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Instituto Nacional de Matemática Pura e Aplicada

Luiz Gustavo Farah Dias

On some Boussinesq-type equations

PhD Thesis

Thesis presented to the Post–graduate Program in Mathematics at IMPA as partial fulfillment of the requirements for the degree of Doctor in Philosophy in Mathematics

Adviser: Prof. Felipe Linares

Rio de Janeiro
February 2008

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Abstract

The purpose of this work is to investigate several questions about the initial value problem (IVP) associated to some Boussinesq-type equations.
In Chapter 1, we study the long-time behavior of solutions (without smallness assumption) of the initial-value problem for a generalized Boussinesq equation. Here we do the reciprocal problem of the scattering theory, we construct a solution $\vec{u}$ with a given scattering state $B(t)\vec{h}$, where $B(\cdot)$ is the unitary group associated to the linear system and $\vec{h}$ is given in suitable spaces.
Next, we study the local well-posedness of the initial-value problem for the nonlinear generalized Boussinesq equation with data in $H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n)$, $s \geq 0$. Under some assumption on the nonlinearity f, local existence results are proved for $H^s(\mathbb{R}^n)$-solutions using an auxiliary space of Lebesgue type. Furthermore, under certain hypotheses on s, n and the growth rate of f these auxiliary conditions can be eliminated. All these results are proved in Chapter 2.
In the sequel, we study the local well-posedness of the (IVP) for the nonlinear "good" Boussinesq equation with data in Sobolev spaces $H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ for negative indices s. Local well-posedness for $s > -1/4$ and ill-posedness (in the sense that the flow-map data solution cannot be C^2 at the origin) for $s < -2$ are proved in Chapters 3 and 4, respectively.
The last chapter is devoted to study the (IVP) for the nonlinear Schrödinger-Boussinesq system. Local existence results are proved for initial data in Sobolev spaces of negative indices. Global results are also obtained with data in $L^2(\mathbb{R}) \times L^2(\mathbb{R}) \times H^{-1}(\mathbb{R})$.

Keywords

Boussinesq equation. Scattering. Large data. Local well-posedness. Ill-posedness. Schrödinger-Boussinesq system. Global well-posedness.

Contents

"Anyone who has never made a mistake has never tried anything new."

Albert Einstein, *1879-1955.*

Introduction

In this work, we consider the Boussinesq Equation (NLB)

$$\begin{cases} u_{tt} - \Delta u + \Delta^2 u + \Delta f(u) = 0, & x \in \mathbb{R}^n, t > 0, \\ u(x,0) = \phi, \quad u_t(x,0) = \psi \end{cases} \tag{0.1}$$

where f is a nonlinear function and ϕ and ψ are real valued functions.

Equations of this type in one dimension, but with the opposite sign in the bilaplacian, were originally derived by Boussinesq [8] in his study of nonlinear, dispersive wave propagation. We should remark that it was the first equation proposed in the literature to describe this kind of physical phenomena. The equation (0.1) was also used by Zakharov [44] as a model of nonlinear string. Finally, Falk *et al* [16] derived an equivalent equation in their study of shape-memory alloys.

In one dimension, equation (0.1) can also be rewritten in the following equivalent system form

$$\begin{cases} u_t &= v_x \\ v_t &= \left(u - u_{xx} - f(u)\right)_x, \quad x \in \mathbb{R}, t > 0. \end{cases} \tag{0.2}$$

Since the generalization to higher dimensions of this system is not straightforward, we, in fact, will work with the system (SNLB)

$$\begin{cases} u_t &= \Delta v \\ v_t &= u - \Delta u - f(u), \quad x \in \mathbb{R}^n, t > 0. \end{cases} \tag{0.3}$$

Concerning the local well-posedness question in one dimension, several results has been obtained for the equation (0.1). Hereafter, we refer to the expression "local well-posedness" in the sense of Kato, that is, the solution uniquely exists in a certain time interval (unique existence), the solution has the same regularity as the initial data in a certain time interval (persistence), and the solution varies continuously depending upon the initial data (continuous dependence). Global well-posedness requires that the same properties hold for all time $t > 0$.

Using Kato's abstract theory for quasilinear evolution equation, Bona and Sachs [5] showed local well-posedness for the system (0.2), where $f \in C^\infty$ and initial data $\phi \in H^{s+2}(\mathbb{R})$, $\psi \in H^{s+1}(\mathbb{R})$ with $s > \frac{1}{2}$. Tsutsumi and Matahashi [39] established similar result when $f(u) = |u|^{p-1}u$, $p > 1$ and $\phi \in H^1(\mathbb{R})$, $\psi = \chi_{xx}$ with $\chi \in H^1(\mathbb{R})$. These results were improved by Linares [29]. Working directly with the equation (0.1) he proved local well-posedness when $f(u) = |u|^{p-1}u$, $p > 1$, $\phi \in H^1(\mathbb{R})$, $\psi = h_x$ with $h \in L^2(\mathbb{R})$ and $f(u) = |u|^{p-1}u$, $1 < p < 5$, $\phi \in L^2(\mathbb{R})$, $\psi = h_x$ with $h \in H^{-1}(\mathbb{R})$. Moreover, assuming smallness in the initial data, it was proved that these solutions can be extended globally in $H^1(\mathbb{R})$. The main tool used in [29] was the Strichartz estimates satisfied by solutions of the linear problem.

Another problem studied in the context of the Boussinesq equation is scattering of small amplitude solutions. Roughly speaking, the problem is as follows: given a initial data with small norm in a suitable space, the outcoming solution $u(t)$ is global in time and there exists initial data $V_\pm$ such that

$$\lim_{t \to \pm\infty} \|u(t) - u_\pm(t)\|_X = 0$$

where $u_\pm(t)$ is the solution of the linear problem associated to the Boussinesq equation (that is, $f \equiv 0$ in (0.1)) with initial data $V_\pm$ and X is an appropriate functional space.

This question was investigated by several authors, see, for instance, Linares and Scialom [32], Liu [33] for results in one dimension and Cho and Ozawa [11] for arbitrary dimension. We should remark that in all the situations above we need some regularity on the initial data to obtain the scattering.

In the present work, we are interested with the reciprocal problem, that is, to construct solutions to (0.1) with a given asymptotic behavior. In other words, given a profile V in a suitable space let $u_V(t)$ be the solution of the linear problem with initial data V. Then there exists a solution $u(t)$ of (0.1), defined for large enough times, such that

$$\lim_{t \to \infty} \|u(t) - u_V(t)\|_W = 0 \tag{0.4}$$

in some functional space W. We refer to this problem as the construction of a wave operator.

In Chapter 1, we construct a wave operator for initial data V in appropriate functional spaces. Our scheme of proof used is based in the one implemented by

Côte [15] in the context of the generalized Korteweg-de Vries equation. The main interesting point in these results is that the smallness assumption can be removed in this context and we are able to construct a wave operator for any possible large profile V in certain functional spaces.

In Chapter 2, we will consider first the local well-posedness problem. Using the integral equation (0.22) below, we prove that (0.1) is locally well-posed for initial data $\phi \in H^s(\mathbb{R})$, $\psi = \eta_{xx}$ with $\eta \in H^s(\mathbb{R})$ and $s \geq 0$. To do this, we observe that the integral formulation (0.22) is very similar to the Schrödinger equation's structure. Therefore applying well known results for this last equation we construct auxiliary spaces such that the integral equation (0.22) is stable and contractive in these spaces. By Banach's fixed point theorem we obtain a unique fixed point to the integral equation in these auxiliary spaces. A natural question arise in this context. Is it possible to remove these auxiliary spaces? In other words, is it possible to prove that the uniqueness holds, in fact, in the whole space $C([0,T]; H^s(\mathbb{R}^n))$? If the answer for these two questions is yes, then we say that (0.1) is unconditionally well-posed in $H^s(\mathbb{R}^n)$.

This question was introduced by Kato [24] in the context of Schrödinger equation and further developed by Furioli and Terraneo [18]. Based in these results, we establish unconditional well-posedness for the generalized Boussinesq equation (0.1), under certain hypotheses on s, n and the growth rate of f.

Another problem considered here is the local well-posedness for the Boussinesq equation (0.1) in one dimension and $f(u) = u^2$. This equation is called "good" Boussinesq equation. For future reference we rewrite this equation below

$$\begin{cases} u_{tt} - u_{xx} + u_{xxxx} + (u^2)_{xx} = 0, & x \in \mathbb{R}, t > 0, \\ u(0) = \phi;\ u_t(x,0) = \psi_x. \end{cases} \tag{0.5}$$

We should notice that all the local well-posedness results found in [5], [39] and [29] also hold for (0.5). A natural question arises in this context: is it possible to prove local well-posedness for less regularity data then L^2?

In this work, we answer partially this question, showing both local well-posedness and ill-posedness for the "good" Boussinesq equation (0.5) with initial data in Sobolev spaces with negative indices of s.

The local well-posedness for dispersive equations with quadratic nonlinearities has been extensively studied in Sobolev spaces with negative indices. The proof of these results is based in the Fourier restriction norm approach introduced by Bourgain [6] in his study of the nonlinear Schrödinger equation (NLS)

$$iu_t + u_{xx} + u|u|^{p-2} = 0, \text{ with } p \geq 3 \tag{0.6}$$

and the Korteweg-de Vries equation (KdV)

$$u_t + u_{xxx} + u_x u = 0. \tag{0.7}$$

This method was further developed by Kenig, Ponce and Vega in [26] for the KdV equation (0.7) and [27] for the quadratics nonlinear Schrödinger equations

$$iu_t + u_{xx} + u^2 = 0 \tag{0.8}$$

$$iu_t + u_{xx} + u\bar{u} = 0 \tag{0.9}$$

$$iu_t + u_{xx} + \bar{u}^2 = 0, \tag{0.10}$$

where $\bar{u}$ denotes the complex conjugate of u, in one spatial dimension and in spatially continuous and periodic case. Using this method, in Chapter 3, we improve the result in [29], proving local well-posedness for the nonlinear "good" Boussinesq equation (0.5) for initial data in Sobolev spaces $H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ with $s > -1/4$.

The next chapter is devoted to the ill-posedness result, which states that the flow-map data solution can not be of class C^2 for $s < -2$. This problem was studied by Bourgain [7] (see also Tzvetkov [40]) in the context of the KdV equation (0.7). The same question was studied by Molinet, Saut and Tzvetkov [35]- [36], for the Benjamin-Ono equation

$$u_t + \mathcal{H}u_{xx} + uu_x = 0 \tag{0.11}$$

and for the Kadomtsev-Petviashvili 1 (KP1) equation

$$(u_t + uu_x + u_{xxx})_x - u_{yy} = 0, \tag{0.12}$$

respectively.

In the last chapter, we consider the initial value problem (IVP) for the Schrödinger-Boussinesq (SB) system

$$\begin{cases} iu_t + u_{xx} = vu, \\ v_{tt} - v_{xx} + v_{xxxx} = (|u|^2)_{xx}, \\ u(x,0) = u_0(x);\ v(x,0) = v_0(x);\ v_t(x,0) = (v_1)_x(x), \end{cases} \tag{0.13}$$

where $x \in \mathbb{R}$ and $t > 0$.

Here u and v are respectively a complex valued and a real valued function defined in space-time $\mathbb{R}^2$. The SB-system is considered as a model of interactions between short and intermediate long waves, which is derived in describing the dynamics of Langmuir soliton formation and interaction in a plasma [34] and diatomic lattice system [42]. The short wave term $u(x,t) : \mathbb{R}\times\mathbb{R} \to \mathbb{C}$ is described by a Schrödinger type equation with a potential $v(x,t) : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ satisfying some sort of Boussinesq equation and representing the intermediate long wave.

The nonlinear Schrödinger (NLS) equation models a wide range of physical phenomena including self-focusing of optical beams in nonlinear media, propagation of Langmuir waves in plasmas, etc. For a introduction on this topic, we refer the reader to [31].

Our principal aim here is to study the well-posedness of the Cauchy problem for the SB-system (0.13). Concerning the local well-posedness question, some results are obtained for the SB-system (0.13). Linares and Navas [30] proved that (0.13) is locally well-posedness for initial data $u_0 \in L^2(\mathbb{R})$, $v_0 \in L^2(\mathbb{R})$, $v_1 = h_x$ with $h \in H^{-1}(\mathbb{R})$ and $u_0 \in H^1(\mathbb{R})$, $v_0 \in H^1(\mathbb{R})$, $v_1 = h_x$ with $h \in L^2(\mathbb{R})$. Moreover, by using some conservations laws, in the latter case the solutions can extended globally. Yongqian [43] established local well-posedness when $u_0 \in H^s(\mathbb{R})$, $v_0 \in H^s(\mathbb{R})$, $v_1 = h_{xx}$ with $h \in H^s(\mathbb{R})$ for $s \geq 0$ and assuming $s \geq 1$ these solutions are global.

Here we considerably improve the previous ones [30]- [43]. Local and global well-posedness for the SB-system is obtained for initial data $(u_0, v_0, v_1) \in H^s(\mathbb{R})\times H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ with $s > -1/4$ and $(u_0, v_0, v_1) \in L^2(\mathbb{R}) \times L^2(\mathbb{R}) \times H^{-1}(\mathbb{R})$, respectively. The scheme of proof used to obtain these results is in the same spirit as the one implemented by Ginibre, Y. Tsutsumi and Velo [21] and Colliander, Holmer, Tzirakis [14] to establish their results for the Zakharov system

$$\begin{cases} iu_t + u_{xx} = vu, \\ \sigma v_{tt} - v_{xx} = (|u|^2)_{xx}, \\ u(0,x) = u_0(x);\ v(x,0) = v_0(x);\ v_t(x,0) = v_1, \end{cases} \tag{0.14}$$

where $x \in \mathbb{R}$ and $t > 0$.

Preliminaries

Notations

In the sequel, c denotes a positive constant which may differ at each appearance.

For any positive numbers a and b, the notation $a \lesssim b$ means that there exists a positive constant θ such that $a \leq \theta b$. We also denote $a \sim b$ when, $a \lesssim b$ and $b \lesssim a$.

In the following, we denote by $a+$ a number slightly larger the a.

Finally we define $\boldsymbol{\langle} a \boldsymbol{\rangle} \equiv (1+|a|^2)^{1/2}$ and $\langle a \rangle \equiv 1+|a|$. Note that $\boldsymbol{\langle} a \boldsymbol{\rangle} \sim \langle a \rangle$. Despite of this fact, we decide to use this two notations in this work. This is justified by the fact that $\boldsymbol{\langle} \cdot \boldsymbol{\rangle}$ (resp. $\langle \cdot \rangle$) is more convenient to prove our results in Chapter 1 (resp. Chapters 3-5).

Functional Spaces

We start with the well-known (generalized) Sobolev spaces.

Definition 0.0.1 *Let $s \in \mathbb{R}$, $1 \leq p \leq \infty$. The homogeneous (generalized) Sobolev space and the inhomogeneous (generalized) Sobolev space are defined respectively as the completion of $S(\mathbb{R}^n)$ with respect to the norms*

$$\|f\|_{\dot{H}^s_p(\mathbb{R}^n)} = \|D^s f\|_{L^p(\mathbb{R}^n)},$$

$$\|f\|_{H^s_p(\mathbb{R}^n)} = \|J^s f\|_{L^p(\mathbb{R}^n)},$$

where $D^s = \mathcal{F}^{-1}|\xi|^s\mathcal{F}$ and $J^s = \mathcal{F}^{-1}\boldsymbol{\langle}\xi\boldsymbol{\rangle}^s\mathcal{F}$.

Remark 0.0.1 *We recall that if $N \geq 1$ is an integer and if $1 < p < \infty$ then there exists $c > 0$, such that for all $g \in H_p^N(\mathbb{R}^n)$*

$$\frac{1}{c}\|g\|_{H_p^N(\mathbb{R}^n)} \leq \sum_{j=1}^{n} \left\| \frac{\partial^N}{\partial x_j^N} g \right\|_{L^p(\mathbb{R}^n)} + \|g\|_{L^p(\mathbb{R}^n)} \leq c\|g\|_{H_p^N(\mathbb{R}^n)}.$$

See [4] Theorem 6.2.3.

For convenience, we denote H_2^s by H^s.

Now we recall the definition of homogeneous Besov spaces. Let $\eta \in C_c^\infty(\mathbb{R}^n)$ such that $\operatorname{supp} \eta \subseteq \{\xi : 2^{-1} \leq \xi \leq 2\}$, $\eta(\xi) > 0$ for $2^{-1} < \xi < 2$ and $\sum_{j\in\mathbb{Z}} \eta(2^{-j}\xi) = 1$ for $\xi \neq 0$. Define a frequency projection operator P_j for $j \in \mathbb{Z}$ by

$$\begin{aligned} P_j\phi &= \mathcal{F}^{-1}\left[\eta\left(\frac{\xi}{2^j}\right)\hat{\phi}\right], \text{ for } j \in \mathbb{Z} - \{0\}, \\ P_0 &= 1 - \sum_{j\geq 1} P_j. \end{aligned}$$

Remark 0.0.2 *For convenience we choose η such that $P_j = \widetilde{P}_j P_j$ where $\widetilde{P}_j \equiv P_{j-1} + P_j + P_{j+1}$.*

We have the following definition

Definition 0.0.2 *Let $s \in \mathbb{R}$, $1 \leq p, q \leq \infty$. The homogeneous Besov space and the inhomogeneous Besov space are defined respectively as follows:*

$$\begin{aligned} \dot{B}^s_{p,q}(\mathbb{R}^n) &= \left\{ f \in S'(\mathbb{R}^n)/\mathcal{P} : \|f\|_{\dot{B}^s_{p,q}} = \left(\sum_{j\in\mathbb{Z}} 2^{js}\|P_j f\|^q_{L^p}\right)^{\frac{1}{q}} < \infty \right\}, \\ B^s_{p,q}(\mathbb{R}^n) &= \left\{ f \in S'(\mathbb{R}^n) : \|f\|_{B^s_{p,q}} = \|P_0 f\|_{L^p} + \left(\sum_{j\geq 1} 2^{js}\|P_j f\|^q_{L^p}\right)^{\frac{1}{q}} < \infty \right\}, \end{aligned}$$

where $\mathcal{P}$ is the space of polynomials in n variables.

It is well-known that $\dot{B}^s_{2,2}(\mathbb{R}^n) = \dot{H}^s_2(\mathbb{R}^n)$. For further details concerning the Besov and (generalized) Sobolev spaces we refer the reader to [4].

Finally, we define the mixed “space-time” spaces

Definition 0.0.3 *Let X be a functional space , $1 \leq r \leq +\infty$ and $T > 0$, the $L^r_{0,T}X$ and $L^r_T X$ spaces are defined, respectively, by*

$$\begin{aligned} L^r_{0,T}X &= \left\{ f : X \times [0,T] \to \mathbb{R} \ \ or \ \ \mathbb{C} : \|f\|_{L^r_{0,T}X} \equiv \left(\int_0^T \|f(\cdot,t)\|^r_X \right)^{\frac{1}{r}} < \infty \right\}. \\ L^r_T X &= \left\{ f : X \times [T,+\infty] \to \mathbb{R} \ \ or \ \ \mathbb{C} : \|f\|_{L^r_T X} \equiv \left(\int_T^\infty \|f(\cdot,t)\|^r_X \right)^{\frac{1}{r}} < \infty \right\}. \end{aligned}$$

Linear equation

First, we consider the linear Boussinesq equation

$$\begin{cases} u_{tt} - u_{xx} + u_{xxxx} = 0, \quad x \in \mathbb{R}, t > 0, \\ u(x,0) = \phi, \quad u_t(x,0) = \psi_x. \end{cases} \tag{0.15}$$

It is well known that the solution of (0.15) is given by

$$u(t) = V_c(t)\phi + V_s(t)\psi_x \tag{0.16}$$

where

$$\begin{aligned} V_c(t)\phi &= \left(\frac{e^{it\sqrt{\xi^2+\xi^4}} + e^{-it\sqrt{\xi^2+\xi^4}}}{2} \widehat{\phi}(\xi) \right)^\vee \\ V_s(t)\psi_x &= \left(\frac{e^{it\sqrt{\xi^2+\xi^4}} - e^{-it\sqrt{\xi^2+\xi^4}}}{2i\sqrt{\xi^2+\xi^4}} \widehat{\psi_x}(\xi) \right)^\vee . \end{aligned}$$

By Duhamel's Principle the solution of (0.5) is equivalent to

$$u(t) = V_c(t)\phi + V_s(t)\psi_x + \int_0^t V_s(t-t')(u^2)_{xx}(t')dt'. \tag{0.17}$$

Another way to write an integral equation associated to (0.1) is as follows. First, we consider the following modified equation

$$u_{tt} + \Delta^2 u + \Delta(g(u)) = 0. \tag{0.18}$$

For the linear equation

$$u_{tt} + \Delta^2 u = 0,$$

the solution for initial data $u(0) = \phi$ and $u_t(0) = \Delta\eta$, is given by

$$u(t) = B_c(t)\phi + B_s(t)\Delta\eta$$

where

$$U(t)u_0 \;=\; \left(e^{-it|\xi|^2}\widehat{u_0}\right)^{\vee}, \tag{0.19}$$

$$B_c(t) \;=\; \frac{1}{2}\left(U(t) + U(-t)\right) \tag{0.20}$$

and

$$B_s(t)\Delta \;=\; \frac{i}{2}\left(U(t) - U(-t)\right). \tag{0.21}$$

Remark 0.0.3 *Note that (0.19) is the unitary group associated to the linear Schrödinger equation (see, for example, [31] Chapter 4).*

By Duhamel's Principle the solution of (0.18) is given by

$$u(t) = B_c(t)\phi + B_s(t)\Delta\eta + B_I(g) \tag{0.22}$$

where $B_I(g) \equiv \int_0^t B_s(t-t')\Delta g(u)(t')dt'$.

We also have this kind of representation when we write the Boussinesq equation as a system. In this case, the linear system is represented by

$$\begin{cases} u_t &= \Delta v \\ v_t &= u - \Delta u, \quad x \in \mathbb{R}^n, t > 0 \end{cases} \tag{0.23}$$

and its solution, for initial data $\vec{u}_0 = (u_{0,1}, u_{0,2})$, is given by

$$B(t)\vec{u}_0 = \int_{-\infty}^{+\infty} e^{ix\xi} \begin{pmatrix} \cos(t|\xi|\langle\xi\rangle) & -\frac{|\xi|}{\langle\xi\rangle}\sin(t|\xi|\langle\xi\rangle) \\ \frac{\langle\xi\rangle}{|\xi|}\sin(t|\xi|\langle\xi\rangle) & \cos(t|\xi|\langle\xi\rangle) \end{pmatrix} \widehat{\vec{u}_0}(\xi)d\xi \tag{0.24}$$

where $\widehat{\vec{u}_0} = (\widehat{u}_{0,1}, \widehat{u}_{0,2})$.

Therefore, using the Duhamel principle, the solutions of (0.3), with initial data $\vec{u}_0$, can be written as

$$\vec{u}(t) = B(t)\vec{u}_0 + \int_0^t B(t-t')\vec{f}(\vec{u}(t'))dt'$$

where $\vec{f}(\vec{u}) = (0, f(u))$.

For the Schrödinger-Boussinesq system (0.13), again by Duhamel's principle, the solution is equivalent to the following system of equations

$$
\begin{aligned}
u(t) =& U(t)u_0 - i\int_0^t U(t-t')(vu)(t')dt', \\
v(t) =& V_c(t)v_0 + V_s(t)(v_1)_x + \int_0^t V_s(t-t')(|u|^2)_{xx}(t')dt'
\end{aligned}
\tag{0.25}
$$

where $U(t)$ is given by (0.19).

Chapter 1
Large data asymptotic behavior

1.1 Introduction

In this chapter, we consider the Boussinesq Equation (0.3), where for a given $\alpha > 1$ the function f satisfies the following assumptions

- $f \in C^{[\alpha]}(\mathbb{R})$ (denoting $[\alpha]$ the integer part of α);
- $|f^{(l)}(v)| \lesssim |v|^{\alpha-l}$ for all integers l varying in the whole range $0 \leq l \leq \alpha$.

As it was mentioned in the introduction, we are interested in constructing solutions to (0.3) with a given asymptotic behavior. In other words, we want to construct a wave operator for a given profile V in suitable spaces (see equation (0.4)).

This problem was studied for other dispersive models. In the case of Schrödinger related equation we refer the reader to Ginibre and Velo [22] for a detailed review. For the generalized Korteweg-de Vries equation it was studied by Côte [15]. The central idea introduced in this last work is that any scheme of proof which allows to prove global well-posedness and linear scattering for small data can be applied successfully to construct solutions with a give linear profile, small or large. In other words, the smallness assumption can be removed in this context and we are able to construct a wave operator for any possible large profile V in certain functional space, where the small data linear scattering holds.

Another feature that the dispersive equation must have in order to apply the arguments of Côte [15] is that the solution associated to the linear equation can be expressed in terms of the action of a unitary group over the initial data. For this reason we use the system formulation (0.3) instead of (0.1).

The plan of this chapter is as follows: in Section 2, we introduce some notation and state our main results. We derive some linear estimates useful in the proof of the main results in Section 3. Section 4 will be devoted to prove Theorems 1.2.1-1.2.4.

1.2 Notations and main results

To give precise statements of the main results we need to introduce some notation. Based in the Sobolev spaces we have the following definition.

Definition 1.2.1 *Let $s \in \mathbb{R}$, $1 \leq p \leq \infty$. The inhomogeneous initial data spaces Y_p^s and $Y_p^{1,1}$ are defined by $Y_p^s = H_p^s \times D^{-1}H_p^{s-1}$ and $Y_p^{1,1} = H_p^1 \times D^{-1}H_p^{-1}$. The norm of these spaces are given respectively by*

$$\begin{aligned} \|\vec{h}\|^2_{Y_p^s} &= \|h_1\|^2_{H_p^s} + \|Dh_2\|^2_{H_p^{s-1}}, \\ \|\vec{h}\|^2_{Y_p^{1,1}} &= \|h_1\|^2_{H_p^1} + \|Dh_2\|^2_{H_p^{-1}}, \end{aligned}$$

where $D = \mathcal{F}^{-1}|\xi|\mathcal{F}$.

For convenience, we denote Y_2^s by Y^s, therefore

$$\|\vec{h}\|^2_{Y^s} = \|h_1\|^2_{H^s} + \|Dh_2\|^2_{H^{s-1}}.$$

Remark 1.2.1 *In view of (0.24) it is easy to see that, for all $s \in \mathbb{R}$*

$$\|B(t)\vec{h}\|_{Y^s} = \|\vec{h}\|_{Y^s}. \tag{1.1}$$

Other spaces which will be useful for our purposes are given in the next definition.

Definition 1.2.2 *Let $s \in \mathbb{R}$, $1 \leq p \leq \infty$. The spaces $\widetilde{L}^p$ and $\widetilde{H}^s_{p+1}$ are defined by $L^p \times L^p$ and $H^s_{p+1} \times H^s_{p+1}$, respectively, with the following norms:*

$$\begin{aligned} \|\vec{h}\|^2_{\widetilde{L}^p} &= \|h_1\|^2_{L^p} + \|h_2\|^2_{L^p}, \\ \|\vec{h}\|^2_{\widetilde{H}^s_{p+1}} &= \|J^s h_1\|^2_{L^{p+1}} + \|J^s h_2\|^2_{L^{p+1}}. \end{aligned}$$

Based in the Besov spaces we introduce some spaces that will be useful to treat the (SNLB) in arbitrary spatial dimension.

Definition 1.2.3 *Let $s \in \mathbb{R}$, $1 \leq p, q \leq \infty$. The nonhomogeneous initial data space $D_1^{-l}\dot{B}^s_{p,q}$ is defined respectively as the completion of $S(\mathbb{R}^n)$ with respect to the norm*

$$\|f\|_{D_1^{-l}\dot{B}^s_{p,q}} = \left(\sum_{j\in\mathbb{Z}} (D_1(2^j)^l (2^j)^s \|P_j f\|_{L^p})^q\right)^{\frac{1}{q}} \tag{1.2}$$

where $D_1 = \mathcal{F}^{-1}\left[\frac{|\xi|}{\langle\xi\rangle}\right]^{\frac{n-2}{2}}\mathcal{F}$.

Remark 1.2.2 *Since $D_1 \in L^\infty$, we have*

$$\dot{B}^s_{p,q} \subseteq D_1^{-(1-\frac{2}{r})} \dot{B}^s_{p,q}.$$

Definition 1.2.4 *Let $s, l \in \mathbb{R}$, $1 \leq p, q \leq \infty$. The initial data spaces $\dot{\mathfrak{B}}^{l,s}_{p,q}$ and $\widetilde{\mathfrak{B}}^{l,s}_{p,q}$ are defined by*

$$\begin{aligned} \dot{\mathfrak{B}}^{l,s}_{p,q} &= \left(\dot{B}^{s+nl}_{p,q} \cap D_1^{-l} \dot{B}^s_{p,q}\right) \times D_2^{-1}\left(\dot{B}^{s+nl}_{p,q} \cap D_1^{-l} \dot{B}^s_{p,q}\right), \\ \widetilde{\mathfrak{B}}^{l,s}_{p,q} &= \left(L^p \cap D_1^{-l} \dot{B}^s_{p,q}\right) \times D_2^{-1}\left(L^p \cap D_1^{-l} \dot{B}^s_{p,q}\right). \end{aligned}$$

Furthermore, the norm of these spaces are given respectively by

$$\|\vec{h}\|_{\dot{\mathfrak{B}}^{l,s}_{p,q}} = \|h_1\|_{\dot{B}^{s+nl}_{p,q}} + \|h_1\|_{D_1^{-l}\dot{B}^s_{p,q}} + \|D_2 h_2\|_{\dot{B}^{s+nl}_{p,q}} + \|D_2 h_2\|_{D_1^{-l}\dot{B}^s_{p,q}} \tag{1.3}$$

and

$$\|\vec{h}\|_{\widetilde{\mathfrak{B}}^{l,s}_{p,q}} = \|h_1\|_{L^p} + \|h_1\|_{D_1^{-l}\dot{B}^s_{p,q}} + \|D_2 h_2\|_{L^p} + \|D_2 h_2\|_{D_1^{-l}\dot{B}^s_{p,q}}.$$

where $D_2 = \mathcal{F}^{-1}\left[\frac{|\xi|}{\langle\xi\rangle}\right]\mathcal{F}$.

Now we are in position to give a precise statement of the main results of this chapter. The first two theorems are concerned with the one dimensional case.

Theorem 1.2.1 *Let $\alpha = 4/\gamma$, $p = \frac{2}{1-\gamma}$ and $\gamma \in (0, 4/5)$. For any $\vec{h} = (h_1, h_2) \in \left(D^{\frac{\gamma}{4}} H^{1+\frac{\gamma}{4}}(\mathbb{R}) \times D^{-1+\frac{\gamma}{4}} H^{\frac{\gamma}{4}}(\mathbb{R})\right) \cap Y^1$ there exist $T_0 = T_0(\vec{h}) \in \mathbb{R}$ and $\vec{u} \in L^\infty([T_0, +\infty), Y^1)$ solution of (0.3) such that*

$$\lim_{t\to\infty} \|\vec{u}(t) - B(t)\vec{h}\|_{Y^1} = 0. \tag{1.4}$$

Moreover, $\vec{u}$ is unique in $L^\alpha_{T_0} Y^{1,1}_p \cap L^\infty_{T_0} Y^1$.

Theorem 1.2.2 *Let $\alpha > \alpha_0 \equiv 2 + \sqrt{7}$, $\beta = 1 - \dfrac{2}{\alpha+1}$ and q such that $1/q + 1/(\alpha+1) = 1$. For any $\vec{h} = (h_1, h_2) \in Y^1_q \cap Y^1$ there exist $T_0 = T_0(\vec{h}) \in \mathbb{R}$ and $\vec{u} \in L^\infty([T_0, +\infty), Y^1_{\alpha+1} \cap Y^1)$ solution of (0.3) such that*

$$\lim_{t\to\infty} \|\vec{u}(t) - B(t)\vec{h}\|_{Y^1} = 0.$$

Moreover, $\vec{u}$ is the unique solution in $L^\infty_{T_0} Y^1$ such that

$$\sup_{t \geq T_0} \|(1+t)^{\frac{\beta}{3}} \vec{u}(t)\|_{Y^1_{\alpha+1}} < \infty.$$

Remark 1.2.3 *To apply Kato's abstract theory for quasilinear evolution equation Bona and Sachs [5] wrote the one dimensional Boussinesq equation in the equivalent system form (0.2). To prove Theorems 1.2.1-1.2.2, we will work with the system (0.3) instead of (0.2), since the generalization to higher dimensions of the latter one is not straightforward. However, it is possible to prove (with the same α) that the existence and convergence statements in the above theorems are valid in $H^1 \times L^2$.*

Remark 1.2.4 *Theorem 1.2.1 generalizes Theorem 1.2.2 in the sense that it holds for more values of α in the nonlinearity.*

The next two theorems treat the construction of the wave operator for (SNLB) in arbitrary dimension.

Theorem 1.2.3 *Let $2 < r < \infty$, $\frac{1}{r} + \frac{1}{r'} = 1$, $s > \frac{n}{r'}$, $\theta = \frac{n}{2}(1 - \frac{2}{r})$ and $\alpha \geq s$, $\alpha > \frac{2}{r'} + \max(1, \frac{1}{\theta})$. For any $\vec{h} \in \dot{\mathfrak{B}}^{1-\frac{2}{r}, \frac{n}{r}}_{r',1} \cap Y^s$ there exist $T_0 = T_0(\vec{h}) \in \mathbb{R}$ and $\vec{u} \in L^\infty([T_0, +\infty), Y^s)$ solution of (0.3) such that*

$$\lim_{t \to \infty} \|\vec{u}(t) - B(t)\vec{h}\|_{Y^s} = 0.$$

Moreover, $\vec{u}$ is the unique solution in $L^\infty_{T_0} Y^s$ such that

$$\sup_{t \geq T_0} \|(1+t)^{\theta} \vec{u}(t)\|_{\widetilde{L}^\infty} < \infty.$$

For the next theorem define $\gamma(n) = 1 + 8/(n - 2 + \sqrt{n^2 + 12n + 4})$ and $\beta(n) = \infty$ if $n = 1, 2$ and $\beta(n) = \frac{n+2}{n-2}$ if $n \geq 3$.

Theorem 1.2.4 *Let $s > 0$, $s \leq \alpha$ and $\theta = \frac{n}{2}(1 - \frac{2}{\alpha+1})$. If $\gamma(n) < \alpha < \beta(n)$ then for any $\vec{h} \in \widetilde{\mathfrak{B}}^{1-\frac{2}{\alpha+1}, s}_{\frac{\alpha+1}{\alpha}, 2} \cap \widetilde{H}^s_{\alpha+1}$ there exist $T_0 = T_0(\vec{h}) \in \mathbb{R}$ and $\vec{u} \in L^\infty([T_0, +\infty), \widetilde{H}^s_{\alpha+1})$ solution of (0.3) such that*

$$\lim_{t \to \infty} \|\vec{u}(t) - B(t)\vec{h}\|_{\widetilde{H}^s_{\alpha+1}} = 0.$$

Moreover, $\vec{u}$ is the unique solution such that

$$\sup_{t\geq T_0}\|(1+t)^{\theta}\vec{u}(t)\|_{\widetilde{H}^s_{\alpha+1}} < \infty.$$

We remark that the appropriate functional spaces where we can construct a wave operator come from the scheme to obtain linear scattering for small data. Moreover, if one can prove that the linear estimates, in Section 3, hold for a large class of functions, then we can construct an associated wave operator following the same arguments given in the proofs of the Theorems 1.2.1-1.2.4.

We recall that the linear scattering for small data obtained in Linares and Scialom [32], Liu [33] and Cho and Ozawa [11] are based in different (but equivalent) ways to write the integral equation associated to the Boussinesq equation. In the paper [33], Liu worked with the system (0.2), while in [32] and [11] the authors worked directly with the equation (0.1) in one and arbitrary dimension, respectively.

To standardize our results we will work only with the system (0.3) which is associated to the unitary group $B(\cdot)$ in Y^s. The existence of such unitary group is essential for our construction of the wave operator (see the proof of Proposition 1.4.1). Therefore, to prove the relevant linear estimates in our context (see Section 3), we need to modify the hypothesis on the initial data given by [32], [33] and [11]. For instance, to obtain the small data linear scattering in [32] and [33], the authors assume that the initial data belongs to $D^{\frac{\gamma}{4}}H^{1+\frac{\gamma}{4}}(\mathbb{R})\times D^{1+\frac{\gamma}{4}}H^{\frac{\gamma}{4}}(\mathbb{R})$ and $(H^1(\mathbb{R})\times L^2(\mathbb{R}))\cap\big(H^1_{\alpha+1}(\mathbb{R})\times L^{\alpha+1}(\mathbb{R})\big)$, respectively. Note that these functional spaces are different from the ones in Theorems 1.2.1-1.2.2. In fact, for the data h_1, they are the same, but for h_2 we do not have any inclusion relation between them. Similarly, the comparison of the Theorems 1.2.3-1.2.4 with the results found in [11] can also be considered, but to do that we will need to introduce much more notation, therefore we decide to omit it.

1.3 Linear estimates

In this section we derive some linear estimates for the one parameter group $B(\cdot)$ introduced in (0.24). First, we treat the one dimensional case. Define the following operator

$$V(t)g(x) \;=\; \int_{-\infty}^{\infty} e^{i(t|\xi|\langle\xi\rangle+x\xi)}\frac{|\xi|\hat{g}(\xi)}{\langle\xi\rangle}d\xi$$

Lemma 1.3.1 *If* $\gamma \in [0,1]$, $p = \frac{2}{1-\gamma}$ *and* $p' = \frac{2}{1+\gamma}$ *then*

$$\|V(t)g\|_{L^p} \leq ct^{-\gamma/2}\|g\|_{L^{p'}}.$$

Proof See [32] Lemma 2.6.

■

Lemma 1.3.2 *Let* $\vec{h} = (h_1, h_2) \in D^{\frac{\gamma}{4}}H^{1+\frac{\gamma}{4}}(\mathbb{R}) \times D^{-1+\frac{\gamma}{4}}H^{\frac{\gamma}{4}}(\mathbb{R})$ *then*

$$\lim_{T\to\infty} \|(B(t)\vec{h})_1\|_{L^\alpha_T H^1_p} = 0.$$

Proof It is sufficient to prove that

$$\|(B(t)\vec{h})_1\|_{L^\alpha([0,\infty):H^1_p)} < \infty.$$

Using the definition of $B(\cdot)$ we have

$$\begin{aligned}(B(t)\vec{h})_1 \quad &= \int_{-\infty}^{\infty} e^{ix\xi}\Big(\frac{e^{it|\xi|\langle\xi\rangle} + e^{-it|\xi|\langle\xi\rangle}}{2}\Big)\widehat{h_1}(\xi)d\xi - \\ &- \int_{-\infty}^{\infty} e^{ix\xi}\Big(\frac{e^{it|\xi|\langle\xi\rangle} - e^{-it|\xi|\langle\xi\rangle}}{2i}\Big)\frac{|\xi|\widehat{h_2}(\xi)}{\langle\xi\rangle}d\xi.\end{aligned}$$

Then, by the proof of Proposition 2.8 in [32] it follows that

$$\|(B(t)\vec{h})_1\|_{L^\alpha([0,\infty):H^1_p)} \leq \|D^{-\frac{\gamma}{4}}h_1\|_{H^{1+\gamma/4}} + \|D^{1-\frac{\gamma}{4}}h_2\|_{H^{\gamma/4}}.$$

■

The next two results are the analogous of Lemmas 2.2 and 2.3 in [33] for the linear system associated to (0.3).

Lemma 1.3.3 *For* $t \neq 0$ *we have*

$$\sup_{x\in\mathbb{R}} \Big|\int_{-\infty}^{+\infty} e^{it(|\xi|\langle\xi\rangle + x\xi)}\Big| \leq c\left(|t|^{-\frac{1}{2}} + |t|^{-\frac{1}{3}}\right).$$

Proof Since the proof is similar to that in [33], we will omit it.

