Instituto de Matemática Pura e Aplicada

Sobre a existência e exemplos de fluxos geodésicos parcialmente hiperbólicos e não hiperbólicos

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A existência de um fluxo geodésico parcialmente hiperbólico é o tema desta tese. Construímos um exemplo de fluxo geodésico parcialmente hiperbólico deformando uma métrica Riemanniana na vizinhança de uma geodésica fechada. Mostramos também que não há fluxos geodésicos parcialmente hiperbólicos entre os que são gerados por uma métrica produto.

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Contents

1 Introduction

The theory of hyperbolic dynamics has been one of the extremely successful stories in dynamical systems. Originated by studying dynamical properties of geodesic flows on manifolds with negative curvature [An] and geometrical properties of homoclinic points [Sm], hyperbolicity is the cornerstone of uniform and robust chaotic dynamics; it characterizes the structural stable systems; it provides the structure underlying the presence of homoclinic points; a large category of rich dynamics are hyperbolic (geodesic flows in negative curvature, billiards with negative curvature, linear automorphisms, some mechanical systems, etc.); the hyperbolic theory has been fruitful in developing a geometrical approach to dynamical systems; and, under the assumption of hyperbolicity one obtains a satisfactory (complete) description of the dynamics of the system from a topological and statistical point of view. Moreover, hyperbolicity has provided paradigms or models of behavior that can be expected to be obtained in specific problems.

Nevertheless, hyperbolicity was soon realized to be a property less universal than it was initially thought: it was shown that there are open sets in the space of dynamics which are nonhyperbolic. To overcome these difficulties, the theory moved in different directions; one being to develop weaker or relaxed forms of hyperbolicity, hoping to include a larger class of dynamics.

There is an easy way to relax hyperbolicity, called partial hyperbolicity, which allows the tangent bundle to split into Df-invariant subbundles $TM = E^s \oplus E^c \oplus E^u$, such that the behavior of vectors in E^s, E^u is similar to the hyperbolic case, but vectors in E^c may be neutral for the action of the tangent map. This notion arose in a natural way in the context of time one maps of Anosov flows, frame flows or group extensions. See [BP], [Sh], [M1], [BD], [BV] for examples of these systems and [HP], [PS] for an overview.

However, and differently to hyperbolic ones, partially hyperbolic systems where unknown in the context of geodesic flows induced by Riemannian metrics. As far as we know, the way to produce partially hyperbolic systems in discrete dynamics are the following: time-one maps of Anosov flows, skew-products over hyperbolic dynamics, products and derived of Anosov deformations (DA). The two last approaches can be adapted to flows.

Our work shows that one is able to deform a specific metric that provides an Anosov geodesic flow to get a partially hyperbolic geodesic flow. This is done inspired by the Mañé's DA construction of a partially hyperbolic diffeomorphism [M1].

We prove the following theorems:

Theorem 1.1. *There is a Riemannian metric such that the induced geodesic flow is partially hyperbolic but not Anosov.*

Theorem 1.2. *For every compact Kahler manifold* (M, ω, J) *of dimension at least* 4*, such that its Kahler metric has constant negative holomorphic curvature* -1*, there is a metric* g^* *in* M *such that its geodesic flow is partially hyperbolic but not hyperbolic.*

The next two corollaries are given by the persistence of quasi-elliptic nondegenerate periodic orbits.

Corollary 1.3. *There is an open set* $\mathcal{U}$ *of metrics in the set of metrics of* (M, ω, J) *such that for* $g \in \mathcal{U}$*, the geodesic flow of* g *is partially hyperbolic but not Anosov.*

Corollary 1.4. *There is an open set* $\mathcal{V}$ *of hamiltonians in the set of hamiltonians of* (TM, ω_{TM})*, near geodesic hamiltonians, such that for* $h \in \mathcal{U}$*, the hamiltonian flow of* h *is partially hyperbolic but not Anosov.*

We also show that product metrics of Anosov geodesic flows are not examples with the partially hyperbolic property:

Proposition 1.5. *The geodesic flow of the product metric of a product manifold of two Riemannian manifolds with Anosov geodesic flows is not partially hyperbolic.*

Roughly speaking, the strategy of the construction is done following the next steps:

1. It is chosen a metric whose geodesic flow is Anosov and whose hyperbolic invariant splitting is of the form $T(SM) = E^{ss} \oplus E^{s} \oplus < X > \oplus E^{u} \oplus E^{uu}$ (section 4);

2. we take a closed geodesic γ_0 without self-intersections (section 5.1);

3. we change the metric in a tubular neighborhood of γ_0 in M, such that along the orbit associated with γ_0 the strong subbundles (E^{ss} and E^{uu}) remain invariant and the weak subbundles dissapear, becoming a central subbundle with no hyperbolic behavior (section 5.1);

3.1. to accomplish the non-hyperbolicity we change the metric in such a way that the directions of small curvature become directions of zero curvature (section 5.1);

3.2. to accomplish that the strong subbundles remain the same along γ_0 we change it in a way that the directions of larger curvature (E^{ss} and E^{uu}) remain (section 5.1);

4. we verify that for the orbits outside the tubular neighborhood the cone fields associated to the extremal subbundles (E^{ss} and E^{uu}) are preserved (sections 5.3,5.4,5.5);

4.1. first, we verify that for orbits of the geodesic flow which are close to the orbit associated with γ_0 (good region) the cones associated with the extremal subbundles are preserved (sections 5.3,5.4);

4.2. second, we verify that for orbits of the geodesic flow which are 'transversal' to the geodesic γ_0 (bad region) we can control the variation of the angle of the cone associated with the extremal subbundles with its own axis under the action of the derivative of the geodesic flow (section 5.5);

4.3. then, we prove that we can control the proportion of time that any geodesic spends in the bad region compared to the time it spends outside the bad region, so that the time spent in the bad region is as small as we need in comparison to the time spent outside it (section 5.5);

4.4 we prove that for vector in the unstable cones there is expansion, and for vectors inside the stable cones there is contraction, under the action of the derivative of the new geodesic flow (section 5.6).

First, a bit of history. Classical examples of Anosov flows are geodesic flows of negative curvature. They are transitive, ergodic, hyperbolic. A canonical way to show that a geodesic flow is Anosov is to look for a splitting in two invariant subbundles, together with the direction of the vector field of the flow, such that each of these subbundles is a Lagrangian subbundle. It is known that if this splitting is dominated, then it is Anosov. So, hyperbolicity is equivalent to domination in a Lagrangian splitting invariant by the geodesic flow [R]. Actually, if one has a dominated Lagragian splitting either for a symplectic flow (a flow which acts on a symplectic bundle), or for a contact flow, then the hyperbolicity follows [Co1].

Newhouse was the first to notice that in the conservative setting [Ne], if there is a dominated splitting, then one can prove hiperbolicity. Mañé then showed that in the symplectic setting the same happens [M2], domination implies hyperbolicity. Ruggiero [R] then used the argument to show that persistently expansive geodesic flows are Anosov. And Contreras [Co1] managed to show that for symplectic flows and contact flows imply that domination is equivalent to hyperbolicity.

Second, there are partially hyperbolic Σ-geodesic flows [CKO], flows which arise in the study of the dynamics of free particles in a system with constrains. These flows are defined in a distribution $\Sigma \subsetneq TM$. Castro, Kobayashi and Oliva [CKO] showed that under some conditions they are partially hyperbolic. But if the distribution Σ is involutive, the conditions imply that the leaves of the distribution are leaves with negative curvature, and we are again in the Anosov geodesic flows case.

The thesis is organized as follows:

In the second section of the thesis, we introduce basic results about the geodesic flow, following the book by Parternain [P]. We also introduce partial hyperbolicity and the equivalent property of the proper invariance of cone fields [HP],[Y].

In the third section we prove that product metrics are not examples of partially hyperbolic non-Anosov geodesic flows.

In the fourth section we introduce the candidate for the deformation, an example of Anosov geodesic flow that one can find in Paternain's article with Dairbekov [DaP], or in Hasselblatt and Katok's book [HK], which is the geodesic flow of the Kahler metric of a Kahler manifold of dimension at least 4 whose holomorphic curvature is -1. We give basic results of Kahler geometry in this section, following Ballmann [Ba] and Goldman [G].

In the fifth section we show that the deformed metric has a partial hyperbolic non-Anosov geodesic flow. We give a proof of the proper invariance of the strong cones following Wojtkowski's [W] tecnique for the proof that some generalized magnetic fields are Anosov, although we do not use quadratic forms, as he does, but we calculate the variation of the opening of the cones of an appropiate cone field.

In the last section we introduce open questions related to the main result of the thesis.

2 Preliminary definitions

In this section, we give some preliminary definitions. In the first two subsections, the definitions are about geodesic flows. In the third subsection, about Jacobi fields and its relation to the derivative of the geodesic flow. The basic reference for these three subsections is the book by Paternain [P]. In the fourth and last subsection, we give the main definitions about partial hyperbolicity and the basic reference is the survey by Hasselblat and Pesin [HP]. The proof of the equivalence between proper invariance of cone fields and the existence of invariant subbundles with dominated splitting is based on the survey by Yoccoz [Y].

2.1 Geodesic flows

A Riemannian manifold (M, g) is a C^∞-manifold with an euclidean inner product g_x in each T_xM which varies smoothly with respect to $x \in M$.

The geodesic flow of the metric g is the flow

$$\phi_t : TM \to TM : (x, v) \to (\gamma_{(x,v)}(t), \gamma'_{(x,v)}(t)),$$

such that $\gamma_{(x,v)}$ is the geodesic for the metric g with initial conditions $\gamma_{(x,v)}(0) = x$ and $\gamma'_{(x,v)}(0) = v$. Since the speed of the geodesics is constant, we can consider the flow restricted to $SM := \{(x, v) \in TM : g_x(v, v) = 1\}$.

Another important definition is the splitting of the tangent bundle of TM into two subbundles, a vertical one and a horizontal one. Let $\pi : TM \to M$ be the tangent bundle of M and $\pi_{TM} : T(TM) \to TM$ the double tangent bundle of M. This splitting helps us to write the derivative of the geodesic flow as a Jacobi field and its first derivative.

Definition 2.1. $\pi_V : V(TM) \to TM$,which is called the vertical subbundle, is the bundle whose fiber at $\theta \in T_xM$, $V(\theta)$, is given by $V(\theta) = ker(d_\theta\pi)$.

Definition 2.2. $K : T(TM) \to TM$, which is called the connection map associated to the metric g, is defined as follows: given $\xi \in T_\theta TM$ let $z : (-\epsilon, \epsilon) \to TM$ be an adapted curve to ξ; let $\alpha : (-\epsilon, \epsilon) \to M : t \to \pi_M \circ z(t)$, and Z the vector field along α such that $z(t) = (\alpha(t), Z(t))$; then $K_\theta(\xi) := (\nabla_{\alpha'} Z)(0)$. $\pi_H : H(TM) \to TM$, the horizontal subbundle, is given by $H(\theta) := ker(K_\theta)$.

Some properties of H and V are:

1. $H(\theta) \cap V(\theta) = 0$,
2. $d_\theta\pi$ and K_θ give identifications of T_xM with $H(\theta)$ and $V(\theta)$,
3. $T_\theta TM = H(\theta) \oplus V(\theta)$.

The decomposition in horizontal and vertical subbundles allows us to define the Sasaki metric on TM:

$$\begin{aligned}\widehat{g}_\theta(\xi,\eta) &:= g_x(d_\theta\pi(\xi), d_\theta\pi(\eta)) + g_x(K_\theta(\xi), K_\theta(\eta)) \\ &= g_x(\xi_h,\eta_h) + g_x(\xi_v,\eta_v)\end{aligned}$$

for ξ and $\eta \in T_\theta TM$, with $\xi = (\xi_h, \xi_v)$ and $\eta = (\eta_h, \eta_v)$ in the decomposition $T_\theta TM = H(\theta) \oplus V(\theta)$, with ξ_h and $\eta_h \in T_xM \cong H(\theta)$, ξ_v and $\eta_v \in T_xM \cong V(\theta)$.

Proposition 2.3. *The geodesic vector $G : TM \to T(TM)$ in this decomposition $H(\theta) \oplus V(\theta) \approx T_xM \oplus T_xM$ is given by $(v, 0)$.*

Proof.

$$G(\theta)_h = d_\theta\pi G(\theta) = d_\theta\pi\frac{\partial}{\partial t}|_{t=0}\phi_t(\theta) = \frac{\partial}{\partial t}|_{t=0}\pi\circ\phi_t(\theta) = \frac{\partial}{\partial t}|_{t=0}\gamma_\theta(t),$$

$$G(\theta)_v = K_\theta G(\theta) = K_\theta\frac{\partial}{\partial t}|_{t=0}\phi_t(\theta) = \nabla_{\gamma'_\theta}\gamma'_\theta(0) = 0.$$

□

2.2 Geodesic flow as a hamiltonian

This decomposition allows us to define a symplectic structure on TM:

$$\begin{aligned}\Omega_\theta(\xi,\eta) &:= g_x(d_\theta\pi(\xi), K_\theta(\eta)) - g_x(K_\theta(\xi), d_\theta\pi(\eta)) \\ &= g_x(\xi_h,\eta_v) - g_x(\eta_h,\xi_v).\end{aligned}$$

Proposition 2.4. *The geodesic flow of g is the hamiltonian vector field of the function $H(x,v) = \frac{1}{2}g_x(v,v)$*

Proof.

$$\begin{aligned}d_\theta H(\xi) &= \frac{\partial}{\partial t}|_{t=0}H(z(t)) = \frac{\partial}{\partial t}|_{t=0}\frac{1}{2}g_{\alpha(t)}(Z(t),Z(t)) \\ &= g_x(\nabla_{\alpha'}Z(0), Z(0)) = g_x(K_\theta(\xi), v),\end{aligned}$$

$$\Omega_\theta(G(\theta),\xi) = g_x(d_\theta\pi(G(\theta)), K_\theta(\xi)) = g_x(v, K_\theta(\xi)) = d_\theta H(\xi).$$

So, we have that $dH = i_G\Omega$. □

The geodesic flow can also be represented by its restriction to the unitary tangent bundle $\phi_t : SM \to SM : (x,v) \to (\gamma_{(x,v)}(t), \gamma'_{(x,v)}(t))$, $SM := \{(x,v) : v \in T_xM, g_x(v,v) = 1\}$. SM also has a structure that is preserved by the geodesic flow, called a contact form.

