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The Cauchy problem for The dispersive Kuramoto-Velarde equation.

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Rio de Janeiro
June, 2006

Abstract.

The purpose of this work is the study of the well-posedness of the initial value problem (IVP) associated to the dispersive Kuramoto-Velarde equation. In the dissipative case, we prove local well-posedness in Sobolev spaces $H^s(\mathbb{R})$ for $s > -1$, and ill-posedness in $H^s(\mathbb{R})$ for $s < -1$. In the purely dispersive case, we first prove an ill-posedness result, which states that the flow map data-solution cannot be of class C^2 in any Sobolev space $H^s(\mathbb{R})$, for $s \in \mathbb{R}$. Then, we prove a well-posedness result in weighted Besov spaces for small initial data.

Contents

Introduction.

The question of the well-posedness for the Cauchy problem associated with a partial differential equation was first raised by Hadamard in [11]. He provided in the case of the Laplace equation, an example of initial data for which the continuous dependence of the map solution of the associated Cauchy problem failed. Later this notion was refined in the case of an *initial value problem* (IVP) (see for example the work of Kato in [15]). We will say that an initial value problem is *locally well-posed* in some functional space X, if for all initial datum ϕ in X, there exists a time $T > 0$ and a unique solution u of the integral equation associated to the IVP (*existence and uniqueness*), such that $u \in C([0,T];X)$ (*persistence*) and the flow map data-solution is (at least) continuous from a neighborhood of ϕ in X into $C([0,T];X)$ (*continuous dependence*). If T can be taken arbitrarily large, we say the well-posedness is *global*.

Then a well-posed initial value problem generates an infinite dimensional dynamical system on the functional space X. Moreover, because of uniqueness and continuous dependence properties, the "well-posed" solutions, even if there are only solutions of the original equation in a weak sense, can be approximated by classical solutions in the topology of $C([0,T];X)$. This allows them to enjoy some conservation laws and other formal identities of the equation, a priori reserved to classical solutions.

Which functional space X will be considered is of fundamental importance: if X is too large, well-posedness fails, if it is too small, the problem lacks physical relevance. The most common choice is the Sobolev space, however in some cases, physical considerations impose more complicated spaces, such as weighted Sobolev or Besov spaces for example. Therefore, when studying an IVP, the first step is to investigate in which spaces X well-posedness occurs.

The purpose of of this work is the study of the well-posedness of the IVP associated to the dispersive Kuramoto-Velarde equation (KdV-KV)

$$\begin{cases} \partial_t u + \delta\partial_x^3 u + \mu(\partial_x^4 u + \partial_x^2 u) + \alpha(\partial_x u)^2 + \gamma u \partial_x^2 u = 0, \\ u(0) = \phi \end{cases} \tag{1}$$

where $x \in \mathbb{R}$, $t \in \mathbb{R}_+$, u is a real-valued function and μ, δ, α and γ are constants such that $\mu \geq 0$ and $\delta \neq 0$. When $\mu > 0$, this equation combines in its linear part dispersive and dissipative effects. It is a generalization of the Kuramoto-Velarde equation (KV)

$$\partial_t u + \mu(\partial_x^4 u + \partial_x^2 u) + \alpha(\partial_x u)^2 + \gamma u \partial_x^2 u = 0, \tag{2}$$

which corresponds to $\delta = 0$, and of the dispersive Kuramoto-Sivashinsky equation (KdV-KS)

$$\partial_t u + \delta\partial_x^3 u + \mu(\partial_x^4 u + \partial_x^2 u) + \alpha(\partial_x u)^2 = 0, \tag{3}$$

which corresponds to $\gamma = 0$. The KdV-KS equation arises in interesting physical situations, for example as a model for long waves on a viscous fluid flowing down an inclined plane (see [33]) and for deriving drift waves in a plasma (see [9]). The KV equation describes slow space-time variations of disturbances at interfaces, diffusion-reaction fronts and plasma instability fronts (see [6], [7]).

In the limit case $\mu = 0$, the linear part of the equation in (1) becomes purely dispersive. The IVP obtained is a particular case of the family of IVPs

$$\begin{cases} \partial_t u + \partial_x^{2j+1} u + P(u, \partial_x u, \dots, \partial_x^{2j} u), \quad x,\ t \in \mathbb{R},\ j \in \mathbb{N} \\ u(0) = \phi, \end{cases} \tag{4}$$

where

$$P : \mathbb{R}^{2j+1} \to \mathbb{R} \quad (\text{or } P : \mathbb{C}^{2j+1} \to \mathbb{C}) \tag{5}$$

is a polynomial having no constant or linear terms. The class of equations (4) contains the KdV hierarchy as well as higher-order models in water waves problems (see [18] for the references).

In the first chapter, we present some basic results and define the functional spaces to be used along this work.

In the second part, we will prove that the Cauchy problem (1) is locally well-posed in the Sobolev space $H^s(\mathbb{R})$ for every $s > -1$, in the dissipative case, *i.e.* when $\mu > 0$. The main idea is to use a fixed point argument in Bourgain's type spaces adapted to both linear parts, dispersive and dissipative, of the equation, as did Molinet and Ribaud for the KdV-Burgers equation in [25]. We also prove that these results are sharp in the sense that when $s < -1$, the IVP (1) cannot be solved in $H^s(\mathbb{R})$ using a fixed point theorem. These results imply in particular that the Cauchy problem associated to (3) is globally well-posed in $H^s(\mathbb{R})$ when $s > -1$, which improves a former result of Biagoni, Bona, Iorio and Scialom [3], who proved the well-posedness of (3) in $H^s(\mathbb{R})$ when $s \geq 1$.

Next, in the third chapter we turn out to the limit case $\mu = 0$ that we will also call the non-dissipative case. The first result we get is a negative one: we prove that the associated IVP problem is ill-posed in every space $H^s(\mathbb{R})$, $s \in \mathbb{R}$, in the sense that the flow map data-solution, when existing, cannot be C^2 at the origin. This means roughly speaking that without the dissipation, the dispersion of the linear part of (1) does not have enough regularizing effect to balance the nonlinearity $u\partial_x^2 u$. Furthermore, we extend this result to other higher-order nonlinear dispersive equations, as for example a higher-order Benjamin-Ono equation derived recently by Craig, Guyenne and Kalisch in [10] using a Hamiltonian perturbation theory. These results are inspired by those from Molinet, Saut and Tzvetkov for the KPI equation [29] and for the Benjamin-Ono equation [28].

Finally, if we want to obtain some well-posedness results for the non-dissipative equation, we have to restrict the functions spaces. For example, Kenig, Ponce and Vega proved the well-posedness of the IVP (1) with $\mu = 0$, in weighted Sobolev spaces [18]. In the fourth part, we improve these results introducing weighted Besov spaces. In particular, these spaces can be considered with fractional derivative exponents, which seems to be difficult for weighted Sobolev spaces. Nevertheless, we only proved well-posedness in these spaces for small initial data.

Chapter 1

Preliminaries.

Notation. For any positive numbers a and b, the notation $a \lesssim b$ means that there exists a positive constant c (independent of the data of the problem) such that $a \leq cb$. We also denote $a \sim b$ when $a \lesssim b$ and $b \lesssim a$.

1.1 Littlewood-Paley multipliers.

Fix a cutoff function χ such that

$$\chi \in C_0^\infty(\mathbb{R}), \quad 0 \leq \chi \leq 1, \quad \chi_{|_{[-1,1]}} = 1 \quad \text{and} \quad \operatorname{supp}(\chi) \subset [-2, 2]. \tag{1.1}$$

Define

$$\psi(\xi) := \chi(\xi) - \chi(2\xi) \quad \text{and} \quad \psi_j(\xi) := \psi(2^{-j}\xi), \tag{1.2}$$

so that

$$\sum_{j \in \mathbb{Z}} \psi_j(\xi) = 1, \ \forall \xi \neq 0 \quad \text{and} \quad \operatorname{supp}(\psi_j) \subset \{2^{j-1} \leq |\xi| \leq 2^{j+1}\}. \tag{1.3}$$

Next define the Littlewood-Paley multipliers by

$$\Delta_j f = \left(\psi_j \widehat{f}\right)^\vee = (\psi_j)^\vee * f \quad \forall f \in \mathcal{S}'(\mathbb{R}), \ \forall j \in \mathbb{Z}, \tag{1.4}$$

and

$$S_j f = \sum_{k \leq j} \Delta_k f \quad \forall f \in \mathcal{S}'(\mathbb{R}), \ \forall j \in \mathbb{Z}. \tag{1.5}$$

More precisely we have that

$$S_0 f = \left(\chi \widehat{f}\right)^{\vee} \quad \forall f \in \mathcal{S}'(\mathbb{R}), \tag{1.6}$$

This means that S_0 is the operator of restriction in the low frequencies. Note also that since $(\psi_j)^{\vee} = 2^j(\psi_j)^{\vee}(2^j\cdot)$, $\|(\psi_j)^{\vee}\|_{L^1} = C$ and then, by Young's inequality we have that for all $j \in \mathbb{Z}$

$$\|\Delta_j f\|_{L^p} \leq C\|f\|_{L^p}, \ \forall \ f \in L^p, \ \forall p \in [1, +\infty]. \tag{1.7}$$

We will need to commute S_0 and Δ_j with the operator of multiplication by x.

Lemma 1.1. *Let $f \in \mathcal{S}(\mathbb{R})$, then*

$$[S_0, x]f = S_0' f \quad \textit{where} \quad S_0' f = \frac{1}{2i\pi}\left((\frac{d}{d\xi}\chi)\widehat{f} \right)^{\vee} \tag{1.8}$$

$$[\Delta_j, x]f = \Delta_j' f \quad \textit{where} \quad \Delta_j' f = \frac{1}{2i\pi}\left(2^{-j}(\frac{d}{d\xi}\psi)(2^{-j}\cdot)\widehat{f} \right)^{\vee} \tag{1.9}$$

Proof. Let $f \in \mathcal{S}(\mathbb{R})$, then we compute using the properties of the Fourier transform

$$\begin{aligned} \left([\Delta_j, x]f\right)^{\wedge}(\xi) &= \psi_j(\xi)(xf)^{\wedge}(\xi) - \frac{1}{-2i\pi}\frac{d}{d\xi}(\psi_j \widehat{f})(\xi) \\ &= \frac{2^{-j}}{2i\pi}(\frac{d}{d\xi}\psi)(2^{-j}\xi)\widehat{f}(\xi), \end{aligned}$$

which leads to (1.9) and (1.8) follows by a similar way. □

Finally let $\tilde{\psi}$ be another smooth function supported in $\{1/4 \leq |\xi| \leq 4\}$ such that $\tilde{\psi} = 1$ on $\mathrm{supp}(\psi)$. We define $\tilde{\Delta}_j$ like Δ_j with $\tilde{\psi}$ instead of ψ which yields in particular the following identity

$$\tilde{\Delta}_j \Delta_j = \Delta_j. \tag{1.10}$$

1.2 Functional spaces.

Let $1 \leq p, q \leq \infty$, $T > 0$, the mixed "space-time" Lebesgue spaces are defined by

$$L^p_x L^q_T := \{u : \mathbb{R} \times [-T, T] \to \mathbb{R} \text{ measurable } : \ \|u\|_{L^p_x L^q_T} < \infty\},$$

and

$$L^q_T L^p_x := \{u : \mathbb{R}\times[-T,T]\to\mathbb{R} \text{ measurable } : \|u\|_{L^q_T L^p_x} < \infty\},$$

where

$$\|u\|_{L^p_x L^q_T} := \left(\int_{\mathbb{R}} \|u(x,\cdot)\|^p_{L^q([-T,T])} dx\right)^{1/p}, \tag{1.11}$$

and

$$\|u\|_{L^q_T L^p_x} := \left(\int_{-T}^{T} \|u(\cdot,t)\|^q_{L^p(\mathbb{R})} dt\right)^{1/q}. \tag{1.12}$$

We will also use the fractional Sobolev spaces. Let $s\in\mathbb{R}$, then

$$H^s(\mathbb{R}) := \{f\in\mathcal{S}'(\mathbb{R}) \ : \ (1+\xi^2)^{\frac{s}{2}}\widehat{f}(\xi)\in L^2(\mathbb{R})\}$$

with the norm

$$\|f\|_{H^s} := \|(1+\xi^2)^{s/2}\widehat{f}(\xi)\|_{L^2}, \tag{1.13}$$

and its homogeneous version

$$\dot{H}^s(\mathbb{R}) := \{f\in\mathcal{S}'(\mathbb{R}) \ : \ |\xi|^s\widehat{f}(\xi)\in L^2(\mathbb{R})\}$$

with the norm

$$\|f\|_{\dot{H}^s} := \||\xi|^s\widehat{\phi}(\xi)\|_{L^2}. \tag{1.14}$$

Consider $H^\infty(\mathbb{R}) = \bigcap_{s=0}^\infty H^s(\mathbb{R})$ with the induced metric. We recall the following of homogeneity identity

$$\|f(\lambda\cdot)\|_{\dot{H}^s} = \lambda^{s-1/2}\|f\|_{\dot{H}^s}, \ \forall\ \lambda>0, \forall\ f\in\dot{H}^s(\mathbb{R}). \tag{1.15}$$

When $s=k\in\mathbb{N}$, it is well known (see for example [32]) that

$$H^k(\mathbb{R}) = \{f\in L^2(\mathbb{R}) \ : \ \partial_x^j f\in L^2(\mathbb{R}), \ \forall 0\le j\le k\},$$

with the equivalent norm

$$\|f\|_{L^2_k} := \sum_{j=0}^k \|\partial_x^j f\|_{L^2} \sim \|f\|_{H^k}. \tag{1.16}$$

Similarly, it is possible to define weighted Sobolev spaces. Let $k \in \mathbb{N}$, then

$$H^k(\mathbb{R}; x^2dx) := \{f \in L^2(\mathbb{R}; x^2dx) \ : \ \partial_x^j f \in L^2(\mathbb{R}; x^2dx), \ \forall 0 \leq j \leq k\},$$

with the norm

$$\|f\|_{H^k(x^2dx)} := \sum_{j=0}^{k} \|x\partial_x^j f\|_{L^2}. \tag{1.17}$$

Finally, we recall the definition of the Besov spaces and define weighted Besov spaces. Let $s \in \mathbb{R}$, $p, q \geq 1$, the non homogeneous Besov space $\mathcal{B}_p^{s,q}(\mathbb{R})$ is the completion of the Schwartz space $\mathcal{S}(\mathbb{R})$ under the norm

$$\|f\|_{\mathcal{B}_p^{s,q}} := \|S_0 f\|_{L^p} + \|\{2^{js}\|\Delta_j f\|_{L^p}\}_{j\geq 0}\|_{l^q(\mathbb{N})}. \tag{1.18}$$

This definition naturally extends (even if $s \in \mathbb{R}$) for weighted spaces. Let $s \in \mathbb{R}$, $p, q \geq 1$, then $\mathcal{B}_p^{s,q}(\mathbb{R}; x^p dx)$ is the completion of the Schwartz space $\mathcal{S}(\mathbb{R})$ under the norm

$$\|f\|_{\mathcal{B}_p^{s,q}(x^pdx)} := \|xS_0 f\|_{L^p} + \|\{2^{js}\|x\Delta_j f\|_{L^p}\}_{j\geq 0}\|_{l^q(\mathbb{N})}. \tag{1.19}$$

It is well known (see [34]) that for all $s \in \mathbb{R}$

$$H^s(\mathbb{R}) = \mathcal{B}_2^{s,2}(\mathbb{R}) \quad \text{and that} \quad \|f\|_{H^s} \sim \|f\|_{\mathcal{B}_2^{s,2}}. \tag{1.20}$$

Next we derive a similar result for weighted spaces in the case $s = k \in \mathbb{N}$.

Lemma 1.2. *Let $k \in \mathbb{N}$, $k \geq 1$ and $f \in \mathcal{S}(\mathbb{R})$, then*

$$\|f\|_{H^k(x^2dx)} + \|f\|_{H^{k-1}} \sim \|f\|_{\mathcal{B}_2^{k,2}(x^2dx)} + \|f\|_{H^{k-1}}. \tag{1.21}$$

Proof. We have using (1.8), (1.9), (1.20), the Plancherel theorem and the fact that the supports of $\frac{d}{d\xi}\psi)(2^{-j}\xi)$ are almost disjoint

$$\begin{aligned}
\|f\|_{\mathcal{B}_2^{k,2}(x^2dx)} &= \|xS_0f\|_{L^2} + \left(\sum_{j\geq 0} 4^{kj}\|x\Delta_j f\|_{L^2}^2\right)^{1/2} \\
&\leq \|S_0(xf)\|_{L^2} + \|S_0'f\|_{L^2} + \left(\sum_{j\geq 0} 4^{kj}(\|\Delta_j(xf)\|_{L^2} + \|\Delta_j' f\|_{L^2})^2\right)^{1/2} \\
&\lesssim \|xf\|_{\mathcal{B}_2^{k,2}} + \left(\int_{\mathbb{R}} |(\frac{d}{d\xi}\chi)(\xi)\widehat{f}(\xi)|^2 d\xi + \sum_{j\geq 0}\int_{\mathbb{R}} 4^{(k-1)j}|(\frac{d}{d\xi}\psi)(2^{-j}\xi)\widehat{f}(\xi)|^2 d\xi\right)^{1/2} \\
&\lesssim \|xf\|_{H^k} + \|\partial_x^{k-1} f\|_{L^2}.
\end{aligned}$$

Then we use (1.16) and the identity

$$\partial_x^j(xf) = j\partial_x^{j-1}f + x\partial_x^j f, \ \forall j \geq 1$$

to obtain that

$$\|f\|_{\mathcal{B}_2^{k,2}(x^2dx)} \lesssim \|f\|_{H^k(x^2dx)} + \|f\|_{H^{k-1}}. \tag{1.22}$$

The other inequality of (1.21) follows exactly by the same way. □

1.3 Strongly continuous group.

Let $U(t) = e^{-t\partial_x^3}$ be the unitary group in $H^s(\mathbb{R})$ associated to the Airy equation, *i.e.*

$$U(t)f = \left(e^{i\xi^3 t}\widehat{f}\right)^{\vee}, \quad \forall\ t \in \mathbb{R}. \tag{1.23}$$

We also denote by $V(t) = e^{-t(\partial_x^3 + \partial_x^4 + \partial_x^2)}$, $t \geq 0$ the semigroup associated to the linear part of the equation (1) with $\delta = \mu = 1$, that we extend to a group on $\mathbb{R}$ by

$$V(t)f = \left(e^{i\xi^3 t - (\xi^4 - \xi^2)|t|}\widehat{f}\right)^{\vee}, \quad \forall t \in \mathbb{R}. \tag{1.24}$$

Moreover, we can generalize the definition of U: for all $j \geq 1$, denote by U_j the unitary group in $H^s(\mathbb{R})$, $U_j(t) = e^{-t\partial_x^{2j+1}}$, *i.e.*

$$U_j(t)f = \left(e^{(-1)^{j+1}i\xi^{2j+1}t}\widehat{f}\right)^{\vee}, \quad \forall\ t \in \mathbb{R}, \quad \forall f \in H^s(\mathbb{R}). \tag{1.25}$$

This means that $U = U_1$. We will need the following lemma to commute U with the operator of multiplication by x.

Lemma 1.3. *Let $f \in \mathcal{S}(\mathbb{R})$, then we have*

$$xU(t)f = U(t)(xf) + 3tU(t)\partial_x^2 f \quad \forall\ t \in \mathbb{R}. \tag{1.26}$$

Proof. We define the following operators

$$L := \partial_t + \partial_x^3 \quad \text{and} \quad \Gamma(x,t) = x - 3t\partial_x^2.$$

Then, a straightforward calculation leads to $\Gamma(x,t)L = L\Gamma(x,t)$ so that

$$L\Gamma(x,t)U(t)f = \Gamma(x,t)LU(t)f = 0.$$

Thus, we deduce that

$$\Gamma(x,t)U(t)f = U(t)(xf)$$

which yields (1.26). □

Finally, let us talk about the properties of continuity of the groups U and V: we know that U and V are strongly continuous group in $H^s(\mathbb{R})$, for all $s \in \mathbb{R}$ (se for example [14]). The same property holds for U in Besov and weighted Besov spaces.

Lemma 1.4. *Let $q \geq 1$, $s \in \mathbb{R}$ and $\phi \in \mathcal{B}_2^{s,q}(\mathbb{R}) \cap \mathcal{B}_2^{s-2,q}(\mathbb{R}; x^2dx)$, define $G(t) := U(t)\phi$, then*

$$G \in C(\mathbb{R}; \mathcal{B}_2^{s,q}(\mathbb{R}) \cap \mathcal{B}_2^{s-2,q}(\mathbb{R}; x^2dx)). \tag{1.27}$$

Proof. Since U is a strongly continuous unitary group in $H^s(\mathbb{R})$ and U commute with the operators Δ_j, we deduce that U is a strongly continuous unitary group in $\mathcal{B}_2^{s,q}(\mathbb{R})$, so that

$$G \in C(\mathbb{R}; \mathcal{B}_2^{s,q}(\mathbb{R})).$$

To prove the continuity of G in $\mathcal{B}_2^{s-2,q}(\mathbb{R};x^2dx)$, it is enough to verify the continuity at $t=0$. In this direction, we use (1.26) to compute

$$\|U(t)\phi-\phi\|_{\mathcal{B}_2^{s-2,q}(x^2dx)} \le 3|t|\|U(t)\phi\|_{\mathcal{B}_2^{s,q}} + \|(U(t)-1)(xS_0\phi)\|_{L^2} + \left(\sum_{j\ge 0} 2^{j(s-2)q}\|(U(t)-1)(x\Delta_j\phi)\|_{L^2}^q\right)^{1/q}.$$

Thus, we deduce, using the Lebesgue dominated convergence theorem, that

$$\lim_{t\to 0}\|U(t)\phi-\phi\|_{\mathcal{B}_2^{s-2,q}(x^2dx)} = 0.$$

□

Chapter 2

The dissipative problem.

2.1 Introduction and statement of the results.

We begin to investigate the well-posedness of the IVP (1) in the dissipative case, *i.e.* when $\mu > 0$. This equation combines in its linear part dispersive and dissipative effects, as the KdV-Burgers equation which was studied by Molinet and Ribaud (see [24], [25]). It is a generalization of the KdV-KS equation (3). We refer to the introduction for the physical motivations.

In [3], using the dissipative effect of the linear part, Biagioni, Bona, Iorio and Scialom showed that the Cauchy problem associated to (3) is globally well-posed in H^s for $s \geq 1$. In [2], Argento used the same techniques to show that (1) is well-posed in H^s when $s \geq 1$.

In these works, no use of the dispersive character of these equations was done. We know for example that the Cauchy problem associated to the equation (3) without dissipation (*i.e.* with $\mu = 0$) is well-posed in H^s for $s > 1/4$. In fact, the derivative of this equation is the KdV equation which was showed in [20], by Kenig, Ponce and Vega to be well-posed in H^s for $s > -3/4$. In [5], Carvajal used this fact to prove that the Cauchy associated to the KdV-KS equation is well-posed for $s > 1/4$. In order to do this he applied a fixed point argument in the Bourgain spaces associated to the KdV equation.

Here, we follow the ideas of Molinet and Ribaud [25] to prove that the Cauchy problem

(1) is locally well-posed in H^s for $s > -1$ which improves the results of Argento [2] for the KdV-KV equation and the results of Biagioni, Bona, Iorio and Scialom [3] and Carvajal [5] for the KdV-KS equation. The main idea is again to use a fixed point argument, but this time in Bourgain's type spaces adapted to both linear parts (dispersive and dissipative) of the equation. We also prove an ill-posedness result for the Cauchy problem (1) in $H^s(\mathbb{R})$ when $s < -1$ which implies that (1) cannot be solved in $H^s(\mathbb{R})$ using a fixed point argument when $s < -1$. Then, in some sense, our well-posedness result turns out to be sharp.

Let us introduce some definitions and notations. We denote by θ a cutoff function satisfying

$$\theta \in C_0^\infty(\mathbb{R}),\ 0 \le \theta \le 1,\ supp(\theta) \subset [-2,2],\ \text{and}\ \theta_{|_{[-1,1]}} = 1 \tag{2.1}$$

and

$$\theta_T = \theta(\cdot/T),\ \forall\, T > 0$$

Next, we define the Bourgain spaces which are "well adapted" to the linear part of the equation. Since $\mu > 0$ and $\delta \neq 0$, we will suppose that $\mu = \delta = 1$ in the rest of this chapter.

