impa Instituto Nacional de Matemática Pura e Aplicada

Heudson Tosta Mirandola

Algumas contribuições à teoria das subvariedades

Tese de Doutorado

Tese apresentada ao Instituto de Matemática Pura e Aplicada como requisito parcial para obtenção do título de doutor em Matemática.

Orientador: Prof. Manfredo Perdigão do Carmo

Rio de Janeiro
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Prof. Manfredo Perdigão do Carmo
Orientador
Instituto de Matemática Pura e Aplicada

Prof. Hilario Alencar
Universidade Federal das Alagoas - UFAL, AL

Prof. Francisco Xavier Fontenele
Universidade Federal Fluminense - UFF,RJ

Prof. Marcos Dajczer
Instituto Nacional de Matemática Pura e Aplicada - IMPA, RJ

Prof. Fernando Codá Marques
Instituto Nacional de Matemática Pura e Aplicada - IMPA, RJ

Prof. Detang Zhou (Suplente)
Universidade Federal Fluminense - UFF, RJ

Rio de Janeiro, 29 de Agosto de 2008

Heudson Tosta Mirandola

Bacharel em Matemática pela Universidade Federal do Espírito Santo–UFES (Vitória, Espírito Santo). Mestre em Matemática pelo Instituto Nacional de Matemática Pura e Aplicada – IMPA (Rio de Janeiro).

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Resumo

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Palavras–chave

Variedade Riemanniana, hipersurperfície, geodésica, conjunto estrelado. Produto torcido, imersão própria, curvatura r-média. Espaço hiperbólico, comportamento assintótico, curvatura média.

Abstract

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Keywords

Riemannian manifold, hypersurface, geodesic, star-shaped set. Warped product, proper immersion, r-mean curvature. Hyperbolic space, asymptotic boundary, mean curvature.

Sumário

Introdução

Esta tese está dividida em cinco partes descritas em forma de capítulos. Apresentaremos a seguir uma síntese de cada um destes capitulos.

Capítulo 1: Hipersuperfícies cujas geodésicas tangentes não cobrem o espaço ambiente.

Este capítulo refere-se ao artigo: *Hypersurfaces whose tangent geodesics do not cover the ambient space*, publicado juntamente com Sergio Mendonça no Proceedings of the American Mathematical Society (veja [MM]).

Dada uma imersão $f : \Sigma^n \to M^{n+1}$ de uma variedade conexa n-dimensional Σ sobre uma variedade completa $(n+1)$-dimensional M, considere $W = W(f)$ o conjunto dos pontos que não pertencem a nenhuma geodésica tangente à f. Estudamos sobre qual situação podemos ter $W \neq \emptyset$. Uma importante resposta a esta questão foi dada por B. Halpern [Hp]. A fim de enunciar o resultado de Halpern precisaremos da seguinte definição:

Definição 1. *Um subconjunto B de uma variedade Riemanniana completa M é chamado estrelado com respeito a um ponto x_0 se $x_0 \in B$ e, para qualquer ponto $p \in B$, existe uma única geodésica normalizada ligando x_0 a p, além disso, a imagem desta única geodésica está contida em B.*

Teorema A. (B. Halpern [Hp]) *Seja $f : \Sigma^n \to \mathbb{R}^{n+1}$ uma imersão de uma variedade fechada Σ com $W \neq \emptyset$. Então, valem as seguintes afirmações:*

(i) *f é um mergulho;*

(ii) *Σ is difeomorfa à esfera unitária Euclideana;*

(iii) *$f(\Sigma)$ é o bordo de um disco diferenciável D estrelado com relação a qualquer ponto de W.*

Teorema A foi estendido por M. Beltagy [Be] para ambientes Riemannianos completos simplesmente conexos e sem pontos conjugados, usando o fato de que a aplicação exponencial de uma tal variedade é um difeomorfismo e, assim, aplicando-se o Teorema A no espaço tangente de qualquer ponto de W.

Hilario Alencar e Katia Frensel [AF] provaram que uma hipersuperfície mínima $f : \Sigma^n \to Q_c^{n+1}$, onde Q_c^{n+1} é uma forma espacial, com W sendo um aberto não-vazio, é totalmente geodésica. Este resultado para superfícies mínimas $f : \Sigma^2 \to Q_c^3$, com $c \geq 0$ e admitindo apenas $W \neq \emptyset$, foi provado por T. Hasanis e D. Koutroufiotis [HK].

O nosso primeiro resultado diz o seguinte:

Teorema 1.1. *Seja $f : \Sigma^n \to M^{n+1}$, $n \geq 2$, uma imersão com $W \neq \emptyset$, onde M é uma variedade Riemanniana completa e sem pontos conjugados. Se o recobrimento universal de Σ é compacto então M é simplesmente conexa.*

A inclusão $T^n \subset T^{n+1}$ de toros planos mostra que a hipótese da compacidade do rebrimento universal de Σ é essencial no Teorema 1.1.

Para o próximo teorema precisamos da seguinte definição:

Definição 1. *Sejam S e X subconjuntos de uma variedade Riemanniana M, onde a inclusão $S \subset M$ é um mergulho. Dizemos que X é um gráfico normal sobre S se existe um homeomorfismo $h : S \to X$ tal que, para qualquer ponto $x \in S$, a imagem $h(x)$ é conectada a x por uma única geodésica minimizante; além disso, esta única geodésica é normal a S.*

Teorema 1.2. *Seja Σ uma hipersuperfície propriamente mergulhada numa variedade Riemanniana completa, simplesmente conexa e sem pontos conjugados M. Se $W \neq \emptyset$ então temos:*

1. *Σ é um gráfico normal sobre um aberto de uma esfera geodésica de M;*
2. *Existe um aberto A tal que A e seu fecho $\overline{A}$ são estrelados com respeito a qualquer ponto de W; além disso, $\overline{A}$ é uma variedade que tem Σ como bordo.*

Se considerarmos a espiral $S \subset \mathbb{R}^2$ dada por $r = 1 + 2^{-\theta}$, $\theta \in \mathbb{R}$, (em coordenadas polares), o produto $S \times \mathbb{R}^n \subset \mathbb{R}^{n+1}$ mostra que a hipótese de Σ ser propriamente mergulhada é essencial no Teorema 1.2.

Capítulo 2: Algumas obstruções topológicas para a existência de hipersuperfícies totalmente geodésicas.

Este capítulo refere-se ao pré-print: *Some topological obstructions to the existence of totally geodesic hypersurfaces*, com a colaboração de Sergio Mendonça e Dethang Zhou.

A existência de uma subvariedade fechada e totalmente geodésica Σ numa variedade fechada M de curvatura positiva implica forte consequência entre as topologias de Σ e M. Por exemplo, um resultado clássico de T. Frankel [Fr1] diz o seguinte:

Teorema A. *Sejam M uma variedade fechada m-dimensional com curvatura seccional positiva, N uma variedade fechada n-dimensional com $n \geq m/2$ e $f : N \to M$ uma imersão totalmente geodésica. Então, o homomorfismo induzido pela inclusão $i_* : \pi_1(f(N)) \to \pi_1(M)$ é sobrejetivo.*

Outros resultados foram obtidos relacionando a topologia de uma subvariedade totalmente geodésica com a topologia de variedades ambientes de curvatura positiva (veja [Fr2],[FMR],[FM],[W]). Outros trabalhos consideraram ainda outras noções de curvatura ou generalizações da condição "totalmente geodésica", veja , por exemplo, [BR], [BRT], [KX].

Nos teoremas a seguir nenhuma condição sobre a curvatura é exigida.

Seja S uma subvariedade mergulhada numa variedade Riemanniana M. Seja $T^{\perp}S$ o fibrado normal de S e seja $T_1^{\perp}S$ o fibrado normal unitário de S. Dado $\epsilon > 0$, relembramos que um aberto W de M é chamado de ϵ-vizinhança tubular de S se $W = \exp^{\perp}(\widetilde{W})$, onde $\widetilde{W} = \left\{(x, v) \in T^{\perp}S \;\middle|\; |v| \leq \epsilon\right\}$, e, além disso, $\exp^{\perp}|_{\widetilde{W}}$ é um difeomorfismo.

Dizemos que um subconjunto $V \subset M$ é uma vizinhança tubular de S quando $V = \exp^{\perp}(\widetilde{V})$, onde $\exp^{\perp}|_{\widetilde{V}}$ é um difeomorfismo e $\widetilde{V}$ é uma subvariedade (possivelmente, com bordo) de $T^{\perp}S$ de dimensão maximal com a seguinte propriedade: se $(p, v) \in \widetilde{V}$ então $(p, tv) \in \widetilde{V}$, para todo $t \in [0, 1]$.

Teorema 2.1. *Seja $f : \Sigma \to M$ uma hipersuperfície, onde Σ é uma variedade fechada e conexa e M é uma variedade Riemanniana completa e conexa. Seja $g : N \to M$ uma imersão sem pontos focais de uma variedade fechada e conexa N de codimensão maior ou igual a três. Se Σ e N têm grupos fundamentais finitos então M é não-compacta com grupo fundamental finito. Admita que M é simplesmente conexo. Então, as seguintes afirmações são* *verdadeiras:*

(i) f *e* g *são mergulhos;*

(ii) M *é difeomorfa ao fibrado normal* $T^{\perp}N$*;*

(iii) N *e* Σ *são simplesmente conexas e* Σ *é difeomorfa ao fibrado normal unitário* $T_1^{\perp}N$*;*

(iv) $f(\Sigma)$ *é o bordo de uma vizinhança tubular compacta de* $g(N)$*.*

Para enunciar o próximo teorema precisamos da seguinte definição:

Definição 2.1. *Sejam* S, X *contidos numa variedade* M*, onde* S *é uma subvariedade mergulhada. Dizemos que* X *é um gráfico normal sobre* S *se existe um homeomorfismo* $h : S \to X$ *tal que para qualquer ponto* $x \in S$*, a imagem* $h(x)$ *é conectada a* x *por uma única geodésica minimizante, e esta única geodésica é ortogonal a* S*.*

Teorema 2.2. *Sejam* Σ *uma hipersuperfície conexa, propriamente mergulhada e totalmente geodésica numa variedade Riemanniana completa e simplesmente conexa* M *e* $g : N \to M$ *uma imersão sem pontos focais, onde* N *é uma variedade compacta. Admita que* $\Sigma \cap g(N) = \emptyset$*. Então, as seguintes afimações são verdadeiras:*

(i) g *é um mergulho;*

(ii) Σ *é um gráfico normal sobre um subconjunto aberto do bordo de uma* ϵ*-vizinhança tubular de* $g(N)$*, para algum* $\epsilon > 0$ *suficientemente pequeno;*

(iii) Σ *é o bordo de uma vizinhança tubular (possivelmente, não-compacta) de* $g(N)$*.*

Extensões tecnicas dos teoremas 2.1 e 2.2 serão provados neste capítulo. Mais especificamente, provaremos os seguintes resultados:

Teorema 2.1'. *Sejam* $f : \Sigma \to M$ *uma imersão de codimensão um de variedade fechada* Σ *numa variedade Riemanniana simplesmente conexa* M *e* $g : N \to M$ *uma imersão sem pontos focais de uma variedade fechada* N *de codimensão maior ou igual a três. Admita que* $f(\Sigma) \cap g(N) = \emptyset$ *e, além disso, nenhuma geodésica de* M *tangente a* f *intersecta* g *ortogonalmente. Então, as afirmações (i), (ii), (iii) e (iv) do Teorema 2.1 permanecem válidos.*

Teorema 2.2'. *Sejam $\Sigma \subset M$ uma hipersuperfície propriamente mergulhada numa variedade Riemanniana completa e simplesmente conexa M e seja $g : N \to M$ uma imersão sem pontos focais de uma variedade compacta N em M. Admita que $f(\Sigma) \cap g(N) = \emptyset$ e nenhuma geodésica de M tangente a f intersecta g ortogonalmente. Então, as afirmações (i), (ii) e (iii) do Teorema 2.2 permanecem válidos.*

Para demonstrar os Teoremas 2.1' e 2.2' será provado a seguinte extensão do Teorema de Hopf-Rinow:

Teorema 2.3. *Seja $g : N \to M$ uma imersão de uma variedade N numa variedade Riemanniana M com $g(N)$ compacta. Admita que $\exp^{\perp}$ é definida em todo fibrado normal $T^{\perp}N$. Então, M é completa.*

Capítulo 3 - Uma nota sobre geodésicas normais a uma imersão.

Este capítulo refere-se ao pré-print: *A note on the normal geodesics of an immersion*, feito em colaboração com Sergio Mendonça.

Um resultado bem conhecido da geometria diferencial que diz o seguinte:

> *Uma hipersuperfície imersa em $\mathbb{R}^{n+1}$ tal que todas as suas retas normais intersectam um ponto fixado de $\mathbb{R}^{n+1}$ é um mergulho isométrico de um aberto de alguma esfera Euclideana de $\mathbb{R}^{n+1}$.*

Generalizamos este resultado para imersões com codimensão arbitrária em quaisquer ambientes Riemannianos. Mais precisamente, provamos o seguinte:

Teorema 1.5. *Seja $f : \Sigma \to M$ uma imersão de uma variedade Σ numa variedade Riemanniana M e seja $\mathcal{C} \subset M$ um subconjunto. Admita que, para todo ponto $p \in \Sigma$, exista uma geodésica $\gamma_p : [0,1] \to M$ satisfazendo as seguintes propriedades:*

(i) $\gamma_p(0) = f(p)$ *e* $\gamma_p(1) \in \mathcal{C}$*;*

(ii) $\gamma_p'(0)$ *é normal a* $f(\Sigma)$*;*

(iii) *o comprimento* $\ell(\gamma_p)$ *de* γ_p *coincide com a distância* $d(f(p), \mathcal{C})$*.*

Então, a função $\psi : p \in \Sigma \to d(f(p), \mathcal{C})$ é constante.

Capítulo 4 - Teoremas do tipo semi-espaço em produtos torcidos com fator unidimensional.

Este capítulo refere-se ao pré-print: *Half-space type theorems in warped product spaces with one-dimensional factor*, do autor desta tese, a ser publicado no Geometriae Dedicata.

O produto torcido [BO] $W = I \times_\rho M$ de um intervalo da reta I por uma variedade Riemanniana M é o produto topológico $I \times M$ munido com a métrica produto torcido:

$$ds^2 = {\pi_I}^* \mathrm{dt}^2 + (\rho \circ \pi_I)^2 {\pi_M}^* g_M,$$

onde $\rho : I \to (0, \infty)$ é uma função suave positiva e π_I, π_M são as projeções ortogonais de $I \times M$ sobre seus fatores correspondentes.

O estudo dessa classe de variedades Riemannianas foi motivada a partir de um artigo de S. Montiel [MO] no qual são classificadas as variedades Riemannianas que são isométricas (localmente ou globalmente) a um produto torcido deste tipo.

Nesta classe de variedades Riemannianas incluem-se as variedades de curvatura constantes. Mais precisamente, o produto torcido $N = I \times_\rho M$ tem curvatura constante κ se, e somente se, M tem curvatura constante c e ρ satisfaz as seguintes equações diferenciais: $\rho'' = -\kappa\rho$ e $(\rho')^2 + c\rho^2 = \kappa$.

As folhas da folheação,

$$t \in I \mapsto \{t\} \times M \subset N,$$

que chamaremos de **fatias** de $I \times_\rho M$, são hipersuperfícies totalmente umbílicas com vetor curvatura média:

$$H_t = \mathcal{H}(t)\, \partial_t, \quad \text{onde } \mathcal{H}(t) = \frac{\rho'(t)}{\rho(t)}, \ t \in I.$$

Antes de enunciar nossos resultados, precisamos das seguintes definições:

Definição 4.1. *Serão denotados por semi-espaços de $N = I \times_\rho M$ os subconjuntos de N dados da seguinte forma:*

I: Semi-espaço superior: $\big([A, \infty) \cap I\big) \times M$, *com* $A \in I$;

II: Semi-espaço inferior: $\big((\infty, A] \cap I\big) \times M$, *com* $A \in I$.

Denotaremos também por **faixa** *de N os subconjuntos da forma $[A, B] \times M$, com $A, B \in I$.*

Definição 4.2. *A função altura de uma imersão* $f : \Sigma \rightarrow I \times_\rho M$ *é a função* $h = \pi_I \circ f$. *Denote:* $h_{\inf} = \inf_{p\in\Sigma} h(p)$ *e* $h_{\sup} = \sup_{p\in\Sigma} h(p)$.

Teorema 4.1. *Seja* $f : \Sigma^2 \rightarrow I \times_\rho \mathbb{R}^2$ *uma imersão isométrica própria de uma superfície conexa* Σ *num produto torcido* $I \times_\rho \mathbb{R}^2$. *Admita que, para alguma constante* $\mu \in (0,1)$, *a curvatura média* H *de* f *satisfaz uma das seguintes afirmações:*

(i) $f(\Sigma)$ *está contida num semi-espaço superior e* $|H| \leq \mu\mathcal{H} \circ h$; *ou*

(ii) $f(\Sigma)$ *está contida num semi-espaço inferior e* $|H| \leq -\mu\mathcal{H} \circ h$.

Então, $f(\Sigma^2)$ *é uma fatia totalmente geodésica, isto é,* $f(\Sigma^2) = \{t\} \times \mathbb{R}^2$, *para algum* $t \in I$ *satisfazendo* $\rho'(t) = 0$.

No caso em que ρ é uma função constante, o Teorema 2.1 reduz-se ao resultado clássico dado Hoffman e Meeks [HM] que diz:

> *Uma superfície mínima propriamente imersa num semi-espaço de* $\mathbb{R}^3$ *é um plano.*

O produto torcido $\mathbb{R} \times_{e^{-t}} \mathbb{R}^n$ é isométrico ao espaço hiperbólico $\mathbb{H}^{n+1}$. Além disso, uma horobola B (parte convexa limitada por uma horosfera) pode ser vista, a menos de uma isometria de $\mathbb{H}^{n+1}$, como um semi-espaço superior de $\mathbb{R} \times_{e^{-t}} \mathbb{R}^n$. Assim, como corolário do Teorema 4.1, segue-se o seguinte resultado:

Corolário 4.1. *Não existe superfície* Σ^2 *propriamente imersa numa horobola de* $\mathbb{H}^3$ *com curvatura média satisfazendo* $\sup_{p\in\Sigma} |H(p)| < 1$.

Pergunta. *As horosferas são as únicas superfícies propriamente imersas numa horobola de* $\mathbb{H}^3$ *com curvatura média* $|H| \leq 1$*?*

Esta pergunta foi parcialmente respondida para superfícies com curvatura média constante em duas direções: primeira, por L. Rodríguez e H. Rosenberg [RR] com a palavra "mergulhada" no lugar de "imersa"; e, posteriormente, por L. Alías e M. Dajczer [AD1] trocando-se a frase "numa horobola" pela frase "entre duas horosferas". A resposta desta questão será afirmativa se pudermos provar que uma superfície conexa propriamente imersa num produto torcido $I \times_\rho \mathbb{R}^2$ que satisfaz as hipóteses do Teorema 2.1, com adição da hipótese: $\mu \in (0,1]$, for uma fatia de $I \times_\rho \mathbb{R}^2$.

Teorema 4.2. *Seja $f : \Sigma \to I \times_\rho M$ uma imersão isométrica própria de uma variedade conexa Σ num produto torcido $I \times M$, onde M é uma variedade Riemanniana compacta. Admita que a imagem $f(\Sigma)$ e o vetor curvatura média H de f satisfaçam uma das seguintes afirmações:*

(i) *$f(\Sigma)$ está contida num semi-espaço superior e $\|H\| \leq \mathcal{H} \circ h$;*

(ii) *$f(\Sigma)$ está contida num semi-espaço inferior e $\|H\| \leq -\mathcal{H} \circ h$.*

Então, $f(\Sigma)$ está contida numa fatia de $I \times_\rho M$.

