Central Limit Theorem for a Tagged Particle in Asymmetric Simple Exclusion

and

Hydrodynamic Limit for a Particle System with degenerate rates

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Ao meu Pai
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Patrícia Gonçalves

Abstract

This work has to do with the study of two different models. The first is the Asymmetric Simple Exclusion in $\mathbb{Z}$. In this process, each particle, independently from the others, waits a mean one exponential time, at the end of which being at x it jumps to $x+1$ at rate p or $x-1$ at rate $1-p$. If the site is occupied the jump is suppressed to respect the exclusion rule. The Bernoulli product measures ν_α are invariant for this process.

We prove a Functional Central Limit Theorem for the position of a Tagged Particle in the one-dimensional setting and in the hyperbolic scaling, for the process starting from a Bernoulli product measure conditioned to have a particle at the origin. We also show that the position of the Tagged Particle at time t depends on the initial configuration, by the number of empty sites in the interval $[0,(p-q)\alpha t]$ divided by α in the hyperbolic and in a longer time scale, namely $N^{4/3}$.

In the second part of the work, we study a conservative particle system with degenerate rates, namely with nearest neighbor exchange rates which vanish for some configurations. Due to this degeneracy the hyperplanes with a fixed number of particles can be decomposed into some irreducible sets of configurations plus isolated configurations that do not evolve under the dynamics.

We show that, for initial profiles smooth enough and bounded away from zero and one, under the diffusive time scaling, the macroscopic density profile evolves according to the porous medium equation. Then we prove the same result for more general profiles for a slightly perturbed microscopic dynamics: we add jumps of the Symmetric Simple Exclusion which remove the degeneracy of rates and are properly slowed down in order not to change the macroscopic behavior. The equilibrium fluctuations and the magnitude of the spectral gap for this perturbed model are also obtained.

Contents

Chapter 1

Central Limit Theorem for a Tagged Particle in Asymmetric Simple Exclusion

1.1 Introduction

The Exclusion process on $\mathbb{Z}^d$ has been extensively studied. In this process, particles evolve on $\mathbb{Z}^d$ according to interacting random walks with an exclusion rule which prevents more than one particle per site. The dynamics can be informally described as follows. Fix a probability $p(\cdot)$ on $\mathbb{Z}^d$. Each particle, independently from the others, waits a mean one exponential time, at the end of which being at x it jumps to $x+y$ at rate $p(y)$. If the site is occupied the jump is suppressed to respect the exclusion rule. In both cases, the particle waits a new exponential time.

The space state of the process is $\{0,1\}^{\mathbb{Z}^d}$ and we denote the configurations by the Greek letter η, so that $\eta(x) = 0$ if the site x is vacant and $\eta(x) = 1$ otherwise. The case in which $p(y) = 0 \forall |y| > 1$ is referred as the Simple Exclusion process and in the Asymmetric Simple Exclusion process (ASEP) the probability p is such that $p(1) = p$, $p(-1) = 1 - p$ with $p \neq 1/2$ while for the Symmetric Simple Exclusion process (SSEP) $p = 1/2$.

For $0 \leq \alpha \leq 1$, denote by ν_α the Bernoulli product measure on $\{0,1\}^{\mathbb{Z}^d}$ with density α. It is known that ν_α is an invariant state for the exclusion process and that all invariant and translation invariant states are convex combinations of ν_α if $p(.)$ is such that $p_t(x,y) + p_t(y,x) > 0$, $\forall x, y \in \mathbb{Z}^d$ and $\sum_x p(x,y) = 1$, $\forall y \in \mathbb{Z}^d$ [15].

Assume that the origin is occupied at time 0. Tag this particle and denote by X_t its position at time t. Applying an invariance principle due to Newman and Wright [17], Kipnis in [13] proved a C.L.T. for the position of the Tagged Particle in the one-dimensional Asymmetric Simple Exclusion process (ASEP), provided the initial configuration is distributed according to ν_α^*, the Bernoulli product measure conditioned to have a particle at the origin. Transforming the exclusion process into a series of queues, an asymmetric Zero-Range process with constant rate, the position of the Tagged Particle becomes the current through the bond $[-1,0]$. Kipnis [13], was able to apply Newman and Wright results to the Zero-Range process and derive the L.L.N. and C.L.T. for the position of the Tagged Particle.

Few years later, Ferrari and Fontes [10] proved that the position at time t of the Tagged Particle, X_t, can be approximated by a Poisson Process. More precisely, they proved that for all $t \geq 0$, if the initial distribution is ν_α^* and $p > q$, $X_t = N_t - B_t + B_0$, where N_t is a Poisson Process with rate $(p-q)(1-\alpha)$ and B_t is a stationary process with bounded exponential moments. As a corollary they obtained the weak convergence of

$$\frac{X_{t\epsilon^{-1}} - (p-q)(1-\alpha)t\epsilon^{-1}}{\sqrt{(p-q)(1-\alpha)t\epsilon^{-1}}}$$

to a Brownian motion. The argument is divided in two steps. The convergence of the finite-dimensional distributions [8] is consequence of the fact that in the scale $t^{\frac{1}{2}}$, the position X_t can be read from the initial configuration: X_t is given (in the L^2-norm) by the initial number of empty sites in the interval $[0,(p-q)\alpha t]$ divided by α. Tightness follows from the sharp approximation of $X_{t\epsilon^{-1}}$ by the Poisson Process and the weak convergence of the Poisson Process to Brownian motion. Using the approximation of X_t by a Poisson Process and Kipnis results for the Tagged Particle, the same authors prove equilibrium density fluctuations for the ASEP in [9]. The density fluctuations for the Totally Asymmetric Simple Exclusion process (the case $p = 1$) have also been obtained by Rezakhanlou in [20] in a more general setting than for the process starting from an equilibrium state.

Recently, Jara and Landim [12] showed that the asymptotic behavior of the Tagged Particle in the one-dimensional nearest neighbor exclusion process, can be recovered from a joint asymptotic behavior of the empirical measure and the current through a bond. From this observation they proved a non-equilibrium C.L.T. for the Tagged Particle in the SSEP, in the diffusive scaling.

In this paper, besides using this general method to reprove Ferrari and Fontes result on the convergence of the rescaled position of the Tagged Particle to a Brownian motion in the hyperbolic time scale, we extended this result by showing that in a longer time scale the position of the Tagged Particle still depends on the initial configuration.

The advantage of our approach is that it relates the C.L.T. for a Tagged Particle to the C.L.T. for the empirical measure, a problem which is relatively well understood, see [14]. In particular, we can expect to apply this approach for a one-dimensional system in contact with reservoirs.

It was shown by Rezakhanlou [19], that in the ASEP the macroscopic particle density profile in the hyperbolic scaling evolves according to the inviscid Burgers equation, namely: $\partial_t\rho(t,u) + (p-q)\nabla(\rho(t,u)(1-\rho(t,u))) = 0$. To establish the C.L.T. for the empirical measure we need to consider the density fluctuation field as defined in (1.2.2) below. We show that, in this time scale, the limit density fluctuation field is deterministic, in the sense that at any given time t, the density field is a translation of the initial one. As mentioned above, this result was previously obtained in [9]. In order to observe fluctuation from the dynamics one has to change to the diffusive scaling.

The translation or velocity of the system is given by $v = (p-q)(1-2\alpha)$ and for $\alpha = 1/2$, the field does not evolve, and one is forced to go beyond the hydrodynamic scaling. We can consider the density fluctuation field in the longer time scale as defined in (1.2.5), where we subtract the velocity of a second class particle and any value of α can be considered in this setting.

It is conjectured that until the time scale $N^{3/2}$ the density fluctuation field does not evolve in time, see chap. 5 of [25] and references therein. The result we obtain is a contribution in this direction, since we can accomplish the result just up to the time scale $N^{4/3}$. The main difficulty in proving the C.L.T. for the empirical measure is the Boltzmann-Gibbs Principle, which we are able to prove for this time scale using a multi-scale argument.

As a consequence of this translation behavior, we show the dependence on the initial configuration of the current through a bond and the position of the Tagged Particle in the longer time scale.

This work is organized as follows. In the second section we introduce some notation and we state the results. The sketch the proof of the C.L.T. for the empirical measure associated to the ASEP in the hyperbolic scaling is exposed in the third section. In section four, we use the same strategy as in [12] to obtain L.L.N. for the position of the Tagged Particle. The convergence of finite-dimensional distributions of the Tagged Particle to those of Brownian motion, is proved following the same arguments as in [12], while tightness is proved by means of the Zero-Range representation as Kipnis in [13]. Both are presented in section five. In this time scale we show that the current and the position of the Tagged Particle at time t, can be read from the initial configuration, in section six.

In the following sections we study the same problem up to the time scale $N^{1+\gamma}$ with $\gamma < 1/3$. We start by showing the C.L.T. for the empirical measure associated to this process, in section seven. Since a Boltzmann-Gibbs Principle is needed, its proof is the content of the eighth section and in the subsequent section we treat the problem of tightness. In the last section we prove the dependence on the initial configuration for the current over a bond and the position of the Tagged Particle, in this longer time scale.

1.2 Statement of Results

The one-dimensional asymmetric simple exclusion process is the Markov process $\eta_t \in \{0,1\}^{\mathbb{Z}}$ with generator given on local functions by

$$Lf(\eta) = \sum_{x\in\mathbb{Z}} \sum_{y=x\pm 1} c(x,y,\eta)[f(\eta^{x,y}) - f(\eta)], \tag{1.2.1}$$

where $c(x,y,\eta) = p(x,y)\eta(x)(1-\eta(y))$, $p(x,x+1) = p$, $p(x,x-1) = q = 1-p$ and

$$\eta^{x,y}(z) = \begin{cases} \eta(z), & \text{if } z \neq x,y \\ \eta(y), & \text{if } z = x \\ \eta(x), & \text{if } z = y \end{cases}.$$

Its description is the following. At most one particle is allowed at each site; if there is a particle at site x, it jumps at rate p to site $x+1$ if there is no particle at that site or to the site $x-1$ at rate q, if it is empty.

Initially, place the particles according to a **Bernoulli product measure** in $\{0,1\}^{\mathbb{Z}}$, of parameter $\alpha \in (0,1)$, denoted by ν_α. This means that the random variables $(\eta(x))_{x\in\mathbb{Z}}$, representing the number of particles in each site x, are independent, and for each site $x \in \mathbb{Z}$, $\eta(x)$ is distributed according to a Bernoulli distribution of parameter α.

For each configuration η, we denote by $\pi^N(\eta, du)$ the **empirical measure**, a measure in $\mathbb{R}$ that assigns mass $\frac{1}{N}$ to each particle:

$$\pi^N(\eta, du) = \frac{1}{N}\sum_{x\in\mathbb{Z}} \eta(x)\delta_{\frac{x}{N}}(du)$$

and let $\pi_t^N(\eta, du) = \pi^N(\eta_t, du)$. First, we state the C.L.T. for the empirical measure, for which we need to introduce some notation.

For each integer $z \geq 0$, let $H_z(x) = (-1)^z e^{x^2}\frac{d^z}{dx}e^{-x^2}$ be the Hermite polynomial, and $h_z(x) = \frac{1}{c_z}H_z(x)e^{-x^2}$ the Hermite function, where $c_z = z!\sqrt{2\pi}$. The set $\{h_z, z \geq 0\}$ is an orthonormal basis of $L^2(\mathbb{R})$. Consider in $L^2(\mathbb{R})$ the operator $K_0 = x^2 - \Delta$. A simple computation shows that $K_0 h_z = \gamma_z h_z$ where $\gamma_z = 2z+1$.

For an integer $k \geq 0$, denote by $\mathcal{H}_k$ the Hilbert space induced by $S(\mathbb{R})$ (the space of smooth rapid decrease functions) and the scalar product $< \cdot,\cdot >_k$ defined by $< f, g >_k = < f, K_0^k g >$, where $< \cdot,\cdot >$ denotes the inner product of $L^2(\mathbb{R})$ and denote by $\mathcal{H}_{-k}$ the dual of $\mathcal{H}_k$ relatively to this inner product.

Fix an integer k and denote by $Y_\cdot^N$ the **density fluctuation field**, a linear functional acting on functions $H \in S(\mathbb{R})$ as

$$Y_t^N(H) = \frac{1}{\sqrt{N}}\sum_{x\in\mathbb{Z}} H\Big(\frac{x}{N}\Big)(\eta_{tN}(x) - \alpha). \tag{1.2.2}$$

Denote by $D(\mathbb{R}^+, \mathcal{H}_{-k})$ (resp. $C(\mathbb{R}^+, \mathcal{H}_{-k})$) the space of H_{-k}-valued functions, right continuous with left limits (resp. continuous), endowed with the uniform weak topology, by Q_N the probability measure on $D(\mathbb{R}^+, \mathcal{H}_{-k})$ induced by the density fluctuation field $Y_\cdot^N$ and the product measure ν_α. Consider $\mathbb{P}_{\nu_\alpha}^N = \mathbb{P}_{\nu_\alpha}$ the probability measure on $D(\mathbb{R}^+, \{0,1\}^{\mathbb{Z}})$ induced by ν_α and the Markov process η_t speeded up by N and denote by $\mathbb{E}_{\nu_\alpha}$ expectation with respect to $\mathbb{P}_{\nu_\alpha}$.

Theorem 1.2.1. *Fix an integer $k > 2$. Let Q be the probability measure on $C(\mathbb{R}^+, \mathcal{H}_{-k})$ corresponding to a stationary Gaussian process with mean 0 and covariance given by*

$$E_Q[Y_t(H)Y_s(G)] = \chi(\alpha)\int_{\mathbb{R}} H(u + v(t-s))G(u)du \tag{1.2.3}$$

for every $0 \leq s \leq t$ and H, G in $\mathcal{H}_k$. Here $\chi(\alpha) = \boldsymbol{Var}(\nu_\alpha, \eta(0)) = \alpha(1-\alpha)$ and $v = (p-q)\chi'(\alpha) = (p-q)(1-2\alpha)$. Then, the sequence $(Q_N)_{N\geq 1}$ converges weakly to the probability measure Q.

We remark that last theorem holds for the ASEP evolving in any $\mathbb{Z}^d$, with the appropriate changes. In this case, the limit density fluctuation field at time t is a translation of the initial density field, since for every $H \in S(\mathbb{R})$:

$$Y_t(H) = Y_0(T_t H), \tag{1.2.4}$$

where $T_t H(u) = H(u + vt)$.

Having the density fluctuations, we can obtain the L.L.N. and the C.L.T. for the current over a bond, as in [12]. For a site x, denote by $J^N_{x,x+1}(t)$ the **current** over the bond $[x, x+1]$, which is the total number of jumps from the site x to the site $x+1$ minus the total number of jumps from the site $x+1$ to the site x in the time interval $[0, tN]$.

Following the same arguments as Jara and Landim in [12] we show the C.L.T. for the current over a fixed bond for the ASEP starting from the invariant measure ν_α.

Denote by ν^*_α the measure ν_α conditioned to have a particle at the origin. By coupling the ASEP starting from ν_α with the ASEP starting from ν^*_α, in such a way that both processes differ at most in one site at any given time, the L.L.N. and the C.L.T. for the empirical measure and for the current over a bond starting from ν^*_α, follows from the L.L.N. and the C.L.T. for the empirical measure and for the current over a bond starting from ν_α.

Assume now that the initial measure is ν^*_α, let $\mathbb{P}^N_{\nu^*_\alpha} = \mathbb{P}_{\nu^*_\alpha}$ be the probability measure on $D(\mathbb{R}^+, \{0,1\}^{\mathbb{Z}})$ induced by ν^*_α and the Markov process η_t speeded up by N and denote by $\mathbb{E}_{\nu^*_\alpha}$ expectation with respect to $\mathbb{P}_{\nu^*_\alpha}$.

Denote by X_{tN} the position at time $tN \geq 0$ of the tagged particle initially at the origin. We reprove the L.L.N. for the position of the Tagged Particle, which was previously obtained by Saada in [22]:

Theorem 1.2.2. *Fix $t \geq 0$. Then,*

$$\frac{X_{tN}}{N} \xrightarrow[N\to+\infty]{} v_t = (p-q)(1-\alpha)t$$

in $\mathbb{P}_{\nu^*_\alpha}$-probability.

and the convergence to the Brownian motion, which was already obtained by Ferrari and Fontes in [10]:

Theorem 1.2.3. *Under $\mathbb{P}_{\nu^*_\alpha}$,*

$$\frac{X_{tN} - v_t N}{\sqrt{N|p-q|(1-\alpha)}} \xrightarrow[N\to+\infty]{} B_t$$

weakly, where B_t denotes the standard Brownian motion.

Another interesting property is the **dependence on the initial configuration** for the position of the Tagged Particle, which was previously obtained by Ferrari in [8]. Suppose $p > q$:

Corollary 1.2.4. *Fix $t \geq 0$. Then for every $\epsilon > 0$,*

$$\lim_{N\to+\infty} \mathbb{E}_{\nu_\alpha^*}\Big[\Big(\frac{X_{tN}}{\sqrt{N}} - \frac{\sum_{x=0}^{(p-q)\alpha tN}(1-\eta_0(x))}{\alpha\sqrt{N}}\Big)^{2-\epsilon}\Big] = 0.$$

In the hyperbolic scaling, we have seen above that for $\alpha = 1/2$ the limit density fluctuation field at time t is the same as the initial one, see relation (1.2.4). In order to observe fluctuations we are forced to consider a longer time scale.

Henceforth, consider the **ASEP evolving in the time scale** $N^{1+\gamma}$, with $\gamma > 0$. In the sequel, we point out the restrictions needed in γ in order to obtain the results. We note here, that there is no particular reason for taking $\alpha = 1/2$, since we can consider a frame of reference moving with velocity $vtN^{1+\gamma}$.

Let $\alpha \in (0,1)$ and redefine the **density fluctuation field** on $H \in S(\mathbb{R})$ by:

$$Y_t^{N,\gamma}(H) = \frac{1}{\sqrt{N}}\sum_{x\in\mathbb{Z}} H\Big(\frac{x - vtN^{1+\gamma}}{N}\Big)(\eta_{tN^{1+\gamma}}(x) - \alpha). \tag{1.2.5}$$

We remark here, than one can define in the hyperbolic scaling of time the density fluctuation field as above. But in that case the current could not be defined through a fixed bond, instead it would have to be defined through a bond that depends on time (see section 9). As we we want to show the C.L.T. for the position of a Tagged Particle using the relation between the density of particles and the current through a fixed bond (1.4.5), we have the need to defined the density fluctuation field as in (1.2.2).

As above, let Q_N^γ be the probability measure on $D(\mathbb{R}^+, \mathcal{H}_{-k})$ induced by the density fluctuation field $Y_\cdot^{N,\gamma}$ and ν_α, let $\mathbb{P}_{\nu_\alpha}^{N,\gamma} = \mathbb{P}_{\nu_\alpha}^\gamma$ be the probability measure on $D(\mathbb{R}^+, \{0,1\}^{\mathbb{Z}})$ induced by ν_α and the Markov process η_t speeded up by $N^{1+\gamma}$ and denote by $\mathbb{E}_{\nu_\alpha}^\gamma$ expectation with respect to $\mathbb{P}_{\nu_\alpha}^\gamma$. Now, we state Theorem 1.2.1 in this longer scaling:

Theorem 1.2.5. *Fix an integer $k > 1$ and $\gamma < 1/3$. Let Q be the probability measure on $C(\mathbb{R}^+, \mathcal{H}_{-k})$ corresponding to a stationary Gaussian process with mean 0 and covariance given by*

$$E_Q[Y_t(H)Y_s(G)] = \chi(\alpha)\int_{\mathbb{R}} H(u)G(u)du \tag{1.2.6}$$

for every $s, t \geq 0$ and H, G in $\mathcal{H}_k$. Again $\chi(\alpha) = \alpha(1-\alpha)$. Then, the sequence $(Q_N^\gamma)_{N\geq 1}$ converges weakly to the probability measure Q.

In order to keep notation simple, here and after, for a random variable X we denote by $\bar{X}$ the centered random variable X.

As we follow the martingale approach, the main difficulty in proving this theorem is the **Boltzmann-Gibbs Principle**, which we can prove for $\gamma < 1/3$ and in this case is stated in the following way:

Theorem 1.2.6. *(Boltzmann-Gibbs Principle)*

Fix $\gamma < 1/3$. For every every $t > 0$ and $H \in S(\mathbb{R})$,

$$\lim_{N\to\infty} \mathbb{E}_{\nu_\alpha}\Big[\Big(\int_0^t \frac{N^\gamma}{\sqrt{N}} \sum_{x\in\mathbb{Z}} H\Big(\frac{x}{N}\Big)\bar{\eta}_s(x)\bar{\eta}_s(x+1)ds\Big)^2\Big] = 0.$$

Let $\mathbb{P}^{N,\gamma}_{\nu^*_\alpha} = \mathbb{P}^{\gamma}_{\nu^*_\alpha}$ be the probability measure on $D(\mathbb{R}^+, \{0,1\}^{\mathbb{Z}})$ induced by ν^*_α and the Markov process η_t speeded up by $N^{1+\gamma}$.

By the results just stated, in this longer time scale the system translates in time at a certain velocity v. This allows us to deduce from the previous results the asymptotic behavior of the Tagged Particle even in the longer scaling:

Corollary 1.2.7. *Fix $t \geq 0$, $\gamma < 1/3$ and suppose that $p > q$. Then,*

$$\frac{X_{tN^{1+\gamma}}}{\sqrt{N}} - \frac{\sum_{x=0}^{(p-q)\alpha t N^{1+\gamma}}(1-\eta_0(x))}{\alpha\sqrt{N}} \xrightarrow[N\to+\infty]{} 0$$

in $\mathbb{P}_{\nu^*_\alpha}$-probability.

1.3 Density Fluctuations in the Hyperbolic Scaling

The aim of this section is to prove Theorem 1.2.1. Fix a positive integer k and recall the definition of the density fluctuation field in (1.2.2). The purpose is to show that $Y^N_\cdot$ converges to a process $Y_\cdot$ whose time evolution is deterministic.

Denote by $\mathfrak{A}$ the operator $v\nabla$ defined on a domain of $L^2(\mathbb{R})$ and by $\{T_t, t \geq 0\}$ the semigroup associated to $\mathfrak{A}$, namely for a function H it holds that $T_t H(u) = H(u + vt)$. For $t \geq 0$, let $\mathcal{F}_t$ be the σ-algebra on $D([0,T], \mathcal{H}_{-k})$ generated by $Y_s(H)$ for $s \leq t$ and H in $S(\mathbb{R})$ and set $\mathcal{F} = \sigma(\bigcup_{t\geq 0}\mathcal{F}_t)$.

To prove the theorem we need to verify that $(Q_N)_{N\geq 1}$ is tight and to characterize the limit field. To check the last assertion, we consider a collection of martingales associated to the density fluctuation field. Fix a function $H \in S(\mathbb{R})$. By lemma A1.5.1 of [14], denoting by $W_{x,x+1}(\eta)$, the instantaneous current between the sites x and $x+1$:

$$W_{x,x+1}(\eta) = c(x, x+1, \eta) - c(x+1, x, \eta)$$

and

$$\nabla^N H\Big(\frac{x}{N}\Big) = N\Big(H\Big(\frac{x+1}{N}\Big) - H\Big(\frac{x}{N}\Big)\Big).$$

then

$$M^{N,H}_t = Y^N_t(H) - Y^N_0(H) - \int_0^t \frac{1}{\sqrt{N}} \sum_{x\in\mathbb{Z}} \nabla^N H\Big(\frac{x}{N}\Big) W_{x,x+1}(\eta_s)ds$$

is a martingale with respect to the filtration $\tilde{\mathcal{F}}_t = \sigma(\eta_s, s \leq t)$, whose quadratic variation is given by:

$$\int_0^t \frac{1}{N^2} \sum_{x\in\mathbb{Z}} \Big(\nabla^N H\Big(\frac{x}{N}\Big)\Big)^2 \Big[c(x, x+1, \eta_s) + c(x+1, x, \eta_s)\Big]ds.$$

Using the fact that $\sum_{x\in\mathbb{Z}}\nabla^N H(\frac{x}{N})=0$, the integral part of the martingale is equal to:

$$\int_0^t \frac{1}{\sqrt{N}}\sum_{x\in\mathbb{Z}}\nabla^N H\Big(\frac{x}{N}\Big)\Big[\bar{W}_{x,x+1}(\eta_s)\Big]ds.$$

As we need to write the expression inside last integral in terms of the fluctuation field Y^N_s, we are able to replace the function $\bar{W}_{x,x+1}(\eta_s)$ by $(p-q)\chi'(\alpha)[\eta_s(x)-\alpha]$, with the use of the:

Theorem 1.3.1. *(Boltzmann-Gibbs Principle)*
For every cylinder function g, for every $H\in S(\mathbb{R})$ and every $t>0$,

$$\lim_{N\to\infty}\mathbb{E}_{\nu_\alpha}\Big[\Big(\int_0^t \frac{1}{\sqrt{N}}\sum_{x\in\mathbb{Z}} H\Big(\frac{x}{N}\Big)\Big\{\tau_x g(\eta_s)-\tilde{g}(\alpha)-\tilde{g}'(\alpha)[\eta_s(x)-\alpha]\Big\}ds\Big)^2\Big]=0,$$

where $\tilde{g}(\alpha)=E_{\nu_\alpha}[g(\eta)]$.

In spite of considering the ASEP in the hyperbolic scaling, the proof of the last result is very close to the one presented for the Zero-Range process in the diffusive scaling, see chap. 11 of [14], and for that reason we have omitted it.

Assume now, that $(Q_N)_{N\geq 1}$ is tight and let Q be one of its limiting points. By the result just stated and since $\lim_{N\to+\infty}\mathbb{E}_{\nu_\alpha}[(M^{N,H}_t)^2]=0$, under Q

$$Y_t(H)=Y_0(H)+\int_0^t Y_s(\mathfrak{A}H)ds. \tag{1.3.1}$$

So, $\frac{d}{dt}Y_t(H)=Y_t(\mathfrak{A}H)$. Take $r<t$, and note that $\frac{d}{dr}<Y_r,T_{t-r}H>=0$. As a consequence, $Y_t(H)=Y_0(T_tH)$ where $T_tH(u)=H(u+vt)$.

Recall that $\mathcal{F}_0$ is the σ-algebra on $D([0,T],\mathcal{H}_{-k})$ generated by $Y_0(H)$ for H in $S(\mathbb{R})$. We start by characterizing the restriction of Q to $\mathcal{F}_0$ as in chap. 11 of [14].