■

Lemma 1.3.4 *Let* $k \in \mathbb{R}$ *and* $\beta = 1 - \dfrac{2}{\alpha+1}$, *then*

$$\|B(t)\vec{h}\|_{Y^k_{\alpha+1}} \leq c(|t|^{-\frac{1}{2}} + |t|^{-\frac{1}{3}})^\beta\|\vec{h}\|_{Y^k_q},$$

where $\dfrac{1}{q}+\dfrac{1}{\alpha+1}=1.$

Proof By definition of $B(\cdot)$ we have for all $k\in\mathbb{R}$

$$\begin{aligned}\|B(t)\vec{h}\|_{Y^k_\infty} &= \|K_1(t,\cdot)*(J^kh_1)(x)-K_2(t,\cdot)*(J^{k-1}Dh_2)(x)\|_{L^\infty}+\\ &\quad+\|K_2(t,\cdot)*(J^kh_1)(x)+K_1(t,\cdot)*(J^{k-1}Dh_2)(x)\|_{L^\infty},\end{aligned}$$

where

$$\begin{aligned}K_1(t,x) &= \int_{\mathbb{R}^n} e^{ix\xi}\left(\frac{e^{it|\xi|\langle\xi\rangle}+e^{-it|\xi|\langle\xi\rangle}}{2}\right)d\xi,\\ K_2(t,x) &= \int_{\mathbb{R}^n} e^{ix\xi}\left(\frac{e^{it|\xi|\langle\xi\rangle}-e^{-it|\xi|\langle\xi\rangle}}{2i}\right)d\xi.\end{aligned}$$

Thus, by Young's inequality and Lemma 1.3.3, we obtain

$$\|B(t)\vec{h}\|_{Y^k_\infty} \le c\left(|t|^{-\frac{1}{2}}+|t|^{-\frac{1}{3}}\right)\|\vec{h}\|_{Y^k_1}.$$

Interpolating this inequality with (1.1) we obtain the desired inequality.

■

Lemma 1.3.5 *Let* $\alpha>2+\sqrt{7}$ *and* $\beta=1-\dfrac{2}{\alpha+1}$ *and*

$I(T)=\sup\limits_{t\ge T}(1+t)^{\frac{\beta}{3}}\int_t^\infty[(|t-t'|^{-\frac{1}{2}}+|t-t'|^{-\frac{1}{3}})(1+t')^{-\frac{\alpha}{3}}]^\beta dt'],$

$H(T)=\sup\limits_{t\ge T}[(|t|^{-\frac{1}{2}}+|t|^{-\frac{1}{3}})(1+t)^{\frac{1}{3}}]^\beta,$

$K(T)=\int_T^\infty(1+t')^{-\frac{\alpha\beta}{3}}dt'.$

Then

(i) $I(T)\longrightarrow 0$ *when* $T\longrightarrow\infty$,

(ii) There exists $M>0$ *such that* $\sup\limits_{T\ge 1}H(T)\le M$,

(iii) $K(T)\longrightarrow 0$ *when* $T\longrightarrow\infty$.

Proof Since this is only a calculation we omit the proof.

■

In the remainder of this section we will consider the n-dimensional Boussinesq equation. To obtain linear estimates in this case we will use Besov spaces.

Lemma 1.3.6 *For all $j \in \mathbb{Z}$ we have*

$$\sup_{x\in\mathbb{R}^n}\left|\int_{\mathbb{R}^n} e^{i(x\cdot\xi+t|\xi|\langle\xi\rangle)}\eta(\frac{\xi}{2^j})d\xi\right| \leq c|t|^{-\frac{n}{2}}D_1(2^j).$$

Proof See [11] Lemma 3.

∎

The next two lemmas are inspired on Lemmas 4-5 of [11]. The difference here is that we are working with system (0.3) while Cho and Ozawa directly worked with equation (0.1). Therefore, we have to state and prove the relevant estimates in this context.

Lemma 1.3.7 *Let $2 \leq r \leq \infty$, $s > \frac{n}{r'}$ and $\theta = n(\frac{1}{2}-\frac{1}{r})$. Then, for $\vec{g}_0 \equiv (0,g)$, we have*

(i) $\|(B(t)\vec{h})_1\|_{L^\infty} \leq c(1+|t|)^{-\theta}\|\vec{h}\|_{\dot{\mathfrak{B}}^{1-\frac{2}{r},\frac{n}{r}}_{r',1}}$,

(ii) $\|B(t)\vec{g}_0\|_{\widetilde{L}^\infty} \leq c(1+|t|)^{-\theta}\|g\|_{B^s_{r',2}}$.

Proof

(i) Using that $P_j = \widetilde{P}_jP_j$, the definition of D_2, Hölder's inequality ($1/r+1/r' = 1$) and Hausdorff-Young's inequality we have for all $t \in \mathbb{R}$ (in particular, for $|t| \leq 1$)

$$\begin{aligned}\|P_j((B(t)\vec{h})_1)\|_{L^\infty_x} &\leq c\int_{\mathbb{R}^n}\left(|(\widetilde{P}_jh_1)\widehat{\ }(\xi)|+|(\widetilde{P}_jD_2h_2)\widehat{\ }(\xi)|\right)|\eta\left(\frac{\xi}{2^j}\right)|d\xi \\ &\leq c(2^j)^{n/r'}\left(\|\widetilde{P}_jh_1\|_{L^{r'}}+\|\widetilde{P}_jD_2h_2\|_{L^{r'}}\right).\end{aligned} \tag{1.5}$$

Then since $\dfrac{n}{r}+n\left(1-\dfrac{2}{r}\right)=\dfrac{n}{r'}$, we obtain

$$\|(B(t)\vec{h})_1\|_{L^\infty} \leq c\left(\|h_1\|_{\dot{B}^{\frac{n}{r'}}_{r',1}}+\|D_2h_2\|_{\dot{B}^{\frac{n}{r'}}_{r',1}}\right). \tag{1.6}$$

On the other hand, by Fubini's theorem we have for $|t| > 1$

$$\|P_j((B(t)\vec{h})_1)\|_{L^\infty_x} \leq c\|\widetilde{P}_jh_1 * K_3(t,\cdot)\|_{L^\infty_x} + c\|\widetilde{P}_jD_2h_2 * K_4(t,\cdot)\|_{L^\infty_x} \tag{1.7}$$

where

$$K_3(t,x) = \int_{\mathbb{R}^n} e^{ix\xi}\left(\frac{e^{it|\xi|\langle\xi\rangle}+e^{-it|\xi|\langle\xi\rangle}}{2}\right)\eta\left(\frac{\xi}{2^j}\right)d\xi,$$

$$K_4(t,x) \;=\; \int_{\mathbb{R}^n} e^{ix\xi}\left(\frac{e^{it|\xi|\langle\xi\rangle}-e^{-it\langle\xi\rangle}}{2}\right)\eta\left(\frac{\xi}{2^j}\right)d\xi.$$

Then using Lemma 1.3.6 and Young's inequality we obtain

$$\|P_j((B(t)\vec{h})_1)\|_{L^\infty_x} \le c|t|^{-\frac{n}{2}}D_1(2^j)(\|\widetilde{P}_jh_1\|_{L^1}+\|\widetilde{P}_jD_2h_2\|_{L^1}). \tag{1.8}$$

Interpolating (1.5) (with $r'=2$) and (1.8), we have for $1\le r'\le 2$

$$\begin{aligned}\|P_j((B(t)\vec{h})_1)\|_{L^\infty_x} \le & c|t|^{-n(\frac{1}{2}-\frac{1}{r})}D_1(2^j)^{1-\frac{2}{r}}(2^j)^{\frac{n}{r}}\left(\|\widetilde{P}_jh_1\|_{L^{r'}}\right.\\ &\left.+\|\widetilde{P}_jD_2h_2\|_{L^{r'}}\right).\end{aligned} \tag{1.9}$$

Since $D_1(2^{j-1})\sim D_1(2^j)\sim D_1(2^{j+1})$ and using definition (1.2), we obtain

$$\|(B(t)\vec{h})_1\|_{L^\infty_x} \le \; c|t|^{-n(\frac{1}{2}-\frac{1}{r})}\left(\|h_1\|_{D_1^{-(1-\frac{2}{r})}\dot{B}^{\frac{n}{r}}_{r',1}}+\|D_2h_2\|_{D_1^{-(1-\frac{2}{r})}\dot{B}^{\frac{n}{r}}_{r',1}}\right) \tag{1.10}$$

Then (1.6), (1.10) and (1.3) yield

$$\|(B(t)\vec{h})_1\|_{L^\infty_x} \le c\,(1+|t|)^{-n(\frac{1}{2}-\frac{1}{r})}\,\|\vec{h}\|_{\dot{\mathfrak{B}}^{1-\frac{2}{r},\frac{n}{r}}_{r',1}}.$$

(ii) By estimates (1.5), (1.9) and the fact that $D_1\in L^\infty$, we have for $l=1,2$

$$\|P_j(B(t)\vec{g}_0)_l\|_{L^\infty} \le \; c(2^j)^{\frac{n}{r'}}\left(\|\widetilde{P}_jg\|_{L^{r'}}+\|\widetilde{P}_jD_2g\|_{L^{r'}}\right)$$

and

$$\|P_j(B(t)\vec{g}_0)_l\|_{L^\infty} \le c|t|^{-n(\frac{1}{2}-\frac{1}{r})}(2^j)^{\frac{n}{r}}\left(\|\widetilde{P}_jg\|_{L^{r'}}+\|\widetilde{P}_jD_2g\|_{L^{r'}}\right).$$

Therefore, summing with respect to j after squaring, we obtain

$$\begin{aligned}\|B(t)\vec{g}_0\|_{\widetilde{L}^\infty} &\le c\left(\|g\|_{\dot{B}^{\frac{n}{r'}}_{r',2}}+\|D_2g\|_{\dot{B}^{\frac{n}{r'}}_{r',2}}\right)\\ \|B(t)\vec{g}_0\|_{\widetilde{L}^\infty} &\le c|t|^{-n(\frac{1}{2}-\frac{1}{r})}\left(\|g\|_{\dot{B}^{\frac{n}{r}}_{r',2}}+\|D_2g\|_{\dot{B}^{\frac{n}{r}}_{r',2}}\right)\end{aligned}$$

Since D_2 is a multiplier in L^1 (see [4] page 149) we known that it is a multiplier in L^p with $1\le p\le\infty$. Therefore, using that D_2 commute with

P_j and $B^s_{r',2} \subseteq \dot{B}^s_{r',2}$ (see [4] Theorem 6.3.2) we have for $s > \frac{n}{r'}$

$$\|B(t)\vec{g}_0\|_{\widetilde{L}^\infty} \le c(1+|t|)^{-n(\frac{1}{2}-\frac{1}{r})}\|g\|_{B^s_{r',2}}$$

■

Lemma 1.3.8 *Let* $2 \le r < \infty$, $s > 0$ *and* $\theta = n(\dfrac{1}{2} - \dfrac{1}{r})$, *then*

(i) $\|(B(t)\vec{g}_0)_i\|_{\dot{B}^0_{r,2}} \le c|t|^{-\theta}\|g\|_{\dot{B}^0_{r',2}}$, *for* $i = 1, 2$,

(ii) $\|(B(t)\vec{g}_0)_i\|_{B^s_{r,2}} \le c|t|^{-\theta}\|g\|_{B^s_{r',2}}$, *for* $i = 1, 2$,

(iii) $\|(B(t)\vec{h})_1\|_{B^s_{r,2}} \le c|t|^{-\theta}\|\vec{h}\|_{\widetilde{\mathfrak{B}}^{1-\frac{2}{r},s}_{r',2}}$.

Proof

(i) Let $B(t)\vec{g}_0 \equiv (B_1(t)g, B_2(t)g)$. Applying the arguments already used in the proof of (1.7), Lemma 1.3.6 and Young's inequality, we obtain

$$\|P_jB_1(t)g\|_{L^\infty} \le c|t|^{-\frac{n}{2}}D_1(2^j)\|\widetilde{P}_jD_2g\|_{L^1} \tag{1.11}$$

and

$$\|P_jB_2(t)g\|_{L^\infty} \le c|t|^{-\frac{n}{2}}D_1(2^j)\|\widetilde{P}_jg\|_{L^1}. \tag{1.12}$$

Since $D_1 \in L^\infty$, D_2 commute with $\widetilde{P}_j$ and is a multiplier in L^1, we obtain for $i = 1, 2$

$$\|P_jB_i(t)g\|_{L^\infty} \le c|t|^{-\frac{n}{2}}\|\widetilde{P}_jg\|_{L^1}. \tag{1.13}$$

On the other hand, by Parseval we have for $i = 1, 2$

$$\|P_jB_i(t)g\|_{L^2} \le c\|\widetilde{P}_jg\|_{L^2}. \tag{1.14}$$

and

$$\|P_jB_1(t)g\|_{L^2} \le c\|\widetilde{P}_jD_2g\|_{L^2}. \tag{1.15}$$

Interpolating (1.13) and (1.14), and using the fact that $2^{j-1} \sim 2^j \sim 2^{j+1}$, we have for all $s \in \mathbb{R}$ and $q \in [1, \infty]$

$$\|B_i(t)g\|_{\dot{B}^s_{r,q}} \le c|t|^{-n\left(\frac{1}{2}-\frac{1}{r}\right)}\|g\|_{\dot{B}^s_{r',q}}. \tag{1.16}$$

Taking $q = 2$ and $s = 0$ in (1.16) we obtain (i).

(ii) By using $\dot{B}^0_{r,2} \hookrightarrow L^r$, for $r \in [2,\infty)$, $L^{r'} \hookrightarrow \dot{B}^0_{r',2}$, for $r \in (1,2]$ (see [37] page 12) and (1.16), we have

$$\|B_i(t)g\|_{L^r} \leq \; c|t|^{-n\left(\frac{1}{2}-\frac{1}{r}\right)}\|g\|_{L^{r'}}. \tag{1.17}$$

Therefore by $B^s_{r,2} = L^r \cap \dot{B}^s_{r,2}$, (1.16) and (1.17), we conclude that for $i = 1,2$

$$\|B_i(t)g\|_{B^s_{r,2}} \leq c|t|^{-n\left(\frac{1}{2}-\frac{1}{r}\right)}\|g\|_{B^s_{r',2}}.$$

(iii) With the notation of the previous items, we have

$$\|(B(t)\vec{h})_1\|_{B^s_{r,2}} \leq \|B_2(t)h_1\|_{B^s_{r,2}} + \|B_1(t)h_2\|_{B^s_{r,2}}. \tag{1.18}$$

Interpolating (1.12) with (1.14) and (1.11) with (1.15), we obtain

$$\begin{aligned} \|B_2(t)h_1\|_{\dot{B}^s_{r,q}} \;&\leq\; c|t|^{-n\left(\frac{1}{2}-\frac{1}{r}\right)}\|h_1\|_{D_1^{-\left(1-\frac{2}{r}\right)}\dot{B}^s_{r',q}}, \\ \|B_1(t)h_2\|_{\dot{B}^s_{r,q}} \;&\leq\; c|t|^{-n\left(\frac{1}{2}-\frac{1}{r}\right)}\|D_2h_2\|_{D_1^{-\left(1-\frac{2}{r}\right)}\dot{B}^s_{r',q}}. \end{aligned} \tag{1.19}$$

Now combining the argument used in the proof of (1.17) with (1.19) and the fact that $D_1 \in L^\infty$, we have

$$\|B_2(t)h_1\|_{L^r} \leq |t|^{-n\left(\frac{1}{2}-\frac{1}{r}\right)}\|h_1\|_{L^{r'}}. \tag{1.20}$$

Since $B^s_{r,2} = L^r \cap \dot{B}^s_{r,2}$ and (1.20) we conclude that

$$\|B_2(t)h_1\|_{B^s_{r,2}} \leq c|t|^{-n\left(\frac{1}{2}-\frac{1}{r}\right)}\left(\|h_1\|_{L^{r'}} + \|h_1\|_{D_1^{-\left(1-\frac{2}{r}\right)}\dot{B}^s_{r',2}}\right). \tag{1.21}$$

By an analogous argument

$$\|B_1(t)h_2\|_{B^s_{r,2}} \leq c|t|^{-n\left(\frac{1}{2}-\frac{1}{r}\right)}\left(\|D_2h_2\|_{L^{r'}} + \|D_2h_2\|_{D_1^{-\left(1-\frac{2}{r}\right)}\dot{B}^s_{r',2}}\right). \tag{1.22}$$

The inequalities (1.21) and (1.22) together with (1.18) prove (iii).

■

Lemma 1.3.9 *For all* $s \in \mathbb{R}$ *we have*

$$\|(B(t)\vec{h})_1\|_{L^\infty H^s} \leq \|\vec{h}\|_{Y^s}.$$

Proof By definition of the space Y^s and since $B(\cdot)$ is a unitary group we have

$$\|(B(t)\vec{h})_1\|_{H^s} \leq \|B(t)\vec{h}\|_{Y^s} = \|\vec{h}\|_{Y^s}.$$

■

Lemma 1.3.10 *Let* $2 < r < \infty$, $\theta = \frac{n}{2}(1 - \frac{2}{r})$, $r' \geq 1$ *such that* $\dfrac{1}{r} + \dfrac{1}{r'} = 1$ *and*

$$J(T) = \sup_{t \geq T}(1+t)^\theta \int_t^\infty (1 + |t - t'|)^{-\theta}(1 + t')^{-\theta(\alpha - \frac{2}{r'})} dt', \tag{1.23}$$

$$L(T) = \sup_{t \geq T} \int_t^\infty (1 + t')^{-\theta(\alpha - 1)} dt'. \tag{1.24}$$

If $\alpha > \frac{2}{r'} + \max(1, \frac{1}{\theta})$ *then*

(i) $J(T) \longrightarrow 0$ *when* $T \to \infty$.

(ii) $L(T) \longrightarrow 0$ *when* $T \to \infty$.

Proof Since this is an elementary calculus fact, we omit the proof.

■

Lemma 1.3.11 *Let* $\theta = \frac{n}{2}(1 - \frac{2}{\alpha+1})$ *and*

$$M(T) = \sup_{t \geq T}(1+t)^\theta \int_t^\infty |t - t'|^{-\theta}(1 + t')^{-\theta\alpha} dt'. \tag{1.25}$$

If $s \leq \alpha$ *and* $\gamma(n) < \alpha < \beta(n)$ *(See Theorem 1.2.4) then* $M(T) \longrightarrow 0$ *when* $T \to \infty$.

Proof Again since this is an elementary calculus fact, we omit the proof.

■

Before finishing this section, we will enunciate a result proved in [24] by Kato (see also [12] and [20]), concerning estimates for fractional derivatives.

Lemma 1.3.12 *Let* $0 \leq s \leq \alpha$, *then*

(i) $\|D^s f(u)\|_{L^r} \leq c\|u\|^{\alpha-1}_{L^{(\alpha-1)r_1}} \|D^s u\|_{L^{r_2}}$,

where $\dfrac{1}{r} = \dfrac{1}{r_1} + \dfrac{1}{r_2}$, $r_1 \in (1, \infty]$, $r_2 \in (1, \infty)$.

(ii) $\|D^s(uv)\|_{L^r} \leq c\,(\|D^s u\|_{L^{r_1}}\|v\|_{L^{q_2}} + \|u\|_{L^{q_1}}\|D^s v\|_{L^{r_2}})$,

where $\dfrac{1}{r} = \dfrac{1}{r_1} + \dfrac{1}{q_2} = \dfrac{1}{q_1} + \dfrac{1}{r_2}$, $r_i \in (1,\infty)$, $q_i \in (1,\infty]$, $i = 1, 2$.

1.4 Proofs of Theorems 1.2.1-1.2.4

Following the ideas introduced by Côte [15], our task is to find a fixed point of the operator

$$\Phi : \vec{w}(t) \longrightarrow -\int_t^\infty B(t-t')\vec{f}(\vec{w}(t') + B(t')\vec{h})dt'.$$

In the next proposition we verify that in fact this fixed point generates a solution of (0.3).

Proposition 1.4.1 *Let $\vec{w}$ be a fixed point of the operator Φ and define*

$$\vec{u}(t) \equiv B(t)\vec{h} + \vec{w}(t). \tag{1.26}$$

Then $\vec{u}$ is a solution of (0.3) in the time interval $[T_0, \infty)$.

Proof We need to verify that

$$\vec{u}(t) = B(t-T_0)\vec{u}(T_0) + \int_{T_0}^t B(t-t')\vec{f}(\vec{u}(t'))dt'.$$

But $\vec{w}(t) = -\int_t^\infty B(t-t')\vec{f}(\vec{w}(t') + B(t')\vec{h})dt'$, then using (1.26)

$$\begin{aligned} B(T_0 - t)\vec{w} = & -\int_t^\infty B(T_0-t')\vec{f}(\vec{w}(t') + B(t')\vec{h})dt' \\ =& \vec{w}(T_0) + \int_{T_0}^t B(T_0-t')\vec{f}(\vec{w}(t') + B(t')\vec{h})dt' \\ =& \vec{w}(T_0) + \int_{T_0}^t B(T_0-t')\vec{f}(\vec{u}(t'))dt'. \end{aligned} \tag{1.27}$$

Now, applying $B(t-T_0)$ in the both sides of (1.27) we obtain

$$\vec{w}(t) = B(t-T_0)\vec{w}(T_0) + \int_{T_0}^t B(t-t')\vec{f}(\vec{u}(t'))dt'$$

then adding the term $B(t)\vec{h}$ we have

$$\begin{aligned} B(t)\vec{h}+\vec{w}(t) &= B(t)\vec{h}+B(t-T_0)\vec{w}(T_0)+\int_{T_0}^{t} B(t-t')\vec{f}(\vec{u}(t'))dt' \\ &= B(t-T_0)[B(T_0)\vec{h}+\vec{w}(T_0)]+\int_{T_0}^{t} B(t-t')\vec{f}(\vec{u}(t'))dt'. \end{aligned}$$

Using again (1.26) we finish the proof.

■

Proof of Theorem 1.2.1 To prove that Φ has a fixed point let us first introduce the following closed subset of a complete metric space

$$B_T(0,a)=\left\{ \begin{array}{c} \vec{w}\in L^\infty([T,+\infty);Y^1)\cap L^\alpha([T,+\infty);Y_p^{1,1}): \\ \Lambda_T(\vec{w})\equiv \|\vec{w}\|_{L_T^\infty Y^1}+\|\vec{w}\|_{L_T^\alpha Y_p^{1,1}}\leq a \end{array} \right\}$$

Lemma 1.4.1 *There exist positive numbers T, a so that Φ maps $B_T(0,a)$ into $B_T(0,a)$ and becomes a contraction map in the $\Lambda_T(\cdot)$-metric.*

Proof To simplify the notation we set $\vec{v}(t)\equiv\vec{w}(t)+B(t)\vec{h}$. Using that $B(\cdot)$ is an unitary group, the definition of $\vec{f}$, Parseval, Hölder's inequality $(1/2=\gamma/2+(\gamma-1)/2)$ and the fact that $H_p^1\subseteq L^{2(\alpha-1)/\gamma}$

$$\begin{aligned} \|\Phi(\vec{w})(t)\|_{Y^1} &\leq c\int_t^\infty \|(f(v_1))_x(t')\|_{L^2}dt' \leq c\int_t^\infty \||v_1|^{\alpha-1}v_{1,x}(t')\|_{L^2}dt' \\ &\leq c\int_t^\infty \|v_1(t')\|^{\alpha-1}_{L^{2(\alpha-1)/\gamma}}\|v_{1,x}(t')\|_{L^p}dt' \leq c\int_t^\infty \|v_1(t')\|^{\alpha}_{H_p^1}dt'. \end{aligned}$$

On the other hand, by Lemma 1.3.1, Hölder's inequality $(1/p'\equiv(1+\gamma)/2=\gamma/2+1/2)$ and the fact that $H_p^1\subseteq L^{2(\alpha-1)/\gamma}$ we have

$$\begin{aligned} \|\Phi(\vec{w})(t)\|_{Y_p^{1,1}} &\leq c\int_t^\infty |t-t'|^{-\gamma/2}(\|f(v_1)\|_{L^{p'}}+\|\partial_x f(v_1)\|_{L^{p'}})dt' \\ &\leq c\sup_{t'\geq t}\|v_1\|_{H^1}\int_t^\infty |t-t'|^{-\gamma/2}\|v_1\|^{\alpha-1}_{H_p^1}dt'. \end{aligned} \tag{1.28}$$

Applying the Hardy-Littlewood-Sobolev Theorem (see [38]) we obtain for all $t\geq T$

$$\|\Phi(\vec{w})(t)\|_{L_T^\alpha Y_p^{1,1}}\leq c\sup_{t'\geq T}\|v_1\|_{H^1}\|v_1\|^{\alpha-1}_{L_T^\alpha H_p^1}.$$

Since $\vec{v_1}(t)=(\vec{w}(t)+B(t)\vec{h})_1$ by Lemma 1.3.9 and the fact that $a^{\alpha-1}b\leq$

$a^\alpha + b^\alpha$ for $a, b, \alpha \geq 0$ we obtain

$$\begin{aligned}\Lambda_T(\Phi(\vec{w})) &\leq c\|v_1\|^{\alpha-1}_{L^\alpha_T H^1_p}(\|v_1\|_{L^\alpha_T H^1_p} + \|v_1\|_{L^\infty_T H^1}) \\ &\leq c[\|(B(t)\vec{h})_1\|^{\alpha-1}_{L^\alpha_T H^1_p}(\|\vec{h}\|_{Y^1} + \|(B(t)\vec{h})_1\|_{L^\alpha_T H^1_p}) \\ &+ \Lambda^{\alpha-1}_T(\vec{w})\|\vec{h}\|_{Y^1} + \Lambda^\alpha_T(\vec{w})].\end{aligned} \tag{1.29}$$

But by Lemma 1.3.2 the term $\|(B(t)\vec{h})_1\|^{\alpha-1}_{L^\alpha_T H^1_p}(\|\vec{h}\|_{Y^1} + \|(B(t)\vec{h})_1\|_{L^\alpha_T H^1_p})$ can be chosen small enough (for T large), so it is possible to choose a small enough such that Φ maps $B_T(0, a)$ into $B_T(0, a)$.

Now we have to prove that Φ is a contraction (for suitable choice of a and T). Set $\vec{v}(t) \equiv \vec{w}(t) + B(t)\vec{h}$ and $\vec{r}(t) \equiv \vec{z}(t) + B(t)\vec{h}$, then for all $t \geq T$

$$\Phi(\vec{w})(t) - \Phi(\vec{z})(t) = -\int_t^\infty B(t-t')(\vec{f}(\vec{v}(t')) - \vec{f}(\vec{r}(t')))dt'.$$

Using that $B(\cdot)$ is an unitary group, the definition of $\vec{f}$, Plancherel and adding the terms $\pm|r_1|^{\alpha-1}v_{1,x}$ we have

$$\|\Phi(\vec{w})(t) - \Phi(\vec{z})(t)\|_{Y_1} \;\leq\; \int_t^\infty \|\vec{f}(\vec{v}(t')) - \vec{f}(\vec{r}(t'))\|_{Y_1}dt'$$

$$\begin{aligned}&\leq\; c\int_t^\infty \|\partial_x(f(v_1) - f(r_1))(t')\|_{L^2}dt' \\ &\leq\; c(\int_t^\infty \|((|v_1|^{\alpha-2} + |r_1|^{\alpha-2})(v_1 - r_1)v_{1,x})(t')\|_{L^2}dt' + \\ &\quad + \int_t^\infty \|(r_1|r_1|^{\alpha-2}(v_{1,x} - r_{1,x}))(t')\|_{L^2}dt') \\ &\equiv\; c(I^1_1 + I^2_1).\end{aligned}$$

Moreover, using Hölder ($1/2 = \gamma/2 + (1-\gamma)/2$), $H^1_p \subseteq L^{2(\alpha-1)/\gamma}$ and Hölder ($1 = (\alpha-1)/\alpha + 1/\alpha$) we have for all $t \geq T$

$$\begin{aligned}I^2_1 &\leq c\int_t^\infty \|r_1(t')\|^{\alpha-1}_{L^{2(\alpha-1)/\gamma}}\|(v_{1,x} - r_{1,x})(t')\|_{L^p}dt' \\ &\leq c\int_t^\infty \|r_1(t')\|^{\alpha-1}_{H^1_p}\|(v_1 - r_1)(t')\|_{H^1_p}dt' \\ &\leq c(\|r_1\|^{\alpha-1}_{L^\alpha_T H^1_p}\|v_1 - r_1\|_{L^\alpha_T H^1_p}).\end{aligned}$$

For the other term, using Hölder ($1/2 = 1/p + 1/q + (\alpha-2)/q$ where $q =$

$\frac{2p(\alpha-1)}{p-2}$), $H_p^1 \subseteq L^q$ and Hölder ($1 = (\alpha-1)/\alpha + 1/\alpha$ and $1 = (\alpha-2)/\alpha + 1/\alpha + 1/\alpha$) we obtain for all $t \geq T$

$$\begin{aligned} I_1^1 &\leq c\int_t^\infty (\|v_1(t')\|_{L^q}^{\alpha-2} + \|r_1(t')\|_{L^q}^{\alpha-2})\|(v_1-r_1)(t')\|_{L^q}\|v_{1,x}(t')\|_{L^p}dt' \\ &\leq c\int_t^\infty (\|v_1(t')\|_{H_p^1}^{\alpha-1} + \|r_1(t')\|_{H_p^1}^{\alpha-2}\|v_1(t')\|_{H_p^1})\|(v_1-r_1)(t')\|_{H_p^1}dt' \\ &\leq c(\|v_1\|_{L_T^\alpha H_p^1}^{\alpha-1} + \|r_1\|_{L_T^\alpha H_p^1}^{\alpha-2}\|v_1\|_{L_T^\alpha H_p^1})\|v_1-r_1\|_{L_T^\alpha H_p^1}. \end{aligned}$$

Then for all $t \geq T$

$$\begin{aligned} \|\Phi(\vec{w}) - \Phi(\vec{z})\|_{L_T^\infty Y_1} \leq c(&\|v_1\|_{L_T^\alpha H_p^1}^{\alpha-1} + \|r_1\|_{L_T^\alpha H_p^1}^{\alpha-1})\|v_1-r_1\|_{L_T^\alpha H_p^1} \\ &+ \|r_1\|_{L_T^\alpha H_p^1}^{\alpha-2}\|v_1\|_{L_T^\alpha H_p^1}\|v_1-r_1\|_{L_T^\alpha H_p^1}. \end{aligned} \tag{1.30}$$

On the other hand, by the same argument used in (1.28) we obtain

$$\begin{aligned} \|\Phi(\vec{w}) - \Phi(\vec{z})\|_{Y_{1,1}^p} &\leq c\int_t^\infty |t-t'|^{-\gamma/2}(\|f(v_1)-f(r_1)\|_{L^{p'}}dt' + \\ &\quad +c\int_t^\infty |t-t'|^{-\gamma/2}(\|\partial_x(f(v_1)-f(r_1))\|_{L^{p'}}dt' \\ &\equiv c(I_2^1 + I_2^2). \end{aligned}$$

Using the Mean Value Theorem, Hölder's inequality ($1/p' \equiv (1+\gamma)/2 = 1/2 + \gamma/2$) and $H_p^1 \subseteq L^{2(\alpha-1)/\gamma}$ we can easily obtain

$$I_2^1 \leq c\|v_1 - r_1\|_{L_T^\infty H^1}\int_t^\infty |t-t'|^{-\gamma/2}(\|v_1\|_{H_p^1}^{\alpha-1} + \|r_1\|_{H_p^1}^{\alpha-1})dt'.$$

Applying the Hardy-Littlewood-Sobolev theorem we have

$$\|I_2^1\|_{L_T^\alpha} \leq c\|v_1-r_1\|_{L_T^\infty H^1}(\|v_1\|_{L_T^\alpha H_p^1}^{\alpha-1} + \|r_1\|_{L_T^\alpha H_p^1}^{\alpha-1}).$$

To estimate I_2^2 we first add $\pm|r_1|^{\alpha-1}v_{1,x}$ to obtain

$$\begin{aligned} I_2^2 \;\leq\; & c\int_t^\infty |t-t'|^{-\gamma/2}\|((|v_1|^{\alpha-2} + |r_1|^{\alpha-2})(v_1-r_1)v_{1,x})(t')\|_{L^{2/(1+\gamma)}}dt' + \\ & +c\int_t^\infty |t-t'|^{-\gamma/2}\|(|r_1|^{\alpha-1}(v_{1,x}-r_{1,x}))(t')\|_{L^{2/(1+\gamma)}}dt'). \end{aligned}$$

Using Hölder's inequality ($1/p' = \gamma(\alpha-2)/2(\alpha-1) + \gamma/2(\alpha-1) + 1/2$), $H_p^1 \subseteq L^{2(\alpha-1)/\gamma}$ for the first term and Hölder's inequality ($1/p' = \gamma/2 + 1/2$),

$H^1_p \subseteq L^{2(\alpha-1)/\gamma}$ for the second term we obtain

$$\begin{aligned} I_2^2 &\le c\int_t^\infty |t-t'|^{-\gamma/2}(\|r_1\|^{\alpha-2}_{H^1_p}+\|v_1\|^{\alpha-2}_{H^1_p})\|v_1-r_1\|_{H^1_p}\|v_{1,x}\|_{L^2}dt' + \\ &+c\int_t^\infty |t-t'|^{-\gamma/2}\|r_1\|^{\alpha-1}_{H^1_p}\|v_1-r_1\|_{H^1}dt'. \end{aligned}$$

The Hardy-Littlewood-Sobolev theorem and Hölder's inequality in the time variable $((\alpha-1)/\alpha = 1/\alpha + (\alpha-2)/\alpha)$ yield

$$\begin{aligned} \|I_2^2\|_{L^\alpha_T} &\le c(\|v_1\|^{\alpha-2}_{L^\alpha_T H^1_p}+\|r_1\|^{\alpha-2}_{L^\alpha_T H^1_p})\|v_1\|_{L^\infty_T H^1}\|v_1-r_1\|_{L^\alpha_T H^1_p} \\ &+c\|r_1\|^{\alpha-1}_{L^\alpha_T H^1_p}\|v_1-r_1\|_{L^\infty_T H^1}. \end{aligned}$$

Thus,

$$\begin{aligned} \|\Phi(\vec{w})-\Phi(\vec{z})\|_{L^\alpha_T Y^p_{1,1}} \le c(\|v_1\|^{\alpha-2}_{L^\alpha_T H^1_p}+\|r_1\|^{\alpha-2}_{L^\alpha_T H^1_p})\|v_1\|_{L^\infty_T H^1}\|v_1-r_1\|_{L^\alpha_T H^1_p} \\ + c(\|v_1\|^{\alpha-1}_{L^\alpha_T H^1_p}+\|r_1\|^{\alpha-1}_{L^\alpha_T H^1_p})\|v_1-r_1\|_{L^\infty_T H^1}. \end{aligned} \tag{1.31}$$

In view of (1.30) and (1.31) we conclude that

$$\Lambda_T(\Phi(\vec{w})-\Phi(\vec{z})) \le$$

$$\begin{aligned} &\le c(\|v_1\|^{\alpha-1}_{L^\alpha_T H^1_p}+\|r_1\|^{\alpha-1}_{L^\alpha_T H^1_p}+\|r_1\|^{\alpha-2}_{L^\alpha_T H^1_p}\|v_1\|_{L^\alpha_T H^1_p})\Lambda_T(\vec{w}-\vec{z}) + \\ &+c(\|v_1\|^{\alpha-2}_{L^\alpha_T H^1_p}+\|r_1\|^{\alpha-2}_{L^\alpha_T H^1_p})\|v_1\|_{L^\infty_T H^1}\Lambda_T(\vec{w}-\vec{z}) + \\ &+c(\|v_1\|^{\alpha-1}_{L^\alpha_T H^1_p}+\|r_1\|^{\alpha-1}_{L^\alpha_T H^1_p})\Lambda_T(\vec{w}-\vec{z}) \\ &\equiv A_1+A_2+A_3. \end{aligned}$$

Using the fact that $a^{\alpha-1}b \le a^\alpha + b^\alpha$ for $a, b, \alpha \ge 0$ we can reduce A_1 into A_3 so we need to treat only the last two terms. By the definition of $\vec{v}$, $\vec{r}$ and Lemma 1.3.9

$$\begin{aligned} A_2 &\le c(\|(B(t)\vec{h})_1\|^{\alpha-2}_{L^\alpha_T H^1_p}+a^{\alpha-2})(a+\|\vec{h}\|_{Y^1})\Lambda_T(\vec{w}-\vec{z}), \\ A_3 &\le c(\|(B(t)\vec{h})_1\|^{\alpha-1}_{L^\alpha_T H^1_p}+a^{\alpha-1})\Lambda_T(\vec{w}-\vec{z}). \end{aligned}$$

Since by Lemma 1.3.2 we can choose T such that the term $\|(B(t)\vec{h})_1\|_{L^\alpha_T L^p_1}$ is small enough. It is straightforward to choose a such that Φ is a contraction. Then Φ has a unique fixed point, which we denote by $\vec{w}$.

■

To finish the proof of Theorem 1.2.1 we need to prove (1.4). In view of Proposition 1.4.1, $\vec{u}$ defined in (1.26) is a solution of (0.3) in the time interval $[T_0, \infty)$. Now by (1.29) we have for all $T \geq T_0$

$$\begin{aligned}\Lambda_T(\vec{w}) = \Lambda_T(\Phi(\vec{w})) &\leq c[\|(B(t)\vec{h})_1\|^{\alpha-1}_{L^\alpha_T H^1_p}(\|\vec{h}\|_{Y^1} + \|(B(t)\vec{h})_1\|_{L^\alpha_T H^1_p}) + \\ &+ \Lambda^{\alpha-1}_T(\vec{w})\|\vec{h}\|_{Y^1} + \Lambda^\alpha_T(\vec{w})].\end{aligned}$$

Since $\alpha > 2$ and $\Lambda_T(\cdot) \leq \Lambda_{T_0}(\cdot)$, we can choose a sufficient small such that for $\vec{w} \in B_{T_0}(0,a)$ we have

$$c(\Lambda_T(\vec{w})^{\alpha-1}\|\vec{h}\|_{Y^1} + \Lambda_T(\vec{w})^\alpha) \leq \frac{1}{2}\Lambda_T(\vec{w}).$$

Therefore, by Lemma 1.3.2, when $T \to \infty$ we obtain

$$\begin{aligned}\|\vec{w}(T)\|_{Y^1} &\leq \|\vec{w}\|_{L^\infty_T Y^1} \\ &\leq c\|(B(t)\vec{h})_1\|^{\alpha-1}_{L^\alpha_T H^1_p}(\|\vec{h}\|_{Y^1} + \|(B(t)\vec{h})_1\|_{L^\alpha_T H^1_p}) \longrightarrow 0.\end{aligned} \tag{1.32}$$

■

Remark 1.4.1 *In fact, we prove that*

$$\|\vec{w}\|_{L^\infty_T Y^1} + \|\vec{w}\|_{L^\alpha_T Y^{1,1}_p} \longrightarrow 0, \text{ when } T \to \infty.$$

Proof of Theorem 1.2.2 Set $\beta = 1 - \dfrac{2}{\alpha+1}$ and define the following closed subset of a complete metric space

$$X_T(0,a) = \left\{ \begin{array}{l} \vec{w} \in L^\infty([T,+\infty); Y^1_{\alpha+1} \cap Y^1) : \\ \Gamma_T(\vec{w}) \equiv \sup\limits_{t \geq T}\{(1+t)^{\beta/3}\|\vec{w}\|_{Y^1_{\alpha+1}} + \|\vec{w}\|_{Y^1}\} \leq a \end{array} \right\}.$$

We first will prove an analog of Lemma 1.4.1.

Lemma 1.4.2 *There exist positive numbers T, a so that Φ maps $X_T(0,a)$ into $X_T(0,a)$ and becomes a contraction map in the $\Gamma_T(\cdot)$-metric.*

Proof Set $\vec{v}(t) \equiv \vec{w}(t) + B(t)\vec{h}$, then by Lemma 1.3.4 and applying Hölder's inequality $(1/q \equiv \alpha/(\alpha+1) = (\alpha-1)/(\alpha+1) + 1/(\alpha+1))$, we obtain

$$\begin{aligned}\|\Phi(\vec{w})(t)\|_{Y^1_{\alpha+1}} &\leq c\int_t^\infty (|t-t'|^{-1/2} + |t-t'|^{-1/3})^\beta \|v_1\|^{\alpha-1}_{L^{\alpha+1}} \|\partial_x v_1\|_{L^{\alpha+1}} dt' \\ &\leq c\int_t^\infty (|t-t'|^{-1/2} + |t-t'|^{-1/3})^\beta \|\vec{v}(t')\|^\alpha_{Y^1_{\alpha+1}} dt'.\end{aligned}$$

Therefore, for all $t \geq T$

$$\sup_{t\geq T}(1+t)^{\beta/3}\|\Phi(\vec{w})(t)\|_{Y^1_{\alpha+1}} \leq cI(T)\sup_{t\geq T}(1+t)^{\alpha\beta/3}\|\vec{v}(t)\|^\alpha_{Y^1_{\alpha+1}}$$

where

$$I(T) = \sup_{t\geq T}(1+t)^{\beta/3}\int_t^\infty [(|t-t'|^{-1/2} + |t-t'|^{-1/3})(1+t')^{-\alpha/3}]^\beta dt'].$$

Then by the definition of $\vec{v}$ and Lemma 1.3.4 we obtain

$$\sup_{t\geq T}(1+t)^{\beta/3}\|\Phi(\vec{w})(t)\|_{Y^1_{\alpha+1}} \leq cI(T)(\Gamma^\alpha_T(\vec{w}) + (H(T)\|\vec{h}\|_{Y^1_q})^\alpha) \tag{1.33}$$

where $1/q + 1/(1+\alpha) = 1$ and

$$H(T) = \sup_{t\geq T}[(|t|^{-1/2} + |t|^{-1/3})(1+t)^{1/3}]^\beta. \tag{1.34}$$

On the other hand, using that $B(\cdot)$ is a unitary group, Parseval and Hölder's inequality $(1/2 = (\alpha-1)/2(\alpha+1) + 1/(\alpha+1))$ we have

$$\|\Phi(\vec{w})(t)\|_{Y^1} \leq c\int_t^\infty \|v_1(t')\|^{\alpha-1}_{L^{2(\alpha+1)}}\|\partial_x v_1(t')\|_{L^{\alpha+1}} dt'.$$

Set $\theta \equiv 1/2(\alpha+1)$, by the Gagliardo-Nirenberg inequality we have

$$\|u\|_{L^{2(\alpha+1)}} \leq \|u_x\|^\theta_{L^{\alpha+1}}\|u\|^{1-\theta}_{L^{\alpha+1}}. \tag{1.35}$$

Therefore we obtain

$$\|\Phi(\vec{w})(t)\|_{Y^1} \leq c\left(K(T)(\Gamma^\alpha_T(\vec{w}) + (H(T)\|\vec{h}\|_{Y^1_q})^\alpha)\right) \tag{1.36}$$

where

$$K(T) = \int_T^\infty (1+t')^{-\alpha\beta/3} dt'. \tag{1.37}$$

By (1.33) and (1.36) we obtain

$$\Gamma_T(\Phi(\vec{w})) \leq c\Big(I(T)(\Gamma_T^\alpha(\vec{w}) + (H(T)\|\vec{h}\|_{Y_q^1})^\alpha) + K(T)(\Gamma_T^\alpha(\vec{w}) + (H(T)\|\vec{h}\|_{Y_q^1})^\alpha)\Big).$$

Hence Lemma 1.3.5 implies that we can choose a and T such that Φ maps $X_T(0,a)$ into $X_T(0,a)$.