Por último, estudamos teoremas do tipo semi-espaços nos produtos torcidos do tipo $I \times_\rho \mathbb{H}^2$. Obtivemos o seguinte resultado:

Teorema 4.3. *Seja $f : \Sigma^2 \to I \times_\rho \mathbb{H}^2$ uma imersão isométrica própria de uma superfície conexa Σ num produto torcido $I \times_\rho \mathbb{H}^2$. Admita que, para alguma constante $\mu \in (0,1)$, a curvatura média H de f satisfaz uma das seguintes afirmações:*

(i) *$f(\Sigma)$ está contida num semi-espaço superior e $|H| \leq \mu\mathcal{H}(h_{\text{inf}})$; ou*

(ii) *$f(\Sigma)$ está contida num semi-espaço inferior e $|H| \leq -\mu\mathcal{H}(h_{\text{sup}})$.*

Então, f é minima. Além disso, considere $h_* = h_{\text{inf}}$ (resp., $h_* = h_{\text{sup}}$) se **(i)** *(resp., se (ii)) ocorre. Então, a folha $\{h_*\} \times \mathbb{H}^2$ é totalmente geodésica.*

Corolário 4.2. *Não existe imersão própria $f : \Sigma^2 \to I \times_\rho \mathbb{H}^2$ de uma superfície Σ no produto torcido $I \times_\rho \mathbb{H}^2$ com curvatura média H satisfazendo uma das seguintes afirmações:*

(i) *$f(\Sigma)$ está contida num semi-espaço superior e $\sup |H| < \inf \mathcal{H} \circ h$;*

(ii) *$f(\Sigma)$ está contida num semi-espaço inferior e $\sup |H| < \inf -\mathcal{H} \circ h$,*

Observação. *B. Nelli e H. Rosenberg [NR] exibiram exemplos de superfícies propriamente mergulhadas (e de rotação) entre duas fatias de $\mathbb{R} \times \mathbb{H}^2$, o que mostra que as desigualdades acimas são necessariamente estritas.*

Capítulo 5 - A influência do comportamento do bordo assintótico sobre imersões isométricas no espaço hiperbólico.

Este capítulo refere-se ao pré-print: *The influence of the boundary behavior on isometric immersions into the hyperbolic space,* feito em colaboração com Feliciano Vitório e Luquésio Jorge. Os resultados deste capítulos foram obtidos pelo autor da tese para o caso de hipersuperfícies. O pré-print em sua versão final deu-se a partir de uma visita de cooperação científica ao professor Marcos Petrucio Cavalcante na Universidade Federal das Alagoas quando, após um seminário sobre este assunto, Feliciano Vitório apresentou-me um princípio de tangência (demonstrado no apêndice desta tese), feito por ele juntamente com Luquésio Jorge com o qual foi possível estender o Teorema 5.1 do presente capítulo para codimensões arbitrárias.

Considere a compactificação natural do espaço hiperbólico m-dimensional,

$$\overline{\mathbb{H}}^m = \mathbb{H}^m \cup \mathbf{S}^{m-1}(\infty)$$

onde $\mathbf{S}^{m-1}(\infty)$ é identificado com as classes assintóticas dos raios geodésicos em $\mathbb{H}^m$ e possui, de maneira natural, a estrutura conforme usual (isometrias de $\mathbb{H}^m$ são dadas por automorfismos conformes de $\mathbf{S}^{m-1}(\infty)$). O **bordo assintótico** de um subconjunto $B \subset \mathbb{H}^m$ é definido por

$$\partial_\infty B = \overline{B} - B$$

onde $\overline{B}$ é o fecho de B em $\overline{\mathbb{H}}^m$. Denotaremos por esferas de $\mathbf{S}^{m-1}(\infty)$ os bordos assintóticos de hipersuperfícies completas e totalmente geodésicas de $\mathbb{H}^m$. Observe que, considerando $\mathbb{H}^m$ no modelo da bola unitária, $\mathbf{S}^{m-1}(\infty)$ é identificada com a esfera Euclideana unitária S^{m-1} e as esferas de $\mathbf{S}^{m-1}(\infty)$ identificadas com as esferas geodésicas de S^{m-1}.

Neste capítulo, estudamos a influência do bordo assintótico sobre imersões isométricas no espaço hiperbólico cujo bordo assintótico (da imagem) está contido numa esfera. Nosso primeiro resultado diz o seguinte:

Teorema 5.1. *Seja $f : \Sigma \to \mathbb{H}^m$ uma imersão isométrica própria de uma variedade conexa Σ cujo bordo assintótico está contido numa esfera S. Seja $\Lambda \subset \mathbb{H}^m$ a hypersuperfície completa e totalmente geodésica que tem S como bordo assintótico. Então, a curvatura média de f satisfaz a seguinte desigualdade:*

$$\sup_{p\in\Sigma} \|H(p)\| \geq \tanh\left(d_{\mathbb{H}}(f(p), \Lambda)\right), \tag{0-1}$$

para todo $p \in \Sigma$, *onde* $d_{\mathbb{H}}$ *é a distância hiperbólica. Além disso, se a igualdade em (0-1) ocorrer para algum ponto de* Σ *então* $f(\Sigma)$ *está contida numa hipersuperfície totalmente umbílica* $\Gamma^{m-1} \subset \mathbb{H}^m$ *com curvatura média* $\mathcal{H}_\Gamma = \sup_{p\in\Sigma} \|H(p)\|$.

Relembramos que uma imersão isométrica $f : \Sigma \to \mathbb{H}^m$ reduz codimensão se a imagem $f(\Sigma)$ estiver contida numa hipersuperfície totalmente geodésica de $\mathbb{H}^m$.

Corolário 5.1. *Seja* $f : \Sigma \to \mathbb{H}^m$ *uma imersão isométrica mínima e própria de um variedade conexa* Σ. *Se o bordo assintótico de* $f(\Sigma)$ *está contida numa esfera então* f *reduz codimensão.*

Corolário 5.1 generaliza um resultado dado por M. do Carmo and B. Lawson [dCL] que diz que uma hipersuperfície mínima e propriamente imersa em $\mathbb{H}^m$ cujo bordo assintótico é uma esfera é totalmente geodésica.

O resultado a seguir foi provado por M. do Carmo, J. Gomes e G. Thorbergsson [dCGT] para hipersuperfícies propriamente mergulhadas com curvatura média constante $H \in [0, 1)$.

Teorema 5.2. *Seja* $f : \Sigma^n \to \mathbb{H}^{n+1}$ *uma hipersuperfície propriamente imersa com curvatura média satisfazendo* $\sup_{p\in\Sigma} |H(p)| < 1$. *Então* $\partial_\infty f(\Sigma)$ *não admite pontos isolados.*

A superfície $\Sigma^2 \subset \mathbb{H}^3$ parametrizada por $\phi(u, v) = (u, v, e^v)$ (considerando $\mathbb{H}^3$ no modelo do semi-espaço superior) é um mergulho próprio tendo como bordo assintótico um único ponto. Além disso, a curvatura média de Σ^2 é dada por $|H(p)| = (2 + 3e^{2v})/(2(1 + e^{2v})^{3/2}) < 1$, para todo $p \in \Sigma$. Este exemplo foi exibido por A. Lluch, em [Al] e mostra que no Teorema 5.2 tanto a norma do supremo quanto a desigualdade estrita são essenciais.

Pergunta. *É possível que uma imersão isométrica própria no espaço hiperbólico com vetor curvatura média paralelo satisfazendo* $\|H\| < 1$ *possua em seu bordo assintótico algum ponto isolado?*

A demonstração do Teorema 5.2 faz uso de um importante resultado. A fim de enunciá-lo, vamos relembrar a noção de "distância"em $\mathbf{S}^n(\infty)$, tal como definida em [dCGT]. Primeiro, dois subconjuntos A_1, $A_2 \subset \mathbf{S}^n(\infty)$ são ditos separados por duas esferas disjuntas $S_1, S_2 \subset \mathbf{S}^n(\infty)$ se A_1 e A_2 estão contidos em distintas componentes conexas do tipo-disco de $\mathbf{S}^n(\infty)-(S_1 \cup S_2)$. Considere $d_\infty(S_1, S_2) := d_{\mathbb{H}}(\Lambda_1, \Lambda_2)$, onde Λ_i é a hipersuperfície totalmente geodésica que tem S_i como bordo assintótico. A distância $d_\infty(A_1, A_2)$ entre dois subconjuntos A_1 e A_2 de $\mathbf{S}^n(\infty)$ será definida do seguinte modo:

$$d_\infty(A_1, A_2) = \begin{cases} 0, & \text{se não existir esferas disjuntas } S_1 \text{ e } S_2 \\ & \text{separando } A_1 \text{ e } A_2; \\ \sup\{d(S_1, S_2) \mid S_1 \text{ e } S_2 \text{ separam } A_1 \,\text{e}\, A_2\}. \end{cases}$$

Existem duas propriedades importantes dessa noção de distância:

(i) a distância d_∞ é conformemente invariante, visto que transformações conformes de $\mathbf{S}^n(\infty)$ são induzidas por isometrias de $\mathbb{H}^{n+1}$;

(ii) se $n \leq 2$ então a distância entre um compacto e um ponto fora deste compacto é infinito.

(iii) se $d_\infty(A_1, A_2) < \infty$ então, por um argumento de compacidade, existem esferas disjuntas S_1 e S_2 satisfazendo $d_\infty(A_1, A_2) = d_\infty(S_1, S_2)$.

Observação. *Apesar de M. do Carmo et al. [dCGT] denotar* d_∞ *por distância em* $\mathbf{S}^n(\infty)$*, eles também observam que a desigualdade triangular não é satisfeita em geral.*

Em [dCGT], M. do Carmo, J. Gomes e G. Thorbergsson provaram o seguinte resultado:

Proposição A (Teorema 1 [dCGT]). *Seja* $\Sigma^n \subset \mathbb{H}^{n+1}$ *uma superfície propriamente mergulhada com curvatura média constante* $H \in [0, 1)$*. Admita que o bordo assintótico* $\partial_\infty \Sigma$ *tenha pelo menos duas componentes conexas e seja* A *uma de suas componentes conexas. Então, existe uma constante* d_H *(dependendo apenas de* H *e computável) tal que a seguinte desigualdade é satisfeita:*

$$d(A, \partial_\infty \Sigma - A) \leq d_H$$

Além disso, a igualdade é satisfeita se, e somente se, Σ *é uma hipersuperfície de rotação do tipo esférico.*

Usando as mesmas técnicas desenvolvidas em [dCGT] para provar Proposição A é possível provar ao seguinte resultado:

Proposição 5.2. *Seja* $f : \Sigma^n \to \mathbb{H}^{n+1}$ *uma hipersuperfície propriamente* *imersa com curvatura média satisfazendo* $\sup_{p\in\Sigma}\|H(p)\| < 1$. *Admita que o bordo assintótico* $\partial_\infty\Sigma$ *tenha pelo menos duas componentes conexas e seja* A *uma de suas componentes conexas. Então, existe uma constante* d *(dependendo apenas de* $\sup\|H\|$ *e computável) tal que a seguinte desigualdade é satisfeita:*

$$d(A, \partial_\infty\Sigma - A) \leq d$$

Além disso, a igualdade é satisfeita se, e somente se, Σ *é uma hipersuperfície de rotação do tipo esférico.*

Teorema 5.1 e Corolário 5.1, para o caso em que f é uma hipersuperfície, generalizam-se para qualquer curvatura r-média

$$H_r = \frac{1}{C_r^n} \sum_{1\leq i_1<\ldots<i_r\leq n} \lambda_{i_1}\cdot\ldots\cdot\lambda_{i_r},$$

onde $\lambda_1 \leq \ldots \leq \lambda_n$ são as curvaturas principais de f. Em particular estes resultados permanecem válidos para as curvaturas escalar e Gauss-Kronecker. A diferença da prova dos teoremas supracitatos para este caso mais geral difere, simplesmente, do uso do seguinte princípio de tangência para curvaturas r-médias, que segue-se como corolário do Teorema 1.1 de [FS] (veja Teorema A do Apêndice A (capítulo seguinte)):

Proposição 5.3. *Sejam* Σ_1 *e* Σ_2 *hipersuperfícies orientáveis de uma variedade Riemanniana* $(n+1)$*-dimensional* M, $p \in \Sigma_1 \cap \Sigma_2$ *satisfazendo* $T_p\Sigma_1 = T_p\Sigma_2$ *e* η *uma orientação de* Σ_2. *Admita que* Σ_2 *é totalmente umbílica e não-totalmente geodésica. Admita que* Σ_1 *está acima de* Σ_2 *com respeito a* $\eta(p)$ *(veja definição em Apêndice A) e a curvatura média* $\mathcal{H}$ *de* Σ_2 *satisfaz:*

$$\mathcal{H} \geq \min_{1\leq r\leq n} |H_r|,$$

onde H_r, $r = 1, \ldots, n$ *são as curvaturas* r*-médias de* Σ_1. *Então,* Σ_1 *e* Σ_2 *coincidem-se numa vizinhança do ponto* p.

Apêndice - Princípios de tangência.

Neste apêndice introduziremos os princípios de tangência que foram fundamentais para provar os teoremas dos quarto e quinto capítulos.

Seja $\Sigma^n \subset M^{n+1}$ uma hipersuperfície numa variedade Riemanniana M. Dado $p_0 \in \Sigma$ e fixado um vetor normal e unitário $\eta_0 \in T_{p_0}^{\perp}\Sigma$ no ponto p_0, sabemos que existe uma parametrização $\psi : W \to U$ de uma vizinhança da origem $W \subset T_{p_0}\Sigma$ numa vizinhanca $U \subset \Sigma$ de p_0, dada da seguinte forma:

$$\psi_\Sigma(x) = \exp_{p_0}(x + \mu_\Sigma(x)\eta_0), \tag{0-2}$$

onde exp é a aplicação exponencial de M. Assim, $\mu_\Sigma : W \to \mathbb{R}$ é única e satisfaz $\mu_\Sigma(0) = 0$.

Definição 1. *Sejam Σ_1 e Σ_2 duas hipersuperfícies de M que se tocam tangencialmente num ponto $p \in \Sigma_1 \cap \Sigma_2$, ou seja, $T_p\Sigma_1 = T_p\Sigma_2$. Assim, fixado um vetor unitário $\eta_0 \in T^\perp\Sigma_1 = T^\perp\Sigma_2$, dizemos que Σ_1* **está localmente acima de Σ_2 no ponto p com respeito ao vetor** *η_0, se as funçoes $\mu_{\Sigma_1}, \mu_{\Sigma_2} : W \to \mathbb{R}$, tais como definidas em (0-2), satisfazem $\mu_{\Sigma_1} \geq \mu_{\Sigma_2}$ numa vizinhança W da origem $0 \in T_p\Sigma_1 = T_p\Sigma_2$.*

O princípio de tangência clássico diz o seguinte:

Princípio de tangência para hipersuperfícies. *Sejam Σ_1 e Σ_2 hipersuperfícies de M que se tocam tangencialmente num ponto $p \in \Sigma_1 \cap \Sigma_2$ e fixe $\eta \in T_p^\perp\Sigma_1 (= T_p^\perp\Sigma_2)$ um vetor unitário e, para cada $i = 1, 2$, considere uma orientação local η_i de Σ_i em p tal que $\eta_i(p) = \eta$. Considere $\psi_1 = \psi_{\Sigma_1}$ e $\psi_2 = \psi_{\Sigma_2}$ as parametrizações tais como definidas em (0-2). Admita que as curvaturas médias, $H^i(\psi_i(x))$, de Σ_i no ponto $\psi_i(x)$, $i = 1, 2$, com relação à η_i satisfaçam $H^1(x) \leq H^2(x)$, para todo x numa vizinhança da origem. Se Σ_1 está localmente acima de Σ_2 em p com respeito ao vetor η, então Σ_1 coincide com Σ_2 numa vizinhança de p.*

Este princípio de tangência foi generalizado por F. Fontenele e S. Silva [FS] para curvaturas r-médias e também por F. Vitório e L. Jorge (cuja prova está no apêndice do quinto capítulo) para o vetor curvatura média de subvariedades. Primeiro, vamos enunciar a generalização dada por Fontenele e Silva. Para isto, temos que fixar algumas notações. Seja $\Sigma^n \subset M^{n+1}$ uma hipersuperfície. Fixado $p \in \Sigma$, considere η uma orientação sobre uma vizinhança de p em Σ. Seja $\psi_\Sigma : W \to \Sigma$ a parametrização de uma vizinhança de p em Σ tal como dada em (0-2), definida sobre um aberto W da origem. Para cada $x \in W$, considere $\lambda_1(x) \leq \ldots \leq \lambda_n(x)$ e $\lambda(x) = (\lambda_1(x), \ldots, \lambda_n(x))$ as curvaturas principais e o vetor curvatura principal de Σ no ponto $\psi(x)$ com relação a η, respectivamente. Dado $r \in \{1, \ldots, n\}$, a curvatura r-média de Σ no ponto $\psi_\Sigma(x)$ com relação a η é dada por:

$$H_r(x) = \frac{1}{C_r^n}\sigma_r(\lambda(x)),$$

onde $\sigma_r : \mathbb{R}^n \to \mathbb{R}$ é a função simétrica r-elementar dada por:

$$\sigma_r(z_1, \ldots, z_n) = \frac{1}{C_k^n} \sum_{1 \le i_1 < \ldots < i_r \le n} z_{i_1} \cdot \ldots \cdot z_{i_r}.$$

Além disso, considere Γ_r a componente conexa do aberto $\{\sigma_r > 0\} \subset \mathbb{R}^n$ que contém o vetor $a_0 = (1, \ldots, 1)$. Observe que o cone

$$\mathcal{O}^n = \{(z_1, \ldots, z_n) \mid z_i > 0, \text{ for all } i = 1, \ldots, n\}$$

está contido em Γ_r , para todo $r \in \{1, \ldots, n\}$.

Teorema A (Fontenele e Silva [FS]) *Sejam Σ_1 e Σ_2 hipersuperfícies que se tocam tangencialmente num ponto $p \in \Sigma_1 \cap \Sigma_2$. Fixemos η_i, $i = 1, 2$, uma orientação local de p em Σ_i tais que $\eta_1(p) = \eta_2(p) = \eta_0$ (note que $T_p^\perp \Sigma_1 = T_p^\perp \Sigma_2$) e sejam $\psi_1 = \psi_{\Sigma_1}$, $\psi_2 = \psi_{\Sigma_2}$ parametrizações tais como dadas em (0-2) definidas num aberto W da origem. Para cada $i = 1, 2$, denote $H_r^i(x)$, $x \in W$, a curvatura r-média de Σ_i no ponto $\psi_i(x)$ com relação a η_i. Assuma que $H_r^1 \le H_r^2$ numa vizinhança da origem e que Σ_1 está localmente acima de Σ_2 em p com respeito a η_0. Se $r \ge 2$, também assuma que o vetor curvatura principal $\lambda^2(0)$ de p em Σ_2 com relação a η_0 pertença a Γ_r. Então, Σ_1 coincide com Σ_2 numa vizinhança de p.*

Vamos enunciar o princípio de tangência para subvariedades dado por F. Vitório e L. Jorge. Necessitamos de uma definição análoga à Definição 1. Para isto, sejam M uma variedade Riemanniana, $f : \Sigma \to M$ uma imersão isométrica e $B \subset M$ uma hipersuperfície.

Definição 2. *Seja $f : \Sigma \to M$ uma imersão isométrica numa variedade Riemanniana M e seja $B \subset M$ uma hipersuperfície tal que $f(p) \in B$, para algum ponto $p \in M$. Fixe η uma orientação de uma vizinhança de p em B. Dizemos que f* **está localmente acima de B em p com respeito a** *η se existem vizinhanças $U \subset \Sigma$, $V \subset M$ de p e $f(p)$, respectivamente, tais que $V - B$ tem exatamente duas componentes conexas e $f(U)$ está contido na componente conexa no qual η aponta.*

Definição 3. *Dados $\epsilon \ge 0$ e $k \in \{1, \ldots, n\}$, dizemos que uma hipersuperfície B numa variedade Riemanniana M é* **(k, ϵ)-média convexa** *com respeito a uma orientação η de B se, considerando $\lambda_1 \le \ldots \le \lambda_n$ as curvaturas principais de B com respeito a η, tem-se $\frac{\lambda_1 + \ldots + \lambda_k}{k} \ge \epsilon$.*

Princípio de tangência (Jorge and Vitorio) *Sejam $f : \Sigma \to M$ uma imersão isométrica numa variedade Riemanniana M e B uma hipersuperfície com $f(p) \in B$, para algum $p \in \Sigma$. Seja η uma orientação de B e assuma que B é (k,ϵ)-média convexa com relação a η, para algum $\epsilon \geq 0$ e $k \in \{1, \ldots, n\}$. Admita também que o vetor curvatura média de f satisfaz $\|H\| \leq \epsilon$. Se f está localmente acima de B em p com respeito a η então $f(U) \subset B$ para alguma vizinhança U de p em Σ.*

Observação. *A proposição acima, com f mínima, foi provada por L. Jorge e F. Tomi [JT]. A demonstração deste resultado foi inspirada na prova deste caso particular.*

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1
Hypersurfaces whose tangent geodesics do not cover the ambient space.