Lemma 1.3.2. *For every $H\in S(\mathbb{R})$, and every $t>0$,*

$$\lim_{N\to\infty}\log\mathbb{E}_{\nu_\alpha}[\exp\{iY_t(H)\}]=-\frac{\chi(\alpha)}{2}<H,H>.$$

Proof. As ν_α is a product invariant and translation invariant measure, we have that:

$$\log\mathbb{E}_{\nu_\alpha}[\exp\{iY_t(H)\}]=\sum_{x\in\mathbb{Z}}\log E_{\nu_\alpha}\Big[\exp\Big\{\frac{i}{\sqrt{N}}H\Big(\frac{x}{N}\Big)[\eta(0)-\alpha]\Big\}\Big].$$

Using Taylor expansion, the right hand side of the last expression is equal to

$$\sum_{x\in\mathbb{Z}}\Big(-\frac{1}{2N}H^2\Big(\frac{x}{N}\Big)\chi(\alpha)\Big)+O(N^{-\frac{1}{2}}),$$

which converges to $-\frac{\chi(\alpha)}{2}<H,H>$ as $N\to\infty$. □

Corollary 1.3.3. *Restricted to $\mathcal{F}_0$, Q is a Gaussian field with covariance given by*

$$E_Q(Y_0(G)Y_0(H)) = \chi(\alpha) < G, H > . \tag{1.3.2}$$

Proof. Fix a positive integer n, $\theta = (\theta_1, \theta_2, .., \theta_n)$ in $\mathbb{R}^n$ and $H_1, H_2, ..., H_n$ in $\mathcal{H}_k$. Since Y_0 is linear, by the weak convergence of Q_N and by the previous Lemma it holds that

$$\log E_Q[\exp\{i\sum_{j=1}^{n} \theta_j Y_0^N(H_j)\}] = \lim_{N\to\infty} \log \mathbb{E}_{\nu_\alpha}[\exp\{iY_0^N(\sum_{j=1}^{n} \theta_j H_j)\}]$$

$$= -\frac{\chi(\alpha)}{2} < \sum_{j=1}^{n} \theta_j H_j, \sum_{j=1}^{n} \theta_j H_j > .$$

So, the Q joint distribution of $(Y_0(H_1), Y_0(H_2), ..., Y_0(H_n))$ is Gaussian with covariance given by (1.3.2). This concludes the proof. □

Since $Y_t(H) = Y_0(T_t H)$ together with the result just proved, it is immediate that the limit density field has covariance given by (1.2.3).

To finish the proof, it remains to show that $(Q_N)_{N\geq 1}$ is tight whose proof follows closely the same arguments as the ones for the Zero-Range process in the diffusive scaling. Lastly, we note that once the process evolves on $\mathbb{Z}$ and the hyperbolic scale is considered, we must take an integer $k > 2$ in order have the density fluctuations field well defined in $\mathcal{H}_{-k}$.

1.4 Law of Large Numbers for the Tagged Particle

In this section we prove Theorem 1.2.2 following the same arguments as Jara and Landim in [12]. Since we are considering the one-dimensional setting, there is an expression that relates the position of the Tagged Particle with the current through a fixed bond and the density of particles. As a consequence, the C.L.T. for the Tagged Particle will follow from the C.L.T. for the empirical measure and the current through a bond. First, we prove convergence of the finite-dimensional distributions of the current through a fixed bond.

If we start with a configuration η with a finite number of particles, we have the following formula

$$J^N_{-1,0}(t) = \sum_{x\geq 0} \Big(\eta_t(x) - \eta_0(x)\Big). \tag{1.4.1}$$

It is easy to see that such current can be written in terms of the density fluctuation field (see 1.2.2) as

$$\frac{1}{\sqrt{N}}\Big\{J^N_{-1,0}(t) - \mathbb{E}_{\nu_\alpha}[J^N_{-1,0}(t)]\Big\} = Y^N_t(H_0) - Y^N_0(H_0),$$

where H_0 is the Heaviside function, $H_0(u) = 1_{[0,\infty)}(u)$. In the general case, the formula (1.4.1) does not makes sense, and the fluctuation field $Y^N_t(H_0)$ is not well defined, but approximating the function H_0 by a sequence $(G_n)_n$, defined for each $u \in \mathbb{R}$ by

$$G_n(u) = (1 - \frac{u}{n})^+ 1_{[0,\infty)}(u), \tag{1.4.2}$$

we obtain:

Proposition 1.4.1. *For every $t \geq 0$,*

$$\lim_{n\to+\infty} \mathbb{E}_{\nu_\alpha}\Big[\Big(\frac{\bar{J}^N_{-1,0}(t)}{\sqrt{N}} - (Y^N_t(G_n) - Y^N_0(G_n))\Big)^2\Big] = 0$$

uniformly in N.

Proof. For a site x, consider the martingale $M_{x,x+1}(t)$ equal to

$$M_{x,x+1}(t) = J^N_{x,x+1}(t) - N\int_0^t W_{x,x+1}(\eta_s)ds, \tag{1.4.3}$$

whose quadratic variation is given by

$$< M_{x,x+1} >_t = N\int_0^t c(x,x+1,\eta_s) + c(x+1,x,\eta_s)ds.$$

Since the number of particles is preserved, it holds that:

$$J^N_{x-1,x}(t) - J^N_{x,x+1}(t) = \eta_t(x) - \eta_0(x)$$

for all $x \in \mathbb{Z}$, $t \geq 0$, and we have that

$$Y^N_t(G_n) - Y^N_0(G_n) = \frac{1}{\sqrt{N}}\sum_{x\in\mathbb{Z}} G_n\Big(\frac{x}{N}\Big)\Big\{\bar{J}^N_{x-1,x}(t) - \bar{J}^N_{x,x+1}(t)\Big\}.$$

Making a summation by parts and using the explicit knowledge of G_n, the right hand side of the last expression is equal to

$$\frac{\bar{J}^N_{-1,0}(t)}{\sqrt{N}} - \frac{1}{\sqrt{N}}\sum_{x=1}^{Nn}\frac{1}{Nn}\bar{J}^N_{x-1,x}(t).$$

So far,

$$\frac{\bar{J}^N_{-1,0}(t)}{\sqrt{N}} - \Big[Y^N_t(G_n) - Y^N_0(G_n)\Big] = \frac{1}{\sqrt{N}}\sum_{x=1}^{Nn}\frac{1}{Nn}\bar{J}^N_{x-1,x}(t).$$

Representing the current $J^N_{x-1,x}(t)$ in terms of the martingales $M_{x-1,x}(t)$, the right hand side of the last expression becomes equal to

$$\frac{1}{\sqrt{N}}\sum_{x=1}^{Nn}\frac{1}{Nn}M_{x-1,x}(t) + \frac{1}{\sqrt{N}}\int_0^t \frac{1}{n}\sum_{x=1}^{Nn}[W_{x-1,x}(\eta_s) - (p-q)\chi(\alpha)]ds. \tag{1.4.4}$$

First we are going to prove that the martingale term converges to 0 in $L^2(\mathbb{P}_{\nu_\alpha})$ as $n \to +\infty$. Estimating their quadratic variation by Nt and using the fact that they are orthogonal, we obtain that

$$\mathbb{E}_{\nu_\alpha}\Big[\Big(\frac{1}{\sqrt{N}}\sum_{x=1}^{Nn}\frac{1}{Nn}M_{x-1,x}(t)\Big)^2\Big] \leq \frac{tC}{Nn}.$$

To prove that the integral term converges to 0 in $L^2(\mathbb{P}_{\nu_\alpha})$, we make an elementary computation to obtain

$$\mathbb{E}_{\nu_\alpha}\Big[\Big(\frac{1}{\sqrt{N}}\int_0^t \frac{1}{n}\sum_{x=1}^{Nn}[W_{x-1,x}(\eta_s)-(p-q)\chi(\alpha)]ds\Big)^2\Big] \leq \frac{t^2\chi(\alpha)(2p^2+2q^2-4pq)}{n}.$$

Taking the limit as $n\to\infty$, our proof is concluded. □

Putting together last result and the C.L.T. for the empirical measure, it holds:

Theorem 1.4.2. *(Central Limit Theorem for the current over a bond)*
Fix $x\in\mathbb{Z}$ and let

$$Z_t^N = \frac{1}{\sqrt{N}}\Big\{J^N_{x,x+1}(t) - \mathbb{E}_{\nu_\alpha}[J^N_{x,x+1}(t)]\Big\}.$$

Then, for every $k\geq 1$ and every $0\leq t_1<t_2<..<t_k$, $(Z^N_{t_1},..,Z^N_{t_k})$ converges in law to a Gaussian vector $(Z_{t_1},..,Z_{t_k})$ with covariance given by

$$E_Q[Z_tZ_s] = \chi(\alpha)|v|s$$

provided $s\leq t$.

Proof. We take $x=0$, but the same argument provides the same result for any $x\in\mathbb{Z}$. Fix $t\geq 0$ and $n\geq 1$. Approximating G_n in $L^2(\mathbb{R})$ by a sequence $(H_{n,k})_{k\geq 1}$ of smooth functions with compact support we have that:

$$\mathbb{E}_{\nu_\alpha}[Y_t^N(H_{n,k}) - Y_t^N(G_n)]^2 \leq \chi(\alpha)||H_{n,k}-G_n||_2.$$

So, $Y_t^N(H_{n,k})$ converges in $L^2(\mathbb{P}_{\nu_\alpha})$ to $Y_t^N(G_n)$. By Theorem 1.2.1, $Y_t^N(H_{n,k})$ converges in law to $Y_t(H_{n,k})$. On the other hand since Y_t is linear and by (1.2.3), $(Y_t(H_{n,k}))_{k\geq1}$ converges to $Y_t(G_n)$, in $L^2(Q)$. Then, we conclude that $Y_t^N(G_n)$ converges in law to $Y_t(G_n)$.

By Proposition 1.4.1, $Y_t^N(G_n)-Y_0^N(G_n)$ is a Cauchy sequence in $L^2(\mathbb{P}_{\nu_\alpha})$ uniformly in N. In particular, $Y_t(G_n)-Y_0(G_n)$ is a Cauchy sequence and converges to a Gaussian limit denoted by $Y_t(H_0)-Y_0(H_0)$.

So, $\frac{\bar{J}^N_{-1,0}(t)}{\sqrt{N}}$ converges in law to $Y_t(H_0)-Y_0(H_0)$.

Using the same argument, we show that $(\frac{\bar{J}^N_{-1,0}(t_1)}{\sqrt{N}},...,\frac{\bar{J}^N_{-1,0}(t_k)}{\sqrt{N}})$ converges in law to $(Y_{t_1}(H_0)-Y_0(H_0),...,Y_{t_k}(H_0)-Y_0(H_0))$.

To compute the variances we do the following:

$$\begin{aligned}
&E_Q[\{Y_t(H_0)-Y_0(H_0)\}\{Y_s(H_0)-Y_0(H_0)\}]\\
&=\lim_{n\to+\infty} E_Q[\{Y_t(G_n)-Y_0(G_n)\}\{Y_s(G_n)-Y_0(G_n)\}]\\
&=\chi(\alpha)\lim_{n\to+\infty}\int_{\mathbb{R}}\Big(T_{t-s}G_nG_n - T_tG_nG_n - T_sG_nG_n + G_n^2\Big)du
\end{aligned}$$

Using that $T_t(G_n)(u)=G_n(u+vt)$ and the form of G_n we get the result stated in the theorem. □

Using the coupling argument described in section two, the L.L.N. and the C.L.T. for the empirical measure and for the current through a fixed bond hold for the ASEP starting from ν_α^*.

Assume now, the initial measure to be ν_α^*. Let X_{tN} be the position of the Tagged Particle at time $tN \geq 0$ initially at the origin. Fix a positive integer n. Since we are considering the one-dimensional setting, particles cannot jumps over other particles, and therefore for a positive integer n the following relation holds:

$$\{X_{tN} \geq n\} = \{J_{-1,0}^N(t) \geq \sum_{x=0}^{n-1} \eta_t(x)\} \tag{1.4.5}$$

which allows, together with the previous results, to obtain L.L.N. and the C.L.T. for the position of the Tagged Particle.

Our attention is now focused in proving **L.L.N. for the Tagged Particle**.

Proof of Theorem 1.2.2.
Denote by $\lceil a \rceil$ the smallest integer larger or equal to a. Fix $u > 0$ and take $n = \lceil uN \rceil$ in the last expression, to obtain

$$\{X_{tN} \geq uN\} = \{J_{-1,0}^N(t) \geq \sum_{x=0}^{\lceil uN \rceil - 1} \eta_t(x)\}.$$

Using the expression for the empirical measure we have that:

$$\{X_{tN} \geq uN\} = \Big\{\frac{J_{-1,0}^N(t)}{N} \geq < \pi_t^N, 1_{[0,u]} > +O(N^{-1})\Big\}.$$

First we prove the L.L.N. for the current. By the martingale decomposition of the current (see expression (1.4.3)), it holds that:

$$\mathbb{E}_{\nu_\alpha}\Big[\frac{J_{-1,0}^N(t)}{N}\Big] = \mathbb{E}_{\nu_\alpha}\Big[\frac{M_{-1,0}(t)}{N}\Big] + \mathbb{E}_{\nu_\alpha}\Big[\int_0^t W_{-1,0}(\eta_s)ds\Big].$$

The first expectation is 0 because $M_{-1,0}(t)$ is a martingale and the second is equal to $(p-q)\chi(\alpha)t$. As a consequence together with Theorem 1.4.2 we have that

$$\frac{J_{-1,0}^N(t)}{N} \xrightarrow[N\to+\infty]{} (p-q)\chi(\alpha)t$$

in $\mathbb{P}_{\nu_\alpha}$-probability.

On the other hand, $< \pi_t^N, 1_{[0,u]} >$ converges in probability to αu, since:

$$\mathbb{E}_{\nu_\alpha}\Big[\Big(< \pi_t^N, 1_{[0,u]} > -\alpha u\Big)^2\Big] = \mathbb{E}_{\nu_\alpha}\Big[\Big(\frac{1}{N}\sum_{x=0}^{Nu}(\eta_t(x) - \alpha)\Big)^2\Big],$$

and the right hand side of the last expression is equal to $\frac{1}{N^2}(Nu+1)\chi(\alpha)$, which vanishes as $N \to \infty$. So, we obtain that:

$$\lim_{N\to+\infty} \mathbb{P}_{\nu_\alpha^*}\Big[\frac{X_{tN}}{N} \geq u\Big] = \begin{cases} 0, & \text{if } (p-q)\chi(\alpha)t < \alpha u \\ 1, & \text{if } (p-q)\chi(\alpha)t \geq \alpha u \end{cases}$$

For $u < 0$ we obtain a similar result. □

1.5 Convergence to Brownian motion

In this section, we prove the convergence of the Tagged Particle process, properly centered and rescaled, to the standard Brownian motion.

Let $W_{tN} = \frac{1}{\sqrt{N}}(X_{tN} - v_t N)$. The result follows from showing the convergence of finite dimensional distributions of W_{tN} to those of Brownian motion together with tightness.

1.5.1 Convergence of finite-dimensional distributions

First, we prove that under $\mathbb{P}_{\nu_\alpha^*}$, for every $k \geq 1$ and every $0 \leq t_1 < ... < t_k$, $(W_{t_1 N}, ..., W_{t_k N})$ converges in law to a Gaussian vector $(W_{t_1}, ..., W_{t_k})$ with mean zero and covariance given by

$$E_Q\Big[W_t W_s\Big] = |p-q|(1-\alpha)s,$$

for $0 \leq s \leq t$.

For that, fix $a > 0$ and let $p > q$. By equation (1.4.5)

$$\{X_{tN} \geq v_t N + a\sqrt{N}\} =$$

$$\Big\{\bar{J}^N_{-1,0}(t) \geq \sum_{x=0}^{v_t N} \bar{\eta}_t(x) + \sum_{x=1}^{a\sqrt{N}-1} \eta_t(x+v_t N) - \Big\{\mathbb{E}_{\nu_\alpha}[J^N_{-1,0}(t)] - \alpha(v_t N+1)\Big\}\Big\}.$$

We now observe that:

$$\mathbb{E}_{\nu_\alpha}\Big[\Big(\frac{\sum_{x=1}^{a\sqrt{N}-1} \eta_t(x+v_t N)}{\sqrt{N}}\Big)^2\Big] = \frac{a\alpha}{\sqrt{N}}. \tag{1.5.1}$$

So, its variance is bounded by $\frac{a\alpha}{\sqrt{N}}$, which implies that $\frac{1}{\sqrt{N}}\sum_{x=1}^{a\sqrt{N}-1} \eta_t(x+v_t N)$ converges in $L^2(\mathbb{P}_{\nu_\alpha})$ to its expectation, which is equal to $a\alpha$.

Using again the martingale decomposition of the current we obtain that

$$\mathbb{E}_{\nu_\alpha}\Big[\frac{J^N_{-1,0}(t)}{\sqrt{N}}\Big] = (p-q)\chi(\alpha)t\sqrt{N},$$

so, $\mathbb{E}_{\nu_\alpha}\Big[\frac{J^N_{-1,0}(t)}{\sqrt{N}}\Big] - \frac{\alpha(v_t N+1)}{\sqrt{N}}$ converges to 0, taking the limit in N.

Finally, by Proposition 1.4.1, for fixed t, $\frac{1}{\sqrt{N}}\Big\{\bar{J}^N_{-1,0}(t)-\sum_{x=0}^{v_tN}\bar{\eta}_t(x)\Big\}$ behaves as $Y^N_t(G_n)-Y^N_0(G_n)-Y^N_t(1_{[0,v_t]})$, as $N\to+\infty$ and $n\to+\infty$. Using the same arguments as in the C.L.T. for the current, we show that $Y^N_t(G_n)-Y^N_0(G_n)-Y^N_t(1_{[0,v_t]})$ converges in law to a centered Gaussian variable, denoted by W_t. So far we have that

$$\lim_{N\to+\infty}\mathbb{P}_{\nu^*_\alpha}\Big[\frac{X_{tN}-v_tN}{\sqrt{N}}\geq a\Big]=\mathbb{P}_{\nu^*_\alpha}[W_t\geq a\alpha]$$

provided that $p>q$. By the symmetry around the origin, we can get the other case: $p<q$ and $a<0$. So, for each fixed t, W_{tN} converges in law to a Gaussian random variable $\frac{W_t}{\alpha}=\frac{Y_t(H_{v_t})-Y_0(H_0)}{\alpha}$, where $H_{v_t}(u)=1_{[v_t,+\infty)}(u)$. By the same argument, it follows that $(W_{t_1N},...,W_{t_kN})$ converges in law to $(\frac{W_{t_1}}{\alpha},...,\frac{W_{t_k}}{\alpha})$.

To compute the variances, do the following:

$$\begin{aligned}
&E_Q[\{Y_t(H_{v_t})-Y_0(H_0)\}\{Y_s(H_{v_s})-Y_0(H_0)\}]\\
&=\lim_{n\to+\infty}E_Q[\{Y_t(F_{n,t})-Y_0(G_n)\}\{Y_s(F_{n,s})-Y_0(G_n)\}]\\
=\chi(\alpha)&\lim_{n\to+\infty}\int_{\mathbb{R}}\Big(T_{t-s}(F_{n,t})G_n-T_t(F_{n,t})G_n-T_s(F_{n,s})G_n+G_n^2\Big)du
\end{aligned}$$

where $F_{n,t}(x)=G_n(x-v_t)$. Using that $T_t(G_n)(u)=G_n(u+vt)$ and the form of G_n we get the result.

1.5.2 Tightness

To end the proof of the theorem it remains to prove tightness. For that we use a relation between the ASEP and a **Zero-Range process**, as Kipnis in [13]. For the latter, the product measures μ_α with marginals given by $\mu_\alpha\{\eta(x)=k\}=\alpha(1-\alpha)^k$ are extremal invariant. This process has space state $\mathbb{N}^{\mathbb{Z}}$ and generator defined on local functions by

$$\Omega f(\eta)=\sum_{x\in\mathbb{Z}}1_{\{\eta(x)\geq1\}}[pf(\eta^{x,x-1})+qf(\eta^{x,x+1})-f(\eta)],$$

where $p+q=1$ and

$$\eta^{x,y}(z)=\begin{cases}\eta(z), & \text{if } z\neq x,y\\ \eta(x)-1, & \text{if } z=x\\ \eta(y)+1, & \text{if } z=y\end{cases}.$$

The process can also be reversed with respect to any μ_α, and the reversed process is denoted by $\hat{\eta}$, whose generator $\hat{\Omega}$ is the same as Ω, except that p is replaced by q and vise-versa.

The position of the Tagged Particle in the Zero-Range representation becomes the current through the bond $[-1,0]$:

$$X_t=-N_t^++N_t^-$$

where N_t^+ (resp. N_t^-) is the number of particles that jumped from site -1 to site 0 during the time interval $[0,t]$ (resp. from site 0 to -1). As a consequence, the proof ends if we show tightness of the distributions of $\frac{v_1(tN)}{\sqrt{N}}$ and $\frac{v_2(tN)}{\sqrt{N}}$, where $v_1(t) = N_t^+ - qt(1-\alpha)$ and $v_2(t) = N_t^- - pt(1-\alpha)$.

With this purpose, we use Theorem 2.1 of [23], with a slightly different definition for weakly positive associated increments given in [24], namely:

Definition 1.5.1. *A process $\{v(t) : t \geq 0\}$ has **weakly positive associated** increments if for all coordinatewise increasing functions $f : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}$*

$$E_{\mu_\alpha}[f(v(t+s) - v(s))g(v(s_1), .., v(s_n))] \geq E_{\mu_\alpha}[f(v(t))]E_{\mu_\alpha}[g(v(s_1), .., v(s_n))],$$

for all $s, t \geq 0$ and $0 \leq s_1 < .. < s_n = s$, (weakly negative associated in the sense of the reversed inequality).

Following the same arguments as in Theorem 2 of [13] we note that the processes N_t^+ and N_t^-, have weakly positive associated increments. For the sake of completeness, we give a sketch of the proof of this result for the process N_t^+.

Let $s, t \geq 0$ and $0 \leq s_1 < ... < s_n = s$, and f, g coordinatewise increasing functions $f : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}$. We have to show that

$$E_{\mu_\alpha}[f(N_{t+s}^+ - N_s^+)g(N_{s_1}^+, .., N_{s_n}^+)] \geq E_{\mu_\alpha}(f(N_t^+))E_{\mu_\alpha}(g(N_{s_1}^+, .., N_{s_n}^+)).$$

Using the Markov property and by reverting the process η_s with respect to μ_α into $\hat{\eta_s}$, we have

$$E_{\mu_\alpha}[f(N_{t+s}^+ - N_s^+)g(N_{s_1}^+, .., N_{s_n}^+)] = \int E_\eta(f(N_t^+))\hat{E}_\eta(g(N_{s_1}^-, .., N_{s_n}^-))d\mu_\alpha.$$

Denote by $\varphi(\eta)$, $\psi(\eta)$ the functions $E_\eta(f(N_t^+))$ and $\hat{E}_\eta(g(N_{s_1}^-, .., N_{s_n}^-))$, respectively. Each one of this functions is increasing in each coordinate $\eta(x)$, because if we add one particle at site x, it can only increase the number of jumps from -1 to 0 (or from 0 to -1). Using Lemma 3 of [13], the right hand side of last expression is bigger than

$$\int E_\eta(f(N_t^+))d\mu_\alpha \int \hat{E}_\eta(g(N_{s_1}^-, ..., N_{s_n}^-))d\mu_\alpha.$$

And reversing the process again we obtain the result. For N_t^- we can use the same argument.

Moreover, both processes have zero-mean and satisfy:

$$\lim_{t\to+\infty} \frac{1}{t} E_{\mu_\alpha}[(v_i(t))^2] = \sigma_i^2$$

for $i = 1, 2$ with $\sigma_i^2 < \infty$, see Theorem 3 of [13]. In particular, the distributions of the processes $\frac{v_1(tN)}{\sqrt{N}}$ and $\frac{v_2(tN)}{\sqrt{N}}$ are tight. The proof, see [23] relies on a maximal inequality [16], which applies to demimartingales. As the processes have weakly positively associated increments and zero-mean, the demimartingale property follows.

1.6 Dependence on the initial configuration

The first result we state concerns the dependence, of the current through a fixed bond, on the initial configuration. Here we suppose that $v > 0$ but for the other case a similar statement holds.

Proposition 1.6.1. *Fix $t \geq 0$ and a site x. Then,*

$$\lim_{N\to+\infty} \mathbb{E}_{\nu_\alpha}\Big[\Big(\frac{\bar{J}^N_{x-1,x}(t)}{\sqrt{N}} - \frac{\sum_{y=x-vtN}^{x-1} \bar{\eta}_0(y)}{\sqrt{N}}\Big)^2\Big] = 0.$$

In the case $\alpha = 1/2$, the normalized current converges to 0 in the $L^2(\mathbb{P}_{\nu_\alpha})$-norm. This result was also obtained before by Ferrari and Fontes in [9].

Proof. Here we consider $x = 0$, but the same argument applied to any site x provides the corresponding result.

By Proposition (4.1), $\frac{\bar{J}^N_{-1,0}(t)}{\sqrt{N}} - (Y^N_t(G_n) - Y^N_0(G_n))$ converges to zero in $L^2(\mathbb{P}_{\nu_\alpha})$, as $n \to +\infty$, uniformly over N.

On the other hand, $Y^N_t(G_n) - Y^N_0(T_t G_n)$ converges to 0 in the $L^2(\mathbb{P}_{\nu_\alpha})$-norm, where $T_t H(u) = H(u + vt)$, taking the limit in N.

The idea is the following: define the density fluctuation field as

$$\tilde{Y}^N_t(H) = \frac{1}{\sqrt{N}} \sum_{x\in\mathbb{Z}} H\Big(\frac{x - vtN}{N}\Big)[\eta_{tN}(x) - \alpha].$$

As before, we have that

$$M^{N,H}_t = \tilde{Y}^N_t(H) - \tilde{Y}^N_0(H) - \int_0^t \Gamma_1(s)ds,$$

where for $U_t H(u) = H(u - vt)$

$$\Gamma_1(s) = \frac{1}{\sqrt{N}} \sum_{x\in\mathbb{Z}} \nabla^N U_s H\Big(\frac{x}{N}\Big)W_{x,x+1}(\eta_s) - \frac{1}{\sqrt{N}} \sum_{x\in\mathbb{Z}} \partial_u U_s H\Big(\frac{x}{N}\Big)v[\eta_s(x) - \alpha],$$

and

$$(M^{N,H}_t)^2 - \int_0^t \frac{1}{N^2} \sum_{x\in\mathbb{Z}} \Big(\nabla^N U_s H\Big(\frac{x}{N}\Big)\Big)^2 [c(x, x+1, \eta_s) + c(x+1, x, \eta_s)]ds$$

are martingales with respect to the natural filtration.

Since $\mathbb{E}_{\nu_\alpha}[(M^{N,H}_t)^2]$ vanishes as $N \to +\infty$ and by the Boltzmann-Gibbs Principle (see Theorem 1.3.1), we have that

$$\lim_{N\to+\infty} \mathbb{E}_{\nu_\alpha}\Big[\Big(\tilde{Y}^N_t(H) - \tilde{Y}^N_0(H)\Big)^2\Big] = 0.$$

But this corresponds to $Y_t^N(H) - Y_0^N(T_tH)$ converging to 0 in the $L^2(\mathbb{P}_{\nu_\alpha})$-norm, for every $H \in S(\mathbb{R})$. The result is accomplished for G_n, by approximating them by functions for which the result holds, just like in the proof of Theorem 1.4.2.

In order to finish the proof it remains to show that

$$\lim_{n\to+\infty} \mathbb{E}_{\nu_\alpha}\Big[\Big(Y_0^N(T_tG_n) - Y_0^N(G_n) - \frac{1}{\sqrt{N}}\sum_{x=-vtN}^{-1}\bar{\eta}_0(x)\Big)^2\Big] = 0,$$

uniformly over N.