Definition 2.5. A 1-form α on an odd dimensional manifold M^{2n-1} is a contact form if the form $\alpha \wedge d\alpha^{n-1}$ is a volume form. To this 1-form we can associate a vector field, called the Reeb vector field of the form α, which is the only vector field such that $\alpha(X) = 1$ and $d\alpha(X) = 0$. By Cartan's formula $L_X\alpha = d(i_X\alpha) + i_X d\alpha = d(1) + 0 = 0$, so the flow of X preserves the contact 1-form α.

We can define a 1-form in TM such that its restriction to SM is a contact 1-form. $\alpha_\theta(\xi) := \widehat{g}_\theta(\xi, G(\theta)) = g_x(d_\theta\pi(\xi), v) = g_x(\xi_h, v)$. Note that $\Omega = -d\alpha$, which implies that α is a contact form on SM. Moreover, the geodesic flow coincides with the Reeb flow:

Proposition 2.6. *The geodesic vector field G is the Reeb vector field of the contact form α.*

Proof.

$$\alpha_\theta(G(\theta)) = g_x(d_\theta\pi(G(\theta)), v) = g_x(v, v) = 1,$$

$$i_G d\alpha_\theta(\xi) = d\alpha_\theta(G(\theta), \xi) = -\Omega_\theta(G(\theta), \xi) = -d_\theta H(\xi) = 0.$$

□

The contact form restricted to SM allows us to restrict the bundle of the action of the derivative of the geodesic flow to $S(SM) := ker\alpha$. For $\theta = (x, v) \in S_xM$, $S(\theta) \oplus \mathbb{R}(v, 0) \oplus \mathbb{R}(0, v)$. The geodesic flow restricted to SM is be partially hyperbolic if it has an invariant splitting of $S(SM)$ in three invariant subbundles $S(SM) = E^u \oplus E^c \oplus E^s$ with non trivial central bundle.

2.3 Jacobi fields

An important property of the derivative of the geodesic flow is that it is related to the Jacobi fields of the metric that generates the flow.

Definition 2.7. A Jacobi field along a geodesic γ_θ, $\theta = (x, v)$ is a vector field obtained by a variation of the geodesic γ_θ through geodesics:

$$\zeta(t) := \frac{\partial}{\partial s}|_{s=0}\pi \circ \phi_t(z(s)),$$

where $z(0) = \theta$, $z'(0) = \xi$ and $z(s) = (\alpha(s), Z(s))$.

It satisfies the following equation:

$$\zeta'' + R(\gamma_\theta', \zeta)\gamma_\theta' = 0.$$

And its initial conditions are:

$$\zeta(0) = \frac{\partial}{\partial s}|_{s=0}\pi \circ z(s) = d_\theta\pi\xi = \xi_h,$$

$$\begin{aligned}\zeta'(0) &= \frac{D}{dt}\frac{\partial}{\partial s}|_{t=0,s=0}\pi \circ \phi_t(z(s)) = \frac{D}{\partial s}\frac{\partial}{\partial t}|_{s=0,t=0}\pi \circ \phi_t(z(s)) \\ &= \frac{D}{\partial s}|_{s=0}Z(s) = K_\theta \xi = \xi_v.\end{aligned}$$

Proposition 2.8. *The derivative of a geodesic flow is:* $d_\theta \phi_t(\xi) = (\zeta_\xi(t), \zeta'_\xi(t))$.

Proof.

$$\zeta_\xi(t) = \frac{\partial}{\partial s}|_{s=0}(\pi \circ \phi_t(z(s))) = d_\theta(\pi \circ \phi_t)(\xi) = d_{\phi_t(\theta)}\pi \circ d_\theta \phi_t(\xi),$$

$$\begin{aligned}\zeta'_\xi(t) &= \frac{D}{dt}\frac{\partial}{\partial s}|_{s=0}\pi \circ \phi_t(z(s)) = \frac{D}{\partial s}|_{s=0}\frac{\partial}{\partial t}\pi \circ \phi_t(z(s)) \\ &= \frac{D}{\partial s}|_{s=0}\phi_t(z(s)) = K_{\phi_t(\theta)}(d\phi_t(\xi)).\end{aligned}$$

□

2.4 Partial hyperbolicity

Definition 2.9. A partially hyperbolic flow $\phi_t : M \to M$ in the manifold M generated by the vector field $X : M \to TM$ is a flow such that its quotient bundle $TM/\langle X \rangle$ have an invariant splitting $TM/\langle X \rangle = E^s \oplus E^c \oplus E^u$ such that these subbundles are non trivial and with the following properties:

$$d\phi_t(x)(E^s(x)) = E^s(\phi_t(x)),$$

$$d\phi_t(x)(E^c(x)) = E^c(\phi_t(x)),$$

$$d\phi_t(x)(E^u(x)) = E^u(\phi_t(x)),$$

$$||d\phi_t(x)|_{E^s}|| \leq C\exp(t\lambda),$$

$$||d\phi_{-t}(x)|_{E^u}|| \leq C\exp(t\lambda),$$

$$C\exp(t\mu) \leq ||d\phi_t(x)|_{E^c}|| \leq C\exp(-t\mu),$$

for $\lambda < \mu < 0 < C$.

Definition 2.10. A splitting $E \oplus F$ of the quotient bundle $TM/\langle X \rangle$ is called a dominated splitting if:

$$d\phi_t(x)(E(x)) = E(\phi_t(x)),$$

$$d\phi_t(x)(F(x)) = F(\phi_t(x)),$$

$$||d\phi_t(x)|_{E(x)}|| \cdot ||d\phi_{-t}(\phi_t(x))|_{F(\phi_t(x))}|| < C\exp(-t\lambda)$$

for some constants C and $\lambda > 0$.

There is a criteria useful for verifying partial hyperbolicity, called the cone criterion:

Given $\theta \in SM$, a subspace $E \subset T_\theta SM$ and a number δ, we define the cone at θ centered around E with angle δ as

$$C(\theta, E, \delta) = \{v \in T_\theta SM : \angle(v, E) < \delta\},$$

where $\angle(v, E)$ is the angle that the vector $v \in T_xM$ makes with its own projection to the subspace $E \subset T_xM$.

One flow is partially hyperbolic if there are $\delta > 0$, some time $T > 0$, and two continuous cone families $C(\theta, E_1(\theta), \delta)$ and $C(\theta, E_2(\theta), \delta)$ such that:

$$d_\theta\phi_{-t}(C(\theta, E_1(\theta), \delta)) \subsetneq C(\theta, E_1(\phi_{-t}(\theta)), \delta),$$

$$d_\theta\phi_t(C(\theta, E_2(\theta), \delta)) \subsetneq C(\theta, E_2(\phi_t(\theta)), \delta),$$

$$||d_\theta\phi_t|_{C(\theta, E_1(\theta), \delta)}|| < K\exp(t\lambda),$$

and

$$||d_\theta\phi_{-t}|_{C(\theta, E_2(\phi_t(\theta)), \delta)}|| < K\exp(t\lambda),$$

for some constant $K > 0$ and all $t > 0$.

Definition 2.11. $\Gamma_c(\mathbb{P}(T(SM))) = \{\sigma : SM \to \mathbb{P}(T(SM))$ continuous function $\}$ is the space of continuous sections of the projective space of the tangent bundle of SM. It is a Banach space. $C(\sigma_0, \delta) = \{\sigma \in \Gamma_c(\mathbb{P}(T(SM))) : \sigma(\theta) \in C(\theta, \sigma_0(\theta), \delta), v \in T_\theta SM\}$, for a continuous section σ_0 and $\delta \in (0, \frac{\pi}{2})$.

Definition 2.12. $\mathcal{F}_t : \Gamma_c(\mathbb{P}(T(SM))) \to \Gamma_c(\mathbb{P}(T(SM))) : \sigma \to \mathcal{F}_t(\sigma)$, such that $\mathcal{F}_t(\sigma)(\phi_t(\theta)) = [d_\theta\phi_t\sigma(\theta)]$, where $[v]$ is the direction of v.

In our case, we deal with one dimensional cones, cones whose axis are one dimensional linear spaces, so we can use the definition above. For cones of more than one dimension, it is the same, but the sections are not in a projective bundle but in a grassmanian bundle.

Proposition 2.13. *The proper invariance of the cones by the derivative of the geodesic flow implies a dominated splitting.*

Proof. First, we notice that $C(\sigma_0, \delta)$ is a convex compact subset of a Banach space. The invariance of the cones in this new setting is written as:

$$\mathcal{F}_{-t} C(\sigma_1, \delta) \subsetneqq C(\sigma_1, \delta),$$

$$\mathcal{F}_{t} C(\sigma_2, \delta) \subsetneqq C(\sigma_2, \delta).$$

So, by fixed point theory, since $C(\sigma_1, \delta)$ and $C(\sigma_2, \delta)$ are compact and convex, there is at least one fixed point for $\mathcal{F}_t$ and one for $\mathcal{F}_{-t}$, for each positive real number t. It must be one and the same for all t because the derivative is linear, and to have two fixed points is the same as having a invariant space of dimension at least two, and this contradicts the proper invariance of the one dimensional cones. So we get two invariant sections σ_+ and σ_-, one for positive t and the other for negative t, respectively. The exponential growth in each other comes from the exponential growth in the family of cones.
For the central direction, we notice that:

$$\mathcal{F}_{t} C(\sigma_1, \delta)^c \subsetneqq C(\sigma_1, \delta)^c,$$

$$\mathcal{F}_{-t} C(\sigma_2, \delta)^c \subsetneqq C(\sigma_2, \delta)^c,$$

for t positive. So, this implies, as in the case of the one dimensional sections, the existence of two more invariant sections of dimension $dim(M) - 2$, $\widehat{\sigma}_+$ and $\widehat{\sigma}_-$, such that $\sigma_+ \subset \widehat{\sigma}_+$ and $\sigma_- \subset \widehat{\sigma}_-$. The central direction is $E^c(\theta) := \widehat{\sigma}_- \cap \widehat{\sigma}_+$, which is a invariant subbundle of dimension one less of the dimension of $\widehat{\sigma}_-$ and $\widehat{\sigma}_+$, because they are different since $\sigma_+ \subset \widehat{\sigma}_+$ and $\sigma_- \subset \widehat{\sigma}_-$. □

For the exponential expansion or contraction in the unstable and stable directions, respectively, we only need to check exponential expansion or contraction inside the unstable and stable cones, respectively.

3 Why the geodesic flow of a metric product is not partially hyperbolic

Now, we are going to show that some simple examples that could be partially hyperbolic geodesic flows but are not, and we are going to prove that product metrics are not Anosov or partially hyperbolic.

Ruggiero [R] (see also [Co1] for proofs of it) shows us that if a geodesic flow has a dominated splitting of lagrangian subbundles, then it is Anosov. But the splitting of $T(SM)$ in this case has two subbundles of the same dimension, which together span a symplectic bundle. So, it does not rule out the existence of partially hyperbolic geodesic flows.

Indeed, if one starts with any symplectic action $\Phi : \mathbb{R} \to Sp(E.\omega)$, $\pi : E \to B$ a symplectic bundle with ω as its symplectic 2-form, one can produce another symplectic action $\Phi^* : \mathbb{R} \to Sp(E, \omega) \oplus Sp(B \times \mathbb{R}^2, \omega_0) : t \to \Phi(t) \oplus Id$. The symplectic flow associated with this symplectic $\mathbb{R}$-action is partially hyperbolic with a central direction of dimension 2. But in the case of geodesic flows, things are not that easy.

Suppose we have a Riemannian manifold (M, g) whose geodesic flow is Anosov. Then, we can say:

Proposition 3.1. *The product Riemannian manifold* $(M \times \mathbb{T}^n, g + g_0)$ *where* $(\mathbb{T}^n, g_0)$ *is* $\mathbb{T}^n$ *with its canonical flat metric, is not partially hyperbolic.*

Proof. $\{x\} \times \mathbb{T}^n$ is a totally geodesic submanifold of $(M \times \mathbb{T}^n, g + g_0)$. So, its second fundamental form is identically zero. Since the metric in $\mathbb{T}^n$ is flat this implies that:

$$R(\gamma'_{(x,y,0,v)}, (0, w))\gamma'_{(x,y,0,v)} = 0.$$

For a product metric in $M_1 \times M_2$ we have the following properties:

i. $R(X, Y, Z, W) = R^1(X, Y, Z, W)$, for X, Y, Z, W tangent to M_1, because of the Gauss' equation and the fact that the second fundamental form is zero [Ca];

ii. $R(X, Y, Z, N)$, for X, Y, Z tangent to M_1 and N tangent to M_2, because of Codazzi's equation and the fact that the second fundamental form is zero [Ca];

iii. $R(X, N, X, \widehat{N}) = 0$, for X, Y tangent to M_1 and $N, \widehat{N}$ tangent to M_2, because $K(X, N) = 0$ [Ca].

R is the curvature tensor of the product Riemannian manifold with the product metric, K its curvature, R^1 the curvature tensor of the Riemannian manifold M_1.

Then, for a submanifold $\{x\} \times \mathbb{T}^n$ with the flat metric:

$$R(\gamma'_{(x,y,0,v)}, \cdot)\gamma'_{(x,y,0,v)} \equiv 0.$$

So, the derivative of the geodesic flow along geodesics in $\{x\} \times \mathbb{T}^n$ does not have any exponential contration or expansion. □

Now, suppose we have two Riemannian manifolds with Anosov geodesic flows: (M_1, g_1) and (M_2, g_2).