This chapter refers to the article [MM] (joint work with Sergio Mendonça).

Abstract Let $x : \Sigma^n \to M^{n+1}$ be an immersion of an n-dimensional connected manifold Σ in an $(n+1)$-dimensional connected complete Riemannian manifold M without conjugate points. Assume that the union of geodesics tangent to x does not cover M. Under these hypotheses we have two results. The first one states that M is simply-connected provided that the universal covering of Σ is compact. The second result says that if x is a proper embedding and M is simply-connected then $x(\Sigma)$ is a normal graph over an open subset of a geodesic sphere. Furthermore, there exists an open star-shaped set $A \subset M$ such that $\bar{A}$ is a manifold with the boundary $x(\Sigma)$.

1.1 Introduction

Let $x : \Sigma^n \to M^{n+1}$ be an immersion of a connected n-dimensional manifold Σ in a connected complete $(n+1)$-dimensional Riemannian manifold M. A very strong condition would be to assume that x is totally geodesic. Here we will make a much weaker assumption, namely, that the union of geodesics tangent to x does not cover M. More precisely, let $W = W(x)$ be the set of points of M that do not lie on any geodesic tangent to x. We could ask in what situations we could have $W \neq \emptyset$. The first important answer to this question was given by Halpern [Hp]. To state it we first recall a well-known definition.

Definition 1 *A subset B of a complete manifold M is said to be star-shaped with respect to x_0 if $x_0 \in B$ and for any point $p \in B$ there exists a unique minimal normal geodesic joining x_0 and p, and the image of that unique geodesic is contained in B.*

Theorem A. (Halpern [Hp]) *Let $x : \Sigma^n \to \mathbb{R}^{n+1}$, $n \geq 2$, be an immersion of a closed manifold Σ with $W \neq \emptyset$. Then, the following conclusions hold:*

1. *x is an embedding;*

2. *Σ^n is diffeomorphic to the Euclidean sphere;*

3. *There exists a diffeomorphic disk D with $\partial D = x(\Sigma)$; furthermore D and its interior $\mathrm{int}(D)$ are star-shaped with respect to any point of W.*

Beltagy [Be] extended this result to the case where M is a complete and simply-connected Riemannian manifold without conjugate points, using the fact that the exponential map is a diffeomorphism and applying Theorem A to the tangent space at any point of W. Alexander [Al] modified Theorem A, restricting the attention to the union V of tangent spaces at saddle points of Σ, and obtained a weaker conclusion.

Remark. Theorem A and Alexander's Theorem can be extended to the case where M is the standard sphere S^{n+1} by using the stereographic projection associated to the antipodal point $-P$ where $P \in W$ in the case of Theorem A and $P \in S^{n+1} - V$ in the case of Alexander's Theorem. Alencar and Frensel proved in [AF] that, if M is a space form $Q^{n+1}(c)$ and x is minimal with W being a nonempty open set, then x is totally geodesic. This result for a minimal immersion $x\colon M^2 \to Q^3_c$, $c \geq 0$, with nonempty W has been proved by Hasanis and Koutroufiotis [HK]. Another result in [AF] says that if $\Sigma^n \subset Q^{n+1}(c)$ is closed and has constant mean curvature with $W \neq \emptyset$ then Σ is a round sphere.

Now we state our first result.

Theorem 1 *Let $x : \Sigma^n \to M^{n+1}$, $n \geq 2$, be an immersion, where M is a connected complete Riemannian manifold without conjugate points and with $W \neq \emptyset$. If the universal covering of Σ is compact then M is simply-connected.*

Remark. The inclusion $T^n \subset T^{n+1}$ of flat tori shows that the hypothesis that $\tilde{\Sigma}$ is compact is essential in Theorem 1.1. The following result is new even in the case that $M = \mathbb{R}^{n+1}$. First we recall the definition of normal graph.

Definition 2 *Let S, X be contained in a manifold M, where S is an embedded submanifold. We say that X is a normal graph over S if there exists a homeomorphism $h : S \to X$ such that for any point $x \in S$, the image $h(x)$ is connected to x by a unique minimal geodesic, and that unique geodesic is orthogonal to S.*

Theorem 2 *Let Σ be a connected properly embedded hypersurface in a simply-connected complete Riemannian manifold M without conjugate points with $W \neq \emptyset$. Then we have:*

1. *Σ is a normal graph over an open subset of a geodesic sphere;*

2. *there exists an open set A such that A and its closure $\bar{A}$ are star-shaped with respect to any point of W; furthermore, $\bar{A}$ is a manifold which has Σ as its boundary.*

Remark. If we consider the spiral S given by $r = 1+2^{-\theta}$ in polar coordinates, the product $S \times \mathbb{R}^{n-1} \subset \mathbb{R}^{n+1}$ shows that the condition that Σ is properly embedded is essential in Theorem 1.2.

1.2 Proof of Theorem 1.1.

Let $x : \Sigma^n \to M^{n+1}$ be an immersion where M is a connected Riemannian manifold without conjugate points. Let $\pi : \widetilde{M} \to M$ be the universal covering of M with the induced metric and let $\nu : \widetilde{\Sigma} \to \Sigma$ be the universal covering of Σ. By the Fundamental Lifting Theorem, for any $p \in \Sigma$, $\bar{p} \in \nu^{-1}(p)$ and $\tilde{q} \in \pi^{-1}(x(p))$, there exists an immersion $y : \widetilde{\Sigma} \to \widetilde{M}$, namely the lifting of x, satisfying $y(\bar{p}) = \tilde{q}$ and such that the diagram bellow commutes.

$$\begin{array}{ccc} \widetilde{\Sigma} & \xrightarrow{\ y\ } & \widetilde{M} \\ \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \pi} \\ \Sigma & \xrightarrow{\ x\ } & M \end{array} \tag{1-1}$$

Let $W(y) \subset \widetilde{M}$ be the set of points that do not lie on any geodesic tangent to the immersion y. Let $\mathcal{A}$ be the set of liftings of x.

Claim 2.1. For $W = W(x)$ it holds that $\pi^{-1}(W) = \bigcap_{y \in \mathcal{A}} W(y)$.

Prova. Assume that $\tilde{q} \notin \pi^{-1}(W)$. Since $q = \pi(\tilde{q}) \notin W$, there exists a geodesic $\gamma : [0, a] \to M$ with $\gamma(0) = q$ which is tangent to the immersion x at $t = a$. The geodesic γ can be lifted to a geodesic $\tilde{\gamma} : [0, a] \to \widetilde{M}$ with $\tilde{\gamma}(0) = \tilde{q}$ and such that $\tilde{\gamma}$ is tangent to some immersion $y \in \mathcal{A}$ at $t = a$. Thus $\tilde{q} \notin W(y)$. Conversely, assume that $\tilde{q} \notin W(y)$ for some $y \in \mathcal{A}$. Then there exists a geodesic $\tilde{\gamma}$ tangent to the immersion y containing $\tilde{q}$. The geodesic $\gamma := \pi \circ \tilde{\gamma}$ is tangent to the immersion x and contains $\pi(\tilde{q})$. Therefore $\pi(\tilde{q}) \notin W$, hence $\tilde{q} \notin \pi^{-1}(W)$. ■

To prove Theorem 1.1 we assume that $\widetilde{\Sigma}$ is compact. Take $x_0 \in W$ and fix some lifting $y \in \mathcal{A}$. By Claim 2.1, $\pi^{-1}(x_0) \subset W(y)$. It follows from Beltagy's Theorem [Be] that $y(\widetilde{\Sigma})$ is an embedded sphere which is the boundary of a compact topological disk D containing the set $\pi^{-1}(x_0)$. We also have that D

and its interior int(D) are star-shaped with respect to any point of $W(y)$, and thus they are star-shaped with respect to any point of $\pi^{-1}(x_0)$. Since $\pi^{-1}(x_0)$ is discrete and contained in a compact disk, the set $\pi^{-1}(x_0)$ and the group $\mathrm{Aut}(\pi)$ of automorphisms of π are both finite. For any $\phi \in \mathrm{Aut}(\pi)$, we have that $\bar{y} := \phi \circ y$ is also a lifting of x since $\pi = \pi \circ \phi$.

$$\begin{array}{ccccc} \widetilde{\Sigma} & \xrightarrow{y} & \widetilde{M} & \xrightarrow{\phi} & \widetilde{M} \\ & & \downarrow{\scriptstyle \pi} & \swarrow{\scriptstyle \pi} & \\ & & M & & \end{array} \qquad \bar{y} = \phi \circ y : \widetilde{\Sigma} \to \widetilde{M} \tag{1-2}$$

Again we have that $\pi^{-1}(x_0) \subset W(\bar{y})$ and that $\bar{y}(\widetilde{\Sigma})$ is embedded as the boundary of the compact topological disk $\phi(D)$ containing the set $\pi^{-1}(x_0)$. We also have that $\phi(D)$ and $\mathrm{int}(\phi(\mathrm{D}))$ are star-shaped with respect to any point of $\pi^{-1}(x_0)$. Thus we have that

$$E = \bigcap_{\phi \in \mathrm{Aut}(\pi)} \phi(D) \tag{1-3}$$

is invariant under $\mathrm{Aut}(\pi)$ and that $\mathrm{int}(E)$ is star-shaped with respect to any point of $\pi^{-1}(x_0)$. In fact, since $\mathrm{Aut}(\pi)$ is finite it is easy to see that $\mathrm{int}(E) = \bigcap_{\phi \in \mathrm{Aut}(\pi)} \mathrm{int}(\phi(D))$. Thus $\mathrm{int}(E)$ is star-shaped with respect to any point in $\pi^{-1}(x_0)$, since it is the intersection of star-shaped sets.

Claim 2.2. *E is a compact topological disk.*

Prova. Fix $\tilde{x}_0 \in \pi^{-1}(x_0)$. We know that $\mathrm{int}(E)$ is star-shaped with respect to $\tilde{x}_0$. Fix a unit vector $v \in T_{\tilde{x}_0}\widetilde{M}$ and $\phi \in \mathrm{Aut}(\pi)$. Since $\tilde{x}_0 \in W(\phi \circ y)$, we claim that the geodesic $\gamma : t \mapsto \exp_{\tilde{x}_0} tv$ meets transversally the hypersurface $\phi \circ y(\widetilde{\Sigma}) = \partial\phi(D)$ at a unique point. In fact, there exists a first time $t_{v,\phi}$ such that γ intersects $\partial\phi(D)$ since $\phi(D)$ is star-shaped with respect to $\tilde{x}_0$. Now we prove that this intersection is unique. Indeed, since $W(\phi\circ y) \neq \emptyset$, the geodesic γ intersects $\partial\phi(D)$ transversely. Since $\phi(D)$ is a manifold with boundary $\partial\phi(D)$, we have that $\gamma(t) \notin \phi(D)$ for $t > t_{v,\phi}$ sufficiently close to $t_{v,\phi}$. If γ intersects $\partial\phi(D)$ a second time this will contradict the fact that $\phi(D)$ is star-shaped with respect to $\tilde{x}_0$. Since the intersection is transversal at $t_{v,\phi}$, the time $t_{v,\phi}$ will depend smoothly on v. Let

$$t(v) = \min_{\phi \in \mathrm{Aut}(\pi)} t_{v,\phi}.$$

Thus $t(v)$ depends continuously on v. Given $w \in E$, set

$$v(w) = \frac{(\exp_{\tilde{x}_0})^{-1} w}{\|(\exp_{\tilde{x}_0})^{-1} w\|} .$$

Let $B \subset T_{\tilde{x}_0}\widetilde{M}$ be the compact unit disk centered at 0. Now we define $F : B \to E$ given by

$$F(z) = \exp_{\tilde{x}_0} t(z/\|z\|)\, z,$$

for $z \neq 0$, and $F(0) = \tilde{x}_0$. It is not difficult to see that F has a continuous inverse $G : E \to B$ given by

$$G(w) = \frac{1}{t(v(w))} (\exp_{\tilde{x}_0})^{-1} w,$$

for $w \neq \tilde{x}_0$ and $G(\tilde{x}_0) = 0$. ■

Finally, for any $\phi \in \mathrm{Aut}(\pi)$, we have that ϕ must have a fixed point in E, since $\phi(E) \subset E$, and E is a compact disk by Claim 2.2. Thus we have that $\mathrm{Aut}(\pi)$ is trivial and M is simply-connected. Theorem 1.1 is proved.

1.3 Proof of Theorem 1.2.

Let M be a complete and simply-connected Riemannian manifold without conjugate points. Let $\Sigma \subset M$ be a connected and properly embedded hypersurface satisfying $W \neq \emptyset$.

Fix $x_0 \in W$. In particular $x_0 \notin \Sigma$. Since Σ is properly embedded there exists some small geodesic sphere S centered at x_0 which does not intersect Σ. We define $F : \Sigma \to S$ as follows. Given $p \in \Sigma$, there exists a unique normal geodesic $\gamma := [x_0, p]$ joining x_0 and p. We know that γ is orthogonal to S at a unique intersection point which we will call $F(p)$. Since $x_0 \in W$ it follows that γ is also transversal to Σ, hence $F : \Sigma \to S$ is a local diffeomorphism onto its open image $F(\Sigma) \subset S$. Thus, to show that $F : \Sigma \to F(\Sigma)$ is a diffeomorphism it is sufficient to show that F is injective.

Define the set

$$\mathcal{C} := \Big\{ p \in \Sigma \;\Big|\; \text{the cardinality } \#\Big([x_0, p] \cap \Sigma\Big) = 1 \Big\} .$$

We need to prove that $\mathcal{C} = \Sigma$.

Claim 3.1. $\mathcal{C} \neq \emptyset$.

In fact, using that Σ is a properly embedded hypersurface, we obtain that there exists a point $p_0 \in \Sigma$, satisfying $d(x_0, p_0) = \min_{p \in \Sigma} d(x_0, p)$. Thus $p_0 \in \mathcal{C}$.

Claim 3.2. $\Sigma - \mathcal{C}$ is open as a subset of Σ.
To prove this we take $x_1 \in \Sigma - \mathcal{C}$. So there exists $x_2 \in \Sigma$ with $x_2 \neq x_1$ and $x_2 \in [x_0, x_1]$. In particular we have $F(x_1) = F(x_2) = q$. Since F is a local diffeomorphism, it is not difficult to see that there exist disjoint neighborhoods of x_1 and x_2 in Σ mapped by F onto the same neighborhood of $q \in S$. Thus we conclude that $\Sigma - \mathcal{C}$ is open in Σ.
Claim 3.3. $\Sigma - \mathcal{C}$ is closed as a subset of Σ.
In fact, take a sequence $x_k \to a \in \Sigma$, with $x_k \in \Sigma - \mathcal{C}$. Since Σ is properly embedded there exists an open neighborhood U of $a \in M$ such that the intersection $\Sigma \cap U$ is connected and the restriction $F|_{\Sigma \cap U}$ is a diffeomorphism onto its open image. By definition of $\mathcal{C}$, for each k there exists a point $y_k \neq x_k$ with $y_k \in [x_0, x_k] \cap \Sigma$. Since (x_k) is bounded, we have that (y_k) is also bounded. So we can assume by passing to a subsequence that (y_k) converges to some point $b \in [x_0, a]$. Since Σ is properly embedded we have that $b \in \Sigma$. Since $F|_{\Sigma \cap U}$ is injective we have that $y_k \notin U$, hence $b \neq a$. So we conclude that $a \in \Sigma - \mathcal{C}$.

Thus $\mathcal{C} = \Sigma$ by connectedness of Σ and $F : \Sigma \to F(\Sigma)$ is a diffeomorphism.

Now we will prove that Σ is the boundary of an open star-shaped set with respect to x_0. Consider the set

$$A := \left\{ z \in M \,\middle|\, \#\Big([x_0, z] \cap \Sigma\Big) = 0 \right\}.$$

Given $z \in A$, the distance between $[x_0, z]$ and Σ is positive, since Σ is properly embedded. This implies that A is open.

We state that $\bar{A} - A = \Sigma$. In fact, given $p \in \Sigma$ the geodesic segment $[x_0, p] - \{p\} \subset A$, hence $\Sigma \subset \bar{A}$. Clearly we have $A \cap \Sigma = \emptyset$, hence $\Sigma \subset \bar{A} - A$. Now take $p \in \bar{A} - A$. Assume by contradiction that $p \notin \Sigma$. Since $p \notin A$ the geodesic $[x_0, p]$ intersects Σ transversely at a unique point $q \neq p$. Consider the unit vector $v \in T_{x_0}M$ such that $\exp_{x_0} t_0 v = p$ for some $t_0 > 0$. Since F is a diffeomorphism onto its open image there exists an open neighborhood U of v in the sphere $S^{n-1} \subset T_{x_0}M$ such that geodesic $t \mapsto \exp_{x_0} tw$ meets Σ at a unique point for any $w \in U$. Since Σ is properly embedded we can choose U and $\epsilon > 0$ sufficiently small such that $\exp_{x_0} tw \notin \Sigma$ if $|t - t_0| < \epsilon$. This defines a neighborhood V of p which $V \subset M - A$. This contradicts the fact that $p \in \bar{A}$.

Clearly we have that A and $\bar{A} = A \cup \Sigma$ are star-shaped with respect to x_0. To conclude the proof we need to show that $\bar{A}$ is a manifold with boundary Σ. In fact, take a point $p \in \Sigma$, with $p = \exp_{x_0} t_0 v$ for some unit vector $v \in T_{x_0}M$. Again there exists an open neighborhood U of v in $S^{n-1} \subset T_{x_0}M$ such that for each $w \in U$ the geodesic $t \mapsto \exp_{x_0} tw$ meets Σ transversely at a unique point $q_w = \exp_{x_0} t_w w$. So the time t_w depends smoothly on w. Thus a small neighborhood W of p in $\bar{A}$ can be defined as

$$W = \exp_{x_0}\{tw \mid w \in U,\, t_w - \epsilon < t \leq t_w\},$$

where $\epsilon > 0$ is small enough. Thus we proved that $\bar{A}$ is a smooth manifold with boundary Σ. ■

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2
Some topological obstructions to the existence of totally geodesic hypersurfaces.

This is a joint work with Sergio Mendonça and Dethang Zhou.

Abstract. One of our results is the following. Let $f : \Sigma \to M$ be a codimension one totally geodesic immersion of a connected closed manifold Σ in a complete connected Riemannian manifold M, and $g : N \to M$ an immersion without focal points of a connected closed manifold N of codimension at least 3. If Σ and N have finite fundamental groups then M is a noncompact manifold with finite fundamental group. In the results of this paper no curvature condition is needed. To prove the above result we use a generalization of the Hopf-Rinow Theorem.

2.1
Introduction

The existence of a closed totally geodesic submanifold in a closed manifold M of positive sectional curvature has strong topological consequences on the relation between the topology of M and the topology of the submanifold. For example, a classical theorem of Frankel ([Fr1]) says:

Theorem A. (Frankel [Fr1]) *Consider an m-dimensional closed manifold M of positive sectional curvature. Let N be an n-dimensional closed manifold and $f : N \to M$ be a totally geodesic immersion with $n \geq m/2$. Then the homomorphism induced by the inclusion $i_* : \pi_1(f(N)) \to \pi_1(M)$ is surjective.*

Several other results were obtained relating the topology of totally geodesic submanifolds with the topology of ambient manifolds of positive curvature. For example, see [Fr2],[FMR],[FM],[W]. Some papers considered other notions of curvature or generalizations of the totally geodesic condition, see, for example, [BR], [BRT], [KX].

In our results below, no curvature condition is needed.

Let S be some embedded submanifold of a Riemmanian manifold M. Let $T^{\perp}S$ be the normal bundle of S and $T_1^{\perp}S$ the unit normal bundle of S.

We recall that an open subset $W \subset M$ is an ϵ-tubular neighborhood of S if $W = \exp^{\perp}(\tilde{W})$, where $\tilde{W} = \{(x,v) \in T^{\perp}S \mid |v| < \epsilon\}$ and $\exp^{\perp}|_{\tilde{W}}$ is a diffeomorphism. Similarly we could define a closed ϵ-tubular neighborhood.