Using the explicit knowledge of G_n and since ν_α is a product measure, the expectation can be bounded by:

$$\mathbb{E}_{\nu_\alpha}\Big[\Big(\frac{1}{\sqrt{N}}\sum_{x=-vtN}^{-1}\Big(\frac{x+vtN}{Nn}\Big)\bar{\eta}_0(x)\Big)^2\Big] + \mathbb{E}_{\nu_\alpha}\Big[\Big(\frac{vt}{n\sqrt{N}}\sum_{x=0}^{-vtN+Nn}\bar{\eta}_0(x)\Big)^2\Big]$$

$$+ \quad \mathbb{E}_{\nu_\alpha}\Big[\Big(\frac{1}{\sqrt{N}}\sum_{x=-vtN+nN+1}^{Nn}\Big(1-\frac{x}{Nn}\Big)\eta_0(x)\Big)^2\Big].$$

It is easily seen that this is of order $O(1/n)$, which concludes the proof.

□

Since both, the current over a bond and the density fluctuation field at time t, can be written in terms of the initial configuration, and since (1.4.5) holds, it is natural that the position of the Tagged Particle also enjoys this property. That is the content of the Corollary 1.2.4, whose proof we start to present.

Proof of Corollary 1.2.4.

We are going to show the convergence in $\mathbb{P}_{\nu_\alpha^*}$-probability to 0 of the random variable appearing in the statement of the Corollary, and then we show that its $L^2(\mathbb{P}_{\nu_\alpha^*})$-norm is finite, which allows to conclude the convergence to 0 in $L^{2-\epsilon}(\mathbb{P}_{\nu_\alpha^*})$, for any $\epsilon > 0$.

With that purpose, start by summing and subtracting the expectation of X_{tN}, namely v_tN, in the expression that appears in the statement of the Corollary, and it becomes as:

$$\frac{\bar{X}_{tN}}{\sqrt{N}} + \frac{\sum_{x=1}^{(p-q)\alpha tN}\bar{\eta}_0(x)}{\alpha\sqrt{N}}.$$

We start by showing that last expression converges to zero in $\mathbb{P}_{\nu_\alpha^*}$-probability as $N \to +\infty$.

At first note that by the rigid transport of the system it holds that:

$$\lim_{N\to+\infty} \mathbb{E}_{\nu_\alpha}\Big[\Big(\frac{\sum_{x=1+vtN}^{v_tN}\bar{\eta}_t(x)}{\alpha\sqrt{N}} - \frac{\sum_{x=1}^{(p-q)\alpha tN}\bar{\eta}_0(x)}{\alpha\sqrt{N}}\Big)^2\Big] = 0.$$

As a consequence we have to show that:

$$\frac{\bar{X}_{tN}}{\sqrt{N}} + \frac{\sum_{x=1+vtN}^{v_tN}\bar{\eta}_t(x)}{\alpha\sqrt{N}},$$

converges to zero in $\mathbb{P}_{\nu_\alpha^*}$-probability as $N \to +\infty$.

In order to keep notation simple we denote by Z_t^N the random variable:

$$Z_t^N = -\sum_{x=1+vtN}^{v_tN} \bar{\eta}_t(x)/\alpha.$$

Notice that $v_tN + Z_t^N$ is a positive random variable since it corresponds to the number of holes in the interval $[1+vtN, v_tN]$.

Fix $a > 0$, take $n = a\sqrt{N} + v_tN + Z_t^N$ in the expression that relates the position of the Tagged Particle with the current through the bond $[-1, 0]$ and the density of particles, see (1.4.5):

$$\{X_{tN} \geq v_tN + a\sqrt{N} + Z_t^N\} = \Big\{J_{-1,0}^N(t) \geq \sum_{x=0}^{v_tN} \eta_t(x) + \sum_{x=1+v_tN}^{a\sqrt{N}-1+v_tN+Z_t^N} \eta_t(x)\Big\}.$$

Introducing the mean of the current, last expression becomes as

$$\{X_{tN} \geq v_tN + a\sqrt{N} + Z_t^N\} = \Big\{\bar{J}_{-1,0}^N(t) \geq \sum_{x=0}^{v_tN} \bar{\eta}_t(x) + \sum_{x=1+v_tN}^{a\sqrt{N}-1+v_tN+Z_t^N} \eta_t(x)\Big\}.$$

Now, we can divide all the terms by $\sqrt{N}$ and then, subtract the mean of the random variable on the right hand side of last inequality to obtain:

$$\{\frac{\bar{X}_{tN}}{\sqrt{N}} - \frac{Z_t^N}{\sqrt{N}} \geq a\} =$$

$$\Big\{\frac{\bar{J}_{-1,0}^N(t)}{\sqrt{N}} \geq \frac{\sum_{x=0}^{v_tN} \bar{\eta}_t(x)}{\sqrt{N}} + \frac{\sum_{x=1+v_tN}^{a\sqrt{N}-1+v_tN+Z_t^N} \bar{\eta}_t(x)}{\sqrt{N}} + \alpha a + \frac{\alpha Z_t^N}{\sqrt{N}}\Big\}.$$

By Proposition 1.6.1, T_t^N converges to zero in $L^2(\mathbb{P}_{\nu_\alpha})$ where

$$T_t^N = \frac{\bar{J}_{-1,0}^N(t)}{\sqrt{N}} - \frac{1}{\sqrt{N}} \sum_{x=-vtN}^{-1} \bar{\eta}_0(x),$$

which together with the Boltzmann-Gibbs Principle gives us that:

$$\mathbb{P}_{\nu_\alpha^*}\Big\{\frac{\bar{X}_{tN}}{\sqrt{N}} + \frac{\sum_{x=1+vtN}^{v_tN} \bar{\eta}_t(x)}{\alpha\sqrt{N}} \geq a\Big\} =$$

$$\mathbb{P}_{\nu_\alpha^*}\Big\{\frac{\sum_{x=0}^{-1+vtN} \bar{\eta}_t(x)}{\sqrt{N}} \geq \frac{\sum_{x=0}^{v_tN} \bar{\eta}_t(x)}{\sqrt{N}} + \frac{\sum_{x=1+v_tN}^{a\sqrt{N}-1+v_tN+Z_t^N} \bar{\eta}_t(x)}{\sqrt{N}} + \alpha a + \frac{\alpha Z_t^N}{\sqrt{N}}\Big\}$$

Now observe that:

$$\mathbb{E}_{\nu_\alpha}\Big[\Big(\frac{1}{\sqrt{N}}\sum_{x=1+v_tN}^{a\sqrt{N}-1+v_tN+Z_t^N}\bar{\eta}_t(x)\Big)^2\Big]=O(N^{-1/2}),$$

whose proof is presented at the end in order to simplify the exposition. Therefore, for N sufficiently big we have that

$$\mathbb{P}_{\nu_\alpha^*}\{\frac{\bar{X}_{tN}}{\sqrt{N}}+\frac{\sum_{x=1+vtN}^{v_tN}\bar{\eta}_t(x)}{\alpha\sqrt{N}}\geq a\}=\mathbb{P}_{\nu_\alpha^*}\Big\{0\geq\frac{\sum_{x=vtN}^{v_tN}\bar{\eta}_t(x)}{\sqrt{N}}+\alpha a-\frac{\sum_{x=1+vtN}^{v_tN}\bar{\eta}_t(x)}{\sqrt{N}}\Big\}$$

which concludes the first step of the proof.

For the $L^{2-\epsilon}(\mathbb{P}_{\nu_\alpha^*})$ convergence, it remains to show that:

$$\sup_N\mathbb{E}_{\nu_\alpha^*}\Big[\Big(\frac{\bar{X}_{tN}}{\sqrt{N}}+\frac{\sum_{x=1}^{(p-q)\alpha tN}\bar{\eta}_0(x)}{\alpha\sqrt{N}}\Big)^2\Big]<+\infty.$$

Last result is a consequence of ν_α being a product measure, which implies that:

$$\mathbb{E}_{\nu_\alpha^*}\Big[\Big(\frac{\sum_{x=1}^{(p-q)\alpha tN}\eta_0(x)}{\sqrt{N}}\Big)^2\Big]\leq(p-q)(1-\alpha)t;$$

together with a result due to De Masi and Ferrari in [5]:

$$\lim_{N\to+\infty}\mathbb{E}_{\nu_\alpha^*}\Big[\Big(\frac{\bar{X}_{tN}}{\sqrt{N}}\Big)^2\Big]=(p-q)(1-\alpha)t.$$

In order to finish the proof it is enough to show that:

$$\mathbb{E}_{\nu_\alpha}\Big[\Big(\frac{1}{\sqrt{N}}\sum_{x=1+v_tN}^{a\sqrt{N}-1+v_tN+Z_t^N}\bar{\eta}_t(x)\Big)^2\Big]=O(N^{-1/2}).$$

To simplify the computations we take $p=1$, nevertheless the case $p\neq 1$ follows the same lines. Since ν_α is an invariant measure, last expectation can be written as:

$$\int\frac{1}{N}\sum_{x,y=1+v_tN}^{a\sqrt{N}-1+v_tN+Z^N}\bar{\eta}(x)\bar{\eta}(y)\nu_\alpha(d\eta)=0,\qquad(1.6.1)$$

where Z^N is equal to:

$$Z^N=-\sum_{x=1+vtN}^{v_tN}\bar{\eta}(x)/\alpha.$$

Notice that Z^N depends on the variables $\eta(x)$ for x depending on the sites from $1+vtN$ to v_tN, while the sum depends on the random variables $\eta(x)$ for x runing through $1+v_tN$

to $a\sqrt{N}-1+v_tN+Z^N$. So, we can separate the sum in (1.6.1) into the sites where the random variables appearing in the sum and Z^N are independent from the sites where they correlate and it becomes as:

$$\int_{\{a\sqrt{N}-1+v_tN+Z^N\geq 1+v_tN\}}\frac{1}{N}\sum_{x,y=1+v_tN}^{a\sqrt{N}-1+v_tN+Z^N}\bar{\eta}(x)\bar{\eta}(y)\nu_\alpha(d\eta)=0$$

$$+\int_{\{a\sqrt{N}-1+v_tN+Z^N<1+v_tN\}}\frac{1}{N}\sum_{x,y=1+v_tN}^{a\sqrt{N}-1+v_tN+Z^N}\bar{\eta}(x)\bar{\eta}(y)\nu_\alpha(d\eta)=0.$$

By independence the first integral is non zero as long as $x=y$ and it equals:

$$\frac{\alpha^2}{N}\int_{\{a\sqrt{N}+Z^N\geq 2\}}\Big(a\sqrt{N}+Z^N-1\Big)\nu_\alpha(d\eta), \tag{1.6.2}$$

while the second can be written as:

$$\frac{1}{N}\int_{\{a\sqrt{N}+Z^N<2\}}\sum_{x,y=a\sqrt{N}-1+v_tN+Z^N}^{1+v_tN}\eta(x)\eta(y)\nu_\alpha(d\eta) \tag{1.6.3}$$

$$-\frac{2\alpha}{N}\int_{\{a\sqrt{N}+Z^N<2\}}\sum_{x,y=a\sqrt{N}-1+v_tN+Z^N}^{1+v_tN}\eta(x)\nu_\alpha(d\eta) \tag{1.6.4}$$

$$+\frac{\alpha^2}{N}\int_{\{a\sqrt{N}+Z^N<2\}}\sum_{x,y=a\sqrt{N}-1+v_tN+Z^N}^{1+v_tN}\nu_\alpha(d\eta). \tag{1.6.5}$$

Now, we give the route to proceed in the computations. For $j=1,2$, let $Z^{N,j}$ be the random variable:

$$Z^{N,j}=-\sum_{x=1+vtN}^{v_tN-j}\bar{\eta}(x)/\alpha.$$

Estimate (1.6.3) by separating the case $x=y$ from the case $x\neq y$. In the first one the integral becomes as:

$$\frac{\alpha}{N}\int_{\{a\sqrt{N}+Z^{N,1}<2+(1-\alpha)/\alpha\}}\Big(2+\frac{(1-\alpha)}{\alpha}-a\sqrt{N}-Z^{N,1}\Big)\nu_\alpha(d\eta),$$

while in the case $x\neq y$ it becomes as:

$$\frac{\alpha^2}{N}\int_{\{a\sqrt{N}+Z^{N,2}<2+2(1-\alpha)/\alpha\}}\Big(2+\frac{2(1-\alpha)}{\alpha}-a\sqrt{N}-Z^{N,2}\Big)\Big(3-a\sqrt{N}-Z^{N,2}\Big)\nu_\alpha(d\eta).$$

On the other hand, (1.6.4) can be written as

$$\frac{-\alpha}{N}\int_{\{a\sqrt{N}+Z^{N,1}<2+(1-\alpha)/\alpha\}}\Big(2+\frac{(1-\alpha)}{\alpha}-a\sqrt{N}-Z^{N,1}\Big)^2\nu_\alpha(d\eta),$$

while (1.6.5) is equal to

$$\frac{\alpha^2}{N}\int_{\{a\sqrt{N}+Z^N<2\}}\Big(2-a\sqrt{N}-Z^N\Big)^2\nu_\alpha(d\eta). \tag{1.6.6}$$

Now, it remains to write all the integrals with respect to the random variable $Z^{N,2}$. Since the Bernoulli product measure is homogenous we condition on $\eta(x)=0$ and $\eta(x)=1$ for some site $x\in[1+vtN, v_tN]$, to write the integrals (1.6.2) and (1.6.6) in terms of $Z^{N,1}$. Them we repeat the same procedure to write the remaining integrals in terms of $Z^{N,2}$. Organizing them all, the result follows. □

1.7 Density Fluctuations in a longer time scale

Here we are focused in proving Theorem 1.2.5. Fix a positive integer k and recall the definition of the density fluctuation field in (1.2.5), namely, the linear functional acting on functions $H\in S(\mathbb{R})$ as:

$$Y_t^N(H)=\frac{1}{\sqrt{N}}\sum_{x\in\mathbb{Z}}H\Big(\frac{x-vtN^{1+\gamma}}{N}\Big)(\eta_{tN^{1+\gamma}}(x)-\alpha).$$

In order to keep notation simple we use $U_t^NH(u)=H(u-vtN^\gamma)$.

Recall that Q_N^γ is the probability measure on $D(\mathbb{R}^+,\mathcal{H}_{-k})$ induced by the density fluctuation field $Y_\cdot^{N,\gamma}$ and ν_α; and $\mathbb{P}_{\nu_\alpha}^\gamma$ is the probability measure on $D(\mathbb{R}^+,\{0,1\}^\mathbb{Z})$ induced by ν_α and the Markov process η_t speeded up by $N^{1+\gamma}$.

As before, we need to prove that the sequence of probability measures $(Q_N^\gamma)_N$ is tight and to characterize the limit field. We start by the latter, while the former is refereed to the ninth section.

Fix a function $H\in S(\mathbb{R})$. By Lemma A1.5.1 of [14]

$$M_t^{N,H}=Y_t^N(H)-Y_0^N(H)-\int_0^t\Gamma_1^H(s)ds, \tag{1.7.1}$$

is a martingale with respect to $\tilde{\mathcal{F}}_t=\sigma(\eta_s, s\le t)$, where $\Gamma_1^H(s)$ is equal to

$$\frac{N^\gamma}{\sqrt{N}}\sum_{x\in\mathbb{Z}}\nabla^NU_s^NH\Big(\frac{x}{N}\Big)W_{x,x+1}(\eta_s)-\frac{N^\gamma}{\sqrt{N}}\sum_{x\in\mathbb{Z}}\partial_uU_s^NH\Big(\frac{x}{N}\Big)v[\eta_s(x)-\alpha], \tag{1.7.2}$$

and whose quadratic variation is given by

$$\int_0^t\frac{N^\gamma}{N^2}\sum_{x\in\mathbb{Z}}\Big(\nabla^NU_s^NH\Big(\frac{x}{N}\Big)\Big)^2\Big[c(x,x+1,\eta_s)+c(x+1,x,\eta_s)\Big]ds. \tag{1.7.3}$$

Easily one shows that the $L^2(\mathbb{P}_{\nu_\alpha}^\gamma)$-norm of $M_t^{N,H}$ vanishes as $N\to+\infty$ as long as $\gamma<1$. Then, under a sub-diffusive time scale regime, the only term contributing to the limit density fluctuation field is its integral part, since its quadratic variation vanishes. The characterization of the limit of the integral part of the martingale is known as the Boltzmann-Gibbs Principle and is the main difficulty when showing the equilibrium fluctuations. In that scaling regime the time evolution of the limit density fluctuation field is given in a similar way to (1.3.1). But when one takes the diffusive scaling a new contribution arises, since the quadratic variation of the martingale does not vanishes, which agrees with the fact that in order to observe fluctuations from the dynamics one has to take this time scale.

Now, we proceed by proving that the integral part of the martingale $M_t^{N,H}$ vanishes in $L^2(\mathbb{P}^{\gamma}_{\nu_\alpha})$ as $N \to +\infty$. Since $\sum_{x\in\mathbb{Z}} \nabla^N U_s^N H\Big(\frac{x}{N}\Big) = 0$, we can introduce it times $E_{\nu_\alpha}[W_{x,x+1}(\eta_s)]$, in the integral part of the martingale $M_t^{N,H}$. Using the **decomposition of the instantaneous current**

$$\bar{W}_{0,1}(\eta) = -(p-q)\bar{\eta}(0)\bar{\eta}(1) - (q(1-\alpha)+p\alpha)[\bar{\eta}(1)-\bar{\eta}(0)] + v[\eta(0)-\alpha], \tag{1.7.4}$$

it becomes as:

$$\int_0^t \frac{N^\gamma}{\sqrt{N}} \sum_{x\in\mathbb{Z}} \nabla^N U_s^N H\Big(\frac{x}{N}\Big)(q-p)\bar{\eta}_s(x)\bar{\eta}_s(x+1)ds \tag{1.7.5}$$

$$+\int_0^t \frac{N^\gamma}{\sqrt{N}} \sum_{x\in\mathbb{Z}} \nabla^N U_s^N H\Big(\frac{x}{N}\Big)(q(1-\alpha)+p\alpha)[\bar{\eta}_s(x+1)-\bar{\eta}_s(x)]ds$$

$$+\int_0^t \frac{N^\gamma}{\sqrt{N}} \sum_{x\in\mathbb{Z}} \Big\{\nabla^N U_s^N H\Big(\frac{x}{N}\Big) - \partial_u U_s^N H\Big(\frac{x}{N}\Big)\Big\} v[\eta_s(x)-\alpha]ds.$$

By a summation by parts, we write the second integral as

$$\int_0^t \frac{N^\gamma}{N^{3/2}} \sum_{x\in\mathbb{Z}} \Delta_N U_s^N H\Big(\frac{x}{N}\Big) C\bar{\eta}_s(x)ds,$$

where

$$\Delta_N H\Big(\frac{x}{N}\Big) = N^2\Big(H\Big(\frac{x+1}{N}\Big) + H\Big(\frac{x-1}{N}\Big) - 2H\Big(\frac{x}{N}\Big)\Big).$$

By Schwarz inequality and since ν_α is a product invariant measure, this expression vanishes in the $L^2(\mathbb{P}^{\gamma}_{\nu_\alpha})$-norm as $N \to +\infty$, while by a Taylor expansion last integral vanishes as $N \to +\infty$. Once more, last results hold as long as $\gamma < 1$.

It remains to show that the $L^2(\mathbb{P}^{\gamma}_{\nu_\alpha})$-norm of the first integral vanishes as $N \to +\infty$. For that, we use the Botzmann-Gibbs Principle, which is proved in the next section. This result is accomplished for $\gamma < 1/3$, but it should hold for $\gamma < 1/2$ as conjectured. We also remark, that almost all the subsequent results rely on the Boltzmann-Gibbs Principle and if one shows that it holds for $\gamma < 1/2$, one can establish the same results up to the time scale $N^{3/2}$.

Assuming that $(Q_N^\gamma)_N$ is tight, it has convergent subsequences. Let Q be one of its limiting points. By the results proved so far, under Q, the density fluctuation field satisfies $Y_t(H) = Y_0(H)$.

As above, for $t \geq 0$ let $\mathcal{F}_t$ be the σ-algebra on $D([0,T],\mathcal{H}_{-k})$ generated by $Y_s(H)$ for $s \leq t$ and H in $S(\mathbb{R})$. It is not hard to show as in chap. 11 of [14], that up to this longer time scale $N^{4/3}$, Q restricted to $\mathcal{F}_0$ is a Gaussian field with covariance given by $E_Q(Y_0(G)Y_0(H)) = \chi(\alpha) < G, H >$ and it is trivial to see that the density field has covariance given by (1.2.6). This concludes the proof of Theorem 1.2.5.

1.8 Boltzmann-Gibbs Principle

In this section we prove Theorem 1.2.6.

Fix $H \in S(\mathbb{R})$ and an integer K. We divide $\mathbb{Z}$ in non overlapping intervals of length K, denoted by $\{I_j, j \geq 1\}$. Then, the expectation that appears in the statement of the Theorem, can be written as:

$$\mathbb{E}^{\gamma}_{\nu_\alpha}\Big[\Big(\int_0^t \frac{N^\gamma}{\sqrt{N}}\sum_{j\geq 1}\sum_{x\in I_j} H\Big(\frac{x}{N}\Big)\bar{\eta}_s(x)\bar{\eta}_s(x+1)ds\Big)^2\Big].$$

In order to have independence of $\bar{\eta}(x)\bar{\eta}(x+1)$ and $\bar{\eta}(y)\bar{\eta}(y+1)$ for x and y in different I_j's, we separate the sum over the intervals I_j for j odd, and j even. Let $\mathcal{O}$ denote the set of odd numbers while $\mathcal{E}$ denote the set of even numbers. So, in fact it remains to bound

$$\mathbb{E}^{\gamma}_{\nu_\alpha}\Big[\Big(\int_0^t \frac{N^\gamma}{\sqrt{N}}\sum_{j\in\mathcal{O}}\sum_{x\in I_j} H\Big(\frac{x}{N}\Big)\bar{\eta}_s(x)\bar{\eta}_s(x+1)ds\Big)^2\Big]. \tag{1.8.1}$$

The case $j \in \mathcal{E}$ follows from the same argument.

Notice that, in this setting, every $x \in I_j$ and $y \in I_l$, for $j \neq l$, are at least at a distance K.

Now, sum and subtract $H\Big(\frac{y_j}{N}\Big)$, where y_j is one point of the interval I_j, inside the summation over x. Since $(x+y)^2 \leq 2x^2 + 2y^2$, expression (1.8.1) can be bounded by

$$2\mathbb{E}^{\gamma}_{\nu_\alpha}\Big[\Big(\int_0^t \frac{N^\gamma}{\sqrt{N}}\sum_{j\in\mathcal{O}}\sum_{x\in I_j}\Big[H\Big(\frac{x}{N}\Big) - H\Big(\frac{y_j}{N}\Big)\Big]\bar{\eta}_s(x)\bar{\eta}_s(x+1)ds\Big)^2\Big]$$
$$+2\mathbb{E}^{\gamma}_{\nu_\alpha}\Big[\Big(\int_0^t \frac{N^\gamma}{\sqrt{N}}\sum_{j\in\mathcal{O}} H\Big(\frac{y_j}{N}\Big)\sum_{x\in I_j}\bar{\eta}_s(x)\bar{\eta}_s(x+1)ds\Big)^2\Big]. \tag{1.8.2}$$

We are going to estimate each term separately and divide the proof in several lemmas, to make the exposition clearer. We start by the former.

Lemma 1.8.1. *For every $H \in S(\mathbb{R})$ and every $t > 0$, if $KN^{\gamma-1} \to 0$ as $N \to +\infty$, then*

$$\lim_{N\to\infty}\mathbb{E}^{\gamma}_{\nu_\alpha}\Big[\Big(\int_0^t \frac{N^\gamma}{\sqrt{N}}\sum_{j\in\mathcal{O}}\sum_{x\in I_j}\Big[H\Big(\frac{x}{N}\Big) - H\Big(\frac{y_j}{N}\Big)\Big]\bar{\eta}_s(x)\bar{\eta}_s(x+1)ds\Big)^2\Big] = 0.$$

Proof. By Schwarz inequality and since ν_α is an invariant product measure, the expectation is bounded by $Ct^2\frac{N^{2\gamma}}{N}\sum_j\sum_{x\in I_j}\Big(H'\Big(\frac{y_j}{N}\Big)\Big)^2\Big(\frac{|x-y_j|}{N}\Big)^2$. Since x and y_j are in the I_j intervals, that has size K, we can bound last expression by $Ct^2N^{2\gamma}||H'||_2^2(\frac{K}{N})^2$ that vanishes as long as $KN^{\gamma-1}\to 0$ when $N\to+\infty$. □

Now, we bound expression (1.8.2). We sum and subtract the expectation of $\sum_{x\in I_j}\bar{\eta}_s(x)\bar{\eta}_s(x+1)$ conditioned on the hyperplanes $M_j=\sigma\Big(\sum_{x\in I_j^*}\eta(x)\Big)$, where $I_j^*=I_j\bigcup\{x_{j+1}\}$, if $I_j=\{x_0,x_1,..,x_j\}$.

Using again the elementary inequality $(x+y)^2\leq 2x^2+2y^2$, the expectation in (1.8.2) is bounded by

$$2\mathbb{E}^\gamma_{\nu_\alpha}\Big[\Big(\int_0^t\frac{N^\gamma}{\sqrt{N}}\sum_{j\in\mathcal{O}}H\Big(\frac{y_j}{N}\Big)V_j(\eta_s)ds\Big)^2\Big] \tag{1.8.3}$$

$$+2\mathbb{E}^\gamma_{\nu_\alpha}\Big[\Big(\int_0^t\frac{N^\gamma}{\sqrt{N}}\sum_{j\in\mathcal{O}}H\Big(\frac{y_j}{N}\Big)E\Big(\sum_{x\in I_j}\bar{\eta}_s(x)\bar{\eta}_s(x+1)\Big|M_j\Big)ds\Big)^2\Big] \tag{1.8.4}$$

where

$$V_j(\eta)=\sum_{x\in I_j}\bar{\eta}(x)\bar{\eta}(x+1)-E\Big(\sum_{x\in I_j}\bar{\eta}(x)\bar{\eta}(x+1)\Big|M_j\Big).$$

Once more, we bound the integrals separately. We start by bounding (1.8.3).

