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ACCESSIBILITY, PROPERTY SH AND ROBUST TRANSITIVITY

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À minha mãezinha Bené.

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ACCESSIBILITY, PROPERTY SH AND ROBUST TRANSITIVITY

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Abstract. We prove results related to robust transitivity of Partially Hyperbolic Diffeomorphisms under conditions involving Accessibility and the Property SH.

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1. Introduction

A very interesting feature of a differentiable dynamical system is transitivity, i.e. existence of a dense orbit. Being a sign of complexity of the underlying dynamics it prevents the possibility of reducing its study to more simple systems.

One of the most important questions in the theory of Differentiable Dynamical Systems regarding a particular dynamical property is to recognize when it is present in all nearby systems (with respect to some topology). When this happens we say that the property is robust or stable under perturbations.

So the search for conditions on a differentiable dynamical system leading to robust transitivity has been a topic of interest for a long time. Many examples exhibiting robust transitivity has been studied, beginning with the transitive Anosov diffeomorphisms.

Robust transitivity is not an exclusive property of Hyperbolic Diffeomorphisms as it has been showed first by the example of Shub on the torus $\mathbb{T}^4$, later by the example of Mañé on the torus $\mathbb{T}^3$ and more recently by the example of Bonatti and Días in [2]. All these examples are Partially Hyperbolic Systems (see section 2). While other example, due to Bonatti and Viana [5], exhibits just dominated splitting.

Ergodicity and its stability are other important properties to study on a dynamical system. A well known conjecture formulated by Pugh and Shub [12] on Stable Ergodicity for Partially Hyperbolic Systems has been the motivation for a lot of research during the last few years. Some progress has been done but the conjecture is still unproved.

One of the conditions appearing on the hypothesis of this conjecture is that of Accessibility (see section 2) which as the work [6] shows is a typical property in the sense that it is C^1 dense among the C^r Partially Hyperbolic Diffeomorphisms of a compact manifold.

Accessibility has also a relation with transitivity according to Brin's Theorem [11] stating that in a Partially Hyperbolic Accessible System, transitivity is equivalent to the fact of the non-wandering set being the whole manifold.

In connection with the mentioned examples of Shub and Mañé, the authors Pujals and Sambarino introduced in [13] an interesting property which they call Property SH, which proves to be there a good mechanism to guarantee that the strong stable foliation is robustly minimal. A key feature of Property SH is its intrinsic robustness which makes it an appealing condition to use for establishing robust transitivity in more general contexts.

The work in this thesis has been motivated by the idea of exploring the consequences, in the sense of robust transitivity, of the combination of these two properties, Property SH and Accessibility, for Partially Hyperbolic Systems.

The first result came naturally when studying the proof of Brin's Theorem, after the observation that accessibility in relation to open sets (as defined in section 2) was enough to guarantee the transitivity. Having at hand the Property SH and its robustness it was a natural step to think on establishing the robustness of accessibility in relation to open sets, conforming to Corollary 3.1 . As a Corollary, it followed the first significant result (see section 3) in the search for robust transitivity:

Corollary 1.1. *Let $\mathcal{PH}_v \subset Diff^r(M)$ be the set of volume-preserving, partially hyperbolic diffeomorphisms. Let $f \in \mathcal{PH}_v$ be accessible, exhibiting Property SH. Then f is robustly transitive in $\mathcal{PH}_v$.*

Abdenur and Crovisier proved in [1] that the fact of a diffeomorphism being robustly transitive implies that it is also topologically mixing modulo an arbitrarily small C^1 perturbation. Then to try to prove that an accessible, transitive diffeomorphism satisfying Property SH is topologically mixing is a natural question. Thus we have the following result in section 3:

Theorem 1.1. *Let f be a partially hyperbolic diffeomorphism, accessible, topologically transitive and satisfying Property SH. Then f is topologically mixing .*

In section 5 we show that Shub's example in $\mathbb{T}^4$ satisfies all the conditions in this Theorem 1.1.

Trying to understand a little more Property SH we found that it is enough to guarantee robust transitivity in the sense given by the following result proved in section 4:

Theorem 1.2. *Let M be a compact Riemannian manifold and let $f \in Diff^r(M)$ be a partially hyperbolic diffeomorphism and transitive. If f and f^{-1} satisfy Property SH then f is robustly transitive.*

The point in this last Theorem is that the hypotheses are given on the tangent bundle. The condition of minimality of the stable foliation assumed in [13] was substituted here by the Property SH for f^{-1}. In section 5 we give an scenario where all the conditions in Theorem 1.2 are realized.

And in Section 6 we prove the density of periodic points under the assumption of Property SH and minimality of the strong stable foliation.

Finally we would like to address a few related and interesting questions:

1) *Does a Partially Hyperbolic, transitive and accessible System exhibiting the Property SH, is necessarily robustly transitive?*

2) *Is it possible to exploit Theorem 1.2 to produce new examples of robustly transitive diffeomorphisms?*

3) *Which robustly transitive and Partially Hyperbolic Diffeomorphism can be approximated by one exhibiting the conditions in Theorem 1.2?*

4) *Given a diffeomorphism satisfying the hypotheses in Theorem 1.2 is it true that the set of its periodic points is dense?*

5) *Given a diffeomorphism exhibiting the Property SH, transitivity and accessibility is it true that the set of its periodic points is dense in the whole manifold?*

6) *Given a diffeomorphism exhibiting the Property SH and transitivity is it true that the set of its periodic points is dense in the whole manifold?*

In the following sections M will denote a compact Riemannian manifold and $Diff^r(M)$ the set of C^r-diffeomorphims defined on M.

2. Preliminaries

In this section we recall some well-known results regarding partially hyperbolic systems. We refer to [7], [11], [14], [13] for a general background on the topics we will review.

2.1. **Partially Hyperbolic Diffeomorphisms.**

Definition 2.1. A diffeomorphism $f : M \to M$ is partially hyperbolic provided the tangent bundle splits into three non-trivial sub-bundles $TM = E^{ss} \oplus E^c \oplus E^{uu}$ which are invariant under the tangent map Df and there are $0 < \lambda < \mu < 1$ such that for all $x \in M$

$$\| Df_{|E^{ss}(x)} \| < \lambda, \quad \| Df^{-1}_{|E^{uu}(x)} \| < \lambda, \quad \mu < \| Df^{-1}_{|E^c(x)} \|, \quad \| Df_{|E^c(x)} \| < \mu^{-1}.$$

Lemma 2.1. *Let $f \in Diff^r(M)$ be a partially hyperbolic diffeomorphism. Then there exist a C^r neighborhood of f, say U, $0 < \lambda < \lambda_1 < \mu_1 < \mu < 1$ and continuous functions $E^{ss} : \mathcal{U} \to C(M, TM)$, $E^c : \mathcal{U} \to C(M, TM)$ and $E^{uu} : \mathcal{U} \to C(M, TM)$ such that, for any $g \in \mathcal{U}$ and $x \in M$, we have the following:*

(1) $TM = E^{ss}(g) \oplus E^c(g) \oplus E^{uu}(g)$, this decomposition is invariant under Dg and no one of these sub-bundles is trivial;

(2) $\| Dg_{|E^{ss}(x)} \| < \lambda_1$, $\| Dg^{-1}_{|E^{uu}(x)} \| < \lambda_1$;

(3) $\mu_1 < \| Dg^{-1}_{|E^c(x)} \|$, $\| Dg_{|E^c(x)} \| < \mu_1^{-1}$.

The sub-bundles $E^{ss}(g)$ and $E^{uu}(g)$ are uniquely integrable and form two foliations $\mathcal{F}^{ss}$ and $\mathcal{F}^{uu}$.

Theorem 2.1. *Let $\mathcal{U}$ be as in Lemma 2.1. Then, for each $g \in \mathcal{U}$ there are two partitions $\mathcal{F}^{ss}(g)$ and $\mathcal{F}^{uu}(g)$ of M such that for each $x \in M$ the elements of the partitions that contain x, denoted by $\mathcal{F}^{ss}(x, g)$ and $\mathcal{F}^{uu}(x, g)$ are C^1 submanifolds such that $T_x\mathcal{F}^{ss}(x, g) = E^{ss}(x, g)$ and $T_x\mathcal{F}^{uu}(x, g) = E^{uu}(x, g)$. These submanifolds depend continuously (on compact subsets) on $x \in M$ and $g \in \mathcal{U}$.*

These submanifolds $\mathcal{F}^{ss}(x, g)$ and $\mathcal{F}^{uu}(x, g)$ inherit the Riemannian metric on M. We shall denote by $\mathcal{F}^{ss}_r(x, g)$ (respectively $\mathcal{F}^{uu}_r(x, g)$) the ball in $\mathcal{F}^{ss}(x, g)$ (respectively $\mathcal{F}^{uu}(x, g)$) of radius r centered at x.

The sub-bundle $E^{cu} = E^c \oplus E^{uu}$ is not integrable in general. However, we can choose a continuous family of locally invariant manifolds tangent to it. Let $dimE^{cu} = l$ and denote by I_ϵ the ball of radius ϵ in R^l.

Lemma 2.2. *Let $\mathcal{U}$ be as in Lemma 2.1. There exists a continuous map $\varphi : M \times \mathcal{U} \to Emb_1(I_1, M)$ such that, if we set $W^{cu}_\epsilon(x, g) = \varphi(x, g)I_\epsilon$, then the following hold:*

(1) $T_xW^{cu}_\epsilon(x, g) = E^{cu}(x, g)$;

(2) given $\epsilon > 0$ there exists $r = r(\epsilon)$ such that $g^{-1}(W^{cu}_r(x, g)) \subset W^{cu}_\epsilon(g^{-1}(x), g)$.

For the sake of simplicity we shall identify $W^{cu}_\epsilon(x, g)$ with the ball of radius ϵ in $W^{cu}_1(x, g)$.

Lemma 2.3. *Let $\mathcal{U}$ be as in Lemmas 2.1 and 2.2. Given $0 < \lambda < \lambda_1 < 1$ there exists r_0 such that if $g \in \mathcal{U}$ and $x \in M$ satisfy*

$$\prod_{j=0}^{n} \| Dg^{-1}_{|_{E^{cu}(g^{-j}(x))}} \| < \lambda^n, \qquad 0 \leq n \leq m,$$

Then $g^{-m}(W^{cu}_{r_0}(x,g)) \subset W^{cu}_{\lambda_1^m r_0}(g^{-m}(x),g)$.

In the following, we will work with partially hyperbolic diffeomorphisms.

2.2. **Accessibility.**

Definition 2.2. Let $f \in Diff^r(M)$ be a partially hyperbolic diffeomorphism. Two points $p,q \in M$ are called accessible , if there are points $z_0 = p, z_1, \dots, z_{l-1}, z_l = q, z_i \in M$, such that $z_i \in \mathcal{F}^\alpha(z_{i-1}, f)$ for $i = 1, \dots, l$ and $\alpha = ss$ or uu.

The collection of points $z_0, z_1, \dots, z_l$ is called the us-path connecting p and q and is denoted by $[p,q,f]$.

Accessibility is an equivalence relation and the collection of points accessible from a given point p is called the accessibility class of p. We will denote this class by $\mathcal{C}(p,f)$.

The diffeomorphism f is said to have the accessibility property if the accessibility class of any point is the whole manifold M, or, in other words, if any two points in M are accessible.

Next we introduce the notion of accessibility in relation to open sets, which we use to give a stronger version of Brin's Theorem (See section 3).

Definition 2.3. Two open sets $P, Q \subseteq M$ are called accessible , if there are points $p \in P$, $q \in Q$, such that p, q are accessible.

We will call a diffeomorphism f accessible in relation to open sets if any two open sets are accessible.

Obviously accessibility implies accessibility in relation to open sets. The converse is not true as can be shown with the following example:

Example 2.1. Take two linear Anosov diffeomorphisms A and B in $\mathbb{T}^2$ with eigenvalues $\frac{1}{8}, 3$ and $\frac{1}{8}, 9$ respectively. Define the diffeomorphism $F : \mathbb{T}^2 \times \mathbb{T}^2 \to \mathbb{T}^2 \times \mathbb{T}^2$ given by $F(x,y) = (A(x), B(y))$. F is partially hyperbolic with $T_{(x,y)}\mathbb{T}^4 = E^s(x,y) \oplus E^s(x,y) \oplus E^c(x,y) \oplus E^{uu}(x,y)$ where $E^c(x,y) = E^u(x,y)$. Observe that $\mathcal{F}^s(0,A) \times \mathcal{F}^s(0,B) \subset \mathcal{F}^{ss}((0,0),F)$ and hence $\mathcal{F}^{ss}((0,0),F)$ is dense in $\mathbb{T}^2 \times \mathbb{T}^2$. Consequently F is accessible in relation to open sets. The fact that F is not accessible follows from the integrability of $E^s \oplus E^s \oplus E^{uu}$ where for every $(x,y) \in \mathbb{T}^2 \times \mathbb{T}^2$ we have that $T_{(x,y)}(\mathbb{T}^2 \times W^s(y,B)) = E^s(x,y) \oplus E^{uu}(x,y) \oplus E^s(x,y)$.

Lemma 2.4. *Assume that a partially hyperbolic diffeomorphism f has the accessibility property. Then for every $\delta > 0$ there exist $l > 0$ and $R > 0$ such that for any $p,q \in M$ one can find a us-path that starts at p, ends within distance $\frac{\delta}{2}$ of q, and has at most l legs, each of them with length at most R.*

Proof. See [11]. □

Lemma 2.5. *Let $f : M \to M$ be a partially hyperbolic accessible diffeomorphism. Given $p_0 \in M$, there is $q_0 \in M$ and a us-path $z_0(q_0) = p_0, z_1(q_0), \dots, z_N(q_0) = q_0$ connecting p_0 to q_0 and satisfying the following property: for any $\epsilon > 0$ there exist $\delta > 0$ and $L > 0$ such that for every $x \in B(q_0, \delta)$ there exists a us-path $z_0(x) = p_0, z_1(x), \dots, z_N(x) = x$ connecting p_0 to x and such that $dist(z_j(x), z_j(q_0)) < \epsilon$ and $dist_{\mathcal{F}^\alpha}(z_{j-1}(x), z_j(x)) < L$ for $j = 1, \dots, N$ where $dist_{\mathcal{F}^\alpha}$ denotes the distance along the strong (either stable or unstable) leaf common to the two points.*

Proof. See [14]. □

Now we give an easy but interesting conquence of the last two Lemmas, useful for our purposes in section 3.