We need to prove that Φ is a contraction. Set $\vec{v}$, $\vec{r}$ like in the proof of Theorem 1.2.1.

$$\|\Phi(\vec{w})(t) - \Phi(\vec{z})(t)\|_{Y^1_{\alpha+1}} \leq$$

$$c\int_t^\infty (|t-t'|^{-\frac{1}{2}} + |t-t'|^{-\frac{1}{3}})^\beta \|(|v_1|^{\alpha-1} - |r_1|^{\alpha-1})\partial_x v_1\|_{L^q} dt' + c\int_t^\infty (|t-t'|^{-\frac{1}{2}} + |t-t'|^{-\frac{1}{3}})^\beta \||r_1|^{\alpha-1}(\partial_x v_1 - \partial_x r_1)\|_{L^q} dt'.$$

By the Mean Value Theorem and the Hölder inequality $(1/q \equiv \alpha/(\alpha+1) = (\alpha-2)/(\alpha+1) + 1/(\alpha+1) + 1/(\alpha+1))$ for the first term and Hölder's inequality $(1/q \equiv \alpha/(\alpha+1) = (\alpha-1)/(\alpha+1) + 1/(\alpha+1))$ for the second, together with Lemma 1.3.4 we obtain

$$\sup_{t\geq T}(1+t)^{\beta/3}\|\Phi(\vec{w})(t) - \Phi(\vec{z})(t)\|_{Y^1_{\alpha+1}} \leq$$

$$\leq cI(T)(\Gamma_T^{\alpha-1}(\vec{w}) + \Gamma_T^{\alpha-1}(\vec{w}) + 2(H(T)\|\vec{h}\|_{Y_q^1})^{\alpha-1})\Gamma_T(\vec{w}-\vec{z}).$$

On the other hand, by Remark 0.0.1

$$\begin{aligned}\|\Phi(\vec{w})(t) - \Phi(\vec{z})(t)\|_{Y^1} &\leq c\int_t^\infty \|(|v_1|^{\alpha-1} - |r_1|^{\alpha-1})\partial_x v_1\|_{L^2} dt' \\ &\quad + c\int_t^\infty \||r_1|^{\alpha-1}(\partial_x v_1 + \partial_x r_1)\|_{L^2} dt' \\ &\equiv (I) + (II).\end{aligned}$$

For (I) we use the Mean Value Theorem and Hölder's inequality $(1/2 =$

$(\alpha-1)/2(\alpha+1)+1/(\alpha+1))$ to obtain

$$\begin{aligned}(I) \quad &\leq \quad c\int_t^\infty \|v_1\|^{\alpha-2}_{L^{\frac{2(\alpha+1)(\alpha-2)}{\alpha-1}}}\|v_1-r_1\|_{L^\infty}\|\partial_x v_1\|_{L^{\alpha+1}}dt' \\ &+c\int_t^\infty \|r_1\|^{\alpha-2}_{L^{\frac{2(\alpha+1)(\alpha-2)}{\alpha-1}}}\|v_1-r_1\|_{L^\infty}\|\partial_x v_1\|_{L^{\alpha+1}}dt'.\end{aligned}$$

Let $q>\alpha+1$ and set $\theta\equiv 1/(\alpha+1)-1/q$, by the Gagliardo-Nirenberg inequality we have

$$\|u\|_{L^q}\leq \|u_x\|^\theta_{L^{\alpha+1}}\|u\|^{1-\theta}_{L^{\alpha+1}}. \tag{1.38}$$

Remark 1.4.2 *The inequality (1.38) is still true for* $q=\infty$.

Since $\alpha>3$ we have $\frac{2(\alpha+1)(\alpha-2)}{\alpha-1}>\alpha+1$ so applying (1.38) it follows that

$$(I)\leq c\int_t^\infty (\|\vec{v}\|^{\alpha-1}_{Y^1_{\alpha+1}}+\|\vec{r}\|^{\alpha-1}_{Y^1_{\alpha+1}})\|\vec{v}-\vec{r}\|_{Y^1_{\alpha+1}}dt'.$$

For (II) we use Hölder's inequality $(1/2=(\alpha-1)/2(\alpha+1)+1/(\alpha+1))$ and (1.35) to obtain

$$(II)\leq c\int_t^\infty \|\vec{r}\|^{\alpha-1}_{Y^1_{\alpha+1}}\|\vec{v}-\vec{r}\|_{Y^1_{\alpha+1}}dt'.$$

Using the last two estimates together with the definitions (1.34) and (1.37) it follows that

$$\begin{aligned}\sup_{t\geq T}\|\Phi(\vec{w})(t)-\Phi(\vec{z})(t)\|_{Y^1} \leq cK(T)\Big(\Gamma^{\alpha-1}_T(\vec{w})+\Gamma^{\alpha-1}_T(\vec{z})+ \\ +2(H(T)\|\vec{h}\|_{Y^1_q})^{\alpha-1}\Big)\,\Gamma_T(\vec{w}-\vec{z}).\end{aligned}$$

Thus by Lemma 1.3.5, we can choose a and T such that Φ is a contraction. Moreover, using a similar argument as the one used in (1.32) we can prove that $\|\vec{w}(T)\|_{Y^1}\to 0$, when $T\to\infty$.

■

Remark 1.4.3 *Notice that, we have prove*

$$\sup_{t\geq T}(1+t)^{\beta/3}\|\vec{w}\|_{Y^1_{\alpha+1}}+\|\vec{w}\|_{L^\infty_T Y^1}\to 0, \ \textit{when } T\to\infty.$$

Proof of Theorem 1.2.3 To prove that Φ has a fixed point we introduce the following metric space

$$\begin{aligned} \Sigma^{s,\theta}_{a,T} &= \left\{ \begin{array}{l} \vec{w} \in L^\infty([T,+\infty); \widetilde{L}^\infty \cap Y^s) : \\ \Gamma_T(\vec{w}) \equiv \sup_{t\geq T}(1+t)^\theta \|\vec{w}(t)\|_{\widetilde{L}^\infty} + \sup_{t\geq T}\|\vec{w}(t)\|_{Y^s} \leq a \end{array} \right\}, \\ d(\vec{u},\vec{v}) &= \sup_{t\geq T} \|\vec{u}-\vec{v}\|_{Y^0}. \end{aligned}$$

Lemma 1.4.3 *$(\Sigma^{s,\theta}_{a,T}, d)$ is a complete metric space.*

Proof This result follows using the same arguments as in [11] page 14. ∎

Thus we need to prove the following result.

Lemma 1.4.4 *There exist T and a so that Φ maps $\Sigma^{s,\theta}_{a,T}$ into $\Sigma^{s,\theta}_{a,T}$ and becomes a contraction map in the metric d.*

Proof Set $\vec{v}(t) \equiv \vec{w}(t) + B(t)\vec{h}$, then using Lemma 1.3.7 (ii), $H^s_{r'} \hookrightarrow B^s_{r',2}$ for $r' \in (1,2]$ (see [4] Theorem 6.4.4), $H^s_{r'} = L^{r'} \cap \dot{H}^s_{r'}$ for $s > 0$ (see [4] Theorem 6.3.2), Hölder's inequality $(1/r' = (r-2)/2r + 1/2)$ and Lemma 1.3.12 (i) with $r_1 = 2r/(r-2)$, $r_2 = 2$, we obtain

$$\begin{aligned} \|\Phi(\vec{w}(t))\|_{\widetilde{L}^\infty} &\leq c\int_t^\infty (1+|t-t'|)^{-\theta} \left(\|f(v_1(t'))\|_{L^{r'}} + \|D^s f(v_1(t'))\|_{L^{r'}}\right) dt' \\ &\leq c\int_t^\infty (1+|t-t'|)^{-\theta} \big(\||v_1|^{\alpha-1}\|_{L^{2r/(r-2)}}\|v_1\|_{L^2} + \\ &\quad + \||v_1|^{\alpha-1}\|_{L^{2r/(r-2)}}\|v_1\|_{H^s}\big)\, dt' \\ &\leq c\int_t^\infty (1+|t-t'|)^{-\theta} \|v_1\|_{L^\infty}^{\alpha-\frac{2}{r'}} \|v_1\|_{L^2}^{\frac{2}{r'}-1} dt'. \end{aligned}$$

Therefore, by Lemma 1.3.7 (i), Lemma 1.3.9 and the fact that $a^\alpha b^\beta \leq a^{\alpha+\beta} + b^{\alpha+\beta}$ for $a, b, \alpha, \beta \geq 0$, we obtain

$$\sup_{t\geq T}(1+t)^\theta \|\Phi(\vec{w}(t))\|_{L^\infty} \leq c\left(\Gamma^\alpha_T(\vec{w}) + \|\vec{h}\|^\alpha_{\dot{\mathfrak{B}}^{1-\frac{2}{r},\frac{n}{r}}_{r',1}} + \|\vec{h}\|^\alpha_{Y^s}\right) J(T)$$

where $J(T)$ was defined in (1.23).

On the other hand, using that $B(\cdot)$ is an unitary group, the definition of Y^s

and Lemma 1.3.12 with $r_1 = \infty$, $r_2 = 2$, we have

$$\begin{aligned} \|\Phi(\vec{w}(t))\|_{Y^s} &\leq c\int_t^\infty \left(\|v_1(t')\|_{L^\infty}^{\alpha-1}\|v_1(t')\|_{L^2} + \|v_1(t')\|_{L^\infty}^{\alpha-1}\|D^s v_1(t')\|_{L^2}\right) dt' \\ &\leq c\int_t^\infty \left(\|v_1(t')\|_{L^\infty}^{\alpha-1}\|v_1(t')\|_{H^s}\right) dt'. \end{aligned}$$

Therefore by the definitions of v_1, $\Sigma_{a,T}^{s,\theta}$ and (1.24), we have

$$\sup_{t\geq T}\|\Phi(\vec{w}(t))\|_{Y^s} \leq c\left(\Gamma_T^\alpha(\vec{w}) + \|\vec{h}\|^\alpha_{\dot{\mathfrak{B}}^{1-\frac{2}{r},\frac{n}{r}}_{r',1}} + \|\vec{h}\|_{Y^s}^\alpha\right) L(T).$$

So, since $\alpha > 2$, by Lemma 1.3.10 it is clear that we can choose a and T such that Φ maps $\Sigma_{a,T}^{s,\theta}$ into $\Sigma_{a,T}^{s,\theta}$.

Now we prove that Φ is a contraction in the metric d. Indeed, for $\vec{v} \equiv \vec{w} + B(\cdot)\vec{h}$ and $\vec{r} \equiv \vec{z} + B(\cdot)\vec{h}$, we have by the Mean Value Theorem, (1.24) and Lemma 1.3.7 (i)

$$\begin{aligned} d(\Phi(\vec{w}),\Phi(\vec{z})) &\leq c\int_T^\infty \left(\|v_1\|_{L^\infty}^{\alpha-1} + \|r_1\|_{L^\infty}^{\alpha-1}\right)\|(v_1 - r_1)(t')\|_{L^2} dt' \\ &\leq cd(\vec{w}-\vec{z})L(T)\left(\Gamma_T^{\alpha-1}(\vec{w}) + \Gamma_T^{\alpha-1}(\vec{z}) + \|\vec{h}\|^{\alpha-1}_{\dot{\mathfrak{B}}^{1-\frac{2}{r},\frac{n}{r}}_{r',1}}\right). \end{aligned}$$

By Lemma 1.3.10, T and a can be chosen such that Φ is a contraction in the d metric. So Φ has a unique fixed point, which we denote by $\vec{w}$. Moreover, using a similar argument as the one in (1.32) we can prove that $\|\vec{w}(T)\|_{Y^s} \to 0$, when $T \to \infty$.

■

Remark 1.4.4 *We actually proved that*

$$\sup_{t\geq T}(1+t)^\theta\|\vec{w}\|_{\widetilde{L}^\infty} + \|\vec{w}\|_{L^\infty_T Y^s} \to 0, \text{ when } T\to\infty.$$

Proof of Theorem 1.2.4 In this case we define the following metric space

$$\begin{aligned} \Xi_{a,T}^{s,\alpha+1} &= \left\{\begin{array}{l} \vec{w}\in L^\infty([T,+\infty);\widetilde{H}^s_{\alpha+1}) : \\ \Delta_T(\vec{w}) \equiv \sup_{t\geq T}(1+t)^\theta\|\vec{w}\|_{\widetilde{H}^s_{\alpha+1}} \leq a \end{array}\right\}, \\ d(\vec{u},\vec{v}) &= \sup_{t\geq T}(1+t)^\theta\|\vec{u}-\vec{v}\|_{\widetilde{L}^{\alpha+1}}. \end{aligned}$$

Lemma 1.4.5 *$(\Xi^{s,\alpha+1}_{a,T}, d)$ is a complete metric space.*

Proof See [11] page 14.

■

Now we will prove an analog of Lemma 1.4.4, that is

Lemma 1.4.6 *There exist $T, a > 0$ such that Φ maps $\Xi^{s,\alpha+1}_{a,T}$ into $\Xi^{s,\alpha+1}_{a,T}$ and becomes a contraction map in the $\Delta_T(\cdot)$-metric.*

Proof Set $\vec{v}(t) \equiv \vec{w}(t) + B(t)\vec{h}$, then using $B^s_{\alpha+1,2} \hookrightarrow H^s_{\alpha+1}$ (see Theorem 6.4.4 of [4]), Lemma 1.3.8 (i),$H^s_{\frac{\alpha+1}{\alpha}} \hookrightarrow B^s_{\frac{\alpha+1}{\alpha},2}$ (see Theorem 6.4.4 of [4]), $H^s_{\frac{\alpha+1}{\alpha}} = L^{\frac{\alpha+1}{\alpha}} \cap \dot{H}^s_{\frac{\alpha+1}{\alpha}}$ (see Theorem 6.2.3 of [4]) and Lemma 1.3.12 (i) with $r = \frac{\alpha+1}{\alpha}, r_1 = \frac{\alpha+1}{\alpha-1}, r_2 = \alpha+1$, it follows that

$$\begin{aligned}
\|\Phi(\vec{w})\|\|_{\tilde{H}^s_{\alpha+1}} &\leq c\int_t^\infty |t-t'|^{-\theta}\|f(v_1)(t')\|_{B^s_{\frac{\alpha+1}{\alpha},2}}dt' \\
&\leq c\int_t^\infty |t-t'|^{-\theta}\left(\|v_1(t')\|^\alpha_{L^{\alpha+1}}+\right. \\
&\quad \left.+\|v_1(t')\|^{\alpha-1}_{L^{\alpha+1}}\|D^s v_1(t')\|_{L^{\alpha+1}}\right)dt' \\
&\leq c\int_t^\infty |t-t'|^{-\theta}\|v_1(t')\|^\alpha_{H^s_{\alpha+1}}dt'.
\end{aligned}$$

Finally, the definition of Δ_T, $B^s_{\alpha+1,2} \hookrightarrow H^s_{\alpha+1}$ (see Theorem 6.4.4 of [4]), Lemma 1.3.8 (iii), the fact that $t^{-\theta\alpha} \leq c(1+t)^{-\theta\alpha}$ for all $t \geq 1$ and definition (1.25) we have

$$\begin{aligned}
\Delta_T(\Phi(\vec{w})) &= \sup_{t\geq T}(1+t)^\theta\|\Phi(\vec{w})\|_{\widetilde{H}^s_{\alpha+1}} \\
&\leq cM(T)\left(\Delta^\alpha_T(\vec{w}) + \|\vec{h}\|^\alpha_{\widetilde{\mathfrak{B}}^{1-\frac{2}{\alpha+1},s}_{\frac{\alpha+1}{\alpha},2}}\right).
\end{aligned}$$

Thus, since $\alpha > 1$, by Lemma 1.3.11 it is clear that we can choose a and T such that $\Phi(\Xi^{s,\alpha+1}_{a,T}) \subseteq \Xi^{s,\alpha+1}_{a,T}$.

Set $\vec{v}$ and $\vec{r}$ as in the proof of Theorem 1.2.3. Using $\dot{B}^0_{\alpha+1,2} \hookrightarrow L^{\alpha+1}$ (See [37]) and Lemma 1.3.8 (i), we have

$$\begin{aligned}
\|\Phi(\vec{w}(t)) - \Phi(\vec{z}(t))\|_{\widetilde{L}^{\alpha+1}} &\leq c\int_t^\infty \|B(t-t')(\vec{f}(\vec{v}_1) - \vec{f}(\vec{r}_1))(t')\|_{\widetilde{L}^{\alpha+1}}dt' \\
&\leq c\int_t^\infty |t-t'|^{-\theta}\|f(v_1) - f(r_1)(t')\|_{\dot{B}^0_{\frac{\alpha+1}{\alpha},2}}dt'
\end{aligned}$$

We know that $L^{\frac{\alpha+1}{\alpha}} \hookrightarrow \dot{B}^0_{\frac{\alpha+1}{\alpha},2}$ (See [37]). Therefore, applying the Mean Value Theorem and Hölder's inequality ($\frac{\alpha}{\alpha+1} = \frac{\alpha-1}{\alpha+1} + \frac{1}{\alpha+1}$), we obtain

$$\begin{aligned}
\|\Phi(\vec{w}(t)) - \Phi(\vec{z}(t))\|_{\widetilde{L}^{\alpha+1}} &\leq c\int_t^\infty |t-t'|^{-\theta}\|f(v_1) - f(r_1)(t')\|_{L^{\frac{\alpha+1}{\alpha}}} dt' \\
&\leq c\int_t^\infty |t-t'|^{-\theta}\| \left(|v_1|^{\alpha-1} + |r_1|^{\alpha-1}\right)|v_1 - r_1|\|_{L^{\frac{\alpha+1}{\alpha}}} dt' \\
&\leq c\int_t^\infty |t-t'|^{-\theta}\|v_1\|^{\alpha-1}_{L^{\alpha+1}}\|v_1 - r_1\|_{L^{\alpha+1}} dt' + \\
&\quad +c\int_t^\infty |t-t'|^{-\theta}\|r_1\|^{\alpha-1}_{L^{\alpha+1}}\|v_1 - r_1\|_{L^{\alpha+1}} dt' \\
&\leq c\sup_{t\geq T}\left((1+t)^\theta\|v_1\|_{L^{\alpha+1}} + (1+t)^\theta\|r_1\|_{L^{\alpha+1}}\right)^{\alpha-1} \\
&\quad d(\vec{w},\vec{z})\int_t^\infty |t-t'|^{-\theta}(1+t')^{-\alpha\theta}.
\end{aligned}$$

Finally, since $H^s_{\alpha+1} \subseteq L^{\alpha+1}$, $B^s_{\alpha+1,2} \subseteq H^s_{\alpha+1}$ (see Theorems 6.3.2 and 6.4.4 of [4]), Lemma 1.3.8 (iii) yields

$$\begin{aligned}
d(\Phi(\vec{w}) - \Phi(\vec{z})) &= \sup_{t\geq T}(1+t)^\theta\|\Phi(\vec{w}(t)) - \Phi(\vec{z}(t))\|_{\widetilde{L}^{\alpha+1}} \\
&\leq c\, d(\vec{w},\vec{z})M(T)\Big(\|(1+t)^\theta w_1\|^{\alpha-1}_{L^\infty_T H^s_{\alpha+1}} + \\
&\quad \|(1+t)^\theta z_1\|^{\alpha-1}_{L^\infty_T H^s_{\alpha+1}} + 2\|(1+t)^\theta (B(t)\vec{h})_1\|^{\alpha-1}_{L^\infty_T H^s_{\alpha+1}}\Big) \\
&\leq c\, d(\vec{w},\vec{z})M(T)\left(\Delta^{\alpha-1}_T(\vec{w}) + \Delta^{\alpha-1}_T(\vec{z}) + \right. \\
&\quad \left. +2\|(1+t)^\theta (B(t)\vec{h})_1\|^{\alpha-1}_{L^\infty_T B^s_{\alpha+1,2}}\right) \\
&\leq c\, d(\vec{w},\vec{z})M(T)\left(\Delta^{\alpha-1}_T(\vec{w}) + \Delta^{\alpha-1}_T(\vec{z}) + \right. \\
&\quad \left. +2\sup_{t\geq T}\left(t^{-\theta}(1+t)^\theta\right)\|\vec{h}\|_{\widetilde{\mathfrak{B}}^{1-\frac{2}{\alpha+1},s}_{\frac{\alpha+1}{\alpha},2}}\right).
\end{aligned}$$

Since $t^{-\theta}(1+t)^\theta \leq c$ for all $t \geq 1$ and $M(T) \to 0$ when $T \to \infty$ (see Lemma 1.3.11) it is clear that a and T can be chosen such that Φ is a contraction in d-metric.

Moreover, by similar estimates as the one used in (1.32) we can show that

$$\sup_{t\geq T}(1+t)^\theta\|\vec{w}\|_{\widetilde{H}^s_{\alpha+1}} \to 0, \text{ when } T \to \infty.$$

■

Chapter 2
Local solutions and unconditional well-posedness

2.1 Introduction

In this chapter we consider the generalized Boussinesq equation (0.1), where the nonlinearity f satisfies the following assumptions

($f1$) $f \in C^{[s]}(\mathbb{C}, \mathbb{C})$, where $s \geq 0$ and $[s]$ denotes the smallest positive integer $\geq s$;

($f2$) $|f^{(l)}(v)| \lesssim |v|^{k-l}$ for all integers l varying in the whole range $0 \leq l \leq [s] \leq k$ with $k > 1$;

($f3$) If $s \leq \dfrac{n}{2}$ then $1 < k \leq 1 + \dfrac{4}{n-2s}$.

Here we will consider first the local well-posedness problem. By Duhamel's principle, one can study the problem by rewriting the differential equation (0.1) in the integral form (0.22). Then we analyze the equation by a fixed point technique. That is, we find $T > 0$ and define a suitable complete subspace of $C([0,T]; H^s(\mathbb{R}^n))$, for instance Ξ_s, such that the integral equation is stable and contractive in this space. By Banach's fixed point theorem, there exists a unique solution in Ξ_s.

However, to define the subset Ξ_s we will need some auxiliary conditions, which is based on the available Strichartz estimates for the linear Schrödinger equation

$$i\partial_t u + \Delta u = 0$$

(see, for example, [31] chapter 4).

Definition 2.1.1 *We call (q, r) an admissible pair if they satisfy the condition:*

$$\frac{2}{q} = n\left(\frac{1}{2} - \frac{1}{r}\right)$$

where

$$\begin{cases} 2 \le r \le \infty & , \text{ if } n = 1, \\ 2 \le r < \infty & , \text{ if } n = 2, \\ 2 \le r \le \dfrac{2n}{n-2} & , \text{ if } n \ge 3. \end{cases}$$

Remark 2.1.1 *We included in the above definition the recent improvement, due to M. Keel and T. Tao [25], to the limiting case for Strichartz's inequalities.*

Now, we can define the following (auxiliary) space

$$\begin{aligned} \mathcal{Y}_s &= (1-\Delta)^{-\frac{s}{2}} \left(\bigcap \{ L^q L^r : \text{ is an admissible pair} \} \right) \\ &= \bigcap \{ L^q H_r^s : (q,r) \text{ is an admissible pair} \} . \end{aligned}$$

With these notations and definitions, we have the following answer to the local existence problem

Theorem 2.1.1 *Assume* $(f1) - (f3)$ *and* $s \ge 0$*. Then for any* $\phi \in H^s(\mathbb{R}^n)$ *and* $\psi = \Delta \eta$ *with* $\eta \in H^s(\mathbb{R}^n)$*, there are* $T > 0$ *and a unique solution* u *to (0.1) with the following properties*

(i) $u \in C([0,T]; H^s(\mathbb{R}^n))$*;*

(ii) $u \in \mathcal{Y}_s$*.*

The next two results are related with the life span and blow-up of the solutions given by Theorem 2.1.1.

Theorem 2.1.2 *Let* $[0, T^*)$ *be the maximal interval of existence for* u *in Theorem 2.1.1. Then* T^* *depends on* ϕ*,* η *in the following way*

(i) Let $s > \dfrac{n}{2}$ *and* $\sigma > 0$ *such that* $\dfrac{n}{2} < \sigma \le s$*. Then* T^* *can be estimate in terms of* $\|\phi\|_{H^\sigma}$ *and* $\|\eta\|_{H^\sigma}$ *only. Moreover,*

$$T^* \to \infty \text{ when } \max\{\|\phi\|_{H^\sigma}, \|\eta\|_{H^\sigma}\} \to 0. \tag{2.1}$$

(ii) Let $s \le \dfrac{n}{2}$ *and* $\sigma \ge 0$ *such that*

$$\sigma \in \left[0, \frac{n}{2}\right) \bigcap \left[\frac{n}{2} - \frac{2}{k-1}, s\right]; \tag{2.2}$$

(iia) *If* $\sigma > \frac{n}{2} - \frac{2}{k-1}$*, Then* T^* *can be estimate in terms of* $\|D^\sigma \phi\|_{L^2}$ *and* $\|D^\sigma \eta\|_{L^2}$ *only. Moreover,*

$$T^* \to \infty \text{ when } \max\{\|D^\sigma \phi\|_{L^2}, \|D^\sigma \eta\|_{L^2}\} \to 0.$$

(iib) *If* $\sigma = \frac{n}{2} - \frac{2}{k-1}$*, the time* T^* *can be estimated in terms of* $D^\sigma \phi, D^\sigma \eta \in L^2$*, but not necessarily of their norms.*

Theorem 2.1.3 *In Theorem 2.1.2, suppose that* $T^* < \infty$*. Then*

(a) *In case (i),* $\max\{\|u(t)\|_{H^\sigma}, \|\Delta^{-1} u_t(t)\|_{H^\sigma}\}$ *blows up at* $t = T^*$ *for all* σ *such that* $\frac{n}{2} < \sigma \le s$*;*

(b) *In case (iia),* $\max\{\|D^\sigma u(t)\|_{L^2}, \|D^\sigma \Delta^{-1} u_t(t)\|_{L^2}\}$ *blows up at* $t = T^*$ *for all* $\sigma \neq \frac{n}{2} - \frac{2}{k-1}$ *and satisfying (2.2).*

Note that part (ii) of Theorem 2.1.1 is essential; without such a condition, uniqueness might not hold. In this case, following [24], we say that (0.1) is conditionally well-posed in $H^s(\mathbb{R}^n)$, with the auxiliary space $\mathcal{Y}_s$.

A natural question arise in this context: Is it possible to remove the auxiliary condition? In other words, is it possible to prove that uniqueness of the solution for (0.1) holds in the whole space $C([0,T]; H^s(\mathbb{R}^n))$? If the answer for these two questions is yes, then we say that (0.1) is unconditionally well-posed in $H^s(\mathbb{R}^n)$.

The next two theorems are concerned with this latter notion.

Theorem 2.1.4 *Assume* $(f1) - (f3)$ *and let* $s \ge 0$*. Uniqueness for (0.1) holds in* $C([0,T]; H^s)$ *in each of the following cases*

(i) $s \ge \frac{n}{2}$;

(ii) $n = 1$, $0 \le s < \frac{1}{2}$ *and* $k \le \frac{2}{1-2s}$;

(iii) $n = 2$, $0 \le s < 1$ *and* $k < \frac{s+1}{1-s}$;

(iv) $n \ge 3$, $0 \le s < \frac{n}{2}$, $k \le \min\left\{1 + \frac{4}{n-2s}, 1 + \frac{2s+2}{n-2s}\right\}$.

The fundamental tool to prove Theorem 2.1.4 are the classic Strichartz estimates satisfied by the solution of the Schrödinger equation. We remark that parts (*i*), (*ii*), and (*iii*) of the above theorem are identical, respectively, to (*i*), (*iii*), and (*ii*) for $n = 2$ of [24], Corollary 2.3. However, for $n \geq 3$, we include the high extreme point for the value of k, in the range of validity of the theorem. This is possible due to the improvement in the Strichartz estimates proved by Keel and Tao [25].

For the particular case where $f(u) = |u|^{k-1}u$, with $k > 1$ we can also improve Theorem 2.1.4 for a large range of values k. This is done in the following theorem.

Theorem 2.1.5 *Let $n \geq 3$, $0 < s < 1$ and $f(u) = |u|^{k-1}u$, with $k > 1$ satisfying $(f3)$. Uniqueness for (0.1) holds in $C([0,T];H^s)$ if k verifies the following conditions*

(1) $k > 2$;

(2) $k > 1 + \dfrac{2s}{n-2s}$, $k < 1 + \min\left\{\dfrac{n+2s}{n-2s}, \dfrac{4s+2}{n-2s}\right\}$;

(3) $k < 1 + \dfrac{4}{n-2s}$;

(4) $k \leq 1 + \dfrac{n+2-2s}{n-2s}$.

Remark 2.1.2 *Note that the restriction $k \leq \dfrac{n+2s}{n-2s}$ seems natural. In fact, this assumption implies $|u|^{k-1}u \in L^1_{loc}(\mathbb{R}^n)$, which ensures that the equation*

$$\begin{cases} u_{tt} - \Delta u + \Delta^2 u + \Delta\left(|u|^{k-1}u\right) = 0, & x \in \mathbb{R}^n, t > 0, \\ u(x,0) = \phi, \quad u_t(x,0) = \Delta\eta \end{cases}$$

makes sense within the framework of the distribution.

Theorem 2.1.5 is inspired on the unconditional well-posed result proved by Furioli and Terraneo [18] for the case of nonlinear Schrödinger equation. As in [18], the proof of this theorem relies in the use of Besov space of negative indices.

The plan of this chapter is as follows: in Section 2, we prove some linear estimates and other preliminary results. The local existence theory is established in Section 3. Finally, the unconditional well-posedness problem is treated in Section 4.

2.2 Preliminary results

In the sequel, we will use the integral formulation (0.22). To treat this integral equation, we need to obtain estimates for the operators $B_c(\cdot)$ and $B_s(\cdot)\Delta$

defined in (0.20) and (0.21), respectively. Let us recall the well-known Strichartz inequalities for solutions of Schrödinger Equation.

Lemma 2.2.1 *If* $t \neq 0$, $\dfrac{1}{p} + \dfrac{1}{p'} = 1$ *and* $p' \in [1,2]$ *then we have that*

$$\|U(t)h\|_{L^p} \leq c|t|^{-\frac{n}{2}\left(\frac{1}{p'}-\frac{1}{p}\right)}\|h\|_{L^{p'}}.$$

Proof See, for instance, [31] Chapter 4.

■

Lemma 2.2.2 *Let* (q,r) *be an admissible pair and* $0 < T \leq \infty$, *then*

$$\sup_{[-T,T]}\left\|\int_0^t U(t')g(\cdot,t')dt'\right\|_{L^2} \leq c\|g\|_{L^{q'}_{0,T}L^{r'}}.$$

Proof Again we refer the reader to [31] Chapter 4.

■

As a consequence of Lemmas 2.2.1-2.2.2 we can prove Strichartz-type inequalities for the operators $B_c(\cdot)$ and $B_s(\cdot)\Delta$. More precisely,

Lemma 2.2.3 *Let* (q,r) *and* (γ,ρ) *be admissible pairs and* $0 < T \leq \infty$. *Then*

(*i*) $\|B_c(\cdot)h\|_{L^q_{0,T}L^r} + \|B_s(\cdot)\Delta h\|_{L^q_{0,T}L^r} \leq c\|h\|_{L^2}$;

(*ii*) $\left\|\int_0^t B_s(t-t')\Delta g(\cdot,t')dt'\right\|_{L^q_{0,T}L^r} \leq c\|g\|_{L^{\gamma'}_{0,T}L^{\rho'}}$,
where (γ',ρ') *denotes the dual of* (γ,ρ).

Proof We will prove only the second inequality (item (*i*) follows from (*ii*) and a duality argument). Let (q,r) be an admissible pair. In view of Lemma 2.2.1, we have

$$\begin{aligned}\left\|\int_0^t B_s(t-t')\Delta g(\cdot,t')dt'\right\|_{L^r} &\leq \left\|\frac{i}{2}\int_0^t \left(U(t-t') - U(t-t')\right)g(\cdot,t')dt'\right\|_{L^r}\\ &\leq c\int_0^t \|U(t-t')g(\cdot,t')\|_{L^r} + \|U(t-t')g(\cdot,t')\|_{L^r}dt'\\ &\leq c\int_{-\infty}^{\infty}\frac{1}{|t-t'|^{\alpha}}\|g(\cdot,t')\|_{L^{r'}}dt'\end{aligned}$$

where $\alpha = \dfrac{n}{2}\left(\dfrac{1}{r'} - \dfrac{1}{r}\right)$.

Thus applying the Hardy-Littlewood-Sobolev theorem we obtain

$$\left\| \int_0^t B_s(t-t')\Delta g(\cdot,t')dt' \right\|_{L^q_{0,T}L^r} \leq c\|g\|_{L^{q'}_{0,T}L^{r'}}, \tag{2.3}$$

where (q', r') denotes the dual of (q, r).

Combining Lemma 2.2.2 with the definition of $B_s(\cdot)\Delta$ we obtain the following inequality

$$\sup_{[0,T]} \left\| \int_0^t B_s(t-t')\Delta g(\cdot,t')dt' \right\|_{L^2} \leq c\|g\|_{L^{q'}_{0,T}L^{r'}}. \tag{2.4}$$

Now, let (γ, ρ) be another admissible pair. Without loss of generality we can assume $\rho \in [2, r)$. Therefore, interpolating (2.3) and (2.4), we have

$$\left\| \int_0^t B_s(t-t')\Delta g(\cdot,t')dt' \right\|_{L^\gamma_{0,T}L^\rho} \leq c\|g\|_{L^{q'}_{0,T}L^{r'}}.$$

To finish the proof, an argument of duality allows us to write

$$\left\| \int_0^t B_s(t-t')\Delta g(\cdot,t')dt' \right\|_{L^q_{0,T}L^r} \leq c\|g\|_{L^{\gamma'}_{0,T}L^{\rho'}}.$$

■

For the question of unconditional well-posedness, we will need Strichartz-type inequalities in Besov spaces, that is;

Lemma 2.2.4 *Let (q, r) and (γ, ρ) be admissible pairs. Then*

(*i*) $\|B_c(\cdot)h\|_{L^q_{0,T}\dot{B}^s_{r,2}} + \|B_s(\cdot)\Delta h\|_{L^q_{0,T}\dot{B}^s_{r,2}} \leq c\|h\|_{\dot{H}^s}$;

(*ii*) $\left\| \int_0^t B_s(t-t')\Delta g(\cdot,t')dt' \right\|_{L^q_{0,T}\dot{B}^s_{r,2}} \leq c\|g\|_{L^{\gamma'}_{0,T}\dot{B}^s_{\rho',2}}$.

Proof Since the above estimates are valid for the Schrödinger group (see [10] Theorem 2.2), using (0.20) and (0.21) the lemma follows.

■

Remark 2.2.1 *These Strichartz inequalities are still valid if we replace $\dot{B}^s_{q,2}$ by the homogeneous Sobolev spaces $\dot{H}^s_q$. (see [10] page 814).*

Another important result are the estimates for the fractional derivatives.

Lemma 2.2.5 *Assume (f1)-(f2) and for $0 \leq s \leq k$, define $D^s = \mathcal{F}^{-1}|\xi|^s\mathcal{F}$, then* *(i)* $\|D^s f(u)\|_{L^r} \le c\|u\|_{L^{(k-1)r_1}}^{k-1}\|D^s u\|_{L^{r_2}}$

where $\dfrac{1}{r} = \dfrac{1}{r_1} + \dfrac{1}{r_2}$, $r_1 \in (1,\infty]$, $r_2 \in (1,\infty)$;

(ii) $\|D^s(uv)\|_{L^r} \le c\left(\|D^s u\|_{L^{r_1}}\|v\|_{L^{q_2}} + \|u\|_{L^{q_1}}\|D^s v\|_{L^{r_2}}\right)$

where $\dfrac{1}{r} = \dfrac{1}{r_1} + \dfrac{1}{q_2} = \dfrac{1}{q_1} + \dfrac{1}{r_2}$, $r_i \in (1,\infty)$, $q_i \in (1,\infty]$, $i = 1, 2$.

Proof See [24] Lemmas A1-A4.

■

Lemma 2.2.6 *Let* $k > 1$, $s \ge 0$, $p \in [1,\infty)$, $s < \min\left\{\dfrac{n}{p}, k\right\}$ *and* $\dfrac{1}{p} - \dfrac{s}{n} \le \dfrac{1}{k}$. *Let* $\alpha = \dfrac{n}{s + k\left(\dfrac{n}{p} - s\right)}$. *Then there exists* $c > 0$ *such that for all* $g \in \dot{H}^s_p(\mathbb{R}^n)$, *we have*

(i) $\||g|^{k-1}g\|_{\dot{H}^s_\alpha} \le c\|g\|^k_{\dot{H}^s_p}$;

(ii) $\||g|^k\|_{\dot{H}^s_\alpha} \le c\|g\|^k_{\dot{H}^s_p}$.

Proof See [18] Lemma 2.3 and the references therein.

■

Before finishing this section, we present some numerical facts that will be important in the local existence result proof.

Lemma 2.2.7 *Let* $k > 1$, *there is* $q \ge 2$ *and* (γ, ρ) *an admissible pair, such that*

$$\frac{1}{\rho'} = \frac{1}{2} + \frac{k-1}{q}.$$

Proof In the case $n \ge 3$, we have to satisfy the following system

$$\begin{cases} \dfrac{1}{\rho} = \dfrac{1}{2} - \dfrac{(k-1)}{q} \\ 2 \le \rho < \dfrac{2n}{n-2}, \end{cases}$$

thus it is enough to choose $q > \max\{n(k-1), 2\}$.

In the case $n = 1, 2$ it is sufficient to satisfy the following system

$$\begin{cases} \dfrac{1}{\rho} = \dfrac{1}{2} - \dfrac{(k-1)}{q} \\ 2 \leq \rho < \infty. \end{cases}$$

It is clearly satisfied for every $q \geq \max\{2(k-1), 2\}$.

■

Now by $(f3)$ we have $\frac{n}{2} - \frac{2}{k-1} \leq s \leq \frac{n}{2}$, then it is always possible to choose $\sigma \geq 0$ satisfying (2.2).

Lemma 2.2.8 *Assume $(f3)$. Then, for all σ satisfying (2.2) there exist (p_1, p_2) and (q_1, q_2) such that*

(i) (p_1, p_2) is an admissible pair;

(ii) There exists an admissible pair (q_1, β_2) such that:

$$\frac{1}{q_2} = \frac{1}{\beta_2} - \frac{\sigma}{n};$$

(iii) $p_1 < q_1$;

(iv) If $\frac{1}{r_i} \equiv \frac{1}{p_i} + \frac{k-1}{q_i}$, $i = 1, 2$, then there exists $s_1 \geq 1$ such that (s_1, r_2) is the dual of an admissible pair and

$$\frac{1}{r_1} < \frac{1}{s_1}, \quad \text{if} \quad \sigma \in \left[0, \frac{n}{2}\right) \bigcap \left(\frac{n}{2} - \frac{2}{k-1}, s\right];$$
$$\frac{1}{r_1} = \frac{1}{s_1}, \quad \text{if} \quad \sigma = s = \frac{n}{2} - \frac{2}{k-1} \geq 0.$$

Proof To obtain the points $p_1, p_2, q_1, q_2, \beta_2, r_1, r_2$ and s_1 we need to solve the system of equations corresponding to conditions $(i) - (iv)$. We consider several cases separately.