Lemma 1.8.2. *For every $H\in S(\mathbb{R})$ and every $t>0$, if $K^2N^{\gamma-1}\to 0$ as $N\to+\infty$, then*

$$\lim_{N\to\infty}\mathbb{E}^\gamma_{\nu_\alpha}\Big[\Big(\int_0^t\frac{N^\gamma}{\sqrt{N}}\sum_{j\in\mathcal{O}}H\Big(\frac{y_j}{N}\Big)V_j(\eta_s)ds\Big)^2\Big]=0.$$

Proof. For $f,g\in L^2(\nu_\alpha)$ define the inner product $<f,-Lg>_{\nu_\alpha}$. Let H_1 be the Hilbert space generated by $L^2(\nu_\alpha)$ and this inner product. Denote by $||\cdot||_1$ the norm induced by this inner product and let $||\cdot||_{-1}$ be its dual norm with respect to $L^2(\nu_\alpha)$:

$$||f||_{-1}=\sup_{g\in L^2(\nu_\alpha)}\Big\{2<f,g>_{\nu_\alpha}-||g||_1\Big\}. \tag{1.8.5}$$

By definition for every $f\in H_{-1}$, $g\in L^2(\nu_\alpha)$ and $A>0$ it holds that:

$$2<f,g>_{\nu_\alpha}\leq\frac{1}{A}||f||_{-1}+A||g||_1. \tag{1.8.6}$$

By proposition A1.6.1 of [14], the expectation in the statement of the Lemma is bounded by

$$Ct\Big|\Big|\frac{N^\gamma}{\sqrt{N}}\sum_{j\in\mathcal{O}}H\Big(\frac{y_j}{N}\Big)V_j\Big|\Big|^2_{-1},$$

where C is a constant. By the variational formula for the H_{-1}-norm (1.8.5), last expression is equal to

$$Ct \sup_{h\in L^2(\nu_\alpha)} \Big\{ 2\int \frac{N^\gamma}{\sqrt{N}} \sum_{j\in\mathcal{O}} H\Big(\frac{y_j}{N}\Big) V_j(\eta)h(\eta)\nu_\alpha(d\eta) - N^{1+\gamma} < h, -L_N^S h >_\alpha \Big\},$$

and is bounded by

$$Ct \sum_{j\in\mathcal{O}} \sup_{h\in L^2(\nu_\alpha)} \Big\{ 2\int \frac{N^\gamma}{\sqrt{N}} H\Big(\frac{y_j}{N}\Big) V_j(\eta)h(\eta)\nu_\alpha(d\eta) - N^{1+\gamma} < h, -L_{I_j^*}^S h >_\alpha \Big\},$$

where $L_{I_j^*}^S$ denotes the restriction of the generator of the one-dimensional Symmetric Simple Exclusion process (SSEP) that we denote by L_N^S, to the set I_j^*, namely:

$$L_{I_j^*}^S f(\eta) = \sum_{\substack{x,y\in I_j^*\\ |x-y|=1}} \frac{1}{2}\eta(x)(1-\eta(y))[f(\eta^{x,y}) - f(\eta)].$$

Since $E(V_j|M_j) = 0$, V_j belongs to the image of the generator $L_{I_j^*}^S$. Therefore, by (1.8.6) for each j and A_j a positive constant it holds that

$$\int V_j(\eta)h(\eta)\nu_\alpha(d\eta) \leq \frac{1}{2A_j} < V_j, (-L_{I_j^*}^S)^{-1}V_j >_\alpha + \frac{A_j}{2} < h, -L_{I_j^*}^S h >_\alpha.$$

Taking for each j, $A_j = \frac{N^{3/2}}{|H(\frac{y_j}{N})|}$, the expectation becomes bounded by

$$Ct \sum_{j\in\mathcal{O}} \frac{N^\gamma}{N^2} H^2\Big(\frac{y_j}{N}\Big) < V_j, (-L_{I_j^*}^S)^{-1}V_j >_\alpha,$$

since the other term cancels with the H_1-norm of h. By the spectral gap inequality for the SSEP, see [18], the last expression can be bounded by

$$Ct \sum_{j\in\mathcal{O}} \frac{N^\gamma}{N^2} H^2\Big(\frac{y_j}{N}\Big) (K+1)^2 Var(V_j, \nu_\alpha).$$

Now we observe that, since we are considering the extended interval I_j^*, it holds that

$$E\Big(\bar\eta(x)\bar\eta(x+1)\Big| M_j\Big) = (\eta^{K+1} - \alpha)^2 - \frac{1}{K}\eta^{K+1}(1-\eta^{K+1}),$$

where $\eta^{K+1} = \frac{1}{K+1}\sum_{x\in I_j^*}\eta(x)$. By a simple computation it is not hard to show that $Var(V_j, \nu_\alpha) \leq KC$. Then, the integral becomes bounded by $Ct\frac{N^\gamma}{N}(K+1)^2||H||_2^2$, which vanishes as long as $K^2N^{\gamma-1} \to 0$ when $N \to +\infty$. $\square$

To conclude the proof of the theorem it remains to bound (1.8.4). The idea we use to proceed consists in doing the following. Fix an integer L and consider bigger disjoint intervals of length $M = LK$, denoted by $\{\tilde{I}_l, l \geq 1\}$. In this setting, we consider L sets of size K together and we are able to write the expectation appearing in (1.8.4) as:

$$\mathbb{E}^{\gamma}_{\nu_\alpha}\Big[\Big(\int_0^t \frac{N^\gamma}{\sqrt{N}} \sum_{l\geq 1}\sum_{j\in\tilde{I}_l} H\Big(\frac{y_j}{N}\Big) E\Big(\sum_{x\in I_j} \bar{\eta}_s(x)\bar{\eta}_s(x+1)\Big| M_j\Big) ds\Big)^2\Big]. \tag{1.8.7}$$

As before, sum and subtract $H\Big(\frac{z_l}{N}\Big)$, where z_l denotes one point of the interval $\tilde{I}_l$, inside the summation over j. Since $(x+y)^2 \leq 2x^2 + 2y^2$, the expectation (1.8.7) can be bounded by

$$2\mathbb{E}^{\gamma}_{\nu_\alpha}\Big[\Big(\int_0^t \frac{N^\gamma}{\sqrt{N}} \sum_{l\geq 1}\sum_{j\in\tilde{I}_l} \Big[H\Big(\frac{y_j}{N}\Big) - H\Big(\frac{z_l}{N}\Big)\Big] E\Big(\sum_{x\in I_j} \bar{\eta}_s(x)\bar{\eta}_s(x+1)\Big| M_j\Big) ds\Big)^2\Big]$$

$$+2\mathbb{E}^{\gamma}_{\nu_\alpha}\Big[\Big(\int_0^t \frac{N^\gamma}{\sqrt{N}} \sum_{l\geq 1} H\Big(\frac{z_l}{N}\Big) \sum_{j\in\tilde{I}_l} E\Big(\sum_{x\in I_j} \bar{\eta}_s(x)\bar{\eta}_s(x+1)\Big| M_j\Big) ds\Big)^2\Big]. \tag{1.8.8}$$

The first expectation can be treated in the way as in the proof of Lemma 1.8.1, and it vanishes if $L^2KN^{2\gamma-2} \to 0$ as $N \to +\infty$.

For the other expectation (1.8.8), inside the sum over l, we sum and subtract $E\Big(\sum_{x\in\tilde{I}_l} \bar{\eta}(x)\bar{\eta}(x+1)\Big|\tilde{M}_l\Big)$ where $\tilde{M}_l = \sigma\Big(\sum_{x\in\tilde{I}^*_l}\eta(x)\Big)$ and $\tilde{I}^*_l$ denotes the extended interval $\tilde{I}_l$. Then, the expectation in (1.8.8) can be bounded by

$$2\mathbb{E}^{\gamma}_{\nu_\alpha}\Big[\Big(\int_0^t \frac{N^\gamma}{\sqrt{N}} \sum_{l\geq 1} H\Big(\frac{z_l}{N}\Big) \tilde{V}_l(\eta_s) ds\Big)^2\Big] \tag{1.8.9}$$

$$+2\mathbb{E}^{\gamma}_{\nu_\alpha}\Big[\Big(\int_0^t \frac{N^\gamma}{\sqrt{N}} \sum_{l\geq 1} H\Big(\frac{z_l}{N}\Big) E\Big(\sum_{x\in\tilde{I}_l} \bar{\eta}_s(x)\bar{\eta}_s(x+1)\Big|\tilde{M}_l\Big) ds\Big)^2\Big], \tag{1.8.10}$$

where

$$\tilde{V}_l(\eta) = \sum_{j\in\tilde{I}_l} E\Big(\sum_{x\in I_j} \bar{\eta}(x)\bar{\eta}(x+1)\Big| M_j\Big) - E\Big(\sum_{x\in\tilde{I}_l} \bar{\eta}(x)\bar{\eta}(x+1)\Big|\tilde{M}_l\Big).$$

We proceed by estimating (1.8.9):

Lemma 1.8.3. *For every $H \in S(\mathbb{R})$ and every $t > 0$, if $L^2KN^{\gamma-1} \to 0$ as $N \to +\infty$, then*

$$\lim_{N\to\infty} \mathbb{E}^{\gamma}_{\nu_\alpha}\Big[\Big(\int_0^t \frac{N^\gamma}{\sqrt{N}} \sum_{l\geq 1} H\Big(\frac{z_l}{N}\Big) \tilde{V}_l(\eta_s) ds\Big)^2\Big] = 0.$$

Proof. Using the same arguments as in the proof of Lemma 1.8.2, the expectation becomes bounded by

$$Ct\sum_{l\geq 1}\sup_{h\in L^2(\nu_\alpha)}\Big\{2\frac{N^\gamma}{\sqrt{N}}H\Big(\frac{z_l}{N}\Big)\int \tilde{V}_l(\eta)h(\eta)\nu_\alpha(d\eta)-N^{1+\gamma}<h,-L^S_{\tilde{I}^*_l}h>_\alpha\Big\}.$$

Using an appropriate A_l and the spectral gap inequality, we can bound last expression by

$$Ct\sum_{l\geq 1}\frac{N^\gamma}{N^2}H^2\Big(\frac{z_l}{N}\Big)(M+1)^2Var(V_l,\nu_\alpha).$$

Since $Var(V_l,\nu_\alpha)\leq LC$, last expression vanishes as long as $L^2KN^{\gamma-1}\rightarrow 0$, when $N\rightarrow +\infty$. □

To treat the remaining expectation (1.8.10) we continue applying the same steps.

The proof of Boltzmann-Gibbs Principle

The idea of the proof was to take intervals of growing size in each step, in a way that the expectation vanishes for certain restrictions on this size. The size of the first intervals taken, was K and the biggest restriction in this size comes from Lemma 1.8.2, namely that K is such that $K^2N^{1-\gamma}\rightarrow 0$ as $N\rightarrow +\infty$. Therefore, we can take $K=N^{\frac{1-\gamma}{2}-\epsilon}$.

In the second step we had intervals of bigger size, namely M, where $M=LK$ and the parameter L has to satisfy $L^2KN^{\gamma-1}\rightarrow 0$ as $N\rightarrow +\infty$. Since in the first step $K=N^{\frac{1-\gamma}{2}-\epsilon}$, we can take $L=N^{\frac{1-\gamma}{4}}$, and as a consequence $M=N^{\frac{1-\gamma}{2}+\frac{1-\gamma}{4}-\epsilon}$.

Continuing the proof applying the same arguments, in the n^{th} step we have intervals, denoted by $\{I^n_p, p\geq 1\geq\}$ of length $K_n=N^{a_n}$, where $a_n=(1-\gamma)(\frac{1}{2}+\frac{1}{2^2}+...+\frac{1}{2^n})-\epsilon$.

Supposing that we stop this induction procedure in the n^{th} step, it remains to bound the following expectation:

$$\mathbb{E}^\gamma_{\nu_\alpha}\Big[\Big(\int_0^t\frac{N^\gamma}{\sqrt{N}}\sum_{p\geq 1}H\Big(\frac{z_p}{N}\Big)E\Big(\sum_{x\in I^n_p}\bar{\eta}_s(x)\bar{\eta}_s(x+1)\Big|M^n_p\Big)ds\Big)^2\Big],$$

where for each p, I^n_p is an interval of size K_n, z_p is one point of it and the hyperplanes are $M^n_p=\sigma\Big(\sum_{x\in(I^n_p)^*}\eta(x)\Big)$, where $(I^n_p)^*$ is taken as above.

Since ν_α is an invariant product measure, last expectation can be bounded by

$$t^2\frac{N^{2\gamma}}{N}\sum_{p\geq 1}\Big(H\Big(\frac{z_p}{N}\Big)\Big)^2E_{\nu_\alpha}\Big[\Big(E\Big(\sum_{x\in I^n_p}\bar{\eta}(x)\bar{\eta}(x+1)\Big|M^n_p\Big)\Big)^2\Big].$$

Now, it is not hard to show that $E_{\nu_\alpha}\Big[\Big(E\Big(\sum_{x\in I^n_p}\bar{\eta}(x)\bar{\eta}(x+1)\Big|M^n_p\Big)\Big)^2\Big]=O(1)$. Then the integral becomes bounded by $\frac{N^{2\gamma}}{K_n}$, and for n sufficiently big, since $K_n\sim N^{1-\gamma}$ and $\gamma<1/3$ this expression vanishes as $N\rightarrow +\infty$. Here is the point in the proof where we need to impose the restriction on the parameter $\gamma<1/3$.

Remark 1.8.4. *Here we give an application of the Boltzmann-Gibbs Principle for the quadratic density fluctuation field associated to the one-dimensional SSEP, in the diffusive scaling.*

Consider a Markov process η_t^{sym} with generator given by (1.2.1), with $p(x,y) = 1/2$ under the diffusive time scale. Consider $\mathbb{P}_{\nu_\alpha}^N = \mathbb{P}_{\nu_\alpha}$ the probability measure on $D(\mathbb{R}^+, \{0,1\}^{\mathbb{Z}})$ induced by the invariant measure ν_α and the Markov process η_t^{sym} speeded up by N^2 and denote by $\mathbb{E}_{\nu_\alpha}$ the expectation with respect to $\mathbb{P}_{\nu_\alpha}$.

Define the quadratic density fluctuation field on $H \in S(\mathbb{R})$ by:

$$\mathcal{Y}_t^N(H) = \frac{1}{\sqrt{N}} \sum_{x \in \mathbb{Z}} H\Big(\frac{x}{N}\Big)[\eta_{tN^2}^{sym}(x) - \alpha][\eta_{tN^2}^{sym}(x+1) - \alpha].$$

Following the same steps as in the proof of the Boltzmann-Gibbs Principle it is easy to show that:

Corollary 1.8.5. *Fix $t > 0$ and $\beta < 1/2$, then*

$$\lim_{N\to\infty} \mathbb{E}_{\nu_\alpha}\Big[\Big(N^\beta \int_0^t \frac{1}{\sqrt{N}} \sum_{x \in \mathbb{Z}} H\Big(\frac{x}{N}\Big)[\eta_{tN^2}^{sym}(x) - \alpha][\eta_{tN^2}^{sym}(x+1) - \alpha]ds\Big)^2\Big] = 0.$$

Therefore, in order to observe fluctuations for the quadratic density fluctuation field, we need to consider $\beta \geq 1/2$. In fact, in [1] it is shown that

$$N^{1/2} \int_0^t \frac{1}{\sqrt{N}} \sum_{x \in \mathbb{Z}} H\Big(\frac{x}{N}\Big)[\eta_{tN^2}^{sym}(x) - \alpha][\eta_{tN^2}^{sym}(x+1) - \alpha]ds$$

converges in law to a non-Gaussian singular functional of an infinite Ornstein-Uhlenbeck process.

1.9 Tightness

Now we prove that the sequence of probability measures $(Q_N^\gamma)_N$ is tight, following chapter 11 of [14].

Definition 1.9.1. *For $\delta > 0$ and a path Y in $D([0,T], \mathcal{H}_{-k})$, the uniform modulus of continuity of Y, is defined by*

$$\omega_\delta(Y) = \sup_{\substack{|s-t|<\delta \\ 0 \leq s,t \leq T}} \|Y_t - Y_s\|_{-k}.$$

The first result gives sufficient conditions for a subset to be weakly relatively compact.

Lemma 1.9.2. *A subset A of $D([0,T], \mathcal{H}_{-k})$ is relatively compact for the uniform weak topology if*

$$\sup_{Y \in A} \sup_{0 \leq t \leq T} \|Y_t\|_{-k} < \infty$$

$$\lim_{\delta\to 0}\sup_{Y\in A}\omega_\delta(Y)=0.$$

From this lemma we can transform the concept of tightness by replacing for compactness its Arzelï¿½Ascoli characterization. So we get a criterium for tightness of a sequence of probability measures defined on $D([0,T],\mathcal{H}_{-k})$.

Lemma 1.9.3. *A sequence $\{P_N, N\geq 1\}$ of probability measures defined on $D([0,T],\mathcal{H}_{-k})$ is tight if this two conditions hold:*

$$a)\lim_{A\to\infty}\limsup_{N\to\infty}P_N\Big[\sup_{0\leq t\leq T}\|Y_t\|_{-k}>A\Big]=0$$

$$b)\lim_{\delta\to 0}\limsup_{N\to\infty}P_N\Big[\omega_\delta(Y)\geq\epsilon\Big]=0$$

for every $\epsilon>0$.

In order to show that $(Q_N^\gamma)_N$ is tight, we must prove that:

$$(1)\qquad \lim_{A\to+\infty}\limsup_{N\to+\infty}\mathbb{P}^\gamma_{\nu_\alpha}\Big(\sup_{0\leq t\leq T}||Y_t||^2_{-k}\Big)<\infty$$

$$(2)\forall\epsilon>0,\qquad \lim_{\delta\to 0}\limsup_{N\to+\infty}\mathbb{P}^\gamma_{\nu_\alpha}\Big[\omega_\delta(Y)\geq\epsilon\Big]=0.$$

We start by showing condition (1). For each integer $z\geq 0$, recall that h_z denotes the Hermite function defined at the beginning of the second section. Denote by $M_t^{N,z}$ the martingale M_t^{N,h_z} as defined in expression (1.7.1), namely:

$$M_t^{N,z}=Y_t^N(h_z)-Y_0^N(h_z)-\int_0^t\Gamma_1^z(s)ds$$

where $\Gamma_1^z(s)$ is equal to:

$$\frac{N^\gamma}{\sqrt{N}}\sum_{x\in\mathbb{Z}}\nabla^N U_s^N h_z\Big(\frac{x}{N}\Big)\Big[\bar{W}_{x,x+1}(\eta_s)\Big]-\frac{N^\gamma}{\sqrt{N}}\sum_{x\in\mathbb{Z}}\partial_u U_s^N h_z\Big(\frac{x}{N}\Big)v[\eta_s(x)-\alpha].$$

Lemma 1.9.4. *There exists a finite constant $C(\alpha,T)$ such that for every $z\geq 0$,*

$$\limsup_{N\to+\infty}\mathbb{E}^\gamma_{\nu_\alpha}\Big(\sup_{0\leq t\leq T}|<Y_t,h_z>|^2\Big)\leq C(\alpha,T)<h_z,h_z>.$$

In this expression $<Y_t,h_z>$ denotes the inner product of $Y_t\in\mathcal{H}_{-k}$ and $h_z\in\mathcal{H}_k$.

Proof. By definition, we have that

$$< Y_t^N, h_z >= M_t^{N,z} + < Y_0^N, h_z > + \int_0^t \Gamma_1^z(s)ds.$$

To prove the Lemma, we estimate separately the $L^2(\mathbb{P}_{\nu_\alpha})$-norm of the terms on the right hand side of last equality. A simple computation shows that

$$\lim_{N\to+\infty} \mathbb{E}_{\nu_\alpha}^\gamma\Big[\Big(\Big| < Y_0^N, h_z > \Big|\Big)^2\Big] = \chi(\alpha) < h_z, h_z > .$$

The $L^2(\mathbb{P}_{\nu_\alpha})$-norm for the martingale term vanishes, combining Doob inequality:

$$\mathbb{E}_{\nu_\alpha}^\gamma\Big(\sup_{0\le t\le T} |M_t^{N,z}|^2\Big) \le 4\mathbb{E}_{\nu_\alpha}^\gamma\Big(|M_T^{N,z}|^2\Big),$$

with the fact that $\mathbb{E}_{\nu_\alpha}^\gamma[(M_T^{N,z})^2]$ vanishes as $N \to +\infty$. This last result is a consequence of estimates in the quadratic variation of the martingale, see (1.7.3).

To end, it remains to bound the other term, namely:

$$\mathbb{E}_{\nu_\alpha}^\gamma\Big[\sup_{0\le t\le T}\Big(\int_0^t \Gamma_1^z(s)ds\Big)^2\Big].$$

The idea to estimate last integral is the same as we used when analyzing the integral part of the martingale, see the expression (1.7.5). By doing so, we have to bound

$$\mathbb{E}_{\nu_\alpha}^\gamma\Big[\sup_{0\le t\le T}\Big(\int_0^t \frac{N^\gamma}{N^{3/2}}\sum_{x\in\mathbb{Z}} \Delta_N U_s^N h_z\Big(\frac{x}{N}\Big)\bar{\eta}_s(x)ds\Big)^2\Big],$$

$$\mathbb{E}_{\nu_\alpha}^\gamma\Big[\sup_{0\le t\le T}\Big(\int_0^t \frac{N^\gamma}{\sqrt{N}}\sum_{x\in\mathbb{Z}} \Big\{\nabla^N U_s h_z\Big(\frac{x}{N}\Big) - \partial_u U_s h_z\Big(\frac{x}{N}\Big)\Big\}v[\eta_s(x)-\alpha]ds\Big)^2\Big],$$

and

$$\mathbb{E}_{\nu_\alpha}^\gamma\Big[\sup_{0\le t\le T}\Big(\int_0^t \frac{N^\gamma}{\sqrt{N}}\sum_{x\in\mathbb{Z}} \nabla^N h_z\Big(\frac{x}{N}\Big)\bar{\eta}_s(x)\bar{\eta}_s(x+1)ds\Big)^2\Big]. \tag{1.9.1}$$

By Schwarz inequality and since ν_α is an invariant product measure, the first integral is bounded by $T^2 N^{2\gamma-2}\frac{1}{4N}\sum_{x\in\mathbb{Z}}\Big(\Delta_N U_s^N h_z\Big(\frac{x}{N}\Big)\Big)^2 \alpha(1-\alpha)$, which vanishes as $N \to +\infty$.

By Taylor expansion, the second expectation vanishes.

In order to bound the last integral, we use the same idea as in the Boltzmann-Gibbs Principle. Then, we can bound the expectation (1.9.1) by

$$\mathbb{E}_{\nu_\alpha}^\gamma\Big[\sup_{0\le t\le T}\Big(\int_0^t \frac{N^\gamma}{\sqrt{N}}\sum_{j\in\mathcal{O}}\sum_{x\in I_j} \nabla^N h_z\Big(\frac{x}{N}\Big)\bar{\eta}_s(x)\bar{\eta}_s(x+1)ds\Big)^2\Big],$$

where the I_j's are taken as in the proof of Theorem 1.2.6, with j odd for example.

By summing and subtracting $h_z\Big(\frac{y_j}{N}\Big)$, where y_j is one point of the interval I_j, we bound last expectation by

$$2\mathbb{E}^{\gamma}_{\nu_\alpha}\Big[\sup_{0\le t\le T}\Big(\int_0^t \frac{N^\gamma}{\sqrt{N}}\sum_{j\in\mathcal{O}}\sum_{x\in I_j}\Big(\nabla^N h_z\Big(\frac{x}{N}\Big)-\nabla^N h_z\Big(\frac{y_j}{N}\Big)\Big)\bar\eta_s(x)\bar\eta_s(x+1)ds\Big)^2\Big]$$

$$+2\mathbb{E}^{\gamma}_{\nu_\alpha}\Big[\sup_{0\le t\le T}\Big(\int_0^t \frac{N^\gamma}{\sqrt{N}}\sum_{j\in\mathcal{O}}\nabla^N h_z\Big(\frac{y_j}{N}\Big)\sum_{x\in I_j}\bar\eta_s(x)\bar\eta_s(x+1)ds\Big)^2\Big]. \tag{1.9.2}$$

By Schwarz inequality and since ν_α is an invariant product measure, the first integral vanishes if K is such that $KN^{\gamma-1}\to 0$ as $N\to+\infty$, see Lemma 1.8.1.

To bound (1.9.2), we sum and subtract inside the sum over j, the expectation of $\sum_{x\in I_j}\bar\eta_s(x)\bar\eta_s(x+1)$ conditioned on the hyperplanes $M_j=\sigma(\sum_{x\in I_j^*}\eta(x))$, where I_j^* denotes the extended interval I_j. Then, we need to bound

$$\mathbb{E}^{\gamma}_{\nu_\alpha}\Big[\sup_{0\le t\le T}\Big(\int_0^t \frac{N^\gamma}{\sqrt{N}}\sum_{j\in\mathcal{O}}\nabla^N h_z\Big(\frac{y_j}{N}\Big)V_j(\eta_s)ds\Big)^2\Big]$$

and

$$\mathbb{E}^{\gamma}_{\nu_\alpha}\Big[\sup_{0\le t\le T}\Big(\int_0^t \frac{N^\gamma}{\sqrt{N}}\sum_{j\in\mathcal{O}}\nabla^N h_z\Big(\frac{y_j}{N}\Big)E\Big(\sum_{x\in I_j}\bar\eta_s(x)\bar\eta_s(x+1)|M_j\Big)ds\Big)^2\Big].$$

where

$$V_j(\eta)=\sum_{x\in I_j}\bar\eta_s(x)\bar\eta_s(x+1)-E\Big(\sum_{x\in I_j}\bar\eta_s(x)\bar\eta_s(x+1)|M_j\Big).$$

By Lemma 4.3 of [4], the first integral is bounded by

$$C_0\int_0^T\Big|\Big|\frac{N^\gamma}{\sqrt{N}}\sum_{j\in\mathcal{O}}\nabla^N h_z\Big(\frac{y_j}{N}\Big)V_j\Big|\Big|^2_{-1}ds,$$

where C_0 is a constant. To bound this H_{-1}-norm we follow the same computations as in Lemma (1.8.2), and it is not hard to show that it vanishes for K, such that $K^2N^{\gamma-1}\to 0$ as $N\to+\infty$. To bound the other integral, we proceed in the same lines as in the Boltzmann-Gibbs Principle. □

Corollary 1.9.5. *For each $k>1$*

$$(a)\quad \limsup_{N\to+\infty}\mathbb{E}^{\gamma}_{\nu_\alpha}\Big(\sup_{0\le t\le T}||Y_t||^2_{-k}\Big)<\infty$$

$$(b)\quad \lim_{n\to+\infty}\limsup_{N\to+\infty}\mathbb{E}^{\gamma}_{\nu_\alpha}\Big[\sup_{0\le t\le T}\sum_{|z|\ge n}(<Y_t,h_z>)^2\gamma_z^{-k}\Big]=0.$$

Proof. Recall the definition of $\mathcal{H}_k$ and the inner product $<,>_k$ at the beginning of the second section. Since $< f, g >_k = \sum_{z\in\mathbb{Z}} < f, h_z >< g, h_z > \gamma_z^{-k}$, then

$$\limsup_{N\to+\infty} \mathbb{E}^{\gamma}_{\nu_\alpha}\Big(\sup_{0\le t\le T} ||Y_t||^2_{-k}\Big) \le \limsup_{N\to+\infty} \sum_{z\in\mathbb{Z}} \gamma_z^{-k} \mathbb{E}^{\gamma}_{\nu_\alpha}\Big(\sup_{0\le t\le T} < Y_t, h_z >^2\Big).$$

and by the previous Lemma it is bounded by $C(\alpha, T)\sum_{z\in\mathbb{Z}} \gamma_z^{-k}$, which is finite as long as $k > 1$. The assertion (b) follows by the same argument. □

We note that this is the place where we need the restriction $k > 1$ in order to have the density fluctuation field well defined in $\mathcal{H}_{-k}$.