Proposition 3.2. *The geodesic flow of the Riemannian manifold* $(M_1 \times M_2, g_1 + g_2)$ *is not Anosov.*

Proof. To see that this geodesic flow is not Anosov is easy. It is a classical result that $(x_0, \gamma_{(y,v)}(t))$ and $(\gamma_{(x,u)}(t), y_0)$ are geodesics of the product metric, $x_0 \in M_1$, $y_0 \in M_2$, $u \in T_xM_1$, $v \in T_yM_2$, $\gamma_{(x,u)}(0) = x$ and $\gamma'_{(x,u)}(0) = u$, $\gamma_{(y,v)}(0) = y$ and $\gamma'_{(y,v)}(0) = v$. So, we choose x_0 and $x_1 \in M_1$ close enough, and $(x_0, \gamma_{(y,v)}(t))$ and $(x_1, \gamma_{(y,v)}(t))$ are two geodesics as close to each other as x_0 and x_1, so the geodesic flow is not expansive, and this implies it is not Anosov. ☐

Proposition 3.3. *The geodesic flow of the product metric of a product manifold of two Riemannian manifolds with Anosov geodesic flows is not partially hyperbolic.*

Proof. Take local coordinates for the geodesic flow of the product metric. $x \in M_1$, $y \in M_2$, $u \in T_xM_1$, $v \in T_yM_2$ Let $\gamma_{(x,y,u,v)}(t)$ be the geodesic with initial conditions $\gamma_{(x,y,u,v)}(0) = (x, y)$ and $\gamma'_{(x,y,u,v)}(0) = (u, v)$. Since the product metric is a sum of the two metrics, we have that $\pi_i : M_1 \times M_2 \to M_i$, $i = 1, 2$, the natural projection from the product manifold to M_i, is a isometric submersion. So $\gamma_{(x,y,u,v)}(t) = (\gamma_{(x,u)}(t), \gamma_{(y,v)}(t))$.

Let us construct an orthonormal basis of parallel vector fields for $\gamma_{(x,y,u,v)}(t)$. Suppose $g^1_x(u, u) = 1$ and $g_y(v, v) = 1$. So, to have (x, y, u, v) in the unitary tangent bundle of $M_1 \times M_2$ we take $(x, y, \alpha u, \beta v)$, and

$$g_{(x,y)}((\alpha u, \beta v), (\alpha u, \beta v)) = \alpha^2 g^1_x(u, u) + \beta^2 g_y(v, v) = \alpha^2 + \beta^2 = 1.$$

Then

$$\gamma_{(x,y,\alpha u,\beta v)}(t) = (\gamma_{(x,\alpha u)}(t), \gamma_{(y,\beta v)}(t))$$

and

$$\gamma'_{(x,y,\alpha u,\beta v)}(t) = (\alpha\gamma'_{(x,u)}(t), \beta\gamma'_{(y,v)}(t)).$$

Take E_i, $i = 2, \ldots, dim(M_1)$, an orthogonal frame of parallel vector fields along the geodesic $\gamma_{(x,u)}$. Take F_j, $j = 2, \ldots, dim(M_2)$, an orthogonal frame of parallel vector fields along the geodesic $\gamma_{(y,v)}$.

Notice that along the geodesic $\gamma_{(x,y,\alpha u,\beta v)}$, since its componentes are $\gamma_{(x,\alpha u)}$ and $\gamma_{(y,\beta v)}$, the following holds:

$$g^1_{\gamma_{(x,\alpha u)}(t)}(\gamma'_{(x,\alpha u)}(t), \gamma'_{(x,\alpha u)}(t)) = \alpha^2,$$

and

$$g^2_{\gamma_{(y,\beta v)}(t)}(\gamma'_{(y,\beta v)}(t), \gamma'_{(y,\beta v)}(t)) = \beta^2,$$

so the proportion (α, β) is preserved along the geodesic.

So $\{(E_i(t),0),(0,F_j(t))\}_{i,j}$, together with $(\alpha\gamma'_{(x,u)}(t),\beta\gamma'_{(y,v)}(t))$ and $(\beta\gamma'_{(x,u)}(t),-\alpha\gamma'_{(y,v)}(t))$, is an orthonormal frame of parallel vector fields along the geodesic $\gamma_{(x,y,\alpha u,\beta v)}(t)$.

The fact that the second fundamental form of the submanifolds $\{p\}\times M_2$ and $M_1\times\{q\}$ is zero, together with Gauss and Codazzi equations, imply that:

$$\begin{aligned}
R((u_1,0),(u_2,0),(u_3,0),(u_4,0)) &= R^1(u_1,u_2,u_3,u_4),\\
R((0,v_1),(0,v_2),(0,v_3),(0,v_4)) &= R^2(v_1,v_2,v_3,v_4),\\
R((u_1,0),(u_2,0),(u_3,0),(0,v_1)) &= 0,\\
R((0,v_1),(0,v_2),(0,v_3),(u_1,0)) &= 0.
\end{aligned}$$

Also the fact that the curvature is zero for planes generated by one vector tangent to M_1 and another tangent to M_2 implies:

$$R((u_1,0),(0,v_1),(u_2,0),(0,v_2)) = 0.$$

All these equations imply that along the geodesic $\gamma_{(x,y,\alpha u,\beta v)}(t)$:

$$R(\gamma'_{(x,y,\alpha u,\beta v)},(E_i,0),\gamma'_{(x,y,\alpha u,\beta v)},(E_k,0)) = \alpha^2 R^1(\gamma'_{(x,u)},E_i,\gamma'_{(x,u)},E_k),$$

$$R(\gamma'_{(x,y,\alpha u,\beta v)},(0,F_j),\gamma'_{(x,y,\alpha u,\beta v)},(0,F_l)) = \beta^2 R^2(\gamma'_{(y,v)},F_j,\gamma'_{(y,v)},F_l),$$

$$R(\gamma'_{(x,y,\alpha u,\beta v)},(E_i,0),\gamma'_{(x,y,\alpha u,\beta v)},(0,F_j)) = 0.$$

Now, we are going to write the system of Jacobi fields. If we have $\zeta(t)=\sum_{i=2} f_iU_i$, then $\zeta''(t)=\sum_{i=2} f_i''U_i$ and

$$0=\sum_{j=2}(f_j''+\sum_{i=2} f_iR(\gamma',U_i,\gamma',U_j))U_j.$$

So, it can be written as:

$$\begin{bmatrix} f\\ f'\end{bmatrix}' = \begin{bmatrix} 0 & I\\ -K & 0\end{bmatrix}\begin{bmatrix} f\\ f'\end{bmatrix}$$

where $K_{ij}=R(\gamma',U_i,\gamma',U_j)$.

In the case of the product metric we have:

$$\begin{bmatrix} f\\ f'\end{bmatrix}' = \begin{bmatrix} 0 & 0 & I & 0\\ 0 & 0 & 0 & I\\ -\alpha^2K^1 & 0 & 0 & 0\\ 0 & -\beta^2K^2 & 0 & 0\end{bmatrix}\begin{bmatrix} f\\ f'\end{bmatrix}.$$

With a change in the order of the basis of parallel vector fields we have:

$$F' = \begin{bmatrix} 0 & I & 0 & 0 \\ -\alpha^2 K^1 & 0 & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & -\beta^2 K^2 & 0 \end{bmatrix} F.$$

So the systems decouples and the solutions are given imediately by the solutions for M_1 and M_2.

Now suppose the geodesic flow of the product metric is partially hyperbolic with splitting $E^s \oplus E^c \oplus E^u$, $dim E^s = p$, $dim E^u = q$. So the geodesic flow of each metric g_1 and g_2 is partially hyperbolic , each geodesic flow inherits a partially hyperbolic splitting:

$$E_1^s \oplus E_1^c \oplus E_1^u,$$

along geodesics of in $M_1 \times \{y\}$ $(\beta = 0)$. $E_1^s \oplus E_1^u \subset T_x M_1 \oplus \{0\} \subset T_x M_1 \oplus T_y M_2$. And

$$E_2^s \oplus E_2^c \oplus E_2^u,$$

along geodesics of in $\{x\} \times M_2$ $(\alpha = 0)$. $E_2^s \oplus E_2^u \subset \{0\} \oplus T_y M_2 \subset T_x M_1 \oplus T_y M_2$.

For geodesics of the product metric which have $\alpha \neq 0 \neq \beta$, we get a splitting into five invariant subbundles $E_1^s \oplus E_2^s \oplus E^c \oplus E_1^u \oplus E_2^u$, without the domination, since α and β multiply the lyapunov exponents of each subbundle. Since we already have an splitting, E^s and E^u are necessarilly one of a combination of subbundles of E_1^s and E_2^s, E_1^u and E_2^u, respectively.

$$E^s \in \{E \oplus F : E \subset E_1^s, F \subset E_2^s, dim E + dim F = p\},$$

$$E^u \in \{E \oplus F : E \subset E_1^u, F \subset E_2^u, dim E + dim F = q\}.$$

So there is no way to go from the case $\alpha = 0$ to $\beta = 0$ without breaking the continuity of the splitting, because one cannot go from the case dim $E = 0$, when $\beta = 0$, to dim $F = 0$, when $\alpha = 0$ continuously.

The proof actually works if the metrics do not have geodesic flows of Anosov type. And it works also for products of more than two manifolds.

$\square$

4 Candidate for the deformation

In the previous section we proved that a product metric is never hyperbolic or partially hyperbolic. So we need to look for examples of partially hyperbolic geodesic flows through deformations of a initial metric. This metric should be hyperbolic and have an invariant splitting in more than two subbundles. It should have an invariant splitting of the tangent bundle of SM in a strong stable, weak stable, weak unstable and strong unstable subbundles, together with the direction of the geodesic vector field G.

So we are going to introduce the candidate for the deformation, an example explored by Dairbekov and Paternain [DaP], which can be also found in Hasselblatt and Katok's book [HK]. It is a Kahler metric of a compact Kahler manifold with constant holomorphic curvature -1 and dimension at least 4. It has a splitting into five subbundles $S(\theta) = E^{uu}(\theta) \oplus E^{u}(\theta) \oplus < G(\theta) > \oplus E^{s}(\theta) \oplus E^{ss}(\theta)$, for $\theta \in SM$. And we are going to break the Anosov condition without destroying the strong unstable and strong stable splitting. But we need some definitions. In the first subsection we define what is a Kahler manifold and what is holomorphic curvature (following [Ba],[G],[KN]). In subsections 4.2 and 4.3 we show how to construct an example of a compact Kahler manifold of constant negative holomorphic curvature [G]. In subsection 4.4 we show that the tangent bundle of SM splits into five invariant subbundles [KN].

4.1 Definitions

Definition 4.1. A symplectic form ω on the smooth manifold M is a closed 2-form on M such that ω_x is non-degenerate for each $x \in M$.

Definition 4.2. An almost complex structure J on the smooth manifold M is an automorphism $J : TM \to TM$ such that $J^2 = -Id$.

Definition 4.3. An almost complex structure J is ω-compatible if $g_x : T_xM \times T_xM \to \mathbb{R} : (X, Y) \to \omega_x(X, J_xY)$ is a Riemannian metric.

Definition 4.4. An almost complex structure J is integrable if there is an atlas $\{U_\alpha, \varphi_\alpha\}$ such that the local charts $\varphi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{C}^n$ satisfy $d\varphi_\alpha \circ J = id\varphi_\alpha$.

Definition 4.5. A Kahler manifold is a triple (M, ω, J) such that M is a smooth manifold, ω is a symplectic form on M, and J is a integrable complex structure compatible with ω.

Definition 4.6. A Kahler structure (M, ω, J) can be defined in the following way:

1. A complex structure J,
2. ω is a close 2-form ($d\omega = 0$),
3. ω is positive ($\omega(JX, X)$ for all non zero real tangent vectors X),
4. ω is a $(1, 1)$-form with respect to J ($\omega(JX, JY) = \omega(X, Y)$).

Proposition 4.7. *Let M be a complex manifold with a compatible Riemannian metric g and Levi-Civita connection ∇. Then $d\omega = 0$ implies $\nabla J = 0$.*

Proof. Since M is a complex manifold we suppose X, Y, Z, JY, JZ commute. Then,

$$d\omega(X,Y,Z) = X\omega(Y,Z) + Y\omega(Z,X) + Z\omega(X,Y),$$

and

$$d\omega(X,JY,JZ) = X\omega(JY,JZ) + JY\omega(JZ,X) + JZ\omega(X,JY).$$

$$\begin{aligned} g((\nabla_X J)Y, Z) &= g(\nabla_X(JY), Z) - g(J(\nabla_X Y), Z) \\ &= g(\nabla_X(JY), Z) + g(\nabla_X Y, JZ). \end{aligned}$$

By the Koszul formula:

$$\begin{aligned} 2g(\nabla_X(JY), Z) &= Xg(JY,Z) + JYg(X,Z) - Zg(X,JY) \\ &= X\omega(Y,Z) - JY\omega(JY,Z) + Z\omega(X,Y), \end{aligned}$$

and

$$2g(\nabla_X Y, JZ) = -X\omega(JY,JZ) + Y\omega(Z,X) - JZ\omega(X,JY).$$

Then

$$2g((\nabla_X J)Y, Z) = d\omega(X,Y,Z) - d\omega(X,JY,JZ).$$

□

Proposition 4.8. *In a Kahler manifold M, if $X(t)$ is a parallel vector field along $c(t)$ then $JX(t)$ is a parallel vector field along $c(t)$.*

Proof. This is easy. Since $\nabla J = 0$ and $(\nabla_X J)Y + J\nabla_X Y = \nabla_X JY$, we have that if $\nabla_{c'(t)} X(t) = 0$, then $\nabla_{c'(t)} JX(t) = (\nabla_{c'(t)} J)X(t) + J\nabla_{c'(t)} X(t) = 0+0$, since $\nabla J = 0$. □

4.2 Ball model

Let $\mathbb{C}^{n,1}$ be the $(n+1)$-dimensional complex vector space

$$\mathbb{C}^{n+1} = \left\{ Z = \begin{bmatrix} Z' \\ Z_{n+1} \end{bmatrix} : Z' \in \mathbb{C}^n, Z_{n+1} \in \mathbb{C} \right\}$$

with the hermitian pairing

$$\begin{aligned} \langle Z, W \rangle &= \langle\langle Z', W' \rangle\rangle - Z_{n+1}\overline{W}_{n+1} \\ &= Z_1\overline{W}_1 + \ldots + Z_n\overline{W}_n - Z_{n+1}\overline{W}_{n+1}, \end{aligned}$$

$$\langle\langle Z', W' \rangle\rangle = Z_1\overline{W}_1 + \ldots + Z_n\overline{W}_n.$$

Definition 4.9. A vector is negative, null or positive if and only if $\langle Z, Z\rangle$ is negative, null or positive, respectively. The complex hyperbolic space with dimension n, $\mathbb{H}^n_{\mathbb{C}}$, is the subset of negative lines of $\mathbb{P}(\mathbb{C}^{n,1})$. The boundary of $\mathbb{H}^n_{\mathbb{C}}$, $\partial\mathbb{H}^n_{\mathbb{C}}$, is the set of null lines of $\mathbb{P}(\mathbb{C}^{n,1})$, lines such that $\langle Z, Z\rangle = 0$.