(a) $\mathbf{n \geq 2}; \sigma \in \left[\mathbf{0}, \frac{\mathbf{n}}{\mathbf{2}}\right) \bigcap \left(\frac{\mathbf{n}}{\mathbf{2}} - \frac{\mathbf{2}}{\mathbf{k-1}}, \mathbf{s}\right]$

Set

$q_1 = \infty$, $\dfrac{1}{q_2} = \dfrac{1}{2} - \dfrac{\sigma}{n}$;

$\dfrac{1}{p_1} = \dfrac{k-1}{4} n \left(\dfrac{1}{2} - \dfrac{\sigma}{n}\right)$, $\dfrac{1}{p_2} = \dfrac{1}{2} - \dfrac{k-1}{2}\left(\dfrac{1}{2} - \dfrac{\sigma}{n}\right)$.

Then, for $\beta_2 = 2$, it is easy to verify properties $(i) - (iii)$. On the other hand, according to (vi), (r_1, r_2) are given by

$$\frac{1}{r_1} = \frac{k-1}{4} n \left(\frac{1}{2} - \frac{\sigma}{n}\right), \ \frac{1}{p_2} = \frac{1}{2} + \frac{k-1}{2}\left(\frac{1}{2} - \frac{\sigma}{n}\right).$$

Setting $\frac{1}{s_1} = 1 - \frac{k-1}{4} n \left(\frac{1}{2} - \frac{\sigma}{n}\right)$, we have that (s_1, r_2) is the dual of (p_1, p_2) and $\frac{1}{r_1} < \frac{1}{s_1}$, if and only if $\sigma > \frac{n}{2} - \frac{2}{k-1}$.

(b) $\mathbf{n \geq 3; \sigma = s = \frac{n}{2} - \frac{2}{k-1} \geq 0}$

In this case we can easily verify properties $(i) - (iv)$ for the points

$q_1 = \infty$, $q_2 = \frac{n(k-1)}{2}$;

$p_1 = 2$, $\frac{1}{p_2} = \frac{1}{2} - \frac{1}{n}$;

$\beta_2 = 2$;

$r_1 = 2$, $\frac{1}{r_2} = \frac{1}{2} + \frac{1}{n}$.

Note that (r_1, r_2) is the dual of (p_1, p_2).

(c) $\mathbf{n = 2; \sigma = s = 1 - \frac{2}{k-1} \geq 0}$

For $n = 2$ the pair $(2, \infty)$ is not admissible. So in this case we choose

$q_1 = q_2 = 2(k-1)$;

$p_1 = 3$, $p_2 = 6$;

$r_1 = \frac{6}{5}$, $r_2 = \frac{3}{2}$.

Now it is easy to verify properties $(i) - (iv)$ hold for $\frac{1}{\beta_2} = \frac{1}{2} - \frac{1}{2(k+1)}$. Note that $k \geq 3$, thus (iii) holds. Moreover, (r_1, r_2) is the dual of the admissible pair $(6, 3)$.

(d) $\mathbf{n = 1; \sigma \in \left[0, \frac{1}{2}\right) \bigcap \left(\frac{1}{2} - \frac{2}{k-1}, s\right]}$

In this case we consider two possibilities.

If $k > 3$ set

$$\frac{1}{q_1} = \frac{1}{4}\left(\frac{1}{2} - \sigma\right), \ \frac{1}{q_2} = \frac{1}{2}\left(\frac{1}{2} - \sigma\right).$$

If $k \leq 3$ then there exists $m \in \mathbb{N} - \{1, 2\}$ such that $1 + \frac{8}{2^{m-1}} \geq k > 1 + \frac{8}{2^m}$. Then, set

$\frac{1}{q_1} = \frac{1}{2^m}\left(\frac{1}{2} - \sigma\right)$, $\frac{1}{q_2} = \left(1 - \frac{1}{2^{m-1}}\right)\left(\frac{1}{2} - \sigma\right)$.

For (p_1, p_2) set, in both cases

$\frac{1}{p_1} = \frac{k-1}{8}\left(\frac{1}{2} - \sigma\right)$, $\frac{1}{p_2} = \frac{1}{2} - \frac{k-1}{4}\left(\frac{1}{2} - \sigma\right)$.

A simple calculation shows that $(i) - (iv)$ hold for

$$\frac{1}{\beta_2} = \begin{cases} \frac{1}{2}\left(\frac{1}{2} - \sigma\right) + \sigma & , \ k > 3, \\ \left(1 - \frac{1}{2^{m-1}}\right)\left(\frac{1}{2} - \sigma\right) + \sigma & , \ \text{otherwise} \end{cases}$$

and

$$\frac{1}{s_1} = \begin{cases} 1 - \frac{k-1}{8}\left(\frac{1}{2} - \sigma\right) & , \ k > 3, \\ 1 - (k-1)\left(\frac{1}{2} - \sigma\right)\left(\frac{3}{8} - \frac{1}{2^m}\right) & , \ \text{otherwise.} \end{cases}$$

(e) $\mathbf{n = 1; \sigma = s = \frac{1}{2} - \frac{2}{k-1} \geq 0}$

Set

$q_1 = \frac{4}{3}(k-1)$, $q_2 = 2(k-1)$;

$p_1 = 5$, $p_2 = 10$.

Therefore

$\beta_2 = \frac{1}{2} - \frac{3}{2(k-1)}$, $r_1 = \frac{20}{19}$ and $r_2 = \frac{5}{3}$.

We have that (r_1, r_2) is the dual of the admissible pair $(20, \frac{5}{2})$. Moreover, (iii) is verified since $k \geq 5$.

∎

2.3 Local well-posedness

Proof of Theorem 2.1.1

Case (i) $\mathbf{s > \frac{n}{2}}$

Choose $\sigma \in \left(\frac{n}{2}, s\right]$ and define

$$X^s = \{u \in L^\infty_{0,T} H^s : \|u\|_{L^\infty_{0,T} H^\sigma} \leq N \text{ and } \|D^s u\|_{L^\infty_{0,T} L^2} \leq K\}.$$

By the Sobolev embedding we have for all $q \geq 2$ and $\gamma = \frac{n}{2} - \frac{n}{q}$ (note that $\gamma < \frac{n}{2} < \sigma$)

$$\|u(t)\|_{L^q} \leq c\|D^\gamma u(t)\|_{L^2} \leq c\|u(t)\|_{H^\sigma}.$$

Then, we obtain

$$\|u\|_{L^\infty_{0,T}L^q} \leq cN.$$

We need to show that N, K and T can be chosen so that the integral operator

$$\Phi(u)(t) = B_c(t)\phi + B_s(t)\Delta\eta + \int_0^t B_s(t-t')\Delta(f(u)-u)(t')dt' \qquad (2.5)$$

maps X^s into X^s and becomes a contraction map in the $L^\infty_{0,T}L^2$-metric.

Remark 2.3.1 *Note that X^s with the $L^\infty_{0,T}L^2$-metric is a complete metric space.*

Since D^σ commute with B_c, B_s and B_I (see (0.22)), we have

$$\begin{aligned}
\|\Phi(u)\|_{L^\infty_{0,T}H^\sigma} &\leq c\|\Phi(u)\|_{L^\infty_{0,T}L^2} + c\|D^\sigma\Phi(u)\|_{L^\infty_{0,T}L^2} \\
&\leq c\Big(\|\phi\|_{H^\sigma} + \|\eta\|_{H^\sigma} + \|B_I(f(u)-u)\|_{L^\infty_{0,T}L^2} + \\
&\quad + \|B_I(D^\sigma(f(u)-u))\|_{L^\infty_{0,T}L^2}\Big) \\
&\leq c\Big(\|\phi\|_{H^\sigma} + \|\eta\|_{H^\sigma} + \|B_I(u)\|_{L^\infty_{0,T}L^2} + \|B_I(D^\sigma u)\|_{L^\infty_{0,T}L^2} \\
&\quad + \|B_I(f(u))\|_{L^\infty_{0,T}L^2} + \|B_I(D^\sigma f(u))\|_{L^\infty_{0,T}L^2}\Big).
\end{aligned}$$

So using Lemma 2.2.3 (i), we have for all (γ, ρ) admissible pair

$$\begin{aligned}
\|\Phi(u)\|_{L^\infty_{0,T}H^\sigma} &\leq c\Big(\|\phi\|_{H^\sigma} + \|\eta\|_{H^\sigma} + \|u\|_{L^1_{0,T}L^2} + \|D^\sigma u\|_{L^1_{0,T}L^2} \\
&\quad + \|f(u)\|_{L^{\gamma'}_{0,T}L^{\rho'}} + \|D^\sigma f(u)\|_{L^{\gamma'}_{0,T}L^{\rho'}}\Big) \\
&\leq c\Big(\|\phi\|_{H^\sigma} + \|\eta\|_{H^\sigma} + T\|u\|_{L^\infty_{0,T}H^\sigma}\Big) + \\
&\quad + cT^{1/\gamma'}\Big(\|f(u)\|_{L^\infty_{0,T}L^{\rho'}} + \|D^\sigma f(u)\|_{L^\infty_{0,T}L^{\rho'}}\Big).
\end{aligned}$$

Let q, γ and ρ be given by Lemma 2.2.7. Then, using $(f2)$, Hölder's inequality

$(\frac{1}{\rho'} = \frac{1}{2} + \frac{k-1}{q})$ and Lemma 2.2.5 we obtain

$$\begin{aligned}
\|\Phi(u)\|_{L^\infty_{0,T}H^\sigma} &\le c\left(\|\phi\|_{H^\sigma} + \|\eta\|_{H^\sigma} + T\|u\|_{L^\infty_{0,T}H^\sigma}\right) + \\
&\quad + cT^{1/\gamma'}\left(\|u\|_{L^\infty_{0,T}L^2}\|u\|^{k-1}_{L^\infty_{0,T}L^q} + \|D^\sigma u\|_{L^\infty_{0,T}L^2}\|u\|^{k-1}_{L^\infty_{0,T}L^q}\right) \\
&\le c\left(\|\phi\|_{H^\sigma} + \|\eta\|_{H^\sigma} + T\|u\|_{L^\infty_{0,T}H^\sigma}\right) + \\
&\quad + cT^{1/\gamma'}\left(\|u\|_{L^\infty_{0,T}H^\sigma}\|u\|^{k-1}_{L^\infty_{0,T}L^q}\right) \\
&\le c\left(\|\phi\|_{H^\sigma} + \|\eta\|_{H^\sigma}\right) + cN\left(T + T^{1/\gamma'}N^{k-1}\right).
\end{aligned} \tag{2.6}$$

By an analogous argument, we obtain

$$\|D^s\Phi(u)\|_{L^\infty_{0,T}L^2} \quad \le \quad c\left(\|D^s\phi\|_{L^2} + \|D^s\eta\|_{L^2}\right) + cK\left(T + T^{1/\gamma'}N^{k-1}\right).$$

Since $\gamma \neq 1$, it is clear that we can choose N, K and T such that Φ maps X^s into X^s.

Now we have to prove that Φ is a contraction in the $L^\infty_{0,T}L^2$-metric. Indeed, using Lemma 2.2.5 (i) and Hölder's inequality we have

$$\begin{aligned}
\|\Phi(u) - \Phi(v)\|_{L^\infty_{0,T}L^2} &\le \|B_I(f(u) - f(v))\|_{L^\infty_{0,T}L^2} + \|B_I(u - v)\|_{L^\infty_{0,T}L^2} \\
&\le c\left(\|f(u) - f(v)\|_{L^{\gamma'}_{0,T}L^{\rho'}} + \|u - v\|_{L^1_{0,T}L^2}\right) \\
&\le c\left(T^{1/\gamma'}\left\|\int_0^1 f'(\lambda u + (1-\lambda)v)(u - v)d\lambda\right\|_{L^\infty_{0,T}L^{\rho'}} + \right. \\
&\quad \left. + T\|u - v\|_{L^\infty_{0,T}L^2}\right) \\
&\le c\left(T^{1/\gamma'}\int_0^1 \|f'(\lambda u + (1-\lambda)v)\|_{L^\infty_{0,T}L^{\frac{q}{k-1}}}d\lambda\right) \\
&\quad \cdot\|u - v\|_{L^\infty_{0,T}L^2} + cT\|u - v\|_{L^\infty_{0,T}L^2} \\
&\le c\left(T^{1/\gamma'}\left(\|u\|^{k-1}_{L^\infty_{0,T}L^q} + \|v\|^{k-1}_{L^\infty_{0,T}L^q}\right) + T\right)\|u - v\|_{L^\infty_{0,T}L^2} \\
&\le c\left(T^{1/\gamma'}N^{k-1} + T\right)\|u - v\|_{L^\infty_{0,T}L^2}.
\end{aligned}$$

Then Φ is a contraction in $L^\infty_{0,T}L^2$-metric for suitable N and T and by standard arguments there is $T > 0$ and a unique solution $u \in C([0,T]; H^s(\mathbb{R}^n)) \cap \mathcal{Y}_s$ to (0.1) with $u(0) = \phi$ and $u_t(0) = \Delta\eta$.

Remark 2.3.2 *If $\Phi(u) = u \in X^s$, then by the proof of (2.6), we have*

$$\|u\|_{L^q_{0,T}H^s_r} \leq c\left(\|\phi\|_{H^s}+\|\eta\|_{H^s}\right)+c\left(T(N+K)+T^{1/\gamma'}N^{k-1}(N+K)\right) \tag{2.7}$$

for all (q,r) admissible pair. Therefore $u \in \mathcal{Y}_s$.

Case (ii) $\mathbf{s \leq \frac{n}{2}, \sigma \in \left[0, \frac{n}{2}\right) \cap \left(\frac{n}{2} - \frac{2}{k-1}, s\right]}$

Consider (p_1, p_2) and (q_1, q_2) given by Lemma 2.2.8 and define the following complete metric space

$$\begin{array}{rcl} Y^s & = & \left\{\begin{array}{l} u \in (1-\Delta)^{-\frac{s}{2}}\left(L^\infty_{0,T}L^2 \cap L^{p_1}_{0,T}L^{p_2}\right): \\ \|u\|_{L^\infty_{0,T}L^2}, \|u\|_{L^{p_1}_{0,T}L^{p_2}} \leq L; \\ \|D^s u\|_{L^\infty_{0,T}L^2}, \|D^s u\|_{L^{p_1}_{0,T}L^{p_2}} \leq K; \\ \|D^\sigma u\|_{L^\infty_{0,T}L^2}, \|D^\sigma u\|_{L^{p_1}_{0,T}L^{p_2}} \leq N \end{array}\right\} \\ d(u,v) & = & \|u\|_{L^\infty_{0,T}L^2} + \|u\|_{L^{p_1}_{0,T}L^{p_2}}. \end{array}$$

By Sobolev embedding, we have

$$\|u\|_{L^{q_1}_{0,T}L^{q_2}} \leq c\|D^\sigma u\|_{L^{q_1}_{0,T}L^{\beta_2}} \text{ ; where } \frac{1}{\beta_2} = \frac{1}{q_2} + \frac{\sigma}{n}.$$

Recall that (q_1, β_2) is an admissible pair. Therefore, in view of Lemma 2.2.8 (iii), we can interpolate between $L^\infty_{0,T}L^2$ and $L^{p_1}_{0,T}L^{p_2}$ and find $0 < \alpha < 1$ such that

$$\|u\|_{L^{q_1}_{0,T}L^{q_2}} \leq c\|D^\sigma u\|^{1-\alpha}_{L^\infty_{0,T}L^2}\|D^\sigma u\|^{\alpha}_{L^{p_1}_{0,T}L^{p_2}} \leq cN. \tag{2.8}$$

Moreover, by Lemma 2.2.8 (iv) together with Lemma 2.2.3 (i) there exists $\theta > 0$ such that

$$\begin{aligned} \|\Phi(u)\|_{L^a_{0,T}L^b} &\leq \|B_c(t)\phi\|_{L^a_{0,T}L^b} + \|B_s(t)\Delta\eta\|_{L^a_{0,T}L^b} + \|B_I(f(u)-u)\|_{L^a_{0,T}L^b} \\ &\leq c\left(\|\phi\|_{L^2} + \|\eta\|_{L^2} + \|B_I(f(u))\|_{L^a_{0,T}L^b} + \|B_I(u)\|_{L^a_{0,T}L^b}\right) \\ &\leq c\left(\|\phi\|_{L^2} + \|\eta\|_{L^2} + T^\theta\|f(u)\|_{L^{r_1}_{0,T}L^{r_2}} + \|u\|_{L^1_{0,T}L^2}\right) \end{aligned}$$

where $(a,b) \in \{(\infty, 2), (p_1, p_2)\}$.

Now using $(f2)$, the definition of (r_1, r_2) in Lemma 2.2.8 and Hölder's inequality, we obtain

$$\begin{aligned} \|\Phi(u)\|_{L^a_{0,T}L^b} &\leq c\left(\|\phi\|_{L^2} + \|\eta\|_{L^2} + T^\theta\|u\|_{L^{p_1}_{0,T}L^{p_2}}\|u\|^{k-1}_{L^{q_1}_{0,T}L^{q_2}} + T\|u\|_{L^\infty_{0,T}L^2}\right) \\ &\leq c\left(\|\phi\|_{L^2} + \|\eta\|_{L^2} + T^\theta N^{k-1}L + TL\right). \end{aligned} \tag{2.9}$$

Following the same arguments, using the estimates for fractional derivatives (remember that $p_2 \neq \infty$) and the fact that D^s and D^σ commute with B_I, B_c and $B_s\Delta$, we have

$$\|D^s\Phi(u)\|_{L^a_{0,T}L^b} \leq c\left(\|D^s\phi\|_{L^2} + \|D^s\eta\|_{L^2} + T^\theta N^{k-1}K + TK\right), \quad (2.10)$$

$$\|D^\sigma\Phi(u)\|_{L^a_{0,T}L^b} \leq c\left(\|D^\sigma\phi\|_{L^2} + \|D^\sigma\eta\|_{L^2} + T^\theta N^{k-1}N + TN\right). \quad (2.11)$$

On the other hand, from an argument analogous to the one used in case (i), we have for $(a,b) \in \{(\infty,2),(p_1,p_2)\}$

$$\begin{aligned}
\|\Phi(u)-\Phi(v)\|_{L^a_{0,T}L^b} &\leq \|B_I(f(u)-f(v))\|_{L^a_{0,T}L^b} + \|B_I(u-v)\|_{L^a_{0,T}L^b} \\
&\leq c\left(T^\theta\|f(u)-f(v)\|_{L^{r_1}_{0,T}L^{r_2}} + \|u-v\|_{L^1_{0,T}L^2}\right) \\
&\leq c\left(T^\theta\left\|\int_0^1 f'(\lambda u+(1-\lambda)v)(u-v)d\lambda\right\|_{L^{r_1}_{0,T}L^{r_2}} + \right. \\
&\quad \left. +T\|u-v\|_{L^\infty_{0,T}L^2}\right) \\
&\leq cT^\theta\left(\|u\|^{k-1}_{L^{q_1}_{0,T}L^{q_2}} + \|v\|^{k-1}_{L^{q_1}_{0,T}L^{q_2}}\right)\|u-v\|_{L^{p_1}_{0,T}L^{p_2}} + \\
&\quad +cT\|u-v\|_{L^\infty_{0,T}L^2} \\
&\leq c\left(T^\theta N^{k-1}+T\right)d(u,v).
\end{aligned}$$

The proof follows by choosing suitable L, N, K and T.

Case (iii) $\mathbf{s \leq \frac{n}{2}, \sigma = \frac{n}{2} - \frac{2}{k-1}}$

Let $\tau < 1$ and (p_1,p_2), (p_1,p_2) be given by Lemma 2.2.8. Define the following complete metric space

$$\begin{aligned}
Y^s_\tau &= \left\{\begin{array}{l}
u \in (1-\Delta)^{-\frac{s}{2}}\left(L^\infty_{0,T}L^2 \cap L^{p_1}_{0,T}L^{p_2}\right): \\
\|u\|_{L^\infty_{0,T}L^2}, \|u\|_{L^{p_1}_{0,T}L^{p_2}} \leq L; \\
\|D^s u\|_{L^\infty_{0,T}L^2}, \|D^s u\|_{L^{p_1}_{0,T}L^{p_2}} \leq K; \\
\|D^\sigma u\|_{L^\infty_{0,T}L^2} \leq N;\ \|D^\sigma u\|_{L^{p_1}_{0,T}L^{p_2}} \leq \tau N < N
\end{array}\right\} \\
d(u,v) &= \|u\|_{L^\infty_{0,T}L^2} + \|u\|_{L^{p_1}_{0,T}L^{p_2}}.
\end{aligned}$$

Then we can show, following the same arguments of (2.8), that there exists

$0 < \alpha < 1$, such that

$$\|u\|_{L^{q_1}_{0,T}L^{q_2}} \leq c\|D^\sigma u\|^{1-\alpha}_{L^\infty_{0,T}L^2}\|D^\sigma u\|^{\alpha}_{L^{p_1}_{0,T}L^{p_2}} \leq c\tau^\alpha N.$$

As in the inequalities (2.9) and (2.10), we have for $(a,b) \in \{(\infty,2),(p_1,p_2)\}$

$$\|\Phi(u)\|_{L^a_{0,T}L^b} \leq c\left(\|\phi\|_{L^2} + \|\eta\|_{L^2} + (\tau^\alpha N)^{k-1}L + TL\right), \tag{2.12}$$

and

$$\|D^s\Phi(u)\|_{L^a_{0,T}L^b} \leq c\left(\|D^s\phi\|_{L^2} + \|D^s\eta\|_{L^2} + (\tau^\alpha N)^{k-1}K + TK\right). \tag{2.13}$$

The inequality (2.11) should be replaced by the following two estimates

$$\|D^\sigma\Phi(u)\|_{L^\infty_{0,T}L^2} \leq c\left(\|D^\sigma\phi\|_{L^2} + \|D^\sigma\eta\|_{L^2} + \tau^{1+\alpha(k-1)}N^k + TN\right) \tag{2.14}$$

and

$$\begin{aligned}\|D^\sigma\Phi(u)\|_{L^{p_1}_{0,T}L^{p_2}} \leq & c\left(\|B_c(\cdot)D^\sigma\phi\|_{L^{p_1}_{0,T}L^{p_2}} + \|B_s\Delta(\cdot)D^\sigma\eta\|_{L^{p_1}_{0,T}L^{p_2}}\right) \\ & + c\left(\tau^{1+\alpha(k-1)}N^k + TN\right).\end{aligned} \tag{2.15}$$

Taking T small the terms $\|B_c(\cdot)D^\sigma\phi\|_{L^{p_1}_{0,T}L^{p_2}}$ and $\|B_s\Delta(\cdot)D^\sigma\eta\|_{L^{p_1}_{0,T}L^{p_2}}$ can be made small enough (note that $p_1 \neq \infty$). So it is clear that the operator Φ maps Y^s_τ into Y^s_τ (choosing suitable L, N, K, T, τ). Since the reminder of the proof follows from a similar argument as the one previously used it will be omitted.

Finally, we remark that once we established that Φ is a contraction appropriate spaces the proof of continuous dependence is straightforward.

■

Proof of Theorem 2.1.2

(i) By (2.6) we have to choose N, T such that

$$c_0\left(\max\{\|\phi\|_{H^\sigma}, \|\eta\|_{H^\sigma}\}\right) + c_0 N\left(T + T^{1/\gamma'}N^{k-1}\right) \leq N. \tag{2.16}$$

Setting $N = 2c_0\left(\max\{\|\phi\|_{H^\sigma}, \|\eta\|_{H^\sigma}\}\right)$ this inequality becomes

$$T + T^{1/\gamma'}\left(2c_0\max\{\|\phi\|_{H^\sigma}, \|\eta\|_{H^\sigma}\}\right)^{k-1} \leq 1/2c_0.$$

This inequality is clearly satisfied for

$$T = \frac{1}{4c_0} \min \left\{ 1, 2 \left(\max\{\|\phi\|_{H^\sigma}, \|\eta\|_{H^\sigma}\} \right)^{\gamma'(1-k)} \right\}.$$

Now setting $c = 1/4c_0$ and $\theta = 1/\gamma'$ we have

$$T^* \geq c \left(\min \left\{ 1, 2 \left(\max \left\{ \|\phi\|_{H^\sigma}, \|\eta\|_{H^\sigma} \right\} \right)^{\frac{1-k}{\theta}} \right\} \right).$$

Note that (2.1) does not follow direct from the inequality above. To prove (2.1) we will use an iterative argument. Set $T = \bar{T} = 1/2c_0$. Thus, inequality (2.16) becomes

$$c_0 \left(\max\{\|\phi\|_{H^\sigma}, \|\eta\|_{H^\sigma}\} \right) + c_1 N^{k-1} \leq N/2 \tag{2.17}$$

for some $c_1 > 0$.

It is clear that (2.17) has a solution N if $\max\{\|\phi\|_{H^\sigma}, \|\eta\|_{H^\sigma}\}$ is sufficiently small. In fact, we have more than that. An application of the implicit function theorem tell us that there are $\bar{\delta} > 0$ and $\lambda > 1$ such that if $\max\{\|\phi\|_{H^\sigma}, \|\eta\|_{H^\sigma}\} \leq \delta \leq \bar{\delta}$ then $N \leq \lambda\delta$, where N is the solution of (2.17).

It follows that if $\max\{\|\phi\|_{H^\sigma}, \|\eta\|_{H^\sigma}\} \leq \lambda^{-n}\bar{\delta}$ then we can find $N_1 \leq \lambda^{-n+1}\bar{\delta}$ such that the solution exists in the interval $[0, \bar{T}]$. Moreover by construction

$$\|u(\bar{T})\|_{H^\sigma} \leq N_1 \leq \lambda^{-n+1}\bar{\delta}.$$

We want to repeat this argument. Therefore, we first need to control the growth of $\|\Delta^{-1}u_t(t)\|_{H^\sigma}$. Since $u(t)$ is given by (0.22) we have that

$$\Delta^{-1}u_t(t) = B_s(t)\Delta\phi - B_c(t)\eta - \int_0^t B_c(t-t')(f(u)-u)(t')dt'.$$

Thus, applying the same argument as the one used in (2.6), we obtain

$$\begin{aligned} \|\Delta^{-1}u_t(\bar{T})\|_{H^\sigma} &\leq \|\Delta^{-1}u_t(\bar{T})\|_{L^\infty_{\bar{T}} H^\sigma} \\ &\leq c_0 \left(\max\{\|\phi\|_{H^\sigma}, \|\eta\|_{H^\sigma}\} \right) + N/2 + c_1 N^{k-1}. \end{aligned}$$

Since N_1 is the solution of (2.17) we also have

$$\|\Delta^{-1}u_t(\bar{T})\|_{H^\sigma} \leq N_1 \leq \lambda^{-n+1}\bar{\delta}.$$

Now, solving equation (0.1) with initial data $u(\bar{T})$ and $\Delta^{-1}u_t(\bar{T})$, we can find $N_2 \leq \lambda^{-n+2}\bar{\delta}$ such that the solution exists in the interval $[\bar{T}, 2\bar{T}]$. Moreover,

$$\max\left\{\|u(2\bar{T})\|_{H^\sigma}, \|\Delta^{-1}u_t(2\bar{T})\|_{H^\sigma}\right\} \leq \lambda^{-n+2}\bar{\delta}.$$

Repeating this process, we can find N_i, $i = 1, \ldots, n$, such that the solution exists on the intervals $[0, \bar{T}], \ldots, [(n-1)\bar{T}, n\bar{T}]$, so that $T^* \geq \bar{T}$. Thus T^* is arbitrarily large if $\max\{\|\phi\|_{H^\sigma}, \|\eta\|_{H^\sigma}\}$ is sufficiently small.

(iia) The proof is essentially the same as (i) using inequality (2.11) instead of (2.6) (that is, just replace $\max\{\|\phi\|_{H^\sigma}, \|\eta\|_{H^\sigma}\}$ by $\max\{\|D^\sigma\phi\|_{L^2}, \|D^\sigma\eta\|_{L^2}\}$).

(iib) In this case, in view of (2.14) and (2.15), we have to choose N, T, τ such that

$$c\left(\max\{\|D^\sigma\phi\|_{L^2}, \|D^\sigma\eta\|_{L^2}\} + \tau^{1+\alpha(k-1)}N^k + TN\right) \leq N$$

and

$$c\left(\max\{B_1, B_2\} + \tau^{1+\alpha(k-1)}N^k + TN\right) \leq \tau N.$$

where $B_1 \equiv \|B_c(\cdot)D^\sigma\phi\|_{L^{p_1}_{0,T}L^{p_2}}$ and $B_2 \equiv \|B_s\Delta(\cdot)D^\sigma\eta\|_{L^{p_1}_{0,T}L^{p_2}}$.

But the sizes of B_1 and B_2 depend on T and $D^\sigma\phi$, $D^\sigma\eta$ (but not necessarily on their norms). That is why T^* cannot be estimated only in terms of $\|D^\sigma\phi\|_{L^2}$ and $\|D^\sigma\eta\|_{L^2}$.

■

Proof of Theorem 2.1.3 We use an argument first used by [41] (see also [10] page 826).

(a) Let T^* be given by Theorem 2.1.2 and $t < T^*$. If we consider $u(t)$ and $\Delta^{-1}u_t(t)$ as the initial data, the solution cannot be extended to a time $\geq T^*$. Setting $D(t) = \max\{\|u(t)\|_{H^\sigma}, \|\Delta^{-1}u_t(t)\|_{H^\sigma}\}$, it follows from (2.6) and the fixed point argument that if for some $N > 0$,

$$cD(t) + cN\left((T-t) + (T-t)^{1/\gamma'}N^{k-1}\right) \leq N$$

then $T < T^*$.

Thus for all $N > 0$, we have

$$cD(t) + cN\left((T^* - t) + (T^* - t)^{1/\gamma'} N^{k-1}\right) \geq N.$$

Now, choosing $N = 2cD(t)$ and letting $t \to T^*$ we have the blow up result.

(b) Since the argument is similar to part (a) it will be omitted.

■

2.4 Unconditional well-posedness

The aim of this section is to prove Theorems 2.1.4 and 2.1.5, but before doing that we need to establish some preliminary lemmas.

Lemma 2.4.1 *Let* (p_1, p_2) *and* (q_1, q_2) *such that*

(*i*) (p_1, p_2) *is an admissible pair;*

(*ii*) *There exists* $\delta \in [0, 1]$ *such that*

$$\frac{1}{p_1} \geq \frac{1-\delta}{q_1} \quad and \quad \frac{1}{p_2} = \frac{1-\delta}{q_2} + \frac{\delta}{2};$$

(*iii*) *If* $\frac{1}{r_i} \equiv \frac{1}{p_i} + \frac{k-1}{q_i}$, $i = 1, 2$, *then there exists* $s_1 \geq 1$ *such that* (s_1, r_2) *is the dual of an admissible pair and* $s_1 \leq r_1$.

Then uniqueness holds in $\mathcal{X} \equiv L^{\infty}_{0,T} L^2 \cap L^{q_1}_{0,T} L^{q_2}$.

Proof The proof follows the same ideas of Lemma 3.1 in [24].

Using Hölder's inequality and interpolation we have, in view of (*ii*), that $\mathcal{X} \subset L^{p_1}_{0,T} L^{p_2}$.

Returning to the uniqueness question, suppose there are two fixed points $u, v \in \mathcal{X}$ of the integral equation (2.5). Then $w \equiv u - v$ may be written as

$$w = B_I(f(u) - f(v)) - B_I(u - v).$$

But for $(a, b) \in \{(\infty, 2), (p_1, p_2)\}$, we have by Lemma 2.2.3 (*ii*) that

$$\begin{aligned} \|B_I(u - v)\|_{L^a_T L^b} &\leq c\|u - v\|_{L^1_T L^2} && (2.18) \\ &\leq cT\|u - v\|_{L^\infty_T L^2}. && (2.19) \end{aligned}$$

It remains to estimate the term $B_I(f(u)-f(v))$. Suppose first that $s_1 < r_1$. In this case, using (iii), Lemma 2.2.3 (ii), the Mean Value Theorem and Hölder's inequality, we obtain for $\theta \equiv \frac{1}{s_1} - \frac{1}{r_1} > 0$

$$\begin{aligned}
\|B_I(f(u)-f(v))\|_{L^a_T L^b} &\leq c\|f(u)-f(v)\|_{L^{s_1}_T L^{r_2}} \\
&\leq cT^\theta \|f(u)-f(v)\|_{L^{r_1}_T L^{r_2}} \\
&\leq cT^\theta \| \left(|u|^{k-1}+|v|^{k-1}\right)(u-v)\|_{L^{r_1}_T L^{r_2}} \\
&\leq cT^\theta \left(\|u\|^{k-1}_{L^{q_1}_T L^{q_2}} + \|u\|^{k-1}_{L^{q_1}_T L^{q_2}}\right) \|u-v\|_{L^{p_1}_T L^{p_2}}.
\end{aligned}$$

When $s_1 = r_1$ we have $\theta = 0$ in the above inequality. To overcome this difficulty we use an argument introduced by Cazenave (see [9] Proposition 4.2.5.). Define

$$\begin{aligned}
f^N &= 1_{\{|u|+|v|>N\}}(f(u)-f(v)), \\
f_N &= 1_{\{|u|+|v|\leq N\}}(f(u)-f(v)).
\end{aligned}$$

Therefore by Lemma 2.2.3 (ii) we have for $(a,b) \in \{(\infty,2),(p_1,p_2)\}$ that

$$\begin{aligned}
\|B_I f_N\|_{L^a_T L^b} \leq& \quad cN^{k-1}\|u-v\|_{L^1_T L^2} \\
\leq& \quad cN^{k-1}T\|u-v\|_{L^\infty_T L^2}.
\end{aligned}$$

On the other hand, using (iii), Lemma 2.2.3 (ii), the Mean Value Theorem and Hölder's inequality, we obtain

$$\|B_I f^N\|_{L^a_T L^b} \leq \; c\left(\left\|1_{\{|u|+|v|>N\}}(|u|+|v|)\right\|_{L^{q_1}_T L^{q_2}}\right)^{k-1} \|u-v\|_{L^{p_1}_T L^{p_2}}.$$

Since $|u|+|v| \in L^{q_1}_T L^{q_2}$, it follows by dominated convergence that

$$\left\|1_{\{|u|+|v|>N\}}(|u|+|v|)\right\|_{L^{q_1}_T L^{q_2}} \to 0, \text{ when } N \to \infty.$$

By choosing N large enough, we can find $\bar{c} > 0$ such that

$$\|u-v\|_{L^{p_1}_T L^{p_2}} + \|u-v\|_{L^\infty_T L^2} \leq \bar{c}TN^{k-1}\|u-v\|_{L^\infty_T L^2}.$$

Set $d(w) = \|w\|_{L_T^{p_1}L^{p_2}} + \|w\|_{L_T^{\infty}L^2}$. Therefore, in both cases we can find a function $H(T)$ such that $H(T) \to 0$ when $T \to 0$ and

$$d(w) \leq H(T)d(w).$$

Taking $T_0 > 0$ small enough such that $H(T_0) < 1$, we conclude that $d(w)$ must be zero in $[0, T_0]$. Now, since the argument does not depend on the initial data, we can reapply this process a finite number of times to extend the uniqueness result in the whole existence interval $[0, T]$.

■

Lemma 2.4.2 *We have three cases:*

(i) If $n = 1$ uniqueness holds in $L^{\infty}_{0,T}(L^2 \cap L^q)$ for all

$$q \geq \max\{k, 2\};$$

(ii) If $n = 2$ uniqueness holds in $L^{\infty}_{0,T}(L^2 \cap L^q)$ for all

$$\frac{1}{q} < \frac{1}{k} \quad \textit{and} \quad \frac{1}{q} \leq \min\left\{\frac{1}{2}, \frac{1}{k-1}\right\};$$

(iii) If $n \geq 3$ uniqueness holds in $L^{\infty}_{0,T}(L^2 \cap L^q)$ for all

$$\frac{1}{q} \leq \min\left\{\left(\frac{1}{2} + \frac{1}{n}\right)\frac{1}{k}, \frac{1}{2}, \frac{2}{n(k-1)}\right\}.$$

Proof Affirmations (i) and (ii) follow from Corollary 2.2 (see also Theorem 2.1) in [24]. On the other hand, the proof of (iii) is a little bit different from Kato's proof since we have one more admissible pair, namely $\left(2, \dfrac{2n}{n-2}\right)$. So we will give a detailed proof of this. We consider several cases separately

(a) $\mathbf{1 < k \leq 1 + \frac{2}{n}}$

Set $(p_1, p_2) = (q_1, q_2) = (\infty, 2)$. It is easy to see that there exists $s_1 \geq 1$ satisfying $(i) - (iii)$ of Lemma 2.4.1 (with $\delta = 0$). Then uniqueness holds in $L^{\infty}_{0,T}L^2$ and therefore in $L^{\infty}_{0,T}L^2 \cap L^{q_1}_{0,T}L^{q_2}$ for all (q_1, q_2). Note that if $k = 1 + \dfrac{2}{n}$, we have that (r_1, r_2) must be given by $r_1 = \infty$ and $\dfrac{1}{r_2} = \dfrac{1}{2} + \dfrac{1}{n}$. Therefore, $(2, r_2)$ is the dual of the admissible pair $\left(2, \dfrac{2n}{n-2}\right)$.

(b) $\mathbf{1+\frac{2}{n}<k<1+\frac{4}{n-2}}$

Let $b_k \equiv \left(\frac{1}{2}+\frac{1}{n}\right)\frac{1}{k}$. By the restriction on k we have $\frac{1}{2}-\frac{1}{n}<b_k<\frac{1}{2}$. Therefore there exists an admissible pair (α_k,β_k) such that $\beta_k=\frac{1}{b_k}$. Let (∞,q) such that $\frac{1}{q}\le b_k$. By interpolation we obtain

$$L^\infty_{0,T}L^2\cap L^\infty_{0,T}L^q\subseteq L^\infty_{0,T}L^2\cap L^\infty_{0,T}L^{\beta_k}.$$

If uniqueness holds on $L^\infty_{0,T}L^2\cap L^\infty_{0,T}L^{\beta_k}$, then it holds, a fortiori, in $L^\infty_{0,T}L^2\cap L^\infty_{0,T}L^q$. Therefore, we just need to verify that $(p_1,p_2)=(\alpha_k,\beta_k)$, $(q_1,q_2)=(\infty,\beta_k)$ satisfy the hypotheses of Lemma 2.4.1. Indeed, in this case $(i)-(ii)$ can be easily verified (for $\delta=0$). On the other hand, (r_1,r_2) must be given by $\frac{1}{r_1}=\frac{1}{\alpha_k}$ and $\frac{1}{r_2}=\frac{k}{\beta_k}$.

Thus, (s_1,r_1), with $s_1=2$ is the dual of the admissible pair $\left(2,\frac{2n}{n-2}\right)$. Moreover

$$s_1<r_1\iff\frac{1}{2}>\frac{n}{2}\left(\frac{1}{2}-\frac{1}{\beta_k}\right)\iff k<1+\frac{4}{n-2}.$$

(c) $\mathbf{k\ge 1+\frac{4}{n-2}}$

In this case

$$\frac{2}{n(k-1)}\le\frac{1}{2}-\frac{1}{n}<\frac{1}{2}. \tag{2.20}$$

Let (∞,q) such that $\frac{1}{q}\le\frac{2}{n(k-1)}$. By the same argument as the one used in item (b) it is sufficient to prove that uniqueness holds in $L^\infty_TL^2\cap L^\infty_TL^{\widetilde{q}}$, where $\frac{1}{\widetilde{q}}=\frac{2}{n(k-1)}$. Therefore, we need to verify that $(p_1,p_2)=(2,\frac{2n}{n-2})$, $(q_1,q_2)=(\infty,\widetilde{q})$ satisfy the hypotheses of Lemma 2.4.1. It is clear that (i) holds. On the other hand, in view of (2.20) we can find $\delta\in[0,1]$ such that (ii) holds. Now, we turn to property (iii). The pair (r_1,r_2) must be given by $r_1=2$ and $\frac{1}{r_2}=\frac{1}{2}+\frac{1}{n}$. Then, (s_1,r_2), with $s_1=2$ is the dual of the admissible pair $\left(2,\frac{2n}{n-2}\right)$.

■

Now we can prove our next main result.

Proof of Theorem 2.1.4 This is an immediate consequence of the last lemma. Using Sobolev Embedding and decreasing T if necessary we have

$$C([0,T];H^s)\subset L^\infty_{0,T}(L^2\cap L^{\bar q})$$

where

$$\bar q=\begin{cases}2n/(n-2s) & \text{if } s<n/2;\\ \text{any } \bar q<\infty & \text{if } s=n/2;\\ \infty & \text{if } s>n/2.\end{cases}$$

So we have only to verify that uniqueness holds in $L^\infty_{0,T}(L^2\cap L^{\bar q})$, but Lemma 2.4.2 tell us when it happens.