Definition 2.1. *We define the space $X^{s,b}$ as the completion of the Schwartz space $\mathcal{S}(\mathbb{R}^2)$ under the norm*

$$\|u\|_{X^{s,b}} = \|\langle i(\tau - \xi^3) + (\xi^4 - \xi^2)\rangle^b \langle \xi \rangle^s \widehat{u}(\xi,\tau)\|_{L^2(\mathbb{R}^2)}, \tag{2.2}$$

where $\langle \xi \rangle := 1 + |\xi|$. And, for all $T > 0$, we define the localized space associated $X_T^{s,b}$ as the set of all functions $u : \mathbb{R} \times [0,T] \to \mathbb{R}$ such that $\|u\|_{X_T^{s,b}} < \infty$, where

$$\|u\|_{X_T^{s,b}} = inf\{\ \|\widetilde{u}\|_{X^{s,b}}\ /\ \widetilde{u} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}\ ,\ \widetilde{u}_{|_{[0,T]}} = u\}. \tag{2.3}$$

Remark 2.1. *Since*

$$(U(-t)u)^\wedge(\xi,\tau) = \int_{\mathbb{R}} e^{it(\xi^3+\tau)} \widehat{u}^x(\xi,t)dt = \widehat{u}(\xi,\tau+\xi^3)$$

note that

$$\|u\|_{X^{s,b}} = \|\langle i\tau + (\xi^4 - \xi^2)\rangle^b \langle \xi \rangle^s \left(U(-t)u\right)^{\wedge}(\xi,\tau)\|_{L^2(\mathbb{R}^2)}. \tag{2.4}$$

Remark 2.2. $\|\cdot\|_{X_T^{s,b}}$ *really defines a norm.*

Proof. The only point to verify is the triangular inequality. Let $u,\, v \in X_T^{s,b}$, we know by the definition of the $\|\cdot\|_{X_T^{s,b}}$ norm, that for all $\epsilon > 0$, there exists two extensions $\widetilde{u}$ and $\widetilde{v}$ of u and v such that $\|\widetilde{u}\|_{X^{s,b}} \leq \|u\|_{X_T^{s,b}} + \epsilon$ and $\|\widetilde{v}\|_{X^{s,b}} \leq \|v\|_{X_T^{s,b}} + \epsilon$. Then, since $\widetilde{u} + \widetilde{v}$ is an extension of $u+v$, we can deduce that $\|u+v\|_{X_T^{s,b}} \leq \|u\|_{X_T^{s,b}} + \|v\|_{X_T^{s,b}} + 2\epsilon$, which leads to the results sending ϵ to zero. □

We are now able to state our results of local and global well-posedness as well as regularity *ad-hoc* of the solution (which comes from the dissipative character of the equation).

Theorem 2.1 (Local well-posedness.). *Let $s > -1$, then for all $\phi \in H^s(\mathbb{R})$, there exists $T = T(\|\phi\|_{H^s})$ (with $T(\rho) \to \infty$ as $\rho \to 0$) and a unique solution u of the Cauchy problem (1), with $\mu > 0$, in the space $X_T^{s,1/2}$. Moreover, u satisfies the additional regularity*

$$u \in C([0,T]; H^s(\mathbb{R})) \cap C((0,T); H^\infty(\mathbb{R})) \tag{2.5}$$

and the map solution

$$S : H^s(\mathbb{R}) \to X_T^{s,1/2} \cap C([0,T]; H^s(\mathbb{R})), \quad \phi \mapsto u, \tag{2.6}$$

is smooth. In addition, if $\phi \in H^{s'}(\mathbb{R})$ with $s' > s$, the result holds with s' instead of s in the same time interval $[0,T]$ with $T = T(\|\phi\|_{H^s})$.

Theorem 2.2 (Global well-posedness.). *Let $s > -1$ and $\phi \in H^s(\mathbb{R})$.*

- *If $\gamma = \alpha/2$, then the local solution u of the Cauchy problem (1), with $\mu > 0$, extends globally in time.*

- *If $\gamma = 0$, then the local solution u of the Cauchy problem (3), with $\mu > 0$, extends globally in time.*

These results are sharp in the following sense

Theorem 2.3 (Ill-posedness.). *We assume that $\alpha \neq \gamma$ in (**??**). Let $s < -1$, if there exists some $T > 0$ such that the problem (1) is locally well-posed in $H^s(\mathbb{R})$, then, the flow-map data solution*

$$S : H^s(\mathbb{R}) \longrightarrow C([0,T]; H^s(\mathbb{R}), \quad \phi \longmapsto u(t) \tag{2.7}$$

is not C^2 at zero.

In section 2.2, we prove the linear estimates, in section 2.3 the bilinear ones, in section 2.4, we give the proofs of Theorems 2.1 and 2.2. Finally we take care of the ill-posedness in section 2.5.

2.2 Linear estimates

The proofs of the linear estimates follow closely the proofs given by Molinet and Ribaud (see [25]) for the KdV-Burgers equation, replacing ξ^2 by $\xi^4 - \xi^2$.

Proposition 2.1 (Homogeneous linear estimate.). *Let $s \in \mathbb{R}$, then*

$$\|\theta(t)V(t)\phi\|_{X^{s,1/2}} \lesssim \|\phi\|_{H^s}, \ \forall \phi \in H^s(\mathbb{R}) \tag{2.8}$$

Proof. Let $\phi \in H^s(\mathbb{R})$, using (1.23), (1.24) and (2.4) we have that

$$\begin{aligned}
\|\theta(t)V(t)\phi\|_{X^{s,1/2}} &= \|\langle i\tau + (\xi^4 - \xi^2)\rangle^{1/2}\langle\xi\rangle^s \left(\theta(t)e^{-(\xi^4-\xi^2)|t|}\widehat{\phi}(\xi)\right)^{\wedge_t}\|_{L^2(\mathbb{R}^2)} \\
&= \|\langle\xi\rangle^s\widehat{\phi}(\xi)\|\langle i\tau + (\xi^4 - \xi^2)\rangle^{1/2}\left(\theta(t)e^{-(\xi^4-\xi^2)|t|}\right)^{\wedge_t}\|_{L^2_\tau}\|_{L^2_\xi} \\
&\lesssim I + II.
\end{aligned} \tag{2.9}$$

where

$$I = \|\langle\xi\rangle^s\langle\xi^4 - \xi^2\rangle^{1/2}\widehat{\phi}(\xi)\|g_\xi(t)\|_{L^2_t}\|_{L^2_\xi},$$

$$II = \|\langle\xi\rangle^s \widehat{\phi}(\xi)\|g_\xi(t)\|_{H_t^{1/2}}\|_{L^2_\xi},$$

and

$$g_\xi(t) = \theta(t)e^{-(|\xi|^4-|\xi|^2)|t|}. \tag{2.10}$$

Contribution of I. When $|\xi| \geq \sqrt{2}$, we have that $\xi^4 - \xi^2 \geq 2$, then we can use (1.15) to see that

$$\|g_\xi\|_{L^2} \leq \|e^{-(\xi^4-\xi^2)|t|}\|_{L^2_t} = \frac{1}{|\xi^4-\xi^2|^{1/2}}\|e^{-|t|}\|_{L^2_t} \lesssim \frac{1}{\langle\xi^4-\xi^2\rangle^{1/2}}.$$

When $|\xi| \leq \sqrt{2}$, then, $-1/4 \leq \xi^4 - \xi^2 \leq 2$ and (2.1) imply that

$$\|g_\xi\|_{L^2} \leq \left(\int_{-2}^{2} e^{|t|/2}dt\right)^{1/2} \lesssim 1 \lesssim \frac{1}{\langle\xi^4-\xi^2\rangle^{1/2}}$$

Then, we deduce that

$$I \lesssim \|\phi\|_{H^s}. \tag{2.11}$$

Contribution of II. When $|\xi| \geq \sqrt{2}$ we use the triangle inequality, Young's inequality and (1.15) to see that

$$\begin{aligned}
\|g_\xi\|_{H^{1/2}} &= \|\langle\tau\rangle^{1/2}\widehat{\theta} * \left(e^{-|t|(\xi^4-\xi^2)}\right)^{\wedge_t}(\tau)\|_{L^2_\tau} \\
&\lesssim \|\langle\tau\rangle^{1/2}\widehat{\theta}(\tau)\|_{L^1_\tau}\|e^{-(\xi^4-\xi^2)|t|}\|_{L^2_t} + \|\widehat{\theta}\|_{L^1}\|e^{-(\xi^4-\xi^2)|t|}\|_{\dot{H}_t^{1/2}} \\
&\lesssim \frac{1}{|\xi^4-\xi^2|^{1/2}} \lesssim 1.
\end{aligned}$$

When $|\xi| \leq \sqrt{2}$, since $|\xi^4 - \xi^2| \leq 2$, we have

$$\|g_\xi\|_{H^{1/2}} \leq \sum_{n\geq 0}\frac{2^n}{n!}\||t|^n\theta(t)\|_{H_t^{1/2}} \lesssim 1,$$

Since for $n \geq 1$, $\||t|^n\theta(t)\|_{H_t^{1/2}} \leq \||t|^n\theta(t)\|_{H_t^1} \lesssim n$. Then, we also have in this case

$$II \lesssim \|\phi\|_{H^s}. \tag{2.12}$$

Then (2.9), (2.11) and (2.12) lead to (2.8). □

Proposition 2.2 (non homogeneous linear estimate.). *Let $s \in \mathbb{R}$, then*

$$\begin{aligned}&\|\theta(t)\int_0^t V(t-t')v(t')dt'\|_{X^{s,1/2}}\\&\lesssim \|v\|_{X^{s,-1/2}} + \left(\int_{\mathbb{R}} \langle\xi\rangle^{2s}\left(\int_{\mathbb{R}} \frac{|\,(U(-t)v)^\wedge\,(\xi,\tau)|}{\langle i\tau+(\xi^4-\xi^2)\rangle}d\tau\right)^2 d\xi\right)^{1/2},\end{aligned} \tag{2.13}$$

for all $v \in \mathbb{S}(\mathbb{R}^2)$. Moreover for any $0<\delta<1/2$,

$$\|\theta(t)\int_0^t V(t-t')v(t')dt'\|_{X^{s,1/2}} \lesssim \|v\|_{X^{s,-1/2+\delta}}, \tag{2.14}$$

for all $v \in \mathbb{S}(\mathbb{R}^2)$.

The inequality (2.14) follows directly from (2.13) and the Cauchy-Schwarz inequality. In order to prove (2.13), we will use the following lemma

Lemma 2.1. *Let $w \in \mathbb{S}(\mathbb{R}^2)$, define k_ξ on $\mathbb{R}$ by*

$$k_\xi(t) = \theta(t)\int_{\mathbb{R}} \frac{e^{it\tau}-e^{-|t|(\xi^4-\xi^2)}}{i\tau+(\xi^4-\xi^2)}\widehat{w}(\xi,\tau)d\tau. \tag{2.15}$$

Then, it holds that for all $\xi \in \mathbb{R}$

$$\begin{aligned}&\|\langle i\tau+(\xi^4-\xi^2)\rangle^{1/2}\widehat{k}_\xi^t(\tau)\|_{L^2_\tau}\\&\lesssim \left(\left(\int_{\mathbb{R}} \frac{|\widehat{w}(\xi,\tau)|}{\langle i\tau+(\xi^4-\xi^2)\rangle}d\tau\right)^2 + \int_{\mathbb{R}} \frac{|\widehat{w}(\xi,\tau)|^2}{\langle i\tau+(\xi^4-\xi^2)\rangle}d\tau\right)^{1/2}.\end{aligned} \tag{2.16}$$

Proof of Lemma 2.1. We decompose k_ξ into

$$\begin{aligned}k_\xi(t) &= \theta(t)\left(\int_{|\tau|\le 1} \frac{e^{it\tau}-1}{i\tau+(\xi^4-\xi^2)}\widehat{w}(\xi,\tau)d\tau + \int_{|\tau|\le 1} \frac{1-e^{-|t|(\xi^4-\xi^2)}}{i\tau+(\xi^4-\xi^2)}\widehat{w}(\xi,\tau)d\tau\right.\\&\quad\left.+\int_{|\tau|\ge 1} \frac{e^{it\tau}}{i\tau+(\xi^4-\xi^2)}\widehat{w}(\xi,\tau)d\tau - \int_{|\tau|\ge 1} \frac{e^{-|t|(\xi^4-\xi^2)}}{i\tau+(\xi^4-\xi^2)}\widehat{w}(\xi,\tau)d\tau\right)\\&= I+II+III+IV,\end{aligned} \tag{2.17}$$

and then we examine the different contributions of (2.17) on the left hand side of (2.16).

Contribution of IV. Since $|\tau| \geq 1$, notice that

$$\begin{aligned}
&\|\langle i\tau + (\xi^4 - \xi^2)\rangle^{1/2}(IV)^{\wedge_t}(\tau)\|_{L^2_\tau}^2 \\
&\leq \left(\int_{\mathbb{R}} \langle i\tau + (\xi^4 - \xi^2)\rangle \left|(g_\xi(t))^{\wedge_t}(\tau)\right|^2 d\tau\right)\left(\int_{\mathbb{R}} \frac{|\widehat{w}(\xi,\tau)|}{\langle i\tau + (\xi^4 - \xi^2)\rangle} d\tau\right)^2,
\end{aligned}$$

where g_ξ is defined in (2.10). Exactly the same computations as in Proposition 2.1 lead to

$$\left(\int_{\mathbb{R}} \langle i\tau + (\xi^4 - \xi^2)\rangle \left|(g_\xi(t))^{\wedge_t}(\tau)\right|^2 d\tau\right) \lesssim 1.$$

We conclude then that

$$\|\langle i\tau + (\xi^4 - \xi^2)\rangle^{1/2}(IV)^{\wedge_t}(\tau)\|_{L^2_\tau}^2 \lesssim \left(\int_{\mathbb{R}} \frac{|\widehat{w}(\xi,\tau)|}{\langle i\tau + (\xi^4 - \xi^2)\rangle} d\tau\right)^2. \tag{2.18}$$

Contribution of III. Since $III = \theta\left(\frac{\widehat{w}(\xi,\tau)}{i\tau + (\xi^4 - \xi^2)}\chi_{\{|\tau|\geq 1\}}\right)^{\vee}$, we use Young's inequality to obtain the following estimate

$$\begin{aligned}
&\|\langle i\tau + (\xi^4 - \xi^2)\rangle^{1/2}(III)^{\wedge_t}(\tau)\|_{L^2_\tau} \\
&= \|\langle i\tau + (\xi^4 - \xi^2)\rangle^{1/2}\widehat{\theta}(\tau') *_{\tau'} \left(\frac{\widehat{w}(\xi,\tau')}{i\tau' + (\xi^4 - \xi^2)}\chi_{|\tau'\geq 1|}\right)(\tau)\|_{L^2_\tau} \\
&\lesssim \|(\langle \tau'\rangle^{1/2}|\widehat{\theta}(\tau')|) *_{\tau'} \left(\frac{|\widehat{w}(\xi,\tau')|}{|i\tau' + (\xi^4 - \xi^2)|}\chi_{|\tau'\geq 1|}\right)\|_{L^2} \\
&+ \||\widehat{\theta}(\tau')| *_{\tau'} \left(\frac{|\widehat{w}(\xi,\tau')|}{|i\tau' + (\xi^4 - \xi^2)|^{1/2}}\chi_{|\tau'\geq 1|}\right)\|_{L^2} \\
&\lesssim \|\theta\|_{H^{1/2}}\|\frac{|\widehat{w}(\xi,\tau')|}{|i\tau' + (\xi^4 - \xi^2)|}\chi_{|\tau'\geq 1|}\|_{L^1} + \|\widehat{\theta}\|_{L^1}\|\frac{|\widehat{w}(\xi,\tau')|}{|i\tau' + (\xi^4 - \xi^2)|^{1/2}}\chi_{|\tau'\geq 1|}\|_{L^2} \\
&\lesssim \int_{\mathbb{R}} \frac{|\widehat{w}(\xi,\tau)|}{\langle i\tau + (\xi^4 - \xi^2)\rangle} d\tau + \left(\int_{\mathbb{R}} \frac{|\widehat{w}(\xi,\tau)|^2}{\langle i\tau + (\xi^4 - \xi^2)\rangle} d\tau\right)^{1/2}.
\end{aligned} \tag{2.19}$$

Contribution of II. First, notice that

$$\begin{aligned}
\|\langle i\tau + (\xi^4 - \xi^2)\rangle^{1/2}(II)^{\wedge_t}(\tau)\|_{L^2_\tau} &\leq \left(\int_{|\tau|\leq 1} \frac{|\widehat{w}(\xi,\tau)|}{|i\tau + (\xi^4 - \xi^2)|} d\tau\right) \\
&\times \|\langle i\tau + (\xi^4 - \xi^2)\rangle^{1/2}\left(\theta(t)(1 - e^{-|t|(\xi^4 - \xi^2)})\right)^{\wedge_t}\|_{L^2_\tau}.
\end{aligned} \tag{2.20}$$

Since

$$\begin{aligned}
&\int_{|\tau|\leq 1}\frac{\langle i\tau+(\xi^4-\xi^2)\rangle}{|i\tau+(\xi^4-\xi^2)|^2}d\tau \\
&=\int_{|\tau|\leq 1}\frac{d\tau}{|i\tau+(\xi^4-\xi^2)|^2}+\int_{|\tau|\leq 1}\frac{d\tau}{|i\tau+(\xi^4-\xi^2)|} \\
&\lesssim \int_0^1 \frac{d\tau}{\tau^2+(\xi^4-\xi^2)^2}+\frac{1}{|\xi^4-\xi^2|} \\
&\lesssim \frac{1}{|\xi^4-\xi^2|}\int_0^1 \frac{1}{1+\left(\frac{\tau}{|\xi^4-\xi^2|}\right)^2}d\left(\frac{\tau}{|\xi^4-\xi^2|}\right)+\frac{1}{|\xi^4-\xi^2|}\lesssim \frac{1}{|\xi^4-\xi^2|}.
\end{aligned}$$

We deduce from (2.20) and the Cauchy-Schwarz inequality that

$$\begin{aligned}
\|\langle i\tau+(\xi^4-\xi^2)\rangle^{1/2}(II)^{\wedge_t}(\tau)\|_{L^2_\tau} &\lesssim \left(\int_{\mathbb{R}}\frac{|\widehat{w}(\xi,\tau)|^2}{\langle i\tau+(\xi^4-\xi^2)\rangle}d\tau\right)^{1/2} \\
&\times\frac{1}{|\xi^4-\xi^2|^{1/2}}\|\langle i\tau+(\xi^4-\xi^2)\rangle^{1/2}\left(\theta(t)(1-e^{-|t|(\xi^4-\xi^2)})\right)^{\wedge_t}\|_{L^2_\tau}. \qquad (2.21)
\end{aligned}$$

Next, as in the proof of Proposition 2.1, we consider two different cases. When $|\xi|\geq\sqrt{2}$, we have that $\xi^4-\xi^2\geq 2$, so that

$$\begin{aligned}
&\|\langle i\tau+(\xi^4-\xi^2)\rangle^{1/2}\left(\theta(t)(1-e^{-|t|(\xi^4-\xi^2)})\right)^{\wedge_t}(\tau)\|_{L^2_\tau} \\
&\lesssim \|\theta\|_{H^{1/2}}+\langle\xi^4-\xi^2\rangle^{1/2}\|\theta\|_{L^2}+\|g_\xi\|_{H^{1/2}}+\langle\xi^4-\xi^2\rangle^{1/2}\|g_\xi\|_{L^2} \\
&\lesssim |\xi^4-\xi^2|^{1/2},
\end{aligned}$$

which implies together with (2.21) that

$$\|\langle i\tau+(\xi^4-\xi^2)\rangle^{1/2}(II)^{\wedge_t}(\tau)\|_{L^2_\tau}\lesssim\left(\int_{\mathbb{R}}\frac{|\widehat{w}(\xi,\tau)|^2}{\langle i\tau+(\xi^4-\xi^2)\rangle}d\tau\right)^{1/2}. \qquad (2.22)$$

When $|\xi|\leq\sqrt{2}$, then $|\xi^4-\xi^2|\leq 2$ and we have

$$\begin{aligned}
&\|\langle i\tau+(\xi^4-\xi^2)\rangle^{1/2}\left(\theta(t)(1-e^{-|t|(\xi^4-\xi^2)})\right)^{\wedge_t}(\tau)\|_{L^2_\tau} \\
&\qquad\lesssim \|\theta(t)(1-e^{-|t|(\xi^4-\xi^2)})\|_{H^{1/2}_t}. \qquad (2.23)
\end{aligned}$$

Then arguing again as in Proposition 2.1, we compute

$$\begin{aligned}\|\theta(t)(1-e^{-|t|(\xi^4-\xi^2)})\|_{H_t^{1/2}} &\le \sum_{n\ge 1}\frac{|\xi^4-\xi^2|^n}{n!}\|t^n\theta(t)\|_{H_t^{1/2}}\\ &\lesssim |\xi^4-\xi^2|\sum_{n\ge 0}\frac{|\xi^4-\xi^2|^n}{n!}\lesssim |\xi^4-\xi^2|,\end{aligned}$$

which together with (2.21) and (2.23) also implies (2.22) in this case.