The first assertion of the previous corollary shows that the condition (1) holds. So, in order to prove that the sequence $(Q^{\gamma}_N)_N$ is tight we only have to show (2). In view of (b) of the previous corollary, this follows from the next Lemma:

Lemma 1.9.6. *For every $n \in \mathbb{N}$ and every $\epsilon > 0$,*

$$\lim_{\delta\to 0}\limsup_{N\to+\infty} \mathbb{P}^{\gamma}_{\nu_\alpha}\Big[\sup_{\substack{|s-t|<\delta \\ 0\le s,t\le T}} \sum_{|z|\le n} (< Y_t - Y_s, h_z >)^2 \gamma_z^{-k} > \epsilon\Big] = 0.$$

Proof. To prove this Lemma it is enough to show that

$$\lim_{\delta\to 0}\limsup_{N\to+\infty} \mathbb{P}^{\gamma}_{\nu_\alpha}\Big[\sup_{\substack{|s-t|<\delta \\ 0\le s,t\le T}} (< Y_t - Y_s, h_z >)^2 > \epsilon\Big] = 0.$$

for every $z \in \mathbb{Z}$ and $\epsilon > 0$.

Fix $z \in \mathbb{Z}$ and recall that $M^z_t =< Y_t, h_z > - < Y_0, h_z > + \int_0^t \Gamma^z_1(s)ds$. The Lemma follows from the next two results. □

Lemma 1.9.7. *Fix a function $H \in S(\mathbb{R})$. For every $\epsilon > 0$*

$$\lim_{\delta\to 0}\limsup_{N\to+\infty} \mathbb{P}^{\gamma}_{\nu_\alpha}\Big[\sup_{\substack{|s-t|<\delta \\ 0\le s,t\le T}} |M^{N,H}_t - M^{N,H}_s| > \epsilon\Big] = 0.$$

Proof. Denote by $\omega'_\delta(M^{N,H})$ the modified modulus of continuity defined as

$$\omega'_\delta(M^{N,H}) = \inf_{\{t_i\}} \max_{0\le i\le r} \sup_{t_i\le s<t\le t_{i+1}} |M^{N,H}_t - M^{N,H}_s|$$

where the infimum is taken over all partitions of $[0, T]$ such that $0 \le i \le r$ $0 = t_0 < t_1 < ... < t_r = T$ with $t_{i+1} - t_i > \delta$.

Since

$$\sup_t |M^{N,H}_t - M^{N,H}_{t_-}| = \sup_t | < Y^N_t, H > - < Y^N_{t_-}, H > | \le \frac{||\nabla H||_\infty}{N^{1+\frac{1}{2}}}$$

and

$$\omega_\delta(M^{N,H}) \leq 2\omega'_\delta(M^{N,H}) + \sup_t |M_t^{N,H} - M_{t_-}^{N,H}|$$

so, the proof ends if we show that

$$\lim_{\delta\to 0}\limsup_{N\to+\infty} \mathbb{P}^\gamma_{\nu_\alpha}\Big[\omega'_\delta(M^{N,H}) > \epsilon\Big] = 0$$

for every $\epsilon > 0$.

By the Aldous criterium, see for example Proposition 4.1.6 of [14], it is enough to show that:

$$\lim_{\delta\to 0}\limsup_{N\to+\infty} \sup_{\substack{\tau\in\mathfrak{T}_\tau\\ 0\leq\theta\leq\delta}} \mathbb{P}^\gamma_{\nu_\alpha}\Big[|M^{N,H}_{\tau+\theta} - M^{N,H}_\tau| > \epsilon\Big] = 0$$

for every $\epsilon > 0$. Here $\mathfrak{T}_\tau$ denotes the family of all stopping times, with respect to the canonical filtration, bounded by T. Using Chebychevs inequality together with the Optional Sampling Theorem, we have that

$$\mathbb{P}^\gamma_{\nu_\alpha}\Big[|M^{N,H}_{\tau+\theta} - M^{N,H}_\tau| > \epsilon\Big] \leq \frac{\mathbb{E}^\gamma_{\nu_\alpha}[(M^{N,H}_{\tau+\theta})^2 - (M^{N,H}_\tau)^2]}{\epsilon^2}.$$

By expression (1.7.3), last expression is bounded by

$$\frac{1}{\epsilon^2}\mathbb{E}^\gamma_{\nu_\alpha}\Big[\int_0^\tau \frac{N^\gamma}{N^2}\sum_{x\in\mathbb{Z}}\Big(\nabla^N U^N_s H\Big(\frac{x}{N}\Big)\Big)^2\Big[c(x,x+1,\eta_s) + c(x+1,x,\eta_s)\Big]ds\Big]$$

which vanishes as $N \to +\infty$. □

Lemma 1.9.8. *Fix $H \in S(\mathbb{R})$. For every $\epsilon > 0$*

$$\lim_{\delta\to 0}\limsup_{N\to+\infty} \mathbb{P}^\gamma_{\nu_\alpha}\Big[\sup_{\substack{|s-t|<\delta\\ 0\leq s,t\leq T}}\Big|\int_s^t \Gamma^H_1(r)dr\Big| > \epsilon\Big] = 0$$

Proof. By using the explicit knowledge of $\Gamma^H_1(r)$ (see 1.7.2), the decomposition of the instantaneous current (1.7.4) and similar computations as the ones performed when analyzing the integral part of the martingale $M^{N,H}_t$, we just need to bound:

$$\mathbb{P}^\gamma_{\nu_\alpha}\Big[\sup_{\substack{|s-t|<\delta\\ 0\leq s,t\leq T}}\Big|\int_s^t \frac{N^\gamma}{\sqrt{N}}\sum_{x\in\mathbb{Z}}\nabla^N U^N_r H\Big(\frac{x}{N}\Big)\bar\eta_r(x)\bar\eta_r(x+1)dr\Big| > \epsilon\Big].$$

Dividing the interval $[0,T]$ in small intervals of length δ, last probability is bounded by

$$\frac{T}{\delta}\mathbb{P}_{\nu_\alpha}\Big[\sup_{0\leq t\leq\delta}\Big|\int_0^t \frac{N^\gamma}{\sqrt{N}}\sum_{x\in\mathbb{Z}}\nabla^N U^N_r H\Big(\frac{x}{N}\Big)\bar\eta_s(x)\bar\eta_s(x+1)dr\Big| > \frac{\epsilon}{2}\Big].$$

Using Chebychev inequality, the last probability is bounded by an expectation analogous to the one that appeared at the end of the proof of Lemma (1.9.4) which we showed to vanish as $N \to +\infty$ □

1.10 Dependence on the initial configuration for the longer time scale

We start by considering the case $\alpha = 1/2$ which implies that $v = 0$. In this case, we can define (as in the hyperbolic scaling) for a site x, the current over the fixed bond $[x, x+1]$ denoted by $J^{N,\gamma}_{x,x+1}(t)$, as the total number of jumps from the site x to the site $x+1$ minus the total number of jumps from the site $x+1$ to the site x during the time interval $[0, tN^{1+\gamma}]$.

In this particular case, the density fluctuation field at time t is the same as at time 0. As a consequence, the current through $[x, x+1]$ converges to 0 in the $L^2(\mathbb{P}^\gamma_{\nu_\alpha})$-norm:

Proposition 1.10.1. *Fix $t \geq 0$, a site $x \in \mathbb{Z}$ and $\gamma < 1/3$. Then,*

$$\lim_{N\to+\infty} \mathbb{E}^\gamma_{\nu_\alpha}\Big[\Big(\frac{\bar{J}^{N,\gamma}_{x-1,x}(t)}{\sqrt{N}}\Big)^2\Big] = 0.$$

The idea of the proof is the same as the one used in the hyperbolic scaling, and it relies on the following result:

Proposition 1.10.2. *For every $t \geq 0$ and $\gamma < 1/3$:*

$$\lim_{n\to+\infty} \mathbb{E}^\gamma_{\nu_\alpha}\Big[\Big(\frac{\bar{J}^{N,\gamma}_{x-1,x}(t)}{\sqrt{N}} - (Y^{N,\gamma}_t(G_n) - Y^{N,\gamma}_0(G_n))\Big)^2\Big] = 0,$$

uniformly over N, where G_n was defined in (1.4.2).

Proof. Recall the proof of Proposition 1.4.1. There is only a slight difference that we need to remark. The expression (1.4.4) in the proof of that Proposition, now becomes:

$$\frac{1}{\sqrt{N}}\sum_{x=1}^{Nn}\frac{1}{Nn}M^{N,\gamma}_{x-1,x}(t) + \frac{N^\gamma}{\sqrt{N}}\int_0^t \frac{1}{n}\sum_{x=1}^{Nn}[W_{x,x+1}(\eta_s) - (p-q)\chi(\alpha)]ds.$$

It is not hard to prove that the martingale term converges to 0 in $L^2(\mathbb{P}^\gamma_{\nu_\alpha})$, since we can estimate their quadratic variation by $N^{1+\gamma}t$ to obtain

$$\mathbb{E}^\gamma_{\nu_\alpha}\Big[\Big(\frac{1}{\sqrt{N}}\sum_{x=1}^{Nn}\frac{1}{Nn}M^{N,\gamma}_{x-1,x}(t)\Big)^2\Big] \leq \frac{tN^{\gamma-1}}{n},$$

which vanishes as $n \to +\infty$.

Now, we need to bound the integral term, namely:

$$\mathbb{E}^\gamma_{\nu_\alpha}\Big[\Big(\frac{N^\gamma}{\sqrt{N}}\int_0^t \frac{1}{n}\sum_{x=0}^{Nn-1}\bar{W}_{x,x+1}(\eta_s)ds\Big)^2\Big].$$

Using the decomposition of the instantaneous current, see (1.7.4), it is enough to bound

$$\mathbb{E}^{\gamma}_{\nu_\alpha}\Big[\Big(\frac{N^\gamma}{\sqrt{N}}\int_0^t \frac{1}{n}\sum_{x=0}^{Nn-1}\bar{\eta}_s(x)\bar{\eta}_s(x+1)ds\Big)^2\Big].$$

Using the inequality $(x+y)^2 \leq 2x^2+2y^2$, last expectation is bounded by

$$2\mathbb{E}^{\gamma}_{\nu_\alpha}\Big[\Big(\frac{(Nn)^\gamma}{\sqrt{nN}}\int_0^t \sum_{x=0}^{Nn}\bar{\eta}_s(x)\bar{\eta}_s(x+1)ds\Big)^2\Big]+2\mathbb{E}^{\gamma}_{\nu_\alpha}\Big[\Big(\frac{N^\gamma}{\sqrt{N}}\int_0^t \frac{1}{n}\bar{\eta}_s(Nn-1)\bar{\eta}_s(Nn)ds\Big)^2\Big].$$

Recall the proof of the Boltzmann-Gibbs Principle (Theorem 1.2.6) when applied to the function $H(u)=1_{[0,1]}(u)$, which gives us that the expectation on the left hand side of last expression vanishes as $n\to+\infty$, uniformly over N.

By Schwarz inequality and since ν_α is an invariant product measure, the other term vanishes as $n\to+\infty$, which concludes the proof. □

Last result is stated for the bond $[-1,0]$ but for $[x,x+1]$ a similar statement holds.

Consider now the case $\alpha\neq 1/2$. In this case, by the definition of the density fluctuation field (see (1.2.5)), as time is going by the position of the particles start to change. So, if there is initially a particle at site x and if it does not move, then at time t, its position is the site $x+[vtN^{1+\gamma}]$, that we denote by y^x_t. By this reason, we cannot consider any longer the current through a fixed bound, but we must consider the current through a bond that depends on time.

Let $J^{N,\gamma}_{y^x_t}(t)$ be the current trough the bond $[y^x_t, y^x_t+1]$, defined as the number of particles that jump from y^x_t to y^x_t+1, minus the number of particles that jump from y^x_t+1 to y^x_t, from time 0 to $tN^{1+\gamma}$. Formally we have that:

$$J^{N,\gamma}_{y^x_t}(t)=\sum_{y\geq 1}\Big(\eta_t(y+y^x_t)-\eta_0(y+x)\Big).$$

As a consequence, it holds that:

Proposition 1.10.3. *Fix $t\geq 0$, a site $x\in\mathbb{Z}$ and $\gamma<1/3$. Then,*

$$\lim_{N\to+\infty}\mathbb{E}^{\gamma}_{\nu_\alpha}\Big[\frac{\bar{J}^{N,\gamma}_{y^x_t}(t)}{\sqrt{N}}\Big]^2=0.$$

As in the hyperbolic scaling, this last results is a consequence of the following:

Proposition 1.10.4. *For every $t\geq 0$ and $\gamma<1/3$:*

$$\lim_{n\to+\infty}\mathbb{E}^{\gamma}_{\nu_\alpha}\Big[\Big(\frac{\bar{J}^{N,\gamma}_{y^x_t}(t)}{\sqrt{N}}-(Y^{N,\gamma}_t(G_n)-Y^{N,\gamma}_0(G_n))\Big)^2\Big]=0,$$

uniformly over N.

Proof. Recall the proof of Proposition 1.10.2. The martingale associated to $J^{N,\gamma}_{y^x_t}(t)$ is now given by

$$M^{N,\gamma}_x(t) = J^{N,\gamma}_{y^x_t}(t) - \int_0^t \Big\{ N^{1+\gamma} W_{y^x_s}(\eta_s) + \partial_s J^{N,\gamma}_{y^x_s}(s) \Big\} ds,$$

where $W_{y^x_s}(\eta)$ denotes the instantaneous current through the bond $[y^x_t, y^x_t + 1]$. Since $\partial_s J^{N,\gamma}_{y^x_t}(s) = -vN^{1+\gamma}\eta_s(y^x_s)$ and repeating the same arguments as in the proof of Proposition 1.10.2, the result follows. □

Proof of Corollary 1.2.7

In this case there is a relation between the position of the Tagged particle and the current through the bond $[y^{-1}_t, y^{-1}_t + 1]$ and the density of particles, which is given by:

$$\Big\{ X_{tN^{1+\gamma}} \geq a \Big\} = \Big\{ J^{N,\gamma}_{y^{-1}_t}(t) \geq \sum_{x=vtN^{1+\gamma}}^{a-1} \eta_t(x) \Big\}.$$

Repeating the same computations as in the proof of Corollary 1.2.4, using the fact that $\mathbb{E}^\gamma_{\nu_\alpha}[J^{N,\gamma}_{y^{-1}_t}(t)] = (p-q)\alpha^2 tN^{1+\gamma}$; that $\frac{\bar{J}^{N,\gamma}_{y^{-1}_t}(t)}{\sqrt{N}}$ converges to 0 in the $L^2(\mathbb{P}^\gamma_{\nu_\alpha})$-norm; that

$$\lim_{N\to+\infty} \mathbb{E}^\gamma_{\nu_\alpha}\Big[\Big(Y^{N,\gamma}_t(H) - Y^{N,\gamma}_0(H) \Big)^2 \Big] = 0$$

for every $H \in S(\mathbb{R})$ and also that

$$\mathbb{E}^\gamma_{\nu_\alpha}\Big[\Big(\frac{1}{\sqrt{N}} \sum_{x=1+v_tN^{1+\gamma}}^{a\sqrt{N}-1+v_tN^{1+\gamma}+Z^{N,\gamma}_t} \bar{\eta}_t(x) \Big)^2 \Big] = O(N^{-1/2}),$$

where $Z^{N,\gamma}_t$

$$Z^{N,\gamma}_t = -\sum_{x=1+vtN^{1+\gamma}}^{v_tN^{1+\gamma}} \bar{\eta}_t(x)/\alpha$$

the result follows. □

Chapter 2

Hydrodynamic Limit for a Particle System with degenerate rates

2.1 Introduction

The purpose of this work is to define a conservative interacting particle system whose macroscopic density profile evolves according to the **porous medium equation**, namely the partial differential equation given by

$$\begin{cases} \partial_t \rho(t,u) = \Delta \rho^m(t,u) \\ \rho(0,\cdot) = \rho_0(\cdot) \end{cases} \tag{2.1.1}$$

where $\Delta = \sum_{1\le j\le d} \partial^2_{u_j}$ and $m \in \mathbb{N} \setminus \{1\}$. This can be rewritten in the divergence form as $\partial_t \rho(t,u) = \nabla(D(\rho(t,u))\nabla(\rho(t,u)))$ with diffusion coefficient $D(\rho(t,u)) = m\rho^{m-1}(t,u)$, which goes to zero as $\rho \to 0$ and thus the equation looses its parabolic character.

One of the most important properties of the porous medium equation is that its solutions can be compactly supported at each fixed time or, in physical terms, to have a finite speed of propagation. This is in strong contrast with the solutions of the classical heat equation. A nonnegative solution of the heat equation is always positive on its domain. A second observation is that the solutions of the equation (2.1.1) can be continuous on the domain of definition without being smooth at the boundary. The existence of this kind of solutions is a direct consequence of the degeneracy of $D(\rho)$ as $\rho \to 0$. For a reference on the mathematical properties of this equation we refer to [26] and references therein. We also recall that such equation is relevant in different contexts in physical literature beyond the description of the density of an ideal gas flowing isothermally through an homogeneous porous medium (corresponding to the choice $m = 2$).

A microscopic derivation of the porous medium equation has been already obtained in [6] and [7], by considering a model in which the occupation number is a continuous variable. Here we consider instead a model with discrete occupation variables: our microscopic dynamics is given by stochastic lattice gases with hard core exclusion, namely systems of interacting particles on the d-dimensional discrete torus $\mathbb{T}^d_N$ with the constraint that on each site there can be at most one particle. A configuration is therefore defined by giving for each site $x \in \mathbb{T}_N^d$ the occupation number $\eta(x) \in \{0,1\}$, which stands for empty or occupied sites, respectively. Evolution is then given by a continuous time Markov process during which the jump of a particle from a site x to a nearest neighbor site y occurs at rate $c(x,y,\eta)$. The choice $c(x,y,\eta) = 1$ corresponds to the Symmetric Simple Exclusion Process (SSEP) and, as is very well known, leads to the heat equation under diffusive re-scaling, namely $D(\rho) = 1$. In order to slow down the low density dynamics and obtain a diffusion coefficient which degenerates for $\rho \to 0$, we impose at the level of rates a local constraint (in addition to hard core exclusion) that must be satisfied in order for a particle jump to be allowed. This constraint requires a minimal number of occupied sites in a proper local neighborhood of the jumping particle so that, since the typical number of particles in a given region decreases as $\rho \to 0$, $D(\rho)$ will also decrease. At the same time the rates are chosen in order to satisfy the detailed balance condition with respect to Bernoulli product measure at any density, as for SSEP, namely the constraints do not introduce additional interactions beyond hard core exclusion.

The models we introduce belong to the class of *kinetically constrained lattice gases* (KCLG), which have been introduced and analyzed in physical literature since the late 1980's to model liquid/glass and more general jamming transitions (see for a review [21, 3] and references therein). For most KCLG a diffusion degenerate coefficient is expected when $\rho \to 0$ but, for very restrictive choices of the constraints, the degeneracy could even occur at non trivial critical density [1]. Here we provide the first derivation of the hydrodynamic limit for the simplest KCLG, namely those that belong to the class of *non cooperative* KCLG. This means that the rates are such that: it is possible to construct a proper finite group of particles that can be moved all over the lattice by a deterministic path (namely one which has strictly positive rates for any configuration); any jump of a particle to a neighboring site can be performed when the special groups of particles is brought in its vicinity. A configuration containing this special group of particles can thus be connected to any other one with the same property by an allowed path. Therefore, very loosely speaking, it is expected that cooperative KCLG behave like a re-scaled SSEP with special groups of particles playing the role of single particles (which are always mobile for SSEP) and their diffusion coefficient should degenerate only when $D(\rho) \to 0$. As we will see the diffusion coefficient indeed degenerates only when $\rho \to 0$ and in order to prove this result the non cooperative property will play a key role since we will use it to provide paths which allow to perform particle exchanges. The use of similar path arguments for KCLG had already been exploited in [2], were the scaling with size of the spectral gap and log Sobolev constant for non cooperative models in contact with particle reservoirs at the boundary were derived. A similar case in which the diffusion coefficient does not vanish has already been studied in [25]. Finally, we stress that all along our proofs of hydrodynamics (both for the Entropy and Relative Entropy method) we use an additional property of the rates:

[1] Note that the role of particles and vacancy is usually exchanged in physical literature with respect to our convention: vacancies rather then particles are needed to facilitate motion. With this notation the diffusion coefficient degenerates at high rather than low density.

the fact that they are of gradient type. A natural development of the present work would be to generalize the present results to non cooperative KCLG with non gradient rates.

Here is an outline of the paper. In the second section we introduce the notation and state the main results. The proof of the hydrodynamic limit by the Relative Entropy method for the original process is presented in the third section. Perturbing slightly the dynamics by adding jumps of the Symmetric simple Exclusion process, we derive the hydrodynamic limit for this perturbed process following the Entropy Method. This is described in section four. In the fifth section, we sketch some inequalities that are used through the sequel. As a Replacement Lemma is needed its proof is refereed to section six. In the last section we treat the problem of the spectral gap.

2.2 Statement of results

The models we consider are continuous time Markov processes η_t with space state $\chi_d^N = \{0,1\}^{\mathbb{T}_N^d}$, where $\mathbb{T}_N^d = \{0,1,..,N-1\}^d$ is the discrete d-dimensional torus. Let η denote a configuration in χ_d^N, x be a site in $\mathbb{T}_N^d$ and $\eta(x) = 1$ if there is a particle at site x, otherwise $\eta(x) = 0$. The models we consider have elementary moves corresponding to jump of particles among nearest neighbor sites, x and y, occurring at a rate $c(x,y,\eta)$ which depends on both x, y and on the value of the configuration η in a finite range neighborhood of x and y. Furthermore these rates are symmetric with respect of an x-y exchange $c(x,y,\eta) = c(y,x,\eta)$ and are translation invariant. More precisely the dynamics is defined by means of an infinitesimal generator, given on local functions $f : \chi_d^N \to \mathbb{R}$ by

$$(\mathcal{L}_P f)(\eta) = \sum_{\substack{x,y\in\mathbb{T}_N^d \\ |x-y|=1}} c(x,y,\eta)\eta(x)(1-\eta(y))(f(\eta^{x,y}) - f(\eta)), \tag{2.2.1}$$

where $|x-y| = \sum_{1\le i\le d} |x_i - y_i|$ is the sum norm in $\mathbb{R}^d$ and

$$\eta^{x,y}(z) = \begin{cases} \eta(z), & \text{if } z \neq x,y \\ \eta(y), & \text{if } z = x \\ \eta(x), & \text{if } z = y \end{cases} . \tag{2.2.2}$$

In the sequel we consider

$$c(x, x+e_j, \eta) = \eta(x-e_j) + \eta(x+2e_j) \tag{2.2.3}$$

where $\{e_j, j = 1,..,d\}$ denotes the canonical basis of $\mathbb{R}^d$, which in the hydrodynamic limits leads to the porous medium equation (2.1.1) for $m = 2$ and we prove all the theorems for this choice of the rates. But we can provide for any m a proper choice of the rates such that all proofs can be readily extended leading in the diffusive re-scaling to the porous medium equation with the correspondent m. For instance in the case $m = 3$, the rates are given by

$$c(x, x+e_j, \eta) = \eta(x-e_j)\eta(x+2e_j) + \eta(x-2e_j)\eta(x-e_j) + \eta(x+2e_j)\eta(x+3e_j).$$

Note that both the choices of the jump rates taken above have the property of defining a **gradient system**, namely one for which the instantaneous current between the sites 0 and e_j:

$$W_{0,e_j}(\eta) = c(0, e_j, \eta)\{\eta(0)(1 - \eta(e_j)) - \eta(e_j)(1 - \eta(0))\}$$

can be rewritten as a function minus its translation. This property will be a key ingredient when deriving the hydrodynamic limit. Also, both the models are non cooperative in any dimension according to the definition given in introduction.

Consider for example the rates (2.2.3) in one dimension. A possible choice for the mobile cluster is given by two particles at distance at most two. Let us describe the deterministic sequence of allowed moves (i.e. with strictly positive exchange rate) which we should perform to shift of one step to the right the mobile cluster when $\eta(x) = \eta(x + e_1) = 1$, i.e. to transform η into η' with $\eta'(x+e_1) = \eta'(x+2e_1) = 1$, $\eta'(x) = \eta(x+2e_1)$ and $\eta'(z) = \eta(z)$ for $z \notin (x, x + e_1, x + 2e_1)$. First we make the move $\eta \to \eta^{x+e_1,x+2e_1}$, which is allowed since $c(x+e_1, x+2e_1, \eta) \geq \eta(x) = 1$. Then we perform the move $\eta^{x+e_1,x+2e_1} \to (\eta^{x+e_1,x+2e_1})^{x,x+e_1}$ which is also allowed, since $c(x, x + e_1, \eta^{x+e_1,x+2e_1}) \geq \eta^{x+e_1,x+2e_1}(x + 2e_1) = \eta(x + e_1) = 1$. The case in which we have instead the particles at distance two, $\eta(x) = \eta(x+2e_1) = 1$, can be treated analogously. The second property which characterizes non cooperative models can also be readily checked: if we are given any two neighboring sites $y, y+e_1$, the exchange of their occupation variables can be performed if we bring the mobile group of two particles in $y - 2e_1, y - e_1$ since $c(y, y + e_1, \eta) \geq \eta(y - e_1)$. It is then possible to verify that any two configurations η and η' with the same number of particles, $\sum \eta(x) = \sum \eta'(x)$ and both containing at least two particles at distance at most two can be connected one to another via a sequence of allowed jumps.

Let ν_α be the Bernoulli product measure in χ_d^N, with $\alpha \in (0, 1)$. Since $c(x, y, \eta) = c(y, x, \eta)$ $\forall x, y \in \mathbb{T}_N^d$, the measures ν_α are reversible for this process $\forall\alpha$, as for the SSEP. By the degeneracy of the rates, other invariant measures arise naturally. For example in the one dimensional setting, any configuration η such that the distance between the position of two consecutive occupied sites is bigger than two has all the exchange rates which vanish. Therefore it is a *blocked configuration* and a Dirac measure supported on it is an invariant measure for this process.

