. We are interested in the two following questions:

(1) Under which conditions topologically trivial deformations of a foliation $\mathcal{F} \in \mathcal{F}(n)$ are analytically trivial?
(2) When does a foliation $\mathcal{F} \in \mathcal{F}(n)$ has dense leaves?

A remarkable result of Y. Ilyashenko states topological rigidity for a residual set of foliations on $\mathcal{X}(n)$ if $n \geq 2$.

More precisely we have:

Theorem 1.1. [9] *For any $n \geq 2$ there exists a residual subset $\mathcal{I}(n) \subset \mathcal{X}(n)$ whose foliations are topologically rigid in the class $\mathcal{X}(n)$.*

This result has been later improved by A. Lins Neto, P. Sad and B. Scardua as follows:

Theorem 1.2. [12] *For each $n \geq 2$, $\mathcal{X}(n)$ contains an open dense subset $\mathcal{R} \subset \mathcal{X}(n)$ whose foliations are topologically rigid in the class $\mathcal{X}(n)$*

We stress the fact that in both theorems above we consider deformations $\{\mathcal{F}_t\}_{t\in\mathbb{D}}$ in the class $\mathcal{X}(n)$, that is, $\mathcal{F}_t$ leaves invariant $L_\infty, \forall t \in \mathbb{D}$; and we assume topological triviality in $\mathbb{C}P(2)$. We relax slightly this last hypothesis by requiring topological triviality for the set of separatrices through the singularities at L_∞:

Theorem 1.3. [12] *For any $n \geq 2$, $\mathcal{X}(n)$ contains an open dense subset $\mathcal{SR}\mathrm{ig}(n)$ whose foliations are s-rigid in the class $\mathcal{X}(n)$.*

According to [12] a foliation $\mathcal{F}_0 \in \mathcal{X}(n)$ is s-rigid if for any deformation $\{\mathcal{F}_t\}_{t\in\mathbb{D}} \subset \mathcal{X}(n)$ of $\mathcal{F}_0$ with the s-triviality property that is: If $S_t \subset \mathbb{C}^2$ denotes the set of separatrices of $\mathcal{F}_t$ which are transverse to L_∞ then there exists a continuous family of maps $\phi_t : S_0 \to \mathbb{C}^2$ such that ϕ_0 is the inclusion map and ϕ_t is a continuous injection map from S_0 to $\mathbb{C}^2$ with $\phi_t(S_0) = S_t$; then $\{\mathcal{F}_t\}$ is analytically trivial.

Remark 1.4. Topological triviality in $\mathbb{C}^2$ implies s-triviality.

Let us change now our point of view.

A deformation $\{\mathcal{F}_t\}_{t\in\mathbb{D}}$ of a foliation $\mathcal{F}_0$ on a manifold M, is an *unfolding* if there exists an analytic foliation $\tilde{\mathcal{F}}$ on $M\times\mathbb{D}$ with the property that: $\tilde{\mathcal{F}}|_{M\times\{t\}} \equiv \mathcal{F}_t, \forall t \in \mathbb{D}$. In other words, *an unfolding is a deformation which embeds into an analytic foliation.* The trivial unfolding of $\mathcal{F}$ is given by the $\mathcal{F}_t := \mathcal{F}, \forall t \in \mathbb{D}$ and $\tilde{\mathcal{F}}$ is the product foliation $\mathcal{F} \times \mathbb{D}$ in $M \times \mathbb{D}$.

Two unfoldings $\{\mathcal{F}_t\}_{t\in\mathbb{D}}$ and $\{\mathcal{F}_t^1\}_{t\in\mathbb{D}}$ of $\mathcal{F}$ are topologically equivalent respectively analytically equivalent if there exists a continuous respectively analytic map $\phi : M\times\mathbb{D} \to M$ such that each map $\phi_t : M \to M$, $\phi_t(p) = \phi(p,t)$, is a topological respectively analytical equivalence between $\mathcal{F}_t$ and $\mathcal{F}_t^1$.

An unfolding $\{\mathcal{F}_t\}_{t\in\mathbb{D}}$ of a foliation $\mathcal{F}_0$ on M is said to be topologically rigid in the class $\mathcal{C} \subset \mathcal{F}(n)$ if any analytic unfolding $\{\mathcal{F}_t^1\}_{t\in\mathbb{D}}$ of $\mathcal{F}$ ($\mathcal{F}_t^1 \in \mathcal{C}$, $\forall t$), which is topologically equivalent to $\{\mathcal{F}_{\rm t}\}_{t\in\mathbb{D}}$, is necessarily analytically equivalent.

These notions rewrite theorems (1.1) and (1.2) as follows:

Theorem 1.5. [9] *For any $n \geq 2$ there exists a residual subset $\mathcal{I}(n) \subset \mathcal{X}(n)$ whose foliations are topologically rigid trivial unfolding in the class $\mathcal{X}(n)$.*

Theorem 1.6. [12] *For each $n \geq 2$, $\mathcal{X}(n)$ contains an open dense subset $\mathcal{R}ig(n) \subset \mathcal{X}(n)$ whose foliations are topologically rigid trivial unfolding in the class $\mathcal{X}(n)$.*

We are now in conditions of stating our main results concerning topological rigidity. We stress the fact that a priori, our deformations are allowed to move the line L_∞ (Theorems A, B and C) which is $\mathcal{F}_0$-invariant by hypothesis. We also state results for deformations which are topological trivial in $\mathbb{C}^2$ not necessarily in $\mathbb{C}P(2)$. Finally, we may relax the hypothesis of hyperbolicity for $\mathrm{Sing}(\mathcal{F}_0) \cap L_\infty$ by allowing quasi-hyperbolic singularities (i.e the exceptional divisor is invariant and all of singularities are of saddle-type) and obtain in this way a link between the analytical classification of the unfolding and the one of its germs at the singularities $p \in \mathrm{Sing}(\mathcal{F}_0) \cap L_\infty$.

Our main results are the following:

Theorem A. *Given $n \geq 2$ there exists an open dense subset $\mathcal{R}ig(n) \subset \mathcal{X}(n)$ such that any foliation in $\mathcal{R}\mathrm{ig}(n)$ is $\mathbb{C}^2$-topological rigid: any deformation $\{\mathcal{F}_t\}_{t\in\mathbb{D}}$ of $\mathcal{F} = \mathcal{F}_0$ which is topologically trivial in $\mathbb{C}^2$ must be analytically trivial in $\mathbb{C}P(2)$ for $t \approx 0$.*

Theorem B. *Let $\{\mathcal{F}_t\}_{t\in\mathbb{D}}$ be topological trivial (in $\mathbb{C}^2$) analytic deformation of a foliation $\mathcal{F}_0$ on $\mathbb{C}^2$ such that:*

1. *$\mathcal{F}_0$ leaves L_∞ invariant,*
2. *$\forall p \in \mathrm{Sing}(\mathcal{F}_0) \cap L_\infty$, p is a quasi-hyperbolic singularity,*
3. *$\mathcal{F}_0$ has degree $n \geq 2$ and exhibits at least two reduced singularities in L_∞.*

Then we have two possibilities:

- *$\mathcal{F}$ is a Darboux (logarithmic) foliation, or*
- *$\{\mathcal{F}_t\}_{t\in\mathbb{D}_\varepsilon}$ is an unfolding.*

In this last case the unfolding is analytically trivial if and only if given a singularity $p \in \mathrm{Sing}(\mathcal{F}_0) \cap L_\infty$ the germ of the unfolding $\{\mathcal{F}_t\}_{t\in\mathbb{D}}$ at p is *analytically trivial for* $t \approx 0$.

Theorem C. *Let* $\mathcal{F}_0$ *be a foliation on* $\mathbb{C}P(2)$ *with the following properties:*

(1) $\mathcal{F}_0$ *leaves* L_∞ *invariant,*
(2) $\forall p \in \mathrm{Sing}(\mathcal{F}_0) \cap L_\infty$, *p is a quasi-hyperbolic singularity,*
(3) $\mathrm{Sing}(\mathcal{F}_0) \cap L_\infty$ *has at least two reduced singularities.*

Given two topologically equivalent unfoldings $\{\mathcal{F}_t\}_{t\in\mathbb{D}}$ *and* $\{\mathcal{F}^1_t\}_{t\in\mathbb{D}}$ *of* $\mathcal{F}_0$ *we have that they are analytically equivalent if and only if the germs of the unfoldings are analytically equivalent at the singular points* $p \in \mathrm{Sing}(\mathcal{F}_0) \cap L_\infty$.

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1.1. **Preliminaries.** Let $\mathcal{F}$ be a (singular) foliation on $\mathbb{C}P(2)$ and $L \subset \mathbb{C}P(2)$ be a projective line, which is not an algebraic solution of $\mathcal{F}$ ($L \setminus \mathrm{Sing}(\mathcal{F})$ is not a leaf of $\mathcal{F}$). We say that $p \in L$ is a tangency point of $\mathcal{F}$ with L, if either $p \in \mathrm{Sing}(\mathcal{F})$ or $p \notin \mathrm{Sing}(\mathcal{F})$ and the tangent spaces of L and of the leaf of $\mathcal{F}$ through p, at p, coincide. We say that L is invariant by $\mathcal{F}$ if $\forall p \in L\setminus\mathrm{Sing}(\mathcal{F})$, p is a tangency point of $\mathcal{F}$ with L. Denote by $T(\mathcal{F}, L)$, the set of tangency points of $\mathcal{F}$ with L. According to [11], if $\mathrm{Sing}(\mathcal{F})$ has codimension ≥ 2 or equivalently the singularities of $\mathcal{F}$ are finitely many points in $\mathbb{C}P(2)$, then there exists an open, dense and connected subset $NI(\mathcal{F})$ of the set of lines in $\mathbb{C}P(2)$, such that every $L \in NI(\mathcal{F})$ satisfies the following properties:

- L is not invariant by $\mathcal{F}$,
- $T(\mathcal{F}, L)$ is an algebraic subset of L defined by a polynomial of degree $k = k(\mathcal{F})$ in L and this number is independent of L.

The integer $k(\mathcal{F})$ is called *the degree of foliation* $\mathcal{F}$. According to [11], a foliation of degree n in $\mathbb{C}P(2)$ can be expressed in an affine coordinate system by a differential equation of the form

$$(P(x, y) + xg(x, y))dy - (Q(x, y) + yg(x, y))dx = 0,$$

where P, Q and g are polynomials such that:

(1) $P + xg$ and $Q + yg$ are relatively prime,
(2) g is homogeneous of degree n,
(3) $\max\{\deg(P),\ \deg(Q)\} \leq n$,
(4) $\max\{\deg(P),\ \deg(Q)\} = n$ if $g \equiv 0$.

Let B_{n+1} be space of polynomials of degree $\leq n + 1$ in two variables. Let $V \subset B_{n+1} \times B_{n+1}$ be the subspace of pairs of polynomials of the form $(p + xg, q + yg)$, where P, Q and g are as in (2) and (3) above. Clearly V is a vector subspace of $B_{n+1} \times B_{n+1}$. Let $\mathbb{P}(V)$ be the projective space of lines through $0 \in V$. Since the differential equations $(P+xg)dy-(Q+yg)dx=0$ and $\lambda(P+xg)dy-\lambda(Q+yg)dx=0$ define the same foliation in $\mathbb{C}^2$, we can identify the set of all foliations of degree n in $\mathbb{C}P(2)$ with a subset $\mathcal{F}(n) \subset \mathbb{P}(V)$. We consider $\mathcal{F}(n)$ with the topology induced by the topology of $\mathbb{P}(V)$. $\mathcal{F}(n)$ is called the *space of foliations of degree n in $\mathbb{C}P(2)$*. We consider the following subsets:

$\mathcal{S}(n) := \{\mathcal{F} \in \mathcal{F}(n) |$ the singularities of $\mathcal{F}$ are non-degenerated$\}$

$\mathcal{T}(n) := \{\mathcal{F} \in \mathcal{S}(n) |$ any characteristic number λ of $\mathcal{F}$ satisfies $\lambda \in \mathbb{C}\backslash\mathbb{Q}_+\} = \{\mathcal{F} \in \mathcal{S}(n) | \mathcal{F}$ has reduced singularities$\}$

$\mathcal{A}(n) := \mathcal{T}(n) \bigcap \mathcal{X}(n)$

$\mathcal{H}(n) := \{\mathcal{F} \in \mathcal{A}(n) |$all singularities of $\mathcal{F}$ in L_∞ are hyperbolic $\}$

Proposition 1.7. [11][12] *$\mathcal{X}(n)$ is an analytic subvariety of $\mathcal{F}(n)$ and also if $n \geq 2$ then:*

(1) *$\mathcal{T}(n)$ contains an open dense subset of $\mathcal{F}(n)$.*
(2) *$\mathcal{H}(n)$ contains an open dense subset $\mathcal{M}_1(n)$ such that if $\mathcal{F} \in \mathcal{M}_1(n)$, $n \geq 2$ then:*
   - *L_∞ is the only algebraic solution of $\mathcal{F}$*
   - *The holonomy group of the leaf $L_\infty\backslash\mathrm{Sing}(\mathcal{F})$ is nonsolvable.*
(3) *$\mathcal{T}(n) \subset \mathcal{H}(n) \subset \mathcal{X}(n)$ are open subsets.*

Lemma 1.8. *Let $\mathcal{F} \in \mathcal{M}_1(n)$, $n \geq 2$; then each leaf $F \neq L_\infty$ is dense in $\mathbb{C}P(2)$.*

Proof. First we notice that F must accumulate L_∞. Since F is a non-algebraic leaf it must accumulate at some regular point $p \in L_\infty\backslash\mathrm{Sing}(\mathcal{F})$. Choose a small transverse disk $\Sigma \pitchfork L_\infty$ with $\Sigma \subset V$, V is a flow-box neighborhood of p. We consider the holonomy group $\mathrm{Hol}(\mathcal{F}, L_\infty, \Sigma)$. Then F accumulates the origin $p \in \Sigma$ and since by [17] (see also theorem (1.15)) G has dense pseudo-orbits in a neighborhood the origin, it follows that F is dense in a neighborhood of p in Σ. Any other leaf L' of $\mathcal{F}$, $L' \neq L_\infty$ must have the same property. Using the continuous dependence of the solutions with respect to the initial conditions we may conclude that F accumulates any point $q \in F'$, $\forall F' \neq L_\infty$. Thus F is dense in $\mathbb{C}^2$ and since L_∞ is $\mathcal{F}$-invariant, F is dense in $\mathbb{C}P(2)$. $\square$

Proposition 1.9. *Let $\{\mathcal{F}_t\}_{t\in\mathbb{D}}$, $\mathcal{F}_0 = \mathcal{F} \in \mathcal{M}_1(n)$ is an unfolding then it is analytically equivalent to the trivial unfolding of $\mathcal{F}$ for $t \approx 0$.*

Proof. Denote by $\tilde{\mathcal{F}}$ the foliation on $\mathbb{C}P(2)\times\mathbb{D}$ such that $\forall t \in \mathbb{D}$, $\tilde{\mathcal{F}}|_{\mathbb{C}P(2)\times\{t\}} = \mathcal{F}_t$,

$\pi : \mathbb{C}^3\backslash\{0\} \to \mathbb{C}P(2)$ the canonical projection and

$\Pi : (\mathbb{C}^3\backslash\{0\}) \times \mathbb{D} \to \mathbb{C}P(2) \times \mathbb{D}$ the map

$\Pi(p,t) := (\pi(p), t)$.

Denote by $\mathcal{F}^* := \Pi^*(\tilde{\mathcal{F}})$, pull-back foliation on $(\mathbb{C}^3\backslash\{0\}) \times \mathbb{D}$. Then $\mathcal{F}^*$ extends to a foliation on $\mathbb{C}^3 \times \mathbb{D}$ by a Hartogs type argument.

Claim. We may choose an integrable holomorphic 1-form Ω which defines $\mathcal{F}^*$ on $\mathbb{C}^3 \times \mathbb{D}$ such that

$$\Omega \quad = \quad A(x,t)dt + \sum_{i=1}^{3} B_j(x,t)dx_j,$$

where B_j is a homogeneous polynomial of degree $n+1$ in x, A is a homogeneous polynomial of degree $n+2$ in x, $\sum_{i=1}^{3} x_j B_j(x,t) \equiv 0$ and $\Omega_t := \sum_{i=1}^{3} B_j(x,t)dx_j$ defines $\pi^*(\mathcal{F}_t)$ on $\mathbb{C}^3$.

Proof of the claim. First we remark that by triviality of Dolbeault and Cech cohomology groups of $\mathbb{C}^3 \times \mathbb{D}$, $\mathcal{F}^*$ is given by an integrable holomorphic 1-form, say, ω in $\mathbb{C}^3 \times \mathbb{D}$.