Lemma 2.6. *Assume that a partially hyperbolic diffeomorphism f has the accessibility property. Then there exist $l_0 > 0$ and $R_0 > 0$ such that for any $p, q \in M$ one can find a us-path that starts at p, ends at q, and has at most l_0 legs, each of them with length at most R_0.*

Proof. Fix $p_0 \in M$. Let $q_0 \in M$ and a us-path $z_0(q_0) = p_0, z_1(q_0), \dots, z_N(q_0) = q_0$ be as in Lemma 2.5. Let $\epsilon > 0$. Take $\delta > 0$ and $L > 0$ as in Lemma 2.5. For this $\delta > 0$ take $l > 0$ and $R > 0$ as in Lemma 2.4. Next, set $l_0 = 2l + 2N$ and $R_0 = max\{R, L\}$. Let $p, q \in M$. From Lemma 2.4 we know that there exists a us-path that starts at p (respectively q), ends within distance δ of q_0, say at p_1 (respectively q_1), and has at most l legs, each of them with length at most R.
From Lemma 2.5 there exist a us-path $z_0(p_1) = p_0, z_1(p_1), \dots, z_N(p_1) = p_1$ connecting p_0 to p_1 and a us-path $z_0(q_1) = p_0, z_1(q_1), \dots, z_N(q_1) = q_1$ connecting p_0 to q_1.
Thus,

$$p_1 = z_N(p_1), z_{N-1}(p_1), \dots, z_0(p_1) = p_0 = z_0(q_1), z_1(q_1), \dots, z_N(q_1) = q_1$$

is a us-path connecting p_1 to q_1, and it has $2N$ legs, each of them with length at most L. Hence, using the us-paths $[p, p_1, f]$ and $[q, q_1, f]$ with at most l legs, each of them with length at most R, we have completed the proof. □

Corollary 2.1. *Let $f : M \to M$ be a partially hyperbolic accessible diffeomorphism. Then there exist $l_1 > 0$ and $R_1 > 0$ such that for any $p, q \in M$ one can find a us-path $z_0 = p, z_1, \dots, z_{l-1}, z_l = q, l \le l_1$, that starts at p, ends at q, such that $q \in \mathcal{F}^{ss}_{R_1}(z_{l-1}, f)$, and each leg has length at most R_1.*

Proof. Let l_0 and R_0 be as in Lemma 2.6. Set $l_1 = l_0 + 1$ and set $R_1 = R_0$. Let $p, q \in M$. Take $q_0 \in \mathcal{F}^{ss}_{R_0}(q, f)$. From Lemma 2.6 we know that one can find a us-path $z_0 = p, z_1, \dots, z_{l-1} = q_0$ that starts at p, ends at q_0, and has at most l_0 legs, each of them with length at most $R_0 = R_1$. Therefore, $z_0 = p, z_1, \dots, z_{l-1} = q_0, z_l = q$ is a us-path that starts at p, ends at q, and has at most l_1 legs, each of them with length at most R_1. □

Now, using the last Corollary, we will prove that if f is accessible then, robustly, for a fixed $r > 0$ and any pair of points $p, q \in M$ there exists a path connecting p to the center unstable disc of radius r centered at q.

Lemma 2.7. *Let $f \in Diff^r(M)$ be a partially hyperbolic diffeomorphism. Given $R_0 > 0$ and $d_0 > 0$ there exist $\delta_0 > 0$ and a neighborhood $\mathcal{U}(f)$ such that for any*

$g \in \mathcal{U}(f)$ and for every $x, y \in M$ such that $d(x,y) < \delta_0$ we have that $\mathcal{F}^{\alpha}_{R_0}(x,g)$ and $\mathcal{F}^{\alpha}_{R_0}(y,f)$ are d_0-close, $\alpha = ss$ or uu.

Proof. From Stable Manifold Theorem we know that for every $x \in M$ there exist $r_x > 0$ and a neighborhood $\mathcal{U}_x(f)$ such that for any $g \in \mathcal{U}_x(f)$ and for every $y \in M$ such that $d(x,y) < r_x$ we have that $\mathcal{F}^{\alpha}_{R_0}(x,f)$ and $\mathcal{F}^{\alpha}_{R_0}(y,g)$ are $\frac{d_0}{2}$-close, $\alpha = ss$ or uu. Thus, for any $g \in \mathcal{U}_x(f)$ and for every $y, z \in B(x, r_x)$ we get $\mathcal{F}^{\alpha}_{R_0}(y,g)$ and $\mathcal{F}^{\alpha}_{R_0}(z,f)$ are d_0-close, $\alpha = ss$ or uu. Since M is compact, there are $x_1, x_2, \ldots, x_n \in M$ such that

$$M \subset \bigcup_{i=1}^{n} B(x_i, r_{x_i}).$$

Let $\delta_0 > 0$ be a Lebesgue number of this cover and take

$$\mathcal{U}(f) = \bigcap_{i=1}^{n} \mathcal{U}_{x_i}(f).$$

Thus, if $d(x,y) < \delta_0$ then $x, y \in B(x_i, r_{x_i})$ for some $i = 1, 2, \ldots, n$. Hence we have that $\mathcal{F}^{\alpha}_{R_0}(x,f)$ and $\mathcal{F}^{\alpha}_{R_0}(y,g)$ are d_0-close, $\alpha = ss$ or uu, for any $g \in \mathcal{U}(f) \subset \mathcal{U}_{x_i}(f)$. □

Lemma 2.8. *Let $f \in Diff^r(M)$ be a partially hyperbolic diffeomorphism. Given $R_0 > 0$ and $r > 0$ there exist $\epsilon > 0$, $\delta_0 > 0$ and a neighborhood $\mathcal{V}(f)$ such that for any $g \in \mathcal{V}(f)$ it follows that for any $x, y \in M$ with $d(x,y) < \delta_0$ the following holds:*

$$\mathcal{F}^{ss}_{R_0+\epsilon}(x,g) \pitchfork W^{cu}_r(z,g) \neq \emptyset \quad \textit{for any} \quad z \in \mathcal{F}^{ss}_{R_0}(y,f).$$

Proof. Take $\epsilon > 0$ given by Stable Manifold Theorem. There exists a neighborhood $\mathcal{U}_1(f)$ such that $\epsilon > 0$ can to be taken for any $g \in \mathcal{U}_1(f)$. Let $d_0 > 0$ be such that for any $g \in \mathcal{U}_1(f)$ it follows that if $d(x,y) < d_0$ then

$$\mathcal{F}^{ss}_{\epsilon}(x,g) \pitchfork W^{cu}_r(y,g) \neq \emptyset.$$

Consider $\mathcal{V}(f) \subset \mathcal{U}_1(f)$ and $\delta_0 > 0$ given by Lemma 2.7. Thus, if $g \in \mathcal{V}(f)$ and $x, y \in M$ with $d(x,y) < \delta_0$ we have that $\mathcal{F}^{ss}_{R_0}(x,g)$ and $\mathcal{F}^{ss}_{R_0}(y,f)$ are d_0-close and therefore

$$\mathcal{F}^{ss}_{R_0+\epsilon}(x,g) \pitchfork W^{cu}_r(z,g) \neq \emptyset \quad \text{for any} \quad z \in \mathcal{F}^{ss}_{R_0}(y,f).$$

□

Lemma 2.9. *Let $f \in Diff^r(M)$ be a partially hyperbolic accessible diffeomorphism. Given $r > 0$ there exist a neighborhood $\mathcal{U}(f)$, $l > 0$ and $R > 0$ such that for any $g \in \mathcal{U}(f)$ it follows that for every $p, q \in M$ there exists $q' \in W^{cu}_r(q,g)$ such that one can find a us-path by g that starts at p, ends at q', and has at most l legs, each of them with length at most R.*

Proof. Let $l_1 > 0$ and $R_1 > 0$ be as in Corollary 2.1. For the sake of simplicity, we will assume that $l_1 = 4$. Given R_1 and r let ϵ, δ_0 and $\mathcal{V}(f)$ be as in Lemma 2.8. From Lemma 2.7 there exist $\delta_1 > 0$ and $\mathcal{U}_1(f) \subset \mathcal{V}(f)$ such that if $g \in \mathcal{U}_1(f)$ and $x, y \in M$ with $d(x,y) < \delta_1$ then $\mathcal{F}^{\alpha}_{R_1}(x,g)$ and $\mathcal{F}^{\alpha}_{R_1}(y,f)$ are δ_0-close, $\alpha = ss$ or uu. Once again, using Lemma 2.7 take $\delta_2 > 0$ and $\mathcal{U}_2(f) \subset \mathcal{U}_1(f)$ such that if $g \in \mathcal{U}_2(f)$ and $x, y \in M$ with $d(x,y) < \delta_2$ then $\mathcal{F}^{\alpha}_{R_1}(x,g)$ and $\mathcal{F}^{\alpha}_{R_1}(y,f)$ are δ_1-close, $\alpha = ss$ or uu. Finally let $\mathcal{U}(f) \subset \mathcal{U}_2(f)$ be a neighborhood such that for any $g \in \mathcal{U}(f)$ and for any $x \in M$ we have that $\mathcal{F}^{\alpha}_{R_1}(x,g)$ and $\mathcal{F}^{\alpha}_{R_1}(x,f)$ are δ_2-close, $\alpha = ss$ or uu.

Let us prove that $\mathcal{U}(f)$, $l = l_1$ and $R = R_1 + \epsilon$ satisfy what we want. Let $g \in \mathcal{U}(f)$ and let $p, q \in M$. We know that there exists a us-path by f that starts at p, ends at q, and has at most l_1 legs, each of them with length at most R_1. Moreover, the last leg lies in $\mathcal{F}^{ss}_{R_1}(q, f)$. Suppose that such a us-path has exactly l_1 legs. Let $p = z_0$, z_1, z_2, z_3, $z_4 = q$ be such a us-path. See the figure.
We have that $\mathcal{F}^{uu}_{R_1}(p, g)$ and $\mathcal{F}^{uu}_{R_1}(p, f)$ are δ_2-close. Then, there exists $x_1 \in \mathcal{F}^{uu}_{R_1}(p, g)$ such that $d(x_1, z_1) < \delta_2$. Thus $\mathcal{F}^{ss}_{R_1}(x_1, g)$ and $\mathcal{F}^{ss}_{R_1}(z_1, f)$ are δ_1-close. Therefore, there exists $x_2 \in \mathcal{F}^{ss}_{R_1}(x_1, g)$ such that $d(x_2, z_2) < \delta_1$. Hence $\mathcal{F}^{uu}_{R_1}(x_2, g)$ and $\mathcal{F}^{uu}_{R_1}(z_2, f)$ are δ_0-close. Take $x_3 \in \mathcal{F}^{uu}_{R_1}(x_2, g)$ with $d(x_3, z_3) < \delta_0$. From Lemma 2.8, since $q \in \mathcal{F}^{ss}_{R_1}(z_3, f)$ we have that

$$\mathcal{F}^{ss}_{R_1+\epsilon}(x_3, g) \pitchfork W^{cu}_r(q, g) \neq \emptyset.$$

Take

$$q' \in \mathcal{F}^{ss}_{R_1+\epsilon}(x_3, g) \pitchfork W^{cu}_r(q, g),$$

and p, x_1, x_2, x_3, q' the us-path by g. The case that such a us-path by f, connecting p to q, has l' legs with $l' < l_1$, is similar.

□

2.3. **Property SH.**

Next, we will define the key property that guarantees the robust transitivity: some hyperbolicity (SH) on the central distribution E^c at some points. Before we do, let us introduce some notation: if $L : V \to W$ is a linear isomorphism between normed vector spaces we denote by $m\{L\}$ the minimum norm of L, i.e. $m\{L\} = \| L^{-1} \|^{-1}$.

Definition 2.4. Let $f \in \mathrm{Diff}^r(M)$ be a partially hyperbolic diffeomorphism. We say that f exhibits the property SH if there exist $\lambda_0 > 1, C > 0$ such that for any $x \in M$ there exists $y^u(x) \in \mathcal{F}^{uu}_1(x, f)$ (the ball of radius 1 in $\mathcal{F}^{uu}(x, f)$ centered at x) satisfying

$$m\{Df^n_{|E^c(f^l(y^u(x)))}\} > C\lambda_0^n \quad \text{for any} \quad n > 0, \quad l > 0.$$

The Property SH persists under slight perturbations and guarantees the robustness of the minimality of a stable foliation for a partially hyperbolic diffeomorphism.

Theorem 2.2. *Let $f \in Diff^r(M)$ be a partially hyperbolic diffeomorphism exhi biting Property SH. Then, there are $\mathcal{U}(f)$, $C' > 0$ and $\sigma > 1$ such that for any $g \in \mathcal{U}$ it follows that for any $x \in M$ there exists $y^u \in \mathcal{F}^{uu}_1(x, g)$ satisfying*

$$m\{Dg^n_{|E^c(g^l(y^u))}\} > C'\sigma^n \quad \textit{for any} \quad n > 0, \quad l > 0.$$

Proof. See [13]. □

Definition 2.5. Let $f : M \to M$ be a C^r partially hyperbolic diffeomorphism. We say that $\mathcal{F}^{ss}(f)$ is minimal when $\mathcal{F}^{ss}(x, f)$, the leave of this foliation passing through the point x, is dense in M for every $x \in M$. We say that $\mathcal{F}^{ss}(f)$ is C^r-robustly minimal if there exist a C^r neighborhood $\mathcal{U}(f)$ such that $\mathcal{F}^{ss}(g)$ is minimal for every diffeomorphism $g \in \mathcal{U}(f)$.

Theorem 2.3. *Let $r \geq 1$ and let $f \in Diff^r(M)$ be a partially hyperbolic diffeomorphism satisfying Property SH and such that the strong stable foliation $\mathcal{F}^{ss}(f)$ is minimal. Then, $\mathcal{F}^{ss}(f)$ is C^1 (and hence C^r) robustly minimal.*

Proof. See [13]. □

2.4. **Blenders and heterodimensional cycles.**

In this subsection we recall the notions of blender and heterodimensional cycle and the relation between them. We also give a condition under which the presence of a blender guarantees the Property SH.