Contribution of I. Since I can be rewritten as

$$I=\theta(t)\int_{\tau\le 1}\sum_{n\ge 1}\frac{(it\tau)^n}{(i\tau+(\xi^4-\xi^2))n!}\widehat{w}(\xi,\tau)d\tau$$

we deduce from the Cauchy-Schwarz inequality that

$$\begin{aligned}&\|\langle i\tau+(\xi^4-\xi^2)\rangle^{1/2}(I)^{\wedge_t}(\tau)\|_{L^2_\tau}\\ &\lesssim \sum_{n\ge 1}\frac{1}{n!}\left(\|t^n\theta(t)\|_{H_t^{1/2}}+\langle\xi^4-\xi^2\rangle^{1/2}\|t^n\theta(t)\|_{L^2_t}\right)\int_{|\tau|\le 1}\frac{|\tau|^n|\widehat{w}(\xi,\tau)|}{|i\tau+(\xi^4-\xi^2)|}d\tau\\ &\lesssim \langle\xi^4-\xi^2\rangle^{1/2}\left(\int_{\mathbb{R}}\frac{|\widehat{w}(\xi,\tau)|^2}{\langle i\tau+(\xi^4-\xi^2)\rangle}d\tau\right)^{1/2}\left(\int_{|\tau|\le 1}\frac{|\tau|^2\langle i\tau+(\xi^4-\xi^2)\rangle}{|i\tau+(\xi^4-\xi^2)|^2}d\tau\right)^{1/2}\\ &\lesssim \left(\int_{\mathbb{R}}\frac{|\widehat{w}(\xi,\tau)|^2}{\langle i\tau+(\xi^4-\xi^2)\rangle}d\tau\right)^{1/2},\end{aligned}\tag{2.24}$$

where we have used that

$$\int_{|\tau|\le 1}\frac{|\tau|^2\langle i\tau+(\xi^4-\xi^2)\rangle}{|i\tau+(\xi^4-\xi^2)|^2}d\tau\lesssim\frac{1}{\langle\xi^4-\xi^2\rangle}.$$

Then, (2.17), (2.18), (2.19), (2.22) and (2.24) lead to (2.16) which concludes the proof of Lemma 2.1. □

Proof of Proposition 2.2. We only have to prove (2.13). Define

$$w(\cdot,t)=U(-t)v(\cdot,t)\in\mathbb{S}(\mathbb{R}).\tag{2.25}$$

We obtain from Fubini's theorem and the Fourier inverse formula, that

$$\theta(t)\int_0^t V(t-t')v(t')dt'=U(t)\left(k_\xi(t)\right)^{\vee_\xi},\tag{2.26}$$

where k_ξ is defined in (2.15). Then, using (2.25), (2.26) and Lemma 2.1, we deduce that

$$\begin{aligned}\|\theta(t)\int_0^t V(t-t')v(t')dt'\|_{X^{s,1/2}} &= \|\langle\xi\rangle^s\|\langle i\tau+(\xi^4-\xi^2)\rangle^{1/2}\widehat{k}_\xi^t(\tau)\|_{L^2_\tau}\|_{L^2_\xi} \\ &\lesssim \|\langle\xi\rangle^s\left(\int_{\mathbb{R}}\frac{|\widehat{w}(\xi,\tau)|^2}{\langle i\tau+(\xi^4-\xi^2)\rangle}d\tau+\left(\int_{\mathbb{R}}\frac{|\widehat{w}(\xi,\tau)|}{\langle i\tau+(\xi^4-\xi^2)\rangle}d\tau\right)^2\right)^{1/2}\|_{L^2_\xi} \\ &\lesssim \|v\|_{X^{s,-1/2}}+\left(\int_{\mathbb{R}}\langle\xi\rangle^{2s}\left(\int_{\mathbb{R}}\frac{|\left(U(-t)v\right)^\wedge(\xi,\tau)|}{\langle i\tau+(\xi^4-\xi^2)\rangle}d\tau\right)^2d\xi\right)^{1/2},\end{aligned}$$

which concludes the proof of (2.13). □

Proposition 2.3 (regularity.)**.** *Let* $s\in\mathbb{R}$ *and* $0<\delta<1/2$*. Then, for all* $f\in X^{s,-1/2+\delta}$*, we have*

$$N : t\longmapsto\int_0^t V(t-t')f(t')dt' \ \in\ C(\mathbb{R}_+;H^{s+4\delta}). \tag{2.27}$$

Moreover

$$\|\int_0^t V(t-t')f(t')dt'\|_{C([0,T];H^s)}\lesssim\|f\|_{X^{s,-1/2+\delta}}. \tag{2.28}$$

Proof. Define $g(x,t):=U(-t)f(\cdot,t)(x)$. Since U is a strongly continuous unitary group in $H^s(\mathbb{R})$ and remembering (1.13), (1.23) and (1.24), it is enough to prove that

$$F(\xi,\cdot) : t\in\mathbb{R}_+\longmapsto\langle\xi\rangle^{s+4\delta}\int_0^t e^{-(t-t')(\xi^4-\xi^2)}(g(\cdot,t))^{\wedge_x}(\xi)dt'$$

is continuous in $L^2_\xi(\mathbb{R})$ when $\langle\xi\rangle^s\langle i\tau+(\xi^4-\xi^2)\rangle^{-1/2+\delta}\widehat{g}(\xi,\tau)\ \in L^2_{\xi,\tau}(\mathbb{R}^2)$. As in (2.26), we can compute, using the Fourier inverse transform in time and Fubini's theorem, that

$$F(\xi,t)=\langle\xi\rangle^{s+4\delta}\int_{\mathbb{R}}\widehat{g}(\xi,\tau)\frac{e^{it\tau}-e^{-t(\xi^4-\xi^2)}}{i\tau+\xi^4-\xi^2}d\tau.$$

Fix $t_0\in\mathbb{R}_+$ and define for all $t\in\mathbb{R}_+$

$$\begin{aligned}H(\xi,t)&:=F(\xi,t)-F(\xi,t_0)\\ &=\langle\xi\rangle^{s+4\delta}\int_{\mathbb{R}}\frac{\widehat{g}(\xi,\tau)}{i\tau+\xi^4-\xi^2}((e^{it\tau}-e^{it_0\tau})-(e^{-t(\xi^4-\xi^2)}-e^{-t_0(\xi^4-\xi^2)}))d\tau.\end{aligned}$$

We will use the Lebesgue dominated convergence theorem to show that

$$\lim_{t\to t_0} \|H(\cdot,t)\|_{L^2(\mathbb{R})} = 0. \tag{2.29}$$

First step.

$$\lim_{t\to t_0} H(\xi,t) = 0 \quad \text{a.e. } \xi \in \mathbb{R}. \tag{2.30}$$

Let

$$h(\xi,\tau,t) = \frac{\widehat{g}(\xi,\tau)}{i\tau+\xi^4-\xi^2}((e^{it\tau}-e^{it_0\tau})-(e^{-t(\xi^4-\xi^2)}-e^{-t_0(\xi^4-\xi^2)})), \tag{2.31}$$

then clearly,

$$\lim_{t\to t_0} h(\xi,\tau,t) = 0 \quad \text{for almost every } (\xi,\tau)\in\mathbb{R}^2 \tag{2.32}$$

Moreover, since $t\to t_0$, we can suppose that $0\le t\le T$, and then,

$$|h(\xi,\tau,t)| \le (2+e^{t/4}+e^{t_0/4})\frac{|\widehat{g}(\xi,\tau)|}{|i\tau+\xi^4-\xi^2|} \lesssim \frac{|\widehat{g}(\xi,\tau)|}{|i\tau+\xi^4-\xi^2|}. \tag{2.33}$$

We deduce from the Cauchy-Schwarz inequality that

$$\begin{aligned}&\int_{\mathbb{R}} \frac{|\widehat{g}(\xi,\tau)|}{|i\tau+\xi^4-\xi^2|}d\tau\\ &\le \left(\int_{\mathbb{R}} \frac{\langle i\tau+\xi^4-\xi^2\rangle^{1-2\delta}}{|i\tau+\xi^4-\xi^2|^2}d\tau\right)^{1/2}\left(\int_{\mathbb{R}} \frac{|\widehat{g}(\xi,\tau)|^2}{\langle i\tau+\xi^4-\xi^2\rangle^{1-2\delta}}d\tau\right)^{1/2}.\end{aligned}$$

By the hypotheses on g, we know that

$$\int_{\mathbb{R}} \langle\xi\rangle^{2s}\left(\int_{\mathbb{R}} \frac{|\widehat{g}(\xi,\tau)|^2}{\langle i\tau+\xi^4-\xi^2\rangle^{1-2\delta}}d\tau\right)d\xi < \infty,$$

so that we deduce

$$\int_{\mathbb{R}} \frac{|\widehat{g}(\xi,\tau)|}{|i\tau+\xi^4-\xi^2|}d\tau \lesssim \left(\int_{\mathbb{R}} \frac{|\widehat{g}(\xi,\tau)|^2}{\langle i\tau+\xi^4-\xi^2\rangle^{1-2\delta}}d\tau\right)^{1/2} < \infty, \tag{2.34}$$

for almost every $\xi\in\mathbb{R}$. We use (2.31), (2.32), (2.33), (2.34) and the Lebesgue dominated convergence theorem to conclude the proof of (2.30).

Second step. There exists $G\in L^2(\mathbb{R})$ such that

$$|H(\xi,t)| \le |G(\xi)| \quad \text{for all } \xi\in\mathbb{R}, \text{ and } t\in\mathbb{R}_+. \tag{2.35}$$

When $|\xi| \geq \sqrt{2}$, we get from the Cauchy-Schwarz inequality and (2.33) that

$$|H(\xi,t)| \lesssim \langle\xi\rangle^{s+4\delta} \left(\int_{\mathbb{R}} \frac{\langle i\tau+\xi^4-\xi^2\rangle^{1-2\delta}}{|i\tau+\xi^4-\xi^2|^2} d\tau\right)^{1/2} \left(\int_{\mathbb{R}} \frac{|\widehat{g}(\xi,\tau)|^2}{\langle i\tau+\xi^4-\xi^2\rangle^{1-2\delta}} d\tau\right)^{1/2}.$$

Since $|\xi^4-\xi^2| \geq 2$,

$$\left(\int_{\mathbb{R}} \frac{\langle i\tau+\xi^4-\xi^2\rangle^{1-2\delta}}{|i\tau+\xi^4-\xi^2|^2} d\tau\right)^{1/2} \lesssim \left(\int_{\mathbb{R}} \frac{1}{|i\tau+\xi^4-\xi^2|^{1+2\delta}} d\tau\right)^{1/2} \lesssim \frac{1}{\langle\xi\rangle^{4\delta}},$$

then using the hypotheses on g, we conclude that for all $t \in \mathbb{R}_+$,

$$|H(\xi,t)| \lesssim \langle\xi\rangle^{s} \left(\int_{\mathbb{R}} \frac{|\widehat{g}(\xi,\tau)|^2}{\langle i\tau+\xi^4-\xi^2\rangle^{1-2\delta}} d\tau\right)^{1/2} \quad \in L^2(\mathbb{R}),$$

which proves (2.35) in the case $|\xi| \geq \sqrt{2}$. When $|\xi| \leq \sqrt{2}$, then we have $|\xi^4-\xi^2| \leq 2$ so that

$$\begin{aligned}|H(\xi,t)| \lesssim &\int_{\mathbb{R}} \frac{|\widehat{g}(\xi,\tau)|}{|i\tau+\xi^4-\xi^2|} |e^{-t(\xi^4-\xi^2)} - e^{-t_0(\xi^4-\xi^2)}| d\tau \\ &+ \int_{\mathbb{R}} \frac{|\widehat{g}(\xi,\tau)|}{|i\tau+\xi^4-\xi^2|} |e^{it\tau} - e^{it_0\tau}| d\tau = I + II.\end{aligned}$$

We first evaluate II, using the Cauchy-Schwarz inequality

$$\begin{aligned}II &\leq |t-t_0| \int_{|\tau|\leq 1} \frac{|\tau||\widehat{g}(\xi,\tau)|}{|i\tau+\xi^4-\xi^2|} d\tau + 2\int_{|\tau|\geq 1} \frac{|\widehat{g}(\xi,\tau)|}{|i\tau+\xi^4-\xi^2|} d\tau \\ &\lesssim \left(\int_{\mathbb{R}} \frac{|\widehat{g}(\xi,\tau)|^2}{\langle i\tau+\xi^4-\xi^2\rangle^{1-2\delta}} d\tau\right)^{1/2} \\ &\quad \times \left[\left(\int_{|\tau|\leq 1} |\tau|^{1-2\delta} d\tau\right)^{1/2} + \left(\int_{|\tau|\geq 1} \langle\tau\rangle^{-1-2\delta} d\tau\right)^{1/2}\right] \\ &\lesssim \left(\int_{\mathbb{R}} \frac{|\widehat{g}(\xi,\tau)|^2}{\langle i\tau+\xi^4-\xi^2\rangle^{1-2\delta}} d\tau\right)^{1/2} \quad \in \ L^2_\xi(\mathbb{R}).\end{aligned}$$

We next turn to I and again use the Cauchy-Schwarz inequality to see that

$$I \leq |t-t_0| \left(\int_{\mathbb{R}} \frac{|\widehat{g}(\xi,\tau)|^2}{\langle i\tau+\xi^4-\xi^2\rangle^{1-2\delta}} d\tau\right)^{1/2} |\xi^4-\xi^2| \left(\int_{\mathbb{R}} \frac{\langle i\tau+(\xi^4-\xi^2)\rangle^{1-2\delta}}{|i\tau+(\xi^4-\xi^2)|^2} d\tau\right)^{1/2}$$

and we compute

$$\begin{aligned}&\left(\int_{\mathbb{R}} \frac{\langle i\tau + (\xi^4-\xi^2)\rangle^{1-2\delta}}{|i\tau + (\xi^4-\xi^2)|^2} d\tau\right)^{1/2}\\ &\sim \left(\int_{\mathbb{R}} \frac{1}{|i\tau + (\xi^4-\xi^2)|^2} d\tau\right)^{1/2} + \left(\int_{\mathbb{R}} \frac{1}{|i\tau + (\xi^4-\xi^2)|^{1+2\delta}} d\tau\right)^{1/2}\\ &\sim \frac{1}{|\xi^4-\xi^2|^{1/2}} + \frac{1}{|\xi^4-\xi^2|^{\delta}}.\end{aligned}$$

Then, since $|\xi^4-\xi^2| \leq 2$, we conclude that

$$II \lesssim \left(\int_{\mathbb{R}} \frac{|\widehat{g}(\xi,\tau)|^2}{\langle i\tau + \xi^4-\xi^2\rangle^{1-2\delta}} d\tau\right)^{1/2} \quad \in L^2_\xi(\mathbb{R}).$$

Thus (2.35) still remains true in the case $|\xi| \leq \sqrt{2}$.

We use (2.30), (2.35) and the dominated convergence theorem to prove (2.29) which concludes the proof of Proposition 2.3. The estimate (2.28) follows exactly by the same computations. □

Then, we will derive a linear estimate to obtain a contraction factor T^μ in the proof of existence in Theorem 2.1 (see for example [23], Lemma (7.31)).

Proposition 2.4. *For all $s \in \mathbb{R}$, for all $T > 0$ and for all $0 < \delta < 1/2$, we have that*

$$\|\theta_T w\|_{X^{s,-1/2+\delta}} \lesssim T^\delta \|w\|_{X^{s,-1/2+2\delta}}, \qquad \forall w \in X^{s,-1/2+2\delta}. \tag{2.36}$$

Proof. Assume that

$$\|\theta_T v\|_{X^{-s,1/2-2\delta}} \lesssim T^\delta \|v\|_{X^{-s,1/2-\delta}}, \qquad \forall v \in X^{-s,1/2-\delta}. \tag{2.37}$$

Then, by duality, we have

$$\begin{aligned}\|\theta_T w\|_{X^{s,-1/2+\delta}} &= \sup_{\|v\|_{X^{-s,1/2-\delta}}=1} |(v,\theta_T w)_{L^2}|\\ &\leq \sup_{\|v\|_{X^{-s,1/2-\delta}}=1} \{\|\theta_T v\|_{X^{-s,1/2-2\delta}} \|w\|_{X^{s,-1/2+2\delta}}\}\\ &\lesssim T^\delta \|w\|_{X^{s,-1/2+2\delta}}.\end{aligned}$$

Therefore to prove (2.36) it is sufficient to show that (2.37) is true. We will proceed by interpolation.

First, we use the definition of the $X^{s,b}$ space (see (2.2)), the fact that U is a unitary group, Hölder's inequality and the Sobolev embedding theorem to compute

$$\begin{aligned}\|\theta_T v\|_{X^{-s,0}} &= \|J^{-s}(\theta_T v)\|_{L^2_{x,t}} = \|\theta_T J^{-s} U(-t)v\|_{L^2_{x,t}} \\ &\lesssim T^{1/2-\delta}\|J^{-s}U(-t)v\|_{L^2_x L^{1/\delta}_t} \\ &\lesssim T^{1/2-\delta}\|J^{-s}U(-t)v\|_{L^2_x H^{1/2-\delta}_t},\end{aligned}$$

which leads, using (2.4), to

$$\|\theta_T v\|_{X^{-s,0}} \lesssim T^{1/2-\delta}\|v\|_{X^{-s,1/2-\delta}}, \quad \forall v \in X^{-s,1/2-\delta}. \tag{2.38}$$

Then, we will prove that

$$\|\theta_T v\|_{X^{-s,1/2-\delta}} \lesssim \|v\|_{X^{-s,1/2-\delta}}, \quad \forall v \in X^{-s,1/2-\delta}. \tag{2.39}$$

By the definition of the $X^{s,b}$ space

$$\begin{aligned}\|\theta_T v\|_{X^{-s,1/2-\delta}} &= \|\langle i(\tau-\xi^3) + (\xi^4-\xi^2)\rangle^{1/2-\delta}\langle\xi\rangle^{-s}(\theta_T v)^\wedge(\xi,\tau)\|_{L^2_{\xi,\tau}} \\ &\lesssim \||\tau-\xi^3|^{1/2-\delta}\langle\xi\rangle^{-s}(\theta_T v)^\wedge(\xi,\tau)\|_{L^2_{\xi,\tau}} \\ &\quad + \|\langle\xi^4-\xi^2\rangle^{1/2-\delta}\langle\xi\rangle^{-s}(\theta_T v)^\wedge(\xi,\tau)\|_{L^2_{\xi,\tau}} \\ &= I \; + \; II.\end{aligned}$$

First, we estimate II using Plancherel's identity

$$II = \|\langle\xi^4-\xi^2\rangle^{1/2-\delta}\langle\xi\rangle^{-s}\|\theta_T(t)\widehat{v}^x(\xi,t)\|_{L^2_t}\|_{L^2_\xi} \leq \|\langle\xi^4-\xi^2\rangle^{1/2-\delta}\langle\xi\rangle^{-s}\widehat{v}(\xi,\tau)\|_{L^2_{\xi,\tau}}.$$

To estimate I, it is enough to show that

$$\int_{\mathbb{R}} |\widehat{\theta_T} *_\tau \widehat{v}(\tau)|^2|\tau-a|^{2(1/2-\delta)}d\tau \lesssim \int_{\mathbb{R}} |\widehat{v}(\tau)|^2|\tau-a|^{2(1/2-\delta)}d\tau, \; \forall \; a \in \mathbb{R}.$$

In this direction, we have using again Plancherel's identity

$$\int_{\mathbb{R}} |\widehat{\theta_T} *_\tau \widehat{v}(\tau)|^2 |\tau - a|^{2(1/2-\delta)} d\tau = \|D_t^{1/2-\delta}(e^{-iat}\theta_T v)\|_{L_t^2}^2,$$

and then, we use the Leibniz rule for fractional derivative derived in [17], recalling that $0 < \delta < 1/2$, and obtain

$$\begin{aligned}
&\|D_t^{1/2-\delta}(e^{-iat}\theta_T v)\|_{L_t^2} \\
&\lesssim \|e^{iat} v D_t^{1/2-\delta}\theta_T\|_{L_t^2} + \|\theta_T\|_{L^\infty}\|D_t^{1/2-\delta}(e^{-iat}v)\|_{L_t^2} \\
&\lesssim \|e^{iat} v D_t^{1/2-\delta}\theta_T\|_{L_t^2} + \left(\int_{\mathbb{R}} |\widehat{v}|^2 |\tau - a|^{2(1/2-\delta)} d\tau|\right)^{1/2}.
\end{aligned}$$

It remains then to estimate $\|e^{iat} v D_t^{1/2-\delta}\theta_T\|_{L_t^2}$, we use the Hölder inequality and the Hardy-Littlewood-Sobolev theorem

$$\begin{aligned}
\|e^{iat} v D_t^{1/2-\delta}\theta_T\|_{L_t^2} &\leq \|e^{iat} v\|_{L_t^{1/\delta}} \|D_t^{1/2-\delta}\theta_T\|_{L_t^{2/(1-2\delta)}} \\
&\lesssim \|D_t^{1/2-\delta}(e^{-iat}v)\|_{L_t^2} \|D_t^{1/2-\delta}\theta_T\|_{L_t^{2/(1-2\delta)}}.
\end{aligned}$$

Finally we use the Hausdorff-Young theorem (which tells that the inverse Fourier transform is bounded from $L^{2/(1+2\delta)}$ in $L^{2/(1-2\delta)}$) to obtain

$$\begin{aligned}
\|D_t^{1/2-\delta}\theta_T\|_{L^{2/(1-2\delta)}} &\lesssim \left(\int_{\mathbb{R}} (|\tau|^{1/2-\delta}|T\widehat{\theta}(T\tau)|)^{2/(1+2\delta)} d\tau\right)^{1/2+\delta} \\
&\lesssim \left(\int_{\mathbb{R}} (|\tau|^{1/2-\delta}|\widehat{\theta}(\tau)|)^{2/(1+2\delta)} d\tau\right)^{1/2+\delta} \lesssim 1.
\end{aligned}$$

which ends with the proof of (2.39).

Since $1/2 - 2\delta = \alpha(1/2 - \delta)$, with $\alpha = \frac{1/2-2\delta}{1/2-\delta}$, we interpolate (2.38) and (2.39) to obtain

$$\|\theta_T v\|_{X^{-s,1/2-2\delta}} \leq \|\theta_T v\|_{X^{-s,1/2-\delta}}^{\alpha} \|\theta_T v\|_{X^{-s,0}}^{1-\alpha} \lesssim T^\delta \|v\|_{X^{-s,1/2-\delta}},$$

which proves (2.37) and (2.36). □

2.3 Bilinear estimates.

Proposition 2.5. *Let $s' > s > -1$, then there exists $\delta > 0$ such that*

$$\|(u)_x(v)_x\|_{X^{s,-1/2+\delta}} \lesssim \|u\|_{X^{s,1/2}}\|v\|_{X^{s,1/2}} \tag{2.40}$$

and

$$\|(u)_x(v)_x\|_{X^{s',-1/2+\delta}} \lesssim \|u\|_{X^{s,1/2}}\|v\|_{X^{s',1/2}} + \|u\|_{X^{s',1/2}}\|v\|_{X^{s,1/2}}. \tag{2.41}$$

Proof. We only treat the case $s = -1+\epsilon$ and $0 < \epsilon < 1/2$. The other cases can be proved using the same argument. Choose δ such that $0 < \delta < \frac{\epsilon}{6}$. By duality to prove (2.40) is equivalent to show that

$$I \lesssim \|h\|_{L^2(\mathbb{R}^2)}\|f_1\|_{L^2(\mathbb{R}^2)}\|f_2\|_{L^2(\mathbb{R}^2)}, \tag{2.42}$$

where

$$I = \int_{\mathbb{R}^4} \frac{h(\xi,\tau)|\xi_1|\langle\xi_1\rangle^{1-\epsilon}f_1(\xi_1,\tau_1)|\xi_2|\langle\xi_2\rangle^{1-\epsilon}f_2(\xi_2,\tau_2)}{\langle\xi\rangle^{1-\epsilon}\langle i\sigma+(\xi^4-\xi^2)\rangle^{1/2-\delta}\langle i\sigma_1+(\xi_1^4-\xi_1^2)\rangle^{1/2}\langle i\sigma_2+(\xi_2^4-\xi_2^2)\rangle^{1/2}}d\nu, \tag{2.43}$$

and

$$\begin{array}{l} d\nu = d\xi d\tau d\xi_1 d\tau_1, \quad \tau_2 = \tau-\tau_1, \quad \xi_2 = \xi-\xi_1 \\ \sigma = \tau-\xi^3, \quad \sigma_1 = \tau_1-\xi_1^3, \quad \text{and} \quad \sigma_2 = \tau_2-\xi_2^3. \end{array}$$

To estimate I, we divide the integral in the following regions

$$A = \{(\xi,\xi_1,\tau,\tau_1) \in \mathbb{R}^4 : |\xi_1| > \sqrt{2} \wedge \ |\xi_2| > \sqrt{2}\} \tag{2.44}$$

$$B = \{(\xi,\xi_1,\tau,\tau_1) \in \mathbb{R}^4 : |\xi_1| \le \sqrt{2} \vee \ |\xi_2| \le \sqrt{2}\}. \tag{2.45}$$

We denote by I_A and I_B the integral I restricted to the regions A and B, respectively.