The restriction $\omega_t := \omega|_{\mathbb{C}^3 \times \{t\}}$ defines $\mathcal{F}_t^* := \pi^*(\mathcal{F}_t)$ in $\mathbb{C}^3$. Thus we may write $\omega = \alpha(x,t)dt + \sum_{k=1}^{3} \beta^k(x,t)dx_k = \alpha(x,t)dt + \omega_t(x)$

Since the radial vector field R is tangent to the leaves of $\mathcal{F}^*$ we have $\omega \circ R = 0$ so that $\omega_t \circ R = 0$, i.e. $\sum_{k=1}^{3} x_k \beta^k(x,t) = 0$. Now we use the Taylor expansion in the variable $x = (x_1, x_2, x_3)$ of ω around a point $(0,t)$ so that $\omega = \sum_{j=\nu}^{+\infty} \omega_j$ where $\omega_j(x,t) := \alpha_j(x,t)dt + \sum_{k=1}^{3} \beta_j^k(x,t)dx_k = \alpha_j(x,t)dt + \omega_j^t$ and α_j, β_j^k are holomorphic in (x,t), polynomial of degree j in x, $\omega_\nu \equiv 0$. Now the main argument is the following:

Lemma 1.10. *$\Omega = \alpha_{\nu+1}dt + \omega_\nu^t$ defines $\mathcal{F}^*$ in $\mathbb{C}^3 \times \mathbb{D}$.*

Proof. Indeed, $\omega \wedge d\omega = 0 \Rightarrow i_R(\omega \wedge d\omega) = i_R(\omega).d\omega - \omega \wedge i_R(d\omega) = 0$

$\omega \wedge i_R(d\omega) = 0$ (since $i_R(\omega) = 0$) $\Rightarrow i_R(d\omega) = f\omega$ for some holomorphic function f (Divisor lemma of Saito). Therefore the Lie derivative of ω with respect to R is

$$L_R(\omega) \quad = \quad i_R(d\omega) + d(i_R(\omega)) = f\omega. \tag{1}$$

On the other hand since $\omega = \sum_{j=\nu}^{+\infty} \omega_j = \sum_{j=\nu}^{+\infty} (\alpha_j(x,t)dt + \omega_j^t)$ we obtain

$$\begin{aligned} L_R(\omega) &= \sum_{j=\nu}^{+\infty} L_R(\alpha_j(x,t)dt + \omega_j^t) \\ &= \sum_{j=\nu}^{+\infty} \frac{d}{dz}[\alpha_j(e^z x,t)dt + \sum_{k=1}^{3} \beta_j^k(e^z x,t)e^z dx_k]|_{z=0} \\ &\quad \text{(The flow of } R \text{ is } R_z(x,t) = (e^z x,t)) \\ &= \sum_{j=\nu}^{+\infty} [j\alpha_j(x,t)dt + (j+1)\omega_j^t]. \end{aligned} \tag{2}$$

Now we write the Taylor expansion also for f in the variable x. $f(x,t) = \sum_{j=0}^{+\infty} f_j(x,t)$ where $f_j(x,t)$ is holomorphic in (x,t) homogeneous polynomial of degree j in x. We obtain from (1) and (2)

$$\begin{aligned}
\sum_{j=\nu}^{+\infty} j\alpha_j dt + (j+1)\omega_j^t &= (\sum_{k=0}^{+\infty} f_k)(\sum_{l=\nu}^{+\infty} \omega_l) \\
&= \sum_{j\geq\nu}(\sum_{l+k=j} f_k\omega_l) j\alpha_j dt + (j+1)\omega_j^t \\
&= \sum_{l+k=j} f_k\omega_l \\
&= \sum_{l+k=j} (f_k\alpha_l dt + f_k\omega_l^t) \quad l \geq \nu \quad \text{and} \quad \forall j \geq \nu
\end{aligned}$$

Then

$$j\alpha_j = \sum_{l+k=j} (f_k\alpha_l) \tag{3}$$

$$(j+1)\omega_j^t = \sum_{l+k=j} (f_k\omega_l^t) \quad \forall j \geq \nu \quad \text{and} \quad l \geq \nu \tag{4}$$

In particular (3) and (4) imply $f_0\alpha_\nu = \nu\alpha_\nu$ and $f_0\omega_\nu^t = (\nu+1)\omega_\nu^t$ then $f_0 = \nu+1, \alpha_\nu = 0$.

An induction argument shows that:

$j \geq \nu \Rightarrow (\alpha_{j+1}dt + \omega_j^t) \wedge \Omega = 0$, $(\Omega := \alpha_{\nu+1}dt + \omega_\nu^t)$

Finally since the degree of the foliation $\mathcal{F} = \mathcal{F}_0$ is n we have $\nu = n+1$. This proves the lemma (1.10). □

Lemma 1.11. *There exists a complete holomorphic vector field X on $\mathbb{C}^3 \times \mathbb{D}_\epsilon$, $\mathbb{D}_\epsilon \subset \mathbb{D}$ small subdisk, such that $X(x,t) = \frac{\partial}{\partial t} + \sum_{j=1}^{3} F_j(x,t)\frac{\partial}{\partial x_j}$, $\Omega \circ X = 0$ and $F_j(x,t)$ is linear on x.*

Proof. We may present $\Omega = A(x,t)dt + \sum_{j=1}^{3} B_j(x,t)dx_j = A(x,t)dt + \omega_t$ where $i_R(\omega_t) = 0$, B_j is a homogeneous polynomial of degree $n+1$ in x, A is a homogeneous polynomial of degree $n+2$ in x.

Claim. $\forall t \in \mathbb{D}_\epsilon$ ($\epsilon \geq 0$ small enough) we have $\mathrm{Sing}(\mathcal{F}_t) \subset \{A(.,t) = 0\}$.

Proof of the claim. Since $\Omega \wedge d\Omega = 0$ we have the coefficients of $dt \wedge dx_i \wedge dx_j$ equal to zero, that is:

$$A(\frac{\partial B_j}{\partial x_i} - \frac{\partial B_i}{\partial x_j}) + B_j\frac{\partial B_j}{\partial t} - B_i\frac{\partial B_j}{\partial t} + B_i\frac{\partial A}{\partial x_j} - B_j\frac{\partial A}{\partial x_i} = 0 \tag{5}$$

Now given $p_0 \in \mathrm{Sing}(\mathcal{F}_{t_0})$, ($t_0 \approx 0$, so that $\mathcal{F}_{t_0} \in \mathcal{M}_1(n)$) we have from (5) that $(B_i(p_0,t_0) = B_i(p_0,t_0) = 0) : A(p_0,t_0)(\frac{\partial B_j}{\partial x_i}(p_0,t_0) - \frac{\partial B_i}{\partial x_j}(p_0,t_0))$. Since $\mathcal{F}_{t_0} \in T(n)$ we have $\frac{\partial B_j}{\partial x_i}(p_0,t_0) \neq \frac{\partial B_i}{\partial x_j}(p_0,t_0)(i \neq j)$ and $A(p_0,t_0) = 0$.

Using now Noether's lemma for foliations we conclude that there exist $F_j(x,t)$ holomorphic in (x,t), homogeneous polynomial of degree $1 = (n+2)-(n+1)$ in x, such that $A(x,t) = \sum_{j=1}^{3} F_j(x,t)B_j(x,t)$. Now we define $X(x,t) := 1\frac{\partial}{\partial t} + \sum_{j=1}^{3} F_j(x,t)\frac{\partial}{\partial x_j}$ so that $\Omega \circ X = A - \sum_{j=1}^{3} F_j B_j = 0$.

In addition X is complete because each F_j is of degree one in x. The flow of X writes $X_z(x,t) = (\Psi_z(x,t), t+z)$. Clearly the $\Psi_z : \mathbb{C}^3\backslash\{0\} \longrightarrow \mathbb{C}^3\backslash\{0\}$ defines an analytic equivalence between $\mathcal{F}$ and $\mathcal{F}_z$. The proposition (1.9) is now proved. □

Another important remark is the following:

Proposition 1.12. *Let $\mathcal{F}$, $\mathcal{G}$ be foliations with hyperbolic singularities on $\mathbb{C}P(2)$. Assume that L_∞ is the only algebraic leaf of $\mathcal{F}$ and that $\mathcal{F}|_{\mathbb{C}^2}$ and $\mathcal{G}|_{\mathbb{C}^2}$ are topologically equivalent. Then L_∞ is also $\mathcal{G}$-invariant.*

Proof. Let $\phi : \mathbb{C}^2 \to \mathbb{C}^2$ be a topological equivalence between $\mathcal{F}$ and $\mathcal{G}$ in $\mathbb{C}^2$. We notice that given a singularity $p \in \mathrm{Sing}(\mathcal{F}) \cap L_\infty$, there exist local coordinates $(x,y) \in U$, $x(p) = y(p) = 0$, $L_\infty \cap U = \{y = 0\}$ such that $\mathcal{F}|_U : xdy - \lambda ydx = 0$, $\lambda \in \mathbb{C}\backslash\mathbb{R}$ and $U \cap \mathrm{Sing}(\mathcal{F}) = \{p\}$. Let $U^* = U\backslash(L_\infty \cap U)$, $V^* = \phi(U^*) \subset \mathbb{C}^2$, $\Gamma := (x = 0)$, $\Gamma^* := \Gamma \cap U^* = \Gamma\backslash\{p\}$.$\Gamma$ is the local separatrix of $\mathcal{F}$ at p, transverse to L_∞. We put $\Gamma_1^* = \phi(\Gamma^*) \subset V^*$. We remark that Γ_1^* is contained in a leaf of $\mathcal{G}$ and it is closed in V^*. On the other hand if we take any local leaf L of $\mathcal{F}|_{U^*}$, $L \neq \Gamma$; then by the hyperbolicity of $p \in \mathrm{Sing}(\mathcal{F})$ we have that L accumulates Γ. Thus the image $L_1 = \phi(L)$ is a leaf of $\mathcal{G}|_{V^*}$ that accumulates $\Gamma_1^* \neq L_1$.

Assume by contradiction that L_∞ is not $\mathcal{G}$-invariant. The curve $\Gamma_1^* \subset \mathbb{C}^2$ accumulates L_∞. By the flow box theorem, a point of accumulation $q \in L_\infty \cap \bar{\Gamma}_1^*$ which is not a singularity of $\mathcal{G}$, must be a point near to which the closure (in $\mathbb{C}P(2)$) $\bar{\Gamma}_1^*$ is analytic.

Thus if there are no singularities of $\mathcal{G}$ in $\bar{\Gamma}_1^* \cap L_\infty$ then $\bar{\Gamma}_1^*$ is an algebraic $\mathcal{G}$-invariant curve in $\mathbb{C}P(2)$. This implies that if $\bar{L}_0$ is the leaf of $\mathcal{F}$ on $\mathbb{C}P(2)$ that contains Γ^* then $\bar{L}_0$ is an algebraic invariant curve and $\mathcal{F}$-invariant. Since $\bar{L}_0 \neq L_\infty$ we have a contradiction to our hypothesis. Therefore Γ_1^* must accumulate to some singularity r of $\mathcal{G}$ in L_∞. Once again by the local behavior of the leaves close to Γ_1^* and due to the fact that r is hyperbolic, it follows that $\bar{\Gamma}_1^*$ is locally a separatrix of $\mathcal{G}$ at r. Since L_∞ is not $\mathcal{G}$-invariant, we have two local separatrices Λ_1, Λ_2 for $\mathcal{G}$ at r with $\Lambda_j \nsubseteq L_\infty$, $j = 1,2$. Thus $\bar{\Gamma}_1^*$ is locally contained in $\Lambda_1 \cup \Lambda_2$ and in particular $\bar{\Gamma}_1^*$ is analytic around r. Since (as we have seen) $\bar{\Gamma}_1^*$ is also analytic around the points $q \in \mathrm{Sing}(\mathcal{G})$, it follows that $\bar{\Gamma}_1^*$

is analytic in $\mathbb{C}P(2)$ and once again it is an algebraic curve. Again we conclude that Γ is contained in an algebraic leaf of $\mathcal{F}$, other than L_∞. Contradiction! $\square$

The proof given above also shows us:

Proposition 1.13. *Let $\mathcal{F}$, $\mathcal{G}$ be foliations on $\mathbb{C}P(2)$ both leaving invariant the line L_∞. Let $\phi : \mathbb{C}^2 \to \mathbb{C}^2$ be a topological invariant equivalence for $\mathcal{F}|_{\mathbb{C}^2}$ and $\mathcal{F}|_{\mathbb{C}^2}$. Then ϕ takes the separatrix set $\mathrm{S}_\mathcal{F}$ onto the separatix set $\mathrm{S}_\mathcal{G}$.*

Here $\mathrm{S}_\mathcal{F}$ and $\mathrm{S}_\mathcal{G}$ are respectively the set of separatrices of $\mathcal{F}$ and $\mathcal{G}$ in $\mathbb{C}^2$ that are transverse to L_∞ at some singular point $p \in \mathrm{Sing}(\mathcal{F})$.

Corollary 1.14. *Let $\mathcal{F}_0 \in \mathcal{H}(n)$, $n \geq 2$. Then any $\mathbb{C}^2$-topologically trivial deformation $\{\mathcal{F}_t\}_{t\in\mathbb{D}}$ of $\mathcal{F}_0$, is a deformation in the class $\mathcal{H}(n)$ and it is also s-trivial if we consider* $\mathrm{t} \approx 0$.

Proof. First we recall that $\mathcal{H}(n)$ is open in $\mathcal{X}(n)$. Thus it remains to use proposition (1.12) to conclude that $\mathcal{F}_t \in \mathcal{H}(n)$, $\forall t \approx 0$ and then we use proposition (1.13) to conclude that $\{\mathcal{F}_t\}_{t\approx 0}$ is s-trivial. $\square$

1.2. **Fixed points and one-parameter pseudogroup.** $\mathrm{Diff}(\mathbb{C}, 0)$ denotes the group of germs of complex diffeomorphisms fixing $0 \in \mathbb{C}$, $f(z) = \lambda z + \sum_{n\geq 2} a_n z^n$; $\lambda \neq 0$.

Let $G \subset \mathrm{Diff}(\mathbb{C}, 0)$ be a finitely generated subgroup with a set of generators $g_1, \cdots, g_r \in G$ defined in a compact disk $\bar{\mathbb{D}}_\epsilon$.

Theorem 1.15. [1], [17] *Suppose G is nonsolvable. Then:*

(1) *The basin of attraction of (the pseudo-orbits of) G is an open neighborhood of the origin Ω.($0 \in \Omega$)*
(2) *Either G has dense pseudo-orbits in some neighborhood $0 \in V \subset \Omega$ or there exists an invariant germ of analytic curve Γ (equivalent to $\mathrm{Im} z^k = 0$ for some $k \in \mathbb{N}$) where G has dense pseudo-orbits and such that G has also dense pseudo-orbits in each component of $V \backslash \Gamma$.*
(3) *G is topologically rigid: Given another nonsolvable subgroup $G' \subset \mathrm{Diff}(\mathbb{C}, 0)$ and a topological conjugation $\phi : \Omega \to \Omega'$ between G and G', then ϕ is holomorphic in a neighborhood of 0.*
(4) *There exists a neighborhood $0 \in W \subset V \subset \Omega$ where G has a dense set of hyperbolic fixed points.*

Remark 1.16. In the case G is nonsolvable and contains some $f \in G$ with $f'(0)^n \neq 1$, $\forall n \in \mathbb{Z}\backslash\{0\}$ (i.e., $f'(0) = e^{2\pi i\lambda}$, $\lambda \notin \mathbb{Q}$) we have the following:

Dense Orbits Property: There exists a neighborhood $0 \in V \subset \Omega$ where the pseudo-orbits of G are dense.

Holomorphic deformations in $\mathrm{Diff}(\mathbb{C}, 0)$**:**

Let $g \in \mathrm{Diff}(\mathbb{C}, 0)$ defined in some open neighborhood $0 \in \Omega$. A *holomorphic (one-parameter) deformation* of g is a map $G : \mathbb{D}_\epsilon \to \mathrm{Diff}(\mathbb{C}, 0)$, $(\epsilon > 0)$ which verifies the four properties:

(1) $G(0) = g$ as germs
(2) The Taylor expansion coefficients of $G(t)$ depend holomorphically on t
(3) The radii of convergence of $G(t)$ and $G(t)^{-1}$ are both uniformly minorated by some constant $R \geq 0$ ($\forall t \in \mathbb{D}_\epsilon$)
(4) The modules of the linear coefficient of $G(t)$ is uniformly minorated by some constant $C \geq 0$. In particular $|(G(t)^{-1})'(0)|$ is uniformly majored by $0 < t < \infty$.

Given a finitely generated pseudo-group $G \subset \mathrm{Diff}(\mathbb{C}, 0)$ with a set of generators $g_1, \cdots g_r \in G$; a holomorphic (one parameter) deformation of G is given by holomorphic deformation of g_j, $j = 1, \cdots, r$. We may restrict ourselves to the following situation:

G_t is an one-parameter analytic deformation of G with $t \in \mathbb{D}$, $G_0 = G$. We have $g_{1,t} \cdots g_{r,t}$ as a set of generators for G_t, all of them defined in a disk $\bar{\mathbb{D}}_\delta$ (uniformly on t). We will consider dynamical and analytical properties of such deformations. The results we state below have their proofs reduced to the following case which is studied in [28].

$g_{1,t}(z) = g_1(z) + tz^{D+1}$ where $D \in \mathbb{N}$ is fixed,
$g_{2,t}(z) = g_2(z), \cdots, g_{r,t}(z) = g_r(z)$.
For such deformations we have:

Theorem 1.17. [28] *Given a hyperbolic fixed point $p \approx 0$ for a word $f = f_n \circ f_{n-1} \circ \cdots \circ f_1$ in G, we consider the corresponding word $f_t = f = f_{n,t} \circ \cdots \circ f_{1,t}$ in G_t. Then f_t has a hyperbolic fixed point $p(t)$ given by the implicit differential equation with initial conditions:*

$$\frac{dp(t)}{p(t)^{D+1} dt} = \frac{f_t'(p(t))}{f_t'(p(t)) - 1} f_{1,t}'(p(t)), \quad p(0) = p.$$

In particular $p(t)$ depends analytically in t as well as its multiplicator ${f_t}'(p(t))$. This holds for $|t| < \epsilon$ if $\epsilon > 0$ is small enough.