∎

Now, we turn to the proof of Theorem 2.1.5. First of all, define $H(u,v)$ by

$$H(u,v)\equiv\int_0^1|\lambda u+(1-\lambda)v|^{k-1}d\lambda. \tag{2.21}$$

We will need the following lemmas.

Lemma 2.4.3 *Let* $n\geq 3$, $0<s<1$, $k>2$ *and* $k\leq 1+\dfrac{2n-2s}{n-2s}$. *Let* $h\in\dot H^s_\tau(\mathbb{R}^n)$ *with* $\tau=\dfrac{n}{s+(k-1)\left(\frac{n}{2}-s\right)}$. *If* k *also verifies the following conditions:*

(*i*) $k>1+\dfrac{2s}{n-2s}$;

(*ii*) $k<1+\min\left\{\dfrac{4s+2}{n-2s},\dfrac{4}{n-2s},\dfrac{n+2s}{n-2s}\right\}$;

(*iii*) $k\leq 1+\dfrac{n+2-2s}{n-2s}$.

Then there exist σ, p *verifying* $\sigma-\dfrac{n}{p}=s-\dfrac{n}{2}$ *and*

(1) $s-1\leq\sigma\leq s$;

(2) $-s<\sigma<0$;

(3) $s-(k-1)\left(\frac{n}{2}-s\right)\leq\sigma\leq\min\left\{s+1,\frac{n}{2}\right\}-(k-1)\left(\frac{n}{2}-s\right)$.

Such that if $g \in \dot{B}^{\sigma}_{p,2}(\mathbb{R}^n)$*, then* $gh \in \dot{B}^{\sigma}_{r',2}(\mathbb{R}^n)$ *with*

$$\|gh\|_{\dot{B}^{\sigma}_{r',2}} \leq c\|g\|_{\dot{B}^{\sigma}_{p,2}}\|h\|_{\dot{H}^s_\tau}$$

where $\frac{1}{r'} = \frac{1}{p} + \frac{(k-1)\left(\frac{n}{2} - s\right)}{n}$ *and* $\frac{2n}{n+2} \leq r' \leq 2$.

Proof See [18] Lemma 3.8.

■

Lemma 2.4.4 *Let* $n \geq 3$*,* $k > 2$*,* $0 \leq s < \frac{n}{2}$ *and* $s < k-1$*. Suppose also that* $(k-1)\left(\frac{1}{2} - \frac{s}{n}\right) \leq 1$ *and define* $\tau = \frac{n}{s + (k-1)(\frac{n}{2} - s)}$*. If* $u, v \in L^\infty_{0,T}\dot{H}^s$*, then* $H(u,v) \in L^\infty_{0,T}\dot{H}^s_\tau$*. Moreover,* $\tau \geq 1$ *if and only if* $k \leq 1 + \frac{2n-2s}{n-2s}$.

Proof By definition of $H(u,v)$, we have

$$\|H(u,v)\|_{\dot{H}^s_\tau} = \int_0^1 \||\lambda u + (1-\lambda)v)|^{k-1}\|_{\dot{H}^s_\tau} \leq c\left(\||u|^{k-1}\|_{\dot{H}^s_\tau} + \||v|^{k-1}\|_{\dot{H}^s_\tau}\right)$$

and using Lemma 2.2.6 (ii) we have the desire estimate.

■

Furthermore, we have the following embedding

Lemma 2.4.5 $\dot{H}^s \hookrightarrow \dot{B}^{\sigma}_{p,2}$ *for all* $\sigma \leq s$ *and* $\sigma - \frac{n}{p} = s - \frac{n}{2}$*. Moreover, there exists* $\gamma \geq 1$ *such that* (γ, p) *is an admissible pair if and only if* $s - 1 \leq \sigma \leq s$.

Proof See [37] for the first part. The second part is an easy consequence of admissible pair's definition.

■

Now we have all tools to prove our last main result of this chapter.

Proof of Theorem 2.1.5 First, we recall that $\dot{B}^{\sigma}_{2,2} = \dot{H}^{\sigma}$, $H^s \subseteq \dot{H}^{\sigma}$ for all $\sigma, s \in \mathbb{R}$ and $\sigma \leq s$. Then, using Lemma 2.4.5, we conclude that $(u-v) \in L^\infty_{0,T}\dot{B}^{\sigma}_{p,2} \cap L^\infty_{0,T}\dot{B}^{\sigma}_{2,2}$, where σ and p satisfy conditions $(1)-(3)$ of Lemma 2.4.3. Moreover, in view of Lemma 2.4.5 and condition (1) of Lemma 2.4.3, there exists $\gamma \geq 1$ such that (γ, p) is an admissible pair.

Thus, by Lemma 2.2.4 (i), we have for $(a,b) \in \{(\infty,2),(\gamma,p)\}$

$$\begin{aligned}\|u-v\|_{L^a_{0,T}\dot{B}^\sigma_{b,2}} &\leq \|B_I(f(u)-f(v))\|_{L^a_{0,T}\dot{B}^\sigma_{b,2}} + \|B_I(u-v)\|_{L^a_{0,T}\dot{B}^\sigma_{b,2}}\\ &\leq c\|f(u)-f(v)\|_{L^{q'}_{0,T}\dot{B}^\sigma_{r',2}} + c\|u-v\|_{L^1_{0,T}\dot{B}^\sigma_{2,2}}\\ &\leq c\|(u-v)H(u,v)\|_{L^{q'}_{0,T}\dot{B}^\sigma_{r',2}} + cT\|u-v\|_{L^\infty_{0,T}\dot{B}^\sigma_{2,2}}\end{aligned}$$

where $\dfrac{1}{r'} = \dfrac{1}{p} + \dfrac{(k-1)\left(\frac{n}{2}-s\right)}{n}$ and $\dfrac{2n}{n+2} \leq r' \leq 2$. Recall that this last condition implies that (q',r') is the dual of an admissible pair.

Then by Lemma 2.4.3, we obtain:

$$\|u-v\|_{L^a_{0,T}\dot{B}^\sigma_{b,2}} \leq c\|\|u-v\|_{\dot{B}^\sigma_{p,2}}\|H(u,v)\|_{\dot{H}^s_\tau}\|_{L^{q'}_{0,T}} + cT\|u-v\|_{L^\infty_{0,T}\dot{B}^\sigma_{2,2}}.$$

But $\dfrac{1}{q'} - \dfrac{1}{\gamma} = 1 - \dfrac{(k-1)}{2}\left(\dfrac{n}{2}-s\right) \equiv \theta > 0$ since $k < 1 + \dfrac{4}{n-2s}$. Thus

$$\|u-v\|_{L^a_{0,T}\dot{B}^\sigma_{b,2}} \leq cT^\theta\|u-v\|_{L^\gamma_{0,T}\dot{B}^\sigma_{p,2}}\|H(u,v)\|_{L^\infty_{0,T}\dot{H}^s_\tau} + cT\|u-v\|_{L^\infty_{0,T}\dot{B}^\sigma_{2,2}}.$$

Set $\omega(u,v) \equiv \|u-v\|_{L^\infty_{0,T}\dot{B}^\sigma_{2,2}} + \|u-v\|_{L^\gamma_{0,T}\dot{B}^\sigma_{p,2}}$, therefore we conclude that

$$\omega(u,v) \leq c\left(T^\theta\|H(u,v)\|_{L^\infty_{0,T}\dot{H}^s_\tau} + T\right)\omega(u,v).$$

Hence, for $T_0 > 0$ small enough, $u(t) = v(t)$ on $[0,T_0]$ and to recover the whole interval we apply the same argument as the one used in the proof of Lemma 2.4.1.

■

Chapter 3
Local solutions in Sobolev spaces with negative indices

3.1 Introduction

In this chapter we consider initial value problem (IVP) for the "good" Boussinesq equation (0.5).

Our principal aim here is to study the local well-posedness for low regularity data. Natural spaces to measure this regularity are the classical Sobolev spaces $H^s(\mathbb{R})$, $s \in \mathbb{R}$. The best result available in the literature was given by Linares [29], who proved local well-posedness for initial data $\phi \in H^1(\mathbb{R})$, $\psi = h_x$ with $h \in L^2(\mathbb{R})$. In this work, we improve the result in [29], proving local well-posedness with $s > -1/4$ for the "good" Boussinesq equation.

To obtain this result we use the Fourier restriction norm method introduced by Bourgain [6] to study the nonlinear Schrödinger equation (0.6) and the KdV equation (0.7). This method was further developed by Kenig, Ponce and Vega in [26] for the KdV equation and [27] for the quadratics nonlinear Schrödinger equation (0.8)-(0.10) in one spatial dimension and in spatially continuous and periodic case.

The original Bourgain method makes extensive use of the Strichartz inequalities in order to derive the bilinear estimates corresponding to the nonlinearity. On the other hand, Kenig, Ponce and Vega simplified Bourgain's proof and improved the bilinear estimates using only elementary techniques, such as Cauchy-Schwartz inequality and simple calculus inequalities.

Both arguments also use some arithmetic facts involving the symbol of the linearized equation. For example, the algebraic relation for quadratic nonlinear Schrödinger equation (0.8) is given by

$$2|\xi_1(\xi - \xi_1)| \leq |\tau - \xi^2| + |(\tau - \tau_1) - (\xi - \xi_1)^2| + |\tau_1 - \xi_1^2|. \tag{3.1}$$

Then splitting the domain of integration in the sets where each term on the right side of (3.1) is the biggest one, Kenig, Ponce and Vega made some cancellation in the symbol in order to use his calculus inequalities (see Lemma 3.3.1) and a clever change of variables to established their crucial estimates.

Here, we shall use this kind of argument, but unfortunately in the Boussinesq case we do not have good cancellations on the symbol. To overcome this difficulty we observe that the dispersion in the Boussinesq case is given by the symbol $\sqrt{\xi^2+\xi^4}$ and this is in some sense related with the Schrödinger symbol (see Lemma 3.3.2 below). Therefore, we can modify the symbols and work only with the algebraic relations for the Schrödinger equation already used in Kenig, Ponce and Vega [27] in order to derive our relevant bilinear estimates. We should remark that in the present case we have to estimate all the possible cases for the relation $\tau \pm \xi^2$ and not only the cases treated in Kenig, Ponce and Vega [27].

To describe our results we define next the $X_{s,b}$ spaces related to our problem. These spaces, with $b=\frac{1}{2}$, were first defined by Fang and Grillakis [17] for the Boussinesq-type equations in the periodic case. Using these spaces and following Bourgain's argument introduced in [6] they proved local well-posedness for (0.5) with the spatial variable in the unit circle (denoted by $\mathbb{T}$) assuming $u(0)\in H^s(\mathbb{T})$, $u_t(0)\in H^{-2+s}(\mathbb{T})$, with $0\le s\le 1$ and $|f(u)|\le c|u|^p$, with $1<p<\frac{3-2s}{1-2s}$ if $0\le s<\frac{1}{2}$ and $1<p<\infty$ if $\frac{1}{2}\le s\le 1$. Moreover, if $u(0)\in H^1(\mathbb{T})$, $u_t(0)\in H^{-1}(\mathbb{T})$ and $f(u)=\lambda|u|^{q-1}u-|u|^{p-1}u$, with $1<q<p$ and $\lambda\in\mathbb{R}$ then the solution is global.

Next we give the precise definition of the $X_{s,b}$ spaces for the Boussinesq-type equation in the continuous case.

Definition 3.1.1 *For $s,b\in\mathbb{R}$, $X_{s,b}$ denotes the completion of the Schwartz class $\mathcal{S}(\mathbb{R}^2)$ with respect to the norm*

$$\|F\|_{X_{s,b}}=\|\langle|\tau|-\gamma(\xi)\rangle^b\langle\xi\rangle^s\widetilde{F}\|_{L^2_{\tau,\xi}}$$

where $\gamma(\xi)\equiv\sqrt{\xi^2+\xi^4}$, $\langle a\rangle\equiv 1+|a|$ and $\sim$ denotes the time-space Fourier transform.

We will also need the localized $X_{s,b}$ spaces defined as follows

Definition 3.1.2 *For $s,b\in\mathbb{R}$ and $T\ge 0$, $X^T_{s,b}$ denotes the space endowed with the norm*

$$\|u\|_{X^T_{s,b}}=\inf_{w\in X_{s,b}}\left\{\|w\|_{X_{s,b}}: w(t)=u(t)\ \text{on}\ [0,T]\right\}.$$

Now we state the main results of this chapter.

Theorem 3.1.1 *Let $s > -1/4$ and $u, v \in X_{s,-a}$. Then, there exists $c > 0$ such that*

$$\left\| \left(\frac{|\xi|^2 \widetilde{uv}(\xi,\tau)}{2i\gamma(\xi)} \right)^{\sim^{-1}} \right\|_{X_{s,-a}} \leq c \, \|u\|_{X_{s,b}} \, \|v\|_{X_{s,b}} \, , \tag{3.2}$$

where $\sim^{-1}$ denotes the inverse time-space Fourier transform, holds in the following cases

(i) $s \geq 0$, $b > 1/2$ *and* $1/4 < a < 1/2$,

(ii) $-1/4 < s < 0$, $b > 1/2$ *and* $1/4 < a < 1/2$ *such that* $|s| < a/2$.

Moreover, the constant $c > 0$ that appears in (3.2) depends only on a, b, s.

Theorem 3.1.2 *For any $s \leq -1/4$ and any $a, b \in \mathbb{R}$, with $a < 1/2$ the estimate (3.2) fails.*

Theorem 3.1.3 *Let $s > -1/4$, then for all $\phi \in H^s(\mathbb{R})$ and $\psi \in H^{s-1}(\mathbb{R})$, there exist $T = T(\|\phi\|_{H^s}, \|\psi\|_{H^{s-1}})$ and a unique solution u of the IVP (0.5) such that*

$$u \in C([0,T] : H^s(\mathbb{R})) \cap X^T_{s,b}.$$

Moreover, given $T' \in (0,T)$ there exists $R = R(T') > 0$ such that giving the set $W \equiv \{(\tilde{\phi}, \tilde{\psi}) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R}) : \|\tilde{\phi} - \phi\|^2_{H^s(\mathbb{R})} + \|\tilde{\psi} - \psi\|^2_{H^{s-1}(\mathbb{R})} < R\}$ the map solution

$$S : W \longrightarrow C([0,T'] : H^s(\mathbb{R})) \cap X^T_{s,b}, \quad (\tilde{\phi}, \tilde{\psi}) \longmapsto u(t)$$

is Lipschitz.

In addition, if $(\phi, \psi) \in H^{s'}(\mathbb{R}) \times H^{s'-1}(\mathbb{R})$ with $s' > s$, then the above results hold with s' instead of s in the same interval $[0,T]$ with

$$T = T(\|\phi\|_{H^s}, \|\psi\|_{H^{s-1}}).$$

The plan of this chapter is as follows: in Section 2, we prove some estimates for the integral equation in the $X_{s,b}$ space introduced above. Bilinear estimates and the relevants counterexamples are proved in Section 3 and 4, respectively. Finally, the local well-posedness question is treated in Section 5.

3.2 Preliminary results

By Duhamel's Principle the solution of (0.5) is equivalent to the integral equation (0.17). Let θ be a cutoff function satisfying $\theta \in C_0^\infty(\mathbb{R})$, $0 \le \theta \le 1$, $\theta \equiv 1$ in $[-1,1]$, $\text{supp}(\theta) \subseteq [-2,2]$ and for $0 < T < 1$ define $\theta_T(t) = \theta(t/T)$. In fact, to work in the $X_{s,b}$ spaces we consider another version of (0.17), that is

$$u(t) = \theta(t)\left(V_c(t)\phi + V_s(t)\psi_x\right) + \theta_T(t)\int_0^t V_s(t-t')(u^2)_{xx}(t')dt'. \tag{3.3}$$

Note that the integral equation (3.3) is defined for all $(x,t) \in \mathbb{R}^2$. Moreover if u is a solution of (3.3) than $\tilde{u} = u|_{[0,T]}$ will be a solution of (0.17) in $[0,T]$.

In the next lemma, we estimate the linear part of the integral equation (3.3).

Lemma 3.2.1 *Let $u(t)$ the solution of the linear equation*

$$\begin{cases} u_{tt} - u_{xx} + u_{xxxx} = 0, \\ u(0,x) = \phi(x); \quad u_t(0,x) = \psi_x(x) \end{cases}$$

with $\phi \in H^s$ and $\psi \in H^{s-1}$. Then there exists $c > 0$ depending only on θ, s, b such that

$$\|\theta u\|_{X_{s,b}} \le c\left(\|\phi\|_{H^s} + \|\psi\|_{H^{s-1}}\right). \tag{3.4}$$

Proof. Taking time-space Fourier transform in $\theta(t)u(x,t)$ and setting $\gamma(\xi) = \sqrt{\xi^2+\xi^4}$, we have

$$\begin{aligned}(\theta(t)u(x,t))^\sim(\xi,\tau) &= \frac{\hat{\theta}(\tau-\gamma(\xi))}{2}\left(\hat{\phi}(\xi) + \frac{\xi\hat{\psi}(\xi)}{\gamma(\xi)}\right) \\ &\quad + \frac{\hat{\theta}(\tau+\gamma(\xi))}{2}\left(\hat{\phi}(\xi) - \frac{\xi\hat{\psi}(\xi)}{\gamma(\xi)}\right).\end{aligned}$$

Thus, setting $h_1(\xi) = \hat{\phi}(\xi) + \frac{\xi\hat{\psi}(\xi)}{\gamma(\xi)}$ and $h_2(\xi) = \hat{\phi}(\xi) - \frac{\xi\hat{\psi}(\xi)}{\gamma(\xi)}$, we have

$$\begin{aligned}&\|\theta u\|_{X_{s,b}}^2 \le \\ &\le \int_{-\infty}^{+\infty}\langle\xi\rangle^{2s}|h_1(\xi)|^2\left(\int_{-\infty}^{+\infty}\langle|\tau|-\gamma(\xi)\rangle^{2b}\left|\frac{\hat{\theta}(\tau-\gamma(\xi))+\hat{\theta}(\tau+\gamma(\xi))}{2}\right|^2 d\tau\right)d\xi \\ &+ \int_{-\infty}^{+\infty}\langle\xi\rangle^{2s}|h_2(\xi)|^2\left(\int_{-\infty}^{+\infty}\langle|\tau|-\gamma(\xi)\rangle^{2b}\left|\frac{\hat{\theta}(\tau-\gamma(\xi))+\hat{\theta}(\tau+\gamma(\xi))}{2}\right|^2 d\tau\right)d\xi.\end{aligned}$$

Since $||\tau| - \gamma(\xi)| \leq \min\{|\tau - \gamma(\xi)|, |\tau + \gamma(\xi)|\}$ and $\hat{\theta}$ is rapidly decreasing, we can bound the terms inside the parentheses, and the claim follows. ■

Next we estimate the integral part of (3.3).

Lemma 3.2.2 *Let* $-\frac{1}{2} < b' \leq 0 \leq b \leq b' + 1$ *and* $0 < T \leq 1$ *then*

(*i*) $\left\|\theta_T(t)\int_0^t g(t')dt'\right\|_{H_t^b} \leq T^{1-(b-b')}\|g\|_{H_t^{b'}}$;

(*ii*) $\left\|\theta_T(t)\int_0^t V_s(t-t')f(u)(t')dt'\right\|_{X_{s,b}} \leq T^{1-(b-b')}\left\|\left(\frac{\widetilde{f(u)}(\xi,\tau)}{2i\gamma(\xi)}\right)^{\sim^{-1}}\right\|_{X_{s,b'}}$.

Proof.

(*i*) See [19] inequality (3.11).

(*ii*) A simple calculation shows that

$$\left(\theta_T(t)\int_0^t V_s(t-t')f(u)(t')dt'\right)^{\wedge(x)}(\xi,t) =$$

$$\begin{aligned} &= e^{it\gamma(\xi)}\left(\theta_T(t)\int_0^t h_1(\xi,t')dt'\right) - e^{-it\gamma(\xi)}\left(\theta_T(t)\int_0^t h_2(\xi,t')dt'\right) \\ &\equiv e^{it\gamma(\xi)}w_1^{\wedge(x)}(\xi,t) - e^{-it\gamma(\xi)}w_2^{\wedge(x)}(\xi,t), \end{aligned}$$

where $h_1(\xi,t') = \dfrac{e^{-it'\gamma(\xi)}f^{\wedge(x)}(\xi,t')}{2i\gamma(\xi)}$ and $h_2(\xi,t') = \dfrac{e^{it'\gamma(\xi)}f^{\wedge(x)}(\xi,t')}{2i\gamma(\xi)}$.

Therefore

$$\left(\theta_T(t)\int_0^t V_s(t-t')f(u)(t')dt'\right)^{\sim}(\xi,\tau) =$$

$$\widetilde{w_1}(\xi,\tau-\gamma(\xi)) - \widetilde{w_2}(\xi,\tau+\gamma(\xi)).$$

Now using the definition of $X_{s,b}$ we have

$$\left\|\theta_T(t)\int_0^t V_s(t-t')f(u)(t')dt'\right\|_{X_{s,b}}^2 \leq$$

$$
\begin{aligned}
&\leq c\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\langle|\tau+\gamma(\xi)|-\gamma(\xi)\rangle^{2b}\langle\xi\rangle^{2s}|\widetilde{w_1}(\xi,\tau)|^2 d\tau d\xi\\
&\quad +c\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\langle|\tau-\gamma(\xi)|-\gamma(\xi)\rangle^{2b}\langle\xi\rangle^{2s}|\widetilde{w_2}(\xi,\tau)|^2 d\tau d\xi\\
&\equiv M.
\end{aligned}
$$

Since $\gamma(\xi)\geq 0$ for all $\xi\in\mathbb{R}$, we have

$$
\max\{||\tau+\gamma(\xi)|-\gamma(\xi)|, ||\tau-\gamma(\xi)|-\gamma(\xi)|\}\leq|\tau|.
$$

Thus applying item (i) we obtain

$$
\begin{aligned}
M &\leq c\sum_{j=1}^{2}\int_{-\infty}^{+\infty}\langle\xi\rangle^{2s}\|w_j^{\wedge(x)}\|^2_{H_t^b}\\
&\leq cT^{1-(b-b')}\sum_{j=1}^{2}\int_{-\infty}^{+\infty}\langle\xi\rangle^{2s}\|h_j\|^2_{H_t^{b'}}\\
&= cT^{1-(b-b')}\left(\int\int_{\mathbb{R}^2}\langle\tau-\gamma(\xi)\rangle^{2b'}\langle\xi\rangle^{2s}\left|\frac{\widetilde{f(u)}(\xi,\tau)}{2i\gamma(\xi)}\right|^2 d\tau d\xi\right.\\
&\quad\left.+\int\int_{\mathbb{R}^2}\langle\tau+\gamma(\xi)\rangle^{2b'}\langle\xi\rangle^{2s}\left|\frac{\widetilde{f(u)}(\xi,\tau)}{2i\gamma(\xi)}\right|^2 d\tau d\xi\right).
\end{aligned}
$$

Since $||\tau|-\gamma(\xi)|\leq\min\{|\tau-\gamma(\xi)|,|\tau+\gamma(\xi)|\}$ and $b'\leq 0$ we obtain the desired inequality.

■

The next lemma says that, for $b>1/2$, $X_{s,b}$ is embedding in $C(\mathbb{R}:H^s)$.

Lemma 3.2.3 *Let $b>\frac{1}{2}$. There exists $c>0$, depending only on b, such that*

$$
\|u\|_{C(\mathbb{R}:H^s)}\leq c\|u\|_{X_{s,b}}.
$$

Proof. First we prove that $X_{s,b}\subseteq L^\infty(\mathbb{R}:H^s)$. Let $u=u_1+u_2$, where $\tilde{u}_1\equiv\tilde{u}\chi_{\{\tau\leq 0\}}$, $\tilde{u}_2\equiv\tilde{u}\chi_{\{\tau>0\}}$ and χ_A denotes the characteristic function of the set

A. Then for all $t \in \mathbb{R}$

$$\begin{aligned} \|u_1(t)\|_{H^s} &= \left\| \left(e^{it\gamma(\xi)}(u_1)^{\wedge(x)}\right)^{\vee(x)}(x,t) \right\|_{H^s} \\ &= \left\| \int_{-\infty}^{+\infty} \left(\left(e^{it\gamma(\xi)}(u_1)^{\wedge(x)}\right)^{\vee(x)}\right)^{\wedge(t)}(x,\tau)e^{it\tau}d\tau \right\|_{H^s} \\ &\leq \int_{-\infty}^{+\infty} \left\| \left(\left(e^{it\gamma(\xi)}(u_1)^{\wedge(x)}\right)^{\vee(x)}\right)^{\wedge(t)}(x,\tau) \right\|_{H^s} d\tau. \end{aligned}$$

Using the Cauchy-Schwarz inequality we obtain

$$\|u_1(t)\|_{H^s} \leq \left(\int_{-\infty}^{+\infty}\langle\tau\rangle^{-2b}\right)^{1/2}\left(\int_{-\infty}^{+\infty}\int_{-\infty}^{0}\langle\tau+\gamma(\xi)\rangle^{2b}\langle\xi\rangle^{2s}|\tilde{u}(\xi,\tau)|^2 d\tau d\xi\right)^{1/2}.$$

On the other hand, similar arguments imply that

$$\|u_2(t)\|_{H^s} \leq \left(\int_{-\infty}^{+\infty}\langle\tau\rangle^{-2b}\right)^{1/2}\left(\int_{-\infty}^{+\infty}\int_{0}^{+\infty}\langle\tau-\gamma(\xi)\rangle^{2b}\langle\xi\rangle^{2s}|\tilde{u}(\xi,\tau)|^2 d\tau d\xi\right)^{1/2}.$$

Now, by the fact that $b > 1/2$, $|\tau+\gamma(\xi)| = ||\tau|-\gamma(\xi)|$ for $\tau \leq 0$ and $|\tau-\gamma(\xi)| = ||\tau|-\gamma(\xi)|$ for $\tau \geq 0$ we have

$$\|u\|_{L^\infty(\mathbb{R}:H^s)} \leq c\|u\|_{X_{s,b}}.$$

It remains to show continuity. Let $t, t' \in \mathbb{R}$ then

$$\|u_1(t)-u_1(t')\|_{H^s} =$$

$$\left\| \int_{-\infty}^{+\infty} \left(\left(e^{it\gamma(\xi)}(u_1)^{\wedge(x)}\right)^{\vee(x)}\right)^{\wedge(t)}(x,\tau)(e^{it\tau}-e^{it'\tau})d\tau \right\|_{H^s}. \tag{3.5}$$

Letting $t' \to t$, two applications of the Dominated Convergence Theorem give that the right hand side of (3.5) goes to zero. Therefore, $u_1 \in C(\mathbb{R} : H^s)$. Of course, the same argument applies to u_2, which concludes the proof. ∎

3.3 Bilinear estimates

Before proceed to the proof of Theorem 3.1.1, we state some elementary calculus inequalities that will be useful later.

Lemma 3.3.1 *For $p, q > 0$ and $r = \min\{p, q, p+q-1\}$ with $p+q > 1$, there exists $c > 0$ such that*

$$\int_{-\infty}^{+\infty} \frac{dx}{\langle x-\alpha\rangle^p \langle x-\beta\rangle^q} \leq \frac{c}{\langle \alpha-\beta\rangle^r}. \tag{3.6}$$

Moreover, for $a_0, a_1, a_2 \in \mathbb{R}$ and $q > 1/2$

$$\int_{-\infty}^{+\infty} \frac{dx}{\langle a_0 + a_1 x + a_2 x^2\rangle^q} \leq c. \tag{3.7}$$

Proof. See Lemma 4.2 in [21] and Lemma 2.5 in [3].

■

Lemma 3.3.2 *There exists $c > 0$ such that*

$$\frac{1}{c} \leq \sup_{x\in\mathbb{R}, y\geq 0} \frac{1+|x-y|}{1+|x-\sqrt{y^2+y}|} \leq c. \tag{3.8}$$

Proof. Since $y \leq \sqrt{y^2+y} \leq y + 1/2$ for all $y \geq 0$ a simple computation shows the desired inequalities.

■

Remark 3.3.1 *In view of the previous lemma we have an equivalent way to compute the $X_{s,b}$-norm, that is*

$$\|u\|_{X_{s,b}} \sim \|\langle |\tau| - \xi^2\rangle^b \langle \xi\rangle^s \widetilde{u}(\xi,\tau)\|_{L^2_{\xi,\tau}}.$$

This equivalence will be important in the proof of Theorem 3.1.1. As we commented in the introduction the Boussinesq symbol $\sqrt{\xi^2+\xi^4}$ does not have good cancelations to make use of Lemma 3.3.1. Therefore, we modify the symbols as above and work only with the algebraic relations for the Schrödinger equation already used in Kenig, Ponce and Vega [27] in order to derive the bilinear estimates.

Now we are in position to prove the bilinear estimate (3.2).

Proof of Theorem 3.1.1. Let $u, v \in X_{s,b}$ and define

$$\begin{aligned} f(\xi,\tau) &\equiv \langle |\tau| - \xi^2\rangle^b \langle \xi\rangle^s \widetilde{u}(\xi,\tau), \\ g(\xi,\tau) &\equiv \langle |\tau| - \xi^2\rangle^b \langle \xi\rangle^s \widetilde{v}(\xi,\tau). \end{aligned}$$

Using Remark 3.3.1 and a duality argument the desired inequality is equivalent to

$$|W(f,g,\phi)| \leq c\|f\|_{L^2_{\xi,\tau}}\|g\|_{L^2_{\xi,\tau}}\|\phi\|_{L^2_{\xi,\tau}} \tag{3.9}$$

where

$$\begin{aligned} W(f,g,\phi) &= \int_{\mathbb{R}^4} \frac{\langle\xi\rangle^s}{\langle\xi_1\rangle^s\langle\xi-\xi_1\rangle^s} \\ &\times \frac{g(\xi_1,\tau_1)f(\xi-\xi_1,\tau-\tau_1)\bar{\phi}(\xi,\tau)}{\langle|\tau|-\xi^2\rangle^a\langle|\tau_1|-\xi_1^2\rangle^b\langle|\tau-\tau_1|-(\xi-\xi_1)^2\rangle^b} d\xi d\tau d\xi_1 d\tau_1. \end{aligned}$$

Therefore to perform the desired estimate we need to analyze all the possible cases for the sign of τ, τ_1 and $\tau-\tau_1$. To do this we split $\mathbb{R}^4$ into the regions

$$\begin{aligned} \Gamma_1 &= \{(\xi,\tau,\xi_1,\tau_1)\in\mathbb{R}^4 : \tau_1<0, \tau-\tau_1<0\}, \\ \Gamma_2 &= \{(\xi,\tau,\xi_1,\tau_1)\in\mathbb{R}^4 : \tau_1\geq 0, \tau-\tau_1<0, \tau\geq 0\}, \\ \Gamma_3 &= \{(\xi,\tau,\xi_1,\tau_1)\in\mathbb{R}^4 : \tau_1\geq 0, \tau-\tau_1<0, \tau<0\}, \\ \Gamma_4 &= \{(\xi,\tau,\xi_1,\tau_1)\in\mathbb{R}^4 : \tau_1<0, \tau-\tau_1\geq 0, \tau\geq 0\}, \\ \Gamma_5 &= \{(\xi,\tau,\xi_1,\tau_1)\in\mathbb{R}^4 : \tau_1<0, \tau-\tau_1\geq 0, \tau<0\}, \\ \Gamma_6 &= \{(\xi,\tau,\xi_1,\tau_1)\in\mathbb{R}^4 : \tau_1\geq 0, \tau-\tau_1\geq 0\}. \end{aligned}$$

Thus, it is sufficient to prove inequality (3.9) with $Z(f,g,\phi)$ instead of $W(f,g,\phi)$, where

$$Z(f,g,\phi) = \int_{\mathbb{R}^4} \frac{\langle\xi\rangle^s}{\langle\xi_1\rangle^s\langle\xi_2\rangle^s} \frac{g(\xi_1,\tau_1)f(\xi_2,\tau_2)\bar{\phi}(\xi,\tau)}{\langle\sigma\rangle^a\langle\sigma_1\rangle^b\langle\sigma_2\rangle^b} d\xi d\tau d\xi_1 d\tau_1$$

with $\xi_2=\xi-\xi_1$, $\tau_2=\tau-\tau_1$ and σ,σ_1,σ_2 belonging to one of the following cases

(I) $\sigma=\tau+\xi^2, \ \ \sigma_1=\tau_1+\xi_1^2, \ \ \sigma_2=\tau_2+\xi_2^2,$

(II) $\sigma=\tau-\xi^2, \ \ \sigma_1=\tau_1-\xi_1^2, \ \ \sigma_2=\tau_2+\xi_2^2,$

(III) $\sigma=\tau+\xi^2, \ \ \sigma_1=\tau_1-\xi_1^2, \ \ \sigma_2=\tau_2+\xi_2^2,$

(IV) $\sigma=\tau-\xi^2, \ \ \sigma_1=\tau_1+\xi_1^2, \ \ \sigma_2=\tau_2-\xi_2^2,$

(V) $\sigma=\tau+\xi^2, \ \ \sigma_1=\tau_1+\xi_1^2, \ \ \sigma_2=\tau_2-\xi_2^2,$

(VI) $\sigma=\tau-\xi^2, \ \ \sigma_1=\tau_1-\xi_1^2, \ \ \sigma_2=\tau_2-\xi_2^2.$

Remark 3.3.2 *Note that the cases* $\sigma=\tau+\xi^2, \ \sigma_1=\tau_1-\xi_1^2, \ \sigma_2=\tau_2-\xi_2^2$ *and* $\sigma=\tau-\xi^2, \ \sigma_1=\tau_1+\xi_1^2, \ \sigma_2=\tau_2+\xi_2^2$ *cannot occur, since* $\tau_1<0, \tau-\tau_1<0$ *implies* $\tau<0$ *and* $\tau_1\geq 0, \tau-\tau_1\geq 0$ *implies* $\tau\geq 0$

Applying the change of variables $(\xi,\tau,\xi_1,\tau_1)\mapsto -(\xi,\tau,\xi_1,\tau_1)$ and observing that the L^2-norm is preserved under the reflection operation, the cases (IV), (V), (VI) can be easily reduced, respectively, to (III), (II), (I). Moreover, making the change of variables $\tau_2=\tau-\tau_1$, $\xi_2=\xi-\xi_1$ and then $(\xi,\tau,\xi_2,\tau_2)\mapsto -(\xi,\tau,\xi_2,\tau_2)$ the case (II) can be reduced (III). Therefore we need only establish cases (I) and (III). We should remark that these are exactly two of the three bilinear estimates that appear in [27], but since it is important to have the inequality (3.9) with $a<1/2<b$ such that $a+b<1$ to make the contraction arguments work (see the proof of Theorem 3.1.3) we reprove these inequalities here.

We first treat the inequality (3.9) with $Z(f,g,\phi)$ in the case (I). We will make use of the following algebraic relation

$$-(\tau+\xi^2)+(\tau_1+\xi_1^2)+((\tau-\tau_1)+(\xi-\xi_1)^2)=2\xi_1(\xi_1-\xi). \tag{3.10}$$

By simmetry we can restrict ourselves to the set

$$A=\{(\xi,\tau,\xi_1,\tau_1)\in\mathbb{R}^4:|(\tau-\tau_1)+(\xi-\xi_1)^2|\le|\tau_1+\xi_1^2|\}.$$

We divide A into three pieces

$$\begin{aligned}A_1 &= \{(\xi,\tau,\xi_1,\tau_1)\in A:|\xi_1|\le 10\},\\ A_2 &= \{(\xi,\tau,\xi_1,\tau_1)\in A:|\xi_1|\ge 10 \text{ and } |2\xi_1-\xi|\ge|\xi_1|/2\},\\ A_3 &= \{(\xi,\tau,\xi_1,\tau_1)\in A:|\xi_1|\ge 10 \text{ and } |\xi_1-\xi|\ge|\xi_1|/2\}.\end{aligned}$$

We have $A=A_1\cup A_2\cup A_3$. Indeed

$$|\xi_1|>|2\xi_1-\xi|+|\xi_1-\xi|\ge|(2\xi_1-\xi)-(\xi_1-\xi)|=|\xi_1|.$$

Next we split A_3 into two parts

$$\begin{aligned}A_{3,1} &= \{(\xi,\tau,\xi_1,\tau_1)\in A_3:|\tau_1+\xi_1^2|\le|\tau+\xi^2|\},\\ A_{3,2} &= \{(\xi,\tau,\xi_1,\tau_1)\in A_3:|\tau+\xi^2|\le|\tau_1+\xi_1^2|\}.\end{aligned}$$

We can now define the sets R_i, $i=1,2$, as follows

$$R_1=A_1\cup A_2\cup A_{3,1} \text{ and } R_2=A_{3,2}.$$

In what follows χ_R denotes the characteristic function of the set R. Using the Cauchy-Schwarz and Hölder inequalities it is easy to see that

$$\begin{aligned}|Z|^2 \;\leq\; & \|f\|^2_{L^2_{\xi,\tau}}\|g\|^2_{L^2_{\xi,\tau}}\|\phi\|^2_{L^2_{\xi,\tau}} \\ & \times \left\| \frac{\langle\xi\rangle^{2s}}{\langle\sigma\rangle^{2a}} \iint \frac{\chi_{R_1} d\xi_1 d\tau_1}{\langle\xi_1\rangle^{2s}\langle\xi_2\rangle^{2s}\langle\sigma_1\rangle^{2b}\langle\sigma_2\rangle^{2b}} \right\|_{L^\infty_{\xi,\tau}} \\ & + \|f\|^2_{L^2_{\xi,\tau}}\|g\|^2_{L^2_{\xi,\tau}}\|\phi\|^2_{L^2_{\xi,\tau}} \\ & \times \left\| \frac{1}{\langle\xi_1\rangle^{2s}\langle\sigma_1\rangle^{2b}} \iint \frac{\chi_{R_2}\langle\xi\rangle^{2s} d\xi d\tau}{\langle\xi_2\rangle^{2s}\langle\sigma\rangle^{2a}\langle\sigma_2\rangle^{2b}} \right\|_{L^\infty_{\xi_1,\tau_1}}.\end{aligned}$$

Noting that $\langle\xi\rangle^{2s} \leq \langle\xi_1\rangle^{2|s|}\langle\xi_2\rangle^{2s}$, for $s \geq 0$, and $\langle\xi_2\rangle^{-2s} \leq \langle\xi_1\rangle^{2|s|}\langle\xi\rangle^{-2s}$, for $s < 0$ we have

$$\frac{\langle\xi\rangle^{2s}}{\langle\xi_1\rangle^{2s}\langle\xi_2\rangle^{2s}} \leq \langle\xi_1\rangle^{\gamma(s)} \tag{3.11}$$

where

$$\gamma(s) = \begin{cases} 0, & \text{if } s > 0 \\ 4|s|, & \text{if } s \leq 0 \end{cases}.$$

Therefore in view of Lemma 3.3.1-(3.6) it suffices to get bounds for

$$\begin{aligned}J_1(\xi,\tau) \;&\equiv\; \frac{1}{\langle\sigma\rangle^{2a}} \int \frac{\langle\xi_1\rangle^{\gamma(s)} d\xi_1}{\langle\tau+\xi^2+2\xi_1^2-2\xi\xi_1\rangle^{2b}} \quad \text{on } R_1, \\ J_2(\xi_1,\tau_1) \;&\equiv\; \frac{\langle\xi_1\rangle^{\gamma(s)}}{\langle\sigma_1\rangle^{2b}} \int \frac{d\xi}{\langle\tau_1-\xi_1^2+2\xi\xi_1\rangle^{2a}} \quad \text{on } R_2.\end{aligned}$$

In region A_1 we have $\langle\xi_1\rangle^{\gamma(s)} \lesssim 1$. Therefore for $a > 0$ and $b > 1/2$ we obtain

$$J_1(\xi,\tau) \lesssim \int_{|\xi_1|\leq 10} d\xi_1 \lesssim 1.$$

In region A_2, by the change of variables $\eta = \tau + \xi^2 + 2\xi_1^2 - 2\xi\xi_1$ and the condition $|2\xi_1 - \xi| \geq |\xi_1|/2$ we have

$$\begin{aligned}J_1(\xi,\tau) \;&\lesssim\; \frac{1}{\langle\sigma\rangle^{2a}} \int \frac{\langle\xi_1\rangle^{\gamma(s)}}{|2\xi_1-\xi|\langle\eta\rangle^{2b}} d\eta \\ &\lesssim\; \frac{1}{\langle\sigma\rangle^{2a}} \int \frac{\langle\xi_1\rangle^{\gamma(s)-1}}{\langle\eta\rangle^{2b}} d\eta \lesssim 1\end{aligned}$$

for $a > 0$, $b > 1/2$ and $s > -1/4$ which implies $\gamma(s) \leq 1$.