Let M be a compact n-dimensional manifold, $n \geq 3$; write $n = k + m + 1$, where $k, m \geq 1$. Let D^k and D^m denote the unitary closed balls in $\mathbb{R}^k$ and $\mathbb{R}^m$, respectively. Consider a C^1-embedding C of $D^k \times [-1,1] \times D^m$. Divide the boundary of C into three parts as follows:

$$\partial^{ss} C = (\partial D^k) \times [-1,1] \times D^m,$$
$$\partial^{u} C = D^k \times \partial([-1,1] \times D^m),$$
$$\partial^{uu} C = D^k \times [-1,1] \times \partial D^m.$$

In C we take coordinates $(x_s, x_c, x_u), x_s \in D^k, x_c \in [-1,1]$ and $x_u \in D^m$. We use the notation $\frac{\partial}{\partial x_s}$ to mean the space spanned by $\{\frac{\partial}{\partial x_s^1}, \cdots, \frac{\partial}{\partial x_s^k}\}$. The definition of $\frac{\partial}{\partial x_u}$ is analogous.

In the manifold M we consider a metric $\| \cdot \|$ that induces in the cube C the product of the usual euclidean metrics in D^k, $[-1,1]$ and D^m.

Given a k-plane Π and $\epsilon > 0$, we define the *ϵ-cone around* Π by $C_\epsilon(\Pi) = \{u \in TM, u = v + w, v \in \Pi, w \in \Pi^\top, \| w \| \leq \epsilon \cdot \| v \|\}$.

Fix $\epsilon \in]0,1[$ and consider conefields $\mathcal{C}^{uu}, \mathcal{C}^u$ and $\mathcal{C}^{ss}$ of size ϵ around the tangent spaces of the families of disks $\{(x_s, x_c)\} \times D^m, \{x_s\} \times [-1,1] \times D^m$ and $D^k \times \{(x_c, x_u)\}$, respectively.

We say that an m-disk Δ in C is a *vertical disk through* C if Δ is tangent to $\mathcal{C}^{uu}$ and its boundary $\partial\Delta$ is contained in $\partial^{uu}\mathcal{C}$.

Definition 2.6. Let $f : M \to M$ be a C^1-diffeomorphism and C a C^1-embedding of $D^k \times [-1,1] \times D^m$. We say that the pair (C, f) is a cs-*blender* if it satisfies the five properties $(H1) - (H5)$ below:

(**H1**) There is a connected component A of $C \cap f(C)$ disjoint from the union $\partial^{ss} C \cup f(\partial^u C)$.

(**H2**) There are $n \in \mathbb{N}*$ and a connected component B of $f^n(C) \cap C$ so that B is disjoint from $D^k \times \{1\} \times D^m$, from $\partial^{ss} C$ and from $f(\partial^{uu} C)$.

(**H3**) There is $\epsilon > 0$ so that the conefields $\mathcal{C}^u$, $\mathcal{C}^{uu}$ and $\mathcal{C}^{ss}$ of size ϵ defined above satisfy:

i) For every $x \in f^{-1}(A)$ (resp. $x \in f^{-n}(B)$) and every vector $v \in \mathcal{C}^u(x)$, the vector $w = Df(v)$ (resp. $w = (Df^n)(v)$) belongs to the interior of $\mathcal{C}^u(f(x))$ (resp. $\mathcal{C}^u(f^n(x))$). In addition, there is $\lambda > 1$ such that $\lambda \cdot \| v \| \leq \| w \|$.

ii) For every $x \in f^{-1}(A)$ (resp. $x \in f^{-n}(B)$) and every vector $v \in \mathcal{C}^{uu}(x)$, the vector $w = Df(v)$ (resp. $w = (Df^n)(v)$) belongs to the interior of $\mathcal{C}^{uu}(f(x))$ (resp. $\mathcal{C}^{uu}(f^n(x))$).

iii) For every $x \in A$ (resp. $x \in B$) and every vector v in $\mathcal{C}^{ss}(x)$, the vector $w = Df^{-1}(v)$ (resp. $w = Df^{-n}(w)$) is in the interior of the cone $\mathcal{C}^{ss}(f^{-1}(x))$ (resp. $\mathcal{C}^{ss}(f^{-n}(x))$). Moreover $\lambda \cdot \| v \| \leq \| w \|$.

In [2] it is proved that the hypotheses (H1) and (H3) imply that the diffeomorphism f has an unique fixed point, say Q, in the component A. This point is hyperbolic and its index is k. Denote by W_0^s the connected component of the intersection $W^s(Q) \cap C$ containing Q; W_0^s is a horizontal k-disk through the cube C.

(**H4**) There is a neighborhood $\mathcal{U}_-$ of the left side $\{x_c = -1\}$ of C so that every vertical disk D through C at the righ of W_0^s does not intersect $\mathcal{U}_-$.

(**H5**) There are neighborhoods $\mathcal{U}$ and $\mathcal{U}_+$ of W_0^s and of the right side $\{x_c = +1\}$ of C, respectively, so that for every vertical disk D through C at the right of W_0^s one of the two following possibilities holds:

i) The intersection $f(D) \cap A$ contains a vertical disk Σ through C at the right of W_0^s and disjoint from $\mathcal{U}_+$;

ii) $f^n(D) \cap B$ contains a vertical disk Σ through C at the right of W_0^s and disjoint from $\mathcal{U}$.

We define cu-*blender* as a cs-*blender* for f^{-1}.

Observation- 1. *In* [2]*, it is proved that if $\mathcal{B}$ is a cs-*blender *there is a conefield $\mathcal{C}^{uu}$ around the strong unstable direction of $\mathcal{B}$ so that every curve σ tangent to $\mathcal{C}^{uu}$ intersects $W^s(\mathcal{B})$. Moreover, such a property is C^1-persistent.*

Proposition 2.1. *Let $f \in Diff^r(M)$ be a partially hyperbolic diffeomorphism with strong unstable minimal foliation such that $\mathcal{B} = (C, f)$ is a cs-*blender. *Then f satisfies Property SH.*
*Analogously if f has a strong stable minimal foliation and it has a cu-*blender *then f^{-1} satisfies Property SH.*

Proof. It is not difficult to see that if the strong unstable foliation is minimal, then there exists $r > 0$ such that $\mathcal{F}_r^{uu}(x, f) \cap C \neq \varnothing, \quad \forall x \in M$. Hence, using the observation above, we have that for some $k > 0$ and for every $x \in M$, there exists

$$y^x \in \mathcal{F}_r^{uu}(x, f) \cap W_k^s(B).$$

From this, it follows that

$$d(f^n(y^x), B) \to 0 \quad \text{when} \quad n \to +\infty.$$

Since $T_zM = E^{ss}(z, f) \oplus E^u(z, f) \oplus E^{uu}(z, f)$ for every $z \in B$, there exists $n_0 \in \mathbb{N}$ such that $Df_{|E^c(z)}$ is uniformly expanding in the future $\forall z \in f^{n_0}(W_k^s(B))$. Therefore, f satisfies Property SH. $\square$

Blenders can be produced by unfolding heterodimensional cycles far from homoclinic tangencies as in next Proposition found in [4].

Definition 2.7. Given a diffeomorphism f with two hyperbolic periodic points P_f and Q_f with different indices, say index$(P_f) >$ index(Q_f), we say that f has a heterodimensional cycle with codimension index$(P_f) -$ index(Q_f) associated to P_f and Q_f if $\mathcal{F}^s(P_f, f)$ and $\mathcal{F}^u(Q_f, f)$ have a (nontrivial) transverse intersection and $\mathcal{F}^u(P_f, f)$ and $\mathcal{F}^s(Q_f, f)$ have a quasi-transverse intersection along the orbit of some point x, i.e., $T_x\mathcal{F}^u(P_f, f) + T_x\mathcal{F}^s(Q_f, f)$ is a direct sum.

Proposition 2.2. *Let f be a C^1 diffeomorphism with a heterodimensional cycle associated to saddles P and Q of indices p and $q = p + 1$. Suppose that the cycle is C^1-far from homoclinic tangencies. Then there is an open set $\mathcal{V} \subset Diff^1(M)$ whose closure contains f such that for every g in $\mathcal{V}$ there are a cs-blender defined for g and a cs-blender defined for g^{-1} such that:*

· The cs-blender for g is associated to a hyperbolic periodic point R_g homoclinically related to Q_g and is activated by P_g.

· The cs-blender for g^{-1} is associated to a hyperbolic periodic point S_g homoclinically related to P_g and is activated by Q_g.

3. A first step towards robust transitivity

From now on we will show our results. This section deals with the first results, already mentioned on the introduction, obtained in colaboration with H. T. Alien.

We begin giving our version of Brin's Theorem. Observe that the condition of accessibility in relation to open sets is weaker than the condition of accessibility in the original version.

Let $g \in Diff^r(M)$. We will denote by $\Omega(g)$ the set of the non-wandering points for g.

Theorem 3.1. *(Brin's Theorem). Let $f \in Diff^r(M)$ be a partially hyperbolic diffeomorphism exhibiting the accessibility property in relation to open sets. If $\Omega(f) = M$ then f is transitive.*

Proof. Let $P, Q \subset M$ be open sets. Take two non-periodic points $p \in P$, $q \in Q$ such that p, q are accessible.
Thus, there exist

$$z_0 = p, z_1, z_2, \cdots, z_{l-1}, z_l = q \in M$$

such that $z_i \in \mathcal{F}^\alpha(z_{i-1}, f)$ for $i = 1, \cdots, l$ and $\alpha = ss$ or uu.

For the sake of simplicity, we will assume that $l = 3$, $z_1 \in \mathcal{F}^{ss}_r(p, f)$, $z_2 \in \mathcal{F}^{uu}_r(z_1, f)$, $q \in \mathcal{F}^{ss}_r(z_2, f)$ for some $r > 0$. Let $\delta_0 > 0$ be such that $B(p, \delta_0) \subset P$. Take $\delta_1 > 0$ such that $\mathcal{F}^{ss}_r(z, f)$ and $\mathcal{F}^{ss}_r(z_1, f)$ are δ_0-close for any $z \in B(z_1, \delta_1)$.

Let $\delta_2 > 0$ be such that $\mathcal{F}^{uu}_r(z, f)$ and $\mathcal{F}^{uu}_r(z_2, f)$ are δ_1-close for any $z \in B(z_2, \delta_2)$.

Finally, choose $\delta > 0$ such that $\mathcal{F}^{ss}_r(z, f)$ and $\mathcal{F}^{ss}_r(q, f)$ are δ_2-close for any $z \in B(q, \delta)$, with $B(q, \delta) \subset Q$. Take l_0 such that

$$f^m(\mathcal{F}^{ss}_r(z, f)) \subset B(f^m(z), \frac{\delta}{2})$$

for any $m \geq l_0$, $z \in M$. Since $\Omega(f) = M$, there are $x_3 \in B(q, \delta)$ and $m_3 > l_0$ such that $f^{m_3}(x_3) \in B(q, \frac{\delta}{2})$.

From this it follows that $\mathcal{F}^{ss}_r(x_3, f)$ and $\mathcal{F}^{ss}_r(q, f)$ are δ_2-close and therefore

$$\mathcal{F}^{ss}_r(x_3, f) \cap B(z_2, \delta_2) \neq \emptyset.$$

Moreover, it also follows that

$$f^{m_3}(\mathcal{F}^{ss}_r(x_3, f)) \subset B(f^{m_3}(x_3), \frac{\delta}{2}).$$

Hence,

$$f^{m_3}(B(z_2,\delta_2)) \cap B(f^{m_3}(x_3), \frac{\delta}{2}) \neq \emptyset.$$

Notice that $B(f^{m_3}(x_3), \frac{\delta}{2}) \subset B(q,\delta) \subset Q$. Thus $\mathcal{V}_2 = B(z_2,\delta_2) \cap f^{-m_3}(Q)$ is non empty.

Let $B(w,\gamma)$ be a ball of radius γ in $\mathcal{V}_2$. Take m_0 such that $f^{-m}(\mathcal{F}_r^{uu}(z,f)) \subset B(f^{-m}(z), \frac{\gamma}{2})$ for any $m \geq m_0$, $z \in M$. Using that $\Omega(f) = M$, we obtain $x_2 \in B(w,\gamma)$ and $m_2 > m_0$ such that $f^{-m_2}(x_2) \in B(w, \frac{\gamma}{2})$. From this it follows that

$$\mathcal{F}_r^{uu}(x_2,f) \quad \text{and} \quad \mathcal{F}_r^{uu}(z_2,f) \quad \text{are} \quad \delta_1\text{-close}$$

and therefore

$$\mathcal{F}_r^{uu}(x_2,f) \cap B(z_1,\delta_1) \neq \emptyset.$$

Moreover, it also follows that

$$f^{-m_2}(\mathcal{F}_r^{uu}(x_2,f)) \subset B(f^{-m_2}(x_2), \frac{\gamma}{2}).$$

Hence,

$$f^{-m_2}(B(z_1,\delta_1)) \cap B(f^{-m_2}(x_2), \frac{\gamma}{2}) \neq \emptyset.$$

Observe that $B(f^{-m_2}(x_2), \frac{\gamma}{2}) \subset \mathcal{V}_2$ because $f^{-m_2}(x_2) \in B(w, \frac{\gamma}{2})$ and $B(w,\gamma) \subset \mathcal{V}_2$.

Thus

$$\begin{aligned}\mathcal{V}_3 &= B(z_1,\delta_1) \cap f^{m_2}(\mathcal{V}_2)\\ &= B(z_1,\delta_1) \cap f^{m_2}(B(z_2,\delta_2)) \cap f^{m_2-m_3}(Q)\end{aligned}$$

is non empty.