We also have:

Theorem 1.18. [27] *Let f and g be two non-commuting complex diffeomorphisms defined in some neighborhood of the origin $0 \in \mathbb{C}$, fixed by f and g. Assume that $f'(0) = e^{2\pi i \lambda}$, $g'(0) = e^{2\pi i \mu}$ with $\lambda, \mu \in \mathbb{C} \backslash \mathbb{R}$, $\mathrm{Re}\lambda, \mathrm{Re}\mu \notin \mathbb{Q}$. Then there exist some bound $K > 0$ and some radius $r_0 > 0$ such that if $r \in (0, r_0)$ and $|t| \leq Kr$ then the orbits of the pseudo-group generated by g and $f_t(z) = t + f(z - t)$ are dense in $\bar{\mathbb{D}}_r$.*

Corollary 1.19. [26] *Let f and g be as above. Any holomorphic deformation of the subgroup $< f, g > \subset \mathrm{Diff}(\mathbb{C}, 0)$ preserves locally at the origin the dense orbits property.*

1.3. **Proof of theorem A.** We use the terminology of [12] and some of the original ideas of [3]. Let therefore $\{\mathcal{F}_t\}_{t\in\mathbb{D}}$ be a $\mathbb{C}^2$-topological trivial deformation of $\mathcal{F}_0 \in \mathcal{H}(n)$, $n \geq 2$. As we have proved in corollary (2.8) there exists $\epsilon > 0$ such that $\{\mathcal{F}_t\}_{t\in\mathbb{D}_\epsilon}$ is a s-trivial deformation of $\mathcal{F}_0$ in the class $\mathcal{H}(n)$. Now we consider the continuous foliation $\tilde{\mathcal{F}}$ on $\mathbb{C}P(2) \times \mathbb{D}_\epsilon$ defined as follows:

- $\mathrm{Sing}(\tilde{\mathcal{F}}) = \bigcup_{|t|<\epsilon} \mathrm{Sing}(\mathcal{F}_t) \times \{t\}$
- The leaves of $\mathcal{F}_t$ are the intersections of the leaves of $\tilde{\mathcal{F}}$ with $\mathbb{C}P(2) \times \{t\}$, $\forall |t| < \epsilon$.

Because of the topological triviality, $\tilde{\mathcal{F}}$ is a continuous foliation on $\mathbb{C}^2 \times \mathbb{D}_\epsilon$. This foliation extends to a continuous foliation on $\mathbb{C}P(2) \times \mathbb{D}_\epsilon$ by adding the leaf with singularities $L_\infty \times \mathbb{D}_\epsilon$. In order to prove that $\tilde{\mathcal{F}}$ is holomorphic we begin by proving that it has holomorphic leaves and then it is transversely holomorphic. This is basically done by the following lemma:

Lemma 1.20. *Let $p_1, \cdots, p_{n+1} \in L_\infty$ be the singularity of $\mathcal{F}_0$ in L_∞. Then*

(1) *There exist analytic functions $p_j(t)$, $t \in \mathbb{D}_\epsilon$ such that* $\{p_1(t), \cdots, p_{n+1}(t)\} = \mathrm{Sing}(\mathcal{F}_t) \cap L_\infty$, $p_j(0) = p_j$, $j = 1, \cdots, n+1$.

Fix $q \in L_\infty \backslash \mathrm{Sing}(\mathcal{F}_0)$ and take small simple loops $\alpha_j \in \pi_1(L_\infty \backslash \mathrm{Sing}(\mathcal{F}_0), q)$ and a small transverse disk $\Sigma \pitchfork L_\infty$. Then for $\epsilon > 0$ small we have:

(2) *The holonomy group $G_t := \mathrm{Hol}(\mathcal{F}_t, L_\infty, \Sigma_i) \subset \mathrm{Diff}(\Sigma, q)$ is generated by the holonomy maps $f_{j,t}$ associated to the loops α_j (α_j is also simple loop around $p_j(t)$).*

In particular we obtain

(3) *$\{G_t\}_{t\in\mathbb{D}_\epsilon}$ is an one-parameter holomorphic deformation of* $G_0 = \mathrm{Hol}(\mathcal{F}_0, L_\infty, \Sigma)$.

(4) *The group G_t is nonsolvable with the density orbits property, a dense set $\eta_t \subset \Sigma \times \{t\}$ of hyperbolic fixed points around the origin (q, t). Moreover, given any $p_0 \in \eta_0$, $p_0 = f_0(p_0)$, there exists an analytic curve $p_t \in \eta_t$ such that $p(0) = p_0$, $f_t(p_t) = p_t$ where $f_t \in G_t$ is the corresponding deformation of f_0.*

Using above lemma we prove that $\tilde{\mathcal{F}}$ is holomorphic close to $L_\infty \times \mathbb{D}_\epsilon$: Given a point $p_0 \in \eta_0$ and $f_0 \in G_0$ as above, the curve $p(t)$ and $f_t \in G_t$ given by (iv) above we have $\{p(t), |t| < \epsilon\} \subset \tilde{L}_{p_0} \cap (\Sigma \times \mathbb{D}_\epsilon)$ where $\tilde{L}_{p_0}$ is the $\tilde{\mathcal{F}}$-leaf through p_0. On the other hand $\tilde{L}_{p_0}$ is already holomorphic along the cuts $\tilde{L}_{p_0} \cap (\mathbb{C}P(2) \times \{t\})$ for $L_{{p'}_0,t}$ for $p_0 = ({p'}_0, 0)$. This implies that $\tilde{L}_{p_0}$ is analytic.

Since the curves $\{p(t), |t| < \epsilon\}$ with $p_0 \in \eta_0$ are analytic and locally dense around $\{q\} \times \mathbb{D}_\epsilon \subset \Sigma \times \mathbb{D}_\epsilon$ it follows that any leaf $\tilde{L}$ of $\tilde{\mathcal{F}}$ is a uniform limit of holomorphic leaves $\tilde{L}_{p_0}$ and it is therefore holomorphic. Thus $\tilde{\mathcal{F}}$ has holomorphic leaves. We proceed to prove that it is transversely holomorphic.

This is in fact a consequence of topological rigidity theorem [17] for nonsolvable groups of $\mathrm{Diff}(\mathbb{C},0)$.

Fix transverse section $\Sigma \pitchfork L_\infty$ as above. We may assume that $\Sigma \subset V$ where V is a flow-box neighborhood for $\mathcal{F}_0$ with $q \in V$. The homeomorphisms $\phi_t : \mathbb{C}^2 \to \mathbb{C}^2$ take the separatrices S_0 of $\mathcal{F}_0$ onto the set of separatrices S_t of $\mathcal{F}_t$. Now we use the following proposition:

Proposition 1.21. *Given $\mathcal{F} \in \mathcal{H}(n)$, $n \geq 2$, the set of separatrices $S_{\mathcal{F}}$ of $\mathcal{F}$ is dense in $\mathbb{C}P(2)$ and it accumulates densely a neighborhood of the origin for any transverse disk $\Sigma \pitchfork L_\infty$, $q \notin \mathrm{Sing}\mathcal{F}$.*

Proof. Indeed, given a separatrix $\Gamma \subset S_{\mathcal{F}}$ the leaf $L \supset \Gamma$ is nonalgebraic for $\mathcal{F} \in \mathcal{H}(n)$. This implies that $L \backslash \Gamma$ accumulates L_∞ and therefore any transverse disk Σ as above is cut by L. Now it remains to use the density of the pseudo-orbits of $\mathrm{Hol}(\mathcal{F}, L_\infty)$ stated in theorem (1.15).

Returning to our argumentation we fix any $p \in \Sigma$, separatrix $(p_0 \in)\Gamma_0 \subset \mathcal{S}_0$ of $\mathcal{F}_0$ and denote by $P(\Gamma_0, p)$ the local plaque of $\mathcal{F}_0|_V$ that is contained in $\Gamma_0 \cap V$ and contains the fixed point p. Put $\Gamma_t = \phi_t(\Gamma_0)$ and consider the map $t \mapsto p(t) := P(\Gamma_t, p)$. Clearly we may write $p(t) = \phi_t(P(\Gamma_0, p)) \cap \Sigma \times \{t\}$ by choosing Σ and $|t|$ small enough. This map $t \mapsto p(t)$ is holomorphic as a consequence of proposition below:

Proposition 1.22. *Given any singularity $p_j^0 \in \mathrm{Sing}\mathcal{F}$ there exists a connected neighborhood $(p_j^0 \in)\mathcal{U}_j$, a neighborhood $\mathcal{U} \ni \mathcal{F}_0$ in $\mathcal{S}(n)$ and a holomorphic map $\psi_j : \mathcal{U} \to \mathcal{U}_j$ such that $\forall \mathcal{F} \in \mathcal{U}$, $\psi_j(\mathcal{F}) = \mathrm{Sing}\mathcal{F} \cap \mathcal{U}_j$, $\psi_j(\mathcal{F}_0) = p_j^0$. In particular, if $\{\mathcal{F}_t\}_{t\in\mathbb{D}}$ is a deformation of $\mathcal{F}_0 \in \mathcal{H}(n)$, $n \geq 2$; then given $\Gamma_0 \in S_0 = S_{\mathcal{F}}$, $\Sigma \pitchfork L_\infty$, V and $p \in \Gamma_0 \cap \Sigma$ as above, there exist analytic curves $p_j(t)$ and $p(t)$ such that: $p_j(t) = \mathrm{Sing}\mathcal{F}_t \cap \mathcal{U}_j$, $p_j(0) = p_j^0$, $p(t) = P(\Gamma_t, p(t))$, $p(0) = p$ and $p(t) \in \Gamma_t \cap \Sigma$.*

Roughly speaking, the proposition says that both the singularities and the separatrices of a foliation with nondegenerate singularities, move analytically under analytic deformations of the foliation.

Finally we define $h_t(p) := p(t)$ obtaining this way an injective map in a dense subset of Σ ($\mathcal{F}_0$ has dense separatrices in (Σ, q)), so that by the λ-lemma for complex mapping we may extend h_t to a map that $h_t : \Sigma \to \Sigma$. Moreover, it is clear that if $f_{j,t}$ is a holonomy map as above then we have

$$h_t(f_{j,0}(p)) \quad = \quad f_{j,t}(h_t(p))$$

Because f_0 and f_t fix the separatrices. Therefore, by density we have $h_t \circ f_{j,0} = f_{j,t} \circ h_t$, $\forall j \in \{1, \cdots, n+1\}$ and the mapping h_t conjugates the holonomy groups $G_t = \mathrm{Hol}(\mathcal{F}_t, L_\infty, \Sigma)$ and G_0. By the topological rigidity theorem h_t is holomorphic which implies that $\tilde{\mathcal{F}}$ is transversely holomorphic close to $L_\infty \times \mathbb{D}_\epsilon$ [17]. The density of $\mathcal{S}_t$, $\forall t$ assures that $\tilde{\mathcal{F}}$ is in fact holomorphic in $\mathbb{C}P(2) \times \mathbb{D}_\epsilon$.

Summarizing the discussion above we have:

Proposition 1.23. *Let* $\{\mathcal{F}_t\}_{t\in\mathbb{D}}$ *be a* $\mathbb{C}^2$*-topologically trivial deformation of* $\mathcal{F}_0 \in \mathcal{H}(n)$, $n \geq 2$. *Then there exists* $\epsilon > 0$ *such that* $\{\mathcal{F}_t\}_{t\in\mathbb{D}_\epsilon}$ *is an unfolding of* $\mathcal{F}_0$ *in* $\mathbb{C}P(2)$.

The end of the proof of Theorem A. The proof is a consequence of propositions (1.9) and (1.23) above.

1.4. **Generalizations.** Theorem (A) may be extended to a more general class of foliations on $\mathbb{C}P(2)$ as well to foliations on other projective spaces. This is the goal of this section. Before going further into generalizations we state a kind of Noether's lemma for foliations.

Lemma 1.24. *Let* $\{\mathcal{F}_t\}_{t\in\mathbb{D}}$ *be a holomorphic unfolding of a foliation* $\mathcal{F}_0$ *of degree* n *on* $\mathbb{C}P(2)$. *Assume that for each singularity* $p \in \mathrm{Sing}(\mathcal{F}_0) \cap L_\infty$ *the germ of unfolding at* p *is analytically trivial. Then there exists* $\epsilon > 0$ *such that* $\{\mathcal{F}_t\}_{|t|<\epsilon}$ *is analytically trivial.*

Proof. Denote by
$\pi : \mathbb{C}^3\backslash\{0\} \to \mathbb{C}P(2)$ the canonical projection and by
$\Pi : (\mathbb{C}^3\backslash\{0\}) \times \mathbb{D} \to \mathbb{C}P(2) \times \mathbb{D}$ the map $\Pi(p, t) := (\pi(p), t)$.
Choose a holomorphic integrable 1-form Ω which defines $\tilde{\mathcal{F}}^*$ extension of $\Pi^*(\mathcal{F})$ to $\mathbb{C}^3 \times \mathbb{D}$, so that we may choose

$$\Omega \quad = \quad A(x, t)dt + \sum_{i=1}^{3} B_j(x, t)dx_j,$$

where A, B_j are holomorphic in $(x, t) \in \mathbb{C}^3 \times \mathbb{D}$, homogeneous polynomial in x of degree $n + 2$, $n + 1$; $\sum_{i=1}^3 x_j B_j = 0$. The foliation $\pi^*(\mathcal{F}_t)$ extends to $\mathbb{C}^3$ and this extension $\mathcal{F}_t^*$ is given by $\Omega_t = 0$ for $\Omega_t := \sum_{i=1}^3 B_j dx_j$.

Claim. Given point $q \in \mathbb{C}^3 \times \mathbb{D}_\epsilon$, $q \notin \{0\} \times \mathbb{D}$, there exist a neighborhood $U(q)$ of q in $\mathbb{C}^3 \times \mathbb{D}_\epsilon$ and local holomorphic vector field $X_q \in \mathfrak{X}(U(q))$ such that $A = \Omega \circ X_q$ in $U(q)$, for ϵ small enough.

Proof of the claim. If $q = (x_1, t_1)$ with $x_1 \notin \mathrm{Sing}(\mathcal{F}_0)$ then $x_1 \notin \mathrm{Sing}(\mathcal{F}_t)$ for $|t|$ small enough and in particular $x_1 \notin \mathrm{Sing}(\mathcal{F}_{t_1})$. Thus the existence of $X_q \in \mathfrak{X}(U(q))$ is assured in this case. On the other hand if $x_1 \notin \mathrm{Sing}(\mathcal{F}_0)$ then we still have the existence of $X_q \in \mathfrak{X}(U(q))$ because of the local analytical triviality hypothesis for the unfolding at x_1.

Using the claim we obtain an open cover $\{U_\alpha\}_{\alpha\in\mathbb{Q}}$ of $M := \mathbb{C}^3\backslash\{0\} \times \mathbb{D}$ with U_α connected and $X_\alpha \in \mathfrak{X}(U_\alpha)$ such that $A = \Omega \circ X_\alpha$ in U_α, $\forall \alpha \in \mathbb{Q}$. Let $U_\alpha \cap U_\beta \neq \varnothing$ then we put $X_{\alpha\beta} := (X_\alpha - X_\beta)|_{U_\alpha\cap U_\beta}$ to obtain $X_{\alpha\beta} \in \mathfrak{X}(U_\alpha \cap U_\beta)$ such that $\Omega \circ X_{\alpha\beta} = 0$. Take now the rotational vector field

$$\begin{aligned} Y &= rot(B_1, B_2, B_3) \\ &= (\frac{\partial B_3}{\partial x_2} - \frac{\partial B_2}{\partial x_3})\frac{\partial}{\partial x_1} + (\frac{\partial B_1}{\partial x_3} - \frac{\partial B_3}{\partial x_1})\frac{\partial}{\partial x_2} + (\frac{\partial B_2}{\partial x_1} - \frac{\partial B_1}{\partial x_2})\frac{\partial}{\partial x_3}. \end{aligned}$$

$Y \in \mathfrak{X}(\mathbb{C}^3 \times \mathbb{D})$ and for each $t \in \mathbb{D}$ we have $i_Y(\text{Vol}) = d\Omega_t$ where $\text{Vol} = dx_1 \wedge dx_2 \wedge dx_3$ is the volume element of $\mathbb{C}^3$ in the x-coordinates. Fixed now $q = (x_1, t_1) \notin \text{Sing}(\Omega_{t_1})$ then the leaf of $\mathcal{F}^*_{t_1}$ through q is spanned by $Y(q)$ the radial vector field $R(q)$, as a consequence of the remark above: actually, we have $i_R i_Y(\text{Vol}) = i_R(d\Omega_t) = (n+1)\Omega_t$.

Given thus $U_{\alpha\beta} := U_\alpha \cap U_\beta \neq \varnothing$, since $\Omega_t(X_{\alpha\beta})$ we have that $X_{\alpha\beta}$ is tangent to $\mathcal{F}^*_t$ outside the points $(x,t) \in \text{Sing}(\Omega_t)$ so that we can write $X_{\alpha\beta} = g_{\alpha\beta}R + h_{\alpha\beta}Y$ for some holomorphic functions $g_{\alpha\beta}$, $h_{\alpha\beta} \in \mathcal{O}(U_{\alpha\beta}\backslash\text{Sing}(\Omega_t))$. Since $\text{Sing}(\Omega_t)$ is an analytic set of codimension ≥ 2, Hartogs extension theorem [11] implies that $g_{\alpha\beta}$, $h_{\alpha\beta}$ extend holomorphically to $U_{\alpha\beta}$. Now if $U_\alpha \cap U_\beta \cap U_\gamma \neq \varnothing$ then

$$0 = X_{\alpha\beta} + X_{\beta\gamma} + X_{\gamma\alpha} = (g_{\alpha\beta} + g_{\beta\gamma} + g_{\gamma\alpha})R + (h_{\alpha\beta} + h_{\beta\gamma} + h_{\gamma\alpha})Y$$

and since R and Y are linearly independent outside $\text{Sing}(\Omega_t)$ we obtain: $g_{\alpha\beta} + g_{\beta\gamma} + g_{\gamma\alpha} = 0$, $h_{\alpha\beta} + h_{\beta\gamma} + h_{\gamma\alpha} = 0$.