Now, by definition of region $A_{3,1}$ and the algebraic relation (3.10) we have

$$\langle\xi_1\rangle^2 \lesssim |\xi_1|^2 \lesssim |\xi_1(\xi_1-\xi)| \lesssim \langle\sigma\rangle.$$

Therefore by Lemma 3.3.1-(3.7)

$$\begin{aligned} J_1(\xi,\tau) &\lesssim \int \frac{\langle\xi_1\rangle^{\gamma(s)-4a}}{\langle\tau+\xi^2+2\xi_1^2-2\xi\xi_1\rangle^{2b}}d\xi_1 \\ &\lesssim \int \frac{1}{\langle\tau+\xi^2+2\xi_1^2-2\xi\xi_1\rangle^{2b}}d\xi_1 \lesssim 1 \end{aligned}$$

for $a>1/4$, $b>1/2$ and $s>-1/4$ which implies $\gamma(s)<4a$.

Next we estimate $J_2(\xi_1,\tau_1)$. Making the change of variables, $\eta=\tau-\xi_1^2+2\xi\xi_1$, using the restriction in the region $A_{3,2}$, we have

$$|\eta| \lesssim |(\tau-\tau_1)+(\xi-\xi_1)^2|+|\tau+\xi^2| \lesssim \langle\sigma_1\rangle.$$

Moreover, in $A_{3,2}$

$$|\xi_1|^2 \lesssim |\xi_1(\xi_1-\xi)| \lesssim \langle\sigma_1\rangle.$$

Therefore, since $|\xi_1|\geq 10$ we have

$$\begin{aligned} J_2(\xi_1,\tau_1) &\lesssim \frac{|\xi_1|^{\gamma(s)}}{\langle\sigma_1\rangle^{2b}} \int_{|\eta|\lesssim\langle\sigma_1\rangle} \frac{d\eta}{|\xi_1|\langle\eta\rangle^{2a}} \\ &\lesssim \frac{|\xi_1|^{\gamma(s)-1}}{\langle\sigma_1\rangle^{2b+2a-1}} \lesssim 1 \end{aligned}$$

for $0<a<1/2$, $b>1/2$ and $s>-1/4$.

Now we turn to the proof of case (III). In the following estimates we will make use of the algebraic relation

$$-(\tau+\xi^2)+(\tau_1-\xi_1^2)+((\tau-\tau_1)+(\xi-\xi_1)^2) = -2\xi_1\xi. \tag{3.12}$$

First we split $\mathbb{R}^4$ into four sets

$$\begin{aligned} B_1 &= \{(\xi,\tau,\xi_1,\tau_1)\in\mathbb{R}^4 : |\xi_1|\leq 10\}, \\ B_2 &= \{(\xi,\tau,\xi_1,\tau_1)\in\mathbb{R}^4 : |\xi_1|\geq 10 \text{ and } |\xi|\leq 1\}, \\ B_3 &= \{(\xi,\tau,\xi_1,\tau_1)\in\mathbb{R}^4 : |\xi_1|\geq 10, |\xi|\geq 1 \text{ and } |\xi|\geq|\xi_1|/2\}, \\ B_4 &= \{(\xi,\tau,\xi_1,\tau_1)\in\mathbb{R}^4 : |\xi_1|\geq 10, |\xi|\geq 1 \text{ and } |\xi|\leq|\xi_1|/2\}. \end{aligned}$$

Next we separate B_4 into three parts

$$\begin{aligned}
B_{4,1} &= \{(\xi,\tau,\xi_1,\tau_1)\in B_4 : |\tau_1-\xi_1^2|, |(\tau-\tau_1)+(\xi-\xi_1)^2| \le |\tau+\xi^2|\},\\
B_{4,2} &= \{(\xi,\tau,\xi_1,\tau_1)\in B_4 : |\tau+\xi^2|, |(\tau-\tau_1)+(\xi-\xi_1)^2| \le |\tau_1-\xi_1^2|\},\\
B_{4,3} &= \{(\xi,\tau,\xi_1,\tau_1)\in B_4 : |\tau_1-\xi_1^2|, |\tau+\xi^2| \le |(\tau-\tau_1)+(\xi-\xi_1)^2|\}.
\end{aligned}$$

We can now define the sets R_i, $i=1,2,3$, as follows

$$S_1 = B_1\cup B_3\cup B_{4,1}, \quad S_2 = B_2\cup B_{4,2} \quad \text{and} \quad S_3 = B_{4,3}.$$

Using the Cauchy-Schwarz and Hölder inequalities and duality it is easy to see that

$$\begin{aligned}
|Z|^2 \le\ & \|f\|^2_{L^2_{\xi,\tau}}\|g\|^2_{L^2_{\xi,\tau}}\|\phi\|^2_{L^2_{\xi,\tau}}\\
&\times \left\| \frac{\langle\xi\rangle^{2s}}{\langle\sigma\rangle^{2a}} \iint \frac{\chi_{S_1} d\xi_1 d\tau_1}{\langle\xi_1\rangle^{2s}\langle\xi_2\rangle^{2s}\langle\sigma_1\rangle^{2b}\langle\sigma_2\rangle^{2b}} \right\|_{L^\infty_{\xi,\tau}}\\
&+\|f\|^2_{L^2_{\xi,\tau}}\|g\|^2_{L^2_{\xi,\tau}}\|\phi\|^2_{L^2_{\xi,\tau}}\\
&\times \left\| \frac{1}{\langle\xi_1\rangle^{2s}\langle\sigma_1\rangle^{2b}} \iint \frac{\chi_{S_2}\langle\xi\rangle^{2s} d\xi d\tau}{\langle\xi_2\rangle^{2s}\langle\sigma\rangle^{2a}\langle\sigma_2\rangle^{2b}} \right\|_{L^\infty_{\xi_1,\tau_1}}\\
&+\|f\|^2_{L^2_{\xi,\tau}}\|g\|^2_{L^2_{\xi,\tau}}\|\phi\|^2_{L^2_{\xi,\tau}}\\
&\times \left\| \frac{1}{\langle\xi_2\rangle^{2s}\langle\sigma_2\rangle^{2b}} \iint \frac{\chi_{\widetilde{S}_3}\langle\xi_1+\xi_2\rangle^{2s} d\xi_1 d\tau_1}{\langle\xi_1\rangle^{2s}\langle\sigma_1\rangle^{2a}\langle\sigma\rangle^{2b}} \right\|_{L^\infty_{\xi_2,\tau_2}}.
\end{aligned}$$

where σ, σ_1, σ_2 were given in the condition (III) and

$$\widetilde{S}_3 \subseteq \left\{ \begin{array}{r} (\xi_2,\tau_2,\xi_1,\tau_1)\in\mathbb{R}^4 : |\xi_1|\ge 10, |\xi_1+\xi_2|\ge 1, |\xi_1+\xi_2|\le|\xi_1|/2 \text{ and}\\ |\tau_1-\xi_1^2|, |(\tau_1+\tau_2)+(\xi_1+\xi_2)^2| \le |\tau_2+\xi_2^2| \end{array} \right\}.$$

Therefore from Lemma 3.3.1-(3.6) and (3.11) it suffices to get bounds for

$$\begin{aligned}
K_1(\xi,\tau) &\equiv \frac{1}{\langle\sigma\rangle^{2a}} \int \frac{\langle\xi_1\rangle^{\gamma(s)} d\xi_1}{\langle\tau+\xi^2-2\xi\xi_1\rangle^{2b}} \quad \text{on } S_1,\\
K_2(\xi_1,\tau_1) &\equiv \frac{\langle\xi_1\rangle^{\gamma(s)}}{\langle\sigma_1\rangle^{2b}} \int \frac{d\xi}{\langle\tau_1-\xi_1^2+2\xi\xi_1\rangle^{2a}} \quad \text{on } S_2,\\
K_3(\xi_1,\tau_1) &\equiv \frac{1}{\langle\sigma_2\rangle^{2b}} \int \frac{\langle\xi_1\rangle^{\gamma(s)} d\xi_1}{\langle\tau_2+\xi_2^2+2\xi_1^2+2\xi_1\xi_2\rangle^{2a}} \quad \text{on } \widetilde{S}_3.
\end{aligned}$$

In region B_1 we have $\langle\xi_1\rangle^{\gamma(s)} \lesssim 1$. Therefore for $a > 0$ and $b > 1/2$ we obtain

$$K_1(\xi,\tau) \lesssim \int_{|\xi_1|\leq 10} d\xi_1 \lesssim 1.$$

In region B_3, the change of variables $\eta = \tau + \xi^2 - 2\xi\xi_1$ and the condition $|\xi| \geq |\xi_1|/2$ give

$$\begin{aligned} K_1(\xi,\tau) &\lesssim \frac{1}{\langle\sigma\rangle^{2a}} \int \frac{\langle\xi_1\rangle^{\gamma(s)}}{|\xi|\langle\eta\rangle^{2b}} d\eta \\ &\lesssim \frac{\langle\xi_1\rangle^{\gamma(s)-1}}{\langle\sigma\rangle^{2a}} \int \frac{1}{\langle\eta\rangle^{2b}} d\eta \lesssim 1 \end{aligned}$$

for $a > 0$, $b > 1/2$ and $s > -1/4$ which implies $\gamma(s) \leq 1$.

Now, by definition of region $B_{4,1}$ and the algebraic relation (3.12) we have

$$\langle\xi_1\rangle \lesssim |\xi_1| \lesssim |\xi_1\xi| \lesssim \langle\sigma\rangle.$$

Therefore the change of variables $\eta = \tau + \xi^2 - 2\xi\xi_1$ and the condition $|\xi| \geq 1$ yield

$$\begin{aligned} K_1(\xi,\tau) &\lesssim \frac{1}{\langle\sigma\rangle^{2a}} \int \frac{\langle\xi_1\rangle^{\gamma(s)}}{|\xi|\langle\eta\rangle^{2b}} d\eta \\ &\lesssim \frac{\langle\xi_1\rangle^{\gamma(s)-2a}}{|\xi|} \int \frac{1}{\langle\eta\rangle^{2b}} d\eta \lesssim 1 \end{aligned}$$

for $s > -1/4$, $b > 1/2$ and $a \in \mathbb{R}$ such that $2|s| < a < 1/2$, if $s < 0$ or $0 < a < 1/2$, if $s \geq 0$.

Next we estimate $K_2(\xi_1,\tau_1)$. Making the change of variables, $\eta = \tau_1 - \xi_1^2 + 2\xi\xi_1$, using the restriction in the region B_2, we have

$$|\eta| \lesssim |\tau_1 - \xi_1^2| + |\xi\xi_1| \lesssim |\sigma_1| + |\xi_1|.$$

Therefore,

$$\begin{aligned} K_2(\xi_1,\tau_1) &\lesssim \frac{|\xi_1|^{\gamma(s)}}{\langle\sigma_1\rangle^{2b}} \int_{|\eta|\lesssim\langle\sigma_1\rangle+|\xi_1|} \frac{d\eta}{|\xi_1|\langle\eta\rangle^{2a}} \\ &\lesssim \frac{|\xi_1|^{\gamma(s)-2a}}{\langle\sigma_1\rangle^{2b}} + \frac{|\xi_1|^{\gamma(s)-1}}{\langle\sigma_1\rangle^{2b+2a-1}} \lesssim 1 \end{aligned}$$

for $s > -1/4$, $b > 1/2$ and $0 < a < 1/2$ such that $\gamma(s) \leq \min\{1, 2a\} = 2a$.

In the region $B_{4,2}$, by the algebraic relation (3.12) we have

$$\langle \xi_1 \rangle \lesssim |\xi_1| \lesssim |\xi_1 \xi| \lesssim \langle \tau_1 - \xi_1^2 \rangle.$$

Moreover, the change of variables $\eta = \tau_1 - \xi_1^2 + 2\xi\xi_1$, the restriction in the region $B_{4,2}$ and (3.12) give

$$|\eta| \lesssim \langle \sigma_1 \rangle.$$

Therefore,

$$\begin{aligned} K_2(\xi_1, \tau_1) &\lesssim \frac{\langle \xi_1 \rangle^{\gamma(s)}}{\langle \sigma_1 \rangle^{2b}} \int_{|\eta| \lesssim \langle \sigma_1 \rangle} \frac{d\eta}{|\xi_1| \langle \eta \rangle^{2a}} \\ &\lesssim \frac{|\xi_1|^{\gamma(s)-1}}{\langle \sigma_1 \rangle^{2b+2a-1}} \lesssim 1 \end{aligned}$$

for $s > -1/4$, $b > 1/2$ and $0 < a < 1/2$ such that $\gamma(s) \leq 1$.

Finally, we estimate $K_3(\xi_1, \tau_1)$. In the region $B_{4,3}$ we have by the algebraic relation (3.12) that

$$\langle \xi_1 \rangle \lesssim |\xi_1| \lesssim |\xi_1(\xi_1 + \xi_2)| \lesssim \langle \sigma_2 \rangle.$$

Therefore Lemma 3.3.1-(3.7) implies that

$$\begin{aligned} K_3(\xi_1, \tau_1) &\lesssim \langle \xi_1 \rangle^{\gamma(s)-2b} \int \frac{1}{\langle \tau_2 + \xi_2^2 + 2\xi_1^2 + 2\xi_1\xi_2 \rangle^{2a}} d\xi_1 \\ &\lesssim 1 \end{aligned}$$

for $a > 1/4$, $b > 1/2$ and $s > -1/4$ which implies $\gamma(s) \leq 2b$.

■

We finish this section with a result that will be useful in the proof of Theorem 3.1.3.

Corollary 3.3.1 *Let $s > -1/4$ and $a, b \in \mathbb{R}$ given in Theorem 3.1.1. For $s' > s$ we have*

$$\left\| \left(\frac{|\xi|^2 \widetilde{uv}(\xi, \tau)}{2i\gamma(\xi)} \right)^{\sim^{-1}} \right\|_{X_{s',-a}} \leq c\, \|u\|_{X_{s',b}} \|v\|_{X_{s,b}} + c\, \|u\|_{X_{s,b}} \|v\|_{X_{s',b}}. \tag{3.13}$$

Proof. The result is a direct consequence of Theorem 3.1.1 and the inequality

$$\langle \xi \rangle^{s'} \leq \langle \xi \rangle^{s} \langle \xi_1 \rangle^{s'-s} + \langle \xi \rangle^{s} \langle \xi - \xi_1 \rangle^{s'-s}.$$

■

3.4 Counterexample to the bilinear estimates (3.2)

Proof of Theorem 3.1.2. Let A_N denote the set

$$A_N = \left\{ \begin{array}{l} (\xi,\tau) \in \mathbb{R}^2 : (\xi,\tau) = (N,N^2) + \alpha\vec{\eta} + \beta\vec{\gamma} \\ 0 \le \alpha \le N, \quad 0 \le \beta \le N^{-1}, \\ \vec{\eta} = \dfrac{(1,2N)}{\sqrt{1+4N^2}}, \quad \vec{\gamma} = \dfrac{(2N,-1)}{\sqrt{1+4N^2}} \end{array} \right\}$$

and define $f_N(\xi,\tau) = \chi_{A_N}$, $g_N(\xi,\tau) = \chi_{-A_N}$ where

$$-A_N = \left\{(\xi,\tau) \in \mathbb{R}^2 : -(\xi,\tau) \in A_N\right\}.$$

It is easy to see that

$$\|f_N\|_{L^2_{\xi,\tau}} = \|g_N\|_{L^2_{\xi,\tau}} = 1. \tag{3.14}$$

Now, let $u_N, v_N \in X_{s,b}$ such that $f_N(\xi,\tau) \equiv \langle|\tau| - \xi^2\rangle^b\langle\xi\rangle^s\widetilde{u}_N(\xi,\tau)$ and $g_N(\xi,\tau) \equiv \langle|\tau| - \xi^2\rangle^b\langle\xi\rangle^s\widetilde{v}_N(\xi,\tau)$.

Therefore, from Lemma 3.3.2-(3.8) and the fact that

$$||\tau| - \xi^2| \le \min\{|\tau - \xi^2|, |\tau + \xi^2|\}$$

we obtain

$$\left\| \left(\frac{|\xi|^2\widetilde{u_N v_N}(\xi,\tau)}{2i\gamma(\xi)} \right)^{\sim^{-1}} \right\|_{X_{s,-a}} \equiv$$

$$\begin{aligned}
&\equiv \left\| \frac{|\xi|^2\langle\xi\rangle^s}{\gamma(\xi)\langle|\tau| - \xi^2\rangle^a} \iint \frac{f_N(\xi_1,\tau_1)\langle\xi_1\rangle^{-s} g_N(\xi-\xi_1,\tau-\tau_1)\langle\xi-\xi_1\rangle^{-s} d\tau_1 d\xi_1}{\langle|\tau-\tau_1| - \gamma(\xi-\xi_1)\rangle^b\langle|\tau_1| - \gamma(\xi_1)\rangle^b} \right\|_{L^2_{\tau,\xi}} \\
&\gtrsim \left\| \frac{|\xi|^2\langle\xi\rangle^s}{\gamma(\xi)\langle\tau - \xi^2\rangle^a} \iint \frac{f_N(\xi_1,\tau_1)\langle\xi_1\rangle^{-s} g_N(\xi-\xi_1,\tau-\tau_1)\langle\xi-\xi_1\rangle^{-s} d\tau_1 d\xi_1}{\langle\tau - \tau_1 + (\xi-\xi_1)^2\rangle^b\langle\tau_1 - \xi_1^2\rangle^b} \right\|_{L^2_{\tau,\xi}} \\
&\equiv B_N.
\end{aligned}$$

From the definition of A_N we have

(i) If $(\xi_1, \tau_1) \in \operatorname{supp} f_N$ and $(\xi - \xi_1, \tau - \tau_1) \in \operatorname{supp} g_N$ then there exists $c > 0$ such that
$$|\tau_1 - \xi_1^2| \le c \text{ and } |\tau - \tau_1 + (\xi - \xi_1)^2| \le c.$$

(ii) $f * g(\xi, \tau) \ge \chi_{R_N}(\xi, \tau)$,

where R_N is the rectangle of dimensions $cN \times (cN)^{-1}$ with a vertice in the origin and longest side pointing in the $(1, 2N)$ direction.

(iii) There exists a positive constant $c > 0$ such that
$$N \le \xi_1 \le N + c, \; N \le \xi_1 - \xi \le N + c$$
and, therefore $|\xi| \sim c$.

Moreover, combining the following algebraic relation
$$(\tau - \tau_1 + (\xi - \xi_1)^2) + (\tau_1 - \xi_1^2) - (\tau - \xi^2) = 2\xi(\xi_1 - \xi)$$
with (i) and (iii) we obtain
$$|\tau - \xi^2| \lesssim N. \tag{3.15}$$

Therefore (3.14), (i), (ii), (iii) and (3.15) imply that
$$\begin{aligned} 1 \;\gtrsim\; B_N &\gtrsim \frac{N^{-2s}}{N^a} \left\| \frac{|\xi|^2}{\gamma(\xi)} \chi_{R_N} \right\|_{L^2_{\xi,\tau}} \\ &\gtrsim \frac{N^{-2s}}{N^a} \left(\iint_{\{|\xi| \ge 1/2\}} \chi^2_{R_N}(\xi, \tau) \right)^{1/2} \\ &\gtrsim N^{-2s-a}. \end{aligned}$$

Letting $N \to \infty$, this inequality is possible only when $-2s - a \le 0$ which yields the result since $a < 1/2$.

■

3.5 Local well-posedness

Proof of Theorem 3.1.3.

1. *Existence.*

For $(\phi, \psi) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$, with $s > -1/4$, and $T \le 1$ we define the integral equation

$$\Gamma_T(u)(t) = \theta(t)\left(V_c(t)\phi + V_s(t)\psi_x\right) + \theta_T(t)\int_0^t V_s(t-t')(u^2)_{xx}(t')dt'. \quad (3.16)$$

Our goal is to use the Picard fixed point theorem to find a solution

$$\Gamma_T(u) = u.$$

Let $s > -1/4$ and $a, b \in \mathbb{R}$ such that Theorem 3.1.1 holds, that is, $1/4 < a < 1/2 < b$ and $1-(a+b) \equiv \delta > 0$.

Therefore using (3.4), Lemma 3.2.2-(ii) with $b' = -a$ and (3.2) we obtain

$$\begin{aligned}\|\Gamma_T(u)\|_{X_{s,b}} &\leq c\,(\|\phi\|_{H^s} + \|\psi\|_{H^{s-1}} + \\ &\quad + T^\delta \left\| \left(\frac{|\xi|^2 \widetilde{u^2}(\xi,\tau)}{2i\gamma(\xi)} \right)^{\sim^{-1}} \right\|_{X_{s,-a}} \Bigg) \\ &\leq c\left(\|\phi\|_{H^s} + \|\psi\|_{H^{s-1}} + T^\delta \|u\|^2_{X_{s,b}} \right), \\ \|\Gamma_T(u) - \Gamma_T(v)\|_{X_{s,b}} &\leq cT^\delta \|u+v\|_{X_{s,b}} \|u-v\|_{X_{s,b}}.\end{aligned} \quad (3.17)$$

We define

$$X_{s,b}(d) = \left\{ u \in X_{s,b} : \|u\|_{X_{s,b}} \leq d \right\}$$

where $d = 2c\,(\|\phi\|_{H^s} + \|\psi\|_{H^{s-1}})$.

Then choosing

$$0 < T < \min\left\{ \frac{1}{(4cd)^{1/\delta}}, 1 \right\}$$

we have that $\Gamma_T : X_{s,b}(d) \to X_{s,b}(d)$ is a contraction and therefore there exists a unique solution $u \in X_{s,b}(d)$ of (3.16).

Moreover, by Lemma 3.2.3, we have that $\tilde{u} = u|_{[0,T]} \in C([0,T] : H^s) \cap X^T_{s,b}$ is a solution of (0.17) in $[0,T]$.

2. *If $s' > s$, the result holds in the time interval $[0,T]$ with* $T = T(\|\phi\|_{H^s}, \|\psi\|_{H^{s-1}})$.

Let $s > -1/4$ and $a, b \in \mathbb{R}$ given in Theorem 3.1.1. For $s' > s$ we consider the closed ball in the Banach space

$$W = \left\{ u \in X_{s',b} : \|u\|_{s'} = \|u\|_{X_{s,b}} + \beta \|u\|_{X_{s',b}} < +\infty \right\}$$

where $\beta = \dfrac{\|\phi\|_{H^s} + \|\psi\|_{H^{s-1}}}{\|\phi\|_{H^{s'}} + \|\psi\|_{H^{s'-1}}}$.

In view of estimate (3.17) we obtain

$$\|\Gamma_T(u)\|_{X_{s,b}} \leq c\left(\|\phi\|_{H^s} + \|\psi\|_{H^{s-1}} + T^\delta \|u\|^2_{X_{s,b}} \right).$$

Now by Corollary 3.3.1 we have

$$\begin{aligned} \|\Gamma_T(u)\|_{X_{s',b}} &\leq c\left(\|\phi\|_{H^{s'}} + \|\psi\|_{H^{s'-1}} + T^\delta \|u\|_{X_{s',b}} \|u\|_{X_{s,b}} \right) \\ &\leq \frac{c}{\beta} \left(\|\phi\|_{H^s} + \|\psi\|_{H^{s-1}} + T^\delta \|u\|^2_{s'} \right). \end{aligned}$$

Therefore

$$\|\Gamma_T(u)\|_{s'} \leq 2c \left(\|\phi\|_{H^s} + \|\psi\|_{H^{s-1}} + T^\delta \|u\|^2_{s'} \right).$$

The same argument gives

$$\|\Gamma_T(u) - \Gamma_T(v)\|_{s'} \leq 2cT^\delta \|u+v\|_{s'} \|u-v\|_{s'}.$$

Then we define in W the closed ball centered at the origin with radius $d' = 4c\left(\|\phi\|_{H^s} + \|\psi\|_{H^{s-1}} \right)$ and choose

$$0 < T < \min \left\{ \frac{1}{(8cd')^{1/\delta}}, 1 \right\}.$$

Thus we have that F_T is a contraction and therefore there exists a solution with $T = T(\|\phi\|_{H^s}, \|\psi\|_{H^{s-1}})$.

3. *Uniqueness.* By the fixed point argument used in item 1 we have uniqueness of the solution of the truncated integral equation (3.16) in the set $X_{s,b}(d)$. We use an argument due to Bekiranov, Ogawa and Ponce [3] to obtain the uniqueness of the integral equation (0.17) in the whole space $X^T_{s,b}$.

Let $T > 0$, $u \in X_{s,b}$ be the solution of the truncated integral equation (3.16) obtained above and $\widetilde{v} \in X_{s,b}^T$ be a solution of the integral equation (0.17) with the same initial data. Fix an extension $v \in X_{s,b}$, therefore, for some $T^* < T < 1$ to be fixed later, we have

$$v(t) = \theta(t)\left(V_c(t)\phi + V_s(t)\psi_x\right) + \theta_T(t)\int_0^t V_s(t-t')(v^2)_{xx}(t')dt'$$

for all $t \in [0, T^*]$.

Fix $M \geq 0$ such that

$$\max\left\{\|u\|_{X_{s,b}}, \|v\|_{X_{s,b}}\right\} \leq M. \tag{3.18}$$

Taking the difference $u - v$, by definition of $X_{s,b}^{T^*}$, we have that for any $\varepsilon > 0$, there exists $w \in X_{s,b}$ such that for all $t \in [0, T^*]$

$$w(t) = u(t) - v(t)$$

and

$$\|w\|_{X_{s,b}} \leq \|u - v\|_{X_{s,b}^{T^*}} + \varepsilon. \tag{3.19}$$

Define

$$\widetilde{w}(t) = \theta_{T^*}(t)\int_0^t V_s(t-t')(w(t')u(t') + w(t')v(t'))_{xx}(t')dt'.$$

We have that, $\widetilde{w}(t) = u(t) - v(t)$, for all $t \in [0, T^*]$. Therefore, from Definition 3.1.2, Lemma 3.2.2-(ii), (3.2) and (3.18) it follows that

$$\begin{aligned} \|u - v\|_{X_{s,b}^{T^*}} &\leq \|\widetilde{w}\|_{X_{s,b}} \\ &\leq 2cM{T^*}^{\delta}\left\|w\right\|_{X_{s,b}}. \end{aligned} \tag{3.20}$$

Choosing $T^* > 0$ such that $2cM{T^*}^{\delta} \leq 1/2$, by (3.19) and (3.20), we have

$$\|u - v\|_{X_{s,b}^{T^*}} \leq \varepsilon.$$

Therefore $u = v$ on $[0, T^*]$. Now, since the argument does not depend on the initial data, we can reapply this process a finite number of times to extend the uniqueness result in the whole existence interval $[0, T]$.

4. *Map data-solution is locally Lipschitz.* Combining an identical argument to the one used in the existence proof with Lemma 3.2.3, one can easily show that the map data-solution is locally Lipschitz.

■

Chapter 4
Ill-posedness for the "good" Boussinesq equation

4.1 Introduction

Since scaling argument cannot be applied to the Boussinesq-type equations to obtain a critically notion it is not clear what is the lower index s where one has local well-posedness for the "good" Boussinesq equation (0.5) with initial data $(\phi, \psi) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$. In this chapter we answer, partially, this question. In fact, our main result is a negative one; it concerns in particular a kind of ill-posedness. We prove that the flow map for the Cauchy problem (0.5) is not smooth (C^2) at the origin for initial data in $H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$, with $s < -2$. Therefore any iterative method applied to the integral formulation of "good" Boussinesq equation (0.5) always fails in this functional setting. In other words, if one can apply the contraction mapping principle to solve the integral equation corresponding to (0.5) thus, by the implicit function Theorem, the flow-map data solution is smooth, which is a contradiction (cf. Theorem 4.1.2).

Tzvetkov [40] (see also Bourgain [7]) established a similar result for the KdV equation (0.7). The same question was studied by Molinet, Saut and Tzvetkov [35]-[36], for the Benjamin-Ono equation (0.11) and Kadomtsev-Petviashvili 1 (0.12)

Before stating the main results let us define the flow-map data solution as

$$\begin{array}{rccc} S: & H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R}) & \rightarrow & C([0,T] : H^s(\mathbb{R})) \\ & (\phi, \psi) & \mapsto & u(t) \end{array} \tag{4.1}$$

where $u(t)$ is given in (0.17).

These are the main results

Theorem 4.1.1 *Let $s < -2$ and any $T > 0$. Then there does not exist any space X_T such that*

$$\|u\|_{C([0,T]:H^s(\mathbb{R}))} \leq c \|u\|_{X_T}, \tag{4.2}$$

for all $u \in X_T$

$$\|V_c(t)\phi + V_s(t)\psi_x\|_{X_T} \le c\left(\|\phi\|_{H^s(\mathbb{R})} + \|\psi\|_{H^{s-1}(\mathbb{R})}\right), \qquad (4.3)$$

for all $\phi \in H^s(\mathbb{R})$, $\psi \in H^{s-1}(\mathbb{R})$ *and*

$$\left\|\int_0^t V_s(t-t')(uv)_{xx}(t')dt'\right\|_{X_T} \le c\,\|u\|_{X_T}\,\|v\|_{X_T}, \qquad (4.4)$$

for all $u, v \in X_T$.

Remark 4.1.1 *We recall that in Chapter 3 we construct a space* X_T *such that the inequalities (4.2), (4.3) and (4.4) hold for* $s > -1/4$*. These are the main tools to prove the local well-posedness result stated in Theorem 3.1.3.*

Theorem 4.1.2 *Let* $s < -2$*. If there exists some* $T > 0$ *such that (0.5) is locally well-posed, then the flow-map data solution* S *defined in (4.1) is not* C^2 *at zero.*

In all the ill-posedness results of Tzvetkov [40], Molinet, Saut and Tzvetkov [35]- [36] it is, in fact, proved that for a fixed $t > 0$ the flow map $S_t : \phi \mapsto u(t)$ is not C^2 differentiable at zero. This, of course, implies that the flow map S is not smooth (C^2) at the origin.

Unfortunately, in our case we cannot fix $t > 0$ since we don't have good cancellations on the symbol $\sqrt{\xi^2 + \xi^4}$. To overcome this difficulty, we allow the variable t to move. Therefore, choosing suitable characteristics functions and sending t to zero we can establish Theorems 4.1.1-4.1.2. We should remark that this kind of argument also appears in the ill-posed result of Bejenaru, Tao [2].

4.2 Proof of Theorems 4.1.1-4.1.2

Proof of Theorem 4.1.1 Suppose that there exists a space X_T satisfying the conditions of the theorem for $s < -2$ and $T > 0$. Let $\phi, \rho \in H^s(\mathbb{R})$ and define $u(t) = V_c(t)\phi$, $v(t) = V_c(t)\rho$. In view of (4.2), (4.3), (4.4) it is easy to see that the following inequality must hold

$$\sup_{1 \le t \le T}\left\|\int_0^t V_s(t-t')(V_c(t')\phi V_c(t')\rho)_{xx}(t')dt'\right\|_{H^s(\mathbb{R})} \le c\,\|\phi\|_{H^s(\mathbb{R})}\,\|\rho\|_{H^s(\mathbb{R})}. \qquad (4.5)$$

We will see that (4.5) fails for an appropriate choice of ϕ, ρ, which would lead to a contradiction.

Define

$$\widehat{\phi}(\xi) = N^{-s}\chi_{[-N,-N+1]} \quad \text{and} \quad \widehat{\rho}(\xi) = N^{-s}\chi_{[N+1,N+2]},$$

where $\chi_A(\cdot)$ denotes the characteristic function of the set A.

We have

$$\|\phi\|_{H^s(\mathbb{R})}, \|\rho\|_{H^s(\mathbb{R})} \sim 1.$$

Recall that $\gamma(\xi) \equiv \sqrt{\xi^2+\xi^4}$. By the definitions of V_c, V_s and Fubini's Theorem, we have

$$\left(\int_0^t V_s(t-t')(V_c(t')\phi V_c(t')\rho)_{xx}(t')dt'\right)^{\wedge(x)}(\xi) =$$

$$\begin{aligned} &= \int_{-\infty}^{+\infty} -\frac{|\xi|^2}{8\gamma(\xi)}\widehat{\phi}(\xi-\xi_1)\widehat{\rho}(\xi_1)K(t,\xi,\xi_1)d\xi_1 \\ &= \int_{A_\xi} -\frac{|\xi|^2}{8\gamma(\xi)}N^{-2s}K(t,\xi,\xi_1)d\xi_1 \end{aligned}$$

where

$$A_\xi = \left\{\xi_1 : \xi_1 \in \text{supp}(\widehat{\rho}) \text{ and } \xi-\xi_1 \in \text{supp}(\widehat{\phi})\right\}$$

and

$$K(t,\xi,\xi_1) \equiv \int_0^t \sin((t-t')\gamma(\xi))\cos(t'\gamma(\xi-\xi_1))\cos(t'\gamma(\xi_1))dt'.$$

Note that for all $\xi_1 \in \text{supp}(\widehat{\rho})$ and $\xi-\xi_1 \in \text{supp}(\widehat{\phi})$ we have

$$\gamma(\xi-\xi_1), \gamma(\xi_1) \sim N^2 \text{ and } 1 \leq \xi \leq 3.$$

On the other hand, since $s<-2$, we can choose $\varepsilon>0$ such that

$$-2s-4-2\varepsilon > 0. \tag{4.6}$$

Let $t = \dfrac{1}{N^{2+\varepsilon}}$, then for N sufficiently large we have

$$\cos(t'\gamma(\xi-\xi_1)), \cos(t'\gamma(\xi_1)) \geq 1/2$$

and

$$\sin((t-t')\gamma(\xi)) \geq c(t-t')\gamma(\xi),$$

for all $0 \leq t' \leq t$, $1 \leq \xi \leq 3$ and $\xi_1 \in \text{supp}(\widehat{\rho})$.

Therefore

$$K(t,\xi,\xi_1) \gtrsim \int_0^t (t-t')\gamma(\xi)dt' \gtrsim \gamma(\xi)\frac{1}{2N^{4+2\varepsilon}}.$$

For $3/2 \leq \xi \leq 5/2$ we have that $\mathrm{mes}(A_\xi) \gtrsim 1$. Thus, from (4.5) we obtain

$$\begin{aligned} 1 &\gtrsim \sup_{1\leq t\leq T}\left\|\int_0^t V_s(t-t')(V_c(t')\phi V_c(t')\rho)_{xx}(t')dt'\right\|_{H^s(\mathbb{R})} \\ &\gtrsim \sup_{1\leq t\leq T}\left(\int_{3/2}^{5/2}\left(1+|\xi|^2\right)^s\left|\int_{A_\xi}\frac{|\xi|^2}{8\gamma(\xi)}N^{-2s}K(t,\xi,\xi_1)d\xi_1\right|^2 d\xi\right)^{1/2} \\ &\gtrsim N^{-2s-4-2\varepsilon}, \text{ for all } N \gg 1 \end{aligned}$$

which is in contradiction with (4.6).

■

Proof of Theorem 4.1.2 Let $s < -2$ and suppose that there exists $T > 0$ such that the flow-map S defined in (4.1) is C^2. When $(\phi,\psi) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$, we denote by $u_{(\phi,\psi)} \equiv S(\phi,\psi)$ the solution of the IVP (0.5), that is

$$u_{(\phi,\psi)}(t) = V_c(t)\phi + V_s(t)\psi_x + \int_0^t V_s(t-t')(u^2_{(\phi,\psi)})_{xx}(t')dt'.$$

The Fréchet derivative of S at (ω,ζ) in the direction $(\phi,\bar{\phi})$ is given by

$$\begin{aligned} d_{(\phi,\bar{\phi})}S(\omega,\zeta) =& V_c(t)\phi + V_s(t)\bar{\phi}_x + \\ &+ 2\int_0^t V_s(t-t')(u_{(\phi,\psi)}(t')d_{(\phi,\bar{\phi})}S(\omega,\zeta)(t'))_{xx}dt'. \end{aligned} \tag{4.7}$$

Using the well-posedness assumption we know that the only solution for initial data $(0,0)$ is $u_{(0,0)} \equiv S(0,0) = 0$. Therefore, (4.7) yields

$$d_{(\phi,\bar{\phi})}S(0,0) = V_c(t)\phi + V_s(t)\bar{\phi}_x.$$

Computing the second Fréchet derivative at the origin in the direction $((\phi,\bar{\phi}),(\rho,\bar{\rho}))$, we obtain

$$d^2_{(\phi,\bar{\phi}),(\rho,\bar{\rho})}S(0,0) =$$

$$= 2\int_0^t V_s(t-t')\left[(V_c(t')\phi + V_s(t')\bar{\phi}_x)(V_c(t')\rho + V_s(t')\bar{\rho}_x)\right]_{xx} dt'.$$

Taking $\bar{\phi}, \bar{\rho} = 0$, the assumption of C^2 regularity of S yields

$$\sup_{1\leq t\leq T}\left\|\int_0^t V_s(t-t')(V_c(t')\phi V_c(t')\rho)_{xx}(t')dt'\right\|_{H^s(\mathbb{R})} \leq c\,\|\phi\|_{H^s(\mathbb{R})}\,\|\rho\|_{H^s(\mathbb{R})}$$

which has been shown to fail in the proof of Theorem 4.1.1.

■

Chapter 5
Local and global solutions for the nonlinear Schrödinger-Boussinesq system

5.1 Introduction

In this chapter we consider the initial value problem (IVP) associated to the Schrödinger-Boussinesq system (hereafter referred to as the SB-system), that is

$$\begin{cases} iu_t + u_{xx} = vu, \\ v_{tt} - v_{xx} + v_{xxxx} = (|u|)_{xx}, \\ u(x,0) = u_0(x);\ v(x,0) = v_0(x);\ v_t(x,0) = (v_1)_x(x), \end{cases} \tag{5.1}$$

where $x \in \mathbb{R}$ and $t > 0$.

Here u and v are respectively a complex valued and a real valued function defined in space-time $\mathbb{R}^2$. The SB-system is considered as a model of interactions between short and intermediate long waves, which is derived in describing the dynamics of Langmuir soliton formation and interaction in a plasma [34] and diatomic lattice system [42]. The short wave term $u(x,t) : \mathbb{R}\times\mathbb{R} \to \mathbb{C}$ is described by a Schrödinger type equation with a potential $v(x,t) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfying some sort of Boussinesq equation and representing the intermediate long wave.

Our principal aim here is to study the well-posedness of the Cauchy problem for the SB-system (5.1) in the classical Sobolev spaces $H^s(\mathbb{R})$, $s \in \mathbb{R}$.

Concerning the local well-posedness question, some results have been obtained for the SB-system (5.1). Linares and Navas [30] proved that (5.1) is locally well-posedness for initial data $u_0 \in L^2(\mathbb{R})$, $v_0 \in L^2(\mathbb{R})$, $v_1 = h_x$ with $h \in H^{-1}(\mathbb{R})$ and $u_0 \in H^1(\mathbb{R})$, $v_0 \in H^1(\mathbb{R})$, $v_1 = h_x$ with $h \in L^2(\mathbb{R})$. Moreover, by using some conservations laws, in the latter case the solutions can be extended globally. Yongqian [43] established similar result when $u_0 \in H^s(\mathbb{R})$, $v_0 \in H^s(\mathbb{R})$, $v_1 = h_{xx}$ with $h \in H^s(\mathbb{R})$ for $s \geq 0$ and assuming $s \geq 1$ these solutions are global.

Since scaling argument cannot be applied to the Boussinesq-type equations to obtain a criticaly notion it is not clear what is the lower Sobolev index s for which one has local (or maybe global) well-posedness. To obtain some idea on which spaces we should expect well-posedness, we recall some results concerning the Schrödinger and Boussinesq equations.

For the single cubic nonlinear Schrödinger (NLS) equation with cubic term $iu_t + u_{xx} + |u|^2u = 0$, Y. Tsutsumi [40] established local and global well-posedness for data in $L^2(\mathbb{R})$. Moreover, by using the scaling and Galilean invariance with the special soliton solutions, it was proved by Kenig, Ponce and Vega [28] that the focusing cubic (NLS) equation is not locally-well posed below $L^2(\mathbb{R})$. This ill-posed result is in the sense that the data-solution map is not uniformly continuous. Recently, Christ, Colliander and Tao [13] have obtained similar results for defocusing (NLS) equations. For the case of quadratics (NLS) Kenig, Ponce and Vega [27] have proved local well-posedness for data in $H^s(\mathbb{R})$ with $s > -3/4$ for (0.8)-(0.10) and $s > -1/4$ for (0.9). This result is sharp, in the sense that we cannot lower these Sobolev indices using the techniques of [27].

Now we turn to the "good" Boussinesq equation (0.5). In Chapter 3, we prove local well-posedness for initial data in $H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ with $s > -1/4$. Again, this last result is sharp in the same as above.