Now, let $B(y,\beta)$ be a ball of radius β in $\mathcal{V}_3$. Take n_0 such that $f^{m}(\mathcal{F}_r^{ss}(z,f)) \subset B(f^{m}(z), \frac{\beta}{2})$ for any $m \geq n_0$, $z \in M$. From $\Omega(f) = M$, there exist $x_1 \in B(y,\beta)$ and $m_1 > n_0$ such that $f^{m_1}(x_1) \in B(y, \frac{\beta}{2})$. From this we have that

$$\mathcal{F}_r^{ss}(x_1,f) \quad \text{and} \quad \mathcal{F}_r^{ss}(z_1,f) \quad \text{are} \quad \delta_0\text{-close}$$

and hence,

$$\mathcal{F}_r^{ss}(x_1,f) \cap B(p,\delta_0) \neq \emptyset.$$

Furthermore, we also have that

$$f^{m_1}(\mathcal{F}_r^{ss}(x_1,f)) \subset B(f^{m_1}(x_1), \frac{\beta}{2}).$$

Thus,

$$f^{m_1}(B(p,\delta_0)) \cap B(f^{m_1}(x_1), \frac{\beta}{2}) \neq \emptyset.$$

Since $B(f^{m_1}(x_1), \frac{\beta}{2}) \subset B(y,\beta) \subset \mathcal{V}_3 \subset f^{m_2-m_3}(Q)$, then

$$f^{m_1}(B(p,\delta_0)) \cap f^{m_2-m_3}(Q) \neq \emptyset$$

and therefore

$$f^{m_1}(P) \cap f^{m_2-m_3}(Q) \neq \emptyset.$$

□

Theorem 3.2. *Let $f \in Diff^r(M)$ be a partially hyperbolic accessible diffeomorphism exhibiting Property SH. Then there is $\mathcal{U}(f)$ such that for every $g \in \mathcal{U}$ and $p \in M$ it follows that $\mathcal{C}(p,g)$ is dense in M.*

Proof. From Theorem 2.2 we know that there exist a neighborhood $\mathcal{U}_0(f)$, $C' > 0$ and $\sigma > 1$ such that for every $g \in \mathcal{U}_0$ and $x \in M$ there exists a point $y^u \in \mathcal{F}_1^{uu}(x, g)$ satisfying

$$m\{Dg^n_{|E^c(g^l(y^u))}\} > C'\sigma^n \quad \text{for any} \quad n > 0, \quad l > 0. \tag{1}$$

We may assume that $C = 1$. Otherwise we take a fixed power of every $g \in \mathcal{U}_0$. Let $\lambda = \sigma^{-1}$ and fix $0 < \lambda < \lambda_1 < 1$ and let r be as in Lemma 2.3. For this $r > 0$ take $\mathcal{U}(f) \subset \mathcal{U}_0(f)$, $l > 0$ and $R > 0$ as in Lemma 2.9. We will prove that for every $g \in \mathcal{U}(f)$ and $p \in M$ we have that $\mathcal{C}(p, g)$ is dense in M.
Let $\mathcal{V} \subset M$ be an open set and let $z \in \mathcal{V}$. Let $\beta > 0$ be such that $\mathcal{F}_\beta^{uu}(z, g) \subset \mathcal{V}$. Take n_0 such that $g^{n_0}(\mathcal{F}_\beta^{uu}(z, g)) \supset \mathcal{F}_1^{uu}(g^{n_0}(z), g)$. Consider the point $y^u \in \mathcal{F}_1^{uu}(g^{n_0}(z), g)$ given by Theorem 2.2 and let $\eta > 0$ be such that

$$g^{-n_0}(W_\eta^{cu}(y^u, g)) \subset \mathcal{V} \tag{2}$$

Choose a positive integer m such that $\lambda_1^m r < \eta$ and set $k = n_0 + m$. From Lemma 2.9 for $q = g^m(y^u)$ there exists $q' \in W_r^{cu}(q, g)$ such that one can find a us-path by g that starts at $g^k(p)$, ends at q', and has at most l legs, each of them with length at most R.
Since $E^{cu} = E^c \oplus E^u$ and this decomposition is dominated, there is $L > 0$ such that $\| Dg^{-n}_{|E^{cu}} \| \leq L \sup\{\| Dg^{-n}_{|E^u} \|, \| Dg^{-n}_{|E^c} \|\}$. For the sake of simplicity, we will assume that $L = 1$. From (1) we know that

$$\prod_{j=0}^{n} \| Dg^{-1}_{|E^c(g^{-j+m}(y^u))} \| < \lambda^n, \quad 0 \leq n \leq m \tag{3}$$

and therefore

$$\prod_{j=0}^{n} \| Dg^{-1}_{|E^{cu}(g^{-j+m}(y^u))} \| < \lambda^n, \quad 0 \leq n \leq m \tag{4}$$

From Lemma 2.3 we conclude that

$$g^{-m}(W_r^{cu}(g^m(y^u), g)) \subset W_{\lambda_1^m r}^{cu}(y^u, g) \subset W_\eta^{cu}(y^u, g) \tag{5}$$

and hence, using (2), we have $g^{-k}(W_r^{cu}(g^m(y^u), g)) \subset \mathcal{V}$. Since $q' \in W_r^{cu}(g^m(y^u), g)$ we get $g^{-k}(q') \in \mathcal{V}$. Thus, there exists a us-path by g that starts at p, ends at $g^{-k}(q') \in \mathcal{V}$. Hence, $g^{-k}(q') \in \mathcal{V} \cap \mathcal{C}(p, g)$ and the proof is completed. □

Corollary 3.1. *Let $f \in Diff^r(M)$ be a partially hyperbolic accessible diffeomorphism exhibiting Property SH. Then there is $\mathcal{U}(f)$ such that for any $g \in \mathcal{U}$ it follows that g is accessible in relation to open sets.*

Corollary 3.2. *Let $\mathcal{PH}_\Omega \subset Diff^r(M)$ be the set of partially hyperbolic diffeomorphisms such that the set of the non-wandering points is M. Let $f \in \mathcal{PH}_\Omega$ be accessible, exhibiting Property SH. Then f is robustly transitive in $\mathcal{PH}_\Omega$.*

Corollary 3.3. *Let $\mathcal{PH}_v \subset Diff^r(M)$ be the set of volume-preserving, partially hyperbolic diffeomorphisms. Let $f \in \mathcal{PH}_v$ be accessible, exhibiting Property SH. Then f is robustly transitive in $\mathcal{PH}_v$.*

As mentioned above our next step will be to prove that f transitive under the conditions of Accessibility and Property SH is topologically mixing.

Proposition 3.1. *Assume that a partially hyperbolic diffeomorphism f has the accessibility property . Then f^k is also accessible for every $k \in \mathbb{N}^*$.*

Proof. For every $x \in M$,

$$\mathcal{F}^{ss}_{\epsilon}(x, f) \subset \mathcal{F}^{ss}_{\epsilon}(x, f^k) \quad \text{for any} \quad k \in \mathbb{N}^*,$$

and therefore,

$$\mathcal{F}^{ss}(x, f) \subset \mathcal{F}^{ss}(x, f^k) \quad \text{for any} \quad k \in \mathbb{N}^*.$$

Analogously

$$\mathcal{F}^{uu}(x, f) \subset \mathcal{F}^{uu}(x, f^k) \quad \text{for any} \quad k \in \mathbb{N}^*.$$

Hence f^k is accessible for every $k \in \mathbb{N}^*$. □

Corollary 3.4. *Let f be a partially hyperbolic accessible diffeomorphism. Then f^k is also accessible for every $k \in \mathbb{Z}^*$.*

Proof. It is enough to notice that f^{-1} is accessible since $\mathcal{F}^{ss}(x, f^{-1}) = \mathcal{F}^{uu}(x, f)$ for any $x \in M$. □

Proposition 3.2. *Let $f \in Diff^r(M)$ be a topologically transitive diffeomorphism. Then $\Omega(f^k) = M$ for any $k \in \mathbb{Z}^*$.*

Proof. Fix $k \in \mathbb{N}^*$ arbitrarily. Take $p \in M$ and let $\mathcal{V}$ be an open neighborhood of p. Since f is transitive, there exists $z \in M$ such that infinitely many iterates of z belong to $\mathcal{V}$. Thus, we have two iterates $f^{m_0}(z)$ and $f^{m_1}(z)$, in $\mathcal{V}$, such that $m_0 \equiv m_1 \pmod{k}$. Therefore, there exist $q_0, q_1 \in \mathbb{N}$ such that $m_0 = kq_0 + r$ and $m_1 = kq_1 + r$ for some $r \in \mathbb{N}^*$ with $0 \leq r \leq k - 1$. Hence,

$$(f^k)^{(q_1 - q_0)}(f^{m_0}(z)) = f^{m_1 - m_0}(f^{m_0}(z)) = f^{m_1}(z)$$

and $p \in \Omega(f^k) \cap \Omega(f^{-k})$. □

Proposition 3.3. *Assume that a partially hyperbolic diffeomorphism f has the accessibility property. If f is topologically transitive then f^k is also topologically transitive for every $k \in \mathbb{Z}^*$.*

Proof. From Corollary 3.4 f^k is accessible for every $k \in \mathbb{Z}^*$. From Proposition 3.2 $\Omega(f^k) = M$ for every $k \in \mathbb{Z}^*$. From Brin's Theorem, f^k is topologically transitive for every $k \in \mathbb{Z}^*$. □

Theorem 3.3. *Let f be a partially hyperbolic diffeomorphism, accessible, topologically transitive and satisfying Property SH. Then f is topologically mixing .*

Proof. Let $\mathcal{U}, \mathcal{W} \subset M$ be open sets. Let $x \in \mathcal{U}$ and let $\beta > 0$ be such that $\mathcal{F}^{uu}_{\beta}(x, f) \subset \mathcal{U}$. Take n_0 such that $f^{n_0}(\mathcal{F}^{uu}_{\beta}(x, f)) \supset \mathcal{F}^{uu}_1(f^{n_0}(x), f)$. Consider the point $y^u \in \mathcal{F}^{uu}_1(f^{n_0}(x), f)$ satisfying

$$m\{Df^n_{|E^c(f^l(y^u))}\} > C\sigma^n \quad \text{for any} \quad n > 0, \quad l > 0, \tag{6}$$

where $C > 0, \sigma > 1$.
We may assume that $C = 1$. Otherwise we take a fixed power of f. Let $\lambda = \sigma^{-1}$, fix $0 < \lambda < \lambda_1 < 1$ and let r be as in Lemma 2.3. Let $\eta > 0$ be such that

$$f^{-n_0}(W^{cu}_{\eta}(y^u, f)) \subset \mathcal{U}. \tag{7}$$

Let $q \in \omega(y^u)$ be a recurrent point.
Consider $\epsilon > 0$ given in Stable Manifold Theorem. Take $\delta > 0$ such that

$$d(z_1, z_2) < \delta \Rightarrow \mathcal{F}^{ss}_\epsilon(z_1, f) \pitchfork W^{cu}_r(z_2, f) \neq \emptyset.$$

From Shadowing Lemma, there exists $p \in M$, periodic hyperbolic point, shadowing a periodic pseudo-orbit in $\omega(y^u)$ defined by recurrent point q, with $d(p, q) < \frac{\delta}{2}$. Since $q \in \omega(y^u)$, take $m \in \mathbb{N}^*$ such that $\lambda_1^m r < \eta$ and $d(f^m(y^u), q) < \frac{\delta}{2}$.
Set $k_0 = n_0 + m$. We have that $d(p, f^m(y^u)) < \delta$ and therefore

$$\mathcal{F}^{ss}_\epsilon(p, f) \pitchfork W^{cu}_r(f^m(y^u), f) \neq \emptyset.$$

Let $\mathcal{P}$ be the period of p. From Proposition 3.3 $f^{\mathcal{P}}$ is topologically transitive. Let $i \in \{0, 1, \ldots, \mathcal{P} - 1\}$. Take $w_i \in f^{-i}(W)$ and $m_i \in \mathbb{N}^*$ such that

$$d((f^{\mathcal{P}})^{-m_i}(w_i), p) < \delta \quad \text{and} \quad (f^{\mathcal{P}})^{m_i}(\mathcal{F}^{ss}_\epsilon((f^{\mathcal{P}})^{-m_i}(w_i), f)) \subset f^{-i}(W). \tag{8}$$

Thus,

$$\mathcal{F}^{ss}_\epsilon((f^{\mathcal{P}})^{-m_i}(w_i), f) \pitchfork W^{cu}_r(p, f) \neq \emptyset.$$

Using λ-Lemma, there exists $l_i \in \mathbb{N}^*$ such that

$$(f^{\mathcal{P}})^{-n}(\mathcal{F}^{ss}_\epsilon((f^{\mathcal{P}})^{-m_i}(w_i), f)) \pitchfork W^{cu}_r(f^m(y^u), f) \neq \emptyset, \quad \forall n \geq l_i.$$

Set $l' = max\{l_i; i = 0, 1, \ldots, \mathcal{P} - 1\}$. Then,

$$(f^{\mathcal{P}})^{-n}(\mathcal{F}^{ss}_\epsilon((f^{\mathcal{P}})^{-m_i}(w_i), f)) \pitchfork W^{cu}_r(f^m(y^u), f) \neq \emptyset, \quad \forall n \geq l'. \tag{9}$$

Assertion: $W^{cu}_r(f^m(y^u), f) \subset f^{k_0}(\mathcal{U})$.

Proof. We know that $E^{cu} = E^c \oplus E^u$ is a dominated decomposition. Thus, there is $L > 0$ such that $\| Df^{-n}_{|E^{cu}} \| \leq L \sup\{\| Df^{-n}_{|E^c} \|, \| Df^{-n}_{|E^u} \|\}$. For the sake of simplicity, we will assume that $L = 1$. From (6) we have that

$$\prod_{j=0}^{n} \| Df^{-1}_{|E^c(f^{-j}(f^m(y^u)))} \| < \lambda^n, \quad 0 \leq n \leq m$$

and therefore

$$\prod_{j=0}^{n} \| Df^{-1}_{|E^{cu}(f^{-j}(f^m(y^u)))} \| < \lambda^n, \quad 0 \leq n \leq m.$$

From Lemma 2.3 we conclude that $f^{-m}(W^{cu}_r(f^m(y^u), f)) \subset W^{cu}_{\lambda_1^m r}(y^u, f) \subset W^{cu}_\eta(y^u, f)$ and hence, using (7), we have $f^{-k_0}(W^{cu}_r(f^m(y^u), f)) \subset \mathcal{U}$ and the assertion is completed. □

Using the assertion, (8) and (10) we obtain

$$\emptyset \neq (f^{\mathcal{P}})^{-n}(\mathcal{F}^{ss}_\epsilon((f^{\mathcal{P}})^{-m_i}(w_i), f)) \pitchfork W^{cu}_r(f^m(y^u), f) \subset$$
$$\subset (f^{\mathcal{P}})^{-n}((f^{\mathcal{P}})^{-m_i}(f^{-i}(W))) \cap f^{k_0}(\mathcal{U}), \quad \forall n \geq l', \quad \forall i \in \{0, 1, \ldots, \mathcal{P} - 1\}.$$

Hence

$$(f^{\mathcal{P}})^{-(n+m_i)}(f^{-i}(W)) \cap f^{k_0}(\mathcal{U}) \neq \emptyset, \quad \forall n \geq l', \quad \forall i \in \{0, 1, \ldots, \mathcal{P} - 1\}.$$

Taking $n' = \max\{l' + m_i; i = 0, 1, \ldots, \mathcal{P} - 1\}$ we have that

$$(f^{\mathcal{P}})^{-n}(f^{-i}(W)) \cap f^{k_0}(\mathcal{U}) \neq \emptyset, \quad \forall n \geq n', \quad \forall i = 0, 1, \ldots, \mathcal{P} - 1.$$

Finally,

$$W \cap f^{p_n+k_0+i}(\mathcal{U}) \neq \emptyset, \quad \forall n \geq n', \quad \forall i = 0, 1, \ldots, \mathcal{P}-1,$$

and the proof is completed. □

4. Robust Transitivity

Unlike the results in preceding section our next Theorem do not have in the hypotheses the condition of Accessibility, leading us to suspect that Property SH might be sufficient to guarantee robust transitivity.