Thus $(g_{\alpha\beta})$, $(h_{\alpha\beta})$ are additive cocycles in M and by Cartan's theorem (for $\mathbb{C}^{n+1}\backslash\{0\}$, $n \geq 2$) these cocycles are trivial, that is, $\exists g_\alpha, h_\alpha \in \mathcal{O}(U_\alpha)$ such that if $U_\alpha \cap U_\beta \neq \phi$ then $g_{\alpha\beta} = g_\alpha - g_\beta h_{\alpha\beta} = h_\alpha - h_\beta$ in $U_\alpha \cap U_\beta$. This gives $X_\alpha - X_\beta = X_{\alpha\beta} = g_{\alpha\beta}R + h_{\alpha\beta}Y(g_\alpha R + h_\alpha Y) - (g_\beta R + h_\beta Y)$ in $U_\alpha \cap U_\beta \neq \phi$. Thus, in $U_\alpha \cap U_\beta \neq \phi$ we obtain $X_\alpha - g_\alpha R - h_\alpha Y = X_\beta - g_\beta R - h_\beta Y$ and this gives a global vector field $\tilde{X} \in \mathfrak{X}(M)$ such that $\tilde{X}|_{U_\alpha} := X_\alpha - g_\alpha R - h_\alpha Y$. This vector field extends holomorphically to $\mathbb{C}^3 \times \mathbb{D}$ and we have $(\Omega_t \circ \tilde{X})|_{U_\alpha} = \Omega_t \circ X_\alpha - g_\alpha \Omega_t \circ R - h_\alpha \Omega_t \circ Y = A$ so that $\Omega_t \circ \tilde{X} = A$.

It remains to prove that we may choose $\tilde{X}$ polynomial in the variable x. Indeed, we write $\tilde{X} = \sum_{k=0}^{\infty} \tilde{X}_k$ for the Taylor expansion of $\tilde{X}$ around the origin, in the variable x.

Then $\tilde{X}_k$ is holomorphic in (x,t) and homogeneous polynomial of degree k in the variable x. We have $A = \Omega_t \circ \tilde{X} = \sum_{k=0}^{+\infty} \Omega_t(\tilde{X}_k)$ and since it is polynomial homogeneous of degree $n+2$ in x it follows that $k \neq 1 \Rightarrow \Omega_t(\tilde{X}_k) = 0$ and $\Omega_t(\tilde{X}_1) = A$. Since $\tilde{X}_1$ is linear, the flow of $\tilde{X}_1$ gives an analytic trivialization for $\{\mathcal{F}_t\}_{t \in \mathbb{D}_\epsilon}$. □

1.5. **Quasi-hyperbolic foliations.** Now we recall some of the features coming from [14]. A germ of a 1-form $\omega = a(x,y)dy - b(x,y)dx$, $a, b \in \mathcal{O}_{(\mathbb{C}^2,0)}$ with an isolated singularity in a neighborhood $U \subset \mathbb{C}^2$ induces a holomorphic foliation $\mathcal{F}$. The main tool in local study is the resolution theorem of Seidenberg [20] that establishes a canonical reduction. More precisely, there is a holomorphic map $\pi : M \to \mathbb{C}^2$ obtained as a composition of a finite number of blowing ups at points over $\{0\}$, such that at each singular point $m \in M$ of $\widetilde{\mathcal{F}}$, foliation with isolated singularity constructed from $\pi^*(\omega)$ is reduced: there are coordinate charts (z, w) such that $z(m) = w(m) = 0$, $\widetilde{\mathcal{F}}$ is given locally by the expression $B(z,w)dz - A(z,w)dw = 0$ and the Jacobian $\frac{\partial(A,B)}{\partial(z,w)}(0,0)$ has at least one nonzero eigenvalue. Moreover if $\lambda_1 \neq 0 \neq \lambda_2$ are eigenvalues of the matrix then $\lambda_1/\lambda_2 \notin \mathbb{Q}$. If one of the eigenvalues of the above matrix of a singularity is zero and another different from zero we call it saddle-node.

Any irreducible component of desingularization (the divisor of normal crossing projective lines appeared after the reduction, $\widetilde{\mathcal{D}} = \cup_{j=1}^{l} D_j$), D_i, may be invariant or not. If it is invariant we call it nondicritical irreducible component and if it isn't we call it dicritical component. The main invariant associated to each nondicritical component D_i, is its holonomy. It is a representation $H : \pi_1(D_i \setminus \mathrm{Sing}(\widetilde{\mathcal{F}})) \to \mathrm{Diff}(\mathbb{C}, 0)$ of the fundamental group of $D_i \setminus \mathrm{Sing}(\widetilde{\mathcal{F}})$ into the group of germs of diffeomorphisms of $\mathbb{C}$ fixing 0.

Definition 1. Let $\mathcal{F}$ be a germ at the origin of $\mathbb{C}^2$ of holomorphic foliation, $\widetilde{\mathcal{F}}$ the foliation obtained by reduction of singularity and $\widetilde{\mathcal{D}}$ the exceptional divisor of this reduction. We say that $\mathcal{F}$ is quasi-hyperbolic if the following properties are satisfied:

(1) the foliation $\mathcal{F}$ is nondicritical, this means that the exceptional divisor $\widetilde{\mathcal{D}}$ is an invariant set of the foliation $\widetilde{\mathcal{F}}$ ($\widetilde{\mathcal{D}}$ is a finite union of leaves and singular points of $\widetilde{\mathcal{F}}$),
(2) all the singular points $m \in \mathrm{Sing}(\widetilde{\mathcal{F}}) \subset \widetilde{\mathcal{D}}$ are of saddle type, i.e. the foliation $\widetilde{\mathcal{F}}$ around m is given by $udv - \lambda v du$ with $\lambda \in (\mathbb{C} - \mathbb{R}) \cup \mathbb{R}_-$. The saddle-node case is excluded.

The *valence of a component* D of $\widetilde{\mathcal{D}}$ is the number $\nu(D)$ of singular points of $\widetilde{\mathcal{F}}$ on D. A *chain* of $\widetilde{\mathcal{D}}$ is either:

- a connected component $\mathcal{C}$ of $\widetilde{\mathcal{D}} - \cup_{\nu(D)\geq 3} D$, the complement of all components of $\widetilde{\mathcal{D}}$ with valence ≥ 3 such that $\mathcal{C}$ joins two components of valence ≥ 3, or
- an intersection point of two components of $\widetilde{\mathcal{D}}$ with valence ≥ 3.

Definition 2. A quasi-hyperbolic foliation $\mathcal{F}$ is generic if it satisfies the following conditions:

(1) $\widetilde{\mathcal{D}}$ possesses an irreducible component D with the nonsolvable holonomy group,
(2) the holonomy group of every component of $\widetilde{\mathcal{D}}$ of valence ≥ 3 is noncommutative,

(3) all germ of holomorphic first integral of $\widetilde{\mathcal{F}}$ at a singular point situated in the intersection of a component of $\widetilde{\mathcal{D}}$ of valence ≥ 3 and of a chain of valence 2, is constant.

Definition 3. let $\mathcal{F}$ be a quasi hyperbolic foliation as above. The dual tree associated to $\mathcal{F}$ is the weighted graph with arrows $\mathbb{A}^*(\mathcal{F})$ constructed in the following way:

- there is a 1-1 correspondence between vertices of $\mathbb{A}^*(\mathcal{F})$ and irreducible components of $\widetilde{\mathcal{D}}$,
- there is an edge between two vertices of $\mathbb{A}^*(\mathcal{F})$ if the corresponding components of $\widetilde{\mathcal{D}}$ intersect,
- we attach an arrow to a vertex of $\mathbb{A}^*(\mathcal{F})$ for each regular point on the corresponding components of $\widetilde{\mathcal{D}}$, which is a singular point of $\widetilde{\mathcal{F}}$,
- the weight at a vertex of $\mathbb{A}^*(\mathcal{F})$ is the Chern class of the normal bundle to the corresponding component.

Roughly speaking a deformation $\{\mathcal{F}_t\}_{t\in(\mathbb{C},0)}$ of a quasi hyperbolic foliation is *equireducible* if for any sufficiently small $t \in (\mathbb{C}, 0)$ one has: $\mathbb{A}^*(\mathcal{F}_t) \equiv \mathbb{A}^*(\mathcal{F})$. For a proper definition of equireducible see [14].

The outstanding result is [14]:

Theorem 1.25. *A topological trivial deformation of a generic quasi-hyperbolic germ of foliation is an equisingular unfolding.*

Proof. Let the foliation $\mathcal{F}_t$ of deformation is induced by a holomorphic 1-form $\omega_t = b_t(x, y)dy - a_t(x, y)dx$ and fix a neighborhood $U = V \times W$ of the origin of $\mathbb{C}^2 \times \mathbb{C}$ where the singular locus

$$S := \{(x, y; t)|a_t(x, y) = b_t(x, y) = 0\}$$

of the deformation $(\omega_t)_{t\in W}$ is $S = \{0\} \times W$. Suppose the trivialization homeomorphism on U is

$$\Phi : V \times W \longrightarrow U' := \bigcup_{t\in W} V_t \times \{t\},\ \Phi(x, y; t) = (\Phi_t(x, y); t)$$

$$\Phi_t : V \longrightarrow \Phi_t(U) =: V_t,$$

so that for every $t \in W$ transforms the restriction of V of foliation $\mathcal{F}_0$ in the foliation $\mathcal{F}_t$ in V_t.

The proof of theorem is in three steps. First of all we show that the deformation is equireducible, next we consider the regular topological codimension one foliation $\mathcal{F}^{top}$ in $U'^* := U' \setminus (\{0\} \times W)$ obtained by direct image by Φ of trivial unfolding $\mathcal{F}_0 \times \mathbb{C}$. We prove that all leaves of $\mathcal{F}^{top}$ are immersed analytic hypersurfaces, then $\mathcal{F}^{top}$ is in fact holomorphic. It isn't hard to see that $\mathcal{F}^{top}$ extends to a holomorphic equisingular unfolding on the point $S = \{0\} \times W$

• First step: *the deformation* $\{\mathcal{F}_t\}_{t\in(\mathbb{C},0)}$ *is equireducible.* For this part we need only the following assumption:

($\star$) after reduction, none of singularities is saddle-node.

Denote by Z_t the union of separatrices (its number is finite by hypothesis) of $\mathcal{F}_t$ in $V \cap V_t$. By taking V small enough, every homeomorphism Φ_t sends $Z_0 \cap V$ in $Z_t \cap V_t$. The Milnor number $\mu(Z_t)$ of Z_t at the origin is a topological invariant by [10] (See also second chapter). Therefore

$$\mu(Z_t) = \mu(Z_0) \ t \in W. \tag{6}$$

One remarkable result due to C. Camacho, A. Lins Neto and P. Sad permits to see that $\mu(Z_0)$ is equal to the Milnor number of $\mathcal{F}_0$ at the origin. Moreover according to [3] page 164 the foliations satisfied the condition ($\star$) are characterized by the equality of the Milnor number of foliation $\mathcal{F}$ and the Milnor number of the union Z of separatrices of $\mathcal{F}$. Another significant property of this type of foliations which are called generalized curves is the same sequence of blowing ups is needed to resolve singularity of foliation $\mathcal{F}$ and its separatrices Z both ([3] page 162). This implies equality of dual trees of reduction:

$$\mathbb{A}^*(\mathcal{F}) \equiv \mathbb{A}^*(Z) \tag{7}$$

On the other hand, according to [3] page 149, the Milnor number of foliation is a topological invariant so:

$$\mu(\mathcal{F}_t) = \mu(\mathcal{F}_0) \ t \in W. \tag{8}$$

In combining the equalities (6) and (8) one obtains $\mu(\mathcal{F}_t) = \mu(Z_t)$ for $t \in W$. Hence for every $t \in W$, $\mathcal{F}_t$ satisfied the condition ($\star$) and therefore the equality (2). This reduces our problem to a classical result of the theory of plane curves and we have $\mathbb{A}^*(Z_t) \equiv \mathbb{A}^*(Z_0)$.

• Second step: *the leaves of $\mathcal{F}^{top}$ are immersed analytic hypersurfaces.*

Denote by

$$E : \widetilde{\mathcal{M}} \longrightarrow U' := \bigcup_{t \in W} V_t \times \{t\}$$

the equireduction of dimension one foliation $\mathcal{F}_{\mathbb{C}}$ in U' defined by the deformation $\{\mathcal{F}_t\}_{t \in (\mathbb{C},0)}$ and $\widetilde{\mathcal{F}_{\mathbb{C}}}$ its strict transform. Above every $t \in W$ fixed, E corresponds to the application $E_t : \widetilde{V_t} \longrightarrow V_t$ of reduction of foliation $\mathcal{F}_t$ on V_t. Denote by $\widetilde{\mathcal{D}} \subset \widetilde{\mathcal{M}}$, $\widetilde{\mathcal{D}}_t \subset \widetilde{\mathcal{M}}_t$, the exceptional divisor and $\widetilde{\mathcal{F}}_{\mathbb{C}}$ and $\widetilde{\mathcal{F}}_t$ the correspond strict transforms. The homeomorphism Φ is lifted outside of $0 \times W$ to a homeomorphism

$$\tilde{\Phi} : (\tilde{V_0} \times W - 0 \times W) \longrightarrow \widetilde{\mathcal{M}}^* := (\widetilde{\mathcal{M}} - \widetilde{\mathcal{D}}),\ \tilde{\Phi} = (\tilde{\Phi}_t)_{t \in W}$$

$$\tilde{\Phi}_t : (\widetilde{V_0}^* \times 0 - \widetilde{\mathcal{D}}_0 \times 0) \longrightarrow \widetilde{V_t}^* := (\tilde{V} - \widetilde{\mathcal{D}}_t).$$

It is clear that $\tilde{\Phi}$ sends the codimension one foliation $\widetilde{\mathcal{F}}_0 \times \mathbb{C}$ in $\widetilde{\mathcal{F}}^{top}$ the lift of foliation $\mathcal{F}^{top}$ by the application E.

Let the component D'_0 of $\widetilde{\mathcal{D}}_0$ has a nonsolvable holonomy group and denote by D' of $\widetilde{\mathcal{D}}$ a corresponding component isomorphic to $D'_0 \times \mathbb{C}$ and more simply we suppose $D'_0 \times \mathbb{C}$. We fix a smooth analytic curve C_m transverse to D'_0 in a point m of $\dot{D}'_0 := (D'_0 - sing(\widetilde{\mathcal{F}}_0))$ with local coordinate z, $z(m) = 0$. By taking W small enough, we suppose that the image by $\tilde{\Phi}_t$ of a annulus

$$C_{\varepsilon,\varepsilon'} := \{q \in C_m | \ \varepsilon' < z(q) < \varepsilon\} \subset C_m$$

with $\varepsilon > \varepsilon' > 0$ arbitrary small, is annulus in C_m.

Let $\gamma_1, \cdots, \gamma_\nu$ be a good system of curves on $\dot{D}'_0$ with the origin m i.e. they generate the holonomy group of D'_0. By taking Wsmall enough, also they induce a good system of curves on $D'_t := D'_0 \times t$ which we show yet with γ_j. The holonomy group $H_t \subset \mathrm{Diff}(C_m \times t; m \times t)$ of the leaf $\dot{D}'_t := D'_t \setminus sing(\widetilde{\mathcal{F}}_t)$ of $\widetilde{\mathcal{F}}_t$ is generated by the diffeomorphisms $h_{j;t}$ corresponding with γ_j. We consider the pseudogroup Γ_{H_0} generated by the restriction of $h_{j,t}$ to $C_m \cap \widetilde{V}^*_0$. After a result [1], [17] for nonsolvable subgroup, the subset $Q_0 \subset C_m$ of attractive fixed points is dense in $C_m \cap \widetilde{V}^*_0$. Let $m' \in Q_0 \cap C_{\varepsilon,\varepsilon'}$. The element g_0 of Γ_{H_0} that fixes m' is obtained by the holonomy relative to a curve β on $\dot{D}'_0$ with the origin m. Denote by β^* the curve with the origin m' obtained by lifting of β in the leaf of $\widetilde{\mathcal{F}}_0$ containing m'. By taking W small enough, the germ

$$h_{\beta;\mathbb{C}} : (C_m \times \mathbb{C}) \longrightarrow (C_m \times \mathbb{C}),\ h_{\beta;\mathbb{C}} = (h_{\beta;t} \times t)_{t\in W}$$

corresponding to the holonomy of dimension one foliation $\widetilde{\mathcal{F}}_{\mathbb{C}}$ induced by β^* makes apparent an analytic family $(m'_t)_{t\in W}$, $m'_0 = m'$ of attractive fixed points of $h_{\beta;t}$. Since the homeomorphism $\tilde{\Phi}$ trivializes $h_{\beta;\mathbb{C}}$, the holomorphic curve $Q : W \longrightarrow \widetilde{\mathcal{M}}$, $t \longmapsto (m'_t, t)$ is contained in the leaf $\widetilde{F}^{top}_{m'}$ containing $(m', 0)$ of the foliation $\widetilde{\mathcal{F}}^{top}$. By local trivialization of the dimension one holomorphic foliation $\widetilde{\mathcal{F}}_{\mathbb{C}}$, one obtains the analyticity of $\widetilde{F}^{top}_{m'}$ in every point of Q. In consideration of the holonomy of foliation $\widetilde{\mathcal{F}}_{\mathbb{C}}$ along of all curves originated from a point of Q and of some end, one sees that $\widetilde{F}^{top}_{m'}$ is an immersed smooth holomorphic manifold in $\widetilde{\mathcal{M}}^*$. It is the same by direct image of the leaf $F_{E(m')}$ of $\mathcal{F}^{top}$ that contains $E(m')$. Finally since the other leaves of $\mathcal{F}^{top}$ are locally uniform limits of $F^{top}_{m'}$, they are therefore immersed holomorphic manifolds.