We say that a subset V of M is a tubular neighborhood of S when $V = \exp^{\perp}(\tilde{V})$, where $\exp^{\perp}|_{\tilde{V}}$ is a diffeomorphism and $\tilde{V}$ is a submanifold (possibly with boundary) of $T^{\perp}S$ with maximal dimension and the following property: if $(p,v) \in \tilde{V}$ then $(p,tv) \in \tilde{V}$ for all $t \in [0,1]$.

Theorem 3 *Let $f : \Sigma \to M$ be a codimension one totally geodesic immersion of a connected closed manifold Σ in a complete connected Riemannian manifold M, and $g : N \to M$ an immersion without focal points of a connected closed manifold N of codimension at least 3. If Σ and N have finite fundamental groups then M is a noncompact manifold with finite fundamental group.*

Furthermore, if M is simply connected then the following conclusions hold:

1. *f and g are embeddings;*
2. *M is diffeomorphic to the normal bundle $T^{\perp}N$;*
3. *N and Σ are simply-connected and Σ is diffeomorphic to the unit normal bundle $T_1^{\perp}N$;*
4. *$f(\Sigma)$ is the boundary of a compact tubular neighborhood of $g(N)$.*

To state our second result we need the following definition.

Definition 3 *Let S, X be contained in a manifold M, where S is an embedded submanifold. We say that X is a normal graph over S if there exists a homeomorphism $h : S \to X$ such that for any point $x \in S$, the image $h(x)$ is connected to x by a unique minimal geodesic, and this unique geodesic is orthogonal to S.*

Theorem 4 *Let Σ be a connected properly embedded totally geodesic hypersurface in a complete simply connected Riemannian manifold M and $g : N \to M$ an immersion without focal points of a closed manifold N. Assume that $\Sigma \cap (g(N)) = \emptyset$. Then we have:*

1. *g is an embedding;*
2. *Σ is a normal graph over an open subset of the boundary of a closed ϵ-tubular neighborhood of $g(N)$ for some small $\epsilon > 0$;*

3. *Σ is the smooth boundary of a tubular neighborhood (possibly noncompact) of $g(N)$.*

In the second section we prove an extension of the Hopf-Rinow Theorem, which will be needed in the following sections. In section 3 we prove Theorem 1.2 and in section 4 we prove Theorem 1.1. Both theorems admit technical extensions, which appear in sections 3 and 4.

2.2 An extension of the Hopf-Rinow Theorem

We present below a generalization of the Hopf-Rinow Theorem.

Theorem 5 *Let $g : N \to M$ be an immersion of a manifold N in a Riemannian manifold M with $g(N)$ compact. Assume that $\exp^{\perp}$ is defined everywhere in $T^{\perp}N$. Then M is complete.*

The proof of Theorem 5 is very similar to that of the Hopf-Rinow Theorem (see for example [dC]). We begin the proof with the following

Lemma 1 *Under the hypotheses of Theorem 5, for all $p \in M$, there exists a geodesic γ joining p to $g(N)$ and satisfying that $L(\gamma) = d(p, g(N))$.*

Prova. Observe that we can assume that $p \notin g(N)$. Since $g(N)$ is compact there exists a point $q \in N$ such that $d(p, g(N)) = d(p, g(q))$. Choose $\delta > 0$ so that the geodesic ball $B_{2\delta} = B_{2\delta}(g(q))$ is a normal ball of M and $p \notin B_{2\delta}$. Let $\widetilde{p} \in \partial B_\delta$ such that $d(p, \partial B_\delta) = d(p, \widetilde{p})$.

Claim. $d(\widetilde{p}, g(N)) = d(\widetilde{p}, g(q)) = \delta$. To see this we consider $\epsilon > 0$. Take a continuous rectifiable curve $c : [0,1] \to M$ with $c(0) = p$, $c(1) = g(q)$, and $L(c) < d(p, g(N)) + \epsilon$. Take $t \in (0,1)$ such that $c(t) \in \partial B_\delta$. Then we have

$$d(p, \widetilde{p}) + \delta \leq L(c|_{[0,t]}) + L(c|_{[t,1]}) = L(c) < d(p, g(N)) + \epsilon.$$

By making $\epsilon \to 0$, we obtain

$$d(p, g(N)) \geq d\,(p, \widetilde{p}) + \delta.$$

Since

$$d(\widetilde{p}, g(N)) \leq d(\widetilde{p}, g(q)) = \delta,$$

we have

$$d(p, g(N)) \geq d(p, \widetilde{p}) + \delta \geq d(p, \widetilde{p}) + d(\widetilde{p}, g(N)) \geq d(p, g(N)),$$

hence $d(\widetilde{p}, g(N)) = \delta = d(\widetilde{p}, g(q))$, which proves this claim. ■

There exists a unique unit vector $v \in T_{g(q)}M$ such that $\widetilde{p} = \exp_{g(q)} \delta v$. By Claim 2.1 the minimal geodesic joining $g(q)$ and $\widetilde{p}$ must be orthogonal to $g(N)$. Thus we have $\widetilde{p} = \exp^{\perp}(g(q), \delta v)$. Now we have by hypothesis that $\gamma(t) = \exp^{\perp}(q, t\delta v)$ is well defined for all $t \geq 0$.

Claim. $p = \gamma(r)$, where $r = d(p, g(N))$.
Prova. Define the set

$$\mathcal{A} = \{s \in [0, r] \mid d(\gamma(s), g(q)) = r - s\}.$$

Notice that $\mathcal{A}$ is closed in $[0, r]$ and $0 \in \mathcal{A}$. Since $\sup \mathcal{A} \in \mathcal{A}$, to prove that $\mathcal{A} = [0, r]$ it suffices to prove that if $s_0 \in \mathcal{A}$ and $s_0 < r$ then there exists $\delta' > 0$ sufficiently small so that $\gamma(s_0 + \delta') \in \mathcal{A}$. In fact, set $s_0 \in \mathcal{A}$, with $s_0 < r$. Let $B' = B_{\delta'}(\gamma(s_0))$ be a normal ball of M centered at $\gamma(s_0)$. Consider $S' = \partial B'$ and choose $x_0 \in S'$ so that $d(p, x_0') = d(p, S')$. We claim that $x_0' = \gamma(s_0 + \delta')$. In fact, using that $r - s_0 = d(\gamma(s_0), p) = \delta' + d(p, S') = \delta' + d(x_0', p)$, by the triangular inequality, it follows that

$$d(q, x_0') \geq d(q, p) - d(p, x_0') = r - (r - s_0 - \delta') = s_0 + \delta'.$$

On the other hand, the branched geodesic joining the geodesic segments: q to $\gamma(s_0)$, by the geodesic γ, and $\gamma(s_0)$ to x_0', by a geodesic ray, has length equal to $s_0 + \delta'$. Thus, this branched geodesic is actually a geodesic; hence, $\gamma(s_0 + \delta') = x_0'$. To finalize the proofs of Claim 2.2 and Lemma 1 notice that

$$r - s_0 = d(\gamma(s_0), p) = \delta' + d(\gamma(s_0 + \delta'), p)$$

which implies that $s_0 + \delta' \in \mathcal{A}$. Therefore, $\mathcal{A} = [0, r]$, which implies that γ is a minimizing geodesic joining q to p. ■

The proof of Lemma 1 follows as a consequence of Claim 2.2. ■

Using Hopf-Rinow's Theorem, Theorem 5 it will be proved if we can prove that the closed and bounded subsets of M are compact. In fact, first we claim the following

Claim. The subset $B(g(N), R) := \{p \in M \mid d(p, g(N)) \leq R\}$ coincides with the image $\exp^{\perp}\left(T_R^{\perp} N\right)$, where $T_R^{\perp} N = \left\{(q, v) \in T^{\perp} N \mid \|v\|_q \leq R\right\}$.
Prova. The inclusion $\exp^{\perp}\left(T_R^{\perp} N\right) \subseteq B(g(N), R)$ is trivial. Now, by Lemma 1, for any $x \in B(g(N), R)$, there exists a geodesic $\gamma(t)$, $t \in [0, 1]$, joining

$g(N)$ to x, such that $L(\gamma) = d(x, g(N))$. Thus $\gamma(t) = \exp^\perp(q, tv)$, for some point $(q, v) \in T^\perp N$, which implies that $x \in \exp^\perp(T^\perp_R N)$ since $\|v\|_q = L(\gamma) = d(x, g(N)) \leq R$. ■

Let B be a closed and bounded subset of M. Then there exists $R > 0$ such that $B \subseteq B(g(N), R) = \exp^\perp\left(T^\perp_R N\right)$. We claim that $\exp^\perp(T^\perp_R N)$ is compact. In fact, let $p_n = \exp^\perp(q_n, v_n)$ be a sequence with $(q_n, v_n) \in T^\perp N$. Since $g(N)$ is compact, there exists $q \in N$ and a subsequence $g(q_{n'})$ that converges to $g(q)$. Let $U \subset N$ be a neighborhood of q whose closure $\overline{U}$ is a compact manifold and the restriction $g|_U$ is an embedding. Then the set

$$T^\perp_R \overline{U} := \left\{(q, v) \in T^\perp N \mid q \in \overline{U} \text{ and } \|v\|_{g(q)} \leq R\right\}$$

is compact since it is a bundle with basis and fibres compact. Since $p_{n'} \in \exp^\perp T^\perp_R \overline{U}$, for all n' sufficiently large, it follows that there exists a subsequence of $p_{n'}$ that converges to a point of $\exp^\perp T^\perp_R \overline{U}$, which proves that $\exp^\perp(T^\perp_R N)$ is compact. Therefore, by Claim 2.2, it follows that B is compact since it is a subset of a compact set.

Remark. If N is compact then $T^\perp_R N$ is compact since it is a bundle with basis and fibres compact and, therefore, $\exp^\perp(T^\perp_R N)$ is compact. ■

Corollary 1 *Let $g : N \to M$ be an immersion without focal points of a compact manifold N in a complete simply connected Riemannian manifold M. The following statements hold:*

(i) $\exp^\perp : T^\perp N \to M$ *is a diffeomorphism;*

(ii) g *is an embedding*

(iii) N *is simply-connected.*

Prova. Since $g : N \to M$ is without focal points and M is complete it follows that $\exp^\perp : T^\perp N \to M$ is a local diffeomorphism. Consider $\widetilde{M}$ the manifold $T^\perp N$ endowed with the induced metric by $\exp^\perp$.

Claim. $\widetilde{M}$ is complete.
Prova. We will prove that the geodesics of $\widetilde{M}$ that start orthogonally from N are well defined for any positive time. In fact, let $\widetilde{\gamma} : [0, \delta) \to \widetilde{M}$ be a geodesic of $\widetilde{M}$, defined in a small interval $[0, \delta)$, satisfying that $q = \widetilde{\gamma}(0) \in N$ and $v = \widetilde{\gamma}'(0) \perp N$. Since $\exp^\perp : \widetilde{M} \to M$ is a local isometry we have that $\gamma = \exp^\perp \circ \widetilde{\gamma}$ is a geodesic of M satisfying $\gamma(0) = g(q)$ and $\gamma'(0) = v \perp g(N)$.

Since M is complete it follows from Theorem 5 that $\gamma(t)$ is well defined for all $t \in \mathbb{R}$. Using that $\exp^{\perp}$ is a local isometry and the uniqueness of geodesics it follows that $\widetilde{\gamma}(t)$ is also well defined for all $t \in \mathbb{R}$. Therefore it follows from Theorem 5 that $\widetilde{M}$ is complete. ■ By Claim 2.2 it

follows that $\exp^{\perp} : T^{\perp}N \to M$ is a covering map. Thus, assuming that M is simply-connected, it follows that $\exp^{\perp}$ is a diffeomorphism, which implies that N is simply-connected since $T^{\perp}N$ is contractible to of N. Furthermore g is an embedding since $g(\cdot) = \exp^{\perp}(\,\cdot\,, 0)$. ■

2.3 Proof of Theorem 4

We will prove the following extension of Theorem 4:

Theorem 4'. *Let Σ be a connected properly embedded hypersurface of a complete simply connected Riemannian manifold M and $g : N \to M$ an immersion without focal points of a compact manifold N. Assume that $\Sigma \cap g(N) = \emptyset$, and the geodesics tangent to Σ do not intersect $g(N)$ orthogonally. Then we have:*

1. *g is an embedding;*
2. *Σ is a normal graph over an open subset of the ϵ-tubular neighborhood of $g(N)$, for some small $\epsilon > 0$;*
3. *there exists an open set A such that A and its closure $\overline{A}$ are $g(N)$-starshaped; furthermore, $\overline{A}$ is a manifold that has Σ as its boundary;*
4. *if Σ is compact then $A = C(\Sigma)$.*

Prova. This proof is similar to proof of Theorem 1.2 of [MM]. Let $g : N \to M$ be an immersion without focal points of a compact manifold N is a complete simply connected M. Let $\Sigma \subset M$ be a connected and properly embedded hypersurface satisfying $\Sigma \cap g(N) = \emptyset$ and the geodesics of M that start tangent from Σ do not intersect N orthogonally. Using that $\exp^{\perp} : T^{\perp}N \to M$ is a diffeomorphism, for all $p \in \Sigma$, there exists a unique normalized geodesic $\gamma_p(t)$ joining p to $g(N)$ and satisfying that the length $L(\gamma_p) = d(p, g(N))$. Define the map $F : \Sigma \to T^{\perp}_1 N$ given by $F(p) = (q_p, v_p)$, where $\gamma_p(t) = \exp^{\perp}(q_p, tv_p)$. Since g is an embedding, γ_p is normal to $g(N)$ and transversal to Σ it follows that $F : \Sigma \to F(\Sigma) \subset T^{\perp}_1 N$ is a local diffeomorphism onto its open image

$F(\Sigma) \subset T_1^{\perp}N$. Thus to show that $F : \Sigma \to F(\Sigma)$ is a diffeomorphism it is sufficient to show that F is injective.

Define the set

$$\mathcal{C} = \left\{ p \in \Sigma \;\middle|\; \text{the cardinality } \#\Big(\big[p, g(N)\big] \cap \Sigma\Big) = 1 \right\},$$

where $[p, g(N)]$ is the trace of γ_p. We need to prove that $\mathcal{C} = \Sigma$.

Claim. $\mathcal{C} \neq \emptyset$. In fact, since Σ is properly embedded and N is compact it follows that there exist points $p \in \Sigma$ and $q \in N$ such that $d(p, g(q)) = d\left(\Sigma, g(N)\right)$. Furthermore, since M is complete there exists a minimizing geodesic segment γ joining p to $g(p)$. Notice that γ is normal to $g(N)$ and satisfies $\gamma \cap \Sigma = \{p\}$ since $L(\gamma) = d(g(q), \Sigma)$ and Σ is embedded in M.

Claim. $\Sigma - \mathcal{C}$ is open as a subset of Σ. To prove this we take $x_1 \in \Sigma - \mathcal{C}$. So there exists $x_2 \in \Sigma$ with $x_2 \neq x_1$ and $x_2 \in \big[x_1, g(N)\big]$. In particular, $F(x_1) = F(x_2) = (q, v)$. Since F is a local diffeomorphism and Σ is embedded in M, it is not difficult to see that there exist disjoint neighborhood of x_1 and x_2 in Σ mapped by F onto the same neighborhood of (q, v) in $T_1^{\perp}N$. Thus we conclude that $\Sigma - \mathcal{C}$ is open in Σ.

Claim. $\Sigma - \mathcal{C}$ is closed as a subset of Σ. In fact, take a sequence $x_k \to a \in \Sigma$, with $x_k \in \Sigma - \mathcal{C}$. Since Σ is properly embedded, there exists an open neighborhood U of a in M such that the intersection $\Sigma \cap M$ is connected and the restriction $F|_{U \cap \Sigma}$ is a diffeomorphism onto its open image. By definition of $\mathcal{C}$, for each k, there exists a point $y_k \neq x_k$ with $y_k \in [x_k, g(N)] \cap \Sigma$. Since (x_k) is bounded and N is compact, we have that (y_k) is also bounded. So we can assume by passing to a subsequence that (y_k) converges to some point $b \in [a, g(N)]$. Since Σ is properly embedded we have that $b \in \Sigma$. Since $F|_{U \cap \Sigma}$ is injective we have that $y_k \notin U$; hence $b \neq a$. So we conclude that $a \in \Sigma - \mathcal{C}$.

Thus $\mathcal{C} = \Sigma$ by connectedness of Σ, which proves that $F : \Sigma \to F(\Sigma)$ is a diffeomorphism.

Now we will prove that Σ is the boundary of an open $\perp$-starshaped set with respect to N. Consider the set

$$A := \left\{ z \in M \,\middle|\, \#\Big(\big[z, g(N)\big] \cap \Sigma\Big) = 0 \right\} \tag{2-1}$$

Given $z \in A$, the distance between $[z, g(N)]$ and Σ, since Σ is properly embedded in M. This implies that A is open.

We state that $\overline{A} - A = \Sigma$. In fact, given $p \in \Sigma$ the geodesic $\big[p, g(N)\big] - \{p\} \subset A$; hence $\Sigma \subset \overline{A}$. Clearly $\Sigma \cap A = \emptyset$, hence $\Sigma \subset \overline{A} - A$. Now take $p \in \overline{A} - A$. Assume, by contradiction, that $p \notin \Sigma$. Since $p \notin A$ the geodesic

$\big[p, g(N)\big]$ intersects Σ transversely at a unique point $q \neq p$. Let $(x, v) \in T_1^\perp N$ such that $\exp^\perp(x, t_0 v) = p$, for some $t_0 > 0$. Since F is a diffeomorphism onto its open image there exists a neighborhood U of (x, v) in $T_1^\perp N$ such that the geodesic $t \mapsto \exp^\perp(y, tw)$ meet Σ at a unique point, for all $(y, w) \in U$. Since Σ is properly embedded we can choose U and $\epsilon > 0$ sufficiently small so that $\exp^\perp(y, tw) \notin \Sigma$ if $|t - t_0| < \epsilon$. This define a neighborhood V of p contained in $M - A$, which contradicts $p \in \overline{A}$.

Clearly A and $\overline{A} = A \cup \Sigma$ are $\perp$-starshaped with respect to N. To conclude the proof we need to show that $\overline{A}$ is a manifold with boundary Σ. In fact, take a point $p \in \Sigma$ and let $(x, v) \in T_1^\perp N$ such that $p = \exp^\perp(q, v)$. Again there exists a neighborhood U of (x, v) in $T_1^\perp N$ such that for each $(y, w) \in U$ the geodesic $t \mapsto \exp^\perp(y, tw)$ meet Σ transversely at a unique point $p_{(y,w)} = \exp^\perp(y, t_{(y,w)} w)$. So the time $t_{(y,w)}$ depends smoothly on (y, w). So a small neighborhood of p in $\overline{A}$ can be defined as

$$W = \left\{ \exp^\perp(y, tw) \;\middle|\; (y, w) \in U \text{ and } t_{(y,w)} - \epsilon < t \leq t_{(y,w)} \right\}$$

where $\epsilon > 0$ is small enough. Thus we prove that $\overline{A}$ is a smooth manifold with boundary Σ. ■

2.4 Proof of Theorem 3

Lemma 2 *Let $x : \Sigma \to M$ be a totally geodesic isometric immersion of a closed manifold Σ into a connected simply connected complete manifold M and $g : N \to M$ be an immersion without focal points of a connected closed manifold N. If $\dim N \leq \dim \Sigma - 1$ then $g(N) \cap f(\Sigma) = \emptyset$.*

Prova. Assume that there exists $p \in g(N) \cap f(\Sigma)$ and write $p = g(q_0) = f(p_0)$. Since $\dim f_* (T_{p_0}\Sigma) + \dim \left(T^\perp_{g(q_0)} g(N)\right) \geq \dim M + 1$, we have that

$$\dim \left(f_* (T_{p_0}\Sigma) \cap T^\perp_{g(q_0)} g(N)\right) \geq 1.$$

Then there exists a unit vector $v \in f_*(T_{p_0}\Sigma) \cap T^\perp_{g(q_0)} g(N)$. By Corollary 1, the geodesic $\gamma : [0, \infty) \to M$ with $\gamma(0) = p$ and $\gamma'(0) = v$ is a ray of M. On the other hand, $\gamma \subset f(\Sigma)$ since $\gamma'(0)$ is tangent to $f(\Sigma)$ and f is totally geodesic. This contradicts the compactness of $f(\Sigma)$. ■

Lemma 2 shows that the result below is a extension of Theorem 3:

Theorem 3'. *Let $f : \Sigma \to M$ be a codimension one immersion of a connected closed manifold Σ in a complete connected Riemannian manifold M, and $g : N \to M$ an immersion without focal points of a connected closed manifold N of codimension at least 3. Assume that $f(\Sigma) \cap g(N) = \emptyset$, and the geodesics tangent to $f(\Sigma)$ do not intersect $g(N)$ orthogonally. If Σ and N have finite fundamental groups then M is a noncompact manifold with finite fundamental group.*
Furthermore, if M is simply connected then the following conclusions hold:

1. *f and g are embeddings;*
2. *M is diffeomorphic to the normal bundle $T^{\perp}N$;*
3. *N and Σ are simply-connected and Σ is diffeomorphic to the unit normal bundle $T_1^{\perp}N$;*
4. *there exists a unique open subset $C(\Sigma)$ of M whose closure is a compact manifold with boundary $f(\Sigma)$; the set $C(\Sigma)$ contains $g(N)$.*

Prova. First assume that M is simply-connected. Using that $f(\Sigma) \cap g(N) = \emptyset$, for all $p \in \Sigma$, it follows from Corollary 1 that there exists a unique $(q_p, v_p) \in T^{\perp}N$, with $v_p \neq 0$, satisfying that $f(p) = \exp^{\perp}(q_p, v_p)$.