Let $\Sigma_{N,k}$ denote the hyperplane of configurations with k particles, namely

$$\Sigma_{N,k} = \{\eta \in \chi_d^N : \sum_{x\in\mathbb{T}_N^d} \eta(x) = k\}, \tag{2.2.4}$$

which is invariant under the dynamics. We say that $\mathcal{O}$ is an irreducible component of $\Sigma_{N,k}$ if for every η, $\xi \in \mathcal{O}$ it is possible to go from η to ξ by a sequence of allowed jumps. For SSEP, the hyperplanes $\Sigma_{N,K}$ are irreducible components for any choice of k and N. In the presence of constraints, a more complicated decomposition in general arises. For example in $d = 1$ with the rates (2.2.3), the above observation on blocked configurations and on the non cooperative character of the model, leads to the following irreducible decomposition for the hyperplanes. If $k > N/3$, $\Sigma_{N,K}$ is irreducible. Instead, if $k \leq N/3$, $\Sigma_{N,k}$ is reducible and decomposable into the irreducible component which contains all configurations with at least one couple of particles at distance at most two plus many irreducible sets each containing only a blocked configuration

The irreducible decomposition in dimension $d > 1$ is more complicated. In this case the model is still non cooperative and a possible mobile cluster is given by a d-dimensional hypercube of particles of linear size 2. For any choice of the spatial dimension d, it is possible to identify a constant $C(d) < \infty$ such that the hyperplane $\Sigma_{N,k}$ is irreducible for $k > C(d)(N/3)^d$, while it is reducible in several components for $k \leq C(d)(N/3)^d$. In this case we have: (i)the irreducible component which contain configurations with at least one d-dimensional hypercube of particles of linear size two plus all configurations that can be connected to these; (ii) irreducible components which contain single (blocked) configuration; (iii) other irreducible components which contain neither blocked configurations nor any configuration belonging to (i). An example of irreducible set of the third kind for the rates (2.2.3) in $d = 2$ is for example the one that contains all configurations which have two particles at distance smaller or equal to two on a given line, for $x = (x_1, x_2)$ such that $\eta(x + e_i) = \eta(x) = 1$ or $\eta(x + 2e_i) = \eta(x) = 1$ and are completely empty $\forall\ y = (y_1, y_2)$ which do not belong to the same line, namely $y_2 \neq x_2$.

In order to investigate the hydrodynamic limit, we need to settle some notation. Define the **empirical measure** by:

$$\pi_t^N(du) = \pi^N(\eta_t, du) = \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} \eta_t(x) \delta_{\frac{x}{N}}(du) \tag{2.2.5}$$

where δ_u denotes the Dirac measure at u.

Let $\mathbb{T}^d$ denote the d-dimensional torus. Fix now, a initial profile $\rho_0 : \mathbb{T}^d \to [0,1]$ and denote by $(\mu^N)_{N\geq 1}$ a sequence of probability measures on χ_d^N.

Definition 2.2.1. *A sequence $(\mu^N)_{N\geq 1}$* ***is associated to*** *ρ_0, if for every continuous function $H : \mathbb{T}^d \to \mathbb{R}$ and for every $\delta > 0$*

$$\lim_{N\to+\infty} \mu^N \Big[\eta : \Big| \frac{1}{N} \sum_{x \in \mathbb{T}_N^d} H\Big(\frac{x}{N}\Big)\eta(x) - \int_{\mathbb{T}^d} H(u)\rho_0(u)du \Big| > \delta \Big] = 0. \tag{2.2.6}$$

Our goal consists in showing that if at time $t = 0$ the empirical measures are associated to some initial profile ρ_0, then at the macroscopic time t they are associated to a profile ρ_t, where ρ_t is the solution of the hydrodynamic equation (2.1.1).

There are at least two methods available in the literature in order to prove the hydrodynamic limit of an interacting particle system. One is known as the Relative Entropy Method and it was first introduced by Yau in [27], when proving the hydrodynamic limit for Ginzburg-Landau models. This method requires the existence of smooth solutions of the hydrodynamic equation. The second one is known as the Entropy Method and it is due to Guo, Papanicolau and Varadhan [11]. In contrast with the first method, this requires the uniqueness of weak solutions of the hydrodynamic equation.

In order to be able to apply the **Relative Entropy Method**, since it requires the existence of a smooth solution of the hydrodynamic equation, fix $\epsilon > 0$ and let $\rho_0 : \mathbb{T}^d \to$

$[0,1]$ be a profile of class $C^{2+\epsilon}(\mathbb{T}^d)$. By Theorem A2.4.1 of [14], equation (2.1.1) admits a solution that we denote by $\rho(t,\cdot)$ which is of class $C^{1+\epsilon,2+\epsilon}(\mathbb{R}_+\times\mathbb{T}^d)$.

Here we also have to impose a bound condition on the initial profile, as the existence of a strictly positive constant δ_0 such that

$$\forall u\in\mathbb{T}^d,\qquad \delta_0\leq\rho_0(u)\leq 1-\delta_0. \tag{2.2.7}$$

Let $\nu^N_{\rho_0(\cdot)}$ be the product measure in χ^N_d such that:

$$\nu^N_{\rho_0(\cdot)}\{\eta,\eta(x)=1\}=\rho_0(x/N).$$

This means that the random variables $(\eta(x))_{x\in\mathbb{T}^d_N}$ are independent and each $\eta(x)$ has Bernoulli distribution of parameter $\rho_0(x/N)$.

For two measures μ and ν in χ^N_d denote by $H(\mu/\nu)$ the **relative entropy** of μ with respect to ν, defined by:

$$H(\mu/\nu)=\sup_f\Big\{\int fd\mu-\log\int e^f d\nu\Big\}, \tag{2.2.8}$$

where the supremum is carried over all continuous functions. In sake of completeness we state two results that concern the relative entropy that will be used in the sequel. For two measures μ,ν in χ^N_d and $\gamma>0$:

$$\int f(\eta)\mu(d\eta)\leq\frac{1}{\gamma}H(\mu/\nu)+\frac{1}{\gamma}\log\int\exp\{\gamma f(\eta)\}\nu(d\eta) \tag{2.2.9}$$

$$\mu(A)\leq\frac{\log(2)+H(\mu/\nu)}{\log(1+1/\nu(A))}. \tag{2.2.10}$$

The first result stated is known as the **Entropy inequality**. The second is an easy consequence of the first, for a proof see Proposition A1.8.2 of [14].

Theorem 2.2.2. *Let $\rho_0:\mathbb{T}^d\to[0,1]$ be a initial profile of class $C^{2+\epsilon}(\mathbb{T}^d)$ that satisfies the bound condition (2.2.7). Let $(\mu^N)_{N\geq1}$ be a sequence of probability measures on χ^N_d such that:*

$$H(\mu^N/\nu^N_{\rho_0(\cdot)})=o(N^d). \tag{2.2.11}$$

Then, for each $t\geq0$

$$\pi^N_t(du)\xrightarrow[N\to+\infty]{}\rho(t,u)du$$

in probability, where $\rho(t,u)$ is a smooth solution of equation (2.1.1).

In last result, we made two assumptions on the initial profile in order to obtain the result: the bound condition (2.2.7) and the restriction of taking class $C^{2+\epsilon}(\mathbb{T}^d)$. This is too restrictive, since we would like to analyze, for instance profiles that are indicator functions over a certain set, and for that reason we would like to consider profiles in which there is a region of their domain in which they vanish. The use of this method does not fulfill our needs in this sense. Another restriction comes from the initial measures: by inequality (2.2.10) it is not hard to show that if μ^N satisfies (2.2.11), then μ^N is associated to the profile ρ_0 as defined in (2.2.6).

On the other hand, the **Entropy Method** relies on the full irreducibility of the Markov process when restricted to a hyperplane. For the Markov process η_t with generator given by (2.2.1), we have seen above, that when restricted to a hyperplane it is not fully irreducible. For example, if one takes an hyperplane with a small density of particles, there exists several irreducible components but also several isolated components - the frozen states. One way of getting over this problem is to perturb slightly the dynamics in such a way that the frozen states are destroyed and the macroscopic hydrodynamic behavior still evolves according to (2.1.1) but in which we can take initial profiles that are not taken into account in the previous Theorem.

With that purpose, consider a Markov process with generator given by

$$\mathcal{L}_\theta^N = \mathcal{L}_P + N^{\theta-2}\mathcal{L}_S \tag{2.2.12}$$

where $\theta > 0$, $\mathcal{L}_P$ was introduced in (2.2.1) and $\mathcal{L}_S$ is the generator of the SSEP, which acts on local functions $f : \chi_d^N \to \mathbb{R}$ as

$$(\mathcal{L}_S f)(\eta) = \sum_{\substack{x,y\in\mathbb{T}_N^d\\|x-y|=1}} \frac{1}{2d}\eta(x)(1-\eta(y))(f(\eta^{x,y}) - f(\eta)),$$

where $\eta^{x,y}$ was defined in (2.2.2). Since the Bernoulli product measures as defined above are invariant measures for each of the processes $\mathcal{L}_P$ and $\mathcal{L}_S$ they are also invariant for the perturbed process $\mathcal{L}_\theta^N$.

Notice that $\mathcal{L}_S$ is multiplied by a factor $N^{\theta-2}$ and since we want to observe the same macroscopic density behavior, we have to restrict ourselves to the case $\theta < 2$. So, considering by one side the Markov process with generator $\mathcal{L}_P$ and by the other, another with generator $\mathcal{L}_\theta^N$, the hydrodynamic equation is the same and given by (2.1.1). However, while the former has for k small, each hyperplane decomposed into several pieces, which is a consequence of the existence of the frozen states, the latter has each hyperplane as a unique irreducible piece. The addition of the jumps of the SSEP destroy the frozen states: the Markov process with generator $\mathcal{L}_\theta^N$ inherits the irreducible properties of $\mathcal{L}_S$. This allows us to apply the Entropy Method to derive the hydrodynamic limit for this process. Henceforth, we consider a Markov process η_t with generator given by $\mathcal{L}_\theta^N$.

For a probability measure μ on χ_d^N, denote by $\mathbb{P}_\mu^{\theta,N} = \mathbb{P}_\mu$ the probability measure on the space $D([0,T], \chi_d^N)$, induced by the Markov process with generator $\mathcal{L}_\theta^N$, speeded up by N^2 and with initial measure μ, and by $\mathbb{E}_\mu$ the expectation with respect to $\mathbb{P}_\mu$.

We start by introducing the definition of weak solutions of equation (2.1.1).

Definition 2.2.3. *Fix a bounded profile $\rho_0 : \mathbb{T}^d \to \mathbb{R}$. A bounded function $\rho : [0,T]\times\mathbb{T}^d \to \mathbb{R}$ is a **weak solution** of equation (2.1.1) if for every function $H : [0,T]\times\mathbb{T}^d \to \mathbb{R}$ of class $C^{1,2}([0,T]\times\mathbb{T}^d)$*

$$\int_0^T dt \int_{\mathbb{T}^d} du \Big\{ \rho(t,u)\partial_t H(t,u) + (\rho(t,u))^2 \sum_{1\leq i\leq d} \partial^2_{u_i} H(t,u) \Big\}$$
$$+ \int_{\mathbb{T}^d} \rho_0(u) H(0,u) du = \int_{\mathbb{T}^d} \rho(T,u) H(T,u) du. \qquad (2.2.13)$$

The Entropy method requires the uniqueness of the weak solution of the hydrodynamic equation. This is a consequence of Theorem A2.4.4 of [14] together with the fact that there is no more than a particle per site.

Theorem 2.2.4. *Let $\rho_0 : \mathbb{T}^d \to [0,1]$ and $(\mu^N)_{N\geq 1}$ be a sequence of probability measures on χ_d^N associated to the profile ρ_0. Then, for every $0 \leq t \leq T$, for every continuous function $H : \mathbb{T}^d \to \mathbb{R}$ and for every $\delta > 0$,*

$$\lim_{N\to+\infty} \mathbb{P}^{\theta}_{\mu^N} \Big[\Big| \frac{1}{N^d} \sum_{x\in\mathbb{T}^d_N} H\Big(\frac{x}{N}\Big)\eta_t(x) - \int_{\mathbb{T}^d} H(u)\rho(t,u)du \Big| > \delta \Big] = 0,$$

where $\rho(t,u)$ is the unique weak solution of equation (2.1.1).

In the last theorem we were able to consider a larger kind of initial profiles since the condition on the smoothness and (2.2.7) were not needed, but also in position of taking initial measures $(\mu^N)_{N\geq 1}$ that do not need to satisfy (2.2.11).

Once we have established the Law of Large Numbers for the empirical measure for the process with generator $\mathcal{L}^N_\theta$, the next step is to obtain the **Central Limit Theorem** starting from the invariant measure ν_ρ, which we proceed to present. For that we need to introduce some notation.

For each $z > 0$ (resp. $z < 0$) define $h_z : \mathbb{T}^d \to \mathbb{R}$ by $h_z(u) = \sqrt{2}\cos(2\pi z\cdot u)$ (resp. $h_z(u) = \sqrt{2}\sin(2\pi z \cdot u)$ and let $h_0 = 1$. Here $\cdot$ denotes the inner product of $\mathbb{R}^d$. It is well known that the set $\{h_z, z \in \mathbb{Z}^d\}$ is an orthonormal basis of $L^2(\mathbb{T}^d)$. In this space consider the operator $\Omega = 1 - \Delta$. A simple computation shows that $\Omega h_z = \gamma_z h_z$ where $\gamma_z = 1 + 4\pi^2||z||^2$.

For a positive integer k, denote by $\mathcal{H}_k$ the space obtained as the completion of $C^\infty(\mathbb{T}^d)$ endowed with the inner product defined by $< f, g >_k = < f, \Omega^k g >$. Let $\mathcal{H}_{-k}$ denote the dual of $\mathcal{H}_k$ with respect to the inner product of $L^2(\mathbb{T}^d)$.

As we want to investigate the fluctuations of the empirical measure, fix $\rho > 0$ and denote by $\mathcal{Y}^N_\cdot$ the **density fluctuation field** that acts on smooth functions H as

$$\mathcal{Y}^N_t(H) = \frac{1}{N^{d/2}} \sum_{x\in\mathbb{T}^d_N} H\Big(\frac{x}{N}\Big)(\eta_{tN^2}(x) - \rho). \qquad (2.2.14)$$

Fix a positive integer k and denote by $D([0,T],\mathcal{H}_{-k})$ (resp. $C([0,T],\mathcal{H}_{-k})$) the space of $\mathcal{H}_{-k}$ functions, that are right continuous with left limits (resp. continuous), endowed with the uniform weak topology. Denote by $\mathcal{Q}_N$ the probability measure on $D([0,T],\mathcal{H}_{-k})$ induced by $\mathcal{Y}^N_\cdot$ and ν_ρ.

Theorem 2.2.5. *Fix an integer $k \geq 3$. Denote by $\mathcal{Q}$ be the probability measure on $C([0,T],\mathcal{H}_{-k})$ corresponding to the stationary generalized Ornstein-Uhlenbeck process with mean 0 and covariance given by*

$$E_{\mathcal{Q}}[Y_t(H)Y_s(G)] = \frac{\boldsymbol{Var}(\nu_\rho,\eta(0))}{\sqrt{8\pi\rho(t-s)}}\int_{\mathbb{R}^d} du \int_{\mathbb{R}^d} dv \bar{H}(u)\bar{G}(v)\exp\Big\{-\frac{(u-v)^2}{8(t-s)\rho}\Big\}$$

for every $0 \leq s \leq t$ and H, G in $\mathcal{H}_k$. Here $\bar{H}$ and $\bar{G}$ are periodic functions equal to H, G in $\mathbb{T}^d$.

Then, $(\mathcal{Q}_N)_{N\geq 1}$ converges weakly to the probability measure $\mathcal{Q}$.

The proof of this theorem is very close to the one presented for the Zero-Range process in [14] and for this reason we have omitted it. We note that, since the proof of the Boltzmann-Gibbs in [14] relies on the ergodicity of the Markov process, we can use the ergodic properties of the generator $\mathcal{L}_S$ to obtain the result. We also remark if one considers the process η_t with generator $\mathcal{L}_P$, we obtain in the limit the same Ornstein-Uhlenbeck process as described above. We only stress that, in this case, we derive the Boltzmann-Gibbs Priciple in a similar way as we do in the proof of the One Block estimate via the Relative Entropy method.

Finally, in section 2.7 we investigate the spectral gap for the process in finite volume for $d = 1$. Our aim is to study the dependence on the size of the system of the spectral gap of the process. This, as is very well known, scales as $1/N^2$ for SSEP on all hyperplanes $\Sigma_{N,k}$, uniformly in k. For our models, due to presence of constraints, the uniformity in k is certainly lost, see remark 2.7.4.

We remark that none of the results stated above need the estimate on the spectral gap. However, if one wants to show the hydrodynamic limit for the non-gradient system: a Markov process in the same space state whose jump rates are given in the one-dimensional setting by $c'(x,x+1,\eta) = 1_{\{\eta:\eta(x-1)+\eta(x+2)=1\}}$, the proof of the hydrodynamic limit relies heavily on the sharp estimate of the spectral gap as of having order N^2.

In order to illustrate our results we need to introduce a few additional notations. Fix an integer N and denote by Λ_N the box of size N, $\Lambda_N = \{1,2\dots N\}$ and by χ^N the configuration space $\chi^N = \{0,1\}^N$. In order to define the generator on χ^N we could use definition 2.2.1 with the sum restricted to $x,y \in \Lambda_N$. However, some care should be put when defining the jump rate for sites close to the boundary of Λ_N, since $c(x,y,\eta)$ as defined in (2.2.3) depend not only on the configuration on x and y but also on their neighbouring sites which can be outside Λ_N. We denote by $\partial\Lambda_N$ the *boundary set* including all sites which do not belong to Λ_N and are nearest neighbour to at least one site in Λ_N, $\partial\Lambda_N = \{x \in \mathbb{Z} : x \notin \Lambda_N, d(x,\Lambda_N) = 1\}$, where as usual the distance between a point x and the set Λ_N is the infimum of the distances between $y \in \Lambda_N$ and x. A possible way to define the finite volume generator is to imagine that the configuration in the boundary set is frozen to a reference configuration, σ, and to define the finite volume rates as $c^\sigma(x,y,\eta) = c(x,y,\eta\cdot\sigma)$ where c are the rates in (2.2.3) and $\eta\cdot\sigma \in \{0,1\}^{|\Lambda_N|+|\partial\Lambda_N|}$ is the configuration equal to $\eta(x)$ on sites $x \in \Lambda_N$ and to $\sigma(x)$ on $x \in \partial\Lambda_N$. In the following we make the choice $\sigma(x) = 0$ in $x \in \partial\Lambda_N$ and we denote by $\mathcal{L}_{P,\Lambda_N}$ and $\mathcal{L}^N_{\theta,\Lambda_N}$ the Markov processes with this choice corresponding to (2.2.1) and (2.2.12). For sake of clarity, we explicitly write the second one

$$
\begin{aligned}
(\mathcal{L}^N_{\theta,\Lambda_N} f)(\eta) = &\sum_{x\in\Lambda_N\setminus\{1,N-1\}} \Big(c(x,x+1,\eta)+N^{\theta-2}\Big)\eta(x)(1-\eta(x+1))(f(\eta^{x,x+1})-f(\eta)) \\
&+\sum_{x\in\Lambda_N\setminus\{2,N\}} \Big(c(x,x-1,\eta)+N^{\theta-2}\Big)\eta(x)(1-\eta(x-1))(f(\eta^{x,x-1})-f(\eta)) \\
&+\Big(\eta(3)+N^{\theta-2}\Big)\Big(\eta(1)-\eta(2)\Big)\Big(f(\eta^{1,2})-f(\eta)\Big) \\
&+\Big(\eta(N-1)+N^{\theta-2}\Big)\Big(\eta(N-1)-\eta(N)\Big)\Big(f(\eta^{N-1,N})-f(\eta)\Big)
\end{aligned}
\tag{2.2.15}
$$

where $c(x,y,\eta)$ and $\eta^{x,y}$ was defined in (2.2.3) and (2.2.2), respectively.

Let, with a slight abuse of notation, $\Sigma_{N,k}$ denote again the hyperplanes with k particles, namely those in (2.2.4) but with the sum running over Λ_N. For each k, the Markov process generated by $\mathcal{L}^N_{\theta,\Lambda_N}$ is irreducible on $\Sigma_{N,k}$. The same holds for the process generated by $\mathcal{L}_{P,\Lambda_N}$ but only for $k > N/3$. This can be again proved by using the fact that the model is non cooperative with two particles at distance at most two being a mobile cluster. In both cases the unique invariant measure is the uniform measure, $\nu_{N,k}$.

For a generator $\mathcal{L}$ with invariant measure μ, denote by $\lambda_N(\mathcal{L})$ its spectral gap, defined by

$$
\lambda_N(\mathcal{L}) = \inf_{f\in L^2(\mu)} \frac{\mathfrak{D}_{\mathcal{L}}(f,\mu)}{\mathbf{Var}(f,\mu)},
$$

where $\mathfrak{D}_{\mathcal{L}}(f,\mu)$ denotes the Dirichlet form defined by

$$
\mathfrak{D}_{\mathcal{L}}(f,\mu) = \int_{\chi_d^N} -f(\eta)\mathcal{L}f(\eta)\mu(d\eta). \tag{2.2.16}
$$

In the following we will also use the shortened notation $\mathfrak{D}_P(f,\mu)$ and $\mathfrak{D}_\theta(f,\mu)$ to denote the Dirichlet form with generator $\mathcal{L}_{P,\Lambda_N}$ and $\mathcal{L}^N_{\theta,\Lambda_N}$, respectively. Let $\rho = k/N$. We obtain that:

Proposition 2.2.6. *Fix $k > N/3$. For the Markov process with generator $\mathcal{L}_{P,\Lambda_N}$, there exists a constant C that does not depend on N nor k such that*

$$
\lambda_N(\mathcal{L}^N_{\theta,\Lambda_N}) \geq \lambda_N(\mathcal{L}_{P,\Lambda_N}) \geq C\frac{(\rho-1/3)}{\rho N^2}.
$$

Proposition 2.2.7. *Fix $k \leq N/3$. For the Markov process with generator $\mathcal{L}^N_{\theta,\Lambda_N}$, there exists a constant C that does not depend on N nor k such that:*

$$
\lambda_N(\mathcal{L}^N_{\theta,\Lambda_N}) \geq C\frac{\rho^3}{\rho^\theta k^{1-\theta} N^2}.
$$

2.3 The Relative Entropy Method

In this section, we prove Theorem 2.2.2. Let $\rho_0 : \mathbb{T}^d \to [0,1]$ be a profile of class $C^{2+\epsilon}(\mathbb{T}^d)$ that satisfies the bound condition (2.2.7) and let $(\mu^N)_{N\geq 1}$ be a sequence of probability measures on χ_d^N that satisfies (2.2.11).

Fix a time $t \geq 0$. Denote by $S_t^{N,P}$, the semigroup associated to the generator $\mathcal{L}_P$ speeded up by N^2 and by μ_t^N the distribution of the process at time t: $\mu_t^N = \mu^N S_t^{N,P}$.

In order to prove the Theorem, we must verify that for $\delta > 0$, $t > 0$ and $H \in C(\mathbb{T}^d)$:

$$\lim_{N\to+\infty} \mu_t^N(\mathcal{A}_{t,\delta}) = 0,$$

where

$$\mathcal{A}_{t,\delta} = \Big\{\eta : \Big|\frac{1}{N}\sum_{x\in\mathbb{T}_N^d} H\Big(\frac{x}{N}\Big)\eta(x) - \int_{\mathbb{T}^d} H(u)\rho(t,u)du\Big| > \delta\Big\}$$

and $\rho(t,u)$ is a smooth solution of (2.1.1).

Denote by $\nu_{\rho(t,\cdot)}^N$ the product measure with slowly varying parameter associated to the profile $\rho(t,\cdot)$. This means that under the measure $\nu_{\rho(t,\cdot)}^N$ the random variables $\Big(\eta(x)\Big)_{x\in\mathbb{T}_N^d}$ are independent and each $\eta(x)$ has Bernoulli distribution of parameter $\rho(t,x/N)$:

$$\nu_{\rho(t,\cdot)}^N\Big\{\eta, \eta(x) = 1\Big\} = \rho(t,x/N).$$

By inequality (2.2.10):

$$\mu_t^N(\mathcal{A}_{t,\delta}) \leq \frac{\log 2 + H(\mu_t^N/\nu_{\rho(t,\cdot)}^N)}{\log(1 + 1/\nu_{\rho(t,\cdot)}^N(\mathcal{A}_{t,\delta}))}.$$

Since $\nu_{\rho_{(t,\cdot)}}^N$ is a product measure and by large deviation estimates, it holds that

$$\lim_{N\to+\infty} \frac{1}{N^d}\log \nu_{\rho(t,\cdot)}^N(\mathcal{A}_{t,\delta}) = -C(\delta).$$

This means that the denominator of the right hand side of last inequality is of order $O(N^d)$. So, the proof is concluded if one can show that $H(\mu_t^N/\nu_{\rho(t,\cdot)}^N)$ is of order $o(N^d)$. In fact, this estimate is the main result when applying the Relative Entropy Method, and as so, we state it as a Theorem:

Theorem 2.3.1. *Let $\rho_0 : \mathbb{T}^d \to \mathbb{R}$ be an initial profile of class $C^{2+\epsilon}(\mathbb{T}^d)$ that satisfies the bound condition (2.2.7) and $(\mu^N)_{N\geq 1}$ a sequence of probability measures in χ_d^N that satisfies the condition (2.2.11). Then, for every $t \geq 0$*

$$H\Big(\mu_t^N/\nu_{\rho(t,\cdot)}^N\Big) = o(N^d),$$

where $\rho(t,u)$ is a smooth solution of (2.1.1).

Proof. Fix $\alpha \in (0,1)$ and an invariant measure ν_α. Let

$$\psi_t^N = \frac{d\nu^N_{\rho(t,\cdot)}}{d\nu_\alpha}, \qquad f_N(t) = \frac{d\mu_t^N}{d\nu_\alpha}, \qquad H_N(t) = H\Big(\mu_t^N / \nu^N_{\rho(t,\cdot)}\Big).$$

Since the measures $\nu^N_{\rho(t,\cdot)}$ and ν_α are product, it is very simple to obtain an expression for ψ_t^N:

$$\psi_t^N(\eta) = \frac{1}{Z_t^N} \exp\Big\{ \sum_{x \in \mathbb{T}_N^d} \eta(x) \lambda(t, x/N) \Big\},$$

where

$$\lambda(t,u) = \log\Big(\frac{\rho(t,u)(1-\alpha)}{\alpha(1-\rho(t,u))} \Big),$$

and Z_t^N is a renormalizing constant. Note that $\lambda(t,u)$ is well defined because the profile $\rho(t,\cdot)$ is bounded away from 0 and 1. This is a consequence of (2.2.7) together with the Maximum Principle, see for example Theorem A2.4.1 of [14].

In order to prove the result, we are going to show that

$$H_N(t) \leq o(N^d) + \frac{1}{\gamma} \int_0^t H_N(s) ds$$

for some $\gamma > 0$, and apply Gronwall inequality to conclude.

There is a celebrated estimate for the entropy production due to Yau [27]:

$$\partial_t H_N(t) \leq \int \Big\{ \frac{N^2 \mathcal{L}_P^* \psi_t^N(\eta)}{\psi_t^N(\eta)} - \partial_t \log \psi_t^N(\eta) \Big\} f_N(t)(\eta) \nu_\alpha(d\eta), \tag{2.3.1}$$

where $\mathcal{L}_P^*$ is the adjoint operator of $\mathcal{L}_P$ in $L^2(\nu_\alpha)$. Now, we compute the right hand side of last inequality.