Lemma 4.1. *Let $f \in Diff^r(M)$ be a partially hyperbolic diffeomorphism. There exist $\epsilon > 0$ such that given $r > 0$ there are $\delta > 0$ and a neighborhood $\mathcal{V}_0$ of f such that for any $x, y \in M$ with $d(x, y) < \delta$ it follows that*

- $\mathcal{F}^{ss}_\epsilon(x, g) \pitchfork \mathcal{W}^{cu}_r(y, g) \neq \emptyset$
- •• $\mathcal{F}^{uu}_\epsilon(x, g) \pitchfork \mathcal{W}^{cs}_r(y, g) \neq \emptyset,$ *for any* $g \in \mathcal{V}_0$.

Proof. The result follows from Stable Manifold Theorem. □

Theorem 4.1. *Let M be a compact Riemannian manifold and let $f \in Diff^r(M)$ be a partially hyperbolic diffeomorphism, non-hyperbolic, transitive. If f and f^{-1} satisfy Property SH then f is robustly transitive .*

Proof. For the sake of clarity we divide the proof in two steps. The first step deals with the construction of an appropiate neighborhood $\mathcal{V}$ of f. In the second step we prove that any diffeomorphism in $\mathcal{V}$ is transitive.

Step 1

From Theorem 2.2 there exist a neighborhood $\mathcal{V}_1(f)$, $C_0 > 0$ and $\sigma_0 > 1$ such that for every $g \in \mathcal{V}_1$ and $x \in M$ there exists a point $y \in \mathcal{F}^{uu}_1(x, g)$ such that

$$m\{Dg^n_{|E^c(g^l(y))}\} > C_0\sigma_0^n \quad \text{for any} \quad n > 0, \quad l > 0. \tag{10}$$

Analogously there exist a neighborhood $\mathcal{V}_2(f^{-1})$, $C_1 > 0$ and $\sigma_1 > 1$ such that for every $h \in \mathcal{V}_2$ and $x \in M$ there exists a point $y \in \mathcal{F}^{uu}_1(x, h)$ such that

$$m\{Dh^n_{|E^c(h^l(y))}\} > C_1\sigma_1^n \quad \text{for any} \quad n > 0, \quad l > 0.$$

Take $C = \min\{C_0, C_1\} > 0$ and $\sigma = \min\{\sigma_0, \sigma_1\} > 1$. Thus, for every $g \in \mathcal{V}_1 \cup \mathcal{V}_2$ and $x \in M$ there exists a point $y \in \mathcal{F}^{uu}_1(x, g)$ such that

$$m\{Dg^n_{|E^c(g^l(y))}\} > C\sigma^n \quad \text{for any} \quad n > 0, \quad l > 0.$$

We may assume that $C = 1$. Otherwise we take a fixed power of every $g \in \mathcal{V}_1 \cup \mathcal{V}_2$. Let $\mathcal{V}_3(f) \subset \mathcal{V}_1$ be a neighborhood of f such that if $g \in \mathcal{V}_3$ then $g^{-1} \in \mathcal{V}_2$. Let $\lambda = \sigma^{-1}$, fix $0 < \lambda < \lambda_1 < 1$ and let $r > 0$ be as in Lemma 2.3. Consider $\epsilon > 0$ given by Stable Manifold Theorem and let $r > 0$ be as above. Take $\delta > 0$ and take $\mathcal{V}_4(f) \subset \mathcal{V}_3$ a neighborhood of f as in Lemma 4.1. Since f is transitive there exists a point $z \in M$ such that $\{f^n(z); n \in \mathbb{N}\}$ and $\{f^{-n}(z); n \in \mathbb{N}\}$ are dense in M. Therefore

$$M = \bigcup_{n\in\mathbb{N}} B(f^n(z), \frac{\delta}{2})$$

and by compactness there exist positive integers $n_1 < \cdots < n_l$ such that

$$\bigcup_{i=1}^{l} B(f^{n_i}(z), \frac{\delta}{2}) = M.$$

Next, choose a positive integer m_0 and a neighborhood $\mathcal{V}_5(f) \subset \mathcal{V}_4$ such that if $m \geq m_0$, $g \in \mathcal{V}_5$ and $q \in M$ then

- $g^m(\mathcal{F}^{ss}_\epsilon(q,g)) \subset B(g^m(q), \dfrac{\delta}{6})$
- $g^{-m}(\mathcal{F}^{uu}_\epsilon(q,g)) \subset B(g^{-m}(q), \dfrac{\delta}{6})$

Affirmation 1. *For each $i = 2, \dots, l$ there exists $m_i \in \mathbb{Z}^*_+$ satisfying:*

(i) $f^{m_i}(z) \in B(f^{n_i}(z), \frac{\delta}{6})$ *for* $i = 2, \dots, l$

(ii) $m_2 > n_1 + m_0$
$m_i > m_{i-1} + m_0$ *for* $i = 3, \dots, l$

Proof. It follows by density of $\{f^n(z); n \in \mathbb{N}\}$ in M. $\square$

Affirmation 2. *For each $i = 2, \dots, l$ there exists $\overline{m_i} \in \mathbb{Z}^*_-$ satisfying:*

(iii) $f^{\overline{m_i}}(z) \in B(f^{n_i}(z), \frac{\delta}{6})$ *for* $i = 2, \dots, l$

(iv) $\overline{m_2} < n_1 - m_0$
$\overline{m_i} < \overline{m_{i-1}} - m_0$ *for* $i = 3, \dots, l$

Proof. It follows by density of $\{f^{-n}(z); n \in \mathbb{N}\}$ in M. $\square$

Set $l_0 = \max\{n_l, m_2, m_3, m_4, \dots, m_l, -\overline{m_2}, -\overline{m_3}, \dots, -\overline{m_l}\}$.
Observe that $l_0 \geq n_l > n_i$ for $i = 1, \dots, l-1$.
Take a neighborhood $\mathcal{V}(f) \subset \mathcal{V}_5$ such that $d_{C^0}(g^n, f^n) < \frac{\delta}{6}$, for any $n \in \mathbb{Z}$ with $|n| \leq l_0$, for any $g \in \mathcal{V}$.

Step 2

We will prove that any $g \in \mathcal{V}$ is transitive. Take two arbitrary open sets $\mathcal{U}, \mathcal{W} \subset M$. Let us prove that there exists a positive integer k_0 such that $g^{k_0}(\mathcal{U}) \cap \mathcal{W} \neq \varnothing$. Let $u \in \mathcal{U}$ and $w \in \mathcal{W}$. Let $\beta > 0$ be such that $\mathcal{F}^{uu}_\beta(u,g) \subset \mathcal{U}$ and $\mathcal{F}^{uu}_\beta(w,g^{-1}) \subset \mathcal{W}$. Take n_0 such that $g^{n_0}(\mathcal{F}^{uu}_{\frac{\beta}{2}}(u,g)) \supset \mathcal{F}^{uu}_1(g^{n_0}(u),g)$ and $g^{-n_0}(\mathcal{F}^{uu}_{\frac{\beta}{2}}(w,g^{-1})) \supset \mathcal{F}^{uu}_1(g^{-n_0}(w),g^{-1})$. Consider $y \in \mathcal{F}^{uu}_1(g^{n_0}(u),g)$ and $x \in \mathcal{F}^{uu}_1(g^{-n_0}(w),g^{-1})$ satisfying:

$$\begin{cases} m\{Dg^n_{|E^c}(g^l(y))\} > \sigma^n & \text{for any} \quad n > 0, \quad l > 0 \\ m\{Dg^{-n}_{|E^c}(g^{-l}(x))\} > \sigma^n & \text{for any} \quad n > 0, \quad l > 0. \end{cases} \tag{11}$$

Observe that

- $\mathcal{F}^{uu}_{\frac{\beta}{2}}(g^{-n_0}(y),g) \subset \mathcal{U}$ because $g^{-n_0}(y) \in \mathcal{F}^{uu}_{\frac{\beta}{2}}(u,g) \subset \mathcal{F}^{uu}_\beta(u,g) \subset \mathcal{U}$
- $\mathcal{F}^{uu}_{\frac{\beta}{2}}(g^{n_0}(x),g^{-1}) \subset \mathcal{W}$ because $g^{n_0}(x) \in \mathcal{F}^{uu}_{\frac{\beta}{2}}(w,g^{-1}) \subset \mathcal{F}^{uu}_\beta(w,g^{-1}) \subset \mathcal{W}$.

Thus, there exist $A \subset \mathcal{U}$ a neighborhood of $g^{-n_0}(y)$ and $B \subset \mathcal{W}$ a neighborhood of $g^{n_0}(x)$ such that

$$\begin{cases} \mathcal{F}^{uu}_{\frac{\beta}{2}}(a,g) \subset \mathcal{U} & \text{for any} \quad a \in A \\ \mathcal{F}^{uu}_{\frac{\beta}{2}}(b,g^{-1}) \subset \mathcal{W} & \text{for any} \quad b \in B. \end{cases} \tag{12}$$

Let $\eta > 0$ be such that

$$\left\{ \begin{array}{l} g^{-n_0}(\mathcal{W}^{cu}_{\eta}(y,g)) \subset A \subset \mathcal{U} \\ g^{n_0}(\mathcal{W}^{cu}_{\eta}(x,g^{-1})) \subset B \subset \mathcal{W} \end{array} \right. \tag{13}$$

Next, choose a positive integer m' such that $\lambda_1^{m'} r < \eta$ and

$$\left\{ \begin{array}{l} g^{m'+n_0}(\mathcal{F}^{uu}_{\frac{\beta}{2}}(q,g)) \supset \mathcal{F}^{uu}_{\epsilon}(g^{m'+n_0}(q),g) \quad \text{for any} \quad q \in M \\ g^{-(m'+n_0)}(\mathcal{F}^{uu}_{\frac{\beta}{2}}(q,g^{-1})) \supset \mathcal{F}^{uu}_{\epsilon}(g^{-(m'+n_0)}(q),g^{-1}) \quad \text{for any} \quad q \in M \end{array} \right. \tag{14}$$

Set $k' = n_0 + m'$. Thus, using (11), we get

$$\prod_{j=0}^{n-1} \| Dg^{-1}_{|_{E^c(g^{-j}(y))}} \| < \lambda^n, \qquad 0 \le n \le m'$$
$$\prod_{j=0}^{n-1} \| Dg_{|_{E^c(g^{j}(x))}} \| < \lambda^n, \qquad 0 \le n \le m'$$

and therefore

$$\prod_{j=0}^{n-1} \| Dg^{-1}_{|_{E^{cu}(g^{-j}(y))}} \| < \lambda^n, \qquad 0 \le n \le m'$$
$$\prod_{j=0}^{n-1} \| Dg_{|_{E^{cu}(g^{j}(x))}} \| < \lambda^n, \qquad 0 \le n \le m'$$

From Lemma 2.3 we conclude that

$$\left\{ \begin{array}{l} g^{-m'}(\mathcal{W}^{cu}_{r}(g^{m'}(y),g)) \subset \mathcal{W}^{cu}_{\lambda_1^{m'} r}(y,g) \subset \mathcal{W}^{cu}_{\eta}(y,g) \\ g^{m'}(\mathcal{W}^{cu}_{r}(g^{-m'}(x),g^{-1})) \subset \mathcal{W}^{cu}_{\lambda_1^{m'} r}(x,g^{-1}) \subset \mathcal{W}^{cu}_{\eta}(x,g^{-1}) \end{array} \right. \tag{15}$$

and hence, using (13), we have

$$\left\{ \begin{array}{l} g^{-k'}(\mathcal{W}^{cu}_{r}(g^{m'}(y),g)) \subset A \subset \mathcal{U} \\ g^{k'}(\mathcal{W}^{cu}_{r}(g^{-m'}(x),g^{-1})) \subset B \subset \mathcal{W}. \end{array} \right. \tag{16}$$

Particularly, it follows

$$\left\{ \begin{array}{l} \mathcal{W}^{cu}_{r}(g^{m'}(y),g) \subset g^{k'}(\mathcal{U}) \\ \mathcal{W}^{cu}_{r}(g^{-m'}(x),g^{-1}) \subset g^{-k'}(\mathcal{W}). \end{array} \right. \tag{17}$$

Moreover, if $p \in \mathcal{W}^{cu}_{r}(g^{m'}(y),g)$, $q \in \mathcal{W}^{cu}_{r}(g^{-m'}(x),g^{-1})$ then $g^{-k'}(p) \in A$ and $g^{k'}(q) \in B$ due to (16). Thus, from (12),

$$\mathcal{F}^{uu}_{\frac{\beta}{2}}(g^{-k'}(p),g) \subset \mathcal{U} \quad \text{and} \quad \mathcal{F}^{uu}_{\frac{\beta}{2}}(g^{k'}(q),g^{-1}) \subset \mathcal{W}$$

and hence, (14) imply

$$\begin{cases} \mathcal{F}^{uu}_{\epsilon}(p,g) \subset g^{k'}(\mathcal{F}^{uu}_{\frac{\beta}{2}}(g^{-k'}(p),g)) \subset g^{k'}(\mathcal{U}) \\ \mathcal{F}^{uu}_{\epsilon}(q,g^{-1}) \subset g^{-k'}(\mathcal{F}^{uu}_{\frac{\beta}{2}}(g^{k'}(q),g^{-1})) \subset g^{-k'}(\mathcal{W}). \end{cases} \tag{18}$$

Finally, from (17) and (18) we conclude that

$$\begin{aligned} &(i)\ \ \mathcal{W}^{cu}_r(g^{m'}(y),g) \subset g^{k'}(\mathcal{U}) \\ &(ii)\ \ \mathcal{F}^{uu}_{\epsilon}(p,g) \subset g^{k'}(\mathcal{U}),\ \forall p \in \mathcal{W}^{cu}_r(g^{m'}(y),g) \\ &(iii)\ \ \mathcal{W}^{cs}_r(g^{-m'}(x),g) \subset g^{-k'}(\mathcal{W}) \\ &(iv)\ \ \mathcal{F}^{ss}_{\epsilon}(q,g) \subset g^{-k'}(\mathcal{W}),\ \forall q \in \mathcal{W}^{cs}_r(g^{-m'}(x),g) \end{aligned}$$

For the sake of simplicity, we will denote $g^{m'}(y)$ for $\bar{y}$ and $g^{-m'}(x)$ for $\bar{x}$.
Since $M = \bigcup_{i=1}^{l} B(f^{n_i}(z), \frac{\delta}{2})$, there are $i, j \in \{1, \ldots, l\}$ such that

$$\bar{y} \in B(f^{n_i}(z), \frac{\delta}{2}) \ \text{ and } \ \bar{x} \in B(f^{n_j}(z), \frac{\delta}{2}).$$

- Case $i = j$

In this case, $d(\bar{x}, \bar{y}) < \delta$. Thus, using Lemma 4.1, $\mathcal{F}^{uu}_{\epsilon}(\bar{y}, g) \pitchfork W^{cs}_r(\bar{x}, g) \neq \varnothing$. Moreover, by (ii) and by (iii), we have that $\mathcal{F}^{uu}_{\epsilon}(\bar{y}, g) \subset g^{k'}(\mathcal{U})$ and $\mathcal{W}^{cs}_r(\bar{x}, g) \subset g^{-k'}(\mathcal{W})$, and hence, $g^{k'}(\mathcal{U}) \cap g^{-k'}(\mathcal{W}) \neq \emptyset$, i.e., $g^{2k'}(\mathcal{U}) \cap \mathcal{W} \neq \emptyset$.