Estimate for I_A. We estimate I_A using the Cauchy-Schwarz inequality

$$I_A \le \sup_{(\xi_1,\tau_1)\in\mathbb{R}^2}\{J_A(\xi_1,\tau_1)^{1/2}\} \times \|h\|_{L^2(\mathbb{R}^2)}\|f_1\|_{L^2(\mathbb{R}^2)}\|f_2\|_{L^2(\mathbb{R}^2)}, \tag{2.46}$$

where

$$\begin{aligned} J_A(\xi_1,\tau_1) = & \frac{\langle\xi_1\rangle^{4-2\epsilon}}{\langle i\sigma_1+(\xi_1^4-\xi_1^2)\rangle} \\ & \times \int_{\tilde{A}(\xi_1,\tau_1)} \frac{\langle\xi_2\rangle^{4-2\epsilon}d\xi d\tau}{\langle i\sigma+(\xi^4-\xi^2)\rangle^{1-2\delta}\langle\xi\rangle^{2-2\epsilon}\langle i\sigma_2+(\xi_2^4-\xi_2^2)\rangle}, \end{aligned}$$

and

$$\tilde{A}(\xi_1,\tau_1)=\{(\xi,\tau)\in\mathbb{R}^2:(\xi,\tau,\xi_1,\tau_1)\in A\}\quad\forall\ (\xi_1,\tau_1)\in\mathbb{R}^2.$$

Then, to obtain (2.46), it is enough to prove that $J_A(\xi_1,\tau_1)\lesssim 1$ for all $(\xi_1,\tau_1)\in\mathbb{R}^2$. Since $|\xi_1|>\sqrt{2}$, we have $|\xi_1^4-\xi_1^2|\gtrsim|\xi_1|^4$ so that

$$\langle i\sigma_1+(\xi_1^4-\xi_1^2)\rangle\gtrsim\langle\xi_1\rangle^4.$$

If we define $|\tilde{\sigma}|=\min\{|\sigma|,|\sigma_2|\}$, then

$$\langle i\sigma+(\xi^4-\xi^2)\rangle^{1-2\delta}\langle i\sigma_2+(\xi_2^4-\xi_2^2)\rangle\geq\langle\tilde{\sigma}\rangle^{1+\delta}\langle\xi_2\rangle^{4(1-3\delta)}.$$

The change of variable $\theta=\tilde{\sigma}$ yields the following estimate of $J_A(\xi_1,\tau_1)$

$$J_A(\xi_1,\tau_1)\lesssim\langle\xi_1\rangle^{-2\epsilon}\int_{\mathbb{R}^2}\frac{\langle\xi_2\rangle^{12\delta-2\epsilon}}{\langle\tilde{\sigma}\rangle^{1+\delta}\langle\xi\rangle^{2-2\epsilon}}d\xi d\tau\lesssim\left(\int_{\mathbb{R}}\frac{d\xi}{\langle\xi\rangle^{2-2\epsilon}}\right)\left(\int_{\mathbb{R}}\frac{d\theta}{\langle\theta\rangle^{1+\delta}}\right)\lesssim 1,\tag{2.47}$$

since $0<\delta<\frac{\epsilon}{6}$ and $0<\epsilon<1/2$. Using (2.46) and (2.47) we conclude that

$$I_A\lesssim\|h\|_{L^2(\mathbb{R}^2)}\|f_1\|_{L^2(\mathbb{R}^2)}\|f_2\|_{L^2(\mathbb{R}^2)}.\tag{2.48}$$

Estimate for I_B. By symmetry we suppose that $|\xi_1|\leq\sqrt{2}$. We use again the Cauchy-Schwarz inequality to get

$$I_B\leq\sup_{(\xi,\tau)\in\mathbb{R}^2}\{J_B(\xi,\tau)^{1/2}\}\times\|h\|_{L^2(\mathbb{R}^2)}\|f_1\|_{L^2(\mathbb{R}^2)}\|f_2\|_{L^2(\mathbb{R}^2)},\tag{2.49}$$

where

$$\begin{aligned}J_B(\xi,\tau)=&\ \frac{1}{\langle\xi\rangle^{2-2\epsilon}\langle i\sigma+(\xi^4-\xi^2)\rangle^{1-2\delta}}\\&\times\int_{|\xi_1|\lesssim 1}\int_{\tau_1\in\mathbb{R}}\frac{\langle\xi_2\rangle^{4-2\epsilon}d\xi_1d\tau_1}{\langle i\sigma_1+(\xi_1^4-\xi_1^2)\rangle\langle i\sigma_2+(\xi_2^4-\xi_2^2)\rangle}.\end{aligned}$$

We want to show that $J_B(\xi,\tau)\lesssim 1$. It is clear when $|\xi_2|\leq 2$. Then we can suppose that $|\xi_2|>2$, and so

$$\langle i\sigma_1+(\xi_1^4-\xi_1^2)\rangle\langle i\sigma_2+(\xi_2^4-\xi_2^2)\rangle\geq\langle\tilde{\sigma}\rangle^{1+\delta}\langle\xi_2\rangle^{4(1-\delta)},$$

where $|\tilde{\sigma}| = \min\{|\sigma_1|, |\sigma_2|\}$. Then

$$J_B(\xi,\tau) \lesssim \int_{|\xi_1|\lesssim 1}\int_{\tau_1\in\mathbb{R}} \frac{\langle\xi_2\rangle^{8\delta-2\epsilon}}{\langle\tilde{\sigma}\rangle^{1+\delta}} d\xi_1 d\tau_1 \lesssim 1, \tag{2.50}$$

From (2.49) and (2.50), we conclude that

$$I_B \lesssim \|h\|_{L^2(\mathbb{R}^2)}\|f_1\|_{L^2(\mathbb{R}^2)}\|f_2\|_{L^2(\mathbb{R}^2)}. \tag{2.51}$$

We use the estimates (2.48), (2.51) and the fact that $I = I_A + I_B$ to prove (2.42). This ends the proof of (2.40).

To prove (2.41), we combine (2.40) and the fact that by the triangular inequality $\langle\xi\rangle^{s'} \leq \langle\xi\rangle^s\langle\xi_1\rangle^{s'-s} + \langle\xi\rangle^s\langle\xi_2\rangle^{s'-s}$, so that

$$\begin{aligned}\|(u)_x(v)_x\|_{X^{s',-1/2+\delta}} &\leq \|(J^{s'-s}u)_x(v)_x\|_{X^{s,-1/2+\delta}} + \|(u)_x(J^{s'-s}v)_x\|_{X^{s,-1/2+\delta}}\\ &\lesssim \|u\|_{X^{s',1/2}}\|v\|_{X^{s,1/2}} + \|u\|_{X^{s,1/2}}\|v\|_{X^{s',1/2}}.\end{aligned}$$

This concludes the proof of Proposition 2.5. □

Proposition 2.6. *Let $s' > s > -1$, then, there exists $\delta > 0$ such that,*

$$\|(uv)_{xx}\|_{X^{s,-1/2+\delta}} \lesssim \|u\|_{X^{s,1/2}}\|v\|_{X^{s,1/2}} \tag{2.52}$$

and

$$\|(uv)_{xx}\|_{X^{s',-1/2+\delta}} \lesssim \|u\|_{X^{s',1/2}}\|v\|_{X^{s,1/2}} + \|u\|_{X^{s,1/2}}\|v\|_{X^{s',1/2}}. \tag{2.53}$$

Proof. We follow the same strategy as in the proof of Proposition 2.5. By duality, (2.52) is equivalent to prove that for all $s > -1$, there exists $\delta > 0$ such that

$$I \lesssim \|h\|_{L^2(\mathbb{R}^2)}\|f_1\|_{L^2(\mathbb{R}^2)}\|f_2\|_{L^2(\mathbb{R}^2)}, \tag{2.54}$$

where

$$I = \int_{\mathbb{R}^4} \frac{|\xi|^2\langle\xi\rangle^s h(\xi,\tau) f_1(\xi_1,\tau_1) f_2(\xi_2,\tau_2)}{\langle i\sigma + (\xi^4-\xi^2)\rangle^{1/2-\delta}\langle\xi_1\rangle^s\langle i\sigma_1 + (\xi_1^4-\xi_1^2)\rangle^{1/2}\langle\xi_2\rangle^s\langle i\sigma_2 + (\xi_2^4-\xi_2^2)\rangle^{1/2}} d\nu \tag{2.55}$$

and

$$d\nu = d\xi d\tau d\xi_1 d\tau_1, \quad \sigma = \tau - \xi^3, \quad \sigma_1 = \tau_1 - \xi_1^3, \quad \text{and} \quad \sigma_2 = \tau_2 - \xi_2^3.$$

Case $s \geq 0$. By the triangle inequality we know that $\langle \xi \rangle^s \leq \langle \xi_1 \rangle^s \langle \xi_2 \rangle^s$ for all $s \geq 0$. Then, it is enough to prove (2.55) for $s = 0$. We choose $0 < \delta < 1/4$. By symmetry, one can suppose that $|\sigma_1| \geq |\sigma_2|$. We can also suppose that we always have $|\xi| > 4$, otherwise (2.54) is trivial. We use the Cauchy-Schwarz inequality to estimate I

$$I \leq \sup_{(\xi,\tau)\in\mathbb{R}^2} \{J(\xi,\tau)^{1/2}\} \times \|h\|_{L^2(\mathbb{R}^2)} \|f_1\|_{L^2(\mathbb{R}^2)} \|f_2\|_{L^2(\mathbb{R}^2)} \tag{2.56}$$

where

$$\begin{aligned} J(\xi,\tau) = & \frac{|\xi|^4}{\langle i\sigma + (\xi^4 - \xi^2)\rangle^{1-2\delta}} \\ & \times \int_{\mathbb{R}^2} \frac{d\xi_1 d\tau_1}{\langle i\sigma_1 + (\xi_1^4 - \xi_1^2)\rangle \langle i\sigma_2 + (\xi_2^4 - \xi_2^2)\rangle}. \end{aligned}$$

Since $|\xi| > 4$, we have $|\xi^4 - \xi^2| \gtrsim |\xi|^4$, so that

$$\langle i\sigma + (\xi^4 - \xi^2)\rangle^{1-2\delta} \gtrsim \langle \xi \rangle^{4(1-2\delta)}.$$

In the case $2|\xi_1| \geq |\xi|$, we have $|\xi_1| \geq 2$, then

$$\langle i\sigma_1 + (\xi_1^4 - \xi_1^2)\rangle \langle i\sigma_2 + (\xi_2^4 - \xi_2^2)\rangle \gtrsim \langle \sigma_2 \rangle^{1+\delta} \langle \xi \rangle^{8\delta} \langle \xi_1 \rangle^{4-8\delta},$$

which yields

$$J(\xi,\tau) \lesssim \int_{\mathbb{R}^2} \frac{d\xi_1 d\tau_1}{\langle \xi_1 \rangle^{4-8\delta} \langle \sigma_2 \rangle^{1+\delta}} \lesssim 1, \tag{2.57}$$

since $0 < \delta < 1/4$. In the case $2|\xi_1| < |\xi|$, we have $|\xi_2| > |\xi|$, so that

$$\langle i\sigma_1 + (\xi_1^4 - \xi_1^2)\rangle \langle i\sigma_2 + (\xi_2^4 - \xi_2^2)\rangle \gtrsim \langle \sigma_2 \rangle^{1+\delta} \langle \xi \rangle^{8\delta} \langle \xi_2 \rangle^{4-8\delta}.$$

We conclude that

$$J(\xi,\tau) \lesssim \int_{\mathbb{R}^2} \frac{d\xi_1 d\tau_1}{\langle \xi_2 \rangle^{4-8\delta} \langle \sigma_2 \rangle^{1+\delta}} \lesssim 1. \tag{2.58}$$

We combine (2.56), (2.57) and (2.58) to obtain the estimate (2.54). This concludes the proof of Proposition 2.6 in the case $s \geq 0$.

Case $-1 < s < 0$. Let $-1 < s < 0$, we can write $s = -1 + \epsilon$, where $0 < \epsilon < 1$. We choose $0 < \delta < \epsilon/6$. We will show that (2.54) still remains true in this case, where I can be rewritten as

$$I = \int_{\mathbb{R}^4} \frac{|\xi|^2 h(\xi,\tau)\langle\xi_1\rangle^{1-\epsilon} f_1(\xi_1,\tau_1)\langle\xi_2\rangle^{1-\epsilon} f_2(\xi_2,\tau_2)}{\langle\xi\rangle^{1-\epsilon}\langle i\sigma + (\xi^4 - \xi^2)\rangle^{1/2-\delta}\langle i\sigma_1 + (\xi_1^4 - \xi_1^2)\rangle^{1/2}\langle i\sigma_2 + (\xi_2^4 - \xi_2^2)\rangle^{1/2}} d\nu, \tag{2.59}$$

where

$$d\nu = d\xi d\tau d\xi_1 d\tau_1, \quad \sigma = \tau - \xi^3, \quad \sigma_1 = \tau_1 - \xi_1^3, \quad \text{and} \quad \sigma_2 = \tau_2 - \xi_2^3.$$

By symmetry we can assume $|\sigma_1| \geq |\sigma_2|$. Once again we divide the domain of integration $\mathbb{R}^4$ in the three regions $A = A_1 \cup A_2$, B and C where

$$\begin{aligned} A_1 &= \{(\xi,\xi_1,\tau,\tau_1) \in \mathbb{R}^4 : |\xi| > 4 \wedge \ |\xi| \leq 2|\xi_1|\} \\ A_2 &= \{(\xi,\xi_1,\tau,\tau_1) \in \mathbb{R}^4 : |\xi| > 4 \wedge \ |\xi| > 2|\xi_1| \wedge |\sigma_1| \geq |\sigma|\} \\ B &= \{(\xi,\xi_1,\tau,\tau_1) \in \mathbb{R}^4 : |\xi| > 4 \wedge \ |\xi| > 2|\xi_1| \wedge |\sigma| > |\sigma_1|\} \\ C &= \{(\xi,\xi_1,\tau,\tau_1) \in \mathbb{R}^4 : |\xi| \leq 4\} \end{aligned}$$

and denote by I_A, I_B and I_C the restriction of I to each one of these regions. Then, we estimate I_A, I_B and I_C respectively.

Estimate for I_A. In the region A, we estimate I_A using the Cauchy-Schwarz inequality

$$I_A \leq \sup_{(\xi_1,\tau_1)\in\mathbb{R}^2} \{J_A(\xi_1,\tau_1)^{1/2}\} \times \|h\|_{L^2(\mathbb{R}^2)}\|f_1\|_{L^2(\mathbb{R}^2)}\|f_2\|_{L^2(\mathbb{R}^2)}, \tag{2.60}$$

where

$$\begin{aligned} J_A(\xi_1,\tau_1) = &\frac{\langle\xi_1\rangle^{2-2\epsilon}}{\langle i\sigma_1 + (\xi_1^4 - \xi_1^2)\rangle} \\ &\times \int_{\tilde{A}(\xi_1,\tau_1)} \frac{|\xi|^4\langle\xi_2\rangle^{2-2\epsilon} d\xi d\tau}{\langle i\sigma + (\xi^4 - \xi^2)\rangle^{1-2\delta}\langle\xi\rangle^{2-2\epsilon}\langle i\sigma_2 + (\xi_2^4 - \xi_2^2)\rangle} \end{aligned}$$

and

$$\tilde{A}(\xi_1,\tau_1) = \{(\xi,\tau) \in \mathbb{R}^2 : (\xi,\tau,\xi_1,\tau_1) \in A\} \quad \forall\ (\xi_1,\tau_1) \in \mathbb{R}^2.$$

Once again, we denote by $J_{A_i}(\xi_1,\tau_1)$, for $i = 1$ or 2, the restriction of the integral $J_A(\xi_1,\tau_1)$ in the two regions

$$\tilde{A}_i(\xi_1,\tau_1) = \{(\xi,\tau) \in \mathbb{R}^2 : (\xi,\tau,\xi_1,\tau_1) \in A_i\}, \quad i = 1,2.$$

Estimate for $J_{A_1}(\xi_1,\tau_1)$. Since $|\xi_1| \geq |\xi|/2 > 2$ in A_1, we have that

$$\langle \xi_1^4 - \xi_1^2 \rangle \gtrsim \langle \xi_1 \rangle^4.$$

Then we get the following estimates in this region

$$\begin{aligned}\langle i\sigma_1 + (\xi_1^4 - \xi_1^2)\rangle &\gtrsim \langle \sigma_1\rangle^\delta \langle \xi_1\rangle^{4(1-\delta)} \\ &\gtrsim \langle \sigma_2\rangle^\delta \langle \xi_1\rangle^{2(1-\epsilon)} \langle \xi_2\rangle^{2(1-\epsilon)} \langle \xi\rangle^{4(\epsilon-\delta)}\end{aligned}$$

and

$$\langle i\sigma + (\xi^4 - \xi^2)\rangle^{1-2\delta} \gtrsim \langle \xi\rangle^{4(1-2\delta)}.$$

This yields

$$J_{A_1}(\xi_1,\tau_1) \lesssim \int_{\mathbb{R}^2} \frac{d\xi d\tau}{\langle \xi\rangle^{2+2\epsilon-12\delta} \langle \sigma_2\rangle^{1+\delta}} \lesssim 1. \tag{2.61}$$

Estimate for $J_{A_2}(\xi_1,\tau_1)$. In A_2, we have $|\xi| \sim |\xi_2| \gtrsim |\xi_1|$ and $|\xi_2| > 2$, then we deduce

$$\langle i\sigma_2 + (\xi_2^4 - \xi_2^2)\rangle \gtrsim \langle \xi_2\rangle^4 \gtrsim \langle \xi_1\rangle^{2(1-\epsilon)} \langle \xi_2\rangle^{2(1-\epsilon)} \langle \xi\rangle^{4\epsilon}.$$

Moreover since $|\sigma_1| > |\sigma|$,

$$\langle i\sigma + (\xi^4 - \xi^2)\rangle^{1-2\delta} \langle i\sigma_1 + (\xi_1^4 - \xi_1^2)\rangle \gtrsim \langle \sigma\rangle^{1+\delta} \langle \xi\rangle^{2(1+\epsilon)} \langle \xi\rangle^{2-12\delta-2\epsilon}.$$

Thus

$$J_{A_2}(\xi_1,\tau_1) \lesssim \int_{\mathbb{R}^2} \frac{d\xi d\tau}{\langle \xi\rangle^{2+2\epsilon-12\delta} \langle \sigma\rangle^{1+\delta}} \lesssim 1. \tag{2.62}$$

Therefore the estimates (2.60), (2.61) and (2.62) imply that

$$I_A \lesssim \|h\|_{L^2(\mathbb{R}^2)}\|f_1\|_{L^2(\mathbb{R}^2)}\|f_2\|_{L^2(\mathbb{R}^2)}. \tag{2.63}$$

Estimate for I_B. We estimate I_B using the Cauchy-Schwarz inequality

$$I_B \le \sup_{(\xi,\tau)\in\mathbb{R}^2}\{J^{1/2}_{B(\xi,\tau)}\} \times \|h\|_{L^2(\mathbb{R}^2)}\|f_1\|_{L^2(\mathbb{R}^2)}\|f_2\|_{L^2(\mathbb{R}^2)}, \tag{2.64}$$

where

$$J_{B(\xi,\tau)} = \frac{\langle\xi\rangle^{2(1+\epsilon)}}{\langle i\sigma + (\xi^4-\xi^2)\rangle^{1-2\delta}} \times \int_{\tilde{B}(\xi,\tau)} \frac{\langle\xi_1\rangle^{2-2\epsilon}\langle\xi_2\rangle^{2-2\epsilon}d\xi_1 d\tau_1}{\langle i\sigma_1+(\xi_1^4-\xi_1^2)\rangle\langle i\sigma_2+(\xi_2^4-\xi_2^2)\rangle}$$

and

$$\tilde{B}(\xi,\tau) = \{(\xi_1,\tau_1)\in\mathbb{R}^2 : (\xi,\tau,\xi_1,\tau_1)\in B\} \quad \forall\ (\xi,\tau)\in\mathbb{R}^2.$$

In B, we have $|\xi| \sim |\xi_2| \gtrsim |\xi_1|$, then

$$\langle i\sigma_2 + (\xi_2^4-\xi_2^2)\rangle \gtrsim \langle\xi_2\rangle^4 \gtrsim \langle\xi_1\rangle^{2(1-\epsilon)}\langle\xi_2\rangle^{2(1-\epsilon)}\langle\xi_1\rangle^{4\epsilon}.$$

Moreover since $|\sigma| > |\sigma_1|$,

$$\langle i\sigma + (\xi^4-\xi^2)\rangle^{1-2\delta}\langle i\sigma_1 + (\xi_1^4-\xi_1^2)\rangle \gtrsim \langle\sigma_1\rangle^{1+\delta}\langle\xi\rangle^{2(1+\epsilon)}\langle\xi_1\rangle^{2-12\delta-2\epsilon}.$$

Then, we get

$$J_B(\xi,\tau) \lesssim \int_{\mathbb{R}^2}\frac{d\xi_1 d\tau_1}{\langle\xi_1\rangle^{2+2\epsilon-12\delta}\langle\sigma_1\rangle^{1+\delta}} \lesssim 1. \tag{2.65}$$

We conclude from (2.64) and (2.65) that

$$I_B \lesssim \|h\|_{L^2(\mathbb{R}^2)}\|f_1\|_{L^2(\mathbb{R}^2)}\|f_2\|_{L^2(\mathbb{R}^2)}. \tag{2.66}$$

Estimate for I_C. We remember that by symmetry we can suppose that $|\sigma_1| \ge |\sigma_2|$ in C. Moreover, if $|\xi_1| \le \sqrt{2}$ and $|\xi_2| \le \sqrt{2}$, it is clear that

$$I_C \lesssim \|h\|_{L^2(\mathbb{R}^2)}\|f_1\|_{L^2(\mathbb{R}^2)}\|f_2\|_{L^2(\mathbb{R}^2)}. \tag{2.67}$$

Then, since $\xi+\xi_1+\xi_2=0$, we can suppose that $|\xi_1|>\sqrt{2}$ and $|\xi_2|>\sqrt{2}$ in C and we use again the Cauchy-Schwarz inequality to deduce that

$$I_C \leq \sup_{(\xi_1,\tau_1)\in\mathbb{R}^2}\{J_C(\xi_1,\tau_1)^{1/2}\}\times\|h\|_{L^2(\mathbb{R}^2)}\|f_1\|_{L^2(\mathbb{R}^2)}\|f_2\|_{L^2(\mathbb{R}^2)}, \tag{2.68}$$

where

$$\begin{aligned} J_C(\xi_1,\tau_1) = &\frac{\langle\xi_1\rangle^{2(1-\epsilon)}}{\langle i\sigma_1+(\xi_1^4-\xi_1^2)\rangle} \\ &\times\int_{|\xi|\lesssim 1}\int_{\tau\in\mathbb{R}}\frac{\langle\xi_2\rangle^{2(1-\epsilon)}d\xi d\tau}{\langle i\sigma+(\xi^4-\xi^2)\rangle^{1-2\delta}\langle i\sigma_2+(\xi_2^4-\xi_2^2)\rangle}. \end{aligned}$$

In the region C, we have

$$\langle i\sigma_1+(\xi_1^4-\xi_1^2)\rangle \gtrsim \langle\sigma_2\rangle^{\delta}\langle\xi_1\rangle^{4(1-\delta)} \gtrsim \langle\sigma_2\rangle^{\delta}\langle\xi_1\rangle^{2(1-\delta)}\langle\xi_2\rangle^{2(1-\delta)}.$$

Then, we obtain

$$J_{\tilde{C}(\xi_1,\tau_1)} \lesssim \langle\xi_1\rangle^{2(\delta-\epsilon)}\int_{|\xi|\lesssim 1}\int_{\tau\in\mathbb{R}}\frac{\langle\xi_2\rangle^{2(\delta-\epsilon)}d\xi d\tau}{\langle\sigma_2\rangle^{1+\delta}} \lesssim 1. \tag{2.69}$$

which together with (2.68) implies (2.67).

A combination of (2.63), (2.66) and (2.67) conclude the proof of the proposition in the case $-1<s<0$.

The proof of (2.53) follows exactly the same argument as in Proposition 2.5. □

2.4 Proofs of Theorems 2.1 and 2.2.

Proof of Theorem 2.1. Since we are in the case $\mu > 0$, we can suppose without loss of generality, that the equation (1) has the form

$$u_t + u_{xxx} + (u_{xxxx} + u_{xx}) + (u_x)^2 + (u^2)_{xx} = 0. \tag{2.70}$$

1. *Existence and persistence.* Let $s > -1$ and $\phi \in H^s(\mathbb{R})$. By Duhamel's principle, solving (2.5) is equivalent to solve the following integral equation

$$u(t) = V(t)\phi - \int_0^t V(t-t')f(u)(t')dt', \tag{2.71}$$

where

$$f(u)(t) = (u_x)^2(t) + (u^2)_{xx}(t). \tag{2.72}$$

In fact, we will solve a weaker version of (2.71), let $0 < T < 1$, θ and θ_T as in (2.1), define

$$F_T(v)(t) = \theta(t)\left(V(t)\phi - \int_0^t V(t-t')(\theta_T f(v))(t')dt'\right). \tag{2.73}$$

We want to use the Picard fixed point theorem to find a solution of

$$v = F_T(v) \tag{2.74}$$

in the space $X^{s,1/2}$, for some $0 < T < 1$. If we define $u = v_{|[0,T]}$, the solution of (2.74) restricted to the interval $[0,T]$, u will be a solution of (2.71) in $[0,T]$ and, by the definition of $X_T^{s,b}$ in (2.3), we will have that $u \in X_T^{s,1/2}$.

Since $s > -1$ is fixed, we choose, using Propositions 2.5 and 2.6, $0 < \delta < 1/2$ such that

$$\|f(v)\|_{X^{s,-1/2+2\delta}} \leq \|(v_x)^2\|_{X^{s,-1/2+2\delta}} + \|(v^2)_{xx}\|_{X^{s,-1/2+2\delta}} \lesssim \|v\|^2_{X^{s,1/2}}. \tag{2.75}$$

Then, using (2.8), (2.14) and (2.36), joint with (2.75), we deduce that, there exists a constant $C > 0$ such that

$$\|F_T(v)\|_{X^{s,1/2}} \leq C\left(\|\phi\|_{H^s(\mathbb{R})} + T^\delta \|v\|^2_{X^{s,1/2}}\right). \tag{2.76}$$

Since

$$f(v)-f(w)=(v+w)_x(v-w)_x+((v+w)(v-w))_{xx}\,,$$

the same computation leads to

$$\|F_T(v)-F_T(w)\|_{X^{s,1/2}}\leq CT^{\delta}\|v+w\|_{X^{s,1/2}}\|v-w\|_{X^{s,1/2}}. \tag{2.77}$$

We define $X^{s,1/2}(a)=\{v\in X^{s,1/2}\ :\ \|v\|_{X^{s,1/2}}\leq a\}$ with $a=2C\|\phi\|_{H^s}$. Then, if we choose

$$0<T<\frac{1}{(2Ca)^{1/\delta}}=\frac{1}{(4C^2\|\phi\|_{H^s})^{1/\delta}}, \tag{2.78}$$

(2.76) and (2.77) imply that F_T is a contraction on the Banach space $X^{s,1/2}(a)$. Thus, we deduce by the fixed point theorem that there exists a unique solution $v\in X^{s,1/2}(a)$ of (2.74).