We can identify $\mathbb{H}^n_{\mathbb{C}}$ with the unit ball $\mathbb{B}^n = \{z \in \mathbb{C}^n | \langle\langle z, z\rangle\rangle < 1\}$ of the n-dimensional complex vector space, by the following biholomorphic map:

$$A : \mathbb{C}^n \to \mathbb{P}(\mathbb{C}^{n,1}),$$

$$z' \to \begin{bmatrix} z' \\ 1 \end{bmatrix}.$$

This biholomorphic embedding of $\mathbb{C}^n$ onto $\mathbb{P}(\mathbb{C}^{n,1}) - \{Z_{n+1} = 0\}$. $Z_{n+1} = 0$ implies Z is positive, so $\mathbb{H}^n_{\mathbb{C}} \subset A(\mathbb{C}^n)$ and A identifies $\mathbb{B}^n$ with $\mathbb{H}^n_{\mathbb{C}}$, and $\partial\mathbb{B}^n$ with $\partial\mathbb{H}^n_{\mathbb{C}}$.

Consider the unitary group $U(n, 1)$ that preserves the hermitian inner product of $\mathbb{C}^{n,1}$. The image of $U(n, 1)$ in $PGL(\mathbb{C}^{n,1})$, which we are going to call $PU(n, 1)$ is the group of biholomorphisms of $\mathbb{H}^n_{\mathbb{C}}$.

Proposition 4.10. *$PU(n, 1)$ acts transitively on $\mathbb{H}^n_{\mathbb{C}}$ and on the unit tangent bundle of $\mathbb{H}^n_{\mathbb{C}}$. $\mathbb{H}^n_{\mathbb{C}}$ is a homogeneous space.*

Proof. First, $PU(n, 1)$ acts transitively on $\mathbb{H}^n_{\mathbb{C}}$:

We can represent two negative lines by two negative vectors X, $Y \in \mathbb{C}^{n,1}$ such that:

$$\langle X, X\rangle = \langle Y, Y\rangle = -1, \langle X, Y\rangle < 0.$$

Notice that $\langle X, Y\rangle$ is not a real number, but we can choose X and Y such that $\langle X, Y\rangle$ is a negative real number.

Define $M := X + Y$, then:

$$\langle M, M\rangle = -2 + 2Re\langle X, Y\rangle < 0.$$

So M is a negative line. Define the map:

$$\rho : Z \to -Z + 2\frac{\langle Z, M\rangle}{\langle M, M\rangle}M.$$

So $\rho \in U(n, 1)$, and $\rho(X) = Y$, $\rho(Y) = X$. This implies $U(n, 1)$ acts transitively on the set of negative lines.

Now we notice that the stabilizer of the origin $0' = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is isomorphic to the unitary group $U(n)$ of $\mathbb{C}^n$:

$$A \in Stab(0') \Leftrightarrow A(0') = 0'.$$

Write $A \in Stab(0') \subset PU(n,1)$ as $A = \begin{bmatrix} A' & b \\ c & d \end{bmatrix}$. $A \in Stab(0') \Rightarrow \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow$ $b = 0, d = 1$. $A \in PU(n,1) \Rightarrow \langle AZ, AZ \rangle = \left\langle \begin{bmatrix} A' & b \\ c & d \end{bmatrix} \begin{bmatrix} z' \\ 1 \end{bmatrix}, \begin{bmatrix} A' & b \\ c & d \end{bmatrix} \begin{bmatrix} z' \\ 1 \end{bmatrix} \right\rangle \langle Z, Z \rangle =$ $\left\langle \begin{bmatrix} z' \\ 1 \end{bmatrix}, \begin{bmatrix} z' \\ 1 \end{bmatrix} \right\rangle \Rightarrow c = 0, A' \in U(n)$.

Since $U(n)$ acts transitively in $S^{2n-1} \subset \mathbb{C}^n$ and $U(n,1)$ acts transitively on the set of negative lines, we conclude that $PU(n,1)$ acts transitively on the unit tangent bundle of $\mathbb{H}^n_{\mathbb{C}}$. Then, $\mathbb{H}^n_{\mathbb{C}}$ is the homogeneous space $PU(n,1)/U(n)$.

Transitivity holds because, if we take $X, Y \in \mathbb{H}^n_{\mathbb{C}}$, $u \in T_X\mathbb{H}^n_{\mathbb{C}}$ and $v \in T_Y\mathbb{H}^n_{\mathbb{C}}$, and $A \in PU(n,1)$ which sends X to Y, then the derivative of A sends u to $v_0 \in T_Y\mathbb{H}^n_{\mathbb{C}}$. If $v_0 = v$, then it is okay. If not, take $B_1, B_2 \in PU(n,1)$, such that B_1 sends $0'$ to X, its derivative sends $u_1 \in T_{0'}\mathbb{H}^n_{\mathbb{C}}$ to $u \in T_X\mathbb{H}^n_{\mathbb{C}}$, B_2 sends $0'$ to Y, its derivative sends $v_1 \in T_{0'}\mathbb{H}^n_{\mathbb{C}}$ to $v \in T_Y\mathbb{H}^n_{\mathbb{C}}$. Take $B_3 \in U(n) = Stab(0')$ which sends u_1 to v_1. Then $B_2 \circ B_3 \circ B_1^{-1}$ sends X to Y, and its derivative sends u to v. □

4.3 Symplectic reduction

The construction of $\mathbb{H}^n_{\mathbb{C}}$ as a Kahler quotient is similar to the construction of the Fubini-Study Kahler structure.

The symplectic structure on $\mathbb{H}^n_{\mathbb{C}}$ comes from the symplectic structure of $\mathbb{C}^{n,1}$ by a symplectic quocient construction. The following 2-form gives a symplectic structure to $\mathbb{C}^{n,1}$:

$$\omega(X,Y) = Im\langle X, Y \rangle.$$

The hamiltonian $f : \mathbb{C}^{n,1} \to \mathbb{R} : X \to -\frac{1}{2}\langle X, X \rangle$ has as his flow $Z \to e^{-it}Z$. The orbits of this flow are periodic of period 4π. For $\kappa \in \mathbb{R}$, the symplectic quotient $f^{-1}(\kappa)/S^1$ inherits a symplectic structure Φ_k. It is of type $(1,1)$:

$$\Phi_k(Jv_1, Jv_2) = \Phi_k(v_1, v_2),$$

v_1, v_2 tangent vectors to $f^{-1}(\kappa)/S^1$.

Then $\mathbb{H}^n_{\mathbb{C}}$ identifies with each level set $f^{-1}(\kappa)/S^1$.

If $Z \in f^{-1}(\kappa)$, $T_{[Z]}f^{-1}(\kappa)/S^1$ naturally identifies with the orthogonal complement $Z^\perp$ with respect to the hermitian metric $\langle , \rangle$. $Z^\perp$ is a positive definite subspace of $T_Z\mathbb{C}^{(n,1)} \cong \mathbb{C}^{(n,1)}$, because Z is negative and the hermitian metric $\langle , \rangle$ has signature 1. Hence Φ_k is a positive 2-form, closed because it is the restriction of a closed symplectic form in $\mathbb{C}^{(n,1)}$. So, together with the complex structure J, which remais a complex structure to $f^{-1}(\kappa)/S^1$ because this is a submanifold of the $\mathbb{C}^{(n,1)}$, they define a Kahler structure on $\mathbb{H}^n_{\mathbb{C}}$.

Now we are going to give an explicit form of the pull-back of Φ_k by the map $A : \mathbb{B}^n \to \mathbb{H}^n_{\mathbb{C}}$. The map A does not map $\mathbb{B}^n$ to a level set of f. So we have to modify the symplectic form. We have to replace it by $\Phi' = 2i\partial\overline{\partial}\log f$. This symplectic form is invariant under scalar multiplication, because $f(\lambda Z) = |\lambda|^2 f(Z)$, and is a constant scalar multiple of Φ_k on $f^{-1}(\kappa)$.

$$\Phi = 2i\partial\overline{\partial}f,$$

$$\partial\overline{\partial}\log f = f^{-1}\partial\overline{\partial}f - (f^{-1}\partial f) \wedge (f^{-1}\overline{\partial}f).$$

The fact that $df = \partial f + \overline{\partial}f$ implies that the restrictions of ∂f and $\overline{\partial}f$ are linearly dependent. So $(f^{-1}\partial f) \wedge (f^{-1}\overline{\partial}f)$ restricted to $f^{-1}(\kappa)$ is zero. So the symplectic form induced on $\mathbb{B}^n$ is equal to $2ik\partial\overline{\partial}\log f$:

$$\begin{aligned}
\Phi_k &= 2ki\partial\overline{\partial}\log(f \circ A) \\
&= 2ki\partial\overline{\partial}\log(1 - \langle\langle z, z\rangle\rangle) \\
&= 2ki\partial\{(1 - \langle\langle z, z\rangle\rangle)^{-1}(-\langle\langle z, dz\rangle\rangle)\} \\
&= \frac{-2ki}{(1 - \langle\langle z, z\rangle\rangle)^2}\left(\langle\langle z, dz\rangle\rangle \wedge \langle\langle dz, z\rangle\rangle - (1 - \langle\langle z, z\rangle\rangle)(\sum_{j=1}^{n} dz_j \wedge d\overline{z}_j)\right) \\
&= \frac{2ki}{(1 - \langle\langle z, z\rangle\rangle)^2}\left((\sum_{j=1}^{n} \overline{z}_j dz_j) \wedge (\sum_{j=1}^{n} z_k d\overline{z}_k) + (1 - \langle\langle z, z\rangle\rangle)\sum_{j=1}^{n} dz_j \wedge d\overline{z}_j\right).
\end{aligned}$$

The metric, $g(,) = \Phi_k(, J.)$ is equal to:

$$2k(1 - \langle\langle z, z\rangle\rangle)^{-2}\{\langle\langle z, dz\rangle\rangle\langle\langle dz, z\rangle\rangle + (1 - \langle\langle z, z\rangle\rangle)\langle\langle dz, dz\rangle\rangle\}.$$

Proposition 4.11. *The metric above has constant holomorphic curvature* $-\frac{2}{k}$.

Proof. If L is a negative complex line, the restriction of this metric to L has constant curvature $-\frac{2}{k}$. Because the space is homogeneous, one only has to check the case $L = L_0$, $L_0 = \{Z_2 = \ldots = Z_n = 0\}$. If we define the coordinate $z := \frac{Z_1}{Z_{n+1}}$, then

$$g_k|L_0 = 2k(1 - |z|^2)^{-2}dzd\overline{z},$$

which is the Poincaré metric of curvature $-\frac{2}{k}$. □

A result of Borel [B] ensures that there are always lattices $\Gamma \subset PU(n,1)/U(n)$ such that $M := \mathbb{H}^n_{\mathbb{C}}/\Gamma$ is a smooth compact manifold. So there are compact Kahler manifolds with constant negative holomorphic curvature.

4.4 The splitting of the geodesic flow in this Kahler manifold

According to Kobayashi and Nomizu, the curvature tensor in this Kahler manifold of constant holomorphic curvature is:

$$\begin{aligned} R(X,Y,Z,W) = &- \frac{1}{4}(g(X,Z)g(Y,W) - g(X,W)g(Y,Z) \\ &+ g(X,JZ)g(Y,JW) - g(X,JW)g(Y,JZ) \\ &+ 2g(X,JY)g(Z,JW)), \end{aligned}$$

which implies:

$$R(X,Y)X = -\frac{1}{4}(g(X,X)Y - g(X,Y)X + 3g(JX,Y)JX),$$

From this we get that

$$R(X,JX,X,JX) = -g(X,X)^2,$$

and

$$R(X,Y,X,Y) = -\frac{1}{4}g(X,X)g(Y,Y),$$

if Y is orthogonal to both X and JX.

Now we are able to write the splitting of this geodesic flow. If W is a parallell vector field along a geodesic γ, suppose $\zeta = fW$. If $W = J\gamma'$, then the Jacobi equation $\zeta'' + R(\gamma',\zeta)\gamma' = 0$ gives us $f''W + g(\gamma',\gamma')fW = 0$. If $g(\gamma',\gamma') = 1$ then $f'' = f$. So if $\zeta(0) = \zeta'(0)$ then $\zeta(t) = \zeta(0)e^t$ and if $\zeta(0) = -\zeta'(0)$ then $\zeta(t) = \zeta(0)e^{-t}$. So, if $W = J\gamma'$ we have $\zeta(t) = \frac{1}{2}(\zeta(0) + \zeta'(0))e^t + \frac{1}{2}(\zeta(0) - \zeta'(0))e^{-t}$. The same calculation for W a parallel vector field orthogonal to γ' and $J\gamma'$ implies that $\zeta(t) = \frac{1}{2}(\zeta(0) + 2\zeta'(0))e^{\frac{t}{2}} + \frac{1}{2}(\zeta(0) - 2\zeta'(0))e^{-\frac{t}{2}}$.