• Third step: *the foliation $\mathcal{F}^{top}$ is holomorphic.*

First of all we recall the following fact:

Lemma 1.26. *There is a fundamental system $(\mathcal{V}_\alpha)_\alpha$ of open neighborhood of $0 \times W$ such that if the foliation $\mathcal{F}^{top}$ is holomorphic on an open set $\Omega \subset \mathcal{V}^*_\alpha := U'^* \cap \mathcal{V}_\alpha$, then it is also holomorphic on*

$$\Omega" := \pi^{-1}(\pi(\Omega)) \cap \mathcal{V}^*,$$

where π is the projection of $\mathbb{C}^2 \times \mathbb{C}$ on $\mathbb{C}$.

Thanks to the lemma (1.26) it is enough to show that for every $t \neq 0$, $\mathcal{F}^{top}$ is holomorphic in a point of $\mathcal{V}_\alpha \cap (\mathbb{C}^2 \times t)$: by homogeneity one obtains

$\mathcal{F}^{top}$ is analytic on $\mathcal{V}_\alpha \setminus (\mathbb{C}^2 \times \{0\})$ and by the extension theorem of Riemann it is holomorphic on $\mathcal{V}_\alpha$.

We remark that every holonomy group H_t is nonsolvable. In fact if H_0 is not solvable then there exists a nontrivial element h_0 of second derivative $D^2 H_0$ of H_0. More precisely the solvable groups of holomorphic diffeomorphisms of $(\mathbb{C}, 0)$ are exactly the metabelian groups. This element corresponds to the holonomy of the foliation $\widetilde{\mathcal{F}}_0$ induced by a curve γ which is in $D^2\pi_1(\dot{D}'_0)$. Image of this curve on $\dot{D}'_t$ defines an element $h_{\gamma,t}$ of H_t. Since the deformation $\{\mathcal{F}_t\}_{t\in(\mathbb{C},0)}$ is equisingular, the family $h_{\gamma,t}$ depends analytically on $t \in W$ and by taking W small enough, $h_{\gamma,t}$ is non-trivial.

On the other hand the existence of non-trivial elements tangent to the identity with different and arbitrary high orders, characterizes also the non-solvable subgroups of Diff$(\mathbb{C}, 0)$. Given three curves in $\dot{D}'_0$ that induce the families f_t, g_t and h_t, $t \in W$, which the order of tangency to the identity p_t, q_t and r_t respectively are not equals: by taking W small enough one may assume that for $t \neq 0$ these orders are constant, we say $p_t = p, q_t = q$ and $r_t = r$ with $p < q < r$. This permit us to make a parameter version of the Nakai's construction of vector fields X_t and Y_t associated to the pairs (f_t, g_t) and (f_t, h_t). More precisely we fix a parameter $t_0 \neq 0$ and t varies in a small neighborhood W_{t_0} of t_0; on the appropriate sectors:

$$\Gamma_t := \Gamma \times t,\ ,\Gamma := \{z \in \mathbb{C} | 0 < |z| < \varepsilon_1,\ \theta_1 < \arg(z) < \theta_2\} \subset C_m$$

eventually by replacing f_t, g_t or h_t by their inverse if it is necessary, and by a holomorphic change of coordinates one obtains:

(1) provided Γ_{t_0} by the coordinate $\tilde{z}_t$ defined by a determination z^{-p} one has the convergence, uniform in $t \in W_{t_0}$, of limits

$$X_t := lim_{n\to\infty} n^{\frac{q-p}{p}} (\tilde{f}_t^{-n} \circ \tilde{g}_t \circ \tilde{f}_t^n(\tilde{z}) - \tilde{z}) \frac{\partial}{\partial \tilde{z}},$$

$$Y_t := lim_{n\to\infty} n^{\frac{r-p}{p}} (\tilde{f}_t^{-n} \circ \tilde{h}_t \circ \tilde{f}_t^n(\tilde{z}) - \tilde{z}) \frac{\partial}{\partial \tilde{z}},$$

where $\tilde{f}_t, \tilde{g}_t$ and $\tilde{h}_t$ design the expresion of diffeomorphisms f_t, g_t and h_t in the coordinate $\tilde{z}_t$ respectively;

(2) for every real s small enough the flows $\Psi_s^{X_t}$ and $\Psi_s^{Y_t}$ of X_t and Y_t respectively are limits, uniform in $t \in W_{t_0}$,

$$\Psi_s^{X_t}(z_t) = lim_{n\to\infty}(f_t^{-n} \circ g_t^{l_n} \circ f_t^n(z_t),\ \text{with } l_n.n^{\frac{p-q}{p}} \to s,$$

$$\Psi_s^{Y_t}(z_t) = lim_{n\to\infty}(f_t^{-n} \circ h_t^{l_n} \circ f_t^n(z_t),\ \text{with } l_n.n^{\frac{p-r}{p}} \to s,$$

(3) for every $t \in W_{t_0}$ the orbit of every point of Γ_t by the pseudogroup generated by the restriction f_t, g_t and h_t in the disc $\{|z| < \varepsilon_1\} \times t$ contains Γ in its closure,

(4) X_t and Y_t are regular, linearly independent and noncommutative.

It is enough to prove that in a point z_0 of Γ_0, the trace $\widetilde{F}^{top}_{\Gamma_{W_{t_0}}}$ of foliation $\widetilde{F}^{top}$ on $\Gamma_{W_{t_0}} := \Gamma_{t_0} \times W_{t_0}$ is a holomorphic foliation in a point. In fact every leaf of $\widetilde{F}^{top}$ intersects transversely $\Gamma_{W_{t_0}}$ since it contains the dimension one foliation $\widetilde{\mathcal{F}}_{\mathbb{C}}$, and this latter induces a local holomorphic rectification $\widetilde{F}^{top} \to \widetilde{F}^{top}_{\Gamma_{W_{t_0}}} \times \mathbb{C}$.

A priori $\widetilde{F}^{top}_{\Gamma_{W_{t_0}}}$ isn't a partition of $\Gamma_{W_{t_0}}$ by analytic curves. By a construction similar to the preceding step, one exhibits a leaf $\mathcal{C} \subset \Gamma_{W_{t_0}}$ given by the projection $\rho : \Gamma_{W_{t_0}} \to W_{t_0}$, formed of fixed point of holonomy application. The families $(f_t)_{t\in W_{t_0}}, (g_t)_{t\in W_{t_0}}$ and $(h_t)_{t\in W_{t_0}}$ induce holomorphic diffeomorphisms f, g and h on $\Gamma_{W_{t_0}}$ that coincide with the holonomy elements of $\mathcal{F}_{\mathbb{C}}$; notably they leave invariant $\widetilde{F}^{top}_{\Gamma_{W_{t_0}}}$. By passing to the limit, the flows Ψ^X_s, Ψ^Y_s by real times s of vertical vector fields X and Y defined on $\Gamma_{W_{t_0}}$ by the families X_t and Y_t preseve also $\widetilde{F}^{top}_{\Gamma_{W_{t_0}}}$. Furthermore the application

$$\mathcal{C} \times B^2_r \longrightarrow \mathcal{W},\ (c; s_1, s_2) \longmapsto \Psi^X_{s_1} \circ \Psi^Y_{s_2}(c),$$

where B^2_r is the ball with radius r of $\mathbb{R}^2$, is a real analytic rectification of $\widetilde{F}^{top}_{\Gamma_{W_{t_0}}}$ on an open neighborhood $\widetilde{W}$ of $\mathcal{C}$. One can therefore define holonomy applications

$$l_t : \Gamma_{t_0} \cap \mathcal{W} \longrightarrow \Gamma_t \cap \mathcal{W}$$

by the condition: $c \in \Gamma_{t_0}$ and $l_{t;a}(c)$ are in a same leaf of $\widetilde{F}^{top}_{\Gamma_{W_{t_0}}}$. It is clear that these applications commute with Ψ^X_s and Ψ^Y_s for $s \in \mathbb{R}$. Thanks to the lemma (5.2) of Nakai [17], that the l_t are holomorphic or antiholomorphic. By a reason of orientation the second is drown aside.

Also the homeomorphisms

$$\Theta : ((\Gamma_{t_0} \cap \mathcal{W}) \times W_{t_0}) \longrightarrow \mathcal{W}\ ((c,t) \longmapsto (l_t(C), t))$$

that trivializes $\widetilde{F}^{top}_{\Gamma_{W_{t_0}}}$ is for every t holomorphic in the variable c. On the other hand the partial application $\Theta(c, 0)$ are analytic since their graph are the leaves of $\widetilde{F}^{top}_{\Gamma_{W_{t_0}}}$.

This achieves the demonstration of theorem (1.25). □

The notion of holonomy of a nondicritical component can be extended to the holonomy of a nondicritical branch. We have restricted ourselves to the quasi-hyperbolic foliations. Therefore there is only one nondicritical branch which is the exceptional divisor all . This notion which is called *singular holonomy* was introduced in [4]. Essentially it is the holonomy over closed paths that are allowed to pass through corners of a branch. Suppose $\widetilde{\mathcal{D}} = \cup_{j=1}^{l} D_j$ is the exceptional divisor of $\widetilde{\mathcal{F}}$, the resolved foliation of $\mathcal{F}$, and assume that all corners are linearizable singular points. That is if $p \in D_i \cap D_j$, then near p there are coordinates (x,y) such that $x(p)=y(p)=0$, $D_i=(x=0)$, $D_j=(y=0)$ and $\widetilde{\mathcal{F}}:\lambda ydx+xdy=0$. We take points $q_i\in D_i$, $q_j\in D_j$ close to p and such that $q_i\neq p\neq q_j$. Let (Σ_i,q_i), (Σ_j,q_j) be small cross sections to $\widetilde{\mathcal{F}}$ such that $\Sigma_i\cap D_i$ and $\Sigma_j\cap D_j$. Suppose moreover that G_i (resp. G_j) is the holonomy group of D_i (resp. D_j) at q_i (resp. q_j). The Dulac correspondence

$$\mathcal{D}:(\Sigma_i,q_i)\longrightarrow(\Sigma_j,q_j)$$

is obtained by following the local leaves of $\widetilde{\mathcal{F}}$. In our situation it is given by $\mathcal{D}(x)=x^\lambda$. To an element $h\in G_i$ we associate an element $h^{\mathcal{D}}\in \mathrm{Diff}(\Sigma_j,q_j)$ verifying the relation

$$h^{\mathcal{D}}\circ\mathcal{D}=\mathcal{D}\circ h,$$

called the *adjunction equation*. This induces a subgroup $G_j*(\mathcal{D}_*G_i)$ generated by G_j and the elements $h^{\mathcal{D}}$, $h\in G_i$. Thus if h is , for instance, linear, i.e., $h(x)=\mu x$, then $h^{\mathcal{D}}(y)=\mu^\lambda(y)$ where μ^λ is any solution of $z^{1/\lambda}=\mu$. Other situations are treated in [4]. The singular holonomy group of a component is the maximal subgroup $\mathrm{Diff}(\Sigma_j,q_j)$ obtained by iteration of the adjunction process made at all the corners in the $\mathbb{D}$, see [4] for a detailed description of this construction. Notice that all of singularities are of Saddle-type, in particular saddle-nodes are excluded.

Using the concept of singular holonomy, we may strength theorem (1.25) as follows:

THEOREM D. *Let $\{\mathcal{F}_t\}_{t\in\mathbb{D}}$ be a topologically trivial analytic deformation of a germ of quasi-hyperbolic foliation $\mathcal{F}_0$ at $0\in\mathbb{C}^2$. We have the following possibilities:*

- **(i):** $\mathcal{F}_0$ admits a Liouvillian first integral and all its projective holonomy groups are solvable,
- **(ii):** $\{\mathcal{F}_t\}_{|t|<\varepsilon}$ is an equisingular unfolding.

Proof. Assume that all the projective singular holonomy groups of $\mathcal{F}_0$ are solvable. In this case according to [4], $\mathcal{F}_0$ has a Liouvillian first integral.(here we use strongly the fact that $\mathcal{F}_0$ is quasi-hyperbolic) We may therefore consider the case where some component of the exceptional divisor has non solvable singular holonomy group. This implies topological rigidity and abundance of hyperbolic fixed points as well as the dense orbits property for this group as well as for all the projective singular holonomy groups, which are the main ingredients in the proof of theorem (1.25) and $\{\mathcal{F}_t\}_{|t|<\varepsilon}$ is an unfolding. □

Proof of Theorem B. First we remark that by the topological triviality on $\mathbb{C}P(2)$ we may assume that L_∞ is an algebraic leaf for $\mathcal{F}_t$ and that $\phi_t(L_\infty)=L_\infty$, $\forall t\in\mathbb{D}$. In fact, we take $S_t=\phi_t(L_\infty)\subset\mathbb{C}P(2)$. Then S_t is compact $\mathcal{F}_t$-invariant and of dimension one, so that S_t is an algebraic leaf of $\mathcal{F}_t$. By a well-known theorem of Zariski S_t is smooth. Since the self-intersection number is a topological invariant we conclude that S_t has self-intersection number one and by Bezout's theorem S_t has degree one, that is, S_t is a straight line in $\mathbb{C}P(2)$.

The problem here is that S_t may do not depend analytically on t. That is where we use the hypothesis that there exist at least two reduced singularities $p_1, p_2 \in \operatorname{Sing}(\mathcal{F}_0) \cap L_\infty$. Since p_j is reduced there exists an analytic curve $p_j(t) \in \operatorname{Sing}(\mathcal{F}_t)$ such that $p_j(t)$ is reduced singularity of $\mathcal{F}_t$ and $p_j(t) = \phi_{t(p_j)}$, $p_j(0) = p_j$. since the line S_t contains $p_1(t) \neq p_2(t)$ it follows that S_t depends analytically on t and there exists a unique automorphism $T_t : \mathbb{C}P(2) \to \mathbb{C}P(2)$ such that $T_t(S_t) = S_0 = L_\infty$; $T_t(p_j(t)) = p_j$, $j = 1, 2$. Thus $\psi_t = T_t \circ \phi_t : \mathbb{C}P(2) \to \mathbb{C}P(2)$ gives a topological trivialization for the deformation $\{\mathcal{F}_t^1\}_{t\in\mathbb{D}}$ of $\mathcal{F}_0$, where $\mathcal{F}_t^1 := T_t(\mathcal{F}_t)$, and L_∞ is an algebraic leaf of $\mathcal{F}_t^1$, $\forall t \in \mathbb{D}$. Thus we may assume that L_∞ is $\mathcal{F}_t$-invariant, $\forall t \in \mathbb{D}$. Now we proceed after performing the reduction of singularities for $\mathcal{F}_0|_{L_\infty}$ we consider the exceptional divisor $\mathcal{D} = \cup_{j=1}^r D_j$, $D_0 \cong L_\infty$, $D_j \cong \mathbb{C}P(1)$, $\forall j \in \{1, \cdots, r\}$ and observe that if the singular holonomy groups of the components D_j are all solvable then according to [7] (using the fact that the singularities $p \in \operatorname{Sing}(\mathcal{F}_0) \cap \mathrm{L}_\infty$ are quasi-hyperbolic) we get that $\mathcal{F}$ is a Darboux (logarithmic) foliation. We assume therefore that some singular holonomy group is nonsolvable, then it follows that by definition of singular holonomy group and due to the fact that the divisor D is invariant and connected and has saddle-singularities at the corners, we can conclude that all components of D has nonsolvable singular holonomy groups. This implies, that each germ of $\{\mathcal{F}_t\}_{t\in\mathbb{D}}$ at a singular point $p \in \operatorname{Sing}(\mathcal{F}_0) \cap L_\infty$ is an unfolding(these germs are evidently topologically trivial). Using now arguments similar to the ones in proof of (1.24) we conclude that $\{\mathcal{F}_t\}_{t\in\mathbb{D}_\epsilon}$ is an unfolding for $\epsilon > 0$ small enough.

If we assume that for any singularity $p \in \operatorname{Sing}(\mathcal{F}_0) \cap L_\infty$ the germs of unfolding is analytically trivial, then as consequence of (1.24) we conclude that $\{\mathcal{F}_t\}_{t\in\mathbb{D}_\epsilon}$ is analytically trivial for $\epsilon > 0$ small enough. Theorem B is now proved

Remark 1.27. Above theorem is still true if one replace condition (3) by the following

(3)'$\phi_t(L_\infty) = L_\infty$, $\forall t \in \mathbb{D}$.

2. DENSITY

The problem of density of the leaves of generic foliations $\mathcal{F}$ on $\mathbb{C}P(2)$ has been considered by several authors,e.g. [9], [16] and [23]. It is also related to the problem of the existence of foliations on $\mathbb{C}P(2)$ having an exceptional minimal set [2].

It is now well-known that given any integer $n \geq 2$ there exists an open dense subset $M_1(n)$ of degree n foliations leaving invariant the line at infinity whose foliations have dense leaves on $\mathbb{C}P(2)$ except the line at infinity.

However $M_1(n)$ is an open subset of $\mathcal{F}(n)$ and any deformation $\mathcal{F}_t$ of an element of $M_1(n)$ is still in $M_1(n)$ for t small enough. Now we recall the construction, due to F. Loray and J. Rebelo, of an open subset $\mathcal{U}$ of $\mathcal{F}(n)$ such that all leaves of all foliations of $\mathcal{U}$ are dense in $\mathbb{C}P(2)$ [31].