Using the bilinear estimates as above, we deduce that there exists $0<\delta<1/2$ such that $f(v)\in X^{s,-1/2+\delta}$. Therefore applying Proposition 2.3 and using the fact that V is a strongly continuous group in $H^s(\mathbb{R})$, we deduce that $v\in C(\mathbb{R};H^s(\mathbb{R}))$. As noticed above, we conclude that

$$u=v_{|[0,T]}\in X_T^{s,1/2}\cap C^0([0,T],H^s(\mathbb{R}))$$

is a solution of the integral equation (2.71).

2. *If $s'>s>-1$, the result holds in the same time interval $[0,T]$ with $T=T(\|\phi\|_{H^s})$.* Let $\phi\in H^{s'}$, in order to show that the time existence of the solution of the integral equation (2.71) only depends on $\|\phi\|_{H^s}$, we have to modify a little bit the argument above. We will again apply the fixed point theorem to solve the equation (2.74) but this time in a closed ball of the Banach space $Z=\{v\in X^{s',1/2}/:\|v\|_Z=\|v\|_{X^{s,1/2}}+\nu\|v\|_{X^{s',1/2}}<\infty\}$, where $\nu=\frac{\|\phi\|_{H^s}}{\|\phi\|_{H^{s'}}}$.

We deduce from (2.76) that there exists $0<\delta<1/2$ and $C>0$ such that

$$\|F_T(v)\|_{X^{s,1/2}}\leq C(\|\phi\|_{H^s}+T^{\delta}\|v\|_Z^2).$$

Using the linear estimates (2.8), (2.14), (2.36) and the bilinear ones (2.41) and (2.53)), we also have

$$\begin{aligned}\|F_T(v)\|_{X^{s',1/2}} &\leq C(\|\phi\|_{H^{s'}} + T^\delta \|v\|_{X^{s',1/2}}\|v\|_{X^{s,1/2}})\\ &\leq \frac{C}{\nu}(\|\phi\|_{H^s} + T^\delta\|v\|_Z^2),\end{aligned}$$

so that

$$\|F_T(v)\|_Z \leq C(2\|\phi\|_{H^s} + T^\delta\|v\|_Z^2)).$$

The same argument gives

$$\|F_T(v) - F_T(w)\|_Z \leq CT^\delta\|v+w\|_Z\|v-w\|_Z.$$

Then if we define $Z(a)$ the closed ball of Z centered at the origin with radius $a = 4C\|\phi\|_{H^s}$ and if we choose

$$0 < T < 1/(8C^2\|\phi\|_{H^s(\mathbb{R})}),$$

we are able to apply the fixed point theorem in the Banach space $Z(a)$ and then we conclude easily that the result holds in the time interval $[0,T]$, T depending only on $\|\phi\|_{H^s(\mathbb{R})}$.

3. *Uniqueness.* Note that the fixed point theorem argument imply the uniqueness of the solution of the truncated integral equation (2.74) in the ball $X^{s,1/2}(a)$. But, we want to have the uniqueness of the integral equation (2.71) in the whole space $X_T^{s,1/2}$.

Let $T > 0$ and $\tilde{u} \in X_T^{s,1/2}$ another solution of the integral equation (2.71). Fix an extension $\tilde{v}$ of $\tilde{u}$ defined on $\mathbb{R}\times\mathbb{R}$. Using the above existence argument and (2.78), we can choose $0 < T_1 < T$ such that the equation (2.74) (with the time T_1 instead of T) admits a unique solution in the ball $X^{s,1/2}(\tilde{a})$, where $\tilde{a} = \|\tilde{v}\|_{X^{s,1/2}}$. Since we used the Picard fixed point theorem to solve (2.74), we deduce in particular that

$$F_{T_1}^n(\tilde{v}) \longrightarrow_{n\to+\infty} v \quad \text{in} \quad X^{s,1/2}. \tag{2.79}$$

Moreover, we know, from the definition of F_T (see (2.73)) and the fact that $\tilde{u}$ is a solution of (2.71) in $X_T^{s,1/2}$, that

$$F_{T_1}(\tilde{v})_{|[0,T_1]} = \tilde{u}. \tag{2.80}$$

Therefore, we can conclude combining (2.79) and (2.80) that

$$\|\tilde{u} - u\|_{X_{T_1}^{s,1/2}} \leq \|F_{T_1}^n(\tilde{v}) - v\|_{X^{s,1/2}} \longrightarrow_{n\to+\infty} 0,$$

so that $u = \tilde{u}$ on $[0, T_1]$. Since T_1 only depends on the extension $\tilde{v}$ of $\tilde{u}$, we can reapply this process a finite number of times to extend the uniqueness result in the whole interval $[0, T]$.

4. *Regularity.* We will show that the solution $u \in X_T^{s,1/2}$ of the integral equation (2.71) that we know to be in $C([0,T]; H^s(\mathbb{R}))$ also belongs to $C((0,T); H^\infty(\mathbb{R}))$.

First we know that

$$L : t \mapsto V(t)\phi \in C((0,T]; H^\infty(\mathbb{R}))$$

(see for example [14] Theorem 4.18). Since our solution u belongs to $X_T^{s,1/2}$, we also know using Proposition 2.3 and the bilinear estimates that there exists $\delta > 0$ such that

$$N : t \mapsto \int_0^t V(t-t')f(u)(t')dt' \in C([0,T]; H^{s+\delta}(\mathbb{R})).$$

Thus, we deduce that $u \in C((0,T]; H^{s+\delta}(\mathbb{R}))$. Now, fix an arbitrary time t_1 in $(0,T)$. Then, since $\lim_{t\to 0} \|u(t) - \phi\|_{H^s} = 0$ we can find $0 < t_0 < t_1$ such that $\tilde{T} = T(\|u(t_0)\|_{H^s}) > t_1$. Thus, if we reapply the existence result with the initial data $u(t_0)$ in the space $H^{s+\delta}(\mathbb{R})$, use the fact that the time existence only depends on $\|u(t_0)\|_{H^s}$ and the uniqueness result, we are able to conclude that $u \in C((t_0, t_0+\tilde{T}); H^{s+2\delta}(\mathbb{R}))$. Therefore, since the time t_1 is arbitrary, we conclude iterating the argument that

$$u \in C^0((0,T); H^\infty(\mathbb{R})).$$

5. *Smoothness of the map solution.* Combining an identical argument to the one use in the existence proof with the estimate (2.28), one can easily show that the map solution

$$\begin{aligned} S: H^s(\mathbb{R}) &\to X_T^{s,1/2} \cap C([0,T]; H^s(\mathbb{R})) \\ \phi \quad &\mapsto u_\phi(t) = S(t)\phi \end{aligned} \tag{2.81}$$

is locally Lipschitz.

In order to prove the smoothness of S, let define

$$\begin{aligned} H: H^s(\mathbb{R}) \times X_T^{s,1/2} \cap C([0,T]; H^s(\mathbb{R})) &\to X_T^{s,1/2} \cap C([0,T]; H^s(\mathbb{R})) \\ (\phi, u) &\mapsto u(t) - V(t)\phi + \int_0^t V(t-t')f(u)(t')dt'. \end{aligned}$$

We define the norm $\|u\|_T := \|u\|_{X_T^{s,1/2}} + \|u\|_{C([0,T];H^s)}$ on the space $X_T^{s,1/2} \cap C([0,T]; H^s(\mathbb{R}))$. Note that H is well defined, H is smooth and that from the existence result, we have $H(\phi, S(t)\phi) = 0$. Moreover, we fix $\phi \in H^s(\mathbb{R})$ and we compute for all $v \in X_T^{s,1/2} \cap C([0,T]; H^s(\mathbb{R}))$,

$$\partial_v H(\phi, S(t)\phi)v(t) = v(t) + 2\int_0^t V(t-t')((S(t)\phi)_x v_x + (S(t)\phi v)_{xx})(t')dt'.$$

Then, we deduce using the estimates (2.14), (2.28), (2.40) and (2.52) that there exists $C > 0$ and $\delta > 0$ such that for all $v \in X_T^{s,1/2} \cap C([0,T]; H^s(\mathbb{R}))$,

$$\|(id - \partial_v H(\phi, u))v(t)\|_T \leq 4C^2 T^\delta \|\phi\|_{H^s} \|v\|_T.$$

This imply that, if we choose T small enough such that $4C^2T^\delta\|\phi\|_{H^s} < 1$, the linear map $\partial_v H(\phi, u) \in \mathcal{L}(X_T^{s,1/2} \cap C([0,T]; H^s(\mathbb{R}))$ is an isomorphism. Thus, by the implicit function theorem, there exists a neighborhood V of ϕ in $H^s(\mathbb{R})$ and a smooth application $h: V \to X_T^{s,1/2} \cap C([0,T]; H^s(\mathbb{R})$ such that $H(\psi, h(\psi)) = 0$, $\forall \psi \in V$. This means that $S_{|V} = h$ is smooth and since smoothness is a local property, we conclude that the flow map data-solution S is smooth. □

Proof of Theorem 2.2. We first deal with the case $\gamma = \alpha/2$. Let $s > -1$, $\phi \in H^s(\mathbb{R})$. Define $T^\star = T^\star(\|\phi\|_{H^s})$ by

$$T^\star = \sup\{T > 0 \;:\; \exists! \text{ solution of (1) in } C([0,T]; H^s(\mathbb{R})) \cap X_T^{s,1/2}\}. \tag{2.82}$$

Let $u \in C([0,T^\star); H^s(\mathbb{R})) \cap C((0,T^\star); H^\infty(\mathbb{R}))$ the local solution of the integral equation associated to (1) in the maximal time interval $[0,T^\star)$. We will make the assumption $T^\star < \infty$ and obtain a contradiction.

Since u is smooth, we deduce that u solves the Cauchy problem (1) in a classical sense, this allows us to take the L^2 scalar product of (1) with u and integrate by parts (recalling that u is a real function), to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 &= -\mu[(u_{xxxx}, u)_{L^2} + (u_{xx}, u)_{L^2}] - \alpha(u_x^2, u)_{L^2} - \frac{\alpha}{2}(uu_{xx}, u)_{L^2} \\ &= -\mu[\|u_{xx}\|_{L^2}^2 + (u_{xx}, u)_{L^2}] \\ &\leq \mu\|u_{xx}\|_{L^2}\|u\|_{L^2} - \mu\|u_{xx}\|_{L^2}^2 \leq \frac{\mu}{4}\|u\|_{L^2}^2,\end{aligned}$$

where we used the Cauchy-Schwarz inequality and the identity $ab - b^2 \leq \frac{a^2}{4}$. Thus, by Gronwall's inequality, we deduce the following *a priori* estimate

$$\|u(t)\|_{L^2} \leq \|\phi\|_{L^2} e^{\frac{\mu t}{4}} \leq \|\phi\|_{L^2} e^{\frac{\mu T^\star}{4}} =: M, \quad \forall\, t \in (0, T^\star). \tag{2.83}$$

Since the time existence $T(\|\psi\|_{H^s})$ is a decreasing function of the norm of the initial data $\|\psi\|_{H^s}$, we know that there exists a time $T_1 > 0$ such that for all $\psi \in L^2(\mathbb{R})$, with $\|\psi\|_{L^2} \leq M$, there exists a unique solution v of (1) satisfying $v(0) = \psi$ and $v \in C([0,T_1]; L^2(\mathbb{R})) \cap C((0,T_1]; H^\infty(\mathbb{R}))$. Now, choose $0 < \epsilon < T_1$, apply this result with $\psi = u(T^\star - \epsilon)$ and define

$$\tilde{u}(t) = \begin{cases} u(t) & \text{when } 0 \leq t \leq T^\star - \epsilon \\ v(t - (T^\star - \epsilon)) & \text{when } T^\star - \epsilon \leq t \leq T^\star - \epsilon + T_1 \end{cases} \tag{2.84}$$

Then $\tilde{u}$ is a solution of (1) in the time interval $[0, T^\star - \epsilon + T_1]$, which contradicts (2.82), since $T^\star - \epsilon + T_1 > T^\star$.

In the case $\gamma = 0$, we only need to prove an *a priori* estimate in $L^2(\mathbb{R})$ (for example) for the solutions of (3), the rest of the proof follows exactly by the same argument as in the case $\gamma = \alpha/2$.

Let u a solution of (3) in the time interval $[0, T]$. Then following [3], we differentiate (3) to get

$$w_t + \delta w_{xxx} + \mu(w_{xxxx} + w_{xx}) + 2\alpha w w_x = 0, \quad \text{with } w = u_x. \tag{2.85}$$

so that, by Gronwall's inequality

$$\|u_x(t)\|_{L^2} = \|w(t)\|_{L^2} \leq \|\phi_x\|_{L^2} e^{\frac{\mu T}{4}}. \tag{2.86}$$

Then, we take the inner product in $L^2(\mathbb{R})$ of (3) with u, integrate by parts, use (2.86), Hölder's, Gagliardo-Nieremberg's and Young's inequalities to obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 &= -\mu(u_{xxxx} + u_{xx}, u)_{L^2} - \alpha(u_x^2, u)_{L^2} \\
&\leq |\alpha|\|u\|_{L^\infty}\|u_x\|_{L^2}^2 + \mu\|u_x\|_{L^2}^2 - \mu\|u_{xx}\|_{L^2}^2 \\
&\leq |\alpha|\|u\|_{L^2}^{\frac{1}{2}}\|u_x\|_{L^2}^{\frac{5}{2}} + \mu\|u_x\|_{L^2}^2 \\
&\leq |\alpha|(\|u\|_{L^2}^2 + \|u_x\|_{L^2}^{\frac{10}{3}}) + \mu\|u_x\|_{L^2}^2 \\
&\leq (|\alpha|\|\phi_x\|_{L^2}^{\frac{10}{3}} + \mu\|\phi_x\|_{L^2}^2)e^{c\mu T} + |\alpha|\|u\|_{L^2}^2,
\end{aligned}$$

where c is a positive constant. This leads, using Gronwall's inequality, to the following *a priori* estimate

$$\|u(t)\|_{L^2} \lesssim (\|\phi\|_{L^2} + (\|\phi_x\|_{L^2}^{\frac{5}{3}} + (\frac{\mu}{|\alpha|})^{1/2}\|\phi_x\|_{L^2})e^{c\mu T})e^{\frac{|\alpha|}{2}T}, \tag{2.87}$$

which ends with the proof of Theorem 2.2. □

2.5 Ill-posedness result.

Without loss of generality, we suppose $\delta = \mu = 1$ in (1). We will first prove that the Cauchy problem (1) cannot be solved in $H^s(\mathbb{R})$ using the fixed point theorem when $s < -1$. Then we will show that this fact implies Theorem 2.3.

Theorem 2.4. *Let $s < -1$ and $T > 0$. Then, there does not exist any space X_T such that X_T is continuously embedded in $C([0,T]; H^s(\mathbb{R}))$, i.e.*

$$\|u\|_{C([0,T];H^s)} \lesssim \|u\|_{X_T}, \quad \forall\ u \in X_T \tag{2.88}$$

and such that

$$\|V(t)\phi\|_{X_T} \lesssim \|\phi\|_{H^s}, \quad \forall\ \phi \in H^s(\mathbb{R}) \tag{2.89}$$

and

$$\|\int_0^t V(t-t')b(u,v)(t')dt'\|_{X_T} \lesssim \|u\|_{X_T}\|v\|_{X_T} \quad \forall\ u,\ v \in X_T, \tag{2.90}$$

where $b(u,v)$ is the nonlinearity defined by

$$b(u,v) = (\alpha-\gamma)u_x v_x + \frac{\gamma}{2}(uv)_{xx}, \quad \text{with } \alpha \neq \gamma. \tag{2.91}$$

Proof. Let $s < -1,\ T > 0$. Suppose that there exists a space X_T as in Theorem 2.4. Take $\phi, \psi \in H^s(\mathbb{R})$, $u(t) = V(t)\phi,\ v(t) = V(t)\psi$, and fix $0 < t < T$, then, we use (2.88), (2.89) and (2.90) to see that

$$\|\int_0^t V(t-t')b(V(t')\phi, V(t')\psi)dt'\|_{H^s} \lesssim \|\phi\|_{H^s}\|\psi\|_{H^s}. \tag{2.92}$$

We will show that (2.92) fails for an appropriate choice of ϕ and ψ, which would lead to a contradiction. Define ϕ and ψ[1] by

$$\phi = r^{-1/2}N^{-s}(\chi_{I_1})^{\vee}, \quad \psi = r^{-1/2}N^{-s}(\chi_{I_2})^{\vee}, \tag{2.93}$$

[1]We can also take $\operatorname{Re}\phi$ and $\operatorname{Re}\psi$ instead of ϕ and ψ if we want to deal with real solutions.

where

$$N \gg 1,\ r \sim 1,\ I_1 = [-N, -N+r] \text{ and } I_2 = [N+r, N+2r].$$

Note first that

$$\|\phi\|_{H^s} \sim 1 \quad \text{and} \quad \|\psi\|_{H^s} \sim 1. \tag{2.94}$$

Then we use the definition of the group V and Fubini's theorem to obtain

$$\begin{aligned} g(\xi,t) &:= \left(\int_0^t V(t-t')b(V(t')\phi, V(t')\psi)dt'\right)^{\wedge_x}(\xi) \\ &= \int_0^t e^{-(\xi^4-\xi^2)(t-t')}e^{i(t-t')\xi^3}[(\alpha-\gamma)(V(t')\phi_x)^{\wedge_x} \\ &\qquad *(V(t')\psi_x)^{\wedge_x}(\xi) + \frac{\gamma}{2}\xi^2(V(t')\phi)^{\wedge_x} * (V(t')\psi)^{\wedge_x}(\xi)]dt' \\ &= e^{it\xi^3}\int_{\mathbb{R}}((\alpha-\gamma)\xi_1\xi_2 + \frac{\gamma}{2}\xi^2)\widehat{\phi}(\xi_1)\widehat{\psi}\xi_2 h(t,\xi,\xi_1)d\xi_1, \\ &= \frac{e^{it\xi^3}}{rN^{2s}}\int_{K_\xi}((\alpha-\gamma)\xi_1\xi_2 + \frac{\gamma}{2}\xi^2)h(t,\xi,\xi_1)d\xi_1 \end{aligned} \tag{2.95}$$

where, $\xi_2 = \xi - \xi_1$ and, by a straightforward computation

$$\begin{aligned} h(t,\xi,\xi_1) &= e^{-(\xi^4-\xi^2)t}\int_0^t e^{-(\xi_1^4-\xi_1^2+\xi_2^4-\xi_2^2-(\xi^4-\xi^2))t'}e^{it'(\xi_1^3+\xi_2^3-\xi^3)}dt' \\ &= \frac{e^{-(\xi_1^4-\xi_1^2+\xi_2^4-\xi_2^2)t}e^{it(\xi_1^3+\xi_2^3-\xi^3)} - e^{-(\xi^4-\xi^2)t}}{-2\xi_1\xi_2(\xi_1^2-\xi\xi_1+2\xi^2-1) - 3i\xi\xi_1\xi_2} \\ &= \frac{a(t,\xi,\xi_1)+ib(t,\xi,\xi_1)}{c(t,\xi,\xi_1)+id(t,\xi,\xi_1)}, \end{aligned}$$

and

$$K_\xi = \{\xi_1 \ / \ \xi_1 \in I_1,\ \xi_2 \in I_2\}.$$

When $\xi_1 \in I_1$ and $\xi_2 \in I_2$, we have that

$$|\xi_1| \sim |\xi_2| \sim N,\ r \le \xi \le 3r \quad \text{and then} \quad |(\alpha-\gamma)\xi_1\xi_2 + \frac{\gamma}{2}\xi^2| \sim N^2,$$

since $\alpha \neq \gamma$. We use the mean value theorem to deduce that there exists $c \in K_\xi$ such that

$$\begin{aligned} |g(\xi,t)| &\ge \frac{1}{rN^{2s}}|\mathrm{Re}\left(\int_{K_\xi}((\alpha-\gamma)\xi_1\xi_2 + \frac{\gamma}{2}\xi^2)h(t,\xi,\xi_1)d\xi_1\right)| \\ &\gtrsim \frac{N^2}{rN^{2s}}\mathrm{mes}(K_\xi)|\mathrm{Re}\, h(t,\xi,c)|. \end{aligned}$$

Then we calculate

$$|\mathrm{Re}\, h(t,\xi,\xi_1)| = |\frac{a(t,\xi,\xi_1)c(t,\xi,\xi_1) + b(t,\xi,\xi_1)d(t,\xi,\xi_1)}{c(t,\xi,\xi_1)^2 + d(t,\xi,\xi_1)^2}|,$$

and since $\xi_1 \in K_\xi$, we have that

$$c(t,\xi,\xi_1)^2 + d(t,\xi,\xi_1)^2 \sim N^8, \;\; |b(t,\xi,\xi_1)d(t,\xi,\xi_1)| \ll 1,$$

and

$$\begin{aligned} |a(t,\xi,\xi_1)c(t,\xi,\xi_1)| &\sim N^4 |e^{-(\xi^4-\xi^2)t} - e^{-(\xi_1^4-\xi_1^2+\xi_2^4-\xi_2^2)t} \cos(3\xi\xi_1\xi_2 t)| \\ &\gtrsim N^4 (e^{-(r^4-r^2)t} - e^{-(\xi_1^4-\xi_1^2+\xi_2^4-\xi_2^2)t}) \gtrsim N^4, \end{aligned}$$

so that

$$|\mathrm{Re}\, h(t,\xi,\xi_1)| \gtrsim \frac{1}{N^4}$$

when $\xi_1 \in K_\xi$. Thus

$$|g(\xi,t)| \gtrsim N^{-2s-2}$$

and

$$\| \int_0^t V(t-t')b(V(t')\phi, V(t')\psi)dt' \|_{H^s} \gtrsim N^{-2s-2}. \tag{2.96}$$

We conclude from (2.92), (2.94) and (2.96) that

$$N^{-2s-2} \lesssim 1 \quad \forall\, N \gg 1, \tag{2.97}$$

which is in contradiction with the assumption $s < -1$. □

Proof of Theorem2.3. Assume that $\alpha \neq \delta$. Let $s < -1$, suppose that there exists $T > 0$ such that the Cauchy problem (1) is locally well-posed in $H^s(\mathbb{R})$ in the time interval $[0,T]$ and that the flow map solution $S : H^s(\mathbb{R}) \longrightarrow C([0,T]; H^s(\mathbb{R}))$ is C^2 at the origin. When

$\phi \in H^s(\mathbb{R})$, we will denote $u_\phi(t) = S(t)\phi$ the solution of the Cauchy problem (1) with initial datum ϕ. This means that u_ϕ is a solution of the integral equation

$$u_\phi(t) = S(t)\phi = V(t)\phi - \int_0^t V(t-t')b(u_\phi(t'), u_\phi(t'))dt', \tag{2.98}$$

where the nonlinearity b is defined in (2.91).