2.3.1 Computation of $N^2 \mathcal{L}_P^* \psi_t^N(\eta) / \psi_t^N(\eta)$

Since the operator $\mathcal{L}_P$ is self-adjoint in $L^2(\nu_\alpha)$, we have that

$$\frac{1}{\psi_t^N(\eta)} N^2 \mathcal{L}_P^* \psi_t^N(\eta) = N^2 \sum_{\substack{x,y \in \mathbb{T}_N^d \\ |x-y|=1}} c(x,y,\eta)\eta(x)(1-\eta(y)) \Big\{ \frac{\psi_t^N(\eta^{x,y})}{\psi_t^N(\eta)} - 1 \Big\}$$

$$= N^2 \sum_{\substack{x,y \in \mathbb{T}_N^d \\ |x-y|=1}} c(x,y,\eta)\eta(x)(1-\eta(y)) \Big\{ \exp\{\lambda(t,y/N) - \lambda(t,x/N)\} - 1 \Big\}.$$

Expanding the exponential up to the second order, last expression can be written as:

$$N^2 \sum_{\substack{x,y \in \mathbb{T}_N^d \\ |x-y|=1}} c(x,y,\eta)\eta(x)(1-\eta(y))[\lambda(t,y/N) - \lambda(t,x/N)]$$

$$+\frac{1}{2}\sum_{\substack{x,y\in\mathbb{T}_N^d\\|x-y|=1}} c(x,y,\eta)\eta(x)(1-\eta(y))[N(\lambda(t,y/N)-\lambda(t,x/N))]^2$$

plus a term of order $O(N^2)$.

The term of order one is equal to:

$$N^2\sum_{x\in\mathbb{T}_N^d}\sum_{j=1}^d c(x,x+e_j,\eta)(\eta(x)-\eta(x+e_j))[\lambda(t,x+e_j/N)-\lambda(t,x/N)].$$

By a Taylor expansion and two summations by parts it is not hard to show that last expression can be written as

$$\sum_{x\in\mathbb{T}_N^d}\sum_{j=1}^d \tau_x h_j(\eta)\partial_{u_j}^2\lambda(t,x/N),$$

plus a term of order $o(1)$, where

$$h_j(\eta)=\eta(0)\eta(e_j)+\eta(0)\eta(-e_j)-\eta(-e_j)\eta(e_j). \tag{2.3.2}$$

By a Taylor expansion, the second order term equals

$$\sum_{x\in\mathbb{T}_N^d}\sum_{j=1}^d \tau_x g_j(\eta)(\partial_{u_j}\lambda(t,x/N))^2,$$

where

$$g_j(\eta)=c(0,e_j,\eta)(\eta(0)-\eta(e_j))^2. \tag{2.3.3}$$

So far we have that $N^2\mathcal{L}_P^*\psi_t^N(\eta)/\psi_t^N(\eta)$ equals to

$$\sum_{x\in\mathbb{T}_N^d}\sum_{j=1}^d \tau_x h_j(\eta)\partial_{u_j}^2\lambda(t,x/N)+\sum_{x\in\mathbb{T}_N^d}\sum_{j=1}^d \tau_x g_j(\eta)(\partial_{u_j}\lambda(t,x/N))^2.$$

2.3.2 Computation of $\partial_t \log\psi_t^N(\eta)$

Using the expression for ψ_t^N, we have that

$$\partial_t\log\psi_t^N(\eta)=\sum_{x\in\mathbb{T}_N^d}\eta(x)\partial_t\lambda(t,x/N)-E_{\psi_t^N}\Big(\sum_{x\in\mathbb{T}_N^d}\eta(x)\partial_t\lambda(t,x/N)\Big).$$

The first term on the right hand side of last expression is equal to

$$\sum_{x\in\mathbb{T}_N^d}\eta(x)\frac{\partial_t\rho(t,x/N)}{\rho(t,x/N)(1-\rho(t,x/N))},$$

while the second equals

$$\sum_{x\in\mathbb{T}_N^d}\frac{\partial_t\rho(t,x/N)}{(1-\rho(t,x/N))}.$$

Using the fact that $\rho(t,\cdot)$ is a solution of (2.1.1) with $m=2$, it holds that

$$\partial_t\log\psi_t^N(\eta)=\sum_{x\in\mathbb{T}_N^d}\sum_{1\le j\le d}\frac{\partial_{u_j}^2\rho^2(t,x/N)}{\rho(t,x/N)(1-\rho(t,x/N))}\Big[\eta(x)-\rho(t,x/N)\Big].$$

Here is the point in the proof where we have used the fact that $\rho(t,.)$ is a classical solution of the hydrodynamic equation.

Here and after, for a local function f, denote $\tilde f(\rho)=E_{\nu_\rho}[f(\eta)]$. A straightforward computation shows that

$$\Big\{\tilde h'(\rho(t,x/N))\partial_{u_j}^2\lambda(t,x/N)+\tilde g'(\rho(t,x/N))(\partial_{u_j}\lambda(t,x/N))^2\Big\}=\frac{\partial_{u_j}^2\rho^2(t,x/N)}{\rho(t,x/N)(1-\rho(t,x/N))}.$$

Then, $\partial_t\log\psi_t^N(\eta)$ equals to

$$\sum_{x\in\mathbb{T}_N^d}\sum_{j=1}^d\tilde h'(\rho(t,x/N))\partial_{u_j}^2\lambda(t,x/N)\Big[\eta(x)-\rho(t,x/N)\Big]$$

$$+\sum_{x\in\mathbb{T}_N^d}\sum_{j=1}^d\tilde g'(\rho(t,x/N))(\partial_{u_j}\lambda(t,x/N))^2\Big[\eta(x)-\rho(t,x/N)\Big].$$

By the computations just made, the expression inside braces of inequality (2.3.1) is equal to

$$\sum_{x\in\mathbb{T}_N^d}\sum_{j=1}^d\partial_{u_j}^2\lambda(t,x/N)\Big\{\tau_xh_j(\eta)-\tilde h'(\rho(t,x/N))\Big[\eta(x)-\rho(t,x/N)\Big]\Big\}$$

$$+\sum_{x\in\mathbb{T}_N^d}\sum_{j=1}^d(\partial_{u_j}\lambda(t,x/N))^2\Big\{\tau_xg_j(\eta)-\tilde g'(\rho(t,x/N))\Big[\eta(x)-\rho(t,x/N)\Big]\Big\},$$

plus a term of order $o(N^d)$. An integration by parts shows that

$$\int_{\mathbb{T}^d}\sum_{j=1}^d\Big\{\tilde h(\rho(t,u))\partial_{u_j}^2\lambda(t,u)+\tilde g(\rho(t,u))(\partial_{u_j}\lambda(t,u))^2\Big\}du=0,$$

so, we can write last expression as

$$\sum_{x\in\mathbb{T}_N^d}\sum_{j=1}^d\partial_{u_j}^2\lambda\Big(t,\frac{x}{N}\Big)\Big\{\tau_xh_j(\eta)-\tilde h\Big(\rho\Big(t,\frac{x}{N}\Big)\Big)-\tilde h'\Big(\rho\Big(t,\frac{x}{N}\Big)\Big)\Big[\eta(x)-\rho\Big(t,\frac{x}{N}\Big)\Big]\Big\}$$

$$+\sum_{x\in\mathbb{T}_N}\sum_{j=1}^{d}\Big(\partial_{u_j}\lambda\Big(t,\frac{x}{N}\Big)\Big)^2\Big\{\tau_x g_j(\eta)-\tilde{g}\Big(\rho\Big(t,\frac{x}{N}\Big)\Big)-\tilde{g}'\Big(\rho\Big(t,\frac{x}{N}\Big)\Big)\Big[\eta(x)-\rho\Big(t,\frac{x}{N}\Big)\Big]\Big\}$$

plus a term of order $o(N^d)$.

Now, we replace the local functions $\tau_x h_j(\eta)$ and $\tau_x g_j(\eta)$ by their expectation with respect to the invariant measure ν_α, where the parameter α is taken equal to the empirical average of particle in a box of size l, namely: $\tilde{h}(\eta_s^l(x))$ and $\tilde{g}(\eta_s^l(x))$, respectively, with the use of the One-Block estimate, see Lemma (2.3.2). It's proof is postponed to the end of this section.

On the other hand, by the continuity of $\partial^2_{u_j}\lambda(t,\cdot)\tilde{h}'(\rho(t,\cdot))$ and $(\partial_{u_j}\lambda(t,\cdot))^2\tilde{g}'(\rho(t,\cdot))$, a summation by parts permits to replace $\eta(x)$ by $\eta^l(x)$, since:

$$\sum_{x\in\mathbb{T}_N^d}\sum_{j=1}^{d}\partial^2_{u_j}\lambda(t,x/N)\tilde{h}'(\rho(t,x/N))[\eta(x)-\eta^l(x)]=o(N^d)$$

and

$$\sum_{x\in\mathbb{T}_N^d}\sum_{j=1}^{d}(\partial_{u_j}\lambda(t,x/N))^2\tilde{g}'(\rho(t,x/N))[\eta(x)-\eta^l(x)]=o(N^d).$$

After all this considerations, we can rewrite $(\psi_t^N(\eta))^{-1}\{N^2\mathcal{L}_P\psi_t^N(\eta)-\partial_t\psi_t^N(\eta)\}$ as

$$\sum_{x\in\mathbb{T}_N^d}\sum_{j=1}^{d}\partial^2_{u_j}\lambda\Big(t,\frac{x}{N}\Big)\Big\{\tilde{h}(\eta^l(x))-\tilde{h}\Big(\rho\Big(t,\frac{x}{N}\Big)\Big)-\tilde{h}'\Big(\rho\Big(t,\frac{x}{N}\Big)\Big)\Big[\eta^l(x)-\rho\Big(t,\frac{x}{N}\Big)\Big]\Big\}$$

plus the term

$$\sum_{x\in\mathbb{T}_N^d}\sum_{j=1}^{d}(\partial_{u_j}\lambda\Big(t,\frac{x}{N}\Big)\Big)^2\Big\{\tilde{g}(\eta^l(x))-\tilde{g}\Big(\rho\Big(t,\frac{x}{N}\Big)\Big)-\tilde{g}'\Big(\rho\Big(t,\frac{x}{N}\Big)\Big)\Big[\eta^l(x)-\rho\Big(t,\frac{x}{N}\Big)\Big]\Big\}.$$

Repeating standard arguments of the relative entropy method, the result follows. We refer the reader to chapter 6 of [14] for details. □

2.3.3 One-Block estimate

The main difficulty in the derivation of the One-Block estimate is the fact that we are not in a erdogic setting. In order to overcome this problem, we separate the set of configurations into two sets: the big irreducible component (the set of all configurations that contain at least one block of particles at distance at most two) and the remaining ones. In the first case the standard proof is easily adapted, while in the remaining set of configurations the important ingredient is the fact that it has small measure with respect to $\nu^N_{\rho(t,\cdot)}$.

Now, we introduce some notation. Denote the also called Dirichlet form associated to a generator Ω and a measure μ in χ_d^N, by $D_\Omega(f,\mu)$ which is defined on positive functions by $D_\Omega(f,\mu)=\mathfrak{D}_\Omega(\sqrt{f},\mu)$ and $\mathfrak{D}_\Omega(f,\mu)$ was defined in (2.2.16). Let $f_t^{N,P}$ denote the Radon-Nikodym density of $\mu^{N,P}(t)=\frac{1}{t}\int_0^t\mu^N S_s^{N,P}ds$ with respect to ν_α.

Lemma 2.3.2. *(One-block Estimate)*
For every local function ψ

$$\limsup_{l\to+\infty}\limsup_{N\to+\infty}\int \frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d}\tau_x V_{l,\psi}(\eta) f_t^{N,P}(\eta)\nu_\alpha(d\eta) \le \frac{1}{\gamma N^d}\int_0^t H_N(s)ds,$$

where

$$V_{l,\psi}(\eta) = \Big|\frac{1}{(2l+1)^d}\sum_{|y|\le l}\tau_y\psi(\eta) - \tilde{\psi}(\eta^l(0))\Big|, \tag{2.3.4}$$

and

$$\eta^l(0) = \frac{1}{(2l+1)^d}\sum_{|y|\le l}\eta(y). \tag{2.3.5}$$

Proof. Fix $x \in \mathbb{T}_N^d$ and denote by $\mathcal{Q}_{x,l}$ the set of configurations in the box of center x and radius l containing at least one d-dimensional hypercube of linear size 2 which is completely filled:

$$\mathcal{Q}_{x,l} = \Big\{\eta : \sum_{y\in C_y}\prod_{z\in Q_y}\eta(z)\ge 1\Big\}.$$

where $Q_y = \{z : |z_i - y_i| \in \{0,1\}\ \forall i \in \{1,\dots,d\}\}$ and $C_y = \{y : |y-x|\le l,\ Q_y \subset \mathbb{T}_N^d\}$. We denote by $\mathcal{E}_{x,l}$ the irreducible set which contains $\mathcal{Q}_{x,l}$ (and all configurations that can be connected via an allowed path to one in $\mathcal{E}_{x,l}$). We can split the integral that appears in the statement of the Lemma into

$$\int \frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d}\tau_x V_{l,\psi}(\eta)1_{\{\mathcal{E}_{x,l}\}}(\eta) f_t^{N,P}(\eta)\nu_\alpha(d\eta) \tag{2.3.6}$$

$$+\int \frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d}\tau_x V_{l,\psi}(\eta)1_{\{\mathcal{E}^c_{x,l}\}}(\eta) f_t^{N,P}(\eta)\nu_\alpha(d\eta). \tag{2.3.7}$$

At first note that since $H(\mu^N/\nu_\alpha) = O(N^d)$ and the entropy decreases with time, it holds that $H(f_t^{N,P}) = O(N^d)$. This implies that the Dirichlet form of $f_t^{N,P}$, namely $D_P(f_t^{N,P},\nu_\alpha)$ is bounded above by CN^{d-2}, see the fifth section for details. Repeating the standard arguments of the One-block estimate, when restricted to the irreducible configurations a stronger result than (2.3.6) holds:

$$\limsup_{l\to+\infty}\limsup_{N\to+\infty}\sup_{D_P(f,\nu_\alpha)\le CN^{d-2}}\int \frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d}\tau_x V_{l,\psi}(\eta)1_{\{\mathcal{E}_{x,l}\}}(\eta) f(\eta)\nu_\alpha(d\eta) = 0.$$

The plan to accomplish this result is the following: first, put the dependence in N that appears in the integrand function, at the density f, by considering the space average of all translations of f that we denote by $\bar{f}$:

$$\bar{f}(\eta) = \frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d}\tau_x f(\eta). \tag{2.3.8}$$

The integral now becomes:

$$\int V_{l,\psi}(\eta)1_{\{\mathcal{E}_{0,l}\}}(\eta)\bar{f}(\eta)\nu_\alpha(d\eta).$$

Since $V_{l,\psi}$ depends on the configuration η through the variables at the box of center 0 and radius l, we can project the density over this box, and denote it by $\bar{f}_l$. Define the Dirichlet form of $\bar{f}_l$ restricted to the set $\mathcal{E}_{0,l}$ by $D_0^l(\bar{f}_l,\nu_\alpha)$. Using the translation invariance property of $\bar{f}$ together with the bound on the Dirichlet form of f, we obtain that $D_0^l(\bar{f}_l,\nu_\alpha)\leq CN^{-2}$. Taking the limit as $N\to+\infty$ along a subsequence, we can restrict last integral to densities f such that $D_0^l(f,\nu_\alpha)=0$. Since we defined this Dirichlet form restricted to the set $\mathcal{E}_{0,l}$, this density has to be constant on each hyperplane. To end the proof, it is enough to apply a simple argument of the equivalence of ensembles, which allows to recover the expectation with respect to the grand canonical measure as the limit of the expectation with respect to the canonical measure and then use the L.L.N. to conclude. For details we refer the reader to chap. 5 of [14].

For the remaining integral (2.3.7), write it as:

$$\frac{1}{t}\int_0^t\int\frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d}\tau_xV_{l,\psi}(\eta)1_{\{\mathcal{E}_{x,l}^c\}}(\eta)\phi_s(\eta)\nu_{\rho(s,\cdot)}^N(d\eta)ds,$$

where $\phi_s=\frac{d\mu_s^N}{d\nu_{\rho(s,\cdot)}^N}$. To keep notation simple, we drop the integral with respect to time. The entropy inequality (2.2.9) allows to bound the integral with respect to $\nu_{\rho(s,\cdot)}^N$ by

$$\frac{H\Big(\mu_s^N/\nu_{\rho(s,\cdot)}^N\Big)}{\gamma N^d}+\frac{1}{\gamma N^d}\log\int\exp\Big\{\gamma\sum_{x\in\mathbb{T}_N^d}\tau_xV_{l,\psi}(\eta)1_{\{\mathcal{E}_{x,l}^c\}}(\eta)\Big\}\nu_{\rho(s,\cdot)}^N(d\eta),$$

for every $\gamma>0$.

The first term is $H_N(s)/\gamma N^d$. On the other hand, since $V_{l,\psi}$ is bounded and by Hölder inequality, the term on the right hand side of last expression can be bounded by

$$\frac{1}{\gamma(2l+1)^dN^d}\log\int\exp\Big\{\gamma(2l+1)^d\sum_{x\in\mathbb{T}_N^d}C1_{\{\mathcal{E}_{x,l}^c\}}(\eta)\Big\}\nu_{\rho(s,\cdot)}^N(d\eta).$$

Since $\nu_{\rho(t,\cdot)}^N$ is a product measure, last expression can be written as:

$$\frac{1}{N\gamma(2l+1)^d}\sum_{x\in\mathbb{T}_N^d}\log\int\exp\Big\{\gamma(2l+1)^dC1_{\{\mathcal{E}_{x,l}^c\}}(\eta)\Big\}\nu_{\rho(s,\cdot)}^N(d\eta)$$

$$=\frac{1}{N\gamma(2l+1)^d}\sum_{x\in\mathbb{T}_N^d}\log\Big(\nu_{\rho(s,\cdot)}^N(\mathcal{E}_{x,l}^c)(\exp\{\gamma(2l+1)^dC\}-1)+1\Big).$$

Once the initial profile is bounded away from zero (see (2.2.7)), this implies that $\forall u \in \mathbb{T}^d$, $\rho(s,u) \geq \delta_0$ and as a consequence $\nu^N_{\rho(s,\cdot)}(\mathcal{E}^c_{x,l}) \leq (1-\delta_0)^{(2l+1)^d}$. On the other hand, $\forall x \in \mathbb{T}^d_N$ $\log(x+1) \leq x$ which implies that last expression is bounded by

$$\frac{1}{\gamma(2l+1)^d}(\exp\{\gamma(2l+1)^d C\}-1)(1-\delta_0)^{(2l+1)^d},$$

and vanishes as $l \to +\infty$ for γ small. □

2.4 The Entropy Method

Now, we prove Theorem 2.2.4. The strategy of the proof is the same as given for the Zero-Range process in [14]. The main step is the derivation of the One-Block and the Two-Blocks estimate, which are presented in section six.

Fix a time $T > 0$. Let $\mathcal{M}_+$ be the space of finite positive measures on $\mathbb{T}^d$ endowed with the weak topology. Consider a sequence of probability measures $(Q_N)_N$ on $D([0,T],\mathcal{M}_+)$ corresponding to the Markov process π^N_t as defined in (2.2.5), speeded up by N^2 and starting from μ^N.

First we prove that $(Q_N)_N$ is a tight sequence. Then we prove the uniqueness of a limit point, by showing that the limit points of $(Q_N)_N$ are concentrated on trajectories of measures absolutely continuous with respect to the Lebesgue measure, equal to $\rho_0(u)du$ at the initial time and whose density is concentrated on weak solutions of the hydrodynamic equation (2.1.1). By the uniqueness of these solutions we conclude that π^N_t has a unique limit point, concentrated on the trajectory with density $\rho(t,u)$, where $\rho(t,u)$ is the weak solution of equation (2.1.1).

We divide the proof in several steps, to make the exposition clearer. Fix a smooth function $H : \mathbb{T}^d \to \mathbb{R}$. Recall the definition of the empirical measure in (2.2.5) and let

$$< \pi^N_t, H >= \frac{1}{N^d}\sum_{x\in\mathbb{T}^d_N} H\Big(\frac{x}{N}\Big)\eta_t(x).$$

By lemma A1.5.1 of [14]

$$M^{N,H}_t =< \pi^N_t, H > - < \pi^N_0, H > - \int_0^t N^2\mathcal{L}_\theta < \pi^N_s, H > ds$$

is a martingale with respect to the filtration $\tilde{\mathcal{F}}_t = \sigma(\eta_s, s \leq t)$, whose quadratic variation is given by

$$\int_0^t \frac{1}{N^{2d}}\sum_{x\in\mathbb{T}^d_N}\sum_{j=1}^d \Big(\partial^N_{u_j} H\Big(\frac{x}{N}\Big)\Big)^2 \tau_x g^N_j(\eta_s)ds, \tag{2.4.1}$$

where $\partial^N_{u_j}H\Big(\frac{x}{N}\Big) = N\Big[H\Big(\frac{x+e_j}{N}\Big) - H\Big(\frac{x}{N}\Big)\Big]$, $g^N_j(\eta) = g_j(\eta) + (\eta(0)-\eta(e_j))^2N^{\theta-2}$ and $g_j(\eta)$ as defined in (2.3.3).

By definition,

$$\mathcal{L}_\theta^N \eta(x) = \sum_{1\le j\le d} W_{x-e_j,x}(\eta) - W_{x,x+e_j}(\eta),$$

and $W_{0,e_j}(\eta) = h_j^N(\eta) - \tau_{e_j} h_j^N(\eta)$ with $h_j^N(\eta) = h_j(\eta) + N^{\theta-2}\eta(0)$ and $h_j(\eta)$ as defined in (2.3.2). This allows us to perform a double summations by parts and rewrite the integral part of the martingale as

$$\int_0^t \frac{1}{N^d} \sum_{x\in\mathbb{T}_N^d} \sum_{j=1}^d \partial_{u_j}^2 H\Big(\frac{x}{N}\Big)\tau_x h_j^N(\eta_s)ds, \tag{2.4.2}$$

where $\partial_{u_j}^2 H\Big(\frac{x}{N}\Big) = N^2\Big[H\Big(\frac{x+e_j}{N}\Big) + H\Big(\frac{x-e_j}{N}\Big) - 2H\Big(\frac{x}{N}\Big)\Big]$.

2.4.1 Relative Compactness

In this section we prove that $(Q_N)_N$ is tight. Recall Aldous criterium:

Lemma 2.4.1. *A sequence $(P_N)_N$ of probability measures defined on $D([0,T],\mathcal{M}_+)$ is relatively compact if this two conditions hold:*

a. *For every $t\in[0,T]$ and every $\epsilon>0$, there exists $K_\epsilon^t \subset \mathcal{M}_+$ compact, such that*

$$\sup_N P_N[\pi_t \notin K_\epsilon^t] \le \epsilon,$$

b. *For every $\epsilon > 0$*

$$\lim_{\gamma\to 0}\limsup_{N\to\infty}\sup_{\substack{\tau\in\mathfrak{T}_T\\ \theta\le\gamma}} P_N[d(\pi_{\tau+\theta},\pi_\tau) > \epsilon] = 0,$$

where $\mathfrak{T}_T$ denotes the set of stopping times with respect to the canonical filtration, bounded by T and d is the metric in the space $\mathcal{M}_+$.

By Proposition 1.7 of chapter 4 in [14] it is enough to show that for every $H\in C^2(\mathbb{T}^d)$, the sequence of measures that corresponds to the real processes $<\pi_t^N, H>$ is relatively compact.

Since the number of particles per site is at most one, we only have to show condition **b**, which can be written in this context as

$$\lim_{\gamma\to 0}\limsup_{N\to\infty}\sup_{\substack{\tau\in\mathfrak{T}_T\\ \theta\le\gamma}} \mathbb{P}_{\mu^N}\Big[\Big|\frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d} H\Big(\frac{x}{N}\Big)\eta_{\tau+\theta}(x) - \frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d} H\Big(\frac{x}{N}\Big)\eta_\tau(x)\Big| > \epsilon\Big] = 0,$$

for every $H\in C^2(\mathbb{T}^d)$ and for every $\epsilon>0$. By the definition of the martingale $M_t^{N,H}$ in order to prove last inequality, we have to show that:

$$\lim_{\gamma\to 0}\limsup_{N\to\infty}\sup_{\substack{\tau\in\mathfrak{T}_T\\ \theta\le\gamma}} \mathbb{P}_{\mu^N}\Big[\Big|M_{\tau+\theta}^{N,H} - M_\tau^{N,H}\Big| > \epsilon\Big] = 0,$$

and

$$\lim_{\gamma\to 0}\limsup_{N\to\infty}\sup_{\substack{\tau\in\mathfrak{T}_T\\ \theta\leq\gamma}}\mathbb{P}_{\mu^N}\Big[\Big|\int_\tau^{\tau+\theta}\frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d}\sum_{j=1}^d\partial_{u_j}^2 H\Big(\frac{x}{N}\Big)\tau_x h_j^N(\eta_s)ds\Big|>\epsilon\Big]=0.$$

We start by the latter. Since $H\in C^2(\mathbb{T}^d)$, $\theta\leq\gamma$ and h is bounded, last integral is bounded by $C(H)\gamma$ and vanishes as $\gamma\to 0$.

By Chebyshev inequality and expression (2.4.1), the former term can be bounded by

$$\frac{1}{\epsilon^2}\mathbb{E}_{\mu^N}\Big[\int_\tau^{\tau+\theta}\frac{1}{N^{2d}}\sum_{x\in\mathbb{T}_N^d}\sum_{j=1}^d\Big(\partial_{u_j}^N H\Big(\frac{x}{N}\Big)\Big)^2\tau_x g_j^N(\eta_s)ds\Big]\leq\frac{C(H)\gamma}{N^d},$$

which vanishes as $N\to\infty$.

So far we have seen that $(Q_N)_N$ is tight, which implies that it has convergent subsequences. In the next section, we show the uniqueness of a limit point.

2.4.2 Uniqueness of Limit Points

Here, we prove at first that all limit points Q of $(Q_N)_N$ are concentrated on absolutely continuous measures with respect to the Lebesgue measure, that are equal to $\rho_0(u)du$ at the initial time and finally that they are concentrated on weak solutions of equation (2.1.1). Let Q be a limit point of $(Q_N)_N$.

Fix a continuous function $H:\mathbb{T}^d\to\mathbb{R}$. Since

$$\sup_{t\in[0,T]}|<\pi_t^N,H>|\leq\frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d}\Big|H\Big(\frac{x}{N}\Big)\Big|,$$

which is a consequence of the fact that of having at most one particle per site, the function that associates to each trajectory $\pi_\cdot$, $\sup_{t\in[0,T]}|<\pi_t,H>|$ is continuous. As a consequence all limit points are concentrated in trajectories π_t such that

$$|<\pi_t,H>|\leq\int_{\mathbb{T}^d}|H(u)|du.$$

In order to show that the measure is absolutely continuous with respect to the Lebesgue measure, that we denote by Leb, we have to show that for each set A such that $Leb(A)=0$, then $\pi_t(A)=0$, for that use last result for a sequence of continuous functions that converge to the indicator function over the set A.