So the invariant subbundles are:

$$E^{uu} =< \zeta(0) = \zeta'(0) = J\gamma' >,$$

$$E^{u} =< \zeta(0) = 2\zeta'(0) = W, W \perp \gamma' and \perp J\gamma' >,$$

$$E^{ss} =< \zeta(0) = -\zeta'(0) = J\gamma' >,$$

$$E^{s} =< \zeta(0) = -2\zeta'(0) = W, W \perp \gamma' and \perp J\gamma' > .$$

The geodesic flow of a Kahler manifold with constant negative holomorphic curvature is not only Anosov but its derivative splits the unitary tangent bundle into five invariant subbundles.

5 Deformation and its tangent bundle properties

In this section we change the metric described in the previous section. It will be a DA-like deformation [M1]. We turn the geodesic flow into a partially hyperbolic one, not Anosov, making it partially hyperbolic along a closed geodesic, and preserving the strong unstable and strong stable cones along the other geodesics.

In the first subsection we change the metric so that the curvature along this geodesic is zero for some vector fields orthogonal to the geodesic. This implies that the geodesic flow is not Anosov anymore, by Corollary 3.4 of Eberlein [E]:

Corollary 5.1. *[E] If the geodesic flow is Anosov, then the following holds: Let any γ be a unit speed geodesic, and $E(t)$ any non-zero perpendicular parallel vector field along γ, then the sectional curvature $K(\gamma', E)(t) < 0$ for some real number t.*

For the geodesic flow of the new metric g^*, $E(t)$ is a non-zero perpendicular parallel vector field along γ, and $K(\gamma', E)(t) = 0$, then the geodesic flow of the metric g^* is not Anosov.

In the subsections 5.2 to 5.5 we show that the new geodesic flow preserves the strong stable and strong unstable cone fields. We first show that along the closed geodesic γ the strong stable and strong unstable cones are properly invariant under the action of the derivative of the geodesic flow (next section). Then, we show that for geodesics which are close to $(v_0, 0, 0, \ldots, 0)$ the strong stable and strong unstable cones are properly invariant too (sections 5.3 and 5.4). Then we show that for geodesics that cross the neighborhood of the deformation of the Kahler metric the strong stable and strong unsable cones are not properly invariant, but we manage to control the lack of this property in such a way that, after crossing the neighborhood, and inside the region where the metric remains the same, where it is equal to the original Kahler metric, proper invariance is obtained (section 5.5). Then we prove that there is expansion for the vectors in the strong unstable cones, and contraction for the vectors in the strong stable cones (section 5.6).

We only need to show the strong unstable cone is properly invariant, because this garantees that we have one unstable subbundle E^u invariant under the flow. For the same reasons there is a properly invariant subbundle under the inverse of the flow, which is the stable subbundle.

5.1 Closed geodesic

First, to make the deformation of the metric, we choose a closed geodesic without self intersections. There is always a geodesic with these properties in a compact Riemannian manifold [HK]. In a first version the deformation was done using Fermi's coordinates but by suggestion done bv R. Ruggiero we are going to work in normal coordinates. Fermi's coordinates are not defined along all the closed geodesic, but normal coordinates are.

Let us call this geodesic $\gamma : [0, T] \to M^{2n}$. Now we introduce normal coordinates along this geodesic. Take an orthonormal basis of vector fields $\{e_0(t) := \gamma'(t), e_1(t) := J\gamma'(t), e_2(t), e_3(t), \ldots, e_{2n-2}(t), e_{2n-1}(t)\}$ in $T_{\gamma(t)}M$. This is possible because the parallel transport preserves orientation and M is orientable. $\Psi : [0,T] \times (-\epsilon_0, \epsilon_0)^{2n-1} \to M : (t,x) \to \exp_{\gamma(t)}(x_1 e_1(t) + x_2 e_2(t) + \ldots + x_{2n-1} e_{2n-1}(t))$ with ϵ_0 less than the injectivity radius, so $\Psi|_U$ is a diffeomorphism, with $U = [0,T] \times (-\epsilon_0, \epsilon_0)^{2n-1}$.

The vector fields $e_0(t)$ and $e_1(t)$ generate a plane with sectional curvature -1. Each vector field $e_i(t), i = 2, \ldots, 2n-1$ together with $e_0(t)$ generate a plane with sectional curvature $-\frac{1}{4}$. The line bundle generated by $e_1(t)$ along the geodesic is invariant by parallel transport. The same holds for the vector bundle generated by $e_i(t), i = 2, \ldots, 2n-1$ along the geodesic γ: the parallel transport along γ leaves it invariant.

Let $g_{ij}(t,x)$ denote the components of the metric in this neighborhood. We define a new Riemannian metric g^* as:

$$g^*_{00}(t,x) := g_{00}(t,x) + \sum_{i,j=1}^{2n-1} \Phi_{ij}(t,x) x_i x_j,$$

$$g^*_{ij}(t,x) := g_{ij}(t,x), (i,j) \neq (0,0),$$

with $\Phi_{ij} : [0,T] \times (-\epsilon_0, \epsilon_0)^{2n-1} \to \mathbb{R}$.

Each Φ_{ij} is a bump function. This kind of deformation allows us to change the curvature (change the second derivative), as γ and the parallel transport along γ (the metric up to its first derivative) remain the same. This becomes clear if we look to the formulas of the metric, the parallel transport and the curvature with respect to a coordinate system.

For this new metric g^*, the coordinates along γ are:

$$g^{*ij}(t,0) = g^{ij}(t,0), 0 \leq i,j \leq 2n-1,$$

$$g^*_{ij}(t,0) = g_{ij}(t,0), 0 \leq i,j \leq 2n-1,$$

$$\partial_k g^{*ij}(t,0) = \partial_k g^{ij}(t,0), 0 \leq i,j,k \leq 2n-1,$$

$$\partial_k g^*_{ij}(t,0) = \partial_k g_{ij}(t,0), 0 \leq i,j,k \leq 2n-1.$$

These equalities imply that the closed geodesic γ still is a closed geodesic for g^*. We are going to use the following deformation:

$$g^*_{00}(t,x) := g_{00}(t,x) + \alpha(t,x),$$

$$\alpha(t,x) = \sum_{k=2}^{2n-1} x_k^2 \Phi_k(x),$$

$$g^*_{ij}(t,x) := g_{ij}(t,x), (i,j) \neq (0,0).$$

The coordinates of the curvature tensor in this neighborhood are:

$$R_{ijkl} = -\frac{1}{2}(\partial^2_{ik} g_{jl} + \partial^2_{jl} g_{ik} - \partial^2_{il} g_{jk} - \partial^2_{jk} g_{il}) - \Gamma^T_{ik} g^{-1} \Gamma_{jl} + \Gamma^T_{il} g^{-1} \Gamma_{jk}, \tag{1}$$

$$\Gamma_{ik} := [\Gamma_{j,ik}]_j,$$

and

$$\Gamma_{j,ik} := \frac{1}{2}(\partial_i g_{jk} + \partial_k g_{ij} - \partial_j g_{ik}).$$

So, at γ, the curvature tensor is:

$$\begin{aligned} R^*_{ijkl}(t,0) = R_{ijkl}(t,0) &- \frac{1}{2}(\delta_{j+l,0}\partial^2_{ik}\alpha(t,0) + \delta_{i+k,0}\partial^2_{jl}\alpha(t,0) \\ &- \delta_{j+k,0}\partial^2_{il}\alpha(t,0) - \delta_{i+l,0}\partial^2_{jk}\alpha(t,0)), \end{aligned}$$

and

$$R^*_{0j0l}(t,0) = R_{0j0l}(t,0) - \frac{1}{2}(\partial^2_{jl}\alpha(t,0)).$$

Then, along γ:

$$R^*_{0i0j}(t,0) = R_{0i0j}(t,0), i \neq j, i, j = 2, \ldots, 2n-1,$$

$$\begin{aligned} R^*_{0k0k}(t,0) &= R_{0k0k}(t,0) - \frac{1}{2}(\partial^2_{kk}\alpha(t,0)) \\ &= R_{0k0k)}(t,0) - \Phi_k(t,0). \end{aligned}$$

For the initial metric and $k = 2, \ldots, 2n-1$:

$$R_{0k0k}(t,0) = g_{00}(t,0) g_{kk}(t,0) K(\gamma'(t), e_k(t)) = -\frac{1}{4}.$$

So, if we choose the bump function Φ_k such that $\Phi_k(t,0) = -\frac{1}{4}$, then $R^*_{0k0k}(t,0) = 0$. Then, Eberlein's corollary applies, and the geodesic flow of g^* is not Anosov. Is it partially hyperbolic?

The new metric g^* has the same coordinates as g along the closed geodesic γ, and not only that, it has the same Christoffel symbols along γ. This implies that g^* has the same parallel transport as g along γ. So, if $\{E_0(t) = \gamma'(t), E_1(t) = J\gamma'(t), \ldots, E_{2n-1}(t)\}$ is a orthonormal basis of parallel vector fields in $T_{\gamma(t)}M$, the Jacobi fields $\zeta(t) = \sum_{i=0}^{2n-1} f_i(t) E_i(t)$ along γ are the solutions of the following equation:

$$\begin{aligned} 0 &= \zeta''(t) + R^*(\gamma'(t), \zeta(t))\gamma'(t) \\ &= \sum_{i,j=0}^{2n-1} (f_i''(t) + R^*(E_0, E_j, E_0, E_i)(t) f_j(t)) E_i(t) \end{aligned}$$

$$\Rightarrow 0 = f_i''(t) + \sum_{j=1}^{2n-1} R^*(E_0, E_j, E_0, E_i)(t) f_i(t), i = 0, 2n-1$$

$$\Rightarrow \begin{bmatrix} f(t) \\ f'(t) \end{bmatrix}' = \begin{bmatrix} 0 & I \\ -K(t) & 0 \end{bmatrix} \begin{bmatrix} f(t) \\ f'(t) \end{bmatrix},$$

$$K_{ij}^*(t) := R^*(E_0, E_j, E_0, E_i)(t).$$

Along γ we have:

$$K_{ij}^*(t) = \begin{bmatrix} -1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}.$$

So there is a central direction spanned by the Jacobi fields related to the curvature $K(\gamma'(t), E_k(t))$, $E_k(t)$ and $tE_k(t)$, for $k = 2, \dots, 2n-1$. This implies we have a central bundle E^c along the geodesic γ. Notice that $\{e_k(t)\}_{k=2}^{2n-1}$ and $\{E_k(t)\}_{k=2}^{2n-1}$ generate the same subspace of $T_{\gamma(t)}M$, invariant by parallel transport because it is orthogonal to $\gamma'(t)$ and $J\gamma'(t)$.The others subbundles are E^{uu}, spanned by $(e^t e_1(t), e^t e_1(t)) = (e^t J\gamma'(t), e^t J\gamma'(t))$ and E^{ss}, spanned by $(e^{-t} e_1(t), -e^{-t} e_1(t)) = (e^{-t} J\gamma'(t), -e^{-t} J\gamma'(t))$.

5.2 The bump function and its properties

So far we have one only property of the function $\alpha : U \to \mathbb{R}$, that $\alpha(t, x) = \sum_{k=2}^{2n-1} x_k^2 \Phi_k(t, x)$ and $\Phi_k(t, 0) = -\frac{1}{4}$. Now we are going to state other properties that are going to help us prove the proper invariance of the cones under the action of the derivative of the geodesic flow.

First, to simplify the problem, we try to perturb the curvature only in the direction of the subspace generated by $\frac{\partial}{\partial x_k}, k = 2, \dots, 2n-1$, at least for some geodesics. This is impossible, but we can construct a bump function such that, as $\epsilon \to 0$, only the term $\partial^2_{x_k x_k}, k = 2, \dots, 2n-1$ perturbs the curvature. How can we do it?

First let us construct

$$\Phi_k(t, x) = \frac{1}{4} \phi_{k,1}(x_1) \phi_{k,2}(x_2) \phi_{k,3}(x_3) \dots \phi_{k,2n-1}(x_{2n-1}),$$

ϕ_i bump functions themselves. So, the second property is that Φ_k does not depend on t.

Third, let us define $\phi_{k,1}, \dots \phi_{k,2n-1}$, except $\phi_{k,k}$, with support on $[-\epsilon, \epsilon]$, such that $\phi_{k,i}(0) = 1$, $\phi_{k,i}(\pm\epsilon) = 0$, with $\epsilon < \epsilon_0$, and $\phi_{k,k}$ with support on $[-\epsilon^2, \epsilon^2]$, $\phi_{k,k}(0) = -1$ and $\phi_{k,k}(\pm\epsilon^2) = 0$. This ensures that the only second order partial derivative of α that does not goes to 0 as $\epsilon \to 0$ is $\partial^2_{k,k}\alpha$. Moreover, α is C^1-close to the constant zero function. Since x_k^2 is of order ϵ^4, we can say that α is of order ϵ^4, $d\alpha$ is of order ϵ^2 and $d^2\alpha$ is of order 1, so that $d^2\alpha$ is limited, with limitation independent of ϵ.

Lema 5.2. *For $\alpha : U \to \mathbb{R} : (t,x) \to x_{2n-1}^2\Phi(t,x)$, the following inequalities are satisfied:*

i. $|\alpha(t,x)| \leq M_0\epsilon^4$,

ii. $|\partial_{x_j}\alpha(t,x)| \leq M_0\epsilon^2$,

iii. $|\partial^2_{x_i x_j}\alpha(t,x)| \leq M_0\epsilon$, *if* $i \neq j$,

iv. $|\partial^2_{x_k x_k}\alpha(t,x)| \leq M_0$, M_0 *independent of* ϵ.