When ϕ and ψ are in $H^s(\mathbb{R})$, we use the fact that b is a bilinear symmetric application to compute the Fréchet derivative of $S(t)$ at ψ in the direction ϕ

$$d_\psi S(t)\phi = V(t)\phi - 2\int_0^t V(t-t')b(u_\psi(t'), d_\psi S(t')\phi)dt'. \tag{2.99}$$

Since the Cauchy problem (1) is supposed to be well-posed, we know using the uniqueness that $S(t)0 = u_0(t) = 0$ so that we deduce from (2.99) that

$$d_0 S(t)\phi = V(t)\phi. \tag{2.100}$$

Using (2.99), we compute the second Fréchet derivative at the origin in the direction (ϕ, ψ)

$$\begin{aligned} d_0^2 S(t)(\phi,\psi) &= d_0(d\ S(t)\phi)\psi = \frac{\partial}{\partial_\beta}(\beta \mapsto d_{\beta\psi}S(t)\phi)_{|\beta=0} \\ &= -2\int_0^t V(t-t')b(d_{\beta\psi}S(t')\psi, d_{\beta\psi}S(t')\phi)dt'_{|\beta=0} \\ &\quad -2\int_0^t V(t-t')b(u_{\beta\psi}(t'), d^2_{\beta\psi}S(t')(\phi,\psi))dt'_{|\beta=0}, \end{aligned}$$

so that we deduce using (2.100) that

$$d_0^2 S(t)(\phi,\psi) = -2\int_0^t V(t-t')b(V(t')\psi, V(t')\phi)dt'. \tag{2.101}$$

The assumption of C^2 regularity of $S(t)$ at the origin would imply that $d_0^2 S(t) \in \mathcal{B}(H^s(\mathbb{R}) \times H^s(\mathbb{R}), H^s(\mathbb{R}))$, which would lead to the following inequality

$$\|d_0^2 S(t)(\phi,\psi)\|_{H^s(\mathbb{R})} \lesssim \|\phi\|_{H^s(\mathbb{R})}\|\psi\|_{H^s(\mathbb{R})}, \quad \forall\ \phi,\ \psi \in H^s(\mathbb{R}). \tag{2.102}$$

But (2.102) is equivalent to (2.92) which has been shown to fail in the proof of Theorem 2.4. $\square$

Chapter 3

The non-dissipative problem: ill-posedness results.

3.1 Introduction and statements of the results.

We now turn our attention to the limit case of the IVP (1) when the dissipation μ tends to zero. In this case, we can suppose $\delta = 1$, so that (1) can be rewritten on the following form

$$\begin{cases} \partial_t u + \partial_x^3 u + \alpha(\partial_x u)^2 + \gamma u \partial_x^2 u = 0, \\ u(0) = \phi \end{cases} \tag{3.1}$$

In fact, we will consider a more general class of higher-order dispersive equations which generalizes (3.1) as well as the KdV equation:

$$\begin{cases} \partial_t u + \partial_x^{2j+1} u = \sum_{0 \le l_1 \le l_2 \le 2j} a_{l_1,l_2} \partial_x^{l_1} u \partial_x^{l_2} u, \quad x,\ t \in \mathbb{R}, \\ u(0) = \phi, \end{cases} \tag{3.2}$$

where u is a real- (or complex-) valued function and a_{l_1,l_2} are constants in $\mathbb{R}$ or $\mathbb{C}$. This class of equation, which was studied by Kenig, Ponce and Vega ([18] and [19]), is a particular case of the family (4), the polynomial considered here being only quadratic. We refer to the introduction for physical aplications.

As mentioned in the introduction, our first result is a negative one: if there exists $k > j$ such that $a_{0,k} \neq 0$, then the IVP (3.2) cannot be solved in any space continuously embedded in $C([-T,T], H^s(\mathbb{R}))$, $s \in \mathbb{R}$, using a fixed point theorem on the corresponding integral equation. We also deduce, as a consequence of this result, that in this case, the flow map data-solution associated to (3.2) cannot be C^2 at the origin from $H^s(\mathbb{R})$ to $H^s(\mathbb{R})$, for any $s \in \mathbb{R}$. Note that these results are valid in particular for our limit IVP (3.1).

These ideas can also be used for systems. For example, consider the following higher-order nonlinear dispersive system:

$$\begin{cases} \partial_t u_1 + \partial_x^3 u_1 + \frac{1}{2}\partial_x(u_1^2) + \frac{1}{2}\partial_x^2(u_2^2) = 0 \\ \partial_t u_2 + \partial_x^3 u_2 + \partial_x(u_1 u_2) = 0 \quad x,\ t \in \mathbb{R}, \\ u_1(0) = \phi_1, \quad u_2(0) = \phi_2, \end{cases} \tag{3.3}$$

where u_1 and u_2 are real-valued functions. This system appears in [22] as a model to particle-like behavior of nonlinear fields and was proved by Angulo and Barros [1] to be well-posed in some weighted Sobolev spaces. We prove here that the flow map of (3.3) cannot be C^2 at the origin from $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ to $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ for any $s \in \mathbb{R}$.

The same kind of argument leads to similar results for other higher-order nonlinear dispersive equations. We consider first an higher order Benjamin-Ono equation.

$$\begin{cases} \partial_t u + a\partial_x^3 u - bH\partial_x^2 u = cu\partial_x u - d\partial_x(uH\partial_x u + H(u\partial_x u)) \\ u(0) = \phi, \end{cases} \tag{3.4}$$

where H is the Hilbert transform, u is a real-valued solution,and $a \in \mathbb{R}$, b, c and d are positive constants. This equation was derived by Craig, Guyenne and Kalisch [10], using an Hamiltonian perturbation theory. It describes, as the Benjamin-Ono equation, the evolution of weakly nonlinear dispersive internal long waves at the interface of a two-layer system, one being infinitely deep.

In the same paper, Craig, Guyenne and Kalisch (always using an Hamiltonian perturbation theory) also derived an higher order intermediate long wave equation.

$$\begin{cases} \partial_t u + (a_1\mathcal{F}_h^2 + a_2)\partial_x^3 u - b\mathcal{F}_h\partial_x^2 u = cu\partial_x u - d\partial_x(u\mathcal{F}_h\partial_x u + \mathcal{F}_h(u\partial_x u)) \\ u(0) = \phi, \end{cases} \tag{3.5}$$

where $\mathcal{F}_h$ is the Fourier multiplier $-i\coth(h\xi)$, u is a real-valued solution, and $a_1,\ a_2,\ b,\ c,\ d$ and h are positive constants. The same ill-posedness results also apply for these equations.

These results are inspired by those from Molinet, Saut and Tzvetkov for the KPI equation [28] and the Benjamin-Ono (and the ILW) equation [29], (see also Bourgain [4] and Tzvetkov [35] for the KdV equation). It is interesting to notice that the equation (3.4) and the BO equation (as well as the equation (3.5) and the ILW equation) share the same property of ill-posedness of the flow in any Sobolev space $H^s(\mathbb{R})$.

This analogy is interesting: however the flow map solution associated to the BO equation cannot be C^2, the BO equation was shown by Iorio [13] to be well-posed in $H^s(\mathbb{R})$, for $s > 3/2$, using parabolic regularization and energy estimates, the flow map solution being only continuous. Later, this result was improved by many authors. The best, as far we know, was obtained recently by Ionescu and Kenig [12] and states the local well-posedness of the BO equation in $H^s(\mathbb{R})$ for $s \geq 0$.

Then one is naturally let to ask if a similar result could also hold for the higher-order BO equation, i.e., do we have well-posedness for the higher-order BO equation in some Sobolev space $H^s(\mathbb{R})$, the flow map remaining of course only continuous? We can also ask the same question for the other higher-order dispersive equations studied here. Unfortunately, we were not able to answer these questions, the difficulty residing here in the energy estimates. We tried to modify the Kato-Ponce commutator estimates (see [16]) to higher-order non-linearities, but without success.

Statement of the results.

Theorem 3.1. *Let $s \in \mathbb{R}$ and $T > 0$, suppose that there exists $k > j$ such that $a_{0,k} \neq 0$, then, there does not exist any space X_T such that X_T is continuously embedded in $C([-T,T]; H^s(\mathbb{R}))$, i.e.,*

$$\|u\|_{C([-T,T];H^s)} \lesssim \|u\|_{X_T}, \quad \forall\ u \in X_T, \tag{3.6}$$

and such that

$$\|U_j(t)\phi\|_{X_T} \lesssim \|\phi\|_{H^s}, \quad \forall\ \phi \in H^s(\mathbb{R}), \tag{3.7}$$

and

$$\| \int_0^t U_j(t-t') \sum_{0\le l_1\le l_2\le 2j} a_{l_1,l_2}\partial_x^{l_1}u(t')\partial_x^{l_2}v(t')dt'\|_{X_T} \lesssim \|u\|_{X_T}\|v\|_{X_T}, \quad \forall\ u,v \in X_T. \tag{3.8}$$

Theorem 3.2. *Let $s\in\mathbb{R}$, suppose that there exists $k>j$ such that $a_{0,k}\neq 0$. Then, if the Cauchy problem (3.2) is locally well-posed in $H^s(\mathbb{R})$, the flow map data-solution*

$$S(t) : H^s(\mathbb{R}) \longrightarrow H^s(\mathbb{R}), \quad \phi \longmapsto u(t) \tag{3.9}$$

is not C^2 at zero.

Theorem 3.3. *Let $s\in\mathbb{R}$. If the Cauchy problem (3.3) is locally well-posed in $H^s(\mathbb{R})\times H^s(\mathbb{R})$, then the flow-map data solution*

$$\begin{array}{rcl} S^{sKdV}(t) : H^s(\mathbb{R})\times H^s(\mathbb{R}) & \longrightarrow & H^s(\mathbb{R})\times H^s(\mathbb{R}) \\ (\phi_1,\phi_2) & \longmapsto & (u_1(t),u_2(t)), \end{array} \tag{3.10}$$

is not C^2 at zero.

Theorem 3.4. *Let $s\in\mathbb{R}$. If the Cauchy problem (3.4) and (3.5) are locally well-posed in $H^s(\mathbb{R})$, then the flow maps data-solution*

$$S^{hoBO}(t) : H^s(\mathbb{R}) \longrightarrow H^s(\mathbb{R}), \quad \phi \longmapsto u(t), \tag{3.11}$$

and

$$S^{hoILW}(t) : H^s(\mathbb{R}) \longrightarrow H^s(\mathbb{R}), \quad \phi \longmapsto u(t) \tag{3.12}$$

are not C^2 at zero.

3.2 Proof of Theorems 3.1, 3.2 and 3.3.

In the proofs of Theorems 3.1 and 3.2, we will suppose, for more simplicity, that the nonlinearity $\sum_{0\le l_1\le l_2\le 2j} a_{l_1,l_2}\partial_x^{l_1}u\partial_x^{l_2}u$ has the form $\partial_x^k(u^2)$ with $k>j$.

Proof of Theorem 3.1. The key point of the proof is the following algebraic relation

Lemma 3.1. *Let $j \in \mathbb{N}$ such that $j \geq 1$ and $\xi, \xi_1 \in \mathbb{R}$, then*

$$\xi_1^{2j+1} + (\xi - \xi_1)^{2j+1} - \xi^{2j+1} = (\xi - \xi_1) Q_{2j}(\xi, \xi_1), \tag{3.13}$$

where

$$Q_{2j}(\xi, \xi_1) = \sum_{l=0}^{2j} ((-1)^l C_{2j}^k - 1) \xi^{2j-l} \xi_1^l \tag{3.14}$$

and $C_n^k = \frac{n!}{k!(n-k)!}$.

Note that $Q_{2j}(\xi, \xi_1)$ and $\xi - \xi_1$ are prime.

Let $s \in \mathbb{R}$, $k, j \in \mathbb{N}$ such that $k > j$ and $T > 0$. Suppose that there exists a space X_T such as in Theorem 3.1. Take ϕ, $\psi \in H^s(\mathbb{R})$, and define $u(t) = U_j(t)\phi$ and $v(t) = U_j(t)\psi$, where U_j was defined in (1.25). Then, we use (3.6), (3.7) and (3.8) to deduce that

$$\| \int_0^t U_j(t - t') \partial_x^k [(U_j(t')\phi)(U_j(t')\psi)] dt' \|_{H^s} \lesssim \|\phi\|_{H^s} \|\psi\|_{H^s}. \tag{3.15}$$

We will show that (3.15) fails for an appropriate pair of ϕ, ψ, which would lead to a contradiction.

Define ϕ and ψ by

$$\phi = (\alpha^{-1/2} \chi_{I_1})^\vee \tag{3.16}$$

and

$$\psi = (\alpha^{-1/2} N^{-s} \chi_{I_2})^\vee \tag{3.17}$$

where

$$N \gg 1, \quad 0 < \alpha \ll 1, \quad I_1 = [\alpha/2, \alpha] \quad \text{and} \quad I_2 = [N, N + \alpha] \tag{3.18}$$

Note first that

$$\|\phi\|_{H^s} \sim \|\psi\|_{H^s} \sim 1. \tag{3.19}$$

Then, we use the algebraic relation (3.13) the definition of the unitary group U_j and the definition of ϕ and ψ to estimate the Fourier transform of the left-hand side of (3.15)

$$\begin{aligned}
&\left(\int_0^t U_j(t-t')\partial_x^k[(U_j(t')\phi)(U_j(t')\psi)]dt'\right)^{\wedge}(\xi) \\
&= \int_0^t e^{(-1)^{j+1}i(t-t')\xi^{2j+1}}(i\xi)^k(e^{(-1)^{j+1}it(\cdot)^{2j+1}}\widehat{\phi}) * (e^{(-1)^{j+1}it(\cdot)^{2j+1}}\widehat{\psi})(\xi)dt' \\
&= \int_{\mathbb{R}} e^{(-1)^{j+1}it\xi^{2j+1}}(i\xi)^k\widehat{\psi}(\xi_1)\widehat{\phi}(\xi-\xi_1)\int_0^t e^{(-1)^{j+1}it'Q_{2j}(\xi,\xi_1)(\xi-\xi_1)}dt'd\xi_1 \\
&= \int_{\mathbb{R}} e^{(-1)^{j+1}it\xi^{2j+1}}(i\xi)^k\widehat{\psi}(\xi_1)\widehat{\phi}(\xi-\xi_1)\frac{e^{(-1)^{j+1}it(\xi-\xi_1)Q_{2j}(\xi,\xi_1)}-1}{(-1)^{j+1}i(\xi-\xi_1)Q_{2j}(\xi,\xi_1)}d\xi_1. \\
&\sim \frac{e^{(-1)^{j+1}it\xi^{2j+1}}\xi^k}{\alpha N^s}\int_{\left\{\substack{\xi_1\in I_2\\ \xi-\xi_1\in I_1}\right.}\frac{e^{(-1)^{j+1}it(\xi-\xi_1)Q_{2j}(\xi,\xi_1)}-1}{(\xi-\xi_1)Q_{2j}(\xi,\xi_1)}d\xi_1. \qquad (3.20)
\end{aligned}$$

When $\xi-\xi_1\in I_1$ and $\xi_1\in I_2$, we have that $|\xi|\sim N$, $|(\xi-\xi_1)Q_{2j}(\xi,\xi_1)|\sim\alpha N^{2j}$. We choose $\alpha=N^{-2j-\epsilon}$, with $0<\epsilon<1$ so that

$$|(\xi-\xi_1)Q_{2j}(\xi,\xi_1)|\sim N^{-\epsilon}\ll 1 \qquad (3.21)$$

and

$$\frac{e^{(-1)^{j+1}it(\xi-\xi_1)Q_{2j}(\xi,\xi_1)}-1}{(\xi-\xi_1)Q_{2j}(\xi,\xi_1)}=ct+o(N^{-\epsilon}) \qquad (3.22)$$

where $c\in\mathbb{C}$. We are now able to give a lower bound for the left-hand side of (3.15)

$$\|\int_0^t U_j(t-t')\partial_x^k[(U_j(t')\phi)(U_j(t')\psi)]dt'\|_{H^s}\gtrsim\frac{N^s}{N^s\alpha}N^k\alpha^{1/2}\alpha. \qquad (3.23)$$

Thus we conclude from (3.15), (3.19) and (3.23) that

$$N^k\alpha^{1/2}=N^{k-j-\epsilon/2}\lesssim 1,\quad \forall\, N\gg 1, \qquad (3.24)$$

which is a contradiction since $k>j$. □

Remark 3.1. *Since the class of equation (3.2) often appears in physical situations where the function u is needed to be real-valued, it is interesting to notice that Theorems 3.1 and 3.2 are also valid if we ask the functions to be real.*

Proof. Actually take $\phi_1 = \operatorname{Re}\phi$ and $\psi_1 = \operatorname{Re}\psi$ instead of ϕ and ψ, then

$$\widehat{\phi_1} = \frac{\alpha^{-1/2}}{2}\chi_{\{\alpha/2\le|\xi|\le\alpha\}} \quad \text{and} \quad \widehat{\psi_1} = \frac{\alpha^{-1/2}N^{-s}}{2}\chi_{\{N\le|\xi|\le N+\alpha\}}, \tag{3.25}$$

and so we can conclude the proof as above. □

Proof of Theorem 3.2. Let $s \in \mathbb{R}$ and $k, j \in \mathbb{N}$ such that $k > j$. Suppose that there exists $T > 0$ such that the IVP (3.2), with the nonlinearity $\partial_x^k(u^2)$, is locally well-posed in $H^s(\mathbb{R})$ in the time interval $[0, T]$ and that the associated flow map solution $S^{j,k} : H^s(\mathbb{R}) \longrightarrow C([0, T]; H^s(\mathbb{R}))$ is C^2 at the origin. When $\phi \in H^s(\mathbb{R})$, we will denote $u_\phi(t) = S^{j,k}(t)\phi$ the solution of the Cauchy problem (3.2) with initial data ϕ. This means that u_ϕ is a solution of the associated integral equation

$$u(t) = U_j(t)\phi + \int_0^t U_j(t-t')\partial_x^k(u^2)(t')dt'. \tag{3.26}$$

When ϕ and ψ are in $H^s(\mathbb{R})$, we use the fact that the nonlinearity $\partial_x^k(uv)$ is a bilinear symmetric application to compute the Fréchet derivative of $S^{j,k}(t)$ at ψ in the direction ϕ

$$d_\psi S^{j,k}(t)\phi = U_j(t)\phi + 2\int_0^t U_j(t-t')\partial_x^k(u_\psi(t')d_\psi S^{j,k}(t')\phi)dt'. \tag{3.27}$$

Since the Cauchy problem (3.2) is supposed to be well-posed, we know using the uniqueness that $S^{j,k}(t)0 = u_0(t) = 0$ and then we deduce from (3.27) that

$$d_0 S^{j,k}(t)\phi = U_j(t)\phi. \tag{3.28}$$

Using (3.27), we compute the second Fréchet derivative at the origin in the direction (ϕ, ψ)

$$\begin{aligned} d_0^2 S^{j,k}(t)(\phi,\psi) &= d_0(d\ S^{j,k}(t)\phi)\psi = \frac{\partial}{\partial_\beta}(\beta \mapsto d_{\beta\psi}S^{j,k}(t)\phi)_{|\beta=0} \\ &= 2\int_0^t U_j(t-t')\partial_x^k(d_{\beta\psi}S^{j,k}(t')\psi d_{\beta\psi}S^{j,k}(t')\phi)dt'_{|\beta=0} \\ &\quad +2\int_0^t U_j(t-t')\partial_x^k(u_{\beta\psi}(t')d_{\beta\psi}^2 S^{j,k}(t')(\phi,\psi))dt'_{|\beta=0}. \end{aligned}$$

Thus we deduce using (3.28) that

$$d_0^2 S^{j,k}(t)(\phi,\psi) = 2\int_0^t U_j(t-t')\partial_x^k[(U_j(t')\psi)(U_j(t')\phi)]dt'. \qquad (3.29)$$

The assumption of C^2 regularity of $S^{j,k}(t)$ at the origin would imply that $d_0^2 S^{j,k}(t) \in \mathcal{B}(H^s(\mathbb{R}) \times H^s(\mathbb{R}), H^s(\mathbb{R}))$, which would lead to the following inequality

$$\|d_0^2 S^{j,k}(t)(\phi,\psi)\|_{H^s(\mathbb{R})} \lesssim \|\phi\|_{H^s(\mathbb{R})}\|\psi\|_{H^s(\mathbb{R})}, \quad \forall\ \phi,\ \psi \in H^s(\mathbb{R}). \qquad (3.30)$$

But (3.30) is equivalent to (3.15) which has been shown to fail in the proof of Theorem 3.1. □

Proof of Theorem 3.3. Let $s \in \mathbb{R}$. Suppose that there exists $T > 0$ such that the Cauchy problem (3.3) is locally well-posed in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ in the time interval $[0,T]$ and that the flow map solution S^{sKdV} is C^2 at the origin. When $(\phi_1,\phi_2) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$, we will denote

$$(u_{1,(\phi_1,\phi_2)}(t), u_{2,(\phi_1,\phi_2)}(t)) = (S_1^{sKdV}(t)(\phi_1,\phi_2), S_2^{sKdV}(t)(\phi_1,\phi_2))$$

the solution of the Cauchy problem (3.3) with initial data(ϕ_1,ϕ_2). Then, we get, performing the same kind of computations as in the proof of Theorem 3.2, that

$$\begin{aligned} &d_{(0,0)}^2 S_2^{sKdV}(t)[(\phi_1,\phi_2),(\psi_1,\psi_2)] \\ &= -\int_0^t U_1(t-t')[\partial_x(U_1(t')\phi_1 U_1(t')\psi_1)) + \partial_x^2(U_1(t')\phi_2 U_1(t')\psi_2))]. \end{aligned}$$

The assumption of C^2 regularity of S^{sKdV} at the origin would imply in particular that

$$\|d_{(0,0)}^2 S_2^{sKdV}(t)[(\phi,\phi),(\psi,\psi)]\|_{H^s(\mathbb{R})} \lesssim \|\phi\|_{H^s(\mathbb{R})}\|\psi\|_{H^s(\mathbb{R})}, \qquad (3.31)$$

for all ϕ and ψ in $H^s(\mathbb{R})$. But exactly the same choice of ϕ and ψ as in the proof of Theorem 3.1 (with $j=1$) shows that (3.31) fails. □

3.3 The higher-order Benjamin-Ono and intermediate long wave equations.

In order to study the Cauchy problems (3.4) (respectively (3.5)), we define V_1 (respectively V_2) the unitary group in $H^s(\mathbb{R})$ associated to the linear part of the equations, *i.e.*

$$V_k(t)\phi = \left(e^{ip_k(\xi)t}\widehat{\phi}\right)^{\vee}, \quad k = 1, 2, \quad \forall\, t \in \mathbb{R}, \quad \forall \phi \in H^s(\mathbb{R}), \tag{3.32}$$

where

$$p_1(\xi) = a\xi^3 + b|\xi|\xi,$$

and

$$p_2(\xi) = (a_1 \coth^2(h\xi) + a_2)\xi^3 + b\coth(h\xi)\xi^2.$$

We denote by f_1 (respectively f_2) the nonlinearity of the equations (3.4) (respectively (3.5)), *i.e.*

$$f_1(u) = cu\partial_x u - d\partial_x(uH\partial_x u + H(u\partial_x u)),$$

and

$$f_2(u) = cu\partial_x u - d\partial_x(u\mathcal{F}_h\partial_x u + \mathcal{F}_h(u\partial_x u)).$$

Then, we have the analogous of Theorem 3.1 for the equations (3.4) and (3.5).

Theorem 3.5. *Let $s \in \mathbb{R}$, $T > 0$ and $k \in \{1, 2\}$. Then, there does not exist any space X_T such that X_T is continuously embedded in $C([-T, T]; H^s(\mathbb{R}))$, i.e.,*

$$\|u\|_{C([-T,T];H^s)} \lesssim \|u\|_{X_T}, \quad \forall\, u \in X_T, \tag{3.33}$$

and such that

$$\|V_k(t)\phi\|_{X_T} \lesssim \|\phi\|_{H^s}, \quad \forall\, \phi \in H^s(\mathbb{R}), \tag{3.34}$$

and

$$\|\int_0^t V_k(t-t')f_k(u)(t')dt'\|_{X_T} \lesssim \|u\|^2_{X_T}, \quad \forall\, u \in X_T. \tag{3.35}$$

Theorem 3.4 is a consequence of Theorem 3.5 (see the proof of Theorem 3.2).

Proof of Theorem 3.5. Let $s \in \mathbb{R}$, $T > 0$ and $k \in \{1, 2\}$. Suppose that there exists a space X_T such as in Theorem 3.5. Take $\phi \in H^s(\mathbb{R})$, and define $u(t) = V_k(t)\phi$. Then, we use (3.33), (3.34) and (3.35) to see that

$$\| \int_0^t V_k(t-t') f_k(V_k(t'))\phi) dt' \|_{H^s} \lesssim \|\phi\|_{H^s}^2. \tag{3.36}$$

We will show that (3.36) fails for an appropriate choice of ϕ, which would lead to a contradiction.