First we recall a result of Lins Neto and Soares:

Theorem 2.1. *Given $n \geq 2$, there exists a Zariski open subset $\mathcal{U}_1 \subset \mathcal{F}(n)$ such that any $\mathcal{F} \in \mathcal{U}_1$ satisfies:*

(1) *$\mathcal{F}$ has exactly $\frac{(n+1)(n+2)}{2}$ hyperbolic singularities and is regular on the complement;*
(2) *$\mathcal{F}$ has no invariant algebraic curve.*

Fix coordinates system (w, z) in $\mathbb{C}^2$. Given $n \geq 2$, we fix pairwais distinct scalars $w_1, \cdots, w_n \in \mathbb{C}$ and consider the family of rational vector fields

$$X(k_1, \cdots, k_n) = \frac{\partial}{\partial w} + \sum_{i=1}^{n} \frac{k_i}{w - w_i} z \frac{\partial}{\partial z}, \quad k_1, \cdots, k_n \in \mathbb{C}.$$

Denote by $\mathcal{F}(k_1, \cdots, k_n)$ the foliation induced on $\mathbb{C}P(2)$.

Lemma 2.2. *Assume that $k_1, \cdots, k_n \neq 0$ and $k_1 + \cdots + k_n \neq 1$. Then the foliation $\mathcal{F} = \mathcal{F}(k_1, \cdots, k_n) \in \mathcal{F}(n)$ (foliations with degree n), is tangent to the projective line $L_0 : \{z = 0\}$ and has $n+1$ isolated singularities $p_i = (w_i, 0), \quad i = 1, \cdots, n$, and $p_{n+1} = (\infty, 0)$ belonging to L_0.*

Denote by $\mathcal{V}$ the smooth complex submanifold of $\mathcal{F}(n)$, which is parametraized by $(k_1, \cdots, k_n) \mapsto \mathcal{F}(k_1, \cdots, k_n)$, satisfying the assumtion of lemma (2.2) above.

A foliation $\mathcal{F} \in \mathcal{V}$ possesses a holonomy group associated to the leaf L_0. Fix a point $p_0 = (w_0, 0)$ in the leaf $L_0^* = L_0 \ \{p_1, \cdots p_{n+1}\}$ and consider the small section Σ at $(w = w_0, 0)$. For a small complex time T, the complex flow Φ_X^T is well defined in a neighborhood of Σ and induces a linear map from Σ to a section at $(w_0 + T, 0)$. This map is also given by integrating the non-autonomous vector field $Y(t) = \sum_{i=1}^{n} \frac{k_i}{w+t-w_i} z \frac{\partial}{\partial z}$ over the segment $t \in [w, w+T]$. Analogously, given any loop $\gamma : [0,1] \to L_0^*$ with extremities at $\gamma(0) = \gamma(1) = p_0$, we define a linear map $f_\gamma : \Sigma \to \Sigma$ by integrating the non-autonomous vector field $Y(t) = \sum_{i=1}^{n} \frac{k_i}{\gamma(t)-w_i} z\frac{\partial}{\partial z}$ over $[0,1]$. The resulting map f_γ depends only on the homotopy class of γ in the fundamental group $\pi_1(L_0^*, p_0)$. Now choose a collection $\gamma_1, \cdots, \gamma_n : [0,1] \to L_0^*$ of generators for $\pi_1(L_0^*, p_0)$ so that each γ_i has index 1 around p_i, is homotopic to 0 in $L_0^* \cup \{p_i\}$, and, moreover, $\gamma_1\gamma_1 \ldots \gamma_n$ is homotopic to 0 in $L_0^* \cup \{p_\infty\}$. Finally denote by $f_i = f_{\gamma_i}(z) = \lambda_i(z)$, $\lambda_i \in \mathbb{C}^*$.

The construction of the return maps f_i remains valid in the context of holonomy pseudo-groups of arbitrary foliations $\mathcal{F}$ sufficiently close to the family $\mathcal{V}$.

Lemma 2.3. *There exists a neighborhood $\mathcal{U}_2 \subset \mathcal{F}(n)$ containing $\mathcal{V}$, such that the return maps constructed above are well defined $f_{i,\mathcal{F}} : \Sigma \hookrightarrow \mathbb{C}^2$ for every $\mathcal{F} \in \mathcal{U}_2$. Furthermore they depend holomorphically on $\mathcal{F}$.*

Denote by $G_\mathcal{F}$ the pseudo-group generated on Σ by the $f_{i,\mathcal{F}}$. Consider $\mathcal{F}_\alpha = \mathcal{F}(\alpha, \cdots, \alpha) \in \mathcal{V}$ where α is in the upper halph-plane. The associated return maps can be computed by explicitely integration in this case and all of them coincide with the linear contraction $A_{i,\mathcal{F}_\alpha}(z) = \lambda(z)$, $\lambda = e^{2i\pi\alpha}$.

Proposition 2.4 (31)**.** *If $\mathcal{F}_\alpha$ is sufficiently close to $\mathcal{F}_1 = \mathcal{F}(1, \cdots, 1)$, then there exists an open neigheborhood $\mathcal{U}_3 \subset \mathcal{F}(n)$ of $\mathcal{F}_\alpha$ such that any $\mathcal{F} \in \mathcal{U}_3$ satisfy the following alternative:*
- either the pseudo-group $G_\mathcal{F}$ has a fixed point in Σ and then, $\mathcal{F}$ has an invariant projective line $L_{0,\mathcal{F}}$ close to L_0.
- or the pseudo-group $G_\mathcal{F}$ accumulate a non-trivial(real) pseudo-flow on Σ.

Moreover, there is a (real analytic) Zariski-open subset $\mathcal{U}_3' \subset \mathcal{U}_3$ such that the pseudo-group $G_\mathcal{F}$ of any $\mathcal{F} \in \mathcal{U}_3'$ has respectively large linear or affine part on Σ.

Proof. The $n+1$ singular points $p_1, \cdots p_{n+1}$ of $\mathcal{F}_\alpha$ belonging to L_0 are hyperbolic. At each of these points p_i, the foliation admits exactly 2 local invariants curves [30]. For a sufficiently small ball W_i centered at p_i, denote by $L_i = L_0 \cap W_i$ the local invariant curve contained in L_0.

Given a foliation $\mathcal{F}$ sufficiently close to $\mathcal{F}_\alpha$, those $n+1$ hyperbolic singularities will persist as singularities $p_{1,\mathcal{F}}, \cdots, p_{n+1,\mathcal{F}}$ of $\mathcal{F}$ and we denote by $L_{i,\mathcal{F}}$ the corresponding persistence invariant curves in W_i. The persistence fixed point of the m^{th} return map $f_{1,\mathcal{F}}$ within Σ necessarily corresponds to the intersection with Σ of the leaf $L_{i,\mathcal{F}}$. Then if the unique fixed point of $f_{1,\mathcal{F}}$, is also fixed by the other return maps, this means the branches $L_{i,\mathcal{F}}$ are parts of a common leaf which turns out to be an embedded sphere close to L_0 and hence a projective line.

On the other hand, if one of the return maps $f_{i,\mathcal{F}}$ does not fix any longer the unique fixed point of $f_{1,\mathcal{F}}$, then we can apply Proposition (2,0) of [31] to $f_{1,\mathcal{F}}$ and $g = f_{i,\mathcal{F}}^{-1} \circ f_{1,\mathcal{F}}$ and the alternative is proved. The last assertion follows from Corollary (5.2) of [31].

□

Theorem 2.5. *Given $n \geq 2$, there exists an open subset $\mathcal{U} \subset \mathcal{F}(n)$ such that every leaf of every foliation $\mathcal{F} \in \mathcal{U}$ is dense in the whole of $\mathbb{C}P^2$.*

Proof. We keep the notations of Proposition (2.4). There exists a neighborhood W_0 of a compact part of $L_0 \setminus (\bigcup_{i=1}^{n+1} L_i)$ such that any leaf intersecting W_0 will also meet the transversal Σ. The existence of W_0 can easily be established by considering a finite covering by trivialization boxes. For each $i = 1, \ldots, n+1$, recall that W_i is a small ball around p_i as in the proof of Proposition (2.4). Without loss of generality we can suppose that $W = W_0 \cup \bigcup_{i=1}^{n+1} W_i$ defines a neighborhood of L_0. Because of hyperbolicity of $p_{i,\mathcal{F}}$, the horizontal and vertical invariant curves $L_{i,\mathcal{F}}$ and $\Gamma_{i,\mathcal{F}}$ in W_i depend holomorphically on $\mathcal{F}$. Hence we can assume that $L_{i,\mathcal{F}}$ intersects W_0. Since $p_{i,\mathcal{F}}$ is non-resonant in Poincare domain, we can also suppose that this singularity is linearizable in the whole of W_i for every $\mathcal{F}$ clsoe to $\mathcal{F}_\alpha$ Thus any leaf in W_i, other than $\Gamma_{i,\mathcal{F}}$, will accumulate on $L_{i,\mathcal{F}}$, and hence intersect W_0 as well. As a consequence it will, indeed, meet Σ. Then we obtain the following alternative: any leaf L of $\mathcal{F}$ either has algebraic closure or meets Σ.

If $\mathcal{F}$ lies in the Zariski-open subset $\mathcal{U}_1 \subset \mathcal{F}(n)$ given by theorem (2.1), then every leaf L of $\mathcal{F}$ meets Σ and is captured by the dynamis generated by the return maps $f_{1,\mathcal{F}}, \ldots, f_{n+1,\mathcal{F}}$ on Σ. Furthermore the second alternative of Proposition (2.4) has to occure. In addition, if $\mathcal{F} \in \mathcal{U}_3'$, then the closure of $G_{\mathcal{F}}$ possesses many translation pseudo-flows. It results that $G_{\mathcal{F}}$ acts minimally on Σ. Thus, if $\mathcal{F} \in \mathcal{U}_1 \cap \mathcal{U}_3'$, then any leaf L of $\mathcal{F}$ is dense in a neighborhood of Σ.

The leaves are, in fact, dense in $\mathbb{C}P^2$ in the following way. Given a leaf L and a regular point $p \in \mathbb{C}P^2$ of $\mathcal{F}$, denoted by L' the leaf passing through p and by $\gamma(t)$ a path in L' joining $\gamma(0) = p$ to $\gamma(1) \in L' \cap \Sigma$. We see that L must accumulate on $\gamma(1)$. By using a simple argument involving flow-boxes along γ, one easily concludes that L accumulates on p as well. So we just need to set $\mathcal{U} \subset \mathcal{U}_1 \cap \mathcal{U}_3'$.

□

3. GROWTH OF FINITELY GENERATED SUBGROUPS OF $\mathrm{Diff}(\mathbb{C}^n, 0)$

Consider a group G generated by $S = \{g_1, \cdots, g_k\}$. Each element $g \in G$ can be represented by a word ${g_{i_1}}^{p_1}{g_{i_2}}^{p_2}\cdots{g_{i_l}}^{p_l}$ and $|p_1| + |p_2| + \cdots + |p_l|$ is called *the length* of the word. *The norm* $\|\gamma\|$ (relative to S) is defined as the minimal length of the words representing g. Let $B(n)$ be the set of elements $g \in G$ with $\|g\| \leq n$. The growth function of g with respect to the set of generators S is defined as $\gamma := \gamma_G(n) := |B(n)|$.

We say that a function $f : \mathbb{N} \longrightarrow \mathbb{R}$ is dominated by a function $g : \mathbb{N} \longrightarrow \mathbb{R}$, denoted by $f \preceq g$, if there is a constant $C > 0$ such that $f(n) \leq g(C.n)$ for all $n \in \mathbb{N}$. Two functions $f, g : \mathbb{N} \longrightarrow \mathbb{R}$ are called equivalent, denoted by $f \sim g$, if $f \preceq g$ and $g \preceq f$. It is known that for any two finite sets of generators S_1 and S_2 of a group, the corresponding two growth functions are equivalent. Note also that if $|S| = k$, then $\gamma(n) \leq k^n$.

Growth of group G is called *exponential* if $\gamma(n) \sim e^n$. Otherwise the growth is said to be *subexponential*. Growth of group G is called *polynomial* if $\gamma(n) \sim n^c$ for some $c > 0$. If $\gamma(n) \succeq n^c$ for all c, the growth of G is said to be *superpolynomial*. If the growth is subexponential and superpolynomial, it is called *intermediate*.

Examples. The finitely generated abelian groups have polynomial growth. More precisely if $S = \{g_1, \cdots, g_k\}$ is a minimal generating set for a free abelian group G then the growth function is

$$\gamma(n) = \sum_{l=0}^{m} 2^l \binom{m}{l}\binom{n}{l}.$$

Also the finitely generated nilpotent groups are of polynomial growth [29].

The free groups with $k \geq 2$ generators have exponential growth. Milnor and Wolf in [15] and [29] showed that a finitely generated solvable group G has exponential growth unless G contains a nilpotent subgroup of finite index.

If G is a finite extension of a group of polynomial growth, then G itself has polynomial growth. So we conclude if a finitely generated group G has a nilpotent subgroup of finite index then G has polynomial growth. Conversely the finitely generated linear groups (Tits [25]), the finitely generated polycyclic groups (Wolf [29]) and the finitely generated subgroups of $\mathrm{Diff}(\mathbb{R}^n, 0)$ (Plante-Thurston [19]) with polynomial growth have nilpotent subgroups with finite index. Finally in an extraordinary work [8], Gromov settled the problem and proved if a finitely generated group G has polynomial growth then G contains a nilpotent subgroup of finite index.

In [7], Grigorchuk constructed a family of groups of intermediate growth which are the only known examples of such groups.

Denote by Diff$(\mathbb{C}^n, 0)$, the group of germs of complex diffeomorphisms fixing the origin. Let $G \subset$ Diff$(\mathbb{C}^n, 0)$ be a finitely generated subgroup. Nonsolvable finitely generated subgroups of complex diffeomorphism in dimension $n = 1$ play a fundamental role in dynamical study of holomorphic vector fields in $(\mathbb{C}^2, 0)$. In fact the holonomy groups of irreducible components of desingularization of the germ of a nondicritical foliation at a singularity in $\mathbb{C}^2$ are finitely generated subgroups of Diff$(\mathbb{C}^n, 0)$. Theorem of Nakai and its consequences [1], [17] provide a new dynamical information for the "generic" foliation whose holonomy groups are nonsolvable as we have seen in the section (1.1). This motivates characterization of solvability of finitely generated subgroups of complex diffeomorphisms. Our objective is to prove a finitely generated subgroup of Diff$(\mathbb{C}^n, 0)$ with polynomial growth is solvable. Moreover if we know there is no finitely generated subgroups with intermediate growth then it implies a nonsolvable subgroup of Diff$(\mathbb{C}^n, 0)$ has exponential growth.

Lemma 3.1. *Let G be a finitely generated (abstract) group of polynomial growth. Then the commutator subgroup $[G, G]$ is also finitely generated.*

Proof. It is enough to prove that the kernel $H \subset G$ of any surjective homomorphism $f : G \to \mathbb{Z}$ is finitely generated. Take a system of generators $g_0, g_1, \ldots, g_k \in G$ such that $f(g_0) = z_0 \in \mathbb{Z}$ where z_0 denotes the generator in $\mathbb{Z}$, and $g_i \in H,\ i = 1, \ldots, k$. Denote by $H_m \subset H$ the subgroup generated by $\{g_0^j g_i g_0^{-j}\},\ i = 1, \ldots, k,\ j = -m, \ldots, m$. We have $\cup_{m=0}^{\infty} H_m = H$. If for some m, $H_m = H_{m+1}$ then $H_m = H$ and the proof is finished. Otherwise, there is a sequence a sequence $\alpha_m \in H,\ m = 0, 1, \ldots$ such that each α_m is of the form $\alpha_m = g_0^m g_i g_0^{-m}$ or $\alpha_m = g_0^{-m} g_i g_0^m$, for some $i = 1, \ldots, k$ and α_m is not contained in the group generated by $\alpha_0, \alpha_1, \ldots, \alpha_{m-1}$.

Consider all the products $\beta = \beta(\varepsilon_0, \ldots, \varepsilon_m) = \alpha_1^{\varepsilon_0}, \ldots, \alpha_m^{\varepsilon_m}$ where $\varepsilon_0 = 0, 1$. The equality $\beta(\varepsilon_0, \ldots, \varepsilon_m) = \beta(\varepsilon'_0, \ldots, \varepsilon'_m)$ implies $\varepsilon_0 = \varepsilon'_0, \ldots, \varepsilon_m = \varepsilon'_m$. So we have 2^{m+1} different $\beta' s$.

On the other hand $\|\beta\| \leq \|\alpha_0\| + \|\alpha_1\| + \cdots + \|\alpha_m\| \leq (m+1)(2m+1)$ and for the ball $B((m+1)(2m+1)) \subset G$ we have $|B((m+1)(2m+1))| \geq 2^{m+1}, m = 1, 2, \ldots$.

This contradicts the polynomial growth. □

The following lemma is proved in [19]:

Lemma 3.2. *Suppose that G has polynomial growth of degree k and that*

$$H_0 \subset H_1 \subset \cdots \subset H_n \subset G$$

is a finite sequence of subgroups such that for each i ($1 \leq i \leq n$) there is non a trivial homomorphism $f_i : H_i \to \mathbb{R}^l$ such that $H_i \subset$ Kerf_i. Then $n \leq k$.

Remark 3.3. A finitely generated solvable group of polynomial growth is polycyclic.