Taking into account the sharp local well-posedness results obtained for the quadratic (NLS) and Boussinesq equations it is natural to ask whether the SB-system is, at least, locally well-posed for initial data $(u_0, v_0, v_1) \in H^s(\mathbb{R}) \times H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ with $s > -1/4$. Here we answer affirmatively this question. Indeed, we obtain local well-posedness for weak initial data $(u_0, v_0, v_1) \in H^k(\mathbb{R}) \times H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ for various values of k and s. The scheme of proof used to obtain our results is in the same spirit as the one implemented by Ginibre, Y. Tsutsumi and Velo [21] to establish their results for the Zakharov system (0.14).

In [1], it was shown that by a limiting procedure, as $\sigma \to 0$, the solution u_σ to (0.14) converges in a certain sense to the unique solution for cubic (NLS). Hence it is natural to expect that the system (0.14) is well-posed for $u_0 \in L^2(\mathbb{R})$. In fact, for the case $\sigma = 1$, in [21] it is shown that (0.14) is local well-posedness for $(u_0, v_0, v_1) \in L^2(\mathbb{R}) \times H^{-1/2}(\mathbb{R}) \times H^{-3/2}(\mathbb{R})$. Moreover, Holmer [23] shows that the one-dimensional local theory of [21] is effectively sharp, in the sense that for (k, s) outside the range given in [21], there exists ill-posedness results for the Zakharov system (0.14). In particular, we cannot have local well-posedness for the initial data in Sobolev spaces of negative index.

Note that the system (0.14) is quite similar to the SB-system. In fact, taking

$\sigma = 1$ and adding v_{xxxx} on the left hand side of the second equation of (0.14) we obtain (5.1). In other words, the intermediate long wave in (0.14) is described by a wave equation instead of a Boussinesq equation.

Despite such similarity, there are strong differences in the local theory. According to Theorem 5.1.1 below, it is possible to prove that the system (5.1) is locally well-posed for initial data $(u_0, v_0, v_1) \in H^s(\mathbb{R}) \times H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ with $s > -1/4$, which is not the case for the system (0.14). Therefore, in the sense of the local theory, we can say that the SB-system (5.1) is better behaved than the Zakharov system (0.14). This is due basically to the fact that (0.13) has more dispersion then (0.14).

To describe our results we define next the $X^S_{s,b}$ and $X^B_{s,b}$ spaces related to the Schrödinger and Boussinesq equations, respectively. The spaces $X^B_{s,b}$ were introduced in Chapter 3. Here we set the indices S, B to emphasize that the spaces are related to the Schrödinger and Boussinesq equations, respectively.

Definition 5.1.1 *For $s, b \in \mathbb{R}$, $X^S_{s,b}$ denotes the completion of the Schwartz class $\mathcal{S}(\mathbb{R}^2)$ with respect to the norm*

$$\|F\|_{X^S_{s,b}} = \|\langle \tau + \xi^2 \rangle^b \langle \xi \rangle^s \widetilde{F}\|_{L^2_{\tau,\xi}}$$

where $\sim$ denotes the space-time Fourier transform and $\langle a \rangle \equiv 1 + |a|$.

Definition 5.1.2 *For $s, b \in \mathbb{R}$, $X^B_{s,b}$ denotes the completion of the Schwartz class $\mathcal{S}(\mathbb{R}^2)$ with respect to the norm*

$$\|F\|_{X^B_{s,b}} = \|\langle |\tau| - \gamma(\xi) \rangle^b \langle \xi \rangle^s \widetilde{F}\|_{L^2_{\tau,\xi}}$$

where $\gamma(\xi) \equiv \sqrt{\xi^2 + \xi^4}$.

We will also need the localized $X^S_{s,b}$ and $X^B_{s,b}$ spaces defined as follows

Definition 5.1.3 *For $s, b \in \mathbb{R}$ and $T \geq 0$, $X^{S,T}_{s,b}$ (resp. $X^{B,T}_{s,b}$) denotes the space endowed with the norm*

$$\|u\|_{X^{S,T}_{s,b}} = \inf_{w \in X^S_{s,b}} \left\{ \|w\|_{X^S_{s,b}} : w(t) = u(t) \text{ on } [0,T] \right\}.$$

(resp. with $X^B_{s,b}$ instead of $X^S_{s,b}$)

Now state the main results of this chapter.

Theorem 5.1.1 *Let $1/4 < a < 1/2 < b$. Then, there exists $c > 0$, depending only on a, b, k, s, such that*

(i) $\|uv\|_{X^S_{k,-a}} \leq c\, \|u\|_{X^S_{k,b}} \|v\|_{X^B_{s,b}}$.

holds for $|k| - s \leq a$.

(ii) $\|u_1\bar{u}_2\|_{X^B_{s,-a}} \leq c\, \|u_1\|_{X^S_{k,b}} \|u_2\|_{X^S_{k,b}}$.

holds for

- $s - k \leq a$, *if* $s > 0$ *and* $k > 0$;
- $s + 2|k| \leq a$, $2|k| > a$, *if* $s > 0$ *and* $k \leq 0$;
- $s + 2|k| \leq 1/2$, $2|k| > a$, *if* $s \leq 0$ *and* $k \leq 0$.

Theorem 5.1.2 *Let $k > -1/4$. Then for any $(u_0, v_0, v_1) \in H^k(\mathbb{R}) \times H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ provided*

(i) $|k| - 1/2 < s < 1/2 + 2k$ *for* $k \leq 0$,

(ii) $k - 1/2 < s < 1/2 + k$ *for* $k > 0$,

there exist $T = T(\|u_0\|_{H^k}, \|v_0\|_{H^s}, \|v_1\|_{H^{s-1}})$, $b > 1/2$ and a unique solution u of the IVP (5.1), satisfying

$$u \in C([0,T] : H^k(\mathbb{R})) \cap X^{S,T}_{k,b} \text{ and } v \in C([0,T] : H^s(\mathbb{R})) \cap X^{B,T}_{s,b}.$$

Moreover, the map $(u_0, v_0, v_1) \mapsto (u(t), v(t))$ is locally Lipschitz from $H^k(\mathbb{R}) \times H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ into $C([0,T] : H^k(\mathbb{R}) \times H^s(\mathbb{R}))$.

Next we obtain bilinear estimates for the case $s = 0$ and $b, b_1 < 1/2$. These estimates will be the main tool to establish global solutions.

Theorem 5.1.3 *Let $a, a_1, b, b_1 > 1/4$, then there exists $c > 0$ depending only on a, a_1, b, b_1 such that*

(i) $\|uv\|_{X^S_{0,-a_1}} \leq c\, \|u\|_{X^S_{0,b_1}} \|v\|_{X^B_{0,b}}$.

(ii) $\|u_1\bar{u}_2\|_{X^B_{0,-a}} \leq c\, \|u_1\|_{X^S_{0,b_1}} \|u_2\|_{X^S_{0,b_1}}$.

These are the essential tools to prove the following global result.

Theorem 5.1.4 *The SB-system (5.1) is globally well-posed for* $(u_0, v_0, v_1) \in L^2(\mathbb{R}) \times L^2(\mathbb{R}) \times H^{-1}(\mathbb{R})$ *and the solution* (u, v) *satisfies for all* $t > 0$

$$\|v(t)\|_{L^2} + \|(-\Delta)^{-1/2} v_t(t)\|_{H^{-1}} \lesssim e^{((\ln 2)\|u_0\|_{L^2}^2 t)} \max\{\|v_0, v_1\|_{\mathfrak{B}}, \|u_0\|_{L^2}\}.$$

The argument used to prove this result follows the ideas introduced by Colliander, Holmer, Tzirakis [14] to deal with the Zakharov system. The intuition for this theorem comes from the fact that the nonlinearity for the second equation of the SB-system (5.1) depends only on the first equation. Therefore, noting that the bilinear estimates given in Theorem 5.1.2 hold for $a, a_1, b, b_1 < 1/2$, it is possible to show that the time existence depends only on the $\|u_0\|_{L^2}$. But since this norm is conserved by the flow, we obtain a global solution.

The plan of this chapter is as follows: in Section 2, we prove some estimates for the integral equation in the $X^S_{s,b}$ and $X^B_{s,b}$ space introduced above. Bilinear estimates are proved in Section 3. Finally, the local and global well-posedness results are treated in Sections 4 and 5, respectively.

5.2 Preliminary results

By Duhamel's Principle the solution of the SB-system is equivalent to (0.25). Let θ be a cutoff function satisfying $\theta \in C_0^\infty(\mathbb{R})$, $0 \leq \theta \leq 1$, $\theta \equiv 1$ in $[-1, 1]$, $\text{supp}(\theta) \subseteq [-2, 2]$ and for $0 < T < 1$ define $\theta_T(t) = \theta(t/T)$. In fact, to work in the $X^S_{s,b}$ and $X^B_{s,b}$ we consider another version of (0.25), that is

$$\begin{aligned} u(t) =& \theta_T(t) U(t) u_0 - i\theta_T(t) \int_0^t U(t - t')(vu)(t') dt' \\ v(t) =& \theta_T(t) \left(V_c(t) v_0 + V_s(t)(v_1)_x\right) + \theta_T(t) \int_0^t V_s(t - t')(|u|^2)_{xx}(t') dt'. \end{aligned} \tag{5.2}$$

Note that the integral equation (5.2) is defined for all $(x, t) \in \mathbb{R}^2$. Moreover if (u, v) is a solution of (5.2) than $(\tilde{u}, \tilde{v}) = (u|_{[0,T]}, v|_{[0,T]})$ will be a solution of (0.25) in $[0, T]$.

Before proceed to the group and integral estimates for (5.2) we introduce the norm

$$\|v_0, v_1\|^2_{\mathfrak{B}^s} \equiv \|v_0\|^2_{H^s} + \|v_1\|^2_{H^{s-1}}.$$

For simplicity we denote $\mathfrak{B}^0$ by $\mathfrak{B}$ and, for functions of t, we use the shorthand

$$\|v(t)\|^2_{\mathfrak{B}^s} \equiv \|v(t)\|^2_{H^s} + \|(-\Delta)^{-1/2} v_t(t)\|^2_{H^{s-1}}.$$

The following lemmas are standard in this context. The difference here is on the exponent of T that appears in the group estimates. This exponent together with the growth control of the solution norm $\|v\|_{\mathfrak{B}}$ will be important for the proof of Theorem 5.1.4 in L^2.

Lemma 5.2.1 (Group estimates) *Let $T \leq 1$.*

(a) Linear Schrödinger equation

(i) $\|S(t)u_0\|_{C(\mathbb{R}:H^s)} = \|u_0\|_{H^s}$.
(ii) If $0 \leq b_1 \leq 1$, *then*

$$\|\theta_T(t)S(t)u_0\|_{X^S_{s,b_1}} \lesssim T^{1/2-b_1}\|u_0\|_{H^s}.$$

(b) Linear Boussinesq equation

(i) $\|V_c(t)v_0 + V_s(t)(v_1)_x\|_{C(\mathbb{R}:H^s)} \leq \|v_0\|_{H^s} + \|v_1\|_{H^{s-1}}$.
(ii) $\|V_c(t)v_0 + V_s(t)(v_1)_x\|_{C(\mathbb{R}:\mathfrak{B})} = \|v_0, v_1\|_{\mathfrak{B}}$.
(iii) If $0 \leq b \leq 1$, *then*

$$\|\theta_T(t)\left(V_c(t)v_0 + V_s(t)(v_1)_x\right)\|_{X^B_{s,b}} \lesssim T^{1/2-b}\left(\|v_0\|_{H^s} + \|v_1\|_{H^{s-1}}\right).$$

Remark 5.2.1 *We should notice that the first inequality of item (a) and the second one of item (b) do not have implicit constant multiplying the right hand side. This will be important in the proof of the global result in L^2 stated in Theorem 5.1.4, since we will make use of an iterated argument to control the growth of the solution norm.*

Proof.

(a) The first inequality comes from the fact that $S(\cdot)$ is a unitary group. The second one with $0 \leq b_1 \leq 1/2$ can be found, for instance, in [14] Lemma 2.1(a). The case $1/2 < b_1 \leq 1$ can be proved using the same arguments as the one used in the previous case. Since in (b) we apply these same arguments in the context of the Boussinesq equation, we omit the proof of (ii).

(b) By the definitions of $V_c(\cdot)$ and $V_s(\cdot)$ it is easy to see that for all $t \in \mathbb{R}$

$$\|V_c(t)v_0\|_{H^s} \leq \|v_0\|_{H^s} \text{ and } \|V_s(t)(v_1)_x\|_{H^s} \leq \|v_1\|_{H^{s-1}}.$$

Let $f(x,t)$ be a solution of the linear Boussinesq equation

$$\begin{cases} f_{tt} - f_{xx} + f_{xxxx} = 0, \\ f(x,0) = v_0, \quad f_t(x,0) = (v_1)_x. \end{cases} \tag{5.3}$$

Recall that $J^s = \mathcal{F}^{-1}(1+|\xi|^2)^{s/2}\mathcal{F}$, for $s \in \mathbb{R}$. Applying the operators $(-\Delta)^{-1}$ and J^{-1} to the equation (5.3), multiplying by $J^{-1}f_t$ and finally integrating with respect to x, we obtain (after an integration by parts) the following

$$\frac{d}{dt}\left\{\|f\|_{L^2}^2 + \|(-\Delta)^{-1/2}f_t\|_{H^{-1}}^2\right\} = 0$$

which implies for all $t \in \mathbb{R}$

$$\|V_c(t)v_0 + V_s(t)(v_1)_x\|_{\mathfrak{B}} = \|v_0, v_1\|_{\mathfrak{B}}.$$

Now we turn to the proof of the second inequality in (b). A simple computation shows that

$$(\theta_T(t)\,(V_c(t)v_0 + V_s(t)(v_1)_x))^{\sim}(\xi,\tau) =$$

$$\frac{\widehat{\theta_T}(\tau-\gamma(\xi))}{2}\left(\widehat{v_0}(\xi) + \frac{i\xi\widehat{v_1}(\xi)}{\gamma(\xi)}\right) + \frac{\widehat{\theta_T}(\tau+\gamma(\xi))}{2}\left(\widehat{v_0}(\xi) - \frac{i\xi\widehat{v_1}(\xi)}{\gamma(\xi)}\right).$$

Thus, setting $h_1(\xi) = \widehat{v_0}(\xi) + \frac{i\xi\widehat{v_1}(\xi)}{\gamma(\xi)}$ and $h_2(\xi) = \widehat{v_0}(\xi) - \frac{i\xi\widehat{v_1}(\xi)}{\gamma(\xi)}$, we have

$$\|\theta_T\,(V_c(t)v_0 + V_s(t)(v_1)_x)\|_{X_{s,b}}^2 \le$$

$$\le \int_{-\infty}^{+\infty}\langle\xi\rangle^{2s}|h_1(\xi)|^2\left(\int_{-\infty}^{+\infty}\langle|\tau|-\gamma(\xi)\rangle^{2b}\left|\frac{\widehat{\theta_T}(\tau-\gamma(\xi))}{2}\right|^2 d\tau\right)d\xi$$
$$+\int_{-\infty}^{+\infty}\langle\xi\rangle^{2s}|h_2(\xi)|^2\left(\int_{-\infty}^{+\infty}\langle|\tau|-\gamma(\xi)\rangle^{2b}\left|\frac{\widehat{\theta_T}(\tau+\gamma(\xi))}{2}\right|^2 d\tau\right)d\xi.$$

Since $||\tau|-\gamma(\xi)| \le \min\{|\tau-\gamma(\xi)|, |\tau+\gamma(\xi)|\}$ we have

$$\begin{aligned} \|\theta_T\,(V_c(t)v_0 + V_s(t)(v_1)_x)\|_{X_{s,b}}^2 &\lesssim \left(\|h_1\|_{H^s}^2 + \|h_2\|_{H^s}^2\right)\|\theta_T\|_{H_t^b}^2 \\ &\lesssim \left(\|v_0\|_{H^s} + \|v_1\|_{H^{s-1}}\right)^2\|\theta_T\|_{H_t^b}^2. \end{aligned}$$

To complete the proof we note that (since $T \leq 1$)

$$\begin{aligned} \|\theta_T\|_{H_t^b} &\lesssim \|\theta_T\|_{L^2} + \|\theta_T\|_{\dot{H}_t^b} \\ &\lesssim T^{1/2}\|\theta_1\|_{L^2} + T^{1/2-b}\|\theta_1\|_{\dot{H}_t^b} \\ &\lesssim T^{1/2-b}\|\theta_1\|_{H_t^b}. \end{aligned}$$

■

Next we estimate the integral parts of (5.2).

Lemma 5.2.2 (Integral estimates) *Let $T \leq 1$.*

(a) Nonhomogeneous linear Schrödinger equation

(i) If $0 \leq a_1 < 1/2$ then

$$\left\| \int_0^t U(t-t')z(t')dt' \right\|_{C([0,T]:H^s)} \lesssim T^{1/2-a_1}\|z\|_{X^S_{s,-a_1}}.$$

(ii) If $0 \leq a_1 < 1/2$, $0 \leq b_1$ and $a_1 + b_1 \leq 1$ then

$$\left\| \theta_T(t) \int_0^t U(t-t')z(t')dt' \right\|_{X^S_{s,b_1}} \lesssim T^{1-a_1-b_1}\|z\|_{X^S_{s,-a_1}}.$$

(b) Nonhomogeneous linear Boussinesq equation

(i) If $0 \leq a < 1/2$ then

$$\left\| \int_0^t V_s(t-t')z_{xx}(t')dt' \right\|_{C([0,T]:\mathfrak{B}^s)} \lesssim T^{1/2-a}\|z\|_{X^B_{s,-a}}.$$

(ii) If $0 \leq a < 1/2$, $0 \leq b$ and $a + b \leq 1$ then

$$\left\| \theta_T(t) \int_0^t V_s(t-t')z_{xx}(t')dt' \right\|_{X^B_{s,b}} \lesssim T^{1-a-b}\|z\|_{X^B_{s,-a}}.$$

Proof.

(a) Again we refer the reader to [14] Lemma 2.2(a). Since in (b) we apply these same arguments in the context of the Boussinesq equation, we omit the proof of this item.

(b) We know that (see [14] inequality (2.13))

$$\left\| \theta_T(t) \int_0^t f(t')dt' \right\|_{L_t^\infty} \lesssim T^{1/2-a} \|f\|_{H_t^{-a}} . \tag{5.4}$$

First, we will prove that

(I) $\left\|\theta_T(t) \int_0^t V_s(t-t')z_{xx}(t')dt'\right\|_{L_t^\infty H^s} \lesssim T^{1/2-a}\|z\|_{X^B_{s,-a}}.$

(II) $\left\|\theta_T(t)(-\Delta)^{-1/2}\partial_t \int_0^t V_s(t-t')z_{xx}(t')dt'\right\|_{L_t^\infty H^s} \lesssim T^{1/2-a}\|z\|_{X^B_{s,-a}}.$

To prove (I), we observe that $\sup_{\xi\in\mathbb{R}} \dfrac{|\xi|^2}{\gamma(\xi)} < \infty$. Therefore, using Minkowski inequality and (5.4) we obtain

$$\begin{aligned}
&\left\| \theta_T(t) \int_0^t V_s(t-t')z_{xx}(t')dt' \right\|_{L_t^\infty H^s} \\
\lesssim\ & \left\| \left\| \theta_T(t) \int_0^t e^{it'\gamma(\xi)} (1+|\xi|^2)^{s/2} z^{\wedge(x)}(\xi,t')dt' \right\|_{L^2_\xi} \right\|_{L_t^\infty} \\
& + \left\| \left\| \theta_T(t) \int_0^t e^{-it'\gamma(\xi)} (1+|\xi|^2)^{s/2} z^{\wedge(x)}(\xi,t')dt' \right\|_{L^2_\xi} \right\|_{L_t^\infty} \\
\lesssim\ & T^{1/2-a} \Big(\left\| \langle \tau+\gamma(\xi)\rangle^{-a} \langle\xi\rangle^s \widetilde{z}(\xi,\tau) \right\|_{L^2_{\xi,\tau}} \\
& + \left\| \langle \tau-\gamma(\xi)\rangle^{-a} \langle\xi\rangle^s \widetilde{z}(\xi,\tau) \right\|_{L^2_{\xi,\tau}} \Big) .
\end{aligned}$$

Since $||\tau| - \gamma(\xi)| \leq \min\{|\tau-\gamma(\xi)|, |\tau+\gamma(\xi)|\}$ and $a \geq 0$ we obtain inequality (I).

To prove (II) we note that

$$\begin{aligned}
&\left\| \theta_T(t)(-\Delta)^{1/2}\partial_t \int_0^t V_s(t-t')z_{xx}(t')dt' \right\|_{L_t^\infty H^{s-1}} \\
=\ & \left\| \left\| |\xi|^{-1}(1+|\xi|^2)^{(s-1)/2} \theta_T(t) \int_0^t \frac{\cos((t-t')\gamma(\xi))}{\gamma(\xi)} \gamma(\xi)|\xi|^2 z^{\wedge(x)}(\xi,t')dt' \right\|_{L^2_\xi} \right\|_{L_t^\infty} .
\end{aligned}$$

Therefore the same arguments used to prove inequality (I) yield (II).

Now, we need to prove the continuity statements. We will prove only for inequality (I), since for (II) it can be obtained applying analogous arguments.

By an $\varepsilon/3$ argument, it is sufficient to establish this statement for z belonging to the dense class $\mathcal{S}(\mathbb{R}^2) \subseteq X^B_{s,-a}$. A simple calculation shows

$$\partial_t \int_0^t V_s(t-t')z_{xx}(t')dt' = \int_0^t V_c(t-t')z_{xx}(t')dt'.$$

Moreover, with essentially the same proof given above, inequality (I) holds for $V_c(t-t')$ and $\|z_{xx}\|_{X^B_{s,-a}}$ instead of $V_s(t-t')$ and $\|z\|_{X^B_{s,-a}}$, respectively. Therefore, by the fundamental Theorem of calculus we have for $t_1, t_2 \in [0,T]$

$$\begin{aligned}
&\left\| \int_0^{t_1} V_s(t_1-t')z_{xx}(t')dt' - \int_0^{t_2} V_s(t_2-t')z_{xx}(t')dt' \right\|_{H^s} \\
&= \left\| \int_{t_1}^{t_2} \left(\int_0^t V_c(t-t')z_{xx}(t')dt' \right) dt \right\|_{H^s} \\
&\lesssim (t_2-t_1) \left\| \theta_T(t) \int_0^t V_c(t-t')z_{xx}(t')dt' \right\|_{L^\infty_t H^s} \\
&\lesssim (t_2-t_1)\|z_{xx}\|_{X^B_{s,-a}}
\end{aligned}$$

which proves the continuity.

It remains to prove the second assertion, but this can be done applying the same arguments as the ones used in the proof of Lemma 3.2.2-(ii) together with the fact that $\sup_{\xi\in\mathbb{R}} \dfrac{|\xi|^2}{\gamma(\xi)} < \infty$.

■

We recall that, for $b > 1/2$, $X^S_{s,b}$ and $X^B_{s,b}$ are embedding in $C(\mathbb{R} : H^s)$. For the spaces associated to the Schrödinger equation this result is well know in the literature. For the $X^B_{s,b}$ spaces, this embedding was proved in Lemma 3.2.3.

We finish this section with the following standard Bourgain-Strichartz estimates.

Lemma 5.2.3 *Let $\bar{X}^S_{s,b}$ denote the space with norm*

$$\|F\|_{\bar{X}^S_{s,b}} = \|\langle \tau - \xi^2\rangle^b \langle \xi\rangle^s \widetilde{F}\|_{L^2_{\tau,\xi}}.$$

Therefore

$$\|u\|_{L^3_{x,t}} \leq c\min\{\|u\|_{X^S_{0,1/4+}}, \|u\|_{\bar{X}^S_{0,1/4+}}\},$$

where $a+$ means that there exists $\varepsilon > 0$ such that $a+ = a + \varepsilon$.

Proof. This estimate is easily obtained by interpolating between

- (Strichartz) $\|u\|_{L^6_{x,t}} \leq c \min\{\|u\|_{X^S_{0,1/2+}}, \|u\|_{\bar{X}^S_{0,1/2+}}\}$.
- (Definition) $\|u\|_{L^2_{x,t}} = \|u\|_{X^S_{0,0}} = \|u\|_{\bar{X}^S_{0,0}}$.

■

5.3 Bilinear estimates

Again, our main tools to obtain the desired estimates are Lemmas 3.3.1-3.3.2 stated in the previous chapter.

Proof of Theorem 5.1.1

(i) For $u \in X^S_{k,b}$ and $v \in X^B_{s,b}$ we define

$$\begin{aligned} f(\xi,\tau) &\equiv \langle \tau + \xi^2 \rangle^b \langle \xi \rangle^k \widetilde{u}(\xi,\tau), \\ g(\xi,\tau) &\equiv \langle |\tau| - \gamma(\xi) \rangle^b \langle \xi \rangle^s \widetilde{v}(\xi,\tau). \end{aligned}$$

By duality the desired inequality is equivalent to

$$|W(f,g,\phi)| \leq c\|f\|_{L^2_{\xi,\tau}} \|g\|_{L^2_{\xi,\tau}} \|\phi\|_{L^2_{\xi,\tau}} \tag{5.5}$$

where

$$W(f,g,\phi) = \int_{\mathbb{R}^4} \frac{\langle \xi \rangle^k}{\langle \xi_1 \rangle^s \langle \xi_2 \rangle^k} \frac{g(\xi_1,\tau_1) f(\xi_2,\tau_2) \bar{\phi}(\xi,\tau)}{\langle \sigma \rangle^a \langle \sigma_1 \rangle^b \langle \sigma_2 \rangle^b} d\xi d\tau d\xi_1 d\tau_1$$

and

$$\begin{gathered} \xi_2 = \xi - \xi_1, \quad \tau_2 = \tau - \tau_1, \\ \sigma = \tau + \xi^2, \quad \sigma_1 = |\tau_1| - \gamma(\xi_1), \quad \sigma_2 = \tau_2 + \xi_2^2. \end{gathered} \tag{5.6}$$

In view of Lemma 3.3.2, we know that $\langle |\tau_1| - \gamma(\xi_1) \rangle \sim \langle |\tau_1| - \xi_1^2 \rangle$. Therefore splitting the domain of integration into the regions $\{(\xi,\tau,\xi_1,\tau_1) \in \mathbb{R}^4 : \tau_1 < 0\}$ and $\{(\xi,\tau,\xi_1,\tau_1) \in \mathbb{R}^4 : \tau_1 \geq 0\}$, it is sufficient to prove inequality (5.5) with $W_1(f,g,\phi)$ and $W_2(f,g,\phi)$ instead of $W(f,g,\phi)$, where

$$W_1(f,g,\phi) = \int_{\mathbb{R}^4} \frac{\langle \xi \rangle^k}{\langle \xi_1 \rangle^s \langle \xi_2 \rangle^k} \frac{g(\xi_1,\tau_1) f(\xi_2,\tau_2) \bar{\phi}(\xi,\tau)}{\langle \sigma \rangle^a \langle \tau_1 + \xi_1^2 \rangle^b \langle \sigma_2 \rangle^b} d\xi d\tau d\xi_1 d\tau_1$$

and

$$W_2(f,g,\phi) = \int_{\mathbb{R}^4} \frac{\langle\xi\rangle^k}{\langle\xi_1\rangle^s\langle\xi_2\rangle^k} \frac{g(\xi_1,\tau_1)f(\xi_2,\tau_2)\bar{\phi}(\xi,\tau)}{\langle\sigma\rangle^a\langle\tau_1-\xi_1^2\rangle^b\langle\sigma_2\rangle^b} d\xi d\tau d\xi_1 d\tau_1.$$

Let us first treat the inequality (5.5) with $W_1(f,g,\phi)$. In this case we will make use of the following algebraic relation

$$-(\tau+\xi^2)+(\tau_1+\xi_1^2)+((\tau-\tau_1)+(\xi-\xi_1)^2) = 2\xi_1(\xi_1-\xi). \tag{5.7}$$

By simmetry we can restrict ourselves to the set

$$A = \{(\xi,\tau,\xi_1,\tau_1)\in\mathbb{R}^4 : |(\tau-\tau_1)+(\xi-\xi_1)^2| \le |\tau_1+\xi_1^2|\}.$$

First we split A into three pieces

$$\begin{aligned} A_1 &= \{(\xi,\tau,\xi_1,\tau_1)\in A : |\xi_1|\le 10\}, \\ A_2 &= \{(\xi,\tau,\xi_1,\tau_1)\in A : |\xi_1|\ge 10 \text{ and } |2\xi_1-\xi|\ge|\xi_1|/2\}, \\ A_3 &= \{(\xi,\tau,\xi_1,\tau_1)\in A : |\xi_1|\ge 10 \text{ and } |\xi_1-\xi|\ge|\xi_1|/2\}. \end{aligned}$$

We have $A = A_1\cup A_2\cup A_3$. Indeed

$$|\xi_1| > |2\xi_1-\xi|+|\xi_1-\xi| \ge |(2\xi_1-\xi)-(\xi_1-\xi)| = |\xi_1|.$$

Next we divide A_3 into two parts

$$\begin{aligned} A_{3,1} &= \{(\xi,\tau,\xi_1,\tau_1)\in A_3 : |\tau_1+\xi_1^2|\le|\tau+\xi^2|\}, \\ A_{3,2} &= \{(\xi,\tau,\xi_1,\tau_1)\in A_3 : |\tau+\xi^2|\le|\tau_1+\xi_1^2|\}. \end{aligned}$$

We can now define the sets R_i, $i=1,2$, as follows

$$R_1 = A_1\cup A_2\cup A_{3,1} \text{ and } R_2 = A_{3,2}.$$

Using the Cauchy-Schwarz and Hölder inequalities it is easy to see that

$$\begin{aligned}
|W_1|^2 \;\leq\; & \|f\|^2_{L^2_{\xi,\tau}}\|g\|^2_{L^2_{\xi,\tau}}\|\phi\|^2_{L^2_{\xi,\tau}} \\
& \times \left\| \frac{\langle\xi\rangle^{2k}}{\langle\sigma\rangle^{2a}} \iint \frac{\chi_{R_1} d\xi_1 d\tau_1}{\langle\xi_1\rangle^{2s}\langle\xi_2\rangle^{2k}\langle\tau_1+\xi_1^2\rangle^{2b}\langle\sigma_2\rangle^{2b}} \right\|_{L^\infty_{\xi,\tau}} \\
& + \|f\|^2_{L^2_{\xi,\tau}}\|g\|^2_{L^2_{\xi,\tau}}\|\phi\|^2_{L^2_{\xi,\tau}} \\
& \times \left\| \frac{1}{\langle\xi_1\rangle^{2s}\langle\tau_1+\xi_1^2\rangle^{2b}} \iint \frac{\chi_{R_2}\langle\xi\rangle^{2k} d\xi d\tau}{\langle\xi_2\rangle^{2k}\langle\sigma\rangle^{2a}\langle\sigma_2\rangle^{2b}} \right\|_{L^\infty_{\xi_1,\tau_1}}.
\end{aligned}$$

Noting that $\langle\xi\rangle^{2k} \leq \langle\xi_1\rangle^{2|k|}\langle\xi_2\rangle^{2k}$, for $k \geq 0$, and $\langle\xi_2\rangle^{-2k} \leq \langle\xi_1\rangle^{2|k|}\langle\xi\rangle^{-2k}$, for $k < 0$ we have

$$\frac{\langle\xi\rangle^{2k}}{\langle\xi_1\rangle^{2s}\langle\xi_2\rangle^{2k}} \leq \langle\xi_1\rangle^{2|k|-2s}. \tag{5.8}$$

Therefore in view of Lemma 3.3.1 it suffices to get bounds for

$$\begin{aligned}
J_1(\xi,\tau) \;&\equiv\; \frac{1}{\langle\sigma\rangle^{2a}} \int \frac{\langle\xi_1\rangle^{2|k|-2s} d\xi_1}{\langle\tau+\xi^2+2\xi_1^2-2\xi\xi_1\rangle^{2b}} \quad \text{on} \quad R_1, \\
J_2(\xi_1,\tau_1) \;&\equiv\; \frac{\langle\xi_1\rangle^{2|k|-2s}}{\langle\tau_1+\xi_1^2\rangle^{2b}} \int \frac{d\xi}{\langle\tau_1-\xi_1^2+2\xi\xi_1\rangle^{2a}} \quad \text{on} \quad R_2.
\end{aligned}$$

In region A_1 we have $\langle\xi_1\rangle^{2|k|-2s} \lesssim 1$ and since $a > 0$, $b > 1/2$ we obtain

$$J_1(\xi,\tau) \lesssim \int_{|\xi_1|\leq 10} d\xi_1 \lesssim 1.$$

In region A_2, by the change of variables $\eta = \tau+\xi^2+2\xi_1^2-2\xi\xi_1$ and the condition $|2\xi_1-\xi| \geq |\xi_1|/2$ we have

$$\begin{aligned}
J_1(\xi,\tau) \;&\lesssim\; \frac{1}{\langle\sigma\rangle^{2a}} \int \frac{\langle\xi_1\rangle^{2|k|-2s}}{|2\xi_1-\xi|\langle\eta\rangle^{2b}} d\eta \\
&\lesssim\; \frac{1}{\langle\sigma\rangle^{2a}} \int \frac{\langle\xi_1\rangle^{2|k|-2s-1}}{\langle\eta\rangle^{2b}} d\eta \lesssim 1
\end{aligned}$$

since $a > 0$, $|k| - s \leq 1/2$ and $b > 1/2$.

Now, by definition of region $A_{3,1}$ and the algebraic relation (5.7) we have

$$\langle\xi_1\rangle^2 \lesssim |\xi_1|^2 \lesssim |\xi_1(\xi_1-\xi)| \lesssim \langle\sigma\rangle.$$

Therefore by Lemma 3.3.1

$$\begin{aligned} J_1(\xi,\tau) &\lesssim \int \frac{\langle \xi_1 \rangle^{2|k|-2s-4a}}{\langle \tau + \xi^2 + 2\xi_1^2 - 2\xi\xi_1 \rangle^{2b}} d\xi_1 \\ &\lesssim \int \frac{1}{\langle \tau + \xi^2 + 2\xi_1^2 - 2\xi\xi_1 \rangle^{2b}} d\xi_1 \lesssim 1 \end{aligned}$$

since $a > 0$, $|k| - s \leq 2a$ and $b > 1/2$.

Next we estimate $J_2(\xi_1, \tau_1)$. Making the change of variables, $\eta = \tau - \xi_1^2 + 2\xi\xi_1$, using the restriction in the region $A_{3,2}$, we have

$$|\eta| \lesssim |(\tau - \tau_1) + (\xi - \xi_1)^2| + |\tau + \xi^2| \lesssim \langle \tau_1 + \xi_1^2 \rangle.$$

Moreover, in $A_{3,2}$

$$|\xi_1|^2 \lesssim |\xi_1(\xi_1 - \xi)| \lesssim \langle \tau_1 + \xi_1^2 \rangle.$$

Therefore, since $|\xi_1| \geq 10$ we have

$$\begin{aligned} J_2(\xi_1,\tau_1) &\lesssim \frac{|\xi_1|^{2|k|-2s}}{\langle \tau_1 + \xi_1^2 \rangle^{2b}} \int_{|\eta| \lesssim \langle \tau_1 + \xi_1^2 \rangle} \frac{d\eta}{|\xi_1| \langle \eta \rangle^{2a}} \\ &\lesssim \frac{|\xi_1|^{2|k|-2s-1}}{\langle \tau_1 + \xi_1^2 \rangle^{2b+2a-1}} \lesssim 1 \end{aligned}$$

in view of $a > 0$, $|k| - s \leq 1/2$ and $b > 1/2$.

Now we turn to the proof of inequality (5.5) with $W_2(f, g, \phi)$. In the following estimates we will make use of the algebraic relation

$$-(\tau + \xi^2) + (\tau_1 - \xi_1^2) + ((\tau - \tau_1) + (\xi - \xi_1)^2) = -2\xi_1\xi. \tag{5.9}$$

First we split $\mathbb{R}^4$ into four sets

$$\begin{aligned} B_1 &= \{(\xi,\tau,\xi_1,\tau_1) \in \mathbb{R}^4 : |\xi_1| \leq 10\}, \\ B_2 &= \{(\xi,\tau,\xi_1,\tau_1) \in \mathbb{R}^4 : |\xi_1| \geq 10 \text{ and } |\xi| \leq 1\}, \\ B_3 &= \{(\xi,\tau,\xi_1,\tau_1) \in \mathbb{R}^4 : |\xi_1| \geq 10, |\xi| \geq 1 \text{ and } |\xi| \geq |\xi_1|/2\}, \\ B_4 &= \{(\xi,\tau,\xi_1,\tau_1) \in \mathbb{R}^4 : |\xi_1| \geq 10, |\xi| \geq 1 \text{ and } |\xi| \leq |\xi_1|/2\}. \end{aligned}$$

Next we separate B_4 into three parts

$$\begin{aligned} B_{4,1} &= \{(\xi,\tau,\xi_1,\tau_1)\in B_4 : |\tau_1-\xi_1^2|, |(\tau-\tau_1)+(\xi-\xi_1)^2| \le |\tau+\xi^2|\},\\ B_{4,2} &= \{(\xi,\tau,\xi_1,\tau_1)\in B_4 : |\tau+\xi^2|, |(\tau-\tau_1)+(\xi-\xi_1)^2| \le |\tau_1-\xi_1^2|\},\\ B_{4,3} &= \{(\xi,\tau,\xi_1,\tau_1)\in B_4 : |\tau_1-\xi_1^2|, |\tau+\xi^2| \le |(\tau-\tau_1)+(\xi-\xi_1)^2|\}. \end{aligned}$$

We can now define the sets R_i, $i=1,2,3$, as follows

$$S_1 = B_1\cup B_3\cup B_{4,1},\quad S_2 = B_2\cup B_{4,2}\quad \text{and}\quad S_3 = B_{4,3}.$$

Using the Cauchy-Schwarz and Hölder inequalities and duality it is easy to see that

$$\begin{aligned} |W_2|^2 \le\ & \|f\|^2_{L^2_{\xi,\tau}}\|g\|^2_{L^2_{\xi,\tau}}\|\phi\|^2_{L^2_{\xi,\tau}}\\ &\times\left\|\frac{\langle\xi\rangle^{2k}}{\langle\sigma\rangle^{2a}}\iint\frac{\chi_{S_1}d\xi_1 d\tau_1}{\langle\xi_1\rangle^{2s}\langle\xi_2\rangle^{2k}\langle\tau_1-\xi_1^2\rangle^{2b}\langle\sigma_2\rangle^{2b}}\right\|_{L^\infty_{\xi,\tau}}\\ &+\|f\|^2_{L^2_{\xi,\tau}}\|g\|^2_{L^2_{\xi,\tau}}\|\phi\|^2_{L^2_{\xi,\tau}}\\ &\times\left\|\frac{1}{\langle\xi_1\rangle^{2s}\langle\tau_1-\xi_1^2\rangle^{2b}}\iint\frac{\chi_{S_2}\langle\xi\rangle^{2k}d\xi d\tau}{\langle\xi_2\rangle^{2k}\langle\sigma\rangle^{2a}\langle\sigma_2\rangle^{2b}}\right\|_{L^\infty_{\xi_1,\tau_1}}\\ &+\|f\|^2_{L^2_{\xi,\tau}}\|g\|^2_{L^2_{\xi,\tau}}\|\phi\|^2_{L^2_{\xi,\tau}}\\ &\times\left\|\frac{1}{\langle\xi_2\rangle^{2k}\langle\sigma_2\rangle^{2b}}\iint\frac{\chi_{\widetilde{S}_3}\langle\xi_1+\xi_2\rangle^{2k}d\xi_1 d\tau_1}{\langle\xi_1\rangle^{2s}\langle\tau_1-\xi_1^2\rangle^{2a}\langle\sigma\rangle^{2b}}\right\|_{L^\infty_{\xi_2,\tau_2}}. \end{aligned}$$

where σ, σ_2, ξ_2, τ_2 were given in (5.6) and

$$\widetilde{S}_3 \subseteq \left\{\begin{array}{r} (\xi_2,\tau_2,\xi_1,\tau_1)\in\mathbb{R}^4 : |\xi_1|\ge 10, |\xi_1+\xi_2|\ge 1, |\xi_1+\xi_2|\le|\xi_1|/2\\ \text{and } |\tau_1-\xi_1^2|, |(\tau_1+\tau_2)+(\xi_1+\xi_2)^2|\le|\tau_2+\xi_2^2| \end{array}\right\}.$$

Noting that $\langle\xi_1+\xi_2\rangle^{2k}\le\langle\xi_1\rangle^{2|k|}\langle\xi_2\rangle^{2k}$, for $k\ge 0$, and $\langle\xi_2\rangle^{-2k}\le\langle\xi_1\rangle^{2|k|}\langle\xi_1+\xi_2\rangle^{-2k}$, for $k<0$ we have

$$\frac{\langle\xi_1+\xi_2\rangle^{2k}}{\langle\xi_1\rangle^{2s}\langle\xi_2\rangle^{2k}}\le\langle\xi_1\rangle^{2|k|-2s}.$$

Therefore in view of Lemma 3.3.1 and (5.8) it suffices to get bounds for

$$\begin{aligned} K_1(\xi,\tau) &\equiv \frac{1}{\langle\sigma\rangle^{2a}}\int\frac{\langle\xi_1\rangle^{2|k|-2s}d\xi_1}{\langle\tau+\xi^2-2\xi\xi_1\rangle^{2b}} \quad \text{on } \ S_1,\\ K_2(\xi_1,\tau_1) &\equiv \frac{\langle\xi_1\rangle^{2|k|-2s}}{\langle\tau_1-\xi_1^2\rangle^{2b}}\int\frac{d\xi}{\langle\tau_1-\xi_1^2+2\xi\xi_1\rangle^{2a}} \quad \text{on } \ S_2,\\ K_3(\xi_1,\tau_1) &\equiv \frac{1}{\langle\sigma_2\rangle^{2b}}\int\frac{\langle\xi_1\rangle^{2|k|-2s}d\xi_1}{\langle\tau_2+\xi_2^2+2\xi_1^2+2\xi_1\xi_2\rangle^{2a}} \quad \text{on } \ \widetilde{S}_3. \end{aligned}$$

In region B_1 we have $\langle\xi_1\rangle^{2|k|-2s}\lesssim 1$ and since $a>0$, $b>1/2$ we obtain

$$K_1(\xi,\tau)\lesssim\int_{|\xi_1|\le 10}d\xi_1\lesssim 1.$$

In region B_3, the change of variables $\eta=\tau+\xi^2-2\xi\xi_1$ and the condition $|\xi|\ge|\xi_1|/2$ imply

$$\begin{aligned} K_1(\xi,\tau) &\lesssim \frac{1}{\langle\sigma\rangle^{2a}}\int\frac{\langle\xi_1\rangle^{2|k|-2s}}{|\xi|\langle\eta\rangle^{2b}}d\eta\\ &\lesssim \frac{\langle\xi_1\rangle^{2|k|-2s-1}}{\langle\sigma\rangle^{2a}}\int\frac{1}{\langle\eta\rangle^{2b}}d\eta\lesssim 1 \end{aligned}$$

since $a>0$, $|k|-s\le 1/2$ and $b>1/2$.