Define ϕ by [1]

$$\phi = \left(\alpha^{-1/2}\chi_{I_1} + \alpha^{-1/2} N^{-s} \chi_{I_2}\right)^{\vee} \tag{3.37}$$

where

$$N \gg 1, \quad 0 < \alpha \ll 1, \quad I_1 = [\alpha/2, \alpha] \quad \text{and} \quad I_2 = [N, N+\alpha] \tag{3.38}$$

Note first that

$$\|\phi\|_{H^s} \sim 1. \tag{3.39}$$

Then, the same computation as for (3.20) leads to

$$\left(\int_0^t V_k(t-t') f_k((V_k(t')\phi) dt'\right)^{\wedge}(\xi) \sim g_1(\xi,t) + g_2(\xi,t) + g_3(\xi,t), \tag{3.40}$$

where,

$$g_1(\xi,t) = \frac{e^{itp(\xi)}}{\alpha} \int_{\substack{\xi_1 \in I_1 \\ \xi-\xi_1 \in I_1}} \tilde{f}_k(\xi,\xi_1) \frac{e^{it(p(\xi_1)+p(\xi-\xi_1)-p(\xi))} - 1}{i(p(\xi_1)+p(\xi-\xi_1)-p(\xi))} d\xi_1,$$

$$g_2(\xi,t) = \frac{e^{itp(\xi)}}{\alpha N^{2s}} \int_{\substack{\xi_1 \in I_2 \\ \xi-\xi_1 \in I_2}} \tilde{f}_k(\xi,\xi_1) \frac{e^{it(p(\xi_1)+p(\xi-\xi_1)-p(\xi))} - 1}{i(p(\xi_1)+p(\xi-\xi_1)-p(\xi))} d\xi_1,$$

$$g_3(\xi,t) = \frac{e^{itp(\xi)}}{\alpha N^{s}} \left(\int_{\substack{\xi_1 \in I_1 \\ \xi-\xi_1 \in I_2}} \tilde{f}_k(\xi,\xi_1) \frac{e^{it(p(\xi_1)+p(\xi-\xi_1)-p(\xi))} - 1}{i(p(\xi_1)+p(\xi-\xi_1)-p(\xi))} d\xi_1, \right.$$
$$\left. + \int_{\substack{\xi_1 \in I_2 \\ \xi-\xi_1 \in I_1}} \tilde{f}_k(\xi,\xi_1) \frac{e^{it(p(\xi_1)+p(\xi-\xi_1)-p(\xi))} - 1}{i(p(\xi_1)+p(\xi-\xi_1)-p(\xi))} d\xi_1 \right),$$

[1] We can also take $\operatorname{Re}\phi$ instead of ϕ (see the remark after the proof of Theorem 3.1).

and

$$\tilde{f}_1(\xi, \xi_1) = c\xi_1 - d(\xi|\xi_1| + |\xi|\xi_1),$$

or

$$\tilde{f}_2(\xi, \xi_1) = c\xi_1 - d(\xi \coth(\xi_1)\xi_1 + \coth(\xi)\xi\xi_1).$$

Since the supports of $g_1(\cdot, t)$, $g_2(\cdot, t)$ and $g_3(\cdot, t)$ are disjoint, we use (3.40) to bound by below the left-hand side of (3.36)

$$\| \int_0^t V_k(t-t') f_k((V_k(t'))\phi) dt' \|_{H^s} \geq \|(g_3)^\vee(\xi, t)\|_{H^s}. \tag{3.41}$$

We notice that the function p_k is smooth and that

$$|p_k'(\xi)| \lesssim 1 + |\xi|^2. \tag{3.42}$$

Thus, when $\xi_1 \in I_1$ and $\xi - \xi_1 \in I_2$ or $\xi - \xi_1 \in I_1$ and $\xi_1 \in I_2$, we have that $|\xi| \sim N$, and we use (3.42) and the mean value theorem to get the estimate

$$|p(\xi_1) + p(\xi - \xi_1) - p(\xi)| \lesssim \alpha N^2. \tag{3.43}$$

Hence we choose $\alpha = N^{-2-\epsilon}$, with $0 < \epsilon < 1$, to get

$$|\frac{e^{it(p(\xi_1)+p(\xi-\xi_1)-p(\xi))} - 1}{p(\xi_1) + p(\xi - \xi_1) - p(\xi)}| = |t| + o(N^{-\epsilon}). \tag{3.44}$$

We are now able to give a lower bound for $\|(g_3)^\vee(\xi, t)\|_{H^s}$

$$\|(g_3)^\vee(\xi, t)\|_{H^s} \gtrsim \frac{N^s}{N^s \alpha} \left(N^2 \alpha^{1/2} \alpha - N\alpha \alpha^{1/2} \alpha \right) \gtrsim N^2 \alpha^{1/2}. \tag{3.45}$$

Thus, we conclude from (3.36), (3.39) and (3.45) that

$$N^2 \alpha^{1/2} = N^{1-\epsilon/2} \lesssim 1, \quad \forall\ N \gg 1, \tag{3.46}$$

which is a contradiction. □

Chapter 4

The non-dissipative problem: well-posedness results.

4.1 Introduction and statements of the results.

In this chapter, we continue investigating the IVP (3.1), associated to the non-dissipative Kuramoto-Velarde equation. Since we showed, in the previous chapter, that this IVP was (in some sense) ill-posed in $H^s(\mathbb{R})$, for any $s \in \mathbb{R}$, we will have to consider smaller functional spaces to obtain well-posedness results.

This IVP was proved (as a particular case of (4)), by Kenig, Ponce and Vega, to be well-posed in some weighted Sobolev spaces for small initial data [19], and for arbitrary initial data [18]. We also refer to Argento's work [2], for more precision on the best exponents of the weighted Sobolev spaces obtained with this technic, in the particular case (3.1): actually, she proved that the IVP (3.1) is well-posed for small initial data in $H^k(\mathbb{R}) \cap H^3(\mathbb{R}; x^2dx)$ for $k \in \mathbb{N}$, $k \geq 5$.

The method used, in the case of small initial data, is an application of a fixed point theorem to the associated integral equation, taking advantage of the smoothing effects associated to the unitary group U of the Airy equation, (see (1.23) for the definition of U). In particular, a maximal (in time) function estimate for $U(t)\phi$ is needed in L^1_x. Actually, as observed in [17], the L^1_x-maximal function estimate fails without weight, and this could be another reason to explain why the problem is ill-posed in the Sobolev spaces $H^s(\mathbb{R})$, for $s \in \mathbb{R}$.

In the case of arbitrary initial data, Kenig, Ponce and Vega performed a gauge transformation on the equation (4) to obtain a dispersive system whose nonlinear terms are independent of the higher-order derivative. This allows to apply the techniques already used in the case of small initial data.

Here, we improved these results for the IVP (3.1), in the case of small initial data, using the weighted Besov spaces defined in the section 1.2. The use of Besov spaces is inspired by the works of Molinet and Ribaud on the Benjamin-Ono equation [27] and on the Korteweg-de Vries equation [26], and Planchon on the nonlinear Schrödinger equation [30]. It permits to refine the L^1_x-maximal function estimate, using the L^4_x-maximal function estimate derived by Kenig and Ruiz [21], and to obtain well-posedness results in fractional weighted Besov spaces (which seems to be difficult with weighted Sobolev spaces).

Unfortunately, we did not achieve to apply this technic in the case of arbitrary initial data. Actually, when performing the gauge transform as in [18], an exponential nonlinearity appears in the new system, which seems difficult to estimate in the Besov spaces.

Statements of the results.

Theorem 4.1. *There exists $\delta > 0$ such that for all $u_0 \in \mathcal{B}_2^{9/4,1}(\mathbb{R}) \cap \mathcal{B}_2^{1/4,1}(\mathbb{R}; x^2dx)$ with*

$$\beta = \|u_0\|_{\mathcal{B}_2^{9/4,1}} + \|u_0\|_{\mathcal{B}_2^{1/4,1}(x^2dx)} \leq \delta, \tag{4.1}$$

there exists $T = T(\beta)$ such that $T(\beta) \nearrow +\infty$ when $\beta \to 0$, a space X_T such that

$$X_T \hookrightarrow C([-T,T]; \mathcal{B}_2^{9/4,1}(\mathbb{R}) \cap \mathcal{B}_2^{1/4,1}(\mathbb{R}; x^2dx)) \tag{4.2}$$

and a unique solution u of (3.1) in X_T. Moreover, the flow map is smooth from $\mathcal{B}_2^{9/4,1}(\mathbb{R}) \cap \mathcal{B}_2^{1/4,1}(\mathbb{R}; x^2dx)$ to X_T near the origin.

Theorem 4.2. *Let $s > 9/4$, then there exists $\delta > 0$ such that for all $u_0 \in H^s(\mathbb{R}) \cap \mathcal{B}_2^{s-2,2}(\mathbb{R}; x^2dx)$ with*

$$\beta = \|u_0\|_{\mathcal{B}_2^{9/4,1}} + \|u_0\|_{\mathcal{B}_2^{1/4,1}(x^2dx)} \leq \delta, \tag{4.3}$$

there exists $T = T(\beta)$ such that $T(\beta) \nearrow +\infty$ when $\beta \to 0$, a space $Y_{T,s}$ such that

$$Y_{T,s} \hookrightarrow C([-T,T]; H^s(\mathbb{R}) \cap \mathcal{B}_2^{s-2,2}(\mathbb{R}; x^2dx)) \tag{4.4}$$

and a unique solution u of (3.1) in $Y_{T,s}$. Moreover, the flow map is smooth from $H^s(\mathbb{R}) \cap \mathcal{B}_2^{s-2,2}(\mathbb{R}; x^2dx)$ to $Y_{T,s}$ near the origin.

Since we have using Lemma 1.2, that $H^3(\mathbb{R}) \cap \mathcal{B}_2^{1,2}(\mathbb{R}; x^2dx) = H^3(\mathbb{R}) \cap H^1(\mathbb{R}; x^2dx)$, we deduce as an application of Theorem 4.2 with $s = 3$, that the Cauchy problem (3.1) is well-posed in $H^3(\mathbb{R}) \cap H^1(\mathbb{R}; x^2dx)$. This improves previous results in [2].

In order to simplify the proof of Theorems 4.1 and 4.2, we will assume that the nonlinearity in (3.1) has the form $\partial_x^2(u^2)$ in the rest of this chapter. However, the proof with the general nonlinearity follows exactly by the same way, rewriting the correspondent nonlinear estimates.

4.2 Linear estimate.

1. Linear estimates for the free and the nonhomogeneous evolutions.

Proposition 4.1 (Kato type smoothing effect.)**.** *If $u_0 \in L^2(\mathbb{R})$, then*

$$\|\partial_x U(t)u_0\|_{L^\infty_x L^2_t} \lesssim \|u_0\|_{L^2}. \tag{4.5}$$

Let $T > 0$, then if $f \in L^1_x L^2_T$

$$\| \int_0^t \partial_x U(t-t')f(\cdot,t')dt'\|_{L^\infty_T L^2_x} \lesssim \|f\|_{L^1_x L^2_T}, \tag{4.6}$$

and

$$\| \int_0^t \partial_x^2 U(t-t')f(\cdot,t')dt'\|_{L^\infty_x L^2_T} \lesssim \|f\|_{L^1_x L^2_T}. \tag{4.7}$$

Proof. See [17].

Proposition 4.2 (Maximal function estimate.)**.** *If $u_0 \in \mathcal{S}(\mathbb{R})$, then*

$$\|U(t)u_0\|_{L^4_x L^\infty_t} \lesssim \|D_x^{1/4}u_0\|_{L^2}, \tag{4.8}$$

and

$$\|U(t)u_0\|_{L^1_x L^\infty_T} \lesssim \|D_x^{1/4}u_0\|_{L^2} + \|D_x^{1/4}(xu_0)\|_{L^2} + T\|D_x^{1/4}\partial_x^2 u_0\|_{L^2}. \tag{4.9}$$

Proof. The estimate (4.8) is due to Kenig and Ruiz (see [21]). We will prove the estimate (4.9) using (1.26), (4.8) and Hölder's inequality

$$\begin{aligned}
\|U(t)u_0\|_{L^1_x L^\infty_T} &= \int_{|x|\le 1} \sup_{[-T,T]} |U(t)u_0(x)|dx + \int_{|x|>1} \frac{1}{|x|} \sup_{[-T,T]} |xU(t)u_0(x)|dx \\
&\lesssim \|U(t)u_0\|_{L^4_x L^\infty_T} + \|U(t)(xu_0)\|_{L^4_x L^\infty_T} + T\|U(t)\partial_x^2 u_0\|_{L^4_x L^\infty_T} \\
&\lesssim \|D_x^{1/4}u_0\|_{L^2} + \|D_x^{1/4}(xu_0)\|_{L^2} + T\|D_x^{1/4}\partial_x^2 u_0\|_{L^2}.
\end{aligned}$$

□

Remark 4.1. *It is interesting to observe that the restriction on the s in Theorem 4.2 ($s > 9/4$) appears in the estimate (4.9).*

2. Linear estimates for phase localized functions.

Following the ideas in [27], we will derive linear estimates for the phase localized free and nonhomogeneous evolutions.

Proposition 4.3. *Let $u_0 \in \mathcal{S}(\mathbb{R})$, then*

$$\|\Delta_j U(t)u_0\|_{L^\infty_T L^2_x} = \|\Delta_j u_0\|_{L^2_x}, \tag{4.10}$$

and

$$\|x\Delta_j U(t)u_0\|_{L^\infty_T L^2_x} \lesssim \|x\Delta_j u_0\|_{L^2_x} + T2^{2j}\|\Delta_j u_0\|_{L^2_x}. \tag{4.11}$$

If $f : \mathbb{R} \times [0,T] \to \mathbb{R}$ *is smooth, then we have for all* $j \geq 0$

$$\| \int_0^t \Delta_j U(t-t')\partial_x^2 f(\cdot,t')dt'\|_{L_T^\infty L_x^2} \lesssim 2^j \|\Delta_j f\|_{L_x^1 L_T^2}, \tag{4.12}$$

and

$$\| \int_0^t x\Delta_j U(t-t')\partial_x^2 f(\cdot,t')dt'\|_{L_T^\infty L_x^2} \lesssim 2^j \|x\Delta_j f\|_{L_x^1 L_T^2} + (1+T)2^{3j}\|\Delta_j f\|_{L_x^1 L_T^2}. \tag{4.13}$$

Proof. The identity (4.10) follows directly from the fact that U is a unitary group in $L^2(\mathbb{R})$. To prove the estimate (4.11), we will use (1.26), (4.10) and Plancherel's theorem

$$\begin{aligned}\|x\Delta_j U(t)u_0\|_{L_T^\infty L_x^2} &\leq \|U(t)(x\Delta_j u_0)\|_{L_T^\infty L_x^2} + 3T\|U(t)\partial_x^2\Delta_j u_0\|_{L_T^\infty L_x^2} \\ &\lesssim \|x\Delta_j u_0\|_{L_T^\infty L_x^2} + T2^{2j}\|\Delta_j U(t)u_0\|_{L_T^\infty L_x^2}.\end{aligned}$$

The estimate (4.12) follows from (4.6), Plancherel's theorem and the fact that Δ_j localize the frequency near $|\xi| \sim 2^j$. Next, we will prove the estimate (4.13). The identity (1.26) imply that

$$\begin{aligned}&\| \int_0^t x\Delta_j U(t-t')\partial_x^2 f(\cdot,t')dt'\|_{L_T^\infty L_x^2} \\ &\lesssim \| \int_0^t U(t-t')(x\partial_x^2\Delta_j f(\cdot,t'))dt'\|_{L_T^\infty L_x^2} \\ &\quad + T\| \int_0^t \Delta_j U(t-t')\partial_x^4 f(\cdot,t')dt'\|_{L_T^\infty L_x^2}.\end{aligned} \tag{4.14}$$

We use again (4.6) to estimate the second term on the right-hand side of (4.14)

$$T\| \int_0^t \Delta_j U(t-t')\partial_x^4 f(\cdot,t')dt'\|_{L_T^\infty L_x^2} \lesssim T2^{3j}\|\Delta_j f\|_{L_x^1 L_T^2}. \tag{4.15}$$

To estimate the first term, we need the following identity

$$x\partial_x^2\Delta_j f = \partial_x^2(x\Delta_j f) - 2\partial_x(\Delta_j f) \tag{4.16}$$

Then, we use (4.6) and the fact that the operator $x\Delta_j$ still localizes the frequency near $|\xi| \sim 2^j$ (see the commutator identity (1.9)).

$$\| \int_0^t U(t-t')(x\partial_x^2\Delta_j f(\cdot,t'))dt'\|_{L_T^\infty L_x^2} \lesssim 2^j\|x\Delta_j f\|_{L_x^1 L_T^2} + \|\Delta_j f\|_{L_x^1 L_T^2}. \tag{4.17}$$

We deduce (4.13) from (4.14), (4.15), (4.17) and the fact that $j \geq 0$. □

Proposition 4.4. *Let $u_0 \in \mathcal{S}(\mathbb{R})$, then*

$$\|\Delta_j U(t)u_0\|_{L^\infty_x L^2_T} \lesssim 2^{-j}\|\Delta_j u_0\|_{L^2_x}, \tag{4.18}$$

and

$$\|x\Delta_j U(t)u_0\|_{L^\infty_x L^2_T} \lesssim 2^{-j}\|x\Delta_j u_0\|_{L^2_x} + T2^j\|\Delta_j u_0\|_{L^2_x}. \tag{4.19}$$

If $f : \mathbb{R} \times [0,T] \to \mathbb{R}$ is smooth, then we have for all $j \geq 0$

$$\|\int_0^t \Delta_j U(t-t')\partial_x^2 f(\cdot,t')dt'\|_{L^\infty_x L^2_T} \lesssim \|\Delta_j f\|_{L^1_x L^2_T}, \tag{4.20}$$

and

$$\|\int_0^t x\Delta_j U(t-t')\partial_x^2 f(\cdot,t')dt'\|_{L^\infty_x L^2_T} \lesssim \|x\Delta_j f\|_{L^1_x L^2_T} + (1+T)2^{2j}\|\Delta_j f\|_{L^1_x L^2_T}. \tag{4.21}$$

Proof. The proof is the same as for the Proposition 4.3 where we use (4.5) and (4.7) instead of (4.6). □

In order to derive a non homogeneous estimate for the localized maximal function, we need the following lemma due to Molinet and Ribaud (see [27]) and inspired by a previous result of Christ and Kiselev (see [8]).

Lemma 4.1. *Let L be a linear operator defined on space-time functions $f(x,t)$ by*

$$Lf(t) = \int_0^T K(t,t')f(t')dt',$$

where $K : \mathcal{S}(\mathbb{R}^2) \to C(\mathbb{R}^3)$ and such that

$$\|Lf\|_{L^{p_1}_x L^\infty_T} \leq C\|f\|_{L^{p_2}_x L^{q_2}_T},$$

with $p_2, q_2 < \infty$. Then,

$$\|\int_0^t K(t,t')f(t')dt'\|_{L^p_x L^\infty_T} \leq C\|f\|_{L^{p_2}_x L^{q_2}_T}.$$

Proposition 4.5. *Let $u_0 \in \mathcal{S}(\mathbb{R})$, then*

$$\|\Delta_j U(t)u_0\|_{L^1_x L^\infty_T} \lesssim 2^{\frac{9}{4}j}(1+T)\|\Delta_j u_0\|_{L^2_x} + 2^{\frac{1}{4}j}\|x\Delta_j u_0\|_{L^2_x}. \tag{4.22}$$

If $f : \mathbb{R}\times[0,T] \to \mathbb{R}$ is smooth, then we have for all $j \geq 0$

$$\begin{aligned}\|\int_0^t \Delta_j U(t-t')\partial_x^2 f(\cdot,t')dt'\|_{L^1_x L^\infty_T}&\\ \lesssim 2^{\frac{5}{4}j}\|x\Delta_j f\|_{L^1_x L^2_T} &+ (1+T)2^{\frac{13}{4}j}\|\Delta_j f\|_{L^1_x L^2_T}.\end{aligned} \tag{4.23}$$

Proof. To obtain the estimate (4.22), we apply (4.9) with $\Delta_j u_0$ instead of u_0, then we use Plancherel's theorem and the fact that the operators Δ_j and $x\Delta_j$ localize the frequency near $|\xi| \sim 2^j$.

In order to prove the estimate (4.23), we first need to derive a "nonretarded" L^4-maximal function estimate. Note first that duality and (4.8) imply that

$$\|\int_0^T \Delta_j U(-t)f(\cdot,t)dt\|_{L^2_x} \lesssim 2^{\frac{1}{4}j}\|\Delta_j f\|_{L^{4/3}_x L^1_T}. \tag{4.24}$$

Then, we deduce combining (4.12), (4.24) and the Cauchy-Schwarz inequality that for all $g \in L^{4/3}_x L^1_T$

$$\begin{aligned}&\int_{\mathbb{R}\times[0,T]}\left(\int_0^T \Delta_j U(t-t')\partial_x^2 f(\cdot,t')dt'\right)g(x,t)dxdt\\ &= \int_{\mathbb{R}}\left(\int_0^T U(-t')\partial_x^2\Delta_j f(\cdot,t')dt'\right)\left(\int_0^T U(-t)\tilde{\Delta}_j\overline{g}(\cdot,t)dt\right)dx\\ &\leq \|\int_0^T U(-t')\partial_x^2\Delta_j f(\cdot,t')dt'\|_{L^2_x}\|\int_0^T U(-t)\tilde{\Delta}_j\overline{g}(\cdot,t)dt\|_{L^2_x}\\ &\lesssim 2^j\|\Delta_j f\|_{L^1_x L^2_T}2^{\frac{1}{4}j}\|g\|_{L^{4/3}_x L^1_T},\end{aligned}$$

so that by duality

$$\|\int_0^T \Delta_j U(t-t')\partial_x^2 f(\cdot,t')dt'\|_{L^4_x L^\infty_T} \lesssim 2^{\frac{5}{4}j}\|\Delta_j f\|_{L^1_x L^2_T}. \tag{4.25}$$

Then, we use Lemma 4.1 to obtain the corresponding "retarded" estimate

$$\|\int_0^t \Delta_j U(t-t')\partial_x^2 f(\cdot,t')dt'\|_{L^4_x L^\infty_T} \lesssim 2^{\frac{5}{4}j}\|\Delta_j f\|_{L^1_x L^2_T}. \tag{4.26}$$

We are now able to derive the $L^1_x L^\infty_T$ estimate for the non homogeneous term. We have by Hölder's inequality

$$
\begin{aligned}
&\| \int_0^t \Delta_j U(t-t')\partial_x^2 f(\cdot,t')dt' \|_{L^1_x L^\infty_T} \\
&= \int_{|x|\le 1} \sup_{t\in[-T,T]} \left| \int_0^t \Delta_j U(t-t')\partial_x^2 f(\cdot,t')dt' \right| dx \\
&\quad + \int_{|x|>1} \frac{1}{|x|} \sup_{t\in[-T,T]} \left| \int_0^t x\Delta_j U(t-t')\partial_x^2 f(\cdot,t')dt' \right| dx \qquad (4.27)\\
&\lesssim \| \int_0^t \Delta_j U(t-t')\partial_x^2 f(\cdot,t')dt' \|_{L^4_x L^\infty_T} + \| \int_0^t x\Delta_j U(t-t')\partial_x^2 f(\cdot,t')dt' \|_{L^4_x L^\infty_T}
\end{aligned}
$$

Thus, we deduce from (1.26), (4.16), (4.26) and (4.27) that

$$
\begin{aligned}
&\| \int_0^t \Delta_j U(t-t')\partial_x^2 f(\cdot,t')dt' \|_{L^1_x L^\infty_T} \\
&\lesssim (2^{\frac{5}{4}j} + T2^{\frac{13}{4}j} + 2^{\frac{1}{4}j}) \|\Delta_j f\|_{L^1_x L^2_T} + 2^{\frac{5}{4}j} \|x\Delta_j f\|_{L^1_x L^2_T}, \qquad (4.28)
\end{aligned}
$$

which leads to (4.23), since $j \ge 0$. □

Remark 4.2. *All the results in Propositions 4.3, 4.4 and 4.5 are still valid with S_0 instead of Δ_j and $j=0$.*

4.3 Proof of Theorems 4.1 and 4.2.

Proof of Theorem 4.1.