Theorem E. *Let G be a finitely generated subgroup of* $\mathrm{Diff}(\mathbb{C}^n, 0)$*. If G has a polynomial growth then G is solvable.*

Proof. Put $G_1 = [G, G]$ and $G_{i+1} = [G_i, G_i]$ for $i \in \mathbb{N}$. By the lemma (3.2) it is enough to show that for each G_i there is a non trivial homomorphism $f_i \in \mathrm{Hom}(G_i, \mathbb{R}^l)$. Notice that $G_i \subset \mathrm{Ker} f_{i+1}$ if there are such f_i's. For simplicity denote by $H := G_i$. By the lemma (3.1) H is finitely generated. Take a symmetric system of generators $S := \{h_1, \cdots, h_k\}$. We may write $h_i(z) = z + \tilde{h}_i(z)$ for $i = 1, \cdots, k$ since elements of H are tangent to the identity. If $u, v \in H$ then

$$\widetilde{(v \circ u)}(z) = \tilde{u}(z) + \tilde{v}(z) + [\tilde{v}(u(z)) - \tilde{v}(z)].$$

Write $\tilde{u}(z) = (\tilde{u}_1(z), \cdots, \tilde{u}_{2n}(z))$ as a real function and $z = (x_1, \cdots, x_{2n}) \in \mathbb{R}^{2n}$. We have

$$(*) \quad \tilde{v}(u(z)) = \tilde{v}(z) + \sum_{i=1}^{2n} \tilde{u}_i(z) \int_0^1 \frac{\partial \tilde{v}}{\partial x_i}(x_1 + t\tilde{u}_1, \cdots, x_{2n} + t\tilde{u}_{2n}) dt.$$

We choose a sequence $\{z_m\}_{m=1}^{\infty}$ in $\mathbb{C}^n$ converges to the origin such that for at least an index $j \in \{1, \ldots, k\}$, all terms of the sequence $\{\tilde{h}_j(z_m)\}$ are different from zero. put $M_m = \max\{\|\tilde{h}_1(z_m)\|, \ldots, \|\tilde{h}_k(z_m)\|\}$. $\forall m \in \mathbb{N}, M_m > 0$ and $\forall i \in \{1, \ldots, k\}$ the sequence $\{\tilde{h}_i(z_m)/M_m\}$ converges to, say, b_i. If h is an arbitrary element of H such that $\tilde{h}(z_m)/M_m$ converges to b, then for each generator h_i the sequence $\{\widetilde{(h_i \circ h)}(z_m)/M_m\}$ converges to $b + b_i$. In fact from $(*)$

$$\lim_{m \to \infty} M_m^{-1}(\tilde{h_i}(h(z_m)) - \tilde{h}(z_m)) = 0.$$

Now define $f : H \to \mathbb{R}^{2n}$ as following:

$$h \mapsto \lim_{m \to \infty} \tilde{h}(z_m)/M_m.$$

f is well defined and non trivial homomorphism.

□

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Part 2. On the Zariski's multiplicity conjecture

4. ON THE ZARISKI'S MULTIPLICITY CONJECTURE

4.1. **Introduction.** Since the early 1960's, O. Zariski developed a comprehensive theory of equisingularity in codimension one. He initiated an equisingularity program with topological, differential geometrical and purely algebrical point of view and proposed a problem list in [19] as an extraction of many possible conjectures in singularity theory [20]. In this part we will be concerned with topological aspects of this program and more specifically with the so-called Zariski's multiplicity conjecture. We first recall some definitions. Let $f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be two germs of holomorphic functions and V_f and V_g be two germs at the origin of the hypersurfaces defined by $f^{-1}(0)$ and $g^{-1}(0)$ respectively. We suppose $0 \in \mathbb{C}^n$ is an isolated singularity of the functions. The *algebraic multiplicity* m_f of the germs of V_f or f is the order of vanishing of function f at $0 \in \mathbb{C}^n$ or equivalently is the order of the first nonzero leading term in the Taylor expansion of f

$$f = f_\nu + f_{\nu+1} + \cdots$$

where f_i is homogeneous polynomial of degree i.

Definition 1. We say V_f and V_g are *topologically equisingular* or topologically V-equivalent if there is a germ of homeomorphism $\phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ sending V_f onto V_g. More precisely, there are neighborhoods U and U' of $0 \in \mathbb{C}^n$ such that f and g are defined and a homeomorphism $\phi : U \to U'$ such that $\phi(f^{-1}(0) \cap U) = g^{-1}(0) \cap U'$ and $\phi(0) = 0$.

Zariski conjecture. *Topological equisingularity of germs of hypersurfaces implies equimultiplicity.*

A well known result by Burau [3] and Zariski [20] states an affirmative answer in the case of curves ($n = 2$). In higher dimension the conjecture is still open despite more than three decades effort to prove it.

We show that to answer affirmatively Zariski's question concerning the topological invariance of the multiplicity of continuously deformation $(f_t)_{t\in[0,1]}$ of complex hypersurfaces at the isolated singular point $0 \in \mathbb{C}^n$ it suffices to prove for every parameter $t \in [0, 1]$, there is line through the origin whose intersection with the tangent cone of f_s is the only origin for s near to t provided that $n \neq 3$ (THEOREM B). Also, in general case, the results of [5] and [16] are sharpened (THEOREM A).

4.2. **Preliminaries.** In [13], Milnor has opened a beautiful account on the complex hypersurfaces. The main achievement of it, is the Milnor fibration which we mention here. Also we recall briefly some generalities about complex hypersurfaces.

Let $f : U \subset \mathbb{C}^n \to \mathbb{C}$ be a holomorphic function on an open neighborhood of 0 in $\mathbb{C}^n$ and $f(0) = 0$. We denote $D_\epsilon = \{z | z \in \mathbb{C}^n : \|z\| \leq \epsilon\}$, $S_\epsilon = \partial D_\epsilon$, $H_0 = \{z \in \mathbb{C}^n | f(z) = 0\}$ and $d_z f = (\frac{\partial f}{\partial z_1}(z), \cdots, \frac{\partial f}{\partial z_n}(z))$.

We say the origin is an isolated singularity of f if $d_0 f = 0$ and $d_z f \neq 0$ for a neighborhood of $0 \in \mathbb{C}^n$ except 0.

Let $\mathcal{O}_n$ be the ring of germs of holomorphic functions defined in some neighborhood of $0 \in \mathbb{C}^n$ and let $< \frac{\partial f}{\partial z_1}, \cdots, \frac{\partial f}{\partial z_n} >$ be the ideal generated by the germs at $0 \in \mathbb{C}^n$ of derivative components of f. We define *Milnor number* μ of the holomorphic function f at $0 \in \mathbb{C}^n$ as

$$\mu = dim_{\mathbb{C}} \mathcal{O}_n / < \frac{\partial f}{\partial z_1}, \cdots, \frac{\partial f}{\partial z_n} >$$

This number is finite and nonzero if and only if $0 \in \mathbb{C}^n$ is an isolated singularity of f, a hypothesis which we will assume from now on. In this case μ coincides with the topological degree of the Gauss mapping induced by $d_z f$ on S_ϵ for ϵ small enough. The following lemma is useful to deal with the Milnor number.

Lemma 4.1. *Let $0 < \mu < \infty$. Given $\epsilon > 0$ there exists $\delta > 0$ such that for any $c \in \mathbb{C}^n$ with $\|c\| < \delta$ the number of solutions of the equation $d_z f = c$ in the ball D_ϵ is at most μ. Moreover, if $p_1, \cdots, p_m$, $m \leqslant \mu$, are such solutions, then $\sum_{i=1}^m \mu(f - \sum_{i=1}^n z_i c_i, p_i) = \mu$.*

The following theorem is called Milnor fibration theorem:

Theorem 4.2. *For ϵ small enough the mapping $\psi_\epsilon : S_\epsilon \setminus H_0 \to S^1$ defined by $\psi_\epsilon(z) = f(z)/\|f(z)\|$ is a smooth fibration which is called Milnor fibration. Moreover the fibers of ψ_ϵ have the homotopy type of a bouquet of μ (the Milnor number of the holomorphic function f at $0 \in \mathbb{C}^n$) spheres of dimension $n-1$.*

Also we call the number of spheres, the number of *vanishing cycles* of f at 0. The following theorem is due to Lê [11]:

Theorem 4.3. *If V_f and V_g are topologically equisingular then the number of vanishing cycles at 0 of f and g are the same.*

Now we recall some definitions and facts about deformations of functions. A *deformation* of an holomorphic function $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is a family $(f_t)_{t \in [0,1]}$ of germs of holomorphic functions with isolated singularities at $0 \in \mathbb{C}^n$ such that the coefficients are holomorphic functions in $t \in [0, 1]$. The *jump of the family* (f_t) is $\mu(f_0) - \mu(f_t)$, where μ the Milnor number at the origin. It is independent of t for t small enough, moreover by the upper semi-continuity of μ this number is a non-negative integer.

We use frequently the following theorem proved by Lê and Ramanujam [12]:

Theorem 4.4. *Let $(f_s)_{s \in [0,1]}$ be a C^∞ family of hypersurfaces having an isolated singularity at the origin. If the Milnor number of singularity does not* *change then the topological type of singularity does not change too provided that $n \neq 3$.*

In [9], theorem (4.4) is generalized which we recall in section (4.5). Finally we recall an interesting result of P. Samuel [17]:

Theorem 4.5. *Every germ V_f is analytically equivalent with V_g in which g is a polynomial.*

Moreover we may choose a polynomial g with cutting the Taylor expansion of f in somewhere. By the theorem of (4.5) it is enough to consider polynomials to prove the conjecture.

4.3. **The topological right equivalent complex hypersurfaces.** In this section we recall several ways to define a topological type of a holomorphic function and relations between them according to [8], [15] and [18].

Let $f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be two germs of holomorphic functions with an isolated singularity at the origin.

Definition 2. f and g are topologically right equivalent if there is a germ of homeomorphism $\varphi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ satisfying $f = g \circ \varphi$

Definition 3. f and g are topologically right-left equivalent if there are germs of homeomorphism $\varphi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ and $\psi : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ satisfying $f = \psi \circ g \circ \varphi$

Put $V_f := f^{-1}(0)$. By [13], $S_\varepsilon^{2n-1} \cap V_f$ is a smooth $(2n-3)$-dimensional manifold for $\varepsilon > 0$ sufficiently small. The pair $(S_\varepsilon^{2n-1}, S_\varepsilon^{2n-1} \cap V_f)$ is called the link of the singularity of f.

Definition 4. f and g are link equivalent if $(S_\varepsilon^{2n-1}, S_\varepsilon^{2n-1} \cap V_f)$ is homeomorphic to $(S_\varepsilon^{2n-1}, S_\varepsilon^{2n-1} \cap V_g)$ for all sufficiently small ε.

By the definitions, the right equivalence implies the right-left equivalence, which in turn implies the V-equivalence. The outstanding result, obtained by King [8] in $n \neq 3$ and by Perron [15] in $n = 3$, is the following:

Theorem 4.6. *The topological V-equivalence implies topologically right-left equivalence. Moreover if f and g are topologically right-left equivalent then g is topologically right equivalent either to f or to $\overline{f}$, the complex conjugate of f.*

Using theorem (4.6), Risler and Trotman in [16] proved that right-left bilipschitz equivalence implies equimultiplicity.

Since $(D_\varepsilon^{2n-1}, D_\varepsilon^{2n-1} \cap V_f)$ is homeomorphic to the cone cover over the link $(S_\varepsilon^{2n-1}, S_\varepsilon^{2n-1} \cap V_f)$ ([13]), the link equivalence implies the V-equivalence. Conversely Saeki [18] showed that the topological V-equivalence implies the link equivalence. This means that there is a homeomorphism $\varphi_1 : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$, sending V_f onto V_g and such that $|\varphi_1(z)| = |z|$. By theorem (4.6) there is a homeomorphism $\varphi_2 : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that $|f(z)| = |g \circ \varphi_2(z)|$. Comte, Milman and Trotman [5] showed that if there is a germ of homeomorphism $\varphi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ having *simultaneously* the properties of φ_1 and of φ_2 then the multiplicity conjecture is true. In fact they proved that it suffices to assume that there are positive constants A, B, C, and D such that:

(1) $A|z| \leq |\varphi(z)| \leq B|z|$, for all z near 0, and

(2) $C|f(z)| \leq |g \circ \varphi(z)| \leq D|f(z)|$, for all z near 0.

Now we prove that it is enough to assume the conditions (1) and (2) are valid for some special sequences converge to the origin. Given two holomorphic function germs $f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$, by an analytic change of coordinates one may assume that the z_1-axis is not contained in the tangent cones $C(V_f), C(V_g)$ (respectively the zero set of first non zero jet of f and g), so that $f(z_1, 0, \cdots, 0) \neq 0$ and $g(z_1, 0, \cdots, 0) \neq 0$ for a neighborhood of 0 in the z_1-axis, and by theorem (4.5) one may assume f and g are polynomials. In this situation we have the following:

THEOREM A. *Suppose there are a germ of homeomorphism* $\varphi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ *with inverse* ψ *and positive constants* A, B, C, *and* D *and two sequences* w_m *and* ${w'}_m$ *in the* z_1*-axis which converge to the origin with the following properties:*

(i) $|\psi({w'}_m)| \leq A|w'_m|$, $|\varphi(w_m)| \leq B|w_m|$ *and*

(ii) $C|f(w_m)| \leq |g \circ \varphi(w_m)|$, $D|g({w'}_m)| \leq |f \circ \psi({w'}_m)|$

then $m_f = m_g$.

The conditions (i) and (ii) are slightly weaker than conditions (1) and (2) above.

Proof. Write

$$f(z) = f_k(z) + f_{k+1}(z) + \cdots + f_{k+r}(z),$$
$$g(z) = g_l(z) + g_{l+1}(z) + \cdots + g_{l+s}(z).$$

f_i and g_j are homogeneous parts of degree i and j of f and g respectively. f_k and g_l are not identically zero. We want to prove $k = l$. By contrary suppose $l > k$. The other case is similar. Let $w_1 = (z_1, 0, \cdots, 0)$ and $w_m = (t_m z_1, 0, \cdots, 0)$, $t_m \neq 0$ and converges to the origin. Also write g in the following form:

$$g(z) = \sum_{j=l}^{l+s} \sum_{|\beta|=j} C_\beta^j z^\beta,$$

where $z = (z_1, \cdots, z_n)$ and $\beta = (\beta_1, \cdots, \beta_n)$, $\beta_i \in \mathbb{N} \cup \{0\}$. Now we have

$$f(w_m) = f_k(w_m) + f_{k+1}(w_m) + \cdots + f_{k+r}(w_m) \ or$$

$$f(w_m) = t_m^k [f_k(w_1) + t_m f_{k+1}(w_1) + \cdots + t_m^r f_{k+r}(w_1)]$$

and

$$|(g\circ\varphi)(w_m)| \leq \sum_{j=l}^{l+s}\sum_{|\beta|=j}|C_\beta^j|B^j|t_m z_1|^j$$

by (i). Now we use condition (ii). It is:

$$C|f(w_m)| \leq |g\circ\varphi(w_m)| \ or$$

$$C|t_m^k[f_k(w_1)+t_m f_{k+1}(w_1)+\cdots+t_m^r f_{k+r}(w_1)]| \leq \sum_{j=l}^{l+s}\sum_{|\beta|=j}|C_\beta^j|B^j|t_m z_1|^j.$$

Divided two sides of above inequality by $|t_m^k|$ we obtain the following:

$$C|f_k(w_1)+t_m f_{k+1}(w_1)+\cdots+t_m^r f_{k+r}(w_1)| \leq \sum_{j=l}^{l+s}\sum_{|\beta|=j}|C_\beta^j|B^j|t_m z_1|^{j-k},$$

or

$$C|f_k(w_1)| \leq \sum_{j=l}^{l+s}\sum_{|\beta|=j}|C_\beta^j|B^j|t_m z_1|^{j-k} + C|t_m^1 f_{k+1}(w_1)+\cdots+t_m^r f_{k+r}(w_1)|.$$

When t_m goes to zero, the right hand of the last inequality goes to zero but the left hand is a positive constant. This contradiction shows $l = k$. □

4.4. **The zeta function of a monodromy.** Now we recall some features from [1] and [2]. Let $f : \mathbb{C}^n \to \mathbb{C}$ be a polynomial so that $f(0) = 0$ and consider the hypersurface defined by it, $V_f = f^{-1}(0)$. The map

$$\pi : z \in S_\varepsilon^{2n-1} \backslash V_f \longmapsto arg(f(z)) \in S^1,$$

defines a Milnor fibration of the hypersurface V_f at the origin. The fiber $F_\theta = \pi^{-1}(\theta)$, $\theta \in S^1$, is a $2(n-1)$-dimensional differential manifold and the characteristic homeomorphism of this fibration

$$h : F_\theta \to F_\theta$$

is the geometric monodromy of V_f at the origin. By definition the zeta function of h is the following:

$$Z(t) = \prod_{q\geq 0}\{\det(Id^* - th^*; H^q(F_\theta,\mathbb{C}))\}^{(-1)^{q+1}}.$$

When the origin of $\mathbb{C}^n$ is an isolated singular point of V_f, one has

$$H^q(F_\theta,\mathbb{C}) = \begin{cases} \mathbb{C} & q = 0 \\ 0 & q \neq 0, q \neq n \\ \mathbb{C}^\mu & q = n, \end{cases}$$

where μ is the Milnor number of f and therefore the characteristic polynomial $\Delta(t)$ of the monodromy at degree n is deduced from the zeta function $Z(t)$ by the formula

$$\Delta(t) = t^{\mu}[\frac{t-1}{t} Z(\frac{1}{t})]^{(-1)^{n+1}}.$$

For an entire $k \geq 1$; let the integer number

$$\Lambda(h^k) = \sum_{q \geq 0} (-1)^q \mathrm{Trace}[(h^*)^k; H^q(F_\theta, \mathbb{C})]$$

the Lefschetz number of the k-th power of h. Let $s_1, s_2, \cdots$ be the integers defined by the following recurrence relations:

$$\Lambda(h^k) = \sum_{i|k} s_i,$$

$k \geq 1$, then the zeta function of h is given by

$$Z(t) = \prod_{q \geq 0} (1 - t^i)^{\frac{-s_i}{i}}.$$

The Lefschetz numbers $\Lambda(h^k)$ are topological invariants of the singularity of V_f, therefore the integers $s_1, s_2, \cdots$ are topological invariants.