Next, we will prove the case $i < j$. The case $i > j$ is similar.

- Case $i < j$

First assume $i > 1$. Consider $j = i + k$ for $k = 1, 2, \ldots, l - i$. In this case we have that

$$\begin{aligned} d(\bar{y}, g^{m_i}(z)) &\leq d(\bar{y}, f^{n_i}(z)) + d(f^{n_i}(z), f^{m_i}(z)) + d(f^{m_i}(z), g^{m_i}(z)) \\ &< \frac{\delta}{2} + \frac{\delta}{6} + \frac{\delta}{6} < \delta \end{aligned}$$

and therefore

$$\mathcal{F}^{ss}_{\epsilon}(g^{m_i}(z), g) \pitchfork W^{cu}_r(\bar{y}, g) \neq \emptyset.$$

Take $p \in \mathcal{F}^{ss}_{\epsilon}(g^{m_i}(z), g) \pitchfork W^{cu}_r(\bar{y}, g)$. Since

$$\begin{aligned} m_j - m_i = m_{i+k} - m_i &= (m_{i+k} - m_{i+(k-1)}) + (m_{i+(k-1)} - m_{i+(k-2)}) \\ &+ \cdots + (m_{i+1} - m_i) > k m_0 > m_0, \end{aligned}$$

$$g^{m_j - m_i}(\mathcal{F}^{ss}_{\epsilon}(g^{m_i}(z), g)) \subset B(g^{m_j}(z), \frac{\delta}{6}),$$

and from this it follows that

$$g^{m_j - m_i}(p) \in B(g^{m_j}(z), \frac{\delta}{6}).$$

Thus,

$$\begin{aligned} d(g^{m_j-m_i}(p),\bar{x}) &\leq d(g^{m_j-m_i}(p),g^{m_j}(z))+d(g^{m_j}(z),f^{m_j}(z)) \\ &+ d(f^{m_j}(z),f^{n_j}(z))+d(f^{n_j}(z),\bar{x}) \\ &< \frac{\delta}{6}+\frac{\delta}{6}+\frac{\delta}{6}+\frac{\delta}{2}=\delta \end{aligned}$$

and from Lemma 4.1, we get

$$(v) \qquad \mathcal{F}^{uu}_{\epsilon}(g^{m_j-m_i}(p),g) \pitchfork \mathcal{W}^{cs}_r(\bar{x},g) \neq \emptyset.$$

Using that $(m_j - m_i) > 0$ and $p \in \mathcal{W}^{cu}_r(\bar{y}, g)$ and using (ii), we have that

$$\mathcal{F}^{uu}_{\epsilon}(g^{m_j-m_i}(p),g) \subset g^{m_j-m_i}(\mathcal{F}^{uu}_{\epsilon}(p,g)) \subset g^{m_j-m_i}(g^{k'}(\mathcal{U})).$$

From (iii) and (v) we conclude that

$$g^{m_j-m_i}(g^{k'}(\mathcal{U})) \cap g^{-k'}(\mathcal{W}) \neq \emptyset.$$

In this case the proof is completed.

Now, assume $i = 1$.
Consider $j = i + k$ for $k = 1, 2, \dots, l - i$. In this case we have that

$$\begin{aligned} d(\bar{y},g^{n_1}(z)) &\leq d(\bar{y},f^{n_1}(z))+d(f^{n_1}(z),g^{n_1}(z)) \\ &< \frac{\delta}{2}+\frac{\delta}{6}<\delta \end{aligned}$$

and therefore

$$\mathcal{F}^{ss}_{\epsilon}(g^{n_1}(z),g) \pitchfork \mathcal{W}^{cu}_r(\bar{y},g) \neq \emptyset.$$

Take $p \in \mathcal{F}^{ss}_{\epsilon}(g^{n_1}(z), g) \pitchfork \mathcal{W}^{cu}_r(\bar{y}, g)$. Since

$$\begin{aligned} m_j - n_1 = m_{1+k} - n_1 &= (m_{1+k}-m_k)+(m_k-m_{k-1})+\cdots+(m_2-n_1) \\ &> km_0 > m_0, \\ g^{m_j-n_1}(\mathcal{F}^{ss}_{\epsilon}(g^{n_1}(z),g)) &\subset B(g^{m_j}(z),\frac{\delta}{6}), \end{aligned}$$

and from this it follows that

$$g^{m_j-n_1}(p) \in B(g^{m_j}(z),\frac{\delta}{6}).$$

Thus,

$$\begin{aligned} d(g^{m_j-n_1}(p),\bar{x}) &\le d(g^{m_j-n_1}(p),g^{m_j}(z))+d(g^{m_j}(z),f^{m_j}(z)) \\ &+d(f^{m_j}(z),f^{n_j}(z))+d(f^{n_j}(z),\bar{x}) \\ &<\frac{\delta}{6}+\frac{\delta}{6}+\frac{\delta}{6}+\frac{\delta}{2}=\delta \end{aligned}$$

and from Lemma 4.1 , we get

$$(vi) \qquad \mathcal{F}^{uu}_{\epsilon}(g^{m_j-n_1}(p),g)\pitchfork \mathcal{W}^{cs}_r(\bar{x},g)\neq\emptyset.$$

However,

$$\mathcal{F}^{uu}_{\epsilon}(g^{m_j-n_1}(p),g)\subset g^{m_j-n_1}(\mathcal{F}^{uu}_{\epsilon}(p,g))\subset g^{m_j-n_1}(g^{k'}(\mathcal{U}))$$

due to

$$m_j-n_1>0,\ \ p\in\mathcal{W}^{cu}_r(\bar{y},g)\ \text{ and }\ (ii).$$

From (iii) and (vi) we conclude that

$$g^{m_j-n_1}(g^{k'}(\mathcal{U}))\cap g^{-k'}(\mathcal{W})\neq\emptyset.$$

Hence, the case $i<j$ is completed. The case $i>j$ follows by symmetry, and the proof of Theorem is completed.

□

5. Examples

5.1. **Shub's example.**

Now we will show that Shub's example satisfies the conditions in Theorems 3.2, 3.3. As a consequence its accessibility classes are robustly dense in $\mathbb{T}^4$ and it is topologically mixing.

Let us remember a few details concerning the construction of Shub's example. We will follow the notation in [13].

Let $f:\mathbb{T}^2\to\mathbb{T}^2$ be an Anosov diffeomorphism having two fixed points p and q. Since f is Anosov, $T\mathbb{T}^2=E^{ss}\oplus E^{uu}$ with $\| Df_{|E^{ss}}\|<\lambda<1$ and $\| Df^{-1}_{|E^{uu}}\|<\lambda$.

Now, consider a smooth family of torus diffeomorphisms $g_x:\mathbb{T}^2\to\mathbb{T}^2$ indexed in $x\in\mathbb{T}^2$ such that the following hold:

- $T\mathbb{T}^2=E^s(g_x)\oplus E^c(g_x)$ invariant under $D(g_x)$ and such that $\| D(g_x)_{|E^s(g_x)}\|<\mu<\mu_1<1$ and $\mu<\mu_1<\| D(g_x)_{|E^c(g_x)}\|\le\mu^{-1}$;
- for all $x\in\mathbb{T}^2, g_x$ preserves a cone field C^s and C^{cu};
- g_p is Anosov and $g_x=g_p$ outside a small disc of $\mathbb{T}^2$;
- g_q is a DA (derived from Anosov) map and $g_x=DA$ inside a smaller disc;
- $g_x(p)=p$ for every x and p is an atractor for g_q.

We assume (taking a power of f if necessary) that $\lambda<\mu$. Next, we define the map on $\mathbb{T}^4$:

$$F:\mathbb{T}^2\times\mathbb{T}^2\to\mathbb{T}^2\times\mathbb{T}^2,\quad F(x,y)=(f(x),g_x(y)).$$

F constructed this way is partially hyperbolic with $T_{(x,y)}\mathbb{T}^4 = E^{ss}(x,y) \oplus E^s(x,y) \oplus E^c(x,y) \oplus E^u(x,y)$. Let us set $E^s = E^{ss} \oplus E^s$.

It is a known fact that F is transitive.

In [13] it is used that

$$W^s(\{p\} \times \mathbb{T}^2) = \bigcup_{z \in \mathbb{T}^2} W^{ss}(p,z) = W^{ss}(p,f) \times \mathbb{T}^2 \tag{19}$$

and hence is dense in $\mathbb{T}^2 \times \mathbb{T}^2$. So for some $\epsilon > 0$ and $r > 0$ we have for every $(z,w) \in \mathbb{T}^2 \times \mathbb{T}^2$ that

$$W^{uu}_\epsilon((z,w)) \cap W^s_r(\{p\} \times \mathbb{T}^2) \neq \varnothing, \tag{20}$$

which guarantees the Property SH because for any point y in the intersection holds that for some uniform n_0 the iterates $F^n(y), n \geq n_0$ are contained in the region where F is the product of two linear Anosovs.

Finally, we will verify that F is accessible.

Relations (19) and (20) reduce our problem to prove that any two points (p,w) and (p,y) are accessible.

Observe that, since g_p is Anosov, we have that

$$W^s(p,z) = \bigcup_{y \in W^s(z,g_p)} W^{ss}(p,y) \tag{21}$$

and

$$W^u(p,z) = \bigcup_{y \in W^u(z,g_p)} W^{uu}(p,y) \tag{22}$$

From (21) it follows that if $x \in W^s(z,g_p)$ then $W^{ss}(p,x) \subset W^s(p,z)$ and so $(p,x) \in W^s(p,z)$. Analogously from (22) it follows that if $x \in W^u(z,g_p)$ then $W^{uu}(p,x) \subset W^u(p,z)$ and so $(p,x) \in W^u(p,z)$.

Since g_p is Anosov we can take $u \in W^s(w,g_p) \cap W^u(y,g_p) \neq \varnothing$. Hence $(p,u) \in W^s(p,w) \cap W^u(p,y)$ and the points (p,w) and (p,y) are accessible.

5.2. **Mañé's example.**

Let $f : \mathbb{T}^3 \to \mathbb{T}^3$ be a partially hyperbolic diffeomorphism, such that there exists an open set of nonhyperbolic diffeomorphisms $\mathcal{V} \subset Diff^1(\mathbb{T}^3)$, containing f, whose elements exhibit no homoclinic tangencies. Suppose that f has two hyperbolic fixed points with different indices, satisfies Property SH and its strong stable foliation $\mathcal{F}^{ss}(f)$ is minimal.

Here we will prove that there exists a transitive diffeomorphism φ as close as we wish to f in C^1 topology, such that φ and φ^{-1} satisfy Property SH.

In fact, from Theorems 2.2 and 2.3, there exists a neighborhood $\mathcal{U}(f) \subset \mathcal{V}$ such that for every $g \in \mathcal{U}(f)$ the strong stable foliation of g is minimal and g satisfies Property SH.

By Hayashi's Connecting Lemma there exists $h \in \mathcal{U}(f)$ with a heterodimensional cycle of codimension one associated to two hyperbolic fixed points P_h and Q_h. (See Lemma 2.5 in [3]).

Consequently, by Proposition 2.2, there exists $\varphi \in \mathcal{U}(f)$ close to h with a *cu*-blender.

As the strong stable foliation $\mathcal{F}^{ss}(\varphi)$ is minimal, because $\varphi \in \mathcal{U}(f)$, then, by the Proposition 2.1, φ^{-1} has the Property SH.

Finally, φ is transitive and exhibits the Property SH, again because $\varphi \in \mathcal{U}(f)$.

Remember that Mañé's example gives an open set $\mathcal{V}$ of nonhyperbolic diffeomorphisms far from homoclinic tangencies (See [10]). In this case, Mañé's example, for $f \in \mathcal{V}$ we have

$\cdot\, f$ has two saddles of different indices whose stable and unstable manifolds are dense in $\mathbb{T}^3$,

$\cdot\, f$ satisfies Property SH and its strong stable foliation is minimal (See [13]).

Hence, there exists a transitive perturbation of Mañé's example in $\mathbb{T}^3$ such that it and its inverse satisfy Property SH.

5.3. **A wider scenario for SH on the inverse.**

Next we generalize the argument used on Mañé's example to get perturbations whose inverses satisfy Property SH, based on some known facts.

Claim 1. *Let $f \in Diff^1(M)$ be a diffeomorphism such that its periodic points are C^1-robustly hyperbolic and $\Omega(f) = M$. Then f is Anosov.*

Proof. See [8].

□

Let M be a smooth compact boundaryless three dimensional manifold and $\mathcal{T}$ the set of non Anosov robustly transitive partially hyperbolic diffeomorphisms in M.

Claim 2. *There is a dense subset $\mathcal{A}$ of $\mathcal{T}$ such that for every $f \in \mathcal{A}$ there exists a pair of hyperbolic periodic points with different indices.*

Proof. It follows from Claim 1 and [9]. □

Claim 3. *There exists a dense subset $\mathcal{B}$ of $\mathcal{T}$ such that every $f \in \mathcal{B}$ has a heterodimensional cycle of codimension one.*

Proof. It follows from Claim 2 and [3]. □

Claim 4. *There exists an open and dense subset $\mathcal{D}$ of $\mathcal{T}$ such that every $g \in \mathcal{D}$ has a cs*-blender *and a cu*-blender.