Claim. The map $\phi : p \in \Sigma \mapsto \left(q_p, \frac{v_p}{|v_p|}\right) \in T_1^{\perp}N$ is a local diffeomorphism.
Prova. Since $v_p \neq 0$ and $\exp^{\perp}$ is a diffeomorphism it follows that ϕ is differentiable. Now, given $(q, v) \in T_1^{\perp}N$, let $t_{(q,v)} > 0$ such that $\exp^{\perp}(q, t_{(q,v)}v)$ touches $f(\Sigma)$ for the first time. Since the intersection of the geodesics of M that start from $T_1^{\perp}N$ intersect $f(\Sigma)$ transversely it follows that the map

$$\psi : (q, v) \in T_1^{\perp}N \mapsto \exp^{\perp}(q, t_{(q,v)}v)$$

is a local diffeomorphism. Fix $p \in \Sigma$ and $(q, v) \in T^{\perp}N$ such that $f(p) = \exp^{\perp}(q, v)$. Let $U \subset \Sigma$ and $V \subset T_1^{\perp}N$ neighborhood of p and $(q, \frac{v}{|v|})$, respectively, such that $f|_U$ is an embedding and the restriction $\psi|_V : V \to f(U)$ is a diffeomorphism. Then for each $(q, v) \in V$ there exists a unique $p \in U$ such that $f(p) = \exp^{\perp}(q, t_{(q,v)}v)$. This implies that $(q, t_{(q,v)}v) = (q_p, v_p)$; hence $\phi \circ \psi(q, v) = (q, v)$. Thus ϕ is a local inverse for ψ, which proves this claim. ■

The fibres $S_q^{\perp} = \left\{(q, v) \in T_q^{\perp}N \mid |v|_q = 1\right\}$ of the unit normal bundle $T_1^{\perp}N$ are simply-connected since they are homeomorphic to the standard unit sphere $S^{n-k\geq 2} \subset \mathbb{R}^{n-k+1}$. Consider the inclusion map $i : (q, v) \in S_q^{\perp} \mapsto (q, v) \in T_1^{\perp}N$ and the projection $\pi : (q, v) \in T_1^{\perp}N \mapsto q \in N$. Since N is simply-connected and the sequence

$$S_q^{\perp} \xrightarrow{\ i\ } T_1^{\perp}N \xrightarrow{\ \pi\ } N, \tag{2-2}$$

is exact it follows that $T_1^{\perp}N$ is simply-connected. Therefore ϕ is a diffeomorphism since $T_1^{\perp}N$ is compact.

Consider the map $F : T^{\perp}N - N \to T_1^{\perp}N$ given by $F(q,v) = \left(q, \frac{v}{|v|}\right)$. Then since $\phi = F \circ (\exp^{\perp})^{-1} \circ f$ is a diffeomorphism it follows that f is an embedding. Thus by Theorem 4' there exists an open set $C(\Sigma)$ whose closure $\overline{C}(\Sigma)$ is a manifold being $f(\Sigma)$ as its boundary; furthermore $C(\Sigma)$ and $\overline{C}(\Sigma)$ are $g(N)$-starshaped. To prove that $\overline{C}(\Sigma)$ is compact notice that, by (2-1),

$$C(\Sigma) = \left\{ z \in M \;\middle|\; \#\Big(\big[z, g(N)\big] \cap \Sigma\Big) = 0 \right\}$$

Furthermore every geodesic starting normal to $g(N)$ intersects $f(\Sigma)$ at a unique point since ϕ is a diffeomorphism. Thus the points of $\overline{C}(\Sigma)$ are contained in the geodesic segments $\big[f(p), g(N)\big]$, where $p \in \Sigma$. Since Σ and N are compact it follows that $\overline{C}(\Sigma)$ is contained in the ball $B(g(N), R)$, where $R = \max_{p \in \Sigma} d\big(g(N), f(p)\big)$. Therefore $\overline{C}(\Sigma)$ is compact since it is a closed and bounded subset of the complete manifold M.

Now, assume that Σ and N have finite fundamental groups. We will prove that M is noncompact and has finite fundamental group. In fact, let $\pi : \widetilde{M} \to M$ be the universal covering of M with the induced metric. Let $\nu : \widetilde{\Sigma} \to \Sigma$ be the universal covering of Σ and let $\mu : \widetilde{N} \to N$ be the universal covering of N. By the Fundamental Lifting Theorem, given $p \in \Sigma$, $\bar{p} \in \nu^{-1}(p)$ and $\widetilde{p} \in \pi^{-1}(f(p))$ there exits a immersion $\widetilde{f} : \widetilde{\Sigma} \to \widetilde{M}$, namely a lifting of f, satisfying that $\widetilde{f}(\bar{p}) = \widetilde{p}$ and $f \circ \nu = \pi \circ \widetilde{f}$. In the same way, for any $q \in N$, $\bar{q} \in \mu^{-1}(g(q))$ and $\widetilde{q} \in \pi^{-1}(g(q))$ there exists an immersion $\widetilde{g} : \widetilde{N} \to N$ satisfying that $\widetilde{g}(\bar{q}) = \tilde{q}$ and $g \circ \mu = \pi \circ \widetilde{g}$. Then the diagram below commutes.

$$\begin{array}{ccccc} \widetilde{\Sigma} & \xrightarrow{\ \widetilde{f}\ } & \widetilde{M} & \xleftarrow{\ \widetilde{g}\ } & \widetilde{N} \\ \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \mu} \\ \Sigma & \xrightarrow{\ f\ } & M & \xleftarrow{\ g\ } & N \end{array} \tag{2-3}$$

Consider on $\widetilde{N}$ the induced metric. Then $\widetilde{g} : \widetilde{N} \to \widetilde{M}$ is a isometric immersion without focal points and thus it follows by Corollary 1 that $\exp^{\perp} : T^{\perp}\widetilde{N} \to \widetilde{M}$ is a diffeomorphism since $\widetilde{M}$ is complete and simply-connected.

Claim. Let $\widetilde{f} : \widetilde{\Sigma} \to \widetilde{M}$ and $\widetilde{g} : \widetilde{N} \to \widetilde{M}$ be any lifting of f and g, respectively. Then the image of $\widetilde{f}$ and $\widetilde{g}$ are disjoint; moreover, the geodesics of $\widetilde{M}$ that start tangentially from $\widetilde{f}(\widetilde{\Sigma})$ do not intersect $\widetilde{g}(\widetilde{N})$ orthogonally.

Prova. Since $f(\Sigma) \cap g(N) = \emptyset$, using diagram (2-3), it follows that $\widetilde{f}(\widetilde{\Sigma}) \cap \widetilde{g}(\widetilde{N}) = \emptyset$. Now, by contradiction, assume that there exist $\bar{q} \in \widetilde{N}$, $\bar{p} \in \widetilde{\Sigma}$ and a geodesic $\gamma : [0,1] \to \widetilde{M}$ satisfying that $\gamma(0) = \widetilde{g}(\bar{q})$, $\gamma'(0) \perp \widetilde{g}_*(T_{\bar{q}}\widetilde{N})$, $\gamma(1) = \widetilde{f}(\bar{p})$ and $\gamma'(1) \in \widetilde{f}_*(T_{\bar{p}}\widetilde{\Sigma})$. Then since π is a local isometry and using diagram (2-3) it follows that $\beta := \pi \circ \gamma : [0,1] \to M$ is a geodesic of M satisfying $\beta(0) = (\pi \circ \widetilde{g})(\bar{q}) = (g \circ \mu)(\bar{q})$, $\beta'(0) \perp g_*(T_{\mu(\bar{q})}N)$, $\beta(1) = (\pi \circ \widetilde{f})(\bar{p}) = (f \circ \nu)(\bar{p})$ and $\beta'(1) \in f_*(T_{\nu(\bar{p})}\Sigma)$, which contradicts the hypothesis of Theorem 3'. ■

Set $f : \widetilde{\Sigma} \to \widetilde{M}$ a lifting of f. Since N and Σ are compact manifolds with finite fundamental groups it follows that $\widetilde{N}$ and $\widetilde{\Sigma}$ are also compact. Given any lifting of g, $\widetilde{g} : \widetilde{N} \to \widetilde{M}$, it follows from the already proved part of Theorem 3 that $\widetilde{f}$ and $\widetilde{g}$ are embedding, $\widetilde{\Sigma}$ is diffeomorphic to the unit normal bundle $T_1^{\perp}\widetilde{N}$, and there exists a unique compact manifold B with boundary $\partial B = \widetilde{f}(\widetilde{\Sigma})$ (which shows that B depends only $\widetilde{f}$); furthermore B is $\widetilde{g}(\widetilde{N})$-starshaped; in particular $\widetilde{g}(\widetilde{N}) \subset B$. Since for all $\widetilde{q} \in \pi^{-1}(g(q))$ there exists a lifting $\widetilde{g}$ such that $\widetilde{q} \in \widetilde{g}(\widetilde{N})$ we have that $\pi^{-1}(g(q)) \subset B$. This proves that $\pi^{-1}(g(q))$ is finite since it is a discrete subset of a compact set; therefore $\pi_1(M)$ is finite. The finiteness of $\pi(M)$ implies that M is noncompact since $T^{\perp}N$ is noncompact and diffeomorphic to $\widetilde{M}$. ■

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3
A note on normal geodesics of an immersion.

This is a joint work with Sergio Mendonça.

Abstract. Let $f : \Sigma^k \to M^m$ be an immersion of a k-dimensional connected manifold Σ in a Riemannian m-dimensional manifold M and let $\mathcal{C} \subset M$ be a subset. Assume that, for each point $p \in \Sigma$, there exists a geodesic joining $x(p)$ to $\mathcal{C}$, which starts orthogonally to $T_x\Sigma$ and whose length equals the distance $d(x(p), \mathcal{C})$. Then the distance $d(x(p), \mathcal{C})$ is constant. This result is a generalization of a well known fact when $k = m - 1$, $M = \mathbb{R}^n$ and $\mathcal{C}$ is reduced to a point.

3.1
Introduction

A well known result in differential geometry says:

Let $f : \Sigma^n \to \mathbb{R}^{n+1}$ be a hypersurface. Assume that all straight lines starting orthogonally from Σ meet a fixed point x_0. Then Σ is part of a sphere centered at x_0.

Given a subset $\mathcal{C}$ of a Riemannian manifold M let

$$S(\mathcal{C}, r) = \{x \in M \mid d(x, \mathcal{C}) = r\}$$

be the sphere centered at $\mathcal{C}$ and radius r. In this paper, we generalize the above result to a Riemannian ambient. In fact we have:

Theorem 6 *Let Σ be a connected manifold, $f : \Sigma \to M$ an immersion and $\mathcal{C} \subset M$ a subset. For each $p \in \Sigma$, assume that there exists a geodesic $\gamma_p : [0, 1] \to M$ satisfying:*

1. *$\gamma_p(0) = f(p)$, $\gamma_p(1) \in \mathcal{C}$ and the length $L(\gamma_p) = d(f(p), \mathcal{C})$;*
2. *$\gamma_p'(0)$ is orthogonal to the tangent space $df_p(T_p\Sigma)$.*

Then, $f(\Sigma)$ is contained in $S(\mathcal{C}, r)$ where $r = d(f(p), \mathcal{C})$, that is, the distance function $x \mapsto d(f(x), \mathcal{C})$ is constant.

3.2 Proof of Theorem 6

Fix $p_0 \in \Sigma$ and a neighborhood V of p_0 such that $f|_V$ is an embedding. Let $S = f(V)$. Fix $p, q \in S$. Consider a curve $\alpha : [a,b] \to S$ with $\alpha(a) = p, \alpha(b) = q$ parameterized by arc length. Let $\rho : [a,b] \to [0,\infty)$ be given by $\rho(s) = d(\alpha(s), \mathcal{C})$. Since

$$|d(q_1, \mathcal{C}) - d(q_2, \mathcal{C})| \leq d(q_1, q_2),$$

for all $q_1, q_2 \in M$, and α is parameterized by arc length, it follows that

$$|\rho(s) - \rho(t)| \leq d(\alpha(s), \alpha(t)) \leq L(\alpha|_{[s,t]}) = |s - t|.$$

Thus ρ is a Lipschitz function. In particular, ρ is differentiable almost everywhere and ρ satisfies the fundamental theorem of calculus. We choose $s_0 \in (a,b)$ so that $\rho'(s_0)$ exists.

Claim. $\rho'(s_0) = 0$. In fact, choose $0 < \delta < \epsilon$, sufficiently small, so that $I = (s_0 - \delta, s_0 + \delta) \subset (a,b)$ and $\alpha([s_0 - \delta, s_0 + \delta])$ is contained in a totally normal neighborhood $U \subset M$ of $q_0 = \alpha(s_0)$. By hypothesis there exists a geodesic $\gamma : [0,1] \to M$ satisfying

1. $\gamma(0) = q_0$, $\gamma(1) \in \mathcal{C}$ and the length $L(\gamma) = d(q_0, \mathcal{C})$;

2. $\gamma'(0) \perp T_{q_0}\Sigma$.

We choose $0 < t_0 < 1$, so that $\gamma([0,t_0]) \subset U$. Then the map $h : I \times [0,1] \to M$ given by

(i) $h(s,t) = \exp_{\alpha(s)}\left(\frac{t}{t_0}\left(\exp^{-1}_{\alpha(s)} \gamma(t_0)\right)\right)$, for all $s \in I$ and all $t \in [0, t_0]$;

(ii) $h(s,t) = \gamma(t)$, for all $s \in I$ and for all $t_0 \leq t \leq 1$.

satisfies $h(s_0, t) = \gamma(t)$, for all $t \in [0,1]$. In particular we have

$$\left\langle \frac{\partial h}{\partial s}(s_0, 0), \frac{\partial h}{\partial t}(s_0, 0) \right\rangle = 0. \tag{3-1}$$

Consider the curve $h_s : t \mapsto h(s,t)$. Then $\rho(s) \leq L(h_s)$ and $\rho(s_0) = L(h_{s_0})$, since $h_{s_0} = \gamma$ is a geodesic satisfying $L(\gamma) = d(q_0, \mathcal{C})$.

The fact that h_{s_0} is a geodesic, together with $\langle \frac{\partial h}{\partial s}(s_0,0), \frac{\partial h}{\partial t}(s_0,0) \rangle = 0$ and $\frac{\partial h}{\partial s}(s,1) = 0$ we have $\frac{d}{ds}\Big|_{s=s_0} L(h_s) = 0$.

Since ρ has derivative at s_0 we have

$$\rho'(s_0) = \lim_{\substack{s \to s_0 \\ s > s_0}} \frac{\rho(s) - \rho(s_0)}{s - s_0} \leq \lim_{\substack{s \to s_0 \\ s > s_0}} \frac{L(h_s) - L(h_{s_0})}{s - s_0} = \frac{d}{ds}\Big|_{s=s_0} L(h_s) = 0$$

and

$$\rho'(s_0) = \lim_{\substack{s \to s_0 \\ s < s_0}} \frac{\rho(s_0) - \rho(s)}{s_0 - s} \geq \lim_{\substack{s \to s_0 \\ s < s_0}} \frac{L(h_{s_0}) - L(h_s)}{s_0 - s} = \frac{d}{ds}\Big|_{s=s_0} L(h_s) = 0.$$

Thus we have $\rho'(s_0) = 0$, which proves Claim 4.

Therefore, $\rho' = 0$ almost everywhere, which implies by the fundamental theorem of calculus that ρ is constant, hence $d(p, \mathcal{C}) = d(q, \mathcal{C})$. Thus $d(., \mathcal{C})$ is constant on S. Then Theorem 6 follows from the connectedness of Σ. ■

4
Half-space type theorems in warped product spaces with one-dimensional factor.

by Heudson Mirandola, to apear in Geometriae Dedicata.

Abstract. This work states some half-space type theorems in a warped product space of the form $I \times_\rho M$, where $I \subseteq \mathbb{R}$ is an open interval and M is either a compact n-manifold, or a complete simply connected surface with constant curvature $c \leq 0$. Such theorems generalize the classical half-space theorem for minimal surfaces in $\mathbb{R}^3$, obtained by D. Hoffmann and W. Meeks [HM], and recent results for surfaces contained in a slab of $\mathbb{R} \times_\rho M$, obtained by M. Dajczer and L. Alías [AD1].

4.1
introduction

The warped product space [BO], $W = I \times_\rho M$, where $I \subseteq \mathbb{R}$ is an open interval and M is a Riemannian manifold, is the topological product $I \times M$, endowed with the warped product metric:

$$\langle\, ,\rangle = \pi_I^* dt^2 + \rho^2(\pi_I)\pi_M^* g \tag{4-1}$$

where $\rho : I \to (0, \infty)$ is a smooth function, g is the Riemannian metric of M and π_I, π_M are the orthogonal projections of W onto its corresponding factors.

Such Riemannian manifolds include the complete simply connected ones which admit a foliation whose leaves are umbilical hypersurfaces. More specifically, the slices $M_t := \{t\} \times M$ form a foliation of W, by totally umbilical leaves with mean curvature vector

$$H_t = \mathcal{H}(t)T \tag{4-2}$$

where $T = \frac{\partial}{\partial_t}$ and $\mathcal{H}(t) = -(\ln\rho)'(t)$ (see Proposition 2 of [MO]). Furthermore, W has constant curvature κ if only if M has constant curvature c and ρ satisfies: $\rho'' = -\kappa\rho$ and $(\rho')^2 + \kappa\rho^2 = c$ (see [ON] p. 345).

Definition 4 *We call an upper (resp., lower) half-space of W, the subset $\widetilde{I} \times M$, where $\widetilde{I} = I \cap [A, \infty)$ (resp., $\widetilde{I} = I \cap (-\infty, A]$), for some $A \in I$; and a slab of W, the subset $[A, B] \times M$, for some $A, B \in I$.*

Definition 5 *The height function of an immersion $x : \Sigma \to W$ is given by $h = \pi_I \circ x$. Consider $h_{\inf} := \inf_{p \in \Sigma} h(p)$ and $h_{\sup} := \sup_{p \in \Sigma} h(p)$.*

In section 4.3, it will be proved:

Theorem 7 *Let $x : \Sigma^2 \to I \times_\rho \mathbb{R}^2$ be a properly immersed surface and set $\mu \in (0, 1)$. Assume that the image $x(\Sigma^2)$ and the mean curvature H of x satisfy either:*

(i) *$x(\Sigma^2)$ lies in an upper half-space and $|H| \leq \mu \mathcal{H} \circ h$; or*

(ii) *$x(\Sigma^2)$ lies in a lower half-space and $|H| \leq -\mu \mathcal{H} \circ h$.*

Then, $x(\Sigma^2)$ is a totally geodesic slice.