Now, by definition of region $B_{4,1}$ and the algebraic relation (5.9) we have

$$\langle\xi_1\rangle\lesssim|\xi_1|\lesssim|\xi_1\xi|\lesssim\langle\sigma\rangle.$$

Therefore the change of variables $\eta=\tau+\xi^2-2\xi\xi_1$ and the condition $|\xi|\ge 1$ we have

$$\begin{aligned} K_1(\xi,\tau) &\lesssim \frac{1}{\langle\sigma\rangle^{2a}}\int\frac{\langle\xi_1\rangle^{2|k|-2s}}{|\xi|\langle\eta\rangle^{2b}}d\eta\\ &\lesssim \frac{\langle\xi_1\rangle^{2|k|-2s-2a}}{|\xi|}\int\frac{1}{\langle\eta\rangle^{2b}}d\eta\lesssim 1 \end{aligned}$$

since $a>0$, $|k|-s\le a$ and $b>1/2$.

Next we estimate $K_2(\xi_1,\tau_1)$. Making the change of variables, $\eta=\tau_1-\xi_1^2+2\xi\xi_1$ and using the restriction in the region B_2, we have

$$|\eta|\lesssim|\tau_1-\xi_1^2|+|\xi\xi_1|\lesssim|\tau_1-\xi_1^2|+|\xi_1|.$$

Therefore,

$$\begin{aligned} K_2(\xi_1,\tau_1) &\lesssim \frac{|\xi_1|^{2|k|-2s}}{\langle\tau_1-\xi_1^2\rangle^{2b}}\int_{|\eta|\lesssim\langle\tau_1-\xi_1^2\rangle+|\xi_1|}\frac{d\eta}{|\xi_1|\langle\eta\rangle^{2a}} \\ &\lesssim \frac{|\xi_1|^{2|k|-2s-2a}}{\langle\tau_1-\xi_1^2\rangle^{2b}}+\frac{|\xi_1|^{2|k|-2s-1}}{\langle\tau_1-\xi_1^2\rangle^{2b+2a-1}}\lesssim 1 \end{aligned}$$

since $a>0$, $|k|-s\le\min\{1/2,a\}$ and $b>1/2$.

In the region $B_{4,2}$, from the algebraic relation (5.9) we obtain

$$\langle\xi_1\rangle\lesssim|\xi_1|\lesssim|\xi_1\xi|\lesssim\langle\tau_1-\xi_1^2\rangle.$$

Moreover, making the change of variables, $\eta=\tau_1-\xi_1^2+2\xi\xi_1$, using the restriction in the region $B_{4,2}$ and (5.9), we obtain

$$|\eta|\lesssim\langle\tau_1-\xi_1^2\rangle.$$

Therefore,

$$\begin{aligned} K_2(\xi_1,\tau_1) &\lesssim \frac{\langle\xi_1\rangle^{2|k|-2s}}{\langle\tau_1-\xi_1^2\rangle^{2b}}\int_{|\eta|\lesssim\langle\tau_1-\xi_1^2\rangle}\frac{d\eta}{|\xi_1|\langle\eta\rangle^{2a}} \\ &\lesssim \frac{|\xi_1|^{2|k|-2s-1}}{\langle\tau_1-\xi_1^2\rangle^{2b+2a-1}}\lesssim 1 \end{aligned}$$

since $a>0$, $|k|-s\le 1/2$ and $b>1/2$.

Finally, we estimate $K_3(\xi_1,\tau_1)$. In the region $B_{4,3}$ we have by the algebraic relation (5.9) that

$$\langle\xi_1\rangle\lesssim|\xi_1|\lesssim|\xi_1(\xi_1+\xi_2)|\lesssim\langle\sigma_2\rangle.$$

Therefore in view of Lemma 3.3.1 we have

$$\begin{aligned} K_3(\xi_1,\tau_1) &\lesssim \langle\xi_1\rangle^{2|k|-2s-2b}\int\frac{1}{\langle\tau_2+\xi_2^2+2\xi_1^2+2\xi_1\xi_2\rangle^{2a}}d\xi_1 \\ &\lesssim 1 \end{aligned}$$

since $a>1/4$, $|k|-s\le b$ and $b>1/2$.

(ii) For $u_1 \in X^S_{k,b}$ and $u_2 \in X^S_{k,b}$ we define

$$\begin{aligned} f(\xi,\tau) &\equiv \langle \tau+\xi^2\rangle^b \langle \xi\rangle^k \widetilde{u}_1(\xi,\tau), \\ g(\xi,\tau) &\equiv \langle \tau+\xi^2\rangle^b \langle \xi\rangle^k \widetilde{u}_2(\xi,\tau). \end{aligned}$$

By duality the desired inequality is equivalent to

$$|Z(f,g,\phi)| \le c\|f\|_{L^2_{\xi,\tau}} \|g\|_{L^2_{\xi,\tau}} \|\phi\|_{L^2_{\xi,\tau}} \tag{5.10}$$

where

$$Z(f,g,\phi) = \int_{\mathbb{R}^4} \frac{\langle \xi\rangle^s}{\langle \xi_1\rangle^k \langle \xi_2\rangle^k} \frac{h(\xi_1,\tau_1) f(\xi_2,\tau_2)\bar{\phi}(\xi,\tau)}{\langle \sigma\rangle^a \langle \sigma_1\rangle^b \langle \sigma_2\rangle^b} d\xi d\tau d\xi_1 d\tau_1$$

and

$$h(\tau_1,\xi_1) = \bar{g}(-\tau_1,-\xi_1), \quad \xi_2 = \xi-\xi_1, \quad \tau_2 = \tau-\tau_1,$$

$$\sigma = |\tau| - \gamma(\xi), \quad \sigma_1 = \tau_1 - \xi_1^2, \quad \sigma_2 = \tau_2 + \xi_2^2.$$

Therefore applying Lemma 3.3.2 and splitting the domain of integration according to the sign of τ it is sufficient to prove inequality (5.10) with $Z_1(f,g,\phi)$ and $Z_2(f,g,\phi)$ instead of $Z(f,g,\phi)$, where

$$Z_1(f,g,\phi) = \int_{\mathbb{R}^4} \frac{\langle \xi\rangle^s}{\langle \xi_1\rangle^k \langle \xi_2\rangle^k} \frac{h(\xi_1,\tau_1) f(\xi_2,\tau_2)\bar{\phi}(\xi,\tau)}{\langle \tau+\xi^2\rangle^a \langle \sigma_1\rangle^b \langle \sigma_2\rangle^b} d\xi d\tau d\xi_1 d\tau_1$$

and

$$Z_2(f,g,\phi) = \int_{\mathbb{R}^4} \frac{\langle \xi\rangle^s}{\langle \xi_1\rangle^k \langle \xi_2\rangle^k} \frac{h(\xi_1,\tau_1) f(\xi_2,\tau_2)\bar{\phi}(\xi,\tau)}{\langle \tau-\xi^2\rangle^a \langle \sigma_1\rangle^b \langle \sigma_2\rangle^b} d\xi d\tau d\xi_1 d\tau_1.$$

Remark 5.3.1 *Note that $Z_1(f,g,\phi)$ is not equal to $W_2(f,g,\phi)$ since the powers of the terms $\langle \xi\rangle$ and $\langle \xi_1\rangle$ are different.*

First we treat the inequality (5.10) with $Z_1(f,g,\phi)$. In this case we will make use of the following algebraic relation

$$-(\tau+\xi^2) + (\tau_1-\xi_1^2) + ((\tau-\tau_1)+(\xi-\xi_1)^2) = -2\xi_1\xi. \tag{5.11}$$

We split $\mathbb{R}^4$ into five pieces

$$\begin{aligned}
A_1 &= \{(\xi,\tau,\xi_1,\tau_1)\in\mathbb{R}^4 : |\xi|\le 10 \text{ and } |\xi_1|\le 100\},\\
A_2 &= \{(\xi,\tau,\xi_1,\tau_1)\in\mathbb{R}^4 : |\xi|\le 10 \text{ and } |\xi_1|\ge 100\},\\
A_3 &= \{(\xi,\tau,\xi_1,\tau_1)\in\mathbb{R}^4 : |\xi|\ge 10 \text{ and } [|\xi_1|\le 1 \text{ or } |\xi_2|\le 1]\},\\
A_4 &= \left\{\begin{array}{r} (\xi,\tau,\xi_1,\tau_1)\in\mathbb{R}^4 : |\xi|\ge 10, |\xi_1|\ge 1, |\xi_2|\ge 1\\ \text{and } [|\xi_1|\ge 2|\xi_2| \text{ or } |\xi_2|\ge 2|\xi_1|]\end{array}\right\},\\
A_5 &= \left\{\begin{array}{r} (\xi,\tau,\xi_1,\tau_1)\in\mathbb{R}^4 : |\xi|\ge 10, |\xi_1|\ge 1, |\xi_2|\ge 1\\ \text{and } |\xi_1|/2\le|\xi_2|\le 2|\xi_1|\end{array}\right\}.
\end{aligned}$$

Next we separate A_5 into three parts

$$\begin{aligned}
A_{5,1} &= \{(\xi,\tau,\xi_1,\tau_1)\in A_5 : |\tau_1-\xi_1^2|, |(\tau-\tau_1)+(\xi-\xi_1)^2|\le|\tau+\xi^2|\},\\
A_{5,2} &= \{(\xi,\tau,\xi_1,\tau_1)\in A_5 : |\tau+\xi^2|, |(\tau-\tau_1)+(\xi-\xi_1)^2|\le|\tau_1-\xi_1^2|\},\\
A_{5,3} &= \{(\xi,\tau,\xi_1,\tau_1)\in A_5 : |\tau_1-\xi_1^2|, |\tau+\xi^2|\le|(\tau-\tau_1)+(\xi-\xi_1)^2|\}.
\end{aligned}$$

Therefore by the same argument as the one used in the proof of (i) it suffices to get bounds for

$$\begin{aligned}
L_1(\xi,\tau) &\equiv \frac{1}{\langle\tau+\xi^2\rangle^{2a}}\int\frac{\langle\xi_1\rangle^{-2k}\langle\xi_2\rangle^{-2k}\langle\xi\rangle^{2s}d\xi_1}{\langle\tau+\xi^2-2\xi\xi_1\rangle^{2b}} \quad\text{on}\quad V_1,\\
L_2(\xi_1,\tau_1) &\equiv \frac{1}{\langle\sigma_1\rangle^{2b}}\int\frac{\langle\xi_1\rangle^{-2k}\langle\xi_2\rangle^{-2k}\langle\xi\rangle^{2s}d\xi}{\langle\tau_1-\xi_1^2+2\xi\xi_1\rangle^{2a}} \quad\text{on}\quad V_2,\\
L_3(\xi_1,\tau_1) &\equiv \frac{1}{\langle\sigma_2\rangle^{2b}}\int\frac{\langle\xi_1\rangle^{-2k}\langle\xi_2\rangle^{-2k}\langle\xi\rangle^{2s}d\xi_1}{\langle\tau_2+\xi_2^2+2\xi_1^2+2\xi_1\xi_2\rangle^{2a}} \quad\text{on}\quad \widetilde{V}_3.
\end{aligned}$$

where

$$V_1 = A_3\cup A_4\cup A_{5,1},\quad V_2 = A_1\cup A_2\cup A_{5,2}$$

and

$$\widetilde{V}_3\subseteq\left\{\begin{array}{r} (\xi_2,\tau_2,\xi_1,\tau_1)\in\mathbb{R}^4 : |\xi_1+\xi_2|\ge 10, |\xi_1|\ge 1,\\ |\xi_2|\ge 1, |\xi_1|/2\le|\xi_2|\le 2|\xi_1|\\ \text{and } |\tau_1-\xi_1^2|, |(\tau_1+\tau_2)+(\xi_1+\xi_2)^2|\le|\tau_2+\xi_2^2|\end{array}\right\}.$$

First we estimate $L_1(\xi,\tau)$. In the regions A_3 or A_4 it is easy to see that

$\max\{|\xi_1|, |\xi_2|\} \sim |\xi|$, therefore

$$\langle \xi_1 \rangle^{-k} \langle \xi_2 \rangle^{-k} \langle \xi \rangle^{s} \lesssim \langle \xi \rangle^{\gamma(k)}$$

where

$$\gamma(k) = \begin{cases} s + 2|k|, & \text{if } k \leq 0 \\ s - k, & \text{if } k > 0. \end{cases}$$

Remark 5.3.2 *Note that $\xi = N + 1$ and $\xi_1 = N$ belong to A_3, for all $N \geq 100$. In all of this cases $|\xi_2| = 1$. Therefore, we cannot expect, in general, that both $|\xi_1|$ and $|\xi_2|$ are equivalent to $|\xi|$. Because of this fact we define $\gamma(k) = s - k$, for $k > 0$.*

Then, making the change of variables $\eta = \tau + \xi^2 - 2\xi\xi_1$, we have

$$L_1(\xi, \tau) \lesssim \frac{\langle \xi \rangle^{2\gamma(k)}}{\langle \tau + \xi^2 \rangle^{2a}} \int \frac{d\eta}{|\xi| \langle \eta \rangle^{2b}} \lesssim 1$$

since $a > 0$, $b > 1/2$, and $\gamma(k) \leq 1/2$, that is, $s - k \leq 1/2$, if $k > 0$ and $s + 2|k| \leq 1/2$, if $k \leq 0$.

In region A_5 we have

$$\langle \xi_1 \rangle^{-k} \langle \xi_2 \rangle^{-k} \langle \xi \rangle^{s} \lesssim \langle \xi_1 \rangle^{\gamma(s,k)} \tag{5.12}$$

where

$$\gamma(s, k) = \begin{cases} 0, & \text{if } s \leq 0, k > 0 \\ 2|k|, & \text{if } s \leq 0, k \leq 0 \\ s - 2k, & \text{if } s > 0, k > 0 \\ s + 2|k|, & \text{if } s > 0, k \leq 0. \end{cases}$$

Moreover, the restriction in the region $A_{5,1}$, the condition $|\xi| > 10$ and the algebraic relation (5.11) give us

$$\langle \xi_1 \rangle \lesssim |\xi_1| \lesssim |\xi_1 \xi| \lesssim \langle \tau + \xi^2 \rangle.$$

Therefore

$$\begin{aligned} L_1(\xi,\tau) &\lesssim \int \frac{\langle \xi_1\rangle^{2\gamma(s,k)-2a} d\eta}{|\xi|\langle \eta\rangle^{2b}} \\ &\lesssim \frac{1}{|\xi|}\int \frac{d\eta}{\langle \eta\rangle^{2b}} \lesssim 1 \end{aligned}$$

if $a>0$, $b>1/2$ and $\gamma(s,k)\leq a$, that is, $2|k|\leq a$, if $s\leq 0$, $k\leq 0$ and $s-2k\leq a$, if $s>0$.

Next we estimate $L_2(\xi_1,\tau_1)$. In region A_1 we have $\langle \xi_1\rangle^{-2k}\langle \xi_2\rangle^{-2k}\langle \xi\rangle^{2s}\lesssim 1$ and since $a,b>0$, we obtain

$$L_2(\xi_1,\tau_2)\lesssim \int_{|\xi|\leq 10} d\xi \lesssim 1.$$

In region A_2, we have $|\xi_1|\sim|\xi_2|$, therefore

$$\langle \xi_1\rangle^{-k}\langle \xi_2\rangle^{-k}\langle \xi\rangle^{2s}\lesssim \langle \xi_1\rangle^{\theta(k)}.$$

where

$$\theta(k)=\begin{cases} 0, & \text{if } k>0\\ 2|k|, & \text{if } k\leq 0.\end{cases}$$

Making the change of variables, $\eta=\tau_1-\xi_1^2+2\xi\xi_1$, using the restriction in the region A_2, we have

$$|\eta|\lesssim|\tau_1-\xi_1^2|+|\xi\xi_1|\lesssim|\tau_1-\xi_1^2|+|\xi_1|.$$

Therefore,

$$\begin{aligned} L_2(\xi_1,\tau_1) &\lesssim \frac{\langle \xi_1\rangle^{2\theta(k)}}{\langle \tau_1-\xi_1^2\rangle^{2b}}\int_{|\eta|\lesssim\langle \tau_1-\xi_1^2\rangle+|\xi_1|}\frac{d\eta}{|\xi_1|\langle\eta\rangle^{2a}} \\ &\lesssim \frac{|\xi_1|^{2\theta(k)-2a}}{\langle \tau_1-\xi_1^2\rangle^{2b}}+\frac{|\xi_1|^{2\theta(k)-1}}{\langle \tau_1-\xi_1^2\rangle^{2b+2a-1}}\lesssim 1 \end{aligned}$$

since $a>0$, $b>1/2$ and $\theta(k)\leq\min\{1/2,a\}$, that is, $|k|\leq\min\{1/4,a/2\}$, if $k\leq 0$.

Now we turn to the region $A_{5,2}$. From (5.11) and the condition $|\xi| > 10$ we have

$$\langle \xi_1 \rangle \lesssim |\xi_1| \lesssim |\xi_1 \xi| \lesssim \langle \tau_1 - \xi_1^2 \rangle$$

and

$$|\eta| \lesssim |\tau_1 - \xi_1^2| + |\xi \xi_1| \lesssim \langle \tau_1 - \xi_1^2 \rangle.$$

Therefore, making the change of variables, $\eta = \tau_1 - \xi_1^2 + 2\xi\xi_1$, and using (5.12), we obtain

$$\begin{aligned} L_2(\xi_1, \tau_1) &\lesssim \frac{\langle \xi_1 \rangle^{2\gamma(s,k)}}{\langle \tau_1 - \xi_1^2 \rangle^{2b}} \int_{|\eta| \lesssim \langle \tau_1 - \xi_1^2 \rangle} \frac{d\eta}{|\xi_1| \langle \eta \rangle^{2a}} \\ &\lesssim \frac{\langle \xi_1 \rangle^{2\gamma(s,k)-1}}{\langle \tau_1 - \xi_1^2 \rangle^{2b+2a-1}} \lesssim 1 \end{aligned}$$

since $a > 0$, $b > 1/2$ and $\gamma(s, k) \leq 1/2$.

Finally, we bound $L_3(\xi_1, \tau_1)$. Again, we use (5.11), so in the region $A_{5,3}$ we have $\langle \xi_1 \rangle \lesssim \langle \sigma_2 \rangle$.From Lemma 3.3.1 it follows that

$$\begin{aligned} L_3(\xi_1, \tau_1) &\lesssim \langle \xi_1 \rangle^{2\gamma(s,k)-2b} \int \frac{1}{\langle \tau_2 + \xi_2^2 + 2\xi_1^2 + 2\xi_1 \xi_2 \rangle^{2a}} d\xi_1 \\ &\lesssim 1 \end{aligned}$$

since $a > 1/4$, $b > 1/2$ and $\gamma(s, k) \leq b$.

Now we turn to the proof of inequality (5.10) with $Z_2(f, g, \phi)$. First we making the change of variables $\tau_2 = \tau - \tau_1$, $\xi_2 = \xi - \xi_1$ to obtain

$$\begin{aligned} Z_2(f, g, \phi) &= \int_{\mathbb{R}^4} \frac{\langle \xi \rangle^s}{\langle \xi - \xi_2 \rangle^k \langle \xi_2 \rangle^k} \\ &\quad \times \frac{h(\xi - \xi_2, \tau - \tau_2) f(\xi_2, \tau_2) \bar{\phi}(\xi, \tau)}{\langle \tau - \xi^2 \rangle^a \langle (\tau - \tau_2) - (\xi - \xi_2)^2 \rangle^b \langle \tau_2 + \xi_2^2 \rangle^b} d\xi d\tau d\xi_2 d\tau_2 \end{aligned}$$

then changing the variables $(\xi, \tau, \xi_2, \tau_2) \mapsto -(\xi, \tau, \xi_2, \tau_2)$ we can rewrite $Z_2(f, g, \phi)$ as

$$\begin{aligned} Z_2(f, g, \phi) &= \int_{\mathbb{R}^4} \frac{\langle \xi \rangle^s}{\langle \xi - \xi_2 \rangle^k \langle \xi_2 \rangle^k} \\ &\quad \times \frac{k(\xi - \xi_2, \tau - \tau_2) l(\xi_2, \tau_2) \bar{\psi}(\xi, \tau)}{\langle \tau + \xi^2 \rangle^a \langle \tau - \tau_2 + (\xi - \xi_2)^2 \rangle^b \langle \tau_2 - \xi_2^2 \rangle^b} d\xi d\tau d\xi_2 d\tau_2 \end{aligned}$$

where

$$k(a,b) = h(-a,-b),\ l(a,b) = f(-a,-b) \text{ and } \psi(a,b) = \phi(-a,-b).$$

But this is exactly $Z_1(f,g,\phi)$ with ξ_1, h, f, ϕ replaced respectively by ξ_2, l, k, ψ. Since the L^2-norm is preserved under the reflection operation the result follows from the estimate for $Z_1(f,g,\phi)$.

■

Now we turn to the proof of the bilinear estimates with $b < 1/2$ and $s = 0$.

Proof of Theorem 5.1.3

(i) For $u \in X^S_{0,b_1}$ and $v \in X^B_{0,b}$ we define

$$\begin{aligned} f(\xi,\tau) &\equiv \langle \tau + \xi^2 \rangle^{b_1} \widetilde{u}(\xi,\tau), \\ g(\xi,\tau) &\equiv \langle |\tau| - \gamma(\xi) \rangle^{b} \widetilde{v}(\xi,\tau). \end{aligned}$$

By duality the desired inequality is equivalent to

$$|R(f,g,\phi)| \le c\|f\|_{L^2_{\xi,\tau}} \|g\|_{L^2_{\xi,\tau}} \|\phi\|_{L^2_{\xi,\tau}} \tag{5.13}$$

where

$$R(f,g,\phi) = \int_{\mathbb{R}^4} \frac{g(\xi_1,\tau_1) f(\xi_2,\tau_2) \bar{\phi}(\xi,\tau)}{\langle \sigma \rangle^{a_1} \langle \sigma_1 \rangle^{b} \langle \sigma_2 \rangle^{b_1}} d\xi d\tau d\xi_1 d\tau_1$$

and

$$\begin{gathered} \xi_2 = \xi - \xi_1, \quad \tau_2 = \tau - \tau_1, \\ \sigma = \tau + \xi^2, \quad \sigma_1 = |\tau_1| - \gamma(\xi_1), \quad \sigma_2 = \tau_2 + \xi_2^2. \end{gathered} \tag{5.14}$$

Without loss of generality we can suppose that f, g, ϕ are real valued and non-negative. Therefore, by Lemma 3.3.2 we have

$$\begin{aligned} R(f,g,\phi) &\le \int_{\mathbb{R}^4} \frac{g(\xi_1,\tau_1) f(\xi_2,\tau_2) \bar{\phi}(\xi,\tau)}{\langle \sigma \rangle^{a_1} \langle \tau_1 + \xi_1^2 \rangle^{b} \langle \sigma_2 \rangle^{b_1}} d\xi d\tau d\xi_1 d\tau_1 \\ &\quad + \int_{\mathbb{R}^4} \frac{g(\xi_1,\tau_1) f(\xi_2,\tau_2) \bar{\phi}(\xi,\tau)}{\langle \sigma \rangle^{a_1} \langle \tau_1 - \xi_1^2 \rangle^{b} \langle \sigma_2 \rangle^{b_1}} d\xi d\tau d\xi_1 d\tau_1 \\ &\equiv R_+ + R_-. \end{aligned}$$

Applying Plancherel's identity and Holder's inequality we obtain

$$R_{\pm} = \int_{\mathbb{R}^2} \left(\frac{g(\xi,\tau)}{\langle \tau \pm \xi^2 \rangle^{b}} \right)^{\sim^{-1}} \left(\frac{f(\xi,\tau)}{\langle \tau + \xi^2 \rangle^{b_1}} \right)^{\sim^{-1}} \left(\frac{\bar{\phi}(\xi,\tau)}{\langle \tau + \xi^2 \rangle^{a_1}} \right)^{\sim^{-1}} d\xi d\tau$$
$$\leq \left\| \left(\frac{g(\xi,\tau)}{\langle \tau \pm \xi^2 \rangle^{b}} \right)^{\sim^{-1}} \right\|_{L^3_{x,t}} \left\| \left(\frac{f(\xi,\tau)}{\langle \tau + \xi^2 \rangle^{b_1}} \right)^{\sim^{-1}} \right\|_{L^3_{x,t}} \left\| \left(\frac{\bar{\phi}(\xi,\tau)}{\langle \tau + \xi^2 \rangle^{a_1}} \right)^{\sim^{-1}} \right\|_{L^3_{x,t}}.$$

Now, the fact that $a_1, b, b_1 > 1/4$ together with Lemma 5.2.3 yields the result.

(ii) For $u_1 \in X^S_{0,b_1}$ and $u_2 \in X^S_{0,b_1}$ we define

$$\begin{aligned} f(\xi,\tau) &\equiv \langle \tau + \xi^2 \rangle^{b_1} \widetilde{u}_1(\xi,\tau), \\ g(\xi,\tau) &\equiv \langle \tau + \xi^2 \rangle^{b_1} \widetilde{u}_2(\xi,\tau). \end{aligned}$$

By duality the desired inequality is equivalent to

$$|S(f,g,\phi)| \leq c \|f\|_{L^2_{\xi,\tau}} \|g\|_{L^2_{\xi,\tau}} \|\phi\|_{L^2_{\xi,\tau}} \tag{5.15}$$

where

$$S(f,g,\phi) = \int_{\mathbb{R}^4} \frac{\bar{g}(\xi_2,\tau_2) f(\xi_1,\tau_1) \bar{\phi}(\xi,\tau)}{\langle \sigma \rangle^{a} \langle \sigma_1 \rangle^{b_1} \langle \sigma_2 \rangle^{b_1}} d\xi d\tau d\xi_1 d\tau_1$$

and

$$\xi_2 = \xi_1 - \xi, \quad \tau_2 = \tau_1 - \tau,$$

$$\sigma = |\tau| - \gamma(\xi), \quad \sigma_1 = \tau_1 + \xi_1^2, \quad \sigma_2 = \tau_2 + \xi_2^2.$$

We note that the estimate above is the same as the stated in item *(i)*, replacing ξ, τ, b, a_1 by ξ_1, τ_1, a, b_1 and $f, g, \bar{\phi}$ by $\bar{g}, \bar{\phi}, f$, respectively. Therefore we need the restriction $a, b_1 > 1/4$.

■

5.4 Local well-posedness

Proof of Theorem 5.1.2. The proof proceeds by a standard contraction principle method applied to the integral equations associated to the IVP (5.1). Given $(u_0, v_0, v_1) \in H^k(\mathbb{R}) \times H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ and $T \leq 1$ we define the integral operators

$$\begin{aligned}
\Gamma_T^S(u,v)(t) =& \theta_T(t)U(t)u_0 - i\theta_T(t)\int_0^t U(t-t')(vu)(t')dt' \\
\Gamma_T^B(u,v)(t) =& \theta_T(t)\left(V_c(t)v_0 + V_s(t)(v_1)_x\right) + \theta_T(t)\int_0^t V_s(t-t')(|u|^2)_{xx}(t')dt'.
\end{aligned} \tag{5.16}$$

Our goal is to use the Picard fixed point theorem to find a solution

$$\begin{aligned}
\Gamma_T^S(u,v) &= u, \\
\Gamma_T^B(u,v) &= v.
\end{aligned}$$

Let k,s satisfy the conditions $(i)-(ii)$ of Theorem 5.1.2. It is easy to see that we can find $\varepsilon>0$ small enough such that for $b=1/2+\varepsilon$ and $a=1/2-2\varepsilon$, Theorem 5.1.1 holds. Therefore using Lemmas 5.2.1-5.2.2, Theorem 5.1.1 and $T\le 1$, we have

$$\begin{aligned}
\|\Gamma_T^S(u,v)\|_{X^S_{k,b}} &\le c\,\|u_0\|_{H^k} + cT^\varepsilon\,\|uv\|_{X^S_{k,-a}} \\
&\le c\,\|u_0\|_{H^k} + cT^\varepsilon\,\|u\|_{X^S_{k,b}}\,\|v\|_{X^B_{s,b}}, \\
\|\Gamma_T^B(u,v)\|_{X^B_{s,b}} &\le c\,\|v_0,v_1\|_{\mathfrak{B}^s} + cT^\varepsilon\,\|u\bar{u}\|_{X^B_{s,-a}} \\
&\le c\,\|v_0,v_1\|_{\mathfrak{B}^s} + cT^\varepsilon\,\|u\|^2_{X^S_{k,b}}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\|\Gamma_T^S(u,v)-\Gamma_T^S(z,w)\|_{X^S_{k,b}} &\le cT^\varepsilon\Big(\|u\|_{X^S_{k,b}}\,\|v-w\|_{X^B_{s,b}} \\
&\quad + \|u-z\|_{X^S_{k,b}}\,\|w\|_{X^B_{s,b}}\Big), \\
\|\Gamma_T^B(u,v)-\Gamma_T^B(z,w)\|_{X^B_{s,b}} &\le cT^\varepsilon\Big(\|u\|_{X^S_{k,b}} + \|z\|_{X^S_{k,b}}\Big) \\
&\quad \times \|u-z\|_{X^S_{k,b}}.
\end{aligned}$$

We define

$$\begin{aligned}
X^S_{k,b}(d_S) &= \Big\{u\in X^S_{k,b} : \|u\|_{X^S_{k,b}} \le d_S\Big\}, \\
X^B_{s,b}(d_B) &= \Big\{v\in X^B_{s,b} : \|v\|_{X^B_{s,b}} \le d_B\Big\},
\end{aligned}$$

where $d_S = 2c\|u_0\|_{H^k}$ and $d_B = 2c\|v_0,v_1\|_{\mathfrak{B}^s}$.

Then choosing

$$0 < T^{\varepsilon} \leq \frac{1}{4} \min \left\{ \frac{1}{cd_B}, \frac{d_B}{cd_S^2}, \frac{1}{c(d_S + d_B)}, \frac{1}{2cd_S} \right\} \tag{5.17}$$

we have that $(\Gamma_T^S, \Gamma_T^B) : X_{k,b}^S(d_S) \times X_{s,b}^B(d_B) \to X_{k,b}^S(d_S) \times X_{s,b}^B(d_B)$ is a contraction mapping and we obtain a unique fixed point which solves the integral equation (5.16) for any T that satisfies (5.17).

Remark 5.4.1 *Note that the choice of suitable values of a, b is essential for our argument. In fact, since $1 - (a + b) = \varepsilon > 0$, the factor T^{ε} can be used directly to obtain a contraction factor for T sufficient small.*

Moreover, by Lemma 3.2.3, we have that $\tilde{u} = u|_{[0,T]} \in C([0, T] : H^s) \cap X_{k,b}^{S,T}$ and $\tilde{v} = v|_{[0,T]} \in C([0, T] : H^s) \cap X_{s,b}^{B,T}$ is a solution of (0.25) in $[0, T]$.

Using the same arguments as the ones in the *Uniqueness* part of Theorem 3.1.3 we one can, in fact, prove that the solution (u, v) of (0.25) obtained above is unique in the whole space $X_{k,b}^{S,T} \times X_{s,b}^{B,T}$. Finally, we remark that since we established the existence of a solution by a contraction argument, the proof that the map $(u_0, v_0, v_1) \mapsto (u(t), v(t))$ is locally Lipschitz follows easily.

■

5.5 Global well-posedness

Proof of Theorem 5.1.4. For $(u_0, v_0, v_1) \in L^2(\mathbb{R}) \times L^2(\mathbb{R}) \times H^{-1}(\mathbb{R})$ and $T \leq 1$ we consider the integral equations given by (5.16). Therefore, applying Lemmas 5.2.1-5.2.2 and Theorem 5.1.3, we obtain

$$\begin{aligned}
\|\Gamma_T^S(u, v)\|_{X_{0,b_1}^S} &\leq cT^{1/2-b_1}\|u_0\|_{L^2} + cT^{1-(a_1+b_1)} \|uv\|_{X_{0,-a_1}^S} \\
&\leq cT^{1/2-b_1}\|u_0\|_{L^2} + cT^{1-(a_1+b_1)} \|u\|_{X_{0,b_1}^S} \|v\|_{X_{0,b}^B}, \\
\|\Gamma_T^B(u, v)\|_{X_{0,b}^B} &\leq cT^{1/2-b}\|v_0, v_1\|_{\mathfrak{B}} + cT^{1-(a+b)} \|u\bar{u}\|_{X_{0,-a}^B} \\
&\leq cT^{1/2-b}\|v_0, v_1\|_{\mathfrak{B}} + cT^{1-(a+b)} \|u\|_{X_{0,b_1}^S}^2
\end{aligned} \tag{5.18}$$

and also

$$\begin{aligned}
\|\Gamma_T^S(u, v) - \Gamma_T^S(z, w)\|_{X_{0,b_1}^S} &\leq cT^{1-(a_1+b_1)} \Big(\|u\|_{X_{0,b_1}^S} \|v - w\|_{X_{0,b}^B} \\
&\quad + \|u - z\|_{X_{0,b_1}^S} \|w\|_{X_{0,b}^B} \Big), \\
\|\Gamma_T^B(u, v) - \Gamma_T^B(z, w)\|_{X_{0,b}^B} &\leq cT^{1-(a+b)} \Big(\|u\|_{X_{0,b_1}^S} + \|z\|_{X_{0,b_1}^S} \Big) \\
&\quad \times \|u - z\|_{X_{0,b_1}^S}.
\end{aligned} \tag{5.19}$$

We define

$$\begin{aligned} X^S_{0,b_1}(d_1) &= \left\{ u \in X^S_{0,b_1} : \|u\|_{X^S_{0,b_1}} \leq d_1 \right\}, \\ X^B_{0,b}(d) &= \left\{ v \in X^B_{0,b} : \|v\|_{X^B_{0,b}} \leq d \right\}, \end{aligned}$$

where $d_1 = 2cT^{1/2-b_1}\|u_0\|_{L^2}$ and $d = 2cT^{1/2-b}\|v_0, v_1\|_{\mathfrak{B}}$.

For (Γ^S_T, Γ^B_T) to be a contraction in $X^S_{0,b_1}(d_1) \times X^B_{0,b}(d)$ it needs to satisfy

$$d_1/2 + cT^{1-(a_1+b_1)}d_1 d \leq d_1 \Leftrightarrow T^{3/2-(a_1+b_1+b)}\|v_0, v_1\|_{\mathfrak{B}} \lesssim 1, \tag{5.20}$$

$$d/2 + cT^{1-(a+b)}d_1^2 \leq d \Leftrightarrow T^{3/2-(a+2b_1)}\|u_0\|^2_{L^2} \lesssim \|v_0, v_1\|_{\mathfrak{B}}, \tag{5.21}$$

$$2cT^{1-(a+b)}d_1 \leq 1/2 \Leftrightarrow T^{3/2-(a+b+b_1)}\|u_0\|_{L^2} \lesssim 1, \tag{5.22}$$

$$2cT^{1-(a_1+b_1)}d_1 \leq 1/2 \Leftrightarrow T^{3/2-(a_1+2b_1)}\|u_0\|_{L^2} \lesssim 1. \tag{5.23}$$

Therefore, we conclude that there exists a solution $(u, v) \in X^S_{0,b_1} \times X^B_{0,b}$ satisfying

$$\|u\|_{X^S_{0,b_1}} \leq 2cT^{1/2-b_1}\|u_0\|_{L^2} \quad \text{and} \quad \|v\|_{X^B_{0,b}} \leq 2cT^{1/2-b}\|v_0, v_1\|_{\mathfrak{B}}. \tag{5.24}$$

On the other hand, applying Lemmas 5.2.1-5.2.2 we have that, in fact, $(u, v) \in C([0, T] : L^2) \times C([0, T] : L^2)$. Moreover, since the L^2-norm of u is conserved by the flow we have $\|u(T)\|_{L^2} = \|u_0\|_{L^2}$.

Now, we need to control the growth of $\|v(t)\|_{\mathfrak{B}}$ in each time step. If, for all $t > 0$, $\|v(t)\|_{\mathfrak{B}} \lesssim \|u_0\|^2_{L^2}$ we can repeat the local well-posedness argument and extend the solution globally in time. Thus, without loss of generality, we suppose that after some number of iterations we reach a time $t^* > 0$ where $\|v(t^*)\|_{\mathfrak{B}} \gg \|u_0\|^2_{L^2}$.

Hence, since $T \leq 1$, condition (5.21) is automatically satisfied and conditions (5.20)-(5.23) imply that we can select a time increment of size

$$T \sim \|v(t^*)\|_{\mathfrak{B}}^{-1/(3/2-(a_1+b_1+b))}. \tag{5.25}$$

Therefore, applying Lemmas 5.2.1(b)-5.2.2(b) to $v = \Gamma_T^B(u,v)$ we have

$$\|v(t^*+T)\|_{\mathfrak{B}} \leq \|v(t^*)\|_{\mathfrak{B}} + cT^{3/2-(a+2b_1)}\|u_0\|_{L^2}^2.$$

Thus, we can carry out m iterations on time intervals, each of length (5.25), before the quantity $\|v(t)\|_{\mathfrak{B}}$ doubles, where m is given by

$$mT^{3/2-(a+2b_1)}\|u_0\|_{L^2}^2 \sim \|v(t^*)\|_{\mathfrak{B}}.$$

The total time of existence we obtain after these m iterations is

$$\begin{aligned} \Delta T = mT \quad &\sim \quad \frac{\|v(t^*)\|_{\mathfrak{B}}}{T^{1/2-(a+2b_1)}\|u_0\|_{L^2}^2} \\ &\sim \quad \frac{\|v(t^*)\|_{\mathfrak{B}}}{\|v(t^*)\|_{\mathfrak{B}}^{-(1/2-(a+2b_1))/(3/2-(a_1+b_1+b))}\|u_0\|_{L^2}^2}. \end{aligned}$$

Taking a, b, a_1, b_1 such that

$$\frac{a+2b_1-1/2}{(3/2-(a_1+b_1+b))} = 1$$

(for instance, $a = b = a_1 = b_1 = 1/3$), we have that ΔT depends only on $\|u_0\|_{L^2}$, which is conserved by the flow. Hence we can repeat this entire argument and extend the solution (u, v) globally in time.

Moreover, since in each step of time ΔT the size of $\|v(t)\|_{\mathfrak{B}}$ will at most double it is easy to see that, for all $\widetilde{T} > 0$

$$\|v(\widetilde{T})\|_{\mathfrak{B}} \lesssim \exp\left((\ln 2)\|u_0\|_{L^2}^2\widetilde{T}\right)\max\{\|v_0, v_1\|_{\mathfrak{B}}, \|u_0\|_{L^2}\}. \tag{5.26}$$

■

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