Proof. It follows from Claim 3 and Proposition 2.2. □

Let us denote by $\mathcal{T}'$ the subset of $\mathcal{T}$ consisting of the diffeomorphisms with strong stable robustly minimal foliation.

Proposition 5.1. *There exists an open and dense subset $\mathcal{D}'$ of $\mathcal{T}'$ such that for every $g \in \mathcal{D}'$ we have that g^{-1} satisfies Property SH.*

Proof. Just apply Claim 4 and Proposition 2.1. □

6. Property SH and density of periodic points

Here we prove that for a diffeomorphism exhibiting Property SH and minimality of the strong stable foliation the set of its periodic points is dense. So both transitivity and density of the periodic points are robust properties under the hypotheses of Property SH and minimality of the strong stable foliation.

As can be seen in the following proof the condition of minimality can be weakened to some kind of d-minimality of the strong stable foliation for some appropriate d.

This result together with those proved before gives us a sense of the potential of the Property SH, raising up a few interesting questions mentioned in the introduction to this work.

Remark 6.1. In the next result we will assume that f is a partially hyperbolic diffeomorphism exhibiting Property SH, as in Definition 2.4. Changing f by a power of itself, we can assume that there is $\sigma > 1$ such that for any $x \in M$ there exists $y^u \in \mathcal{F}_1^{uu}(x, f)$ such that

$$m\{Df^n_{|E^c(f^l(y^u))}\} > \sigma^n \text{ for any } n > 0, \;\; l > 0. \tag{23}$$

Theorem 6.1. *Let $f \in Diff^r(M)$ be a partially hyperbolic diffeomorphism exhibiting Property SH and such that the strong stable foliation is minimal. Then, $\overline{Per(f)} = M$.*

Proof. Let SH be defined by:

$$SH = \{y \in M : m\{Df^n_{|E^c(f^l(y))}\} > \sigma^n \text{ for any } n > 0, \;\; l > 0\}. \tag{24}$$

Lemma 6.1. *If $SH \subset \overline{Per(f)}$ then $M \subset \overline{Per(f)}$.*

Proof. Let $x \in M$ and V an open set containing x. Let $\beta > 0$ be such that $\mathcal{F}^{uu}_\beta(x, f) \subset V$. Take l_0 such that $\mathcal{F}^{uu}_1(f^{l_0}(x), f) \subset f^{l_0}(\mathcal{F}^{uu}_\beta(x, f))$. Consider $h \in \mathcal{F}^{uu}_1(f^{l_0}(x), f) \cap SH$. Let U be an open set containing h such that $f^{-l_0}(U) \subset V$. It is enough to take p_0 a periodic point in U and consequently $f^{-l_0}(p_0)$ in V. □

Analogously if $f^n(SH) \subset \overline{Per(f)}$ for some $n \in \mathbb{N}^*$ then $M \subset \overline{Per(f)}$.
From now on our goal will be to prove that $SH \subset \overline{Per(f)}$.

Lemma 6.2. *Let $\epsilon > 0$ be given by the Stable Manifold Theorem and $r > 0$ suficiently small. For any $\epsilon' < \epsilon, r' < r$ there exists $d' = d'(\epsilon', r') > 0$ such that for any pair of points $x, y \in M$ with $dist(x, y) < d'$ the manifolds $W^{cu}_{r'}(x, f)$ and $\mathcal{F}^{ss}_{\epsilon'}(y, f)$ intersect transversely in exactly one point.*

Let us fix ϵ', r' and d' as in Lemma 6.2.

Definition 6.1. We will call a cylinder any open set $W \subset M$, with $diam(W) < d'$, which is the domain of some local chart $\eta : M \to \mathbb{R}^n$ trivializing the strong stable foliation such that $W^{cu}_{r'}(y, f) \nsubseteq W$ and $\mathcal{F}^{ss}_{\epsilon'}(y, f) \nsubseteq W$ for any $y \in W$.

Lemma 6.3. *For every $x \in M$ there exists a cylinder containing x.*

Proof. First observe that there exists a local chart $(\widetilde{W}, \widetilde{\eta})$, trivializing the strong stable foliation, with $x \in \widetilde{W}$ and such that $W^{cu}_{r'}(x, f) \nsubseteq \widetilde{W}, \mathcal{F}^{ss}_{\epsilon'}(x, f) \nsubseteq \widetilde{W}$. Now by the continuous dependence of the manifolds $W^{cu}_{r'}(y, f)$ and $\mathcal{F}^{ss}_{\epsilon'}(y, f)$ on the point y, follows the existence of an open set $\widetilde{\widetilde{W}} \subset \widetilde{W}$ containing x and such that $W^{cu}_{r'}(z, f) \nsubseteq \widetilde{W}, \mathcal{F}^{ss}_{\epsilon'}(z, f) \nsubseteq \widetilde{W}$ for any $z \in \widetilde{\widetilde{W}}$.

Finally take a local chart trivializing the strong stable foliation (W, η), with $x \in W \subset \widetilde{\widetilde{W}}$ and $diam(W) < d'$. $\square$

Notice that there exists a base B of open sets of the topology of manifold M whose elements are cylinders.

Let $\mathcal{C}$ be an open covering of cylinders of the manifold M and L its Lebesgue number.

Lemma 6.4. *Let C be a cylinder and let $\eta : C \to U^{cu} \times V^{ss}$ be a local chart trivializing the strong stable foliation, where U^{cu}, V^{ss} are open sets in $\mathbb{R}^{c+uu}, \mathbb{R}^{ss}$ respectively and $0 \in \eta(C)$. Let $\pi : U^{cu} \times V^{ss} \to U^{cu} \times \{0\}$ be the projection of $\mathbb{R}^{c+uu+ss}$ on $\mathbb{R}^{c+uu} \times \{0\}$. Let $h \in C$ and $\hat{r} > 0$ be such that $\overline{\mathcal{F}^{ss}_{\hat{r}}(h, f)} \subset C$ and let $\hat{\delta} > 0$ be such that $\overline{W^{cu}_{\hat{\delta}}(y, f)} \subset C$ for any $y \in \overline{\mathcal{F}^{ss}_{\hat{r}}(h, f)}$. Denote $\pi(\eta(h))$ by (h_{cu}, h_{ss}). Then the following hold:*

(i) $\pi_{|\eta(\overline{W^{cu}_{\hat{\delta}}(y,f)})}$ is an homeomorphism on its image for any $y \in \overline{\mathcal{F}^{ss}_{\hat{r}}(h, f)}$.

(ii) There exists an open ball $B \subset U^{cu}$ centered at h_{cu} such that

$$B \times \{0\} \subset \bigcap_{y \in \overline{\mathcal{F}^{ss}_{\hat{r}}(h,f)}} \pi(\eta(W^{cu}_{\hat{\delta}}(y, f))).$$

(iii) There exists $0 < \overline{\delta} < \hat{\delta}$ such that for any $y_1, y_2 \in \overline{\mathcal{F}^{ss}_{\hat{r}}(h, f)}$ there exists a continuous map $\pi_{y_1,y_2} : W^{cu}_{\overline{\delta}}(y_1, f) \to W^{cu}_{\hat{\delta}}(y_2, f)$ and if $t' = \pi_{y_1,y_2}(t)$ then $t' \in \mathcal{F}^{ss}(t, f)$.

Proof. (i) As C is a cylinder it follows that if $x, y \in C, W^{cu}_{\overline{r}}(y, f) \subset C, \mathcal{F}^{ss}_{\overline{\epsilon}}(x, f) \subset C$ then $\overline{r} < r', \overline{\epsilon} < \epsilon'$ and by Lemma 6.2 if $W^{cu}_{\frac{\overline{r}+r'}{2}}(y, f) \cap \mathcal{F}^{ss}_{\overline{\epsilon}}(x, f) \neq \varnothing$ then this intersection is exactly one point. So if $\overline{W^{cu}_{\overline{r}}(y, f)} \cap \mathcal{F}^{ss}_{\overline{\epsilon}}(x, f) \neq \varnothing$ then this intersection is exactly one point and so $\pi_{|\eta(\overline{W^{cu}_{\overline{r}}(y,f)})}$ is injective. As $\eta(\overline{W^{cu}_{\overline{r}}(y, f)})$ is compact and Hausdorff then $\pi(\eta(\overline{W^{cu}_{\overline{r}}(y, f)}))$ is Hausdorff and we get that $\pi_{|\eta(\overline{W^{cu}_{\overline{r}}(y,f)})}$ is an homeomorphism on its image. Taking $\overline{r} = \hat{\delta}$ it follows that $\pi_{|\eta(\overline{W^{cu}_{\hat{\delta}}(y,f)})}$ is an homeomorphism on its image.

(ii) As $\pi_{|\eta(\overline{W^{cu}_{\hat{\delta}}(y,f)})}$ is an homeomorphism on its image by the Invariance of Domain Theorem it is easy to see that for any $y \in \overline{\mathcal{F}^{ss}_{\hat{r}}(h, f)}$ there exists an open ball $B_y \subset U^{cu}$ centered at h_{cu} such that $B_y \times \{0\} \subset \pi(\eta(W^{cu}_{\hat{\delta}}(y, f)))$. Now by the compacity of $\overline{\mathcal{F}^{ss}_{\hat{r}}(h, f)}$ and by the continuous dependence of the manifolds $W^{cu}_{\hat{\delta}}(y, f)$ on the points y follows the existence of the open ball B.

(iii) It is enough to take $\overline{\delta} < \hat{\delta}$ suficiently small such that $\eta(W^{cu}_{\overline{\delta}}(y, f)) \subset B \times V^{ss}$ for any $y \in \overline{\mathcal{F}^{ss}_{\hat{r}}(h, f)}$. Then $\pi(\eta(W^{cu}_{\overline{\delta}}(y_1, f))) \subset B \times \{0\} \subset \pi(\eta(W^{cu}_{\hat{\delta}}(y_2, f)))$ for every $y_1, y_2 \in \overline{\mathcal{F}^{ss}_{\hat{r}}(h, f)}$. Since $\pi_{|\eta(\overline{W^{cu}_{\hat{\delta}}(y_1,f)})}$ and $\pi_{|\eta(\overline{W^{cu}_{\hat{\delta}}(y_2,f)})}$ are homeomorphisms, the function:

$$\pi_{y_1,y_2} = (\eta)^{-1} \circ (\pi_{|\eta(\overline{W^{cu}_{\hat{\delta}}(y_2,f)})})^{-1} \circ \pi_{|\eta(\overline{W^{cu}_{\hat{\delta}}(y_1,f)})} \circ \eta : W^{cu}_{\overline{\delta}}(y_1, f) \to W^{cu}_{\hat{\delta}}(y_2, f)$$

is well defined and continuous.

Moreover if $t' = \pi_{y_1,y_2}(t)$ then

$$\pi_{|\eta(\overline{W^{cu}_{\hat{\delta}}(y_2,f)})} \circ \eta(t') = \pi_{|\eta(\overline{W^{cu}_{\hat{\delta}}(y_1,f)})} \circ \eta(t)$$

and using that η is a chart trivializing the strong stable foliation we conclude that $t' \in \mathcal{F}^{ss}(t, f)$.

$\square$

Choose $\delta > 0$ such that if $dist(z, SH) < \delta$ then

$$\| Df^{-1}_{|E^c(f(z))} \| < (\sigma')^{-1} < 1 \tag{25}$$

for some $1 < \sigma' < \sigma$.
Let us define the set SH$'$ by

$$\mathrm{SH}' = \bigcup_{z\in \mathrm{SH}} \mathcal{F}^{ss}_{\delta}(z, f).$$

Lemma 6.5. *If $x \in SH'$ then $m\{Df^n_{|E^c(f^l(x))}\} > (\sigma')^n$ for any $n > 0, l > 0$.*

Proof. It follows by induction, using (25) and the fact that $f(\mathrm{SH}) \subset \mathrm{SH}$. $\square$

Let $\alpha = (\sigma')^{-1}$, fix $0 < \alpha < \alpha_1 < 1$ and let r_0 be as in Lemma 2.3.

Lemma 6.6. *There exist $0 < d < r_1 < r_0$ and $\epsilon_1 > 0$ such that for every $x \in M, z \in W^{cu}_d(x, f)$ and*

$$A_{x,z} = W^{cu}_{r_1}(x, f) \cup \Big(\bigcup_{y\in W^{cu}_d(z,f)} \mathcal{F}^{ss}_{\epsilon_1}(y, f)\Big)$$

then $diam(A_{x,z}) < L$ and for any $y \in W^{cu}_d(z, f)$ the intersection of $\mathcal{F}^{ss}_{\epsilon_1}(y, f)$ with $W^{cu}_{r_1}(x, f)$ is exactly one point.

Proof. Take $r_1 < min\{\frac{L}{8}, r, r_0\}$ and $\epsilon_1 < min\{\frac{L}{8}, \epsilon\}$. From Lemma 6.2 there exist d_1 such that if $dist(x, y) < d_1$ then the manifolds $W^{cu}_{r_1}(x, f)$ and $\mathcal{F}^{ss}_{\epsilon_1}(y, f)$ intersect in exactly one point.

Take $d < min\{\frac{d_1}{4}, r_1\}$. Let now $x \in M$ be an arbitrary point and $z \in W^{cu}_d(x, f)$. Observe that if $y \in W^{cu}_d(z, f)$ then

$$dist(x, y) \le dist(x, z) + dist(z, y) \le dist_{W^{cu}}(x, z) + dist_{W^{cu}}(z, y) \le 2d < \frac{d_1}{2} < d_1$$

so the manifolds $W^{cu}_{r_1}(x, f)$ and $\mathcal{F}^{ss}_{\epsilon_1}(y, f)$ intersect at exactly one point.

Also if $t \in \mathcal{F}^{ss}_{\epsilon_1}(y, f)$ for some $y \in W^{cu}_d(z, f)$ then

$$dist(t, x) \le dist(t, y)+dist(y, x) \le dist_{\mathcal{F}^{ss}}(t, y)+dist(y, x) \le \epsilon_1+2d < \frac{L}{8}+2r_1 < \frac{L}{2}$$

and if $\bar{t} \in W^{cu}_{r_1}(x, f)$ then

$$dist(\bar{t}, x) \le dist_{W^{cu}}(\bar{t}, x) \le r_1 < \frac{L}{8}$$

so it follows easily that $diam(A) < L$. $\square$

Consider $\lambda < 1$ the contraction factor of the strong stable subbundle.
Let $h \in \mathrm{SH}$ and let $U \subset M$ be an open set containing h. We will prove that there exists a periodic point in U.
Let $C \in B$ be a cylinder contained in U such that $h \in C$.
Take $0 < \hat{r} < \delta$ such that $\overline{\mathcal{F}^{ss}_{2\hat{r}}(h, f)} \subset C$ and $K > \delta + \epsilon_1$ such that

$$\mathcal{F}^{ss}_K(x, f) \pitchfork W^{cu}_d(y, f) \neq \varnothing, \forall x, y \in M. \tag{26}$$

Remark 6.2. Observe that this is the only step where the condition of minimality of the strong stable foliation is used and that the number d is uniform depending only of the diffeomorphism. In fact the condition of minimality of the strong stable foliation can be substituted by the fact that for any $x \in M$ there exists $K > 0$ such that (26) holds for any $y \in M$.