Proof. Item i. $|\alpha(x)| \leq \frac{1}{4}\epsilon^4$. Item ii.: $|\partial_{x_j}\alpha(x)| \leq \frac{1}{4}\epsilon^4 2\epsilon^{-2}$. Item iii.: $|\partial^2_{x_j x_i}\alpha(x)| \leq \frac{n}{4}\epsilon^4 4\epsilon^{-2}$ if $j \neq i$. Item iv.: $|\partial^2_{x_k x_k}\alpha| \leq \frac{1}{4}\epsilon^4 3\epsilon^{-4} \leq 1$. □

Lema 5.3. *For every $\delta > 0$ there is a bump function ϕ, such that its minimum value is at $x = 0$, $\phi(\pm\epsilon^2) = 0$, and $F(\phi)(x) := x^2\phi''(x) + 4x\phi'(x) + 2\phi(x) \in [(-2-\delta)F(\phi)(0), (2+\delta)F(\phi)(0)]$.*

Proof. To prove the lemma, first we construct a C^2 function ϕ such that the property stated in the lemma holds for $\frac{\delta}{2}$. Then, there will be a C^∞ function ϕ such that it holds for δ. To construct this C^2 function is easy. We define the following function φ_τ, continuous and piecewise-C^1 in $(0, \frac{1}{2})$ and $(\frac{1}{2}, 1)$ (See Figure 1):

- $\varphi_\tau(0) = \varphi_\tau(1) = \varphi(\frac{1}{2}) = 0$,
- $\varphi'_\tau(x) = \frac{h_\tau}{\tau}$, if $x \in (0, \tau) \cup (1-\tau, 1)$,
- $\varphi'_\tau(x) = -\frac{h_\tau}{\tau}$, if $x \in (\frac{1}{2} - \tau, \frac{1}{2} + \tau)$,
- $\varphi_\tau(x) = h_\tau$ for $x \in (\tau, \frac{1}{2} - \tau)$, $\varphi_\tau(x) = -h_\tau$, for $x \in (\frac{1}{2} - \tau, 1 - \tau)$.

Then we define ϕ_τ such that $\phi_\tau(1) = 0$, $\phi'_\tau(0) = \phi'_\tau(1) = 0$ and $\phi''_\tau = \varphi_\tau$. Then ϕ_τ is C^2 and $\phi_\tau(0) = -\frac{h_\tau}{4}(1 - 2\tau)$. We use the fact that it holds for $\tau = 0$ and then show that it holds for τ small enough.

For $\tau = 0$, ϕ_0 is not C^2 but this is not a problem. For ϕ_0, we have that $F(\phi_0)(x) = (-\frac{1}{2} + 6x^2)h_0$ for $x \in (0, \frac{1}{2})$ and $F(\phi_0)(x) = (-1 + 6x + 6x^2)h_0$ for $x \in (\frac{1}{2}, 1)$. Then it is simple to see that $F(\phi)(0) = -\frac{h_0}{2}$ and $F(\phi_0)(x) \in [-h_0, h_0]$ (See Figure 2). Then, for ϕ_0 we have that $F(\phi_0)(x) \in [-2F(\phi)(0), 2F(\phi)(0)]$. So, why does it holds for ϕ_τ, with τ small enough?

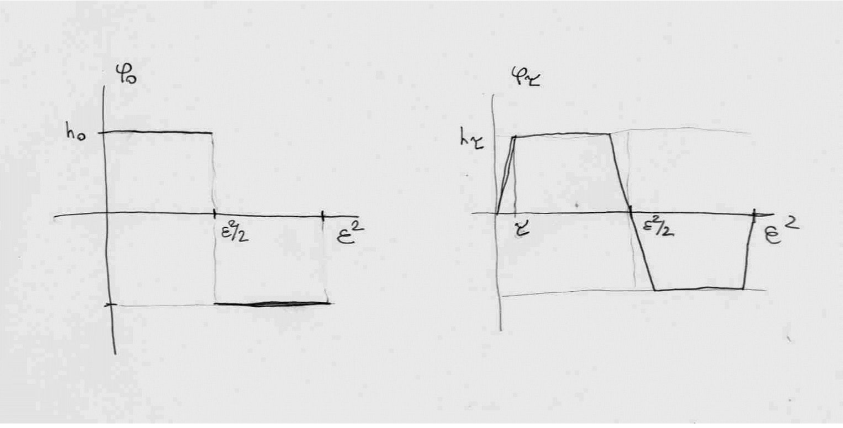

Figure 1: ϕ_0 and ϕ_τ

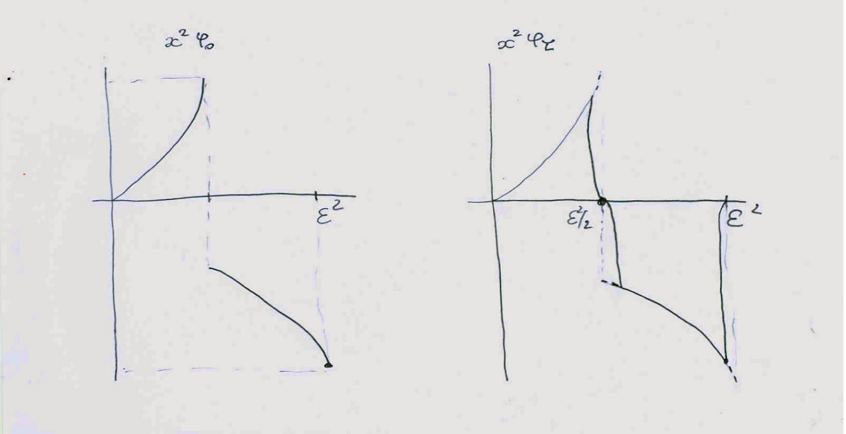

Figure 2: $x^2\phi_0$ and $x^2\phi_\tau$

First, we notice that the first term of $F(\phi_\tau)$ is the only one that does not varies continuosly as τ varies. The other two do vary continuosly because ϕ_τ is C^1-close to ϕ_0. So we have to analyse only $x^2\phi_\tau''(x)$. But $\phi_\tau''(x) \in [-h_\tau, h_\tau]$, which implies $\phi_\tau''(x) \in [-\frac{1}{1-2\tau}h_0, \frac{1}{1-2\tau}h_0]$. Then $x^2\phi_\tau''(x) \in [-\frac{1}{1-2\tau}x^2h_0, \frac{1}{1-2\tau}x^2h_0]$. This, in turn, implies that $F(\phi_\tau)(x) \in [-\frac{2}{1-2\tau}F(\phi_0)(0) - \delta'(\tau), \frac{2}{1-2\tau}F(\phi)(0) + \delta'(\tau)] = [-2F(\phi_\tau)(0) - \delta'(\tau), 2F(\phi_\tau)(0) + \delta'(\tau)]$. Then, for τ small enough, the lemma holds for a C^2 ϕ (See Figure 3). This implies it holds for a C^∞ ϕ.

Our bump functions are defined in an interval of lenght ϵ^2, so let us notice that if the lemma holds for ϕ with support in $[0,1]$, then it holds for ϕ^λ such that $\phi^\lambda(x) := \phi(\lambda x)$. It holds also if ϕ is multiplied by a constant.

$\square$

Remark 5.4. This is an important lemma because it amounts to say that if the curvature

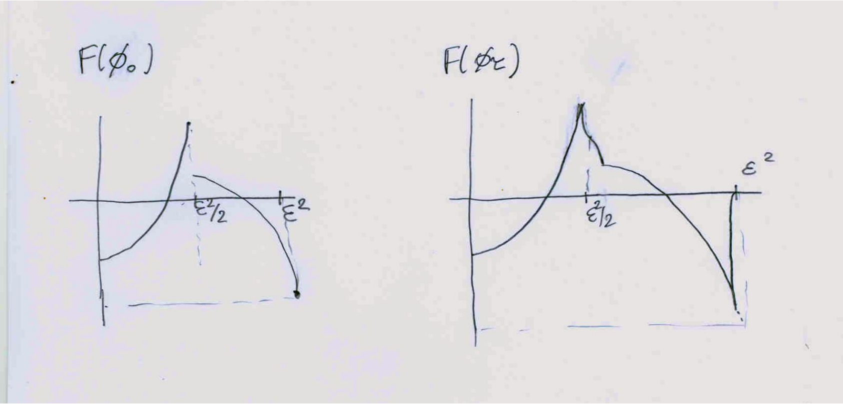

Figure 3: $F(\varphi_0)$ and $F(\varphi_\tau)$

is changed by $\frac{1}{4}$ along the closed geodesic γ, then the curvature is deformed by $\pm\frac{1}{2}$ in the weak directions of the splitting of the geodesic flow, so the curvature for the strong directions is still greater than in the other directions. This explains in a rough way why the geodesic flow still preserves the strong directions.

5.3 Extension of the cone property for some vectors

First, we calculate the preservation of the cones in the initial case, the case of the original geodesic flow, the geodesic flow of the Kahler Riemannian manifold of constant and negative holomorphic curvature.

We use the following family of trajectories for the system:

$$q(t,u) = \pi \circ \phi_t(z(u)),$$

$q(t,u), |u| < \epsilon$.
The Jacobi field is given by

$$\xi = \frac{dq}{du}|_{u=0}, \eta = \frac{Dv}{du}|_{u=0} = \frac{D}{du}|_{u=0}\frac{dq}{dt}.$$

So the following equations hold:

$$\frac{D\xi}{dt} = \eta, \frac{D\eta}{dt} = -R(v,\xi)v.$$

The quantity

$$\frac{(g(\xi,Jv)+g(\eta,Jv))^2}{g(\xi,\xi)+g(\eta.\eta)}$$

indicates twice the cosine of the angle between the vector $(\xi,\eta) \in T_\theta SM$ and $(Jv,Jv) \in T_\theta SM$, $\theta = (x,v)$. So, it is the same to prove that the cone fields are properly invariant or to prove that the cosine of this angle increases under the action of the derivative of the geodesic flow, for vector in the boundary of the cone fields, or $(\xi, \eta) \in T_\theta SM$ such that

$$\frac{(g(\xi, Jv) + g(\eta, Jv))^2}{g(\xi, \xi) + g(\eta.\eta)} = C \in (1, 2).$$

Remember that if g is a Kahler metric then:

$$\frac{d}{dt} g(u, v) = g\left(\frac{Du}{dt}, v\right) + g\left(u, \frac{Dv}{dt}\right),$$

$$\frac{D}{dt} Jv = 0.$$

Then, for

$$\frac{(g(\xi, Jv) + g(\eta, Jv))^2}{g(\xi, \xi) + g(\eta.\eta)} = C \in (1, 2),$$

the following holds:

$$\begin{aligned} \frac{d}{dt} \frac{(g(\xi, Jv) + g(\eta, Jv))^2}{g(\xi, \xi) + g(\eta.\eta)} &= 2\frac{(g(\xi, Jv) + g(\eta, Jv))}{g(\xi, \xi) + g(\eta.\eta)}(g(\eta, Jv) - R(v, \xi, v, Jv)) \\ &- 2\frac{(g(\xi, Jv) + g(\eta, Jv))^2}{(g(\xi, \xi) + g(\eta.\eta))^2}(g(\xi, \eta) - R(v, \xi, v, \eta)). \end{aligned}$$

But for the Kahler metric of constant holomorphic curvature, the curvature tensor is [KN]:

$$R(v, \xi, v, \eta) = -\frac{1}{4} g(\xi, \eta) - \frac{3}{4} g(\xi, Jv) g(\eta, Jv).$$

So, we have:

$$\begin{aligned} &\frac{d}{dt} \frac{(g(\xi, Jv) + g(\eta, Jv))^2}{g(\xi, \xi) + g(\eta.\eta)} = 2\frac{(g(\xi, Jv) + g(\eta, Jv))^2}{g(\xi, \xi) + g(\eta.\eta)} - 2\frac{(g(\xi, Jv) + g(\eta, Jv))^2}{(g(\xi, \xi) + g(\eta.\eta))^2} \\ &(\frac{5}{4} g(\xi, \eta) + \frac{3}{4} g(\xi, Jv) g(\eta, Jv)) = 2\frac{(g(\xi, Jv) + g(\eta, Jv))^2}{(g(\xi, \xi) + g(\eta.\eta))^2}(g(\xi, \xi) + g(\eta, \eta) - \\ &\frac{5}{4} g(\xi, \eta) + \frac{3}{4} g(\xi, Jv) g(\eta, Jv)) = 2\frac{(g(\xi, Jv) + g(\eta, Jv))^2}{(g(\xi, \xi) + g(\eta.\eta))^2}(\frac{5}{8} g(\xi - \eta, \xi - \eta) + \\ &\frac{3}{8} g(\xi, \xi) - \frac{3}{4} g(\xi, Jv) g(\eta, Jv) - \frac{3}{8} g(\eta, \eta)). \end{aligned}$$

Since $|g(\xi, Jv)| \leq g(\xi, \xi)$ and $|g(\eta, Jv)| \leq g(\eta, \eta)$, with equality only if (ξ, η) is a multiple of (Jv, Jv), the derivative above is positive, and this means that the cones are properly invariant under the action of the derivative of the geodesic flow.