Corollary 2 *Let $x : \Sigma^2 \to I \times_\rho \mathbb{R}^2$ be a properly immersed minimal surface. Assume that the image $x(\Sigma^2)$ and the function ρ satisfy either:*

(i) *$\rho(t)$ is nonincreasing and $x(\Sigma^2)$ lies in an upper half-space; or*

(ii) *$\rho(t)$ is nondecreasing and $x(\Sigma^2)$ lies in a lower half-space.*

Then, $x(\Sigma^2)$ is a totally geodesic slice.

Theorem 7 and Corollary 2 generalize the classical half-space theorem, obtained by D. Hoffman and W. Meeks in [HM], which states that a properly immersed minimal surface contained in a half-space of $\mathbb{R}^3$ is a plane. Furthermore, since a horoball, i.e. the mean convex part of a horosphere, of $\mathbb{H}^3$ can be thought as an upper half-space of $\mathbb{R} \times_{e^{-t}} \mathbb{R}^2$ (see Example 1 of the present paper), another consequence of Theorem 7 is:

Corollary 3 *There is no properly immersed surface Σ with mean curvature satisfying $\sup_{p \in \Sigma} \|H(p)\| < 1$ and image contained in a horoball of $\mathbb{H}^3$.*

Remark. This corollary is also a consequence of Theorem 1.1 of [M], which states that a hypersurface Σ^n properly immersed in $\mathbb{H}^{n+1}$ with mean curvature satisfying $\sup_{p \in \Sigma^n} |H(p)| < 1$ has no isolated points in its asymptotic boundary.

An interesting question is if the horospheres are the only surfaces properly immersed in a horoball of $\mathbb{H}^3$ and mean curvature $|H| \leq 1$. The answer of this question will be positive if we can prove that a surface satisfying the hypothesis of Theorem 7 with $\mu = 1$ must be a slice. Recently, L. Alías and M. Dajczer proved that a properly immersed surface with constant mean curvature $|H| \leq 1$ and image contained between two equidistant horospheres of $\mathbb{H}^3$ is a horosphere (see Theorem 4 of [AD1]). Furthermore, L. Rodrígues and H. Rosenberg proved that a properly embedded surface with constant mean curvature $H = 1$ and image contained in a horoball of $\mathbb{H}^3$ is also a horosphere (see Theorem 1 of [RR]). In Example 2 of the present paper, we will give an example of a (rotational) properly embedded hypersurface $\Sigma^{n\geq 3} \subset \mathbb{H}^{n+1}$ with mean curvature $|H| < 1$ and contained between two equidistant horospheres, which shows the negative answer to this question in higher dimensions.

Our next theorem, proved in section 4.3, states the following:

Theorem 8 *Let $x : \Sigma^2 \to I \times_\rho \mathbb{H}^2$ be a properly immersed surface; set $\mu \in (0,1)$. Assume that the image and mean curvature H of x satisfy either:*

(i) *$x(\Sigma^2)$ is contained in an upper half-space and $|H| \leq \mu\mathcal{H}(h_{\inf})$; or*

(ii) *$x(\Sigma^2)$ is contained in a lower half-space and $|H| \leq -\mu\mathcal{H}(h_{\sup})$.*

Then, x is minimal. Moreover, considering $h_* = h_{\inf}$, if (i) occur, or $h_* = h_{\sup}$, if (ii) occur, the slice $\{h_*\} \times M^2(c)$ is totally geodesic.

Corollary 4 *There is no properly immersed surface $x : \Sigma^2 \to I \times_\rho \mathbb{H}^2$ satisfying either:*

(i) *$x(\Sigma^2)$ lies in an upper half-space and* $\sup|H| < \inf \mathcal{H} \circ h$*; or*

(ii) *$x(\Sigma^2)$ lies in a lower half-space and* $\sup|H| < \inf(-\mathcal{H} \circ h)$.

The inequality in Theorem 8 (and Corollary 4) must be strict, since B. Nelli and H. Rosenberg showed that a catenoid of $\mathbb{R} \times \mathbb{H}^2$ is contained in a slab (Theorem 1 of [NR]). Furthermore, let M^2 be a complete surface with nonnegative Gaussian curvature, satisfying that the geodesic curvature of all its geodesic circles (from some fixed point) of radius at last one is bounded by a constant. Then, the same version of Corollary 4 for the warped product $\mathbb{R} \times_\rho M^2$, was proved by Alías and Dajczer in [AD1]. Notice that, in the special case that $M^2 = \mathbb{R}^2$, this result also follows as a consequence of Theorem 7.

Alías and Dajczer also proved that if a warped product $\mathbb{R} \times_\rho M$ admits a compact hypersurface Σ with mean curvature satisfying either:

$$|H| \leq \mathcal{H} \circ h, \text{ or } |H| \leq -\mathcal{H} \circ h$$

then M is compact and Σ is a slice (see Proposition 2 of [AD2]). In section 4.3, it will be proved the following:

Theorem 9 *Let $x : \Sigma \to I \times_\rho M$ be a properly immersed hypersurface, with M compact. Assume that the image $x(\Sigma)$ and the mean curvature H of x satisfy either:*

(i) *$x(\Sigma)$ lies in an upper half-space and $|H| \leq \mathcal{H} \circ h$; or*

(ii) *$x(\Sigma)$ lies in a lower half-space and $|H| \leq -\mathcal{H} \circ h$.*

Then, $x(\Sigma)$ is a slice.

Remark. Theorem 9, assuming M complete and Σ parabolic, was proved by Alías and Dajczer (see Proposition 2.12 of [AD3]).

4.2 Preliminaries

It is proved in section 2.3 of [AD2] that a warped product $I \times_\rho M$ is conformal to a product space $J \times M$, where J is the image of the function $s : I \to J$ given by

$$s(t) = s_0 + \int_0^t \rho^{-1}(\gamma) d\gamma \tag{4-3}$$

for any $s_0 \in \mathbb{R}$ fixed. More specifically, the map

$$\tau : I \times_\rho M \to J \times M \tag{4-4}$$

defined by $\tau(t, x) = (s(t), x)$ is a conformal diffeomorphism and its pull-back τ^* satisfies:

$$\tau^*\Big((\lambda \circ \pi_J)^2 \big(ds^2 + g\big)\Big) = \langle\,,\rangle \tag{4-5}$$

where $\lambda = \rho \circ s^{-1}$ and g is the Riemannian metric of M.

Example 1 *Given a horoball $B \subset \mathbb{H}^{n+1}$, it is a well known fact that $\mathbb{H}^{n+1}$ admits a half-space model of the form $(0, \infty) \times \mathbb{R}^n$, where B coincides with an upper half-space. On the other hand, the map $\Psi : \mathbb{R} \times_{e^{-t}} \mathbb{R}^n \to \mathbb{H}^{n+1}$ given by $\Psi(t, x) = (e^t, x)$ is an isometry (see (4-5)). Therefore, $\mathbb{H}^{n+1}$ admits a warped product model, $\mathbb{H}^{n+1} = \mathbb{R} \times_{e^{-t}} \mathbb{R}^n$, in which has B as an upper half-space.*

Let $x : (\Sigma, g) \to I \times_\rho M$ be an isometric immersion and consider $h = \pi_I \circ x$ its height function. Set N a unit normal vector field. Then

$$y := \tau \circ x : \Sigma \to J \times M \tag{4-6}$$

is an immersion whose induced metric, $\tilde{g} = y^*(ds^2 + g)$, satisfies: $\tilde{g} = (\rho \circ h)^{-2} g$. Furthermore, the vector field $\widetilde{N} = \tau_*\big((\rho \circ h) N\big)$ is unit and normal to y.

Denote by $\langle\!\langle\, ,\rangle\!\rangle = ds^2 + g$ (the product metric of $J \times M$) and $\widetilde{T} = \partial/\partial_s$ (the vector field tangent to the first factor of $J \times M$). Then, (4-5) implies that:

$$\left\langle\!\!\left\langle \widetilde{N}, \widetilde{T} \circ y \right\rangle\!\!\right\rangle = \langle N, T \circ x \rangle\,, \tag{4-7}$$

since $\widetilde{T} \circ y = \tau_*\big((\rho \circ h)\, T \circ x\big)$.

Proposition 1 *Let A, $\widetilde{A}$ be the shape operators of the isometric immersions $x : (\Sigma, g) \to I \times_\rho M$ and $y : (\Sigma, \tilde{g}) \to J \times M$, relative to the unit normal vectors N and $\widetilde{N}$ (as given above), respectively. Then*

$$\widetilde{A} = (\rho \circ h)\big(A - (\mathcal{H} \circ h)\langle T \circ x, N\rangle I\big), \tag{4-8}$$

where $I : p \in \Sigma \mapsto I_p =$ Identity operator of $T_p\Sigma$.

Prova. Set $\{v_1, \ldots, v_n\}$ an orthonormal frame of Σ. Let $\{a_1, \ldots, a_{n+1}\}$ be an orthonormal frame of $I \times_\rho M$ adapted to x, where $a_i \circ x = x_* v_i$, for all $i = 1, \ldots, n$ and $N = a_{n+1} \circ x$ is normal to x. Furthermore, consider $\{\theta_1, \ldots, \theta_{n+1}\}$ its associated coframe. Then

$$\{e_i = \tau_*\big((\rho \circ \pi_I) a_i\big) \mid i = 1, \ldots, n+1\}$$

is an orthonormal frame of $J \times M$, adapted to y. Notice that $e_i \circ y = y_*\big((\rho \circ h) v_i\big)$, for all $i = 1, \ldots, n$ and $\widetilde{N} = e_{n+1} \circ y = \tau_*\big((\rho \circ h) N\big)$ is unit and normal to y. Consider

$$\tau^* w_i = (\rho \circ \pi_I)^{-1} \theta_i$$

for $i = 1, \ldots, n+1$. Then $\{w_1, \ldots, w_{n+1}\}$ is the associated coframe of $\{e_1, \ldots, e_{n+1}\}$. Thus, for all $j = 1, \ldots, n+1$, we have

$$\begin{aligned} \tau^* dw_j &= \sum_{k=1}^{n+1} \left(-\frac{\rho'(\pi_I)}{\rho^2(\pi_I)} \langle T, a_k \rangle \theta_k \wedge \theta_j + \rho^{-1}(\pi_I) \theta_k \wedge \theta_{kj} \right) \\ &= \sum_{i=1}^{n+1} \tau^* w_k \wedge \left(\theta_{kj} - \frac{\rho'(\pi_I)}{\rho(\pi_I)} \Big(\langle T, a_k \rangle \theta_j - \langle T, a_j \rangle \theta_k \Big) \right) \end{aligned} \tag{4-9}$$

where θ_{kj} are the connection forms of (Σ^n, g). Consider w_{kj} the connection forms of $(\Sigma^n, \tilde{g})$. Then (4-9) implies that

$$\tau^* w_{kj} = \left(\theta_{kj} - \frac{\rho'(\pi_I)}{\rho(\pi_I)} \Big(\langle T, a_k \rangle \theta_j - \langle T, a_j \rangle \theta_k \Big) \right) \tag{4-10}$$

Thus, using that $y = \tau \circ x$, we have

$$y^* w_{kj} = x^* \left(\theta_{kj} - (\ln \circ \rho)'(\pi_I) \Big(\langle T, a_k \rangle \theta_j - \langle T, a_j \rangle \theta_k \Big) \right). \tag{4-11}$$

Notice that $y^*w_i = \rho^{-1}(h)x^*\theta_i$ and $x^*\theta_{n+1} = 0$. Furthermore, write $x^*\theta_{k\,n+1} = \sum_{i=1}^n \alpha_{ij}x^*\theta_j$ and $y^*w_{k\,n+1} = \sum_{i=1}^n \tilde{\alpha}_{ij}y^*w_j$, where α_{ij}, $\tilde{\alpha}_{ij}$ are the coefficients of the second fundamental forms of $x : (\Sigma^n, g) \to I \times_\rho M$ and $y : (\Sigma, \tilde{g} = (\rho \circ h)^{-2}g) \to J \times M$, relative to the unit normal vectors N and $\widetilde{N}$, respectively. Then, (4-11) implies that

$$\sum_{l=1}^n \tilde{\alpha}_{kl}\rho^{-1}(h)x^*\theta_l = \sum_{l=1}^n \alpha_{kl}x^*\theta_l + \frac{\rho'(h)}{\rho(h)}\langle T \circ x, N\rangle x^*\theta_k \qquad (4\text{-}12)$$

Thus,

$$\tilde{\alpha}_{kl} = (\rho \circ h)\left(\alpha_{kl} + (\ln \circ \rho)'(h)\langle T \circ x, N\rangle \delta_{kl}\right), \qquad (4\text{-}13)$$

and therefore,

$$\begin{aligned}
\left\langle x_*(\widetilde{A}v_k), x_*v_l \right\rangle &= g\left(\widetilde{A}v_k, v_l\right) = (\rho \circ h)^2 \widetilde{g}\left(\widetilde{A}v_k, v_l\right) \\
&= \left\langle\!\left\langle y_*\left(\widetilde{A}((\rho \circ h)v_k)\right), y_*((\rho \circ h)v_l)\right\rangle\!\right\rangle = \widetilde{\alpha}_{kl} \\
&= \rho \circ h\left(\alpha_{kl} + (\ln \circ \rho)'(h)\langle T \circ x, N\rangle \delta_{kl}\right) \\
&= \rho \circ h\Big(\Big\langle x_*(Av_k) - (\mathcal{H} \circ h)\left\langle T \circ x, N\right\rangle x_*v_k, x_*v_l\Big\rangle\Big),
\end{aligned}$$

for all $k, l = 1, \ldots, n$, which proves the proposition. ■

Corollary 5 *Let H, $\widetilde{H}$ be the mean curvatures of of the isometric immersions $x : (\Sigma, g) \to I \times_\rho M$ and $y : (\Sigma, \tilde{g}) \to J \times M$, in the directions of the unit normal vectors N and $\widetilde{N}$ (as given above), respectively. Then, we have:*

$$\widetilde{H} = (\rho \circ h)\big(H - \big(\mathcal{H} \circ h\big)\langle T \circ x, N\rangle\big) \qquad (4\text{-}14)$$

Corollary 6 *An isometric immersion $x : (\Sigma, g) \to I \times_\rho M$ is umbilical with respect to a unit normal vector N (resp., totally umbilical) if only if the isometric immersion $y : (\Sigma, \widetilde{g}) \to J \times M$ (as given above) is umbilical with respect to the unit norma vector $\widetilde{N} = \tau_*((\rho \circ h)N)$ (resp., totally umbilical).*

Lemma 3 *If $x : (\Sigma, g) \to I \times_\rho M$ is a proper immersion then $y := \tau \circ x : (\Sigma, \tilde{g}) \to J \times M$ is also a proper immersion.*

Prova. let C be a compact set of $J \times M$. Then

$$K = y^{-1}(C) = x^{-1}(\tau^{-1}(C))$$

is a compact set of Σ, since τ is a diffeomorphism and x is a proper immersion. ■

Example 2 *It is a well known fact that any $(n \geq 3)$-catenoid of $\mathbb{R}^{n+1}$ lies between two parallel hyperplanes. Consider now a $(n \geq 3)$-catenoid $\Sigma \subset \mathbb{R}^{n+1}$ contained between two parallel hyperplanes $P_1 = \{t_1\} \times \mathbb{R}^n$ and $P_2 = \{t_2\} \times \mathbb{R}^n$, with $0 < t_1 < t_2$. On the other hand, the map $\tau : \mathbb{H}^{n+1} = \mathbb{R} \times_{e^{-t}} \mathbb{R}^n \to (0, \infty) \times \mathbb{R}^n$ given by $\tau(t, x) = (e^t, x)$ is a conformal diffeomorphism (see Example 1) for which the planes $\{t\} \times \mathbb{R}^n$, with $t > 0$, are image of equidistant horospheres. Then, the hypersurface*

$$\Sigma' := \tau^{-1}(\Sigma) \subset \mathbb{H}^{n+1}$$

endowed with the induced metric, is properly embedded (by Lemma 3) and lies between two equidistant horospheres. Since Σ is minimal, by Corollary 5, the mean curvature of τ^{-1} is satisfies: $|H| = |\langle T, N \rangle|$, where $T = \frac{\partial}{\partial_t}$ is the vertical vector and N is a unit normal vector of Σ. Thus $|H| < 1$, since T is nowhere tangent to N.

4.3 Proof of Theorems 7, 8 and 9

First, we claim that item **(ii)** of Theorems 7, 8 and 9 may be omitted (without loss of generality) in the proofs of this theorems. In fact, consider $J = \{-t \mid t \in I\}$ and define $\widetilde{\rho}(t) = \rho(-t)$, for all $t \in J$. Then, the map

$$\psi : I \times_\rho M \to J \times_{\widetilde{\rho}} M$$

given by $\psi(t, x) = (-t, x)$ is an isometry which transforms each lower half-space of $I \times_\rho M$ in an upper half-space of $J \times_{\widetilde{\rho}} M$. Each slice M_s of $J \times_{\widetilde{\rho}} M$ is the image by ψ, of the slice M_t of $I \times_\rho M$, with $t = -s$ and the mean curvature of M_s, in the direction of $\frac{\partial}{\partial_s}$, satisfies:

$$\widetilde{\mathcal{H}}(s) = -(\ln \widetilde{\rho})'(s) = (\ln \rho)'(t) = -\mathcal{H}(t).$$

Given an isometric immersion $x : \Sigma \to I \times_\rho M$, let h and $\widetilde{h}$ be the height functions of x and $y = \psi \circ x$, respectively. Then $h_{\inf} = \widetilde{h}_{\sup}$ and $h_{\sup} = \widetilde{h}_{\inf}$. Therefore, if a hypersurface $x : \Sigma \to I \times M$ satisfies item **(ii)** of one of the Theorems 7, 8 and 9, then the hypersurface $\widetilde{x} = \psi \circ x : \Sigma \to J \times_{\widetilde{\rho}} M$ satisfies item **(i)** of the same theorem. Furthermore, $x(\Sigma)$ is a (totally geodesic) slice of $I \times_\rho M$ if only if $\widetilde{x}(\Sigma)$ is also a (totally geodesic) slice of $J \times_{\widetilde{\rho}} M$. This proves this claim.

Let $x : \Sigma \to I \times_\rho M$ be a properly immersed hypersurface, where M is a complete Riemannian manifold. Suppose that $x(\Sigma)$ is contained in an upper half-space. Then, the height function h of x satisfies either:

(a) There exists $p_0 \in \Sigma$ such that $h(p_0) = h_{\inf} \;\; (= \inf_{p\in\Sigma} h(p))$, or

(b) $h > h_{\inf}$

As a consequence of the tangency principle (see this principle, for instance, in Theorem 1.1 of [FS]) it follows:

Claim. Assume that **(a)** occur. If the mean curvature of x satisfies either:

(i) $|H| \leq \mathcal{H} \circ x$, or

(ii) $|H| \leq \mathcal{H}(h_{\inf})$.

Then $x(\Sigma)$ is a slice of $I \times_\rho M$.

Proof. Choose an unit normal vector N of x, defined on a neighborhood of p, so that $N(p_0) = (T \circ x)(p_0)$. First, assume that **(i)** occur. Since $x(\Sigma)$ remains above the slice $\{h_{\inf}\} \times M$ (see definition in [FS]), the tangency principle implies that the $x(\Sigma)$ coincides with $\{h_{\inf}\} \times M$ in a neighborhood of p_0. Then, the set of all points $q \in \Sigma$ such that $x(q) \in \{h_{\inf}\} \times M$ is a nonempty closed and open subset of Σ. By the connectedness and completeness of Σ, we have that $x(\Sigma) = \{h_{\inf}\} \times M$.