Choose $n_0 \in \mathbb{N}$ such that

$$\mathcal{F}^{ss}_K(f^{-n_0}(x), f) \subset f^{-n_0}(\mathcal{F}^{ss}_{\hat{r}}(x, f)), \quad \forall x \in M.$$

Take $\hat{\delta}$ satisfying simultaneously the following three conditions:

1) $\overline{W^{cu}_{\hat{\delta}}(y, f)} \subset C, \quad \forall y \in \overline{\mathcal{F}^{ss}_{\hat{r}}(h, f)}$
2) $f^{-n_0}(W^{cu}_{\hat{\delta}}(y, f)) \subset W^{cu}_d(f^{-n_0}(y), f), \quad \forall y \in \overline{\mathcal{F}^{ss}_{\hat{r}}(h, f)}$
3) If $dist(z, \overline{\mathcal{F}^{ss}_{\hat{r}}(h, f)}) < \hat{\delta}$ then $\overline{\mathcal{F}^{ss}_{\hat{r}}(z, f)} \subset C$.

To see that there exists $\hat{\delta}$ satisfying 3) observe that for each $y \in \overline{\mathcal{F}^{ss}_{\hat{r}}(h, f)}$ there exists θ_y such that if $dist(z, y) < \theta_y$ then $\overline{\mathcal{F}^{ss}_{\hat{r}}(z, f)} \subset C$ because $\overline{\mathcal{F}^{ss}_{\hat{r}}(y, f)} \subset \overline{\mathcal{F}^{ss}_{2\hat{r}}(h, f)} \subset C$.

Let now $\overline{\delta}$ and π_{y_1,y_2} be like in Lemma 6.4. Observe that $\overline{\delta}$ can be selected arbitrarily small. Take then $\overline{\delta}$ such that

$$\mathcal{F}^{ss}_{\hat{r}}(\pi_{h,y_2}(q), f) \subset \mathcal{F}^{ss}_{2\hat{r}}(q, f), \quad \forall q \in W^{cu}_{\overline{\delta}}(h, f), \quad \forall y_2 \in \overline{\mathcal{F}^{ss}_{\hat{r}}(h, f)} \tag{27}$$

This is possible because $\mathcal{F}^{ss}_{\hat{r}}(h, f)$ intersects transversely $W^{cu}_{\hat{\delta}}(y_2, f)$ in y_2. Thus, using the compacity of $\overline{\mathcal{F}^{ss}_{\hat{r}}(h, f)}$, we conclude that $\mathcal{F}^{ss}_{\hat{r}}(q, f)$ intersects $W^{cu}_{\hat{\delta}}(y_2, f)$ for any $y_2 \in \overline{\mathcal{F}^{ss}_{\hat{r}}(h, f)}$ when q is in some suficiently small open ball centered at h. As C is a cylinder any non empty intersection of a strong stable disc and a center unstable disc, both of them contained in C, consists of exactly one point. Furthermore, $\pi_{h,y_2}(q) \in W^{cu}_{\hat{\delta}}(y_2, f) \cap \mathcal{F}^{ss}_{\tilde{r}}(q, f)$ for some $\tilde{r}$. Hence $\pi_{h,y_2}(q) \in \mathcal{F}^{ss}_{\hat{r}}(q, f)$, which clearly implies (27).

Let $N \in \mathbb{N}$ be such that $(\alpha_1)^N r_0 < \overline{\delta}$ and $f^N(\overline{\mathcal{F}^{ss}_{2\hat{r}}(y, f)}) \subset \mathcal{F}^{ss}_{\delta}(f^N(y), f), \quad \forall y \in M$. From Lemma 6.5 it follows that

$$\prod_{j=0}^{n} \| Df^{-1}_{|E^c(f^{-j}(z))} \| < \alpha^n, \quad 0 \leq n \leq N, \quad \forall z \in f^N(\overline{\mathcal{F}^{ss}_{\hat{r}}(h, f)})$$

and therefore

$$\prod_{j=0}^{n} \| Df^{-1}_{|E^{cu}(f^{-j}(z))} \| < \alpha^n, \quad 0 \leq n \leq N, \quad \forall z \in f^N(\overline{\mathcal{F}^{ss}_{\hat{r}}(h, f)}).$$

Then by Lemma 2.3 we conclude that:

$$f^{-N}(W^{cu}_{r_1}(f^N(y), f)) \subset f^{-N}(W^{cu}_{r_0}(f^N(y), f)) \subset W^{cu}_{\alpha_1^N r_0}(y, f) \subset W^{cu}_{\overline{\delta}}(y, f)$$

for any $y \in \overline{\mathcal{F}^{ss}_{\hat{r}}(h, f)}$.
Put $y_1 = h$. We know that $\mathcal{F}^{ss}_K(f^{-n_0}(h), f)$ intersects $W^{cu}_d(f^N(h), f)$ in some point z. Then there exists $y_2 \in \mathcal{F}^{ss}_{\hat{r}}(h, f)$ such that $z = f^{-n_0}(y_2)$ and a continuous function $\pi_{h,y_2} : W^{cu}_{\overline{\delta}}(h, f) \to W^{cu}_{\hat{\delta}}(y_2, f)$ such that if $t' = \pi_{h,y_2}(t)$ then $t' \in \mathcal{F}^{ss}(t, f)$.

Set $x = f^N(h)$ in Lemma 6.6. From this it follows that there exists a cylinder $\hat{C}$ containing

$$A = W^{cu}_{r_1}(x,f) \cup (\bigcup_{y \in W^{cu}_d(z,f)} \mathcal{F}^{ss}_{\epsilon_1}(y,f))$$

and for any $y \in W^{cu}_d(z,f)$ the intersection of the manifolds $\mathcal{F}^{ss}_{\epsilon_1}(y,f)$ and $W^{cu}_{r_1}(x,f)$ is exactly one point.

Now let $\phi : \hat{C} \to \hat{U}^{cu} \times \hat{V}^{ss}$ be the trivializing local chart of the strong stable foliation with $0 \in \phi(\hat{C})$ and $\hat{\pi} : \hat{U}^{cu} \times \hat{V}^{ss} \to \hat{U}^{cu} \times \{0\}$ the projection. By the same argument used while proving (i) in Lemma 6.4 we know that if $\pi_1 = \hat{\pi}_{|\phi(\overline{W^{cu}_{r_1}(x,f)})}$ then π_1 is a homeomorphism on its image.

On the other side if $\pi_2 = \hat{\pi}_{|\phi(W^{cu}_d(z,f))}$, as the intersection between the manifolds $\mathcal{F}^{ss}_{\epsilon_1}(y,f)$ and $W^{cu}_{r_1}(x,f)$ is exactly one point, it follows

$$\pi_2(\phi(W^{cu}_d(z,f))) \subset \pi_1(\phi(W^{cu}_{r_1}(x,f))).$$

Then the function

$$g : \pi_1(\phi(\overline{W^{cu}_{r_1}(x,f)})) \to \pi_2(\phi(W^{cu}_d(z,f)))$$

defined by $g = \pi_2 \circ \phi \circ f^{-n_0} \circ \pi_{h,y_2} \circ f^{-N} \circ (\phi)^{-1} \circ (\pi_1)^{-1}$ is well defined and it is continuous. Hence by Brower's fixed point Theorem there exists a fixed point $p \in \pi_1(\phi(W^{cu}_{r_1}(x,f)))$, that is

$$\pi_2 \circ \phi \circ f^{-n_0} \circ \pi_{h,y_2} \circ f^{-N} \circ (\phi)^{-1} \circ (\pi_1)^{-1}(p) = p. \tag{28}$$

Observe that $\hat{\pi}(\pi_1^{-1}(p)) = \hat{\pi}(\pi_2^{-1}(p)) = p$ and therefore the stable manifolds of $p_{r_1} = (\phi)^{-1} \circ \pi_1^{-1}(p)$, $p_{-n_0} = (\phi)^{-1} \circ \pi_2^{-1}(p)$ coincide ($\mathcal{F}^{ss}(p_{r_1},f) = \mathcal{F}^{ss}(p_{-n_0},f)$). Also

$$\mathcal{F}^{ss}(f^{-N}(p_{-n_0}),f) = \mathcal{F}^{ss}(f^{-N}(p_{r_1}),f) \text{ and } \mathcal{F}^{ss}(f^{n_0}(p_{-n_0}),f) = \mathcal{F}^{ss}(f^{n_0}(p_{r_1}),f).$$

From (28)

$$(\phi)^{-1} \circ \pi_2^{-1} \circ \pi_2 \circ \phi \circ f^{-n_0} \circ \pi_{h,y_2} \circ f^{-N}(p_{r_1}) = (\phi)^{-1} \circ \pi_2^{-1}(p) = p_{-n_0}$$

and hence

$$\pi_{h,y_2} \circ f^{-N}(p_{r_1}) = f^{n_0}(p_{-n_0}).$$

So

$$\begin{aligned}\mathcal{F}^{ss}(f^{n_0}(p_{r_1}),f) &= \mathcal{F}^{ss}(f^{n_0}(p_{-n_0}),f) = \mathcal{F}^{ss}(\pi_{h,y_2} \circ f^{-N}(p_{r_1}),f) \\ &= \mathcal{F}^{ss}(f^{-N}(p_{r_1}),f) = \mathcal{F}^{ss}(f^{-N}(p_{-n_0}),f)\end{aligned}$$

which implies $\mathcal{F}^{ss}(p_{r_1},f) = \mathcal{F}^{ss}(f^{-n_0-N}(p_{-n_0}),f)$.

Observe that $p_{-n_0} \in W^{cu}_d(z,f)$, $p_{r_1} \in W^{cu}_{r_1}(x,f) = W^{cu}_{r_1}(f^N(h),f)$ and $p_{r_1} \in \mathcal{F}^{ss}_{\epsilon_1}(p_{-n_0},f)$. Thus $dist_{\mathcal{F}^{ss}}(p_{r_1},p_{-n_0}) \leq \epsilon_1$ and $p_{-n_0} = f^{-n_0} \circ \pi_{h,y_2} \circ f^{-N}(p_{r_1}) \in f^{-n_0}(W^{cu}_{\hat{\delta}}(y_2,f))$.

Take $\theta \in \mathcal{F}^{ss}_{\delta}(p_{r_1},f)$ arbitrary. Then

$$dist_{\mathcal{F}^{ss}}(\theta,p_{-n_0}) \leq dist_{\mathcal{F}^{ss}}(\theta,p_{r_1}) + dist_{\mathcal{F}^{ss}}(p_{r_1},p_{-n_0}) \leq \delta + \epsilon_1 < K$$

and from there

$$\mathcal{F}^{ss}_{\delta}(p_{r_1},f) \subset \mathcal{F}^{ss}_K(p_{-n_0},f) \subset f^{-n_0}(\mathcal{F}^{ss}_{\hat{r}}(f^{n_0}(p_{-n_0}),f)).$$

Remember that

$$\begin{aligned} f^N(\overline{\mathcal{F}^{ss}_{\hat r}(f^{n_0}(p_{-n_0}),f)}) \subset f^N(\overline{\mathcal{F}^{ss}_{2\hat r}(f^{-N}(p_{r_1}),f)}) \subset \mathcal{F}^{ss}_{\delta}(f^N(f^{-N}(p_{r_1})),f) \\ = \mathcal{F}^{ss}_{\delta}(p_{r_1},f) \end{aligned}$$

we have that

$$f^N(\overline{\mathcal{F}^{ss}_{\hat r}(f^{n_0}(p_{-n_0}),f)}) \subset \mathcal{F}^{ss}_{\delta}(p_{r_1},f) \subset f^{-n_0}(\mathcal{F}^{ss}_{\hat r}(f^{n_0}(p_{-n_0}),f)).$$

So

$$\begin{aligned} f^{n_0+N}(f^{-n_0}(\overline{\mathcal{F}^{ss}_{\hat r}(f^{n_0}(p_{-n_0}),f)}) &= f^N(\overline{\mathcal{F}^{ss}_{\hat r}(f^{n_0}(p_{-n_0}),f)}) \\ &\subset f^{-n_0}(\overline{\mathcal{F}^{ss}_{\hat r}(f^{n_0}(p_{-n_0}),f)}). \end{aligned}$$

Again by Brower's fixed point Theorem there exists $Q \in f^{-n_0}(\overline{\mathcal{F}^{ss}_{\hat r}(f^{n_0}(p_{-n_0}),f)}) \subset f^{-n_0}(C)$ a fixed point by the function f^{n_0+N}, and hence $f^{n_0}(Q)$ a periodic point in C.

□

We will understand by unstable index of a point the dimension of its subbundle where vectors are backward contracted. Remember that two points P and Q are homoclinically related when the stable manifold of the orbit of Q transversely meets the unstable manifold of the orbit of P and vice versa.

Corollary 6.1. *Let $f \in Diff^r(M)$ be a partially hyperbolic diffeomorphism exhibiting Property SH and such that the strong stable foliation is minimal. Then the set of periodic points with unstable index $c+u$ is dense in M.*

Proof. It follows from the proof of Theorem 6.1. □

Corollary 6.2. *Let $f \in Diff^r(M)$ be a partially hyperbolic diffeomorphism exhibiting Property SH and such that the strong stable foliation is minimal. Then the periodic points with unstable index $c+u$ are homoclinically related. Moreover, if p is a periodic point with unstable index $c+u$ and*

$$H(p) = \overline{\bigcup_{i=1}^{Per(p)} (\mathcal{F}^{ss}(f^i(p)) \pitchfork \mathcal{W}^{cu}(f^i(p)))}$$

is the homoclinic class of p, then $H(p) = M$.

Proof. The first assertion follows from the minimality of the strong stable foliation. The second follows from Corollary 6.1 and the fact that $H(p)$ coincides with the closure of the saddles homoclinically related with p. □

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