Remark 4.7. In [2], A'Campo has calculated $\Lambda(h)$ as following:

$$\Lambda(h) = \begin{cases} 0 & \text{if } d_0 f = 0 \\ 1 & \text{if } d_0 f \neq 0. \end{cases}$$

This tells us that if f is regular and g is singular at the origin there is no topological equivalence between germs of V_f and V_g at the origin.

Remark 4.8. More generally Deligne has explained in a letter to A'Campo (see [1], [7]) that

$$\Lambda(h^k) = 0, \text{ if } 0 < k < \text{ multiplicity of } V_f \text{ at the origin.}$$

The Lefschetz numbers $\Lambda(h^k)$ are topological invariants of the singularity of V_f, therefore the entires $s_1, s_2, \cdots$ are topological invariants. A'Campo discovered the meaning of the topological invariants $s_1, s_2, \cdots$ as following:

Let $\pi : X \to \mathbb{C}^n$ be a proper modification such that in all points of $S := \pi^{-1}(0)$, the divisor $V'_f := \pi^{-1}(V_f)$ has normal crossings. Such a local resolution of $(\mathbb{C}^n, V_f)$ at the origin exists by the theorem of resolution of singularities due to Hironaka [6]. For every $m \in \mathbb{N}$, let S_m be all points $s \in S$ such that the equation of V'_f at s is of the form $z_1^m = 0$ for a local coordinate z of X at s and denote by $\chi(S_m)$ the Euler-Poincaré characteristic of S_m. A'Campo proved that $s_m = m\chi(S_m)$. More precisely:

Theorem 4.9. *One has*

(1) $\Lambda(h^k) = \sum_{m|k} m\chi(S_m)$, $k \geq 1$,

(2) $\Lambda(h^0) = \chi(F_\theta) = \sum_{m\geq 1} m\chi(S_m)$,

(3) $\mu = \dim H^{n-1}(F_\theta, \mathbb{C}) = (-1)^{n-1}[-1 + \sum_{m\geq 1} m\chi(S_m)]$.

Therefore the numbers $\chi(S_m)$ don't depend on the chosen resolution and are topological invariants of the singularity. As a consequence we have the following result that may be useful for resolving the multiplicity conjecture.

Proposition 4.10. *If $f(z) = f_k(z) + f_{k+1}(z) + \cdots + f_{k+r}(z)$, $g(z) = g_l(z) + g_{l+1}(z) + \cdots + g_{l+s}(z)$ and $k + r < l$ then there is no topological equivalence between germs of V_f and V_g at the origin.*

Proof. Let h_1 and h_2 be the monodromies associated to f and g and $s_1, s_2, \cdots$ and $s'_1, s'_2, \cdots$ the two related sequences of f and g respectively as above. If there exists such an equivalence then $\Lambda(h_1^j) = \Lambda(h_2^j) = 0$ and $s_j = s'_j = 0$ for every j. Hence $\mu_f = \mu_g = (-1)^{n-1}[-1 + \sum_{j\geq 1} s_j] = (-1)^n$. If n is odd this is impossible and if n is even, then $\mu_f = \mu_g = 1$. In this case $k = l = 2$. Contradiction! □

The second result is the following [1], [7]:

Theorem 4.11. *Given two germs of hypersurfaces V_f and V_g. Let $\mathbb{P}C(V_f)$, respectively $\mathbb{P}C(V_g)$, denote the projectivized tangent cone which is a subvariety of $\mathbb{C}P^{n-1}$. If $\chi(\mathbb{C}P^{n-1} \setminus \mathbb{P}C(V_f)) \neq 0$ and $\chi(\mathbb{C}P^{n-1} \setminus \mathbb{P}C(V_g)) \neq 0$, then topological equisingularity of V_f and V_g implies $m_f = m_g$.*

The key point of the proof is that: if $\chi(\mathbb{C}P^{n-1} \setminus \mathbb{P}C(V_f)) \neq 0$ then by theorem (4.9), $m_f = inf\{s \in \mathbb{N} | \Lambda(h^s) \neq 0\}$.

It is unknown whether $\chi(\mathbb{C}P^{n-1} \setminus \mathbb{P}C(V_f))$ is a topological invariant or not.

Example 4.12. Let $g = z_1^l + z_2^l + \cdots + z_n^l$, $V_g = g^{-1}(0) \subset \mathbb{C}^n$ and $C(V_g) \subset \mathbb{C}P^{n-1}$ and F_θ be the fiber of the Milnor fibration of g at the origin. By (4.15) in the appendix we have

$$\mu_g = (l-1)^n.$$

With one explosion at the origin, the singularity of g may be resolved and then we may apply the theorem (4.9): $S = \mathbb{C}P^{n-1}$ and

$$S_m = \begin{cases} \phi & \text{if } m \neq l \\ \mathbb{C}P^{n-1} \setminus V_g & \text{if } m = l. \end{cases}$$

By theorem (4.9),

$$\mu_g = (-1)^{n-1}[-1 + l\chi(S_l)].$$

The numbers $\chi(S_l)$ and $\chi(C(V_g))$ are related by

$$\chi(S_l) + \chi(C(V_g)) = \chi(\mathbb{C}P^{n-1}) = n.$$

Therefore we obtain the following well known formula

$$\chi(C(V_g)) = n - \frac{1-(1-l)^n}{l}.$$

Example 4.13. Let $\mathcal{A}$ be the set of all holomorphic functions g such that $0 \in \mathbb{C}^n$ is an isolated singularity for the first nonzero homogeneous part of the Taylor expansion of g. Then by an argument (see (4.17) in the appendix) the origin is an isolated singularity of g. Let $g \in \mathcal{A}$ with algebraic multiplicity l and the leading term g_l. Since g and g_l have the same projevtivized tangent cones and V_{g_l} is topologically equivalent to $V_{z_1^l+z_2^l+\cdots+z_n^l}$ then by the previous example

$$\chi(\mathbb{C}P^{n-1} \setminus \mathbb{P}C(V_g)) = \frac{1-(1-l)^n}{l} \neq 0.$$

Hence by theorem (4.11) any topological equivalence between two elements of $\mathcal{A}$ preserves multiplicities.

Still it is unknown whether there is any topological equivalence between $g \in \mathcal{A}$ and $f \notin \mathcal{A}$. By contrary if there exists such an equivalence then $k < l$, where k and l are the multiplicities of f and g respectively. The reason is that by (4.16) the Milnor number $\mu_f > (k-1)^n$ and $\mu_g = (l-1)^n$ and by theorem (4.3), Milnor number is a topological invariant. Therefore it remains to show that: Let $g = z_1^l + z_2^l + \cdots + z_n^l$ and $f = f_k + \cdots + f_{k+r}$ with $k < l$ and $k + r \geq l$ then the germs V_f and V_g at the origin are not topologically equisingular.

4.5. **On the deformation of complex hypersurfaces.** Let us, instead of dealing with a pair of hypersurfaces, consider families of hypersurfaces, V_{f_t}, all having an isolated singular point at the origin and depending continuously in $t \in [0,1]$ and $f_0 = f$ and $f_1 = g$. We denote by $C(V_{f_t})$, the tangent cone at 0 of V_{f_t}, that is, the zero set of the initial polynomial of f_t. H. King generalized theorem (4.4) as follows:

Theorem 4.14. *Suppose $f_t : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$, $t \in [0,1]$ is a continuous family of holomorphic germs with the same Milnor number and $n \neq 3$. Then there is a continuous family of germs of homeomorphisms $h_t : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ so that $f_0 = f_t \circ h_t$*

Now we have the following result:

THEOREM B. *If for every $t_0 \in [0,1]$ there exist a neighborhood I_{t_0} of t_0 in $[0,1]$ and a line L_{t_0} through 0 in $\mathbb{C}^n$ such that $L_{t_0} \cap C(V_{f_s}) = \{0\}$ for $s \in I_{t_0}$, then topological equisingularity of the family implies equimultiplicity provided that $n \neq 3$.*

Proof. By theorem (4.14) there exists a continuous family of homeomorphisms φ_t such that $f_t = f \circ \varphi_t$. Therefore for every $t_0 \in [0,1]$ we may write

$$f_s = f_{t_0} \circ \varphi_{t_0 s}$$

where $\varphi_{t_0s} = \varphi_{t_0}^{-1} \circ \varphi_s$.

Since f_{t_0} is uniformly continuous on a compact small ball $B_r \subset \mathbb{C}^n$ around 0, there exists $\eta > 0$ such that, for anf $z, w \in B_r$,

$$|z - w| < \eta \implies |f_{t_0}(z) - f_{t_0}(w)| < \min_{u \in S_\delta} |f_{t_0}(u)|,$$

where S_δ is the boundary of $\overline{D_\delta}$, the closed disc with radius $\delta < r/2$ in L_{t_0} around 0. Let $\varepsilon := \min\{\eta, \delta\}$. By continuity of φ_s, if I_{t_0} is sufficiently small then $|\varphi_{t_0s}(z) - z| < \varepsilon$ for $s \in I_{t_0}$. Then for all z in the closed ball $B_\delta \subset \mathbb{C}^n$, $\varphi_{t_0s}(z) \in B_r$ and

$$|f_{t_0}(z) - f_{t_0} \circ \varphi_{t_0s}(z)| < \min_{u \in S_\delta} |f_{t_0}(u)|.$$

In particular for all $z \in S_\delta$ we have

$$|f_{t_0}(z) - f_s(z)| < |f_{t_0}(z)|, \text{ for } s \in I_{t_0}.$$

By hypothesis $L_{t_0} \cap C(V_{f_s}) = \{0\}$ for $s \in I_{t_0}$, then m_{f_s} is the order at 0 of $f_s|_{L_{t_0}}$ for $s \in I_{t_0}$. By Roché's theorem, [10, Ch.VI], $f_{t_0}|_{L_{t_0}}$ and $f_s|_{L_{t_0}}$ have the same number of zeros, counted with their multiplicities in the interior of $\overline{D_\delta}$. As $f_{t_0}|_{L_{t_0}}$ and $f_s|_{L_{t_0}}$ vanish only at 0 on $\overline{D_\delta}$, the orders at 0 of $f_{t_0}|_{L_{t_0}}$ and $f_s|_{L_{t_0}}$ are equal. So $m_{f_{t_0}} = m_{f_s}$ for $s \in I_{t_0}$. This tells us that the multiplicity of the deformation is constant. □

4.6. **Appendix.** Let $\mathcal{A}$ be the set of all holomorphic functions f such that $0 \in \mathbb{C}^n$ is an isolated singularity not only for f but also for the first nonzero homogeneous polynomial of the Taylor expansion of the f. Actually if the origin is an isolated singularity of the leading term of f then the same holds for f. In this case we have the following relation between multiplicity and Milnor number of f:

Proposition 4.15.

$$\mu_f = (m_f - 1)^n.$$

Proof. Let

$$d_z f = \sum_{j=m_f-1}^{\infty} F_j(z)$$

be the Taylor expansion of the derivative of f at z where $F_j(z)$ is the homogeneous polynomial of degree j. For simplicity we put $F(z) := d_z f$. There exists $N > 0$ such that $F(z)$ is defined in the ball $B_{1/N}(0)$. Let $k = inf\{\|F_{m_f-1}(z)\|; \|z\| = 1/2N\}$ and $k_j = sup\{\|F_j(z)\|; \|z\| = 1/2N\}$. We have $\|F_j(z)\| \leqslant k_j(2N)^j\|z\|^j$ for $j \geqslant m_f$ and $\|F_{m_f-1}(z)\| \geqslant k(2N)^{m_f-1}\|z\|^{m_f-1}$. These imply that

$$\|F(z)\| \geqslant (k - \sum_{j \geqslant m_f} (2N\|z\|)^{j-m_f+1})(2N\|z\|)^{m_f-1}.$$

By the convergence of series, there exists $\epsilon > 0$ such that for $\|z\| < \epsilon$ we have $k - \sum_{j \geqslant m_f} (2N\|z\|)^{j-m_f+1} \geqslant k/2$ and so $\|F(z)\| \geqslant kN\|z\|^{m_f-1}$ for $\|z\| < \epsilon$. Let $0 < r < \epsilon$ and consider the homotopy $G : [0,1] \times \mathbb{S}_r^{2n-1} \to \mathbb{S}^{2n-1}$ given by

$$G(t,z) = \frac{F(tz)}{\|F(tz)\|} = \sum_{j \geqslant m_f - 1} t^j F_j(z) / \| \sum_{j \geqslant m_f - 1} t^j F_j(z) \|$$
$$= \sum_{j \geqslant m_f - 1} t^{j-m_f+1} F_j(z) / \| \sum_{j \geqslant m_f - 1} t^{j-m_f+1} F_j(z) \|.$$

Then $G(1,z) = F(z)/\|F(z)\|$ and $G(0,z) = F_{m_f-1}(z)/\|F_{m_f-1}(z)\|$. Therefore $\mu_f = \deg G(1,z) = \deg G(0,z)$. Now, if $c \neq 0$ is a regular value of the map $F_{m_f-1}(z) = c$ is exactly $(m_f-1)^n$. So by the lemma (4.1) we have $\deg G(0,z) = (m_f-1)^n$. □

Remark 4.16. If $0 \in \mathbb{C}^n$ is not an isolated singularity of the first nonzero homogeneous polynomial then $\mu_f > (m_f-1)^n$.

The following proposition is true in any dimension. But the following proof is based on the theorem of Lé and Ramanujam and the hypothesis $n \neq 3$ comes from there.

Proposition 4.17. *The germ at the origin of the hypersurface defined by an element $f \in \mathcal{A}$ with the algebraic multiplicity k is topologically equivalent with the germ at the origin of the hypersurfaces defined by $z_1^k + \cdots + z_n^k$ provided that $n \neq 3$.*

Proof. Let

$$f = f_k + f_{k+1} + \cdots$$

be the Taylor expansion of f, where f_i is homogeneous polynomial of degree i. We consider the following family $(H_t)_{t \in [0,1]} \in \mathcal{A}$:

$$H_t = f_k + t f_{k+1} + t^2 f_{k+2} + \cdots$$

The origin is an isolated singularity for $H_0 = f_k$, so we can choose r such that $d_z f_k \neq 0$ for $0 < \|z\| < r$. Therefor for t close enough to 0, $d_z H_t(z) \neq 0$ on $\|z\| = r$. By the lemma [4.15] the Milnor number $\mu_{H_t} = (k-1)^n$. So the topological type of H_t for t near 0 does not change. Now we may write $f(tz_1, \cdots, tz_n) = t^k f_k + t^{k+1} f_{k+1} + \cdots = t^k H_t(z)$ or $H_t(z) = t^{-k} f(tz_1, \cdots, tz_n)$. Hence $H_t(z)$ derives from f by the coordinate transformation $(z_1, \cdots, z_n) \mapsto (tz_1, \cdots, tz_n)$. Therefore the topological type of f_k and f are the same. Also the above argument shows that if 0 is an isolated singularity of f_k then the same is true for f.

By a symbol $f \sim g$ between two germs of holomorphic functions at the origin we mean V_f and V_g, two germs of hypersurfaces defined by f and g

respectively, are topological equivalent. Obviously we have $f \sim cf$ for any nonzero constant $c \in \mathbb{C}$.

Now our task is to show $P(z) \sim (z_1^k + \cdots + z_n^k)$ where $P(z)$ is a homogeneous polynomial of degree k.

Claim: *There is a non zero complex number α such that 0 is an isolated singularity of $F_t(z) := (1-t)(z_1^k + \cdots + z_n^k) + t\alpha P(z)$ for $t \in [0,1]$.*

By the above claim $F_t(z)$ defines a family of constant Milnor number and by theorem (3.4), $(z_1^k + \cdots + z_n^k) \sim \alpha P(z) \sim P(z)$.

The proof of claim: The partial derivatives of $F_t(z)$ form a system of bihomogeneous polynomials of bidegree $(1, k-1)$:

$$\frac{\partial F_t}{\partial z_1} = k(1-t)z_1^{k-1} + t\alpha \frac{\partial P}{\partial z_1}$$

$$\vdots$$

$$\frac{\partial F_t}{\partial z_n} = k(1-t)z_n^{k-1} + t\alpha \frac{\partial P}{\partial z_n}$$

and $V := \text{Zero}(\frac{\partial F_t}{\partial z_1}, \cdots, \frac{\partial F_t}{\partial z_n})$ is an algebraic subset of $\mathbb{C}P(1) \times \mathbb{C}P(n-1)$. Now consider the projection $\pi : \mathbb{C}P(1) \times \mathbb{C}P(n-1) \to \mathbb{C}P(1)$. Image of V, $\pi(V)$, is a Zariski-closed subset of $\mathbb{C}P(1)$ (see for instance [14] Pg. 33). Since $F_t(z)$ for $t = 0$ has the isolated singularity, $(1:0)$ is not in the $\pi(V)$. Therefore $\pi(V)$ is finite and there are infinitely many lines in the complement of $\pi(V)$ in $\mathbb{C}P(1)$. Since $P(z)$ has an isolated singularity at $0 \in \mathbb{C}^n$ we may choose lines passing through the origin. This means that there is α such that the claim is true for every $t \in \mathbb{R}$.

□

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