Now, assume that **(ii)** occur. Consider the function

$$t \in I \mapsto \alpha(t) = \int_{h_{\inf}}^{t} \rho(\tau) d\tau.$$

In Proposition 1 of [AD3] is proved that the function $p \in \Sigma \mapsto (\alpha \circ h)(p)$ satisfies:

$$\Delta(\alpha \circ h) = n(\rho \circ h)\big(H \langle T \circ x, N\rangle - \mathcal{H} \circ h\big) \tag{4-15}$$

Then $\Delta(\alpha \circ h) \leq 0$. Since $\alpha \circ h \geq 0$ and $(\alpha \circ h)(p_0) = 0$ we have that $(\alpha \circ h)(p) = 0$, for all p in a neighborhood of p_0. This implies that the set of all points $q \in \Sigma$ such that $x(q) \in \{h_{\inf}\} \times M$ is a nonempty closed and open subset of Σ. By the connectedness and completeness of Σ, we have that $x(\Sigma) = \{h_{\inf}\} \times M$. ■

4.3.1
Proof of Theorem 9

By Claim 4, it suffices to prove that **(a)** occur. In fact, let (p_n) be a sequence of Σ such that $\lim h(p_n) = h_{\inf}$. Write $x(p_n) = \big(h(p_n), q_n\big) \in I \times_\rho M$. Since M is compact, after a subsequence, we can assume that (q_n) converges to a point $q_0 \in M$. Since x is proper and

$$\mathcal{C} = \big\{\{h(p_n)\} \cup \{h_{\inf}\}\big\} \times M$$

is a compact set, the set $x^{-1}(\mathcal{C})$ is compact. Thus the sequence $(p_n) \in x^{-1}(\mathcal{C})$ admits a subsequence $(p_{n'})$ convergent to a point $p_0 \in \Sigma$; thus $x(p_{n'})$ converges to $x(p_0)$. This implies that $x(p_0) = (h_{\inf}, q_0)$, which proves that **(a)** occur. ■

4.3.2
Proof of Theorem 7

Let $x : \Sigma \to I \times_\rho \mathbb{R}^2$ be a properly immersed surface such that $x(\Sigma)$ is contained in a upper half-space of $I \times_\rho \mathbb{R}^2$. Set $0 < \mu < 1$ and assume that the mean curvature of x satisfies:

$$|H| \leq \mu \mathcal{H} \circ h.$$

By Claim 4, we may suppose that **(b)** occur. Consider $s(t) = \int_{h_{\inf}}^{t} \frac{1}{\rho(\tau)} d\tau$. Using that $s \circ h = \pi_J \circ y$ and $s'(t) = \rho^{-1}(t) > 0$, we have that $y(\Sigma) \subset [0, \infty) \times \mathbb{H}^2$. Then the height function $\widetilde{h} = \pi_J \circ y$ satisfies:

(b') $\widetilde{h} > 0$ and $\inf_{p \in \Sigma} \widetilde{h}(p) = 0$.

By the properness of y (Lemma 3), there exists $R > 0$, sufficiently small, so that $[-R, R] \subset J$ and the solid cilinder $B = [-R, R] \times \overline{B}_{\mathbb{R}^2}(0, 3R)$ does not intersect $y(\Sigma)$.

Given $0 < \epsilon < R$, consider the family $\{\Sigma_\lambda\}_{\lambda \geq 1}$ of catenoids contained in $J \times \mathbb{R}^2$ defined by

$$\Sigma_\lambda = \left\{ \left(t, 0, \frac{\epsilon}{\lambda} \cosh\left(\frac{\lambda(t - \epsilon)}{\epsilon} \right) \right) \mid t \in (-\epsilon, \epsilon) \right\}$$

Notice that $\Sigma_1 \cap \big((0, \infty) \times \mathbb{R}^2\big) \subset B$ and $\partial \Sigma_\lambda \subset B \cup \{-\epsilon\} \times \mathbb{R}^2$, for all $\lambda \geq 1$. Let N_λ be the unit normal vector of Σ_λ satisfying: $\left\langle\!\left\langle N_\lambda, \widetilde{T} \right\rangle\!\right\rangle > 0$, for all $\lambda \geq 1$. By a straightforward computation, we may choose $\epsilon > 0$, sufficiently small, so that

$$\left\langle\!\left\langle N_\lambda, \widetilde{T} \circ y \right\rangle\!\right\rangle > \mu \tag{4-16}$$

everywhere in $(\Sigma_\lambda \cap ((0,\infty)\times\mathbb{R}^2)) - B$. When $\lambda \to \infty$, we have that Σ_λ converges to the double covering of $\{\epsilon\}\times\mathbb{R}^2$ with a singularity at $(\epsilon, 0)$; thus there exists $\lambda_0 > 1$ such that Σ_{λ_0} touches $y(\Sigma)$ for the first time, say at a point $y(p)$. Let $\widetilde{N}$ be an unit normal vector, defined on an oriented neighborhood U of p, such that $\widetilde{N}(p) = N_{\lambda_0}(y(p))$. By (4-16), we may consider U, sufficiently small, satisfying:

$$\left\langle\!\left\langle \widetilde{N}, \widetilde{T}\circ y \right\rangle\!\right\rangle \geq \mu$$

Thus, by Corollary 5,

$$\begin{aligned} \widetilde{H} &= (\rho\circ h)\left(H - \mathcal{H}\circ h\left\langle\!\left\langle \widetilde{N}, \widetilde{T}\circ y \right\rangle\!\right\rangle\right) \\ &\leq (\rho\circ h)\left(H - \mu\mathcal{H}\circ h\right) \leq 0 \end{aligned} \tag{4-17}$$

Then, since $y(\Sigma)$ remains above Σ_{λ_0}, by the tangency principle, $y(\Sigma)$ coincides with Σ_{λ_0} in a neighborhood of $y(p)$. By the connectedness of Σ and completeness of y, it follows that $y(\Sigma) = \Sigma_{\lambda_0}$, which contradicts **(b')**. ■

4.3.3 Catenoidal type surfaces

Let $\gamma : \mathbb{R} \to \mathbb{H}^2$ be a normalized geodesic of $\mathbb{H}^2$ and let $x_0 = \gamma(0)$. Then the product map $\sigma := \gamma \times \mathrm{id} : (0,\infty)\times\mathbb{R} \to \mathbb{H}^2\times\mathbb{R}$ is a totally geodesic surface. For each $0 < H \leq 1/2$, U. Abresch and H. Rosenberg [AR] exhibited H-surfaces $C_H^2 \subset \mathbb{R}\times\mathbb{H}^2$. Such surfaces have the following properties:

(i) They are properly embedded annulus, with two asymptotic conical ends, given by rotating a strictly concave curve:

$$s \in \mathbb{R} \to \psi_\epsilon^H(s) := \sigma\left(r^H(s), \xi^H(s) + \epsilon\right)$$

around the axis $\ell_0 := \mathbb{R}\times\{x_0\}$;

(ii) the curve $\psi^H(s) = \sigma\left(r^H(s), \xi^H(s)\right)$ has two asymptote with slopes:

$$\frac{d(r^H)}{d\xi} = \pm\tan(\arccos(2H));$$

(iii) the mean curvature vector of C_H^2 points to the connected component containing the axis of rotation ℓ_0;

(iv) the distance $d(\ell_0, \psi_\epsilon^H)$ occur when $\xi^H(s) = 0$;

(v) when H approaches zero, C_H^2 converges to the double covering of $\{\epsilon\}\times\mathbb{H}^n$ with a singularity at (ϵ, x_0).

4.3.4 Proof of Theorem 8

Let $x : \Sigma \to I \times_\rho \mathbb{H}^2$ be a properly immersed surface and $\mu \in (0,1)$. Assume that the mean curvature satisfies:

$$|H| < \mu\mathcal{H}(h_{\inf}) \tag{4-18}$$

and assume that $x(\Sigma)$ is contained in an upper half-space of $I \times_\rho \mathbb{H}^2$. By Claim 4, we may suppose that **(b)** occur. Choose s_0 in (4-3), so that $s(\mathcal{A}) = 0$. Then, using that $s \circ h = \pi_J \circ y$ and $s'(t) = \rho^{-1}(t) > 0$, we have that $y(\Sigma) \subset [0,\infty) \times \mathbb{H}^2$. Then, the height function $\widetilde{h} = \pi_J \circ y$ satisfies:

(b') $\widetilde{h} > 0$ and $\inf_{p\in\Sigma} \widetilde{h}(p) = 0$.

Fix $A > h_{\inf}$ and consider $\rho_0 = \inf \rho|_{[h_{\inf},A]}$. Set $\mu' \in (\mu, 1)$ and choose $0 < \kappa < \frac{1}{2}$ so that $\mu\mathcal{H}(h_{\inf}) < \mu'\big(\mathcal{H}(h_{\inf}) - \frac{\kappa}{\rho_0}\big)$. Then there exists $\delta > 0$ such that $\mu\mathcal{H}(h_{\inf}) < \mu'\left(\mathcal{H}(t) - \frac{\kappa}{\rho_0}\right)$, for all $t \in (h_{inf}, h_{inf} + \delta)$. Then

$$|H(p)| \le \mu'\left((\mathcal{H} \circ h)(p) - \frac{\kappa}{\rho_0}\right), \tag{4-19}$$

for all $p \in \Sigma$ such that $h(p) \in (h_{inf}, h_{inf} + \delta)$.

By the properness of y (Lemma 3), there exists $0 < R < \delta$, sufficiently small, so that $[-R, R] \subset J$ and the solid cylinder $B = [-R,R] \times \overline{B}_{\mathbb{H}^2}(x_0, 3R)$ does not intersect $y(\Sigma)$ and set $0 < \epsilon < R$.

Consider the family $\{\Sigma_\lambda\}_{\lambda>2}$ of surfaces in $J \times \mathbb{H}^2$ defined by:

$$\Sigma_\lambda := C^2_{\lambda^{-1}} \cap \big((-\epsilon,\epsilon) \times \mathbb{H}^2\big) \tag{4-20}$$

oriented by the unit normal vector $N_\lambda = -\lambda\overrightarrow{H}_\lambda$, where $\overrightarrow{H}_\lambda$ is the mean curvature vector of $C^2_{\lambda^{-1}}$. Then Σ_λ has constant mean curvature $H_\lambda = \frac{-1}{\lambda}$ and the vector field N_λ, satisfies: $\left\langle\left\langle N_\lambda, \widetilde{T}\right\rangle\right\rangle > 0$ everywhere.

Set $\lambda_0 = \max\{2, (\kappa\mu)^{-1}\}$, and choose $\epsilon > 0$, sufficiently small, so that the family $\{\Sigma_\lambda\}_{\lambda>2}$ satisfies: $\Sigma_{\lambda_0} \cap ((0,\infty) \times \mathbb{H}^2) \subset B$, and, for all $\lambda \ge \lambda_0$,

$$\left\langle\left\langle N_\lambda, \widetilde{T} \circ y\right\rangle\right\rangle > \mu \tag{4-21}$$

everywhere in $(\Sigma_\lambda \cap ((0,\infty) \times \mathbb{H}^2)) - B$. Then, using that, $\partial\Sigma_\lambda \subset B \cup \big(\{-\epsilon\} \times \mathbb{H}^2\big)$, there exists $\lambda_1 > \lambda_0$ such that Σ_{λ_1} touches $y(\Sigma)$ for the first time, say at a point $y(p)$. By (4-21), choose an unit normal vector field $\widetilde{N}$ of y, defined in an oriented neighborhood U of p, satisfying: $N(p) = N_{\lambda_1}(y(p))$ and

$$\left\langle\left\langle \widetilde{N}, \widetilde{T} \circ y\right\rangle\right\rangle > \mu \tag{4-22}$$

everywhere in U. Therefore, by (4-22) and Corollary 5,

$$\begin{aligned} \widetilde{H} &= (\rho\circ h)\left(H-\mathcal{H}\circ h\left\langle\!\left\langle\widetilde{N},\widetilde{T}\circ y\right\rangle\!\right\rangle\right) \\ &\leq (\rho\circ h)\left(H-\left((\mu')^{-1}|H|+\frac{\kappa}{\rho_0}\right)\left\langle\!\left\langle\widetilde{N},\widetilde{T}\circ y\right\rangle\!\right\rangle\right) \\ &\leq (\rho\circ h)\left(H-\left((\mu')^{-1}|H|+\frac{\kappa}{\rho_0}\right)\mu'\right) \\ &\leq -\kappa\mu' \leq -\frac{1}{\lambda_0} < -\frac{1}{\lambda_1} \end{aligned} \tag{4-23}$$

everywhere in U. By the tangency principle, $y(\Sigma)$ coincides with Σ_{λ_1}, in a neighborhood of $x(p)$, which contradicts (4-23). ∎

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5
The influence of the boundary behavior on isometric immersions into the hyperbolic space.

The results of this chapter have been obtained by the author of this thesis. The Appendix is a work by L. Jorge and F. Vitorio.

Abstract. We prove that a minimal proper isometric immersion whose asymptotic boundary is contained in a sphere reduces codimension. This result is a corollary of a more general one that states a sharp lower bound for the sup-norm of the mean curvature vector of proper isometric immersions in the hyperbolic space whose asymptotic boundary is contained in a sphere. We also prove that if $f : \Sigma^n \to \mathbb{H}^{n+1}$ is proper hypersurface with mean curvature satisfying $\sup_{p\in\Sigma} \|H(p)\| < 1$ then the asymptotic boundary of $f(\Sigma)$ has no isolated point.

5.1
Introduction

This paper studies how the behavior of a proper isometric immersion into the hyperbolic space is influenced by its behavior at infinity. Many works ([dCL], [dCGT], [LR], [Lo]) show, motivated principally by the Alexandrov reflection method, that a properly embedded hypersurface in the hyperbolic space with constant mean curvature inherit the symmetry of its boundary.

The m-dimensional hyperbolic space $\mathbb{H}^m$ carries a natural compactification:

$$\overline{\mathbb{H}}^m = \mathbb{H}^m \cup \mathbf{S}^{m-1}(\infty)$$

where $\mathbf{S}^{m-1}(\infty)$ is identified with the asymptotic classes of geodesic rays in $\mathbb{H}^m$ and carries, in a natural way, the standard conformal structure (isometries of $\mathbb{H}^m$ become conformal automorphisms of $\mathbf{S}^{m-1}(\infty)$). The *asymptotic boundary* of a subset $B \subset \mathbb{H}^m$ is defined by:

$$\partial_\infty B = \overline{B} \cap \mathbf{S}^{m-1}(\infty)$$

where $\overline{B}$ is the closure of B in $\overline{\mathbb{H}}^m$. By a *sphere* in $\mathbf{S}^{m-1}(\infty)$ we will denote the asymptotic boundary of a complete totally geodesic hypersurface of $\mathbb{H}^m$. Notice that, considering $\mathbb{H}^m$ in its Poincaré ball model, $\mathbf{S}^{m-1}(\infty)$ is identified with the standard unit sphere $S_1^{m-1} \subset \mathbb{R}^m$ and the spheres of $\mathbf{S}^{m-1}(\infty)$ by the geodesic spheres of S_1^{m-1}.

Theorem 10 *Let $f : \Sigma \to \mathbb{H}^m$ be a proper isometric immersion of a connected Riemannian manifold Σ into $\mathbb{H}^m$. Assume that $\partial_\infty f(\Sigma)$ is contained in a sphere S and let $\Lambda \subset \mathbb{H}^m$ be the totally geodesic hypersurface such that $\partial_\infty \Lambda = S$. Then the mean curvature H of f satisfies:*

$$\sup_{p\in\Sigma} \|H(p)\| \geq \tanh(d_{\mathbb{H}}(f(p), \Lambda)), \tag{5-1}$$

for all $p \in \Sigma$, where $d_{\mathbb{H}}$ is the hyperbolic distance. Furthermore if, for some $p \in \Sigma$, the equality in (5-1) is satisfied then $f(\Sigma)$ is contained in a totally umbilical hypersurface $\Gamma^{m-1} \subset \mathbb{H}^m$ with mean curvature $\mathcal{H}_\Gamma = \sup_{p\in\Sigma} \|H(p)\|$.

We recall that an immersion $f : \Sigma \to \mathbb{H}^m$ reduces codimension if its image $f(\Sigma)$ is contained in a totally geodesic hypersurface of $\mathbb{H}^m$.

Corollary 7 *Let $f : \Sigma \to \mathbb{H}^m$ be a proper minimal isometric immersion. If the asymptotic boundary $\partial_\infty f(\Sigma)$ is contained in a sphere then f reduces codimension.*

A beautiful theorem by M. do Carmo, J. Gomes and G. Thorbergsson [dCGT] states that a properly embedded hypersurface in $\mathbb{H}^m$ with constant mean curvature $H \in [0, 1)$ has no isolated point in its asymptotic boundary. We generalize this result of the following manner.

Theorem 11 *Let $f : \Sigma^{m-1} \to \mathbb{H}^m$ be a proper immersed hypersurface with mean curvature satisfying $\sup_{p\in\Sigma} |H(p)| < 1$. Then the asymptotic boundary $\partial_\infty f(\Sigma)$ has no isolated points.*

The proof of Theorem 11 uses an important result (Proposition 2) that to be presented we need to recall the concept of distance between two compact sets in $\mathbf{S}^{m-1}(\infty)$, as defined in [dCGT]. First, two subsets A_1, $A_2 \subset \mathbf{S}^{m-1}(\infty)$ are separated by two disjoint spheres S_1, S_2 if they are contained in distinct disk-type connected components of $\mathbf{S}^{m-1}(\infty) - (S_1 \cup S_2)$. Consider $d(S_1, S_2) := d_{\mathbb{H}}(\Lambda_1, \Lambda_2)$, where $\Lambda_i \subset \mathbb{H}^m$, $i = 1, 2$, are the totally geodesic hypersurfaces with $\partial_\infty \Lambda_i = S_i$. The distance $d(A_1, A_2)$ from A_1 to A_2 is defined by:

$$d(A_1, A_2) := \begin{cases} 0, \text{ if there are no disjoint spheres } S_1 \text{ and } S_2 \\ \quad \text{that separate } A_1 \text{ and } A_2; \\ \sup\left\{d(S_1, S_2) \mid S_1 \text{ and } S_2 \text{ separate } A_1 \text{ and } A_2\right\}. \end{cases} \tag{5-2}$$

The distance $d(\cdot,\cdot)$ is conformally invariant, since conformal transformations of $\mathbf{S}^{m-1}(\infty)$ are induced by isometries of $\mathbb{H}^m$. Furthermore, for $m \geq 2$, the distance of a compact set from a point away from this set is infinite. Notice also that if $d(A_1, A_2) < \infty$ then, by compactness, there exist two disjoint sphere S_1 and S_2 satisfying: $d(A_1, A_2) = d(S_1, S_2)$.

Remark. Although M. do Carmo et al. to refer d as a distance, they also observe that the triangle inequality does not hold in general.

In [dCGT], M. do Carmo, J. Gomes and G. Thorbergsson proved:

Theorem A. [Theorem 1 of [dCGT]] *Let $\Sigma^n \subset \mathbb{H}^{n+1}$ be a properly embedded hypersurface with constant mean curvature $H \in [0,1)$. Assume that the asymptotic boundary $\partial_\infty \Sigma$ has at least two connected components and let A be any such component. Then there exists a constant d_H (depending only H, and computable) such that*

$$d(A, \partial_\infty \Sigma - A) \leq d_H \tag{5-3}$$

and the equality holds if only if Σ^n is a rotation hypersurface of spherical type.

In the present paper, we observe that the techniques, developed in [dCGT] to prove Theorem A, are suffice to prove the following result:

Proposition 2 *Let $f : \Sigma^n \to \mathbb{H}^{n+1}$ a properly immersed hypersurface with mean curvature satisfying $\sup_{p\in\Sigma} |H(p)| < 1$. Assume that the asymptotic boundary $\partial_\infty f(\Sigma)$ has at least two connected components and let A be any such component. Then there exists a constant d (depending only $\sup_{p\in\Sigma} |H(p)|$, and computable) such that*

$$d(A, \partial_\infty f(\Sigma) - A) \leq d$$

and the equality holds if only if $f(\Sigma)$ is a rotation hypersurface of spherical type.

Remark. Theorem 10 (consequently, Corollary 7), assuming that f has codimension one, remain true considering, in the place of the mean curvature, any normalized symmetric function of the principal curvatures (namely r-mean curvatures), in particular, escalar and Gauss-Kronecker curvatures. The proof of Theorem 10, in this more general context, differ simply of the use of the following tangency principle which follows as a consequence of Theorem 1.1 of [FS].