1. *Existence.* Consider the integral equation associated to (3.1)

$$u(t) = F(u)(t) := U(t)u_0 + \int_0^t U(t-t')\partial_x^2(u^2)(t')dt'. \tag{4.29}$$

Let $T > 0$, define the following semi norms:

$$N_1^T(u) = \|S_0 u\|_{L_T^\infty L_x^2} + \sum_{j=0}^{\infty} 2^{\frac{9}{4}j}\|\Delta_j u\|_{L_T^\infty L_x^2}, \tag{4.30}$$

$$N_2^T(u) = \|xS_0 u\|_{L_T^\infty L_x^2} + \sum_{j=0}^{\infty} 2^{\frac{1}{4}j}\|x\Delta_j u\|_{L_T^\infty L_x^2}, \tag{4.31}$$

$$P_1^T(u) = \|S_0 u\|_{L_x^\infty L_T^2} + \sum_{j=0}^{\infty} 2^{\frac{13}{4}j}\|\Delta_j u\|_{L_x^\infty L_T^2}, \tag{4.32}$$

$$P_2^T(u) = \|xS_0 u\|_{L_x^\infty L_T^2} + \sum_{j=0}^{\infty} 2^{\frac{5}{4}j}\|x\Delta_j u\|_{L_x^\infty L_T^2}, \tag{4.33}$$

$$M^T(u) = \|S_0 u\|_{L_x^1 L_T^\infty} + \sum_{j=0}^{\infty} \|\Delta_j u\|_{L_x^1 L_T^\infty}. \tag{4.34}$$

Then, we define the Banach space

$$X_T = \{u \in C([-T,T]; \mathcal{B}_2^{9/4,1}(\mathbb{R}) \cap \mathcal{B}_2^{1/4,1}(\mathbb{R}; x^2 dx)) \;:\; \|u\|_{X_T} < \infty\}, \tag{4.35}$$

where

$$\|u\|_{X_T} = N_1^T(u) + N_2^T(u) + P_1^T(u) + P_2^T(u) + M^T(u). \tag{4.36}$$

We deduce from (4.10), (4.11), (4.18), (4.19) and (4.22) that

$$\|U(t)u_0\|_{X_T} \lesssim (1+T)\left(\|u_0\|_{\mathcal{B}_2^{9/4,1}} + \|u_0\|_{\mathcal{B}_2^{1/4,1}(x^2dx)}\right), \tag{4.37}$$

and from (4.12), (4.13), (4.20), (4.21) and (4.23) that

$$\begin{aligned}
&\| \int_0^t U(t-t')\partial_x^2(uv)(t')dt' \|_{X_T} \\
&\lesssim (1+T) \left(\|S_0(uv)\|_{L^1_x L^2_T} + \sum_{j=0}^{\infty} 2^{\frac{13}{4}j} \|\Delta_j(uv)\|_{L^1_x L^2_T} \right. \\
&\quad \left. + \|xS_0(uv)\|_{L^1_x L^2_T} + \sum_{j=0}^{\infty} 2^{\frac{5}{4}j} \|x\Delta_j(uv)\|_{L^1_x L^2_T} \right). \qquad (4.38)
\end{aligned}$$

In order to estimate the nonlinear term $\sum_{j=0}^{\infty} 2^{\frac{13}{4}j}\|\Delta_j(uv)\|_{L^1_x L^2_T}$, we perform the following calculation

$$\begin{aligned}
\Delta_j(uv) &= \Delta_j(\lim_{r\to\infty} S_r u S_r v) \\
&= \Delta_j \left(S_0 u S_0 v + \sum_{r=0}^{\infty} (S_{r+1}uS_{r+1}v - S_r u S_r v) \right) \qquad (4.39) \\
&= \Delta_j \left(S_0 u S_0 v + \frac{1}{2} \sum_{r=0}^{\infty} (\Delta_r u(S_r v + S_{r+1}v) + \Delta_r v(S_r u + S_{r+1}u)) \right).
\end{aligned}$$

First, since $\Delta_j(S_0 u S_0 v) = 0$ for $j \geq 3$ and since the operators Δ_j are uniformly bounded (in j) in L^1, we have by Hölder's inequality

$$\sum_{j\geq 0} 2^{\frac{13}{4}j} \|\Delta_j(S_0uS_0v)\|_{L^1_x L^2_T} \lesssim \|S_0u\|_{L^\infty_x L^2_T} \|S_0v\|_{L^1_x L^\infty_T} \lesssim \|u\|_{X_T}\|v\|_{X_T}. \qquad (4.40)$$

In order to estimate the second term on the right-hand side of (4.39), we notice, since the term $\Delta_r u(S_r v + S_{r+1}v)$ is localized in frequency in the set $|\xi| \leq 2^{r+3}$ and the operator Δ_j only see the frequency in the set $2^{j-1} \leq |\xi| \leq 2^{j+1}$, that

$$\begin{aligned}
&\Delta_j \left(\sum_{r=0}^{\infty} (\Delta_r u(S_r v + S_{r+1}v) + \Delta_r v(S_r u + S_{r+1}u)) \right) \\
&= \Delta_j \left(\sum_{r\geq j-3} (\Delta_r u(S_r v + S_{r+1}v) + \Delta_r v(S_r u + S_{r+1}u)) \right). \qquad (4.41)
\end{aligned}$$

Then, we only have to estimate terms of the form $\Delta_j(\sum_{r\geq j} \Delta_r u S_r v)$. By Fubini's theorem, we get

$$\begin{aligned}
&\sum_{j\geq 0} 2^{\frac{13}{4}j}\|\Delta_j(\sum_{r\geq j}\Delta_r u S_r v)\|_{L^1_x L^2_T}\\
&\qquad \leq \sum_{j\geq 0} 2^{\frac{13}{4}j}\sum_{r\geq j}\|\Delta_r u\|_{L^\infty_x L^2_T}\|S_r v\|_{L^1_x L^\infty_T}\\
&\qquad \leq M^T(v)\sum_{r\geq 0}\left(\sum_{j=0}^{r} 2^{\frac{13}{4}j}\right)\|\Delta_r u\|_{L^\infty_x L^2_T}\\
&\qquad \lesssim M^T(v)P_1^T(u) \leq \|u\|_{X_T}\|v\|_{X_T}, \qquad (4.42)
\end{aligned}$$

where we used the fact that

$$\|S_r v\|_{L^1_x L^\infty_T} \leq \|S_0 v\|_{L^1_x L^\infty_T} + \sum_{j=0}^{r}\|\Delta_j v\|_{L^1_x L^\infty_T} \leq M^T(v). \qquad (4.43)$$

Thus, we obtain, gathering (4.39), (4.40), (4.41) and (4.42) that

$$\sum_{j=0}^{\infty} 2^{\frac{13}{4}j}\|\Delta_j(uv)\|_{L^1_x L^2_T} \lesssim \|u\|_{X_T}\|v\|_{X_T}. \qquad (4.44)$$

We apply exactly the same strategy to estimate the other bilinear term $\sum_{j=0}^{\infty} 2^{\frac{5}{4}j}\|x\Delta_j(uv)\|_{L^1_x L^2_T}$. Then, we have only to estimate terms of the form $\sum_{j=0}^{\infty} 2^{\frac{5}{4}j}\|x\Delta_j(\sum_{r\geq j}\Delta_r u S_r v)\|_{L^1_x L^2_T}$, and we use the commutator identity (1.9) and the fact that the operators Δ_j' are also uniformly bounded (in j) in L^1 to deduce that

$$\begin{aligned}
&\sum_{j\geq 0} 2^{\frac{5}{4}j}\|x\Delta_j(uv)\|_{L^1_x L^2_T}\\
&\lesssim \sum_{j\geq 0}\left(2^{\frac{5}{4}j}\sum_{r\geq j}\|x\Delta_r u S_r v\|_{L^\infty_x L^2_T} + 2^{\frac{1}{4}j}\sum_{r\geq j}\|\Delta_r u S_r v\|_{L^\infty_x L^2_T}\right)\\
&\lesssim M^T(v)(P_1^T(u)+P_2^T(u)) \leq \|u\|_{X_T}\|v\|_{X_T}. \qquad (4.45)
\end{aligned}$$

Thus, we deduce from (4.38), (4.44) and(4.45) that

$$\|\int_0^t U(t-t')\partial_x^2(uv)(t')dt'\|_{X_T} \lesssim (1+T)\|u\|_{X_T}\|v\|_{X_T}. \qquad (4.46)$$

Then, we use (4.37) and (4.46) to deduce that there exists a constant $C>0$ such that

$$\|F(u)\|_{X_T} \leq C(1+T)\left(\|u_0\|_{\mathcal{B}_2^{9/4,1}} + \|u_0\|_{\mathcal{B}_2^{1/4,1}(x^2dx)} + \|u\|_{X_T}^2\right), \quad \forall\, u\in X_T, \qquad (4.47)$$

and

$$\|F(u) - F(v)\|_{X_T} \leq C(1+T)(\|u\|_{X_T} + \|v\|_{X_T})\|u - v\|_{X_T}, \quad \forall\, u, v \in X_T. \tag{4.48}$$

Let $X_T(a) := \{u \in X_T \ : \ \|u\|_{X_T} \leq a\}$ the closed ball of X_T with radius a. $X_T(a)$ equipped with the metric induced by the norm $\|\cdot\|_{X_T}$ is a complete metric space. If we choose

$$\beta = \|u_0\|_{\mathcal{B}_2^{9/4,1}} + \|u_0\|_{\mathcal{B}_2^{1/4,1}(x^2dx)} \leq \delta < \min\{(\frac{1}{4C})^2, 1\}, \tag{4.49}$$

$$a = \sqrt{\beta}, \quad \text{and} \quad T = \frac{1}{4C\sqrt{\beta}}, \tag{4.50}$$

we have that

$$2C(1+T)a < 1. \tag{4.51}$$

Then, we deduce from (4.47) and (4.48) that the operator F is a contraction in $X_T(a)$ (up to the persistence property) and so, by the Picard fixed point theorem, there exists a unique solution of (4.29) in $X_T(a)$.

2. *Persistence.* We want to show that the solution u of the integral equation (4.29) is in $C([-T,T]; \mathcal{B}_2^{9/4,1}(\mathbb{R}) \cap \mathcal{B}_2^{1/4,1}(\mathbb{R}; x^2dx))$. By Lemma 1.4, we only need to prove that

$$\lim_{t\to 0} \| \int_0^t U(t-t')\partial_x^2(u^2)(t')dt'\|_{\mathcal{B}_2^{9/4,1}} = 0, \tag{4.52}$$

and

$$\lim_{t\to 0} \| \int_0^t U(t-t')\partial_x^2(u^2)(t')dt'\|_{\mathcal{B}_2^{1/4,1}(x^2dx)} = 0. \tag{4.53}$$

Let $0 < t < t_0 < T$, then we obtain, using the same computation as for (4.46) that

$$\begin{aligned} \| \int_0^t U(t-t')\partial_x^2(u^2)(t')dt'\|_{\mathcal{B}_2^{9/4,1}} &\leq N_1^{t_0}(\int_0^t U(t-t')\partial_x^2(u^2)(t')dt') \\ &\leq C(1+t_0)aP_1^{t_0}(u). \end{aligned} \tag{4.54}$$

Since $P_1^T(u) \leq \|u\|_{X^T} \leq a$, we deduce by the monotone convergence theorem that

$$\lim_{t_0\to 0} P_1^{t_0}(u) = 0. \tag{4.55}$$

We deduce (4.52) combining (4.54) and (4.55). The identity (4.53) follows by a similar argument.

3. *Uniqueness.* In order to prove the uniqueness of the solution in the whole space X_T, suppose that there exists another solution $\tilde{u}$ of the integral equation (4.29) such that $\tilde{u}$ also belongs to X_T. then, we deduce from (4.37), (4.38), (4.49) and (4.50) that for all $0 < T_1 < T$,

$$\|\tilde{u}\|_{X_{T_1}} < a/2 + O^{T_1}(\int_0^t U(t-t')\partial_x^2(\tilde{u}^2)(t')dt') \tag{4.56}$$

where

$$O^T(u) = \|S_0 u\|_{L^1_x L^2_T} + \sum_{j=0}^{\infty} 2^{\frac{13}{4}j}\|\Delta_j u\|_{L^1_x L^2_T} + \|xS_0 u\|_{L^1_x L^2_T} + \sum_{j=0}^{\infty} 2^{\frac{5}{4}j}\|x\Delta_j u\|_{L^1_x L^2_T}.$$

Since by the estimate (4.46),

$$O^T(\int_0^t U(t-t')\partial_x^2(\tilde{u}^2)(t')dt') \le C(1+T)\|\tilde{u}\|^2_{X_T} < \infty,$$

we deduce by the monotone convergence theorem that

$$\lim_{T_1\to 0}(O^{T_1}(\int_0^t U(t-t')\partial_x^2(\tilde{u}^2)(t')dt') = 0. \tag{4.57}$$

Hence, by (4.56) and (4.57), we can fix a $0 < T_1 < T$ such that $\tilde{u} \in X_{T_1}(a)$, and consequently $u \equiv \tilde{u}$ for $(x,t) \in \mathbb{R} \times [-T_1, T_1]$. We observe that T_1 only depends on $\tilde{u}$ and then, we can reapply this process (a finite number of times) to extend the uniqueness result in the whole interval $[-T, T]$.

4. *Smoothness of the flow map data-solution.* We denote by S the flow map of the equation (3.1). By the existence and uniqueness part of Theorem 4.1, S is well defined in the ball $B(0,\delta)$ of $C([-T,T]; \mathcal{B}_2^{9/4,1}(\mathbb{R}) \cap \mathcal{B}_2^{1/4,1}(\mathbb{R}; x^2dx))$,

$$S : B(0,\delta) \longrightarrow X^T(a), \quad u_0 \longmapsto S(t)u_0. \tag{4.58}$$

Then, using (4.37), (4.46) and (4.49), we deduce that

$$\begin{aligned}\|S(t)u_0 - S(t)v_0\|_{X_T} \le C(1+T)\Big(&\|u_0 - v_0\|_{\mathcal{B}_2^{9/4,1}} + \|u_0 - v_0\|_{\mathcal{B}_2^{1/4,1}(x^2dx)} \\ &+2a\|S(t)u_0 - S(t)v_0\|_{X_T}\Big),\end{aligned} \tag{4.59}$$

so that (4.51) imply that S is Lipschitz.

To prove the smoothness of S, let define

$$\begin{aligned} H \,:\, & B(0,\delta) \times X_T \cap C^0([0,T], H^s(\mathbb{R})) \to X_T \\ & (\phi, v) \mapsto v(t) - U(t)\phi - \int_0^t U(t-t')\partial_x^2(v^2)(t')dt'. \end{aligned}$$

Note that H is well defined, H is smooth and $H(\phi, S(t)\phi) = 0$. Moreover, we fix $\phi \in B(0,\delta)$ and we compute for all $w \in X_T$

$$\partial_v H(\phi, S(t)\phi)w(t) = w(t) - 2\int_0^t U(t-t')\partial_x^2(S(t)\phi w)(t')dt'.$$

Then, we deduce using (4.46), that

$$\|(id - \partial_v H(\phi, S(t)\phi))w(t)\|_{X_T} \leq 2C(1+T)a\|w\|_{X_T},$$

so that, by the choice of T in (4.51), $\partial_v H(\phi, S(t)\phi) \in \mathcal{L}(X_T)$ is an isomorphism. Thus, we conclude by the implicit function theorem that there exists a neighborhood V of ϕ in $B(0,\delta)$ and a smooth application $h : V \longrightarrow X_T$ such that $H(\psi, h(\psi)) = 0$, for all $\psi \in V$. This means that $S_{|_V} = h$ is smooth, and since smoothness is a local property, we conclude that the flow map S is smooth in $B(0,\delta)$. □

Proof of Theorem 4.2. We will need the following lemma:

Lemma 4.2. *Let $s > 9/4$, then the injection*

$$H^s(\mathbb{R}) \cap \mathcal{B}_2^{s-2,2}(\mathbb{R}; x^2dx) \hookrightarrow \mathcal{B}_2^{9/4,1}(\mathbb{R}) \cap \mathcal{B}_2^{1/4,1}(\mathbb{R}; x^2dx) \tag{4.60}$$

is continuous.

Proof. Let $s > 9/4$ and $f \in H^s(\mathbb{R})$. We obtain using the Cauchy-Schwarz inequality that

$$\begin{aligned}
\|f\|_{\mathcal{B}_2^{9/4,1}} &= \|S_0 f\|_{L^2} + \sum_{j\geq 0} 2^{js}\|\Delta_j f\|_{L^2} 2^{j(9/4-s)} \\
&\leq \|S_0 f\|_{L^2} + \left(\sum_{j\geq 0} 4^{j(9/4-s)}\right)^{1/2} \left(\sum_{j\geq 0} 4^{js}\|\Delta_j f\|_{L^2}^2\right)^{1/2} \\
&\lesssim \|f\|_{\mathcal{B}_2^{s,2}} \sim \|f\|_{H^s}. \qquad (4.61)
\end{aligned}$$

Similarly, we get

$$\|f\|_{\mathcal{B}_2^{1/4,1}(x^2dx)} \lesssim \|f\|_{\mathcal{B}_2^{s-2,2}(x^2dx)}, \qquad (4.62)$$

when $s > 9/4$ and then, (4.61) and (4.62) yield (4.60). □

Now, let $s > 9/4$. Exactly as in the proof of Theorem 4.1, we want to apply a fixed point theorem to solve the integral equation (4.29) in some good function space. In this way, define the following semi-norm

$$\|u\|_{X_{T,s}} = N_{1,s}^T(u) + N_{2,s}^T(u) + P_{1,s}^T(u) + P_{2,s}^T(u), \qquad (4.63)$$

where

$$N_{1,s}^T(u) = \|S_0 u\|_{L_T^\infty L_x^2} + \left(\sum_{j=0}^{\infty} 4^{js}\|\Delta_j u\|_{L_T^\infty L_x^2}^2\right)^{1/2}, \qquad (4.64)$$

$$N_{2,s}^T(u) = \|x S_0 u\|_{L_T^\infty L_x^2} + \left(\sum_{j=0}^{\infty} 4^{j(s-2)}\|x\Delta_j u\|_{L_T^\infty L_x^2}^2\right)^{1/2}, \qquad (4.65)$$

$$P_{1,s}^T(u) = \|S_0 u\|_{L_x^\infty L_T^2} + \left(\sum_{j=0}^{\infty} 4^{j(s+1)}\|\Delta_j u\|_{L_x^\infty L_T^2}^2\right)^{1/2}, \qquad (4.66)$$

$$P_{2,s}^T(u) = \|x S_0 u\|_{L_x^\infty L_T^2} + \left(\sum_{j=0}^{\infty} 4^{j(s-1)}\|x\Delta_j u\|_{L_x^\infty L_T^2}^2\right)^{1/2}. \qquad (4.67)$$

If $u_0 \in H^s(\mathbb{R}) \cap \mathcal{B}_2^{s-2,2}(\mathbb{R}; x^2dx)$, by Lemma 4.2, it makes sense to define

$$\lambda_s := \frac{\|u_0\|_{\mathcal{B}_2^{9/4,1}} + \|u_0\|_{\mathcal{B}_2^{1/4,1}(x^2dx)}}{\|u_0\|_{H^s} + \|u_0\|_{\mathcal{B}_2^{s-2,2}(x^2dx)}}. \qquad (4.68)$$

Then, let $Y_{T,s}$ be the Banach space

$$Y_{T,s} = \{u \in C([-T,T]; H^s(\mathbb{R}) \cap \mathcal{B}_2^{s-2,2}(\mathbb{R}; x^2dx)) \text{ such that } \|u\|_{Y_{T,s}} < \infty\}, \tag{4.69}$$

where

$$\|u\|_{Y_{T,s}} = \|u\|_{X_T} + \lambda_s \|u\|_{X_{T,s}}. \tag{4.70}$$

We deduce from (4.10), (4.11), (4.18), (4.19), (4.22), (4.68) and (4.70) that

$$\|U(t)u_0\|_{Y_{T,s}} \lesssim (1+T)\left(\|u_0\|_{\mathcal{B}_2^{9/4,1}} + \|u_0\|_{\mathcal{B}_2^{1/4,1}(x^2dx)}\right). \tag{4.71}$$

In order to estimate the nonlinear term of (4.29) in the norm $\|\cdot\|_{Y_{T,s}}$, we remember (4.46), and then it only remains to derive an estimate of the form

$$\|\int_0^t U(t-t')\partial_x^2(uv)(t')dt'\|_{X_{T,s}} \lesssim (1+T)\|u\|_{Y_{T,s}}\|v\|_{Y_{T,s}}. \tag{4.72}$$

In this way, we use (4.12), (4.13), (4.20), (4.21) and (4.23) to deduce that

$$\begin{aligned}
&\|\int_0^t U(t-t')\partial_x^2(uv)(t')dt'\|_{X_{T,s}} \\
&\lesssim (1+T)\left(\|S_0(uv)\|_{L_x^1L_T^2} + \left(\sum_{j=0}^{\infty} 4^{j(s+1)}\|\Delta_j(uv)\|_{L_x^1L_T^2}^2\right)^{1/2}\right. \\
&\quad \left. + \|xS_0(uv)\|_{L_x^1L_T^2} + \left(\sum_{j=0}^{\infty} 4^{j(s-1)}\|x\Delta_j(uv)\|_{L_x^1L_T^2}^2\right)^{1/2}\right).
\end{aligned} \tag{4.73}$$

And arguing as in the proof of Theorem 4.1, we estimate the right-hand side of (4.73) by some terms of the form

$$A = \left(\sum_{j\geq 0} 4^{j(s+1)}\|\Delta_j(\sum_{r=j}^{\infty}\Delta_r u S_r v)\|_{L_x^1L_T^2}^2\right)^{1/2}, \tag{4.74}$$

$$B = \left(\sum_{j\geq 0} 4^{j(s-1)}\|x\Delta_j(\sum_{r=j}^{\infty}\Delta_r u S_r v)\|_{L_x^1L_T^2}^2\right)^{1/2}, \tag{4.75}$$

and some others harmless terms. We next estimate A, we get from (1.7), Hölder's inequality and (4.43), the inequality

$$A \leq M^T(v) \left(\sum_{j\geq 0} 4^{j(s+1)} \left(\sum_{r=j}^{\infty} \|\Delta_r u\|_{L^\infty_x L^2_T} \right)^2 \right)^{1/2}. \tag{4.76}$$

Then, define

$$\gamma_r = 2^{r(s+1)} \|\Delta_r u\|_{L^\infty_x L^2_T} \quad \text{and note that} \quad \|\{\gamma_r\}_r\|_{l^2(\mathbb{N})} \leq P^T_{1,s}(u). \tag{4.77}$$

We deduce by (4.76), a change of index and Minkowski's inequality that

$$\begin{aligned} A &\leq M^T(v) \|\{\sum_{r=j}^{\infty} 2^{(j-r)(s+1)} \gamma_r\}_j\|_{l^2(\mathbb{N})} = M^T(v)\|\{\sum_{l\geq 0} 2^{-l(s+1)} \gamma_{l+j}\}_j\|_{l^2(\mathbb{N})} \\ &\leq M^T(v) \sum_{l\geq 0} 2^{-l(s+1)} \|\{\gamma_{l+j}\}_j\|_{l^2(\mathbb{N})} \leq M^T(v) \|\{\gamma_j\}\|_{l^2(\mathbb{N})} \sum_{l\geq 0} 2^{-l(s+1)}, \end{aligned}$$

and then, (4.77) imply that

$$A \lesssim P^T_{1,s}(u) M^T(v). \tag{4.78}$$

Analogously, we obtain a similar estimate for B

$$B \lesssim P^T_{2,s}(u) M^T(v). \tag{4.79}$$

Thus, (4.73)-(4.79) yield (4.72) and we conclude the proof of Theorem 4.2 as for Theorem 4.1 using (4.71) and (4.72) instead of (4.37) and (4.46). □

Conclusion.

In conclusion, we point out some open problems connected with this work:

- In the second chapter, we proved that the IVP (1) associated to the dispersive Kuramoto-Velarde equation was well-posed in $H^s(\mathbb{R})$ for $s > -1$, and ill-posed (in some sense) for $s < -1$. What does happen in in the case $s = -1$?

- In the third chapter, when studying the non-dissipative limit case (3.1), we derived some ill-posedness results [1] for higher-order nonlinear dispersive equations, as for example a higher-order Benjamin-Ono equation. Doing an analogy with the Benjamin-Ono equation, one can ask if these equations still could be well-posed in some Sobolev spaces, admitting in this case a flow map data-solution only continuous.

- Another interesting problem would be to investigate the existence of solitary waves for the higher-order Benjamin-Ono equation (3.4).

- Finally, the results of well-posedness in weighted Besov spaces for the IVP (3.1), derived in the fourth chapter, were only obtained for small initial data. Is it possible to generalize the well-posedness in weighted Besov spaces for arbitrary initial data?

[1] the flow map data-solution, when existing in $H^s(\mathbb{R})$, $s \in \mathbb{R}$, fails to be C^2.

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