The value of the derivative is the same if (ξ, η) is multiplied by a scalar. Suppose $g(\xi,\xi) + g(\eta,\eta) = 1$, then the derivative is:

$$\begin{aligned}\frac{d}{dt}\frac{(g(\xi, Jv) + g(\eta, Jv))^2}{g(\xi,\xi) + g(\eta.\eta)} &\geq 2C(\frac{3}{8}g(\xi,\xi) - \frac{3}{4}g(\xi, Jv)g(\eta, Jv) + \frac{3}{8}g(\eta,\eta)) \\ &\geq \frac{3}{8}C(2 - C).\end{aligned}$$

To get the exponential growth, we need to calculate:

$$\frac{d}{dt}(g(\xi, Jv) + g(\eta, Jv))^2 = 2(g(\xi, Jv) + g(\eta, Jv))(g(\eta, Jv) - R(v, \xi, v, Jv))$$

$$= 2(g(\xi, Jv) + g(\eta, Jv))^2.$$

This implies that the vectors inside the cone grow at the rate of e^t.

Now we do the same to the new metric g^*, first for vectors $v = (v_0, 0, \ldots, 0) \in S^*M$. By the formula of the bump function Φ_k we have that, as ϵ goes to zero, the partial derivatives of second order of α which do not involve the direction of $\frac{\partial}{\partial_{x_k}}$ go to zero. The only one that does not shrink is $\partial^2_{k,k}\Phi_k$.

So, the following holds:

$$R^*_{010k} \approx R_{010k}, k = 2, \ldots, 2n - 1,$$

$$R^*_{0k0k} \approx R_{0k0k} - \frac{1}{2}\partial^2_{k,k}\alpha$$

If $v = (v_0, 0, \ldots, 0)$ then:

$$R^*_{v\xi v\eta} \approx R_{v\xi v\eta} - \frac{1}{2}\partial^2_{\xi\eta}\alpha v_0^2$$

$$\approx R_{v\xi v\eta} - \frac{1}{2}\sum_{k=2}^{2n-1}\partial^2_{kk}\alpha v_0^2\xi_k\eta_k.$$

When we use the symbol $\approx$ we mean that the difference between the left side and the right side is of order ϵ. It depends on the size of $|\alpha|, |\partial\alpha|, |\partial^2_{ij}\alpha|, i \neq j$, and the size of $supp(\Phi_i), i = 1, \ldots, 2n-1$. To calculate the derivative of the closing of the cone precisely:

$$\begin{aligned}
&\frac{d}{dt}\frac{(g^*(\xi,Jv)+g^*(\eta,Jv))^2}{g^*(\xi,\xi)+g^*(\eta.\eta)}-2\frac{(g^*(\xi,Jv)+g^*(\eta,Jv))^2}{(g^*(\xi,\xi)+g^*(\eta.\eta))^2}\biggl(\frac{5}{8}g^*(\xi-\eta,\xi-\eta)\\
&+\frac{3}{8}g^*(\xi,\xi)-\frac{3}{4}g^*(\xi,Jv)g^*(\eta,Jv)+\frac{1}{2}\sum_{k=2}^{2n-1}\partial^2_{kk}\alpha v_0^2\xi_k\eta_k+\frac{3}{8}g^*(\eta,\eta)\biggr)\\
&=2\frac{(g^*(\xi,Jv)+g^*(\eta,Jv))^2}{(g^*(\xi,\xi)+g^*(\eta,\eta))^2}\left(\left(\frac{g^*(\eta,Jv)-R^*(v,\xi,v,Jv)}{g^*(\xi,Jv)+g^*(\eta,Jv)}\right)(g^*(\xi,\xi)\right.\\
&+g^*(\eta,\eta))+\left(\frac{g^*(\xi,\frac{D^*}{dt}Jv)+g^*(\eta,\frac{D^*}{dt}Jv)}{g^*(\xi,Jv)+g^*(\eta,Jv)}\right)(g^*(\xi,\xi)+g^*(\eta,\eta))\\
&-g^*(\xi,\eta)+R^*(v,\xi,v,\eta)-\frac{5}{8}g^*(\xi-\eta,\xi-\eta)-\frac{3}{8}g^*(\xi,\xi)-\frac{3}{8}g^*(\eta,\eta)\\
&\left.+\frac{3}{4}g^*(\xi,Jv)g^*(\eta,Jv)-\frac{1}{2}\sum_{k=2}^{2n-1}\partial^2_{kk}\alpha v_0^2\xi_k\eta_k\right)\\
&=2\frac{(g^*(\xi,Jv)+g^*(\eta,Jv))^2}{(g^*(\xi,\xi)+g^*(\eta.\eta))^2}\left(\left(\frac{g^*(\xi,\frac{D^*}{dt}Jv)+g^*(\eta,\frac{D^*}{dt}Jv)}{g^*(\xi,Jv)+g^*(\eta,Jv)}\right)(g^*(\xi,\xi)\right.\\
&+g^*(\eta.\eta))-\left(\frac{g^*(\xi,Jv)+R^*(v,\xi,v,Jv)}{g^*(\xi,Jv)+g^*(\eta,Jv)}\right)(g^*(\xi,\xi)+g^*(\eta.\eta))+\frac{1}{4}g^*(\xi,\eta)\\
&\left.+\frac{3}{4}g^*(\xi,Jv)g^*(\eta,Jv)-\frac{1}{2}\sum_{k=2}^{2n-1}\partial^2_{kk}\alpha v_0^2\xi_k\eta_k+R^*(v,\xi,v,\eta)\right).
\end{aligned}$$

If C is the opening of the cone and $g^*(\xi,\xi)+g^*(\eta,\eta)=1$, because the derivative does not depend on the norm of the (ξ,η), the equation above is:

$$=2C(C^{-\frac{1}{2}}(g^*\left(\xi,\frac{D^*}{dt}Jv\right)+g^*\left(\eta,\frac{D^*}{dt}Jv\right))-C^{-\frac{1}{2}}(g^*(\xi,Jv)+R^*(v,\xi,v,Jv))$$

$$+\frac{1}{4}g^*(\xi,\eta)+\frac{3}{4}g^*(\xi,Jv)g^*(\eta,Jv)-\frac{1}{2}\sum_{k=2}^{2n-1}\partial^2_{kk}\alpha v_0^2\xi_k\eta_k+R^*(v,\xi,v,\eta)).$$

Then:

$$|g^*(\xi,Jv)+R^*(v,\xi,v,Jv)|\le|R^*(v,\xi,v,Jv)-R(v,\xi,v,Jv)|+|g^*(\xi,Jv)-g(\xi,Jv)|\,.$$

Since $|g^*(\xi,Jv)-g(\xi,Jv)|$ is dependent on $|\alpha|$, and $|R^*(v,\xi,v,Jv)-R(v,\xi,v,Jv)|+|g^*(\xi,Jv)-g(\xi,Jv)|$ is dependent on $|\alpha|$, $|\partial\alpha|$, and $|\partial^2_{1\xi}\alpha|$, and these terms are limited by $M\epsilon$, we can say that, for some big enough M_1 independent of ϵ:

$$|g^*(\xi, Jv) + R^*(v, \xi, v, Jv)| \leq |R^*(v, \xi, v, Jv) - R(v, \xi, v, Jv)| + \\ |g^*(\xi, Jv) - g(\xi, Jv)| \leq M_1 \epsilon.$$

For the same reasons:

$$\left| g^*\left(\xi, \frac{D^*}{dt} Jv\right) + g^*\left(\eta, \frac{D^*}{dt} Jv\right) \right| \leq M_0 \left\| g^* - g \right\|_{C^1} (|\xi|^* + |\eta|^*) \leq M_1 \epsilon.$$

$$\left| \frac{1}{4} g^*(\xi, \eta) + \frac{3}{4} g^*(\xi, Jv) g^*(\eta, Jv) - \frac{1}{2} \sum_{k=2}^{2n-1} \partial^2_{kk} \alpha v_0^2 \xi_k \eta_k + R^*(v, \xi, v, \eta) \right| \leq M_1 \epsilon.$$

Suppose M_1 sufficiently big to be the same in the three inequalities above. So we have:

$$\left| \frac{d}{dt} \frac{(g^*(\xi, Jv) + g^*(\eta, Jv))^2}{g^*(\xi, \xi) + g^*(\eta.\eta)} - 2 \frac{(g^*(\xi, Jv) + g^*(\eta, Jv))^2}{(g^*(\xi, \xi) + g^*(\eta.\eta))^2} \left(\frac{5}{8} g^*(\xi - \eta, \xi - \eta) + \right. \right. \\ \left. \left. \frac{3}{8} g^*(\xi, \xi) - \frac{3}{4} g^*(\xi, Jv) g^*(\eta, Jv) + \frac{1}{2} \sum_{k=2}^{2n-1} \partial^2_{kk} \alpha v_0^2 \xi_k \eta_k + \frac{3}{8} g^*(\eta, \eta) \right) \right| \\ \leq 2C(2C^{-\frac{1}{2}} M_1 + M_1)\epsilon = M_2 \epsilon.$$

Let us analyse the following expression over the initial closed geodesic:

$$\left(\frac{3}{8} g^*(\xi, \xi) - \frac{3}{4} g^*(\xi, Jv) g^*(\eta, Jv) + \frac{1}{2} \sum_{k=2}^{2n-1} \partial^2_{kk} \alpha v_0^2 \xi_k \eta_k + \frac{3}{8} g^*(\eta, \eta) \right) = \\ \frac{3}{8} (\xi_1^2 + \xi_2^2 + \xi_3^2 + \eta_1^2 + \eta_2^2 + \eta_3^2 - 2\xi_1 \eta_1 + \frac{4}{3} \sum_{k=2}^{2n-1} \partial^2_{kk} \alpha v_0^2 \xi_k \eta_k).$$

The expression $\xi_1^2 + \eta_1^2 + \xi_2^2 + \eta_2^2 + \ldots + \xi_{2n-1}^2 + \eta_{2n-1}^2 - 2\xi_1\eta_1 + \frac{4}{3} \sum_{k=2}^{2n-1} \partial^2_{kk} \alpha v_0^2 \xi_k \eta_k$ equals $(\xi_1 - \eta_1)^2 + \xi_2^2 - \frac{2}{3}\xi_2\eta_2 + \eta_2^2 + \ldots + \xi_{2n-1}^2 - \frac{2}{3}\xi_{2n-1}\eta_{2n-1} + \eta_{2n-1}^2 = (\xi_1 - \eta_1)^2 + (\xi_2 - \frac{1}{3}\eta_2)^2 + \frac{8}{9}\eta_2^2 + \ldots + (\xi_{2n-1} - \frac{1}{3}\eta_{2n-1})^2 + \frac{8}{9}\eta_{2n-1}^2$, which is positive in the border of the cone with opening C. This implies that along the closed geodesic γ the cone is preserved, but that we already knew. We need to prove the positivity of the derivative along the other geodesics of the flow. So, we need the following:

$$inf_{a \in [-1-\frac{\delta}{2}, 1+\frac{\delta}{2}]} inf \{ \xi_2^2 + \eta_2^2 + \ldots \xi_{2n-1}^2 - \frac{4a}{3} \sum_{k=2}^{2n-1} \xi_k \eta_k + \eta_{2n-1}^2 \} \geq L(A, B) > 0,$$

for any (ξ, η) in the boundary of the cone with opening $C \in [A, B] \subset (1, 2)$.

Because g^* is a C^∞ metric, and its coordinates along γ are δ_{ij}, if the neighborhood of γ is sufficiently small, if ϵ is small enough, we can conclude:

$$inf_{x\in supp(\alpha)} inf\{(g^*(\xi,\xi) - 2g^*(\xi,Jv)g^*(\eta,Jv) +$$
$$\frac{4}{3}\sum_{k=2}^{2n-1}\partial^2_{kk}\alpha v_0^2\xi_k\eta_k + g^*(\eta,\eta))\} \geq \frac{1}{2}L(A,B) > 0.$$

So:

$$inf_{x\in supp(\alpha)} inf\{\frac{3}{8}g^*(\xi,\xi) - \frac{3}{4}g^*(\xi,Jv)g^*(\eta,Jv) + \frac{1}{2}\sum_{k=2}^{2n-1}\partial^2_{kk}\alpha v_0^2\xi_k\eta_k$$
$$+\frac{3}{8}g^*(\eta,\eta))\} \geq L'(A,B) = \frac{3}{16}L(A,B) > 0.$$

This implies that, if $\epsilon < \frac{1}{2M_2}L'(A,B)$, for (ξ,η) in the boundary of the cone with opening $C \in [A,B] \subset (1,2)$, and for $v = (v_0,0,\ldots,0)$, then the derivative is positive.

5.4 Extension of the cone property to a band

Now we are going to show that this derivative is positive not only for vectors of the type $v = (v_0,0,\ldots,0)$, but for vectors which are close to $(1,0,0,\ldots,0)$.

We are going to consider $v \in S^*M$ such that $|v_i| < \theta, i = 1,2,\ldots,2n-1$. For this vectors we have:

$$R^*(v,\xi,v,\eta) - R(v,\xi,v,\eta) \approx -\frac{1}{2}\sum_{k=2}^{2n-1}\partial^2_{kk}\alpha(v_k^2\xi_0\eta_0 + v_0^2\xi_k\eta_k - v_0v_k(\xi_0\eta_k + \xi_k\eta_0)).$$

This is so because (1) implies the following relation:

$$R^*_{ijkl} - R_{ijkl} \approx -\frac{1}{2}(\partial^2_{ik}\Delta g_{jl} + \partial^2_{jl}\Delta g_{ik} - \partial^2_{il}\Delta g_{jk} - \partial^2_{jk}\Delta g_{il}), \qquad (2)$$

where $\approx$ means that the rest of the equation depends on α and $\partial\alpha$, and $\Delta g_{ij} := g^*_{ij} - g_{ij}$. So we can say that:

$$\left|R^*(v,\xi,v,\eta) - R(v,\xi,v,\eta) + \frac{1}{2}\sum_{k=2}^{2n-1}\partial^2_{kk}\alpha v_0^2\xi_k\eta_k\right| \leq M_1\epsilon + M_0|\theta|(\|\xi\|^*\|\eta\|^*).$$

So, for the derivative we have:

$$\begin{aligned}&\left|\frac{d}{dt}\frac{(g^*(\xi,Jv)+g^*(\eta,Jv))^2}{g^*(\xi,\xi)+g^*(\eta.\eta)}-2\frac{(g^*(\xi,Jv)+g^*(\eta,Jv))^2}{(g^*(\xi,\xi)+g^*(\eta.\eta))^2}\left(\frac{5}{8}g^*(\xi-\eta,\xi-\eta)\right.\right.\\&\left.\left.+\frac{3}{8}g^*(\xi,\xi)-\frac{3}{4}g^*(\xi,Jv)g^*(\eta,Jv)+\frac{1}{2}\sum_{k=2}^{2n-1}\partial^2_{kk}\alpha v_0^2\xi_k\eta_k+\frac{3}{8}g^*(\eta,\eta)\right)\right|\\&\leq M_2\epsilon+M_0|\theta|(\|\xi\|^*\|\eta\|^*).\end{aligned}$$

So, if we calculate for (ξ,η) in $g^*(\xi,\xi)+g^*(\eta,\eta)=1$, we have that if $|\theta|<\frac{1}{4M_0}L'(A,B)$ and $\epsilon<\frac{1}{2M_2}L'(A,B)$, then:

$$\frac{d}{dt}\frac{(g^*(\xi,Jv)+g^*(\eta,Jv))^2}{g^*(\xi,\xi)+g^*(\eta.\eta)}\geq\frac{1}{2}L'(A,B)>0$$

Then we conclude that, in the band $\{v\in S^*M : v\ \theta\text{-close to }(1,0,\ldots,0)\}$ the cones are properly invariant for the geodesic flow.