Proposition 3 *Let M_1 and M_2 be hypersurfaces of a Riemannian manifold N and a point $p \in M_1 \cap M_2$ satisfying $T_pM_1 = T_pM_2$. Let η be a unit normal* *vector field of* M_2*. Assume that* M_2 *is umbilical and nontotally geodesic and* M_1 *remains above* M_2 *with respect to* $\nu(p)$*. Furthermore, assume that, near* p*, the mean curvature* H *of* M_2 *with respect to* ν *satisfies:*

$$H \geq \min_{1\leq r\leq n} |H_r^1|,$$

where H_r^1*,* $r = 1, \ldots, n$*, are the* r*-mean curvatures of* M_1*. Then, near* p*,* M_1 *coincides with* M_2*.*

5.2 Proofs of Proposition 2 and Theorem 11

The steps of the proof of Proposition 2 are inspired in the proof of Theorem A (see [dCGT]). Assume that

$$d(A, \partial_\infty f(\Sigma) - A) > d, \tag{5-4}$$

where d depends only $\sup_{p\in\Sigma} |H(p)|$ and it will be given soon. Then we will derive a contradiction. In fact, set $B = \partial_\infty f(\Sigma) - A$ and choose totally geodesic hypersurfaces Λ_A, Λ_B of $\mathbb{H}^{n+1}$ with hyperbolic distance $d_{\mathbb{H}}(\Lambda_A, \Lambda_B) > d$, so that A and B are contained in distinct disk-type connected components, D_A and D_B, of $\mathbf{S}^n(\infty) - (\partial_\infty\Lambda_A \cup \partial_\infty\Lambda_B)$, where $A \subset D_A$ and $\partial_\infty D_A = \partial_\infty\Lambda_A$.

Consider the Poincaré half-space model for the hyperbolic space

$$\mathbb{H}^{n+1} = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_{n+1} > 0\}$$

such that $\partial_\infty f(\Sigma) \subset \{x_{n+1} = 0\}$ and D_A is a disk centered at the origin $0 \in \mathbb{R}^{n+1}$. Let γ be the geodesic of $\mathbb{H}^{n+1}$, represented in $\mathbb{R}^{n+1}$, as the half-line emanating from the origin, and p_A, p_B the intersection of γ with Λ_A and Λ_B, respectively. Consider p the middle point of the segment $\overline{p_A p_B}$ and let g be a geodesic of $\mathbb{H}^{n+1}$, orthogonal to γ at p.

Consider the one-parameter family $\{M_\lambda\}_{\lambda>0}$ of rotational properly embedded hypersurfaces of spherical type, with constant mean curvature

$$H_\lambda = \sup_{p\in\Sigma} \|H(p)\| < 1$$

such as obtained in Section 2 of [dCGT]. Each hypersurface M_λ satisfies the following statements:

1. The generating curve of each hypersurface M_λ (in the vertical hyperplane containing γ and g) is symmetric relative to g and intersect g at a distance $\lambda = d_{\mathbb{H}}(M_\lambda, \gamma)$ of p;

2. the mean curvature vector of each hypersurface M_λ points to the connected component of $\mathbb{H}^{n+p} - M_\lambda$ containing the rotation axis γ;

3. the asymptotic boundary $\partial_\infty M_\lambda$ consists of two disjoint n-spheres S_1^λ, S_2^λ. Furthermore, the function $d(\lambda) = d(S_1^\lambda, S_2^\lambda)$, $\lambda \in (0, \infty)$, satisfies: $\lim_{\lambda\to 0} d(\lambda) = 0$, increases initially, reaches a maximum d_{max}, and decreases asymptotically to zero as $\lambda \to \infty$, and its maximum value d_{max} depends only $\sup_{p\in\Sigma} |H(p)|$ and can be given in terms of an integral; thus, it can be explicitly computed to any degree of accuracy (see Proposition 5.6 of [G]).

Consider $d := d_{max}$ in the inequality (5-4). Notice that $\partial_\infty M_\lambda$ does not intersect either A or B, since

$$d(S_1^\lambda, S_2^\lambda) \leq d < d_{\mathbb{H}}(p_A, p_B).$$

Using that $d_{\mathbb{H}}(M_\lambda, \gamma) = \lambda$ and $M_\lambda \cap \Sigma = \emptyset$, for λ sufficiently large, we have that there exists λ_0 such that M_{λ_0} touches $x(\Sigma)$ for the first time (by the properness of f), say at a point $f(q)$.

Consider M_λ oriented by its mean curvature vector and consider a local orientation at p such that the unit normal vector $N(p)$ of f at p coincides with the unit normal vector $N_{\lambda_0}(f(p))$ of M_{λ_0} at $f(p)$. Then $f(\Sigma)$ lies above M_{λ_0}, near $x(p)$, with respect to $N(p)$. Since the mean curvature of x satisfies:

$$|H| \leq H_{\lambda_0} = \sup_{p\in\Sigma} |H|$$

by the tangency principle, we have that $f(\Sigma)$ coincides with M_{λ_0}, near $f(p)$. Then, by the connectedness of Σ, it follows that $f(\Sigma) \subset M_\lambda$, which contradicts $\partial_\infty f(\Sigma) \cap \partial_\infty M_{\lambda_0} = \emptyset$. Therefore $d(A, B) \leq d$, and it proves the first part.

Now, suppose that the equality (5-4) holds. Choose Λ_A and Λ_B, as above, such that $d_{\mathbb{H}}(\Lambda_A, \Lambda_B) = d$. Proceeding as in the the first part of the proof, we obtain that $f(\Sigma) = M_{\lambda_0}$ (by the connectedness of Σ and the completeness of f). This proves the second part, and completes the proof. ∎

Lemma 4 *Let $f : \Sigma \to \mathbb{H}^m$ be a proper isometric immersion of a connected Riemannian manifold Σ into $\mathbb{H}^m$. Assume that $\partial_\infty f(\Sigma)$ has a single point then $\sup_{p\in\Sigma} |H(p)| \geq 1$.*

Prova. Consider $\mathbb{H}^m$ in the Poincaré half-space model,

$$\mathbb{H}^m = \{(x_1, \dots, x_m) \in \mathbb{R}^m \mid x_m > 0\},$$

such that $\partial_\infty f(\Sigma) = \{0\}$. Given $R > 0$, let $\Gamma_R \subset \mathbb{H}^m$ be the totally umbilical hypersurface with mean curvature

$$\mathcal{H}_0 > \sup_{p\in\Sigma} |H(p)|, \tag{5-5}$$

whose asymptotic boundary of its convex part is the ball $B_R(0) \subset \{x_{n+1} = 0\}$ of radius R and centered at the origin 0. Since $\partial_\infty f(\Sigma) = \{0\}$ choose $R > 0$, sufficiently large, so that $f(\Sigma)$ is contained in the convex part of Γ_R. Notice that the convex part of Γ_R tends to $\emptyset$ as $R \to 0$. Then, by the properness of f, there exists $R_0 > 0$ such that Γ_{R_0} touches $f(\Sigma)$ for the first time, say at a point $f(q)$. Since $f(\Sigma)$ is contained in the convex part of Γ_{R_0} it follows from $(5-5)$ and Proposition 4 that there exists a neighborhood U of q in Σ satisfying that $f(U) \subset \Gamma_{R_0}$. By the connectedness of Σ it follows that $f(\Sigma)$ is contained in Γ_{R_0}, which contradicts $\partial_\infty f(\Sigma) = \{0\}$. ■

Now, we prove Theorem 11. Notice that the distance $d(\cdot\,,\cdot)$ in $\mathbf{S}^{m-1}(\infty)$, $m \geq 2$, of a compact set to a point away from this set is infinite. Then, it follows from Proposition 2 that if $\partial_\infty f(\Sigma)$ has an isolated point then $\partial_\infty f(\Sigma)$ reduces to a single point. Therefore, Theorem 11 follows from Proposition 2 join Lemma 4. ■

5.3 Proof of Theorem 10

We can assume that $\sup_{p\in\Sigma} \|H(p)\| < 1$. Consider $\mathbb{H}^m$ in the Poincaré ball model,

$$\mathbb{H}^m = \left\{(x_1,\ldots,x_m) \in \mathbb{R}^m \mid x_1^2 + \ldots + x_m^2 < 1\right\},$$

such that $\partial_\infty f(\Sigma)$ is contained in the asymptotic boundary of the following totally geodesic hypersurface

$$\Lambda = \{(x_1,\ldots,x_m) \in \mathbb{H}^m \mid x_m = 0\};$$

consider $P_N = (0,\ldots,0,1)$ and $P_S = (0,\ldots,0,-1)$ and let $\gamma : \mathbb{R} \to \mathbb{H}^m$ be the normalized geodesic such that $\gamma(0) = 0$, $\lim_{t\to-\infty} \gamma(t) = P_S$ and $\lim_{t\to\infty} \gamma(t) = P_N$.

Given $R \geq 0$, let Λ_R^+ and Λ_R^- be the complete totally geodesic hypersurfaces of $\mathbb{H}^m$ that intersect γ orthogonally at the points $\gamma(R)$ and $\gamma(-R)$, respectively. Let B_R be the connected component of $\mathbb{H}^m - (\Lambda_R^+ \cup \Lambda_R^-)$ that contains $\gamma(0)$. Since $\partial_\infty f(\Sigma) \subset \partial_\infty \Lambda$ and f is proper, the family:

$$\Big\{ f(\Sigma) \cap (\mathbb{H}^m - B_R) \Big\}_{R>0}$$

is a decreasing chain of compact sets converging to $\emptyset$, as $R \to \infty$. Thus, there exists $R_0 > 0$, sufficiently large, such that $f(\Sigma) \subset B_{R_0}$. Choose $d_0 > 0$, sufficiently large, so that

$$\tanh(d_0) > \sup_{p\in\Sigma} \|H(p)\| \tag{5-6}$$

and consider, for each $R \geq 0$, the points $q_R^+ = \gamma(R + d_0)$ and $q_R^- = \gamma(-R - d_0)$.

For each $0 \leq R \leq R_0$, let $\Gamma_R^+ = \Gamma_R^+(d_0)$ and $\Gamma_R^- = \Gamma_R^-(d_0)$ be the complete totally umbilical hypersurfaces of $\mathbb{H}^m$ equidistant to Λ_R^+ and Λ_R^-, and intersect γ orthogonally at q_R^+ and q_R^-, respectively. Since Γ_0^+ and Γ_0^- satisfy $\partial_\infty \Gamma_0^+ = \partial_\infty \Gamma_0^- = \partial_\infty \Lambda$ we have that Γ_0^+ and Γ_0^- are hypersurfaces equidistant to Λ.

Claim. $f(\Sigma)$ lies between Γ_0^+ and Γ_0^-. By contradiction, assume that Claim 5.3 is false. Since the region between Γ_0^+ and Γ_0^- is given by the intersection of the convex parts of Γ_0^+ and Γ_0^-, assume that $f(\Sigma)$ is not contained in the convex part of Γ_0^+ (the other part is analogue). Since $f(\Sigma) \subset B_{R_0}$ we have that $f(\Sigma)$ is contained in the connected component of $\mathbb{H}^{n+p} - (\Gamma_{R_0}^+ \cup \Gamma_{R_0}^-)$ that contains $\gamma(0)$. Since f is proper there exists $0 < R_1 < R_0$ such that $f(\Sigma)$ touches $\Gamma_{R_1}^+$ for the first time, say at a point $f(q)$. Notice that $f(\Sigma)$ is contained in the convex part of $\Gamma_{R_1}^+$. Since $\Gamma_{R_1}^+$ and $\Lambda_{R_1}^+$ are equidistant each other and intersect γ orthogonally at the points $q_{R_1}^+ = \gamma(R_1 + d_0)$ and $\gamma(R_1)$, respectively, the mean curvature $\mathcal{H}_{R_1}$ of $\Gamma_{R_1}^+$ is given by:

$$\mathcal{H}_{R_1} = \tanh(d_0) > \sup_{p\in\Sigma} \|H(p)\|. \tag{5-7}$$

It follows from Propostion 4 that there exists a neighborhood U of q in Σ such that $f(U) \subset \Gamma_{R_1}^+$. By the connectedness of Σ it follows that $f(\Sigma) \subset \Gamma_{R_1}^+$, which contradicts the fact $\partial_\infty f(\Sigma) \subset \partial_\infty \Lambda$. Then, $f(\Sigma)$ is contained in the convex part of Γ_0^+ and Γ_0^- (by an analogue argument), which proves the claim.

Since $d(\Lambda, \Gamma_0^+) = d(\Lambda, \Gamma_0^-) = d_0$ and $\sup_{p\in\Sigma} \|H(p)\| = \tanh(d_0)$ it follows from Claim 5.3 that

$$\sup_{p\in\Sigma} \|H(p)\| \geq \tanh\left(d_{\mathbb{H}}(f(p), \Lambda)\right), \tag{5-8}$$

for all $p \in \Sigma$.

To prove the second part of Theorem 10 assume that, for some $p \in \Sigma$, the equality in (5-8) occur. It follows from (5-8) that if f is minimal then

$f(\Sigma) \subset \Lambda$. Now, assume that f is not minimal. By the arbitrariely of $d_0 > 0$ (satisfying $\tanh(d_0) > \sup_{p\in\Sigma} \|H(p)\|$) it follows that $f(\Sigma)$ lies between two totally umbilical hypersurfaces Γ^+ and Γ^- equidistant to Λ and with constant mean curvatures equal to $\mathcal{H} = \sup_{p\in\Sigma} \|H(p)\|$. Since $\sup_{p\in\Sigma} \|H(p)\| = \tanh(d(f(p), \Lambda))$ it follows from (5-8) that $d(f(p), \Lambda) = \sup_{p\in\Sigma} d(f(p), \Lambda)$. Then $f(\Sigma)$ touches either Γ^+ or Γ^- and it is contained in the convex parts of Γ^+ and Γ^-. By Proposition 4 and the connectedness of Σ if follows that either $f(\Sigma)$ is contained in Γ^+ or Γ^-. ■

5.4 Appendix - Tangency principles for submanifolds of arbitrary codimension.

by Luquesio Jorge and Feliciano Vitorio.

Abstract. In this chapter, we prove the tangency principle for submanifold of arbitrary codimension. This result was obtained by F. Vitorio and L. Jorge (send to me by correspondence). This result generalizes a previous result by L. Jorge and F. Tomi [JT] for minimal submanifold and their proof inspired the one of the present work.

Proposition 4 (Tangency Principle) *Let Σ be a k-dimensinal Riemannian manifold, $f : \Sigma \to M$ an isometric immersion and $B \subset M$ be a (k, ϵ)-mean convex hypersurface with respect to a normal direction ν on B. Assume that for some point $p \in \Sigma$, $f(p) \in B$ and $f(\Sigma)$ remains above B near p with respect to $\nu(p)$. Furthermore, assume that the mean curvature H of f satisfies $\|H\| \leq \epsilon$, near p. Then $f(U) \subset B$, for some neighborhood $U \subset \Sigma$ of p.*

Remark. Proposition 4 with f minimal was proved by Jorge and Tomi [JT]; furthermore, the proof of Proposition 4 was inspired in the proof of this particular case (see section 2 of [JT]).

5.5 Preliminaries

To prove Proposition 4 we will need of the following lemmas.

Lemma 5 *Let $f : M \to N$ be as in Proposition 4. Let d be a C^2-function on N such that $\|\nabla d\| = 1$ and $\|\nabla(d \circ f)\| < 1$. Then we have the inequality*

$$\left|\Delta(d \circ f) + \mathrm{Trace}(A|(f_*TM)^T)\right| \leq k\|H\| + \|(\mathrm{Hess}\, d) \circ f\| \frac{\|\nabla(d \circ f)\|^2}{1 - \|\nabla(d \circ f)\|^2}, \tag{5-9}$$

where, at a point $x \in N$, A is the second fundamental form of the hypersurface $d^{-1}(d(x))$ in the direction of the normal ∇d, and, for a subspace $W \subset T_xN$, W^T denotes the orthogonal projection of W on $T_x(d^{-1}(d(x)))$.

Prova. Fix $\{e_1, \dots, e_k\}$ an orthonormal frame on M. By the Gauss Formula:

$$\overline{\nabla}_{f_*e_i} f_*e_i = f_*\left(\nabla_{e_i} e_i\right) + \alpha(e_i, e_i), \tag{5-10}$$

it follows that

$$\Delta(d \circ f) = \sum_{i=1}^{k} \mathrm{Hess}\,(d \circ f)(e_i, e_i) = k\,\langle H, \nabla d\rangle + \sum_{i=1}^{k} \mathrm{Hess}\, d(f_*e_i, f_*e_i). \tag{5-11}$$

Now, for any vector $v \in T_xN$, we denote by v^T its tangential component with respect to the hypersurface $d^{-1}(d(x))$. Then, using that $\|\nabla d(x)\| = 1$, it follows that $v^T = v - \langle v, \nabla d(x)\rangle \nabla d(x)$ and $\mathrm{Hess}\, d(v, \nabla d) = 0$, for all $v \in T_xN$. Thus

$$\mathrm{Hess}\, d(f_*e_i, f_*e_j) = \mathrm{Hess}\, d\left((f_*e_i)^T, (f_*e_j)^T\right) = -A\left((f_*e_i)^T, (f_*e_j)^T\right). \tag{5-12}$$

In order to relate the equation (5-12) to a partial trace of A we compute

$$b_{ij} := \left\langle (f_*e_i)^T, (f_*e_j)^T \right\rangle = \delta_{ij} - e_i(d \circ f)e_j(d \circ f),$$

for which we get the inverse matrix

$$b^{ij} = \delta_{ij} + \frac{e_i(d \circ f)e_j(d \circ f)}{1 - \|\nabla(d \circ f)\|^2}.$$

Then, using equation (5-12), it follows that

$$\begin{aligned} A\left((f_*e_i)^T, (f_*e_i)^T\right) &= \sum_{j=1}^{k} b^{ij} A\left((f_*e_i)^T, (f_*e_j)^T\right) \\ &+ \sum_{j=1}^{k} \mathrm{Hess}\, d(f_*e_i, f_*e_j) \frac{e_i(d \circ f)e_j(d \circ f)}{1 - \|\nabla(d \circ f)\|^2}. \end{aligned}$$

Since

$$\mathrm{Trace}(A|(f_*T_xM)^T) = \sum_{i,j=1}^{k} b^{ij} A\left((f_*e_i)^T, (f_*e_j)^T\right),$$

using equations (5-11) and (5-12), we have

$$\begin{aligned} \Delta(d \circ f) - \mathrm{Trace}(A|(f_*T_xM)^T) &= k\,\langle H, \nabla d\rangle + \sum_{i,j=1}^{k} \mathrm{Hess}\, d(f_*e_i, f_*e_j) \frac{e_i(d \circ f)e_j(d \circ f)}{1 - \|\nabla(d \circ f)\|^2} \\ &\leq k\|H\| + \|(\mathrm{Hess}\, d) \circ f\| \frac{\|\nabla(d \circ f)\|^2}{1 - \|\nabla(d \circ f)\|^2}, \end{aligned}$$

which proves Lemma 5. ■

Lemma 6 *Let A be a quadratic form on an n-dimensional Euclidean vector space V with the eigenvalues $\lambda_1 \leq \ldots \leq \lambda_n$. Then,*

$$\text{Trace}(A|W) \geq \lambda_1 + \ldots + \lambda_k,$$

for any k-dimensional subspace $W \subset V$.

Remark. This lemma was proved in [JT].

5.5.1 Proof of Proposition 4

Consider the signed distance function

$$d : x \in V \mapsto d(x) = \begin{cases} \text{dist}(x, B), & \text{if } x \text{ belongs to the connected component} \\ & \text{of } V - B \text{ for which } \nu \text{ points;} \\ -\text{dist}(x, B), & \text{otherwise} \end{cases}$$

Then d is of class C^2 (considering V sufficiently small) and satisfies $\|\nabla d\| = 1$. Furthermore, $\nabla(d \circ f) < 1$ (considering U sufficiently small), since $\nu = \nabla d$ and $f_*(T_pM)$ is a subspace of $T_{f(p)}B$.

Since $\sum_{j=1}^{k} \lambda_j \geq k\epsilon$ on B (by hypothesis) and each eigenvalue λ_j of A is a Lipschitz continuous function, in a sufficiently small neighborhood of B, we have the estimative:

$$\sum_{j=1}^{k} \lambda_j(x) \geq k\epsilon - C_1|d(x)|,$$

with a suitable constant $C_1 \geq 0$. Then, since $\|H\| \leq \epsilon$, (5-9) implies the differential inequality

$$\Delta(d \circ f) - C_1(d \circ f) - C_2\|\nabla(d \circ f)\|^2 \leq 0, \tag{5-13}$$

with a further constant C_2. By hypothesis it follows that $d \circ f \geq 0$, in V, and $d \circ f(p) = 0$. Therefore, Hopf's maximum principle (see, for instance, [GT]) is immediately applicable to (5-13), which proves proves Proposition 4. ■

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