5.5 The cone property outside the band

For vectors that are not θ-close to $(1,0,\ldots,0)$, for θ as defined in the previous subsection, which we are going to call 'transversal' to γ', we do not have preservation of the cones. But this is not at all a problem if we choose an ϵ small enough such that the cone with openning B stays inside the cone with opening A. This is possible because α is C^1 close to zero, the second derivative of α is limited and this limitation does not depend on ϵ. So:

$$\frac{d}{dt}\frac{(g^*(\xi,Jv)+g^*(\eta,Jv))^2}{g^*(\xi,\xi)+g^*(\eta.\eta)}\geq M$$

As ϵ goes to 0, the support of the deformation of the metric shrinks. As it shrinks, the time that the geodesics take to cross this neighborhood of the geodesic γ goes to zero. So, as we can control the time which these geodesics spend inside the neighborhood, we choose an ϵ such that the cone with opening B stays inside the cone of openning A.

Let us be more precise:

Proposition 5.5. *The time which transversal geodesics cross the neighborhood of the deformation of the metric g is comparable to ϵ.*

Proof. To see that the time spent is comparable to ϵ we need to express the geodesic vector field in Fermi coordinates of the neighborhood. We can use Fermi coordinates now because we don't need the coordinates in the whole neighborhood of the closed geodesic γ in this case. The maps $d\pi$ and K are:

$$d\pi\xi=(\xi_0,\xi_1,\ldots,\xi_{2n-1}),$$

$$K\xi = \left(\xi_{2n+k} + \sum_{i,j=0}^{2n-1} \Gamma_{ij}^{*k} v_i \xi_j\right)_{k=0}^{2n-1}.$$

So, the pre-image of $(v, 0)$ by the map $(d\pi, K)$ is:

$$\left(v_0, v_1, \ldots, v_{2n-1}, -\sum_{i,j=0}^{2n-1} \Gamma_{ij}^{*0} v_i v_j, -\sum_{i,j=0}^{2n-1} \Gamma_{ij}^{*1} v_i v_j, \ldots, -\sum_{i,j=0}^{2n-1} \Gamma_{ij}^{*2n-1} v_i v_j\right).$$

Since g^* is C^∞ and along the geodesic γ, $\Gamma_{ij}^{*k} = 0$, then, if ϵ is sufficiently small, the geodesic vector field is approximately $(v_0, v_1, \ldots, v_{2n-1}, 0, 0, \ldots, 0)$.

Since the second part of the geodesic vector field is small as ϵ is small, we can say that geodesics such that $|v_i| \geq \theta$ for some $i = 1, \ldots, 2n-1$ cross the neighborhood in at most $\frac{\epsilon}{\theta}$, and they leave the neighborhood at least $\frac{\theta}{2}$ far from $(1, 0, \ldots, 0)$, or, better said, outside the set $\{v \in S^*M : |v_i| < \frac{\theta}{2}, i = 1, 2, \ldots, 2n-1\}$.

□

After these orbits leave the neighborhood, they spent some time outside it. As the set of these orbits is a compact set, the infimum is positive. Let us say they spend at least T_ϵ outside the neighborhood. As ϵ goes to zero, T_ϵ does not goes to zero. If it did, we could get a sequence of geodesics outside $\{v \in S^*M : |v_i| < \frac{\theta}{2}, i = 1, 2, \ldots, 2n-1\}$ which would spend very little time outside the neighborhood of γ before enter it again. So, in the limit, there would be a contradiction with the unicity of the solutions of the ordinary differential equations of the geodesic flow. So the time spent outside the neighborhood of γ is bounded from below - let us say it's bounded from below by T. This means that we can choose ϵ so that the quotient between the time spent inside and the time spent outside of the neighborhood of γ is as small as we want. As small as it is necessary for the preservation of the strong unstable and strong stable cones.

Outside the neighborhood of the deformation the following holds:

$$\frac{d}{dt}\frac{(g^*(\xi, Jv) + g^*(\eta, Jv))^2}{g^*(\xi, \xi) + g^*(\eta, \eta)} = \frac{d}{dt}\frac{(g(\xi, Jv) + g(\eta, Jv))^2}{g(\xi, \xi) + g(\eta, \eta)} \geq \frac{3}{8}C(2 - C),$$

for (ξ, η) in the boundary of the cone of openning C. So, for cones with border in $[A, B]$, we have:

$$\frac{d}{dt}\frac{(g^*(\xi, Jv) + g^*(\eta, Jv))^2}{g^*(\xi, \xi) + g^*(\eta, \eta)^*} \geq \frac{3}{8}B(2 - B).$$

So we choose A' such that $|A' - B| < \frac{3}{16}B(2 - B)T$. This ensures that outside the neighborhood the geodesic flow sends the cone with opening A' inside the cone with opening B in time $\frac{T}{2}$. For ϵ sufficiently small, with the inferior limit of the derivative not depending on ϵ, the cone with opening B is not sent outside the cone with opening A'.

So, we do not have exactly the proper invariance of the cones, it fails in an interval of length as small as we want for each geodesic of length T. But this is enough. It is enough because after that it takes an interval of length $\frac{T}{2}$ for the cones to be properly contained, or, for the map $\mathcal{F}_t^\epsilon$, it takes an interval of length $\frac{T}{2}$ for the cone in the projective to be properly invariant. So we get an fixed section for an interval of lenght at least $\frac{T}{3}$, and, by the same reasoning of the proposition 2.12, the invariant section is unique and invariant for all t positive. The same happens for the stable invariant direction, because for a geodesic flow the 'past' of the orbit of $v \in S(SM)$ is the future of the orbit of $-v$.

5.6 Exponential growth of the Jacobi fields

So, the strong unstable cone is preserved by the new geodesic flow. By simmetry, or by the reversibility of geodesic flows, the strong stable cone is preserved too. But preservation of these cones only proves that there are invariant subbundles with domination. We have to show that there is exponential growth along these strong directions.

Outside the neighborhood of γ where we deform the metric, the following holds:

$$\begin{aligned}
&\frac{d}{dt}(g^*(\xi, Jv) + g^*(\eta, Jv))^2 = \frac{d}{dt}(g(\xi, Jv) + g(\eta, Jv))^2 \\
&= 2(g(\xi, Jv) + g(\eta, Jv))(g(\eta, Jv) - R(v, \xi, v, Jv)) \\
&= 2(g(\xi, Jv) + g(\eta, Jv))^2 = 2(g^*(\xi, Jv) + g^*(\eta, Jv))^2.
\end{aligned}$$

For vectors $v \in \{v \in S^*M : v\ \theta\text{-close to } (1, 0, \dots, 0)\}$:

$$\begin{aligned}
&\frac{d}{dt}(g^*(\xi, Jv) + g^*(\eta, Jv))^2 = 2(g^*(\xi, Jv) + g^*(\eta, Jv)) \\
&(g^*(\eta, Jv) + g^*\left(\xi, \frac{D^*}{dt}Jv\right) + g^*\left(\eta, \frac{D^*}{dt}Jv\right) - R^*(v, \xi, v, Jv)) \\
&\geq 2(g^*(\xi, Jv) + g^*(\eta, Jv))(g^*(\xi, Jv) + g^*(\eta, Jv) - L\epsilon(|\xi|^* + |\eta|^*)) \\
&\geq 2(1 - 2L\epsilon)(g^*(\xi, Jv) + g^*(\eta, Jv))^2.
\end{aligned}$$

So for ϵ sufficiently small we have exponential growth in this case. Now, in the case of v 'transversal' to γ:

$$\begin{aligned}
&\frac{d}{dt}(g^*(\xi, Jv) + g^*(\eta, Jv))^2 = 2(g^*(\xi, Jv) + g^*(\eta, Jv))(g^*(\eta, Jv) + \\
&g^*\left(\xi, \frac{D^*}{dt}Jv\right) + g^*\left(\eta, \frac{D^*}{dt}Jv\right) - R^*(v, \xi, v, Jv)) \\
&\approx K(g^*(\xi, Jv) + g^*(\eta, Jv))^2,
\end{aligned}$$

for some $K \in \mathbb{R}$ which does not depend on ϵ.

So, if we take any geodesic $c : [0, T] \to M$, we have that it takes only ϵ inside the neighborhood, and 'transversal' to γ'. So, if we call $f(t) := (g^*(\xi, Jv) + g^*(\eta, Jv))^2$, we have that $f'(t) \approx 2f(t)$ for time $T - \epsilon$ and $f'(t) \geq Kf(t)$ for time ϵ, at most. This implies:

$$\begin{aligned} \int_0^T (logf)'(s)ds &\geq 2(T-\epsilon) + K\epsilon = 2T + (K-2)\epsilon \Rightarrow \\ logf(T) - logf(0) &\geq 2T + (K-2)\epsilon \Rightarrow f(T) \geq f(0)e^{(2T+(K-2)\epsilon)}. \end{aligned}$$

So for ϵ sufficiently small, we have that f grows exponentially for the (ξ, η) inside the unstable cone we have exponential growth.

5.7 Conclusion

So, we proved the proper invariance of the unstable and stable cones. And we proved the exponential expansion or contraction respectively, in the previous subsection. Then we conclude:

Theorem 5.6. *For every compact Kahler manifold* (M, ω, J) *of dimension at least* 4, *such that its Kahler metric has constant negative holomorphic curvature* -1, *there is a metric* g^* *in* M *such that its geodesic flow is partially hyperbolic but not hyperbolic.*

Corollary 5.7. *There is an open set* $\mathcal{U}$ *of metrics in the set of metrics of* (M, ω, J) *such that for* $g \in \mathcal{U}$, *the geodesic flow of* g *is partially hyperbolic but not Anosov.*

Proof. We can make the closed geodesic γ, which is a geodesic for both metrics g and g^*, a quasi-elliptic nondegenerate closed geodesic. The linearized Poincare map of a quasi-elliptic nondegenerate orbit has eigenvalues on the unit circle but they are different than one. We only need to multiply the bump function by a constant greater but sufficiently close to 1 such that the geodesic flow remains partially hyperbolic. Since quasi-elliptic nondegenerate closed geodesics are persistent, there is an open neighborhood of g^* in the set of metrics of M such that all metrics in this open set are partially hyperbolic, and are far away from the set of Anosov metrics. □

Corollary 5.8. *There is an open set* $\mathcal{V}$ *of hamiltonians in the set of hamiltonians of* (TM, ω), *near geodesic hamiltonians, such that for* $h \in \mathcal{U}$, *the hamiltonian flow of* h *is partially hyperbolic but not Anosov.*

Proof. For the same reasons of the previous corollary there is an open set of hamiltonians with the same property, near geodesic hamiltonians. □

6 Further considerations

6.1 Open problems

We should look for more properties of this example. Two natural questions are:

Problem 1. Is this example transitive? Is it robustly transitive?

Statistical properties are as important as topological ones, so we could ask too:

Problem 2. Is this example of geodesic flow ergodic?

We could think of this deformation as an one parameter family of deformations:

Definition 6.1. g_s is a one parameter family such that outside the neighborhood of the closed geodesic γ of g, g_s and g coincide. As we defined above, in the ball of radius ϵ_0 around γ, with the injectivity radius of (M, g) bigger than ϵ_0, and $\{E_i\}$ an orthornormal frame of parallel vector fields along γ:

$$\Psi : [0, T] \times (-\epsilon_0, \epsilon_0)^{2n-1} \to M : (t, x) \to \exp_{\gamma(t)}(\sum_{i=1}^{2n-1} x_i E_i(t))$$

is a coordinate system where g has the following coordinates:

$$g_{ij} = \delta_{ij}, \partial_k g_{ij} = 0,$$

g^* was defined as:

$$g^*_{00} = g_{00} + x^2_{2n-1}\Phi(t, x), g^*_{ij} = g_{ij}, (i, j) \neq (0, 0).$$

For $s \in [0, 1]$ we define g_s as:

$$(g_s)_{00} = g_{00} + s x^2_{2n-1}\Phi(t, x), (g_s)_{ij} = g_{ij}, (i, j) \neq (0, 0).$$

For this one parameter family, the geodesic flow of g_s is hyperbolic for $s \in [0, s_0)$. The geodesic flow of the metric g_{s_0} is partially hyperbolic, for the same reason as for g^*, and it is not hyperbolic. So, we could ask:

Problem 3. Is the geodesic flow of the metric g_{s_0} transitive? Is it ergodic? Is it robustly transitive? Is it conjugate to the geodesic flow of g_s for $s < s_0$?

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