Now we prove that Q is concentrated on a dirac measure equal to $\rho_0(u)du$ at time 0. Fix $\epsilon>0$. By the weak convergence over a subsequence and Portmanteau's Theorem, it holds that:

$$Q\Big[\Big|\frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d}H\Big(\frac{x}{N}\Big)\eta_0(x)-\int_{\mathbb{T}^d}H(u)\rho_0(u)du\Big|>\epsilon\Big]$$

$$\leq \liminf_{K\to+\infty} Q_{N_k}\Big[\Big|\frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d} H\Big(\frac{x}{N}\Big)\eta_0(x) - \int_{\mathbb{T}^d} H(u)\rho_0(u)du\Big| > \epsilon\Big]$$

$$= \liminf_{K\to+\infty} \mu^{N_k}\Big[\Big|\frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d} H\Big(\frac{x}{N}\Big)\eta(x) - \int_{\mathbb{T}^d} H(u)\rho_0(u)du\Big| > \epsilon\Big].$$

This last line is equal to zero, by the hypothesis of μ^N being associated to the profile ρ_0, see (2.2.6).

Finally, we show that all limit points of $(Q_N)_N$ are concentrated on weak solutions of (2.1.1). Fix a smooth function $H : [0,T]\times\mathbb{T}^d \to \mathbb{R}$ of class $C^{1,2}$,

$$M_t^{N,H} = \frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d} H\Big(t,\frac{x}{N}\Big)\eta_t(x) - \frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d} H\Big(t,\frac{x}{N}\Big)\eta(x) \tag{2.4.3}$$

$$-\int_0^t \frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d}\Big\{\partial_s H\Big(s,\frac{x}{N}\Big) + \sum_{j=1}^d \partial_{u_j}^2 H\Big(s,\frac{x}{N}\Big)\tau_x h_j^N(\eta_s)\Big\}ds$$

is an $\tilde{\mathcal{F}}_t$-martingale, whose quadratic variation is given by

$$\int_0^t \frac{1}{N^{2d}}\sum_{x\in\mathbb{T}_N^d}\sum_{j=1}^d \Big(\partial_{u_j} H\Big(s,\frac{x}{N}\Big)\Big)^2 \tau_x g_j^N(\eta_s)ds,$$

where $g_j^N(\eta) = g_j(\eta) + (\eta(0)-\eta(e_j))^2 N^{\theta-2}$ with $g_j(\eta)$ as defined in (2.3.3), and $h_j^N(\eta) = h_j(\eta) + N^{\theta-2}\eta(0)$ with $h_j(\eta)$ as defined in (2.3.2).

Using at first Chebyshev and then Doob inequalitities, we have that

$$\lim_{N\to+\infty}\mathbb{P}_{\mu^N}\Big[\sup_{0\leq t\leq T}\Big| < \pi_t^N, H_t > - < \pi_0^N, H_0 > - \int_0^t < \pi_s^N, \partial_s H_s > ds$$

$$-\int_0^t \frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d}\sum_{j=1}^d \partial_{u_j}^2 H\Big(s,\frac{x}{N}\Big)\tau_x h_j^N(\eta_s)ds\Big| > \delta\Big]$$

$$\leq \frac{1}{\delta^2}\mathbb{E}_{\mu^N}\Big[\Big(\sup_{0\leq t\leq T}\Big| < \pi_t^N, H_t > - < \pi_0^N, H_0 > - \int_0^t < \pi_s^N, \partial_s H_s > ds$$

$$-\int_0^t \frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d}\sum_{j=1}^d \partial_{u_j}^2 H\Big(s,\frac{x}{N}\Big)\tau_x h_j^N(\eta_s)ds\Big|\Big)^2\Big]$$

$$\leq \frac{4}{\delta^2}\mathbb{E}_{\mu^N}\Big[\Big(M_T\Big)^2\Big]$$

which vanishes as $N\to+\infty$.

But, in order to prove that the limit points are concentrated on weak solutions of equation (2.1.1), we need to write the integral part of the martingale (2.4.3) as a function of the empirical measure. This is the main difficulty in the proof of an hydrodynamical limit for a gradient system and we state it as a lemma:

Lemma 2.4.2. *(Replacement Lemma)*
For every $\delta > 0$ and every local function ψ

$$\limsup_{\epsilon\to 0}\limsup_{N\to+\infty}\mathbb{P}_{\mu^N}\Big[\int_0^T \frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d}\tau_x V_{\epsilon N,\psi}(\eta_s)ds \geq \delta\Big] = 0,$$

where $V_{\epsilon N,\psi}$ was defined in (2.3.4).

We postpone it's proof to other section in order to make the exposition more clear. To keep notation simple, in the previous statement and hereafter we write ϵN for $[\epsilon N]$, it's integer part.

This Lemma states that

$$\int_0^t \frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d}\sum_{j=1}^d H\Big(s,\frac{x}{N}\Big)\Big\{\tau_x h_j(\eta_s) - \tilde{h}(\eta_s^{\epsilon N}(x))\Big\}ds$$

vanishes in probability as $N\to+\infty$ and then as $\epsilon\to 0$, for every continuous function H.

This replacement permits to close the integral part of the martingale in terms of the empirical measure, see (2.4.3). For that we need to show that $\tilde{h}(\eta_s^{\epsilon N}(x)) = (\eta_s^{\epsilon N}(x))^2$ is a function of the empirical measure. For each $\epsilon > 0$, denote by ι_ϵ, the approximation of the identity

$$\iota_\epsilon(u) = \frac{1}{2\epsilon^d}1_{[-\epsilon,\epsilon]^d}(u).$$

It is easy to see that

$$\eta^{\epsilon N}(x) = C_{N,\epsilon}(\pi^N * \iota)\Big(\frac{x}{N}\Big),$$

where $C_{N,\epsilon} = 1 + O(\frac{1}{N})$. This lead us to replace $\eta^{\epsilon N}(x)$ by $(\pi^N * \iota_\epsilon)(\frac{x}{N})$. Since $|\alpha^2-\beta^2| \leq 2|\alpha-\beta|$, we can also replace the discrete Laplacian by the continuous one. So far, we have that

$$\limsup_{\epsilon\to 0}\limsup_{N\to+\infty}Q^N\Big[\Big| < \pi_T, H_T > - < \pi_0, H_0 > - \int_0^T < \pi_s, \partial_s H_s > ds$$
$$- \int_0^T \frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d}\Delta H\Big(s,\frac{x}{N}\Big)\Big(\pi_s^N * \iota_\epsilon\Big(\frac{x}{N}\Big)\Big)^2 ds\Big| \geq \delta\Big] = 0.$$

Since H is of class $C^{1,2}$, we may replace

$$\frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d}\Delta H\Big(s,\frac{x}{N}\Big)\Big(\pi_s^N * \iota_\epsilon\Big(\frac{x}{N}\Big)\Big)^2$$

by the integral

$$\int_{\mathbb{T}^d}\Delta H(s,u)(\pi^N * \iota_\epsilon(u))^2 du.$$

For each $\epsilon > 0$, consider the function that associates to a trajectory π the number

$$< \pi_T, H_T > - < \pi_0, H_0 > - \int_0^T < \pi_s, \partial_s H_s > ds$$
$$- \int_0^T ds \int_{\mathbb{T}^d} \Delta H(s,u)(\pi_s * \iota_\epsilon(u))^2 du.$$

By the dominated convergence theorem this function is continuous, which implies that every limit point Q of $(Q_N)_N$ is such that

$$\limsup_{\epsilon \to 0} Q\Big[\Big| < \pi_T, H_T > - < \pi_0, H_0 > - \int_0^T < \pi_s, \partial_s H_s > ds$$
$$- \int_0^T ds \int_{\mathbb{T}^d} \triangle H(s,u)(\pi_s * \iota_\epsilon(u))^2 du\Big| \geq \delta\Big] = 0.$$

To end the proof it remains to show that:

$$\limsup_{\epsilon \to 0} Q\Big[\Big| \int_0^T ds \int_{\mathbb{T}^d} \Delta G(s,u)\{(\pi_s * \iota_\epsilon(u))^2 - \pi(s,u)^2\} du\Big| \geq \delta\Big] = 0.$$

We have already seen that π_s has a density with respect to the Lebesgue measure, then

$$(\pi_s * \iota_\epsilon)(u) = \frac{1}{2\epsilon} \int_{[u-\epsilon, u+\epsilon]} \pi(s,r) dr$$

for all $0 \leq s \leq T$. On the other hand, for each integrable function $f : \mathbb{T}^d \to \mathbb{R}$,

$$f * \iota_\epsilon \underset{\epsilon \to 0}{\longrightarrow} f \tag{2.4.4}$$

in the $L^1(\mathbb{T}^d)$-norm. Then, we have that

$$Q\Big[\Big| \int_0^T ds \int_{\mathbb{T}^d} \Delta H(s,u)\{(\pi_s * \iota_\epsilon(u))^2 - \pi(s,u)^2\} du\Big| \geq \delta\Big]$$
$$\leq \frac{1}{\delta} E_Q\Big[\int_0^T ds \int_{\mathbb{T}^d} \Delta H(s,u)|(\pi_s * \iota_\epsilon)(u)^2 - (\pi(s,u))^2| du\Big]$$
$$\leq \frac{2}{\delta} \int_0^T E_Q\Big[\int_{\mathbb{T}^d} \Delta H(s,u)|\pi_s * \iota_\epsilon(u) - \pi(s,u)| du\Big] ds,$$

which vanishes as $\epsilon \to 0$, since π^N is integrable and by (2.4.4). Then, we have that

$$Q\Big[\Big| < \pi_T, H_T > - < \pi_0, H_0 > - \int_0^T < \pi_s, \partial_s H_s > ds$$
$$- \int_0^T ds \int_{\mathbb{T}^d} \triangle H(s,u)\pi(s,u)^2 du\Big| = 0\Big] = 1,$$

for every $H : [0,T] \times \mathbb{T}^d \to \mathbb{R}$ of class $C^{1,2}$. This is the weak form of the hydrodynamic equation (2.1.1) with $m = 2$, see expression (2.2.13). Since it has at most one weak solution, then $\pi(t,u) = \rho(t,u)\ Q - a.s.$, which ends the proof of Theorem 2.2.4.

2.5 Entropy and Dirichlet form

Fix a density α and an invariant measure ν_α as reference. Let μ^N be a sequence of probability measures on χ_d^N and denote by S_t^N the semigroup associated to the generator $\mathcal{L}_\theta^N$ accelerated by N^2. Denote by f_t^N the density of $\mu^N S_t^N$ with respect to ν_α.

Since the process evolves on a torus, for each N, μ^N is a convex combination of dirac measures. By the convexity of the entropy, it holds that

$$H(\mu^N/\nu_\alpha) \leq \max_\eta H(\delta_\eta/\nu_\alpha(\eta)).$$

By the definition of the entropy and the explicitly knowledge of ν_α

$$H(\delta_\eta/\nu_\alpha(\eta)) = -\log(\nu_\alpha(\eta)) = N^d \log\Big(\frac{1}{1-\alpha}\Big) - \sum_{x\in\mathbb{T}_N^d} \eta(x)\log\Big(\frac{\alpha}{1-\alpha}\Big),$$

which is bounded by $C(\alpha)N^d$. On the other hand, since the entropy decreases with time, it holds that $H(\mu^N S_t^N/\nu_\alpha^N) \leq C(\alpha)N^d$, see Proposition A1.9.2 of [14]. Denote by $(\mathcal{L}_\theta^N)^*$ the adjoint operator of $\mathcal{L}_\theta^N$ in $L^2(\nu_\alpha)$. Since the process is reversible with respect to ν_α, f_t^N satisfies the following equality:

$$\partial_t f_t^N = N^2(\mathcal{L}_\theta^N)^* f_t^N = N^2\mathcal{L}_\theta^N f_t^N$$

with initial condition $f_0^N = \frac{d\mu^N}{d\nu_\alpha^N}$.

For $f : \chi_d^N \to \mathbb{R}_+$, recall the definition of the Dirichlet form $D(f,\nu_\alpha) = \mathfrak{D}(\sqrt{f},\nu_\alpha)$ and $\mathfrak{D}(f,\nu_\alpha)$ defined in (2.2.16). Since there exists a constant C, such that $H_N(f_0^N) \leq CN^d$, it can be proved for all $t > 0$ (see [14], section 5.2), that

$$H_N\Big(\frac{1}{t}\int_0^t f_s^N ds\Big) \leq CN^d, \qquad D_\theta\Big(\frac{1}{t}\int_0^t f_s^N ds, \nu_\alpha\Big) \leq \frac{CN^{d-2}}{2t}. \tag{2.5.1}$$

2.6 Proof of Replacement Lemma

The proof relies on the well-known One-block and Two-blocks estimates. At first we reduce the dynamical problem, since the function depends on a trajectory, to a static one using the estimates of the previous section. For that, we need to introduce some notation.

Let $\mu^N(T)$ be the Cesaro mean of $\mu^N S_t^N$, namely:

$$\mu^N(T) = \frac{1}{T}\int_0^T \mu^N S_t^N dt$$

and $\bar{f}_T^N$ the Radon-Nikodym density of $\mu^N(T)$ with respect to ν_α. In the last section, we have obtained estimates for the entropy and the Dirichlet form of $\bar{f}_T^N$. To prove replacement it is enough to show that

$$\limsup_{\epsilon\to 0}\limsup_{N\to+\infty} \sup_{D_\theta(f,\nu_\alpha)\leq CN^{d-2}} \int \frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d} \tau_x V_{\epsilon N,\psi}(\eta) f(\eta)\nu_\alpha(d\eta) = 0, \tag{2.6.1}$$

for every $C < \infty$. Indeed, by the Markov inequality,

$$\mathbb{P}_{\mu^N}\Big[\frac{1}{N^d}\int_0^T \sum_{x\in\mathbb{T}_N^d} \tau_x V_{\epsilon N,\psi}(\eta_s)ds \geq \delta\Big] \leq \frac{1}{\delta}\mathbb{E}_{\mu^N}\Big[\frac{1}{N^d}\int_0^T \sum_{x\in\mathbb{T}_N^d} \tau_x V_{\epsilon N,\psi}(\eta_s)ds\Big]$$

which is equal to

$$\frac{1}{\delta}T\int \frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d} \tau_x V_{\epsilon N,\psi}(\eta)\bar{f}_T^N \nu_\alpha(d\eta).$$

In view of (2.5.1) having established (2.6.1), Replacement Lemma follows. The proof of (2.6.1) is divided in two steps, the first is known as the One-block estimate and it allows to replace the empirical average of a local function ψ by $\tilde{\psi}(\eta^l(\cdot))$, over boxes of length l. While, the second is know as the Two-blocks estimate and it allows to substitute $\tilde{\psi}(\eta^l(\cdot))$ by $\tilde{\psi}(\eta^{\epsilon N}(\cdot))$.

Lemma 2.6.1. *(One-block Estimate)*
For every finite constant C,

$$\limsup_{l\to+\infty}\limsup_{N\to+\infty}\sup_{D_\theta(f,\nu_\alpha)\leq CN^{d-2}}\int \frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d}\tau_x V_{l,\psi}(\eta)f(\eta)\nu_\alpha^N(d\eta) = 0,$$

where $V_{l,\psi}$ was defined in (2.3.4).

Lemma 2.6.2. *(Two-blocks Estimate)*
For every finite constant C,

$$\limsup_{l\to+\infty}\limsup_{\epsilon\to 0}\limsup_{N\to+\infty}\sup_{D_\theta(f,\nu_\alpha)\leq CN^{d-2}}\sup_{|y|\leq\epsilon N}$$

$$\int \frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d}\Big|\eta^l(x+y)-\eta^{\epsilon N}(x)\Big|f(\eta)\nu_\alpha(d\eta) = 0.$$

Before proceeding, we make a small stop in order to see that Replacement Lemma is a consequence of this two last results. With that purpose, add and subtract the expression

$$\frac{1}{(2\epsilon N+1)^d}\sum_{|y|\leq\epsilon N}\Big\{\frac{1}{(2l+1)^d}\sum_{|z-y|\leq l}\tau_z\psi(\eta)-\tilde{\psi}(\eta^l(y))\Big\}$$

inside the absolute value that appears in the definition of $V_{\epsilon N,\psi}$. The first term is equal to

$$\int \frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d}\tau_x\Big|\frac{1}{(2\epsilon N+1)^d}\sum_{|y|\leq\epsilon N}\Big\{\tau_y\psi(\eta)-\frac{1}{(2l+1)^d}\sum_{|z-y|\leq l}\tau_z\psi(\eta)\Big\}\Big|f(\eta)\nu_\alpha(d\eta).$$

It is not hard to show that this term is bounded above by:

$$C(d)\frac{l}{\epsilon N}\int \frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d}\tau_x\psi(\eta)f(\eta)\nu_\alpha(d\eta),$$

and vanishes as $N\to+\infty$, for details see [14].

The second term is bounded by

$$\int \frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d}\tau_x\Big|\frac{1}{(2l+1)^d}\sum_{|y|\le l}\tau_y\psi(\eta)-\tilde{\psi}(\eta^l(0))\Big|f(\eta)\nu_\alpha(d\eta)$$

and by Lemma (2.6.1) vanishes as $N\to+\infty$ and $l\to+\infty$.

The third term is bounded by

$$\sup_{|y|\le\epsilon N}\int\frac{1}{N}\sum_{x\in\mathbb{T}_N}|\tilde{\psi}(\eta^l(x+y))-\tilde{\psi}(\eta^{\epsilon N}(x))|f(\eta)\nu_\alpha^N(d\eta)$$

$$\le 2\sup_{|y|\le\epsilon N}\int\frac{1}{N}\sum_{x\in\mathbb{T}_N}|\eta^l(x+y)-\eta^{\epsilon N}(x)|f(\eta)\nu_\alpha^N(d\eta)$$

which vanishes as $N\to+\infty$, $\epsilon\to 0$ and $l\to+\infty$, by Lemma (2.6.2).

2.6.1 One-Block Estimate

At first, we note that when applying the Relative Entropy method, we derived the One-Block estimate for the process with generator $\mathcal{L}_P$ by using strongly the assumption that the density f had entropy of order $o(N^d)$. Instead of examining the evolution of entropy of the process with respect to the measure $\nu^N_{\rho(t,\cdot)}$, the entropy method analyzes the entropy of the process with respect to a fixed invariant measure, namely ν_α. This implies that the entropy of f is no longer of $o(N^d)$ but of order $O(N^d)$. For that reason another proof has to be accomplished.

Since in this case we are considering the perturbed process $\mathcal{L}^N_\theta$, we can use the One-Block estimate for the SSEP speeded up by N^θ, which is enough to conclude the One-Block estimate for the perturbed process, the idea to proceed is the following: by the definition of the Dirichlet form (2.2.16) and the generator $\mathcal{L}^N_\theta$, it holds that

$$D_\theta(f,\nu_\alpha)=D_P(f,\nu_\alpha)+N^{\theta-2}D_S(f,\nu_\alpha), \tag{2.6.2}$$

where D_θ, (resp. D_P and D_S) denotes the Dirichlet form as defined in (2.2.16). Since the Dirichlet form is always positive, and we are restricted to densities f for which $D_\theta(f,\nu_\alpha)\le CN^{d-2}$, we have that:

$$D_S(f,\nu_\alpha)\le N^{2-\theta}D_\theta(f,\nu_\alpha)\le CN^{d-\theta}.$$

Following the same arguments as for the Zero-Range in section 5 of [14], it is not hard to show that the modified One-Block estimate

$$\limsup_{l\to+\infty}\limsup_{N\to+\infty}\sup_{D_S(f,\nu_\alpha)\le CN^{d-\theta}}\int\frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d}\tau_x V_{l,\psi}(\eta)f(\eta)\nu_\alpha(d\eta)=0,$$

holds for the SSEP. The main difference between the proofs, comes from the bounds on the Dirichlet forms. Having the bound $D_S(f,\nu_\alpha)\le CN^{d-2}$ it provides the estimate $D_S^l(f_l,\nu_\alpha)\le C/N^2$, where f_l is the conditional expectation of f with respect to σ-algebra generated by $\{\eta(x),|x|\le l\}$, while the bound $D_S(f,\nu_\alpha)\le CN^{d-\theta}$, provides the estimate $D_S^l(f_l,\nu_\alpha)\le C/N^\theta$, which is enough to conclude the standard proof of the One-block estimate, since $\theta>0$.

2.6.2 Two-Blocks Estimate

Now we are focused in showing that

$$\limsup_{l\to+\infty}\limsup_{\epsilon\to 0}\limsup_{N\to+\infty}\sup_{D_\theta(f,\nu_\alpha)\le CN^{d-2}}\sup_{|y|\le\epsilon N}$$

$$\int\frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d}|\eta^l(x+y)-\eta^{\epsilon N}(x)|f(\eta)\nu_\alpha(d\eta)=0.$$

This last integral is bounded above by

$$\int\frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d}\frac{1}{(2\epsilon N+1)^d}\sum_{\substack{|z|\le\epsilon N\\|z-y|>2l}}|\eta^l(x+y)-\eta^l(x+z)|f(\eta)\nu_\alpha(d\eta)$$

$$+C(d)\frac{l}{\epsilon N}\int\frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d}\eta(x)f(\eta)\nu_\alpha(d\eta).$$

The second integral in the last expression vanishes as $N\to+\infty$. So, it is enough to prove that

$$\limsup_{l\to+\infty}\limsup_{\epsilon\to+\infty}\limsup_{N\to+\infty}\sup_{D_\theta(f,\nu_\alpha)\le CN^{d-2}}\sup_{2l\le|y|\le\epsilon N}$$

$$\int\frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d}|\eta^l(x)-\eta^l(x+y)|f(\eta)\nu_\alpha(d\eta)=0.$$

We can write this integral as

$$\int|\eta^l(0)-\eta^l(y)|\bar{f}(\eta)\nu_\alpha(d\eta),\qquad(2.6.3)$$

where $\bar{f}$ denotes the average of all space translations of f (see 2.3.8).

In this case, we are able to separate the integral over configurations with small density of particles from the ones with high density. The first case is easily treated. For remaining, since we are taking configurations with high density of particles, we are able to construct a coupled process, in which each marginal evolves according to the SSEP and they are connected by jumps of $\mathcal{L}_P$. The main difference between the proof for the Zero-Range process in [14] comes from the estimate in the Dirichlet form of this process, which in this case we are able to show that is bounded by $C(d,l)\epsilon^\theta$ and is enough to conclude. We divide the proof in several steps, in order to make the exposition clearer.

1. Cut off of small densities

At first we note that if we restrict the integral (2.6.3) to the set

$$\Omega_{0,y} = \{\eta : \sum_{|x|\leq l} \eta(x) \leq 2^d\} \cap \{\eta : \sum_{|x-y|\leq l} \eta(x) \leq 2^d\},$$

it is bounded above by $\frac{C(d)}{(2l+1)^d}$, which vanishes as $l \to +\infty$. So, in fact we just have to consider the integral over the set $\Omega_{0,y}^c$.

2. Reduction to microscopic cubes

Here we need to introduce some notation. Fix a positive integer l, denote by $\wedge_l(0)$ the box centered at 0 with radius l. Let $\chi^{2,l}$ denote the configuration space $\{0,1\}^{\wedge_l(0)} \times \{0,1\}^{\wedge_l(0)}$, $\xi = (\xi_1, \xi_2)$ the configurations of $\chi^{2,l}$ and by $\nu_\alpha^{2,l}$ the product measure ν_α restricted to $\chi^{2,l}$. Denote by $f_{y,l}$ the conditional expectation of f with respect to the σ-algebra generated by $\{\eta(z), z \in \wedge_l(0,y)\}$, where

$$\wedge_l(0,y) = \wedge_l(0) \cup \wedge_l(y).$$

Since, $\eta^l(0)$ and $\eta^l(y)$ depend on $\eta(x)$, for $x \in \wedge_l(0,y)$, we are able to replace $\bar{f}$ by $\bar{f}_{y,l}$ and rewrite our desired limit as

$$\limsup_{l\to+\infty} \limsup_{\epsilon\to 0} \limsup_{N\to+\infty} \sup_{D_\theta(f,\nu_\alpha)\leq CN^{d-2}} \sup_{2l<|y|\leq 2\epsilon N}$$

$$\int_{\Omega_{0,y}^c} |\xi_1^l(0) - \xi_2^l(0)| \bar{f}_{y,l}(\xi) \nu_\alpha^{2,l}(d\xi) = 0.$$

3.Estimates on the Dirichlet form

At first we need to introduce some notation since we are working in the space $\Omega_{0,y}^c$. Recall from (2.2.16) the definition of the Dirichlet form of f. Since the Bernoulli product measures are homogeneous, we can define the Dirichlet form of f as:

$$D_\theta(f,\nu_\alpha) = \sum_{\substack{x,z\in\mathbb{T}_N^d \\ |x-z|=1}} I_{x,z}(f,\nu_\alpha),$$

where

$$I^{\theta}_{x,z}(f,\nu_\alpha) = I^{P}_{x,z}(f,\nu_\alpha) + N^{\theta-2} I^{S}_{x,z}(f,\nu_\alpha), \tag{2.6.4}$$

$$I^{P}_{x,z}(f,\nu_\alpha) = \int_{\Omega^c_{0,y}} c(x,z,\eta)\Big[\sqrt{f(\eta^{x,z})} - \sqrt{f(\eta)}\Big]^2 \nu_\alpha(d\eta)$$

and

$$I^{S}_{x,z}(f) = \int_{\Omega^c_{0,y}} \Big[\sqrt{f(\eta^{x,z})} - \sqrt{f(\eta)}\Big]^2 \nu_\alpha(d\eta).$$

By the definition of $\bar{f}$, see (2.3.8) and since the Dirichlet form is convex, it implies that $D_\theta(\bar{f},\nu_\alpha) \le D_\theta(f,\nu_\alpha)$.

The main step in the proof, consists in obtaining an upper bound for the Dirichlet form of $\bar{f}_{y,l}$ from the upper bound of the Dirichlet form of $\bar{f}$, in such a way that we can use the ergodic properties of the Markov process. The idea is to obtain a limit density with Dirichlet form in $\wedge_l(0,y)$ equal to 0, and then by using the irreducibility we can decompose that density along the hyperplanes. As the boxes $\wedge_l(0)$ and $\wedge_l(y)$ have no communication, we cannot define the Dirichlet form in $\wedge_l(0,y)$ as the sum of the Dirichlet forms in $\wedge_l(0)$ and in $\wedge_l(y)$, we must add a term that connects both boxes. Define on positive densities $f : \chi^{2,l} \to \mathbb{R}_+$:

$$D^{2,l}_\theta(f,\nu^{2,l}_\alpha) = I^{l}_{0,0}(f,\nu^{2,l}_\alpha) + \sum_{\substack{x,z\in\wedge_l(0)\\ |x-z|=1}} I^{1,l}_{x,z}(f,\nu^{2,l}_\alpha) + \sum_{\substack{x,z\in\wedge_l(0)\\ |x-z|=1}} I^{2,l}_{x,z}(f,\nu^{2,l}_\alpha), \tag{2.6.5}$$

where

$$I^{1,l}_{x,z}(f,\nu^{2,l}_\alpha) = \int_{\Omega^c_{0,y}} \Big[\sqrt{f(\xi^{x,z}_1,\xi_2)} - \sqrt{f(\xi_1,\xi_2)}\Big]^2 \nu^{2,l}_\alpha(d\xi_1,d\xi_2),$$

$$I^{2,l}_{x,z}(f,\nu^{2,l}_\alpha) = \int_{\Omega^c_{0,y}} \Big[\sqrt{f(\xi_1,\xi^{x,z}_2)} - \sqrt{f(\xi_1,\xi_2)}\Big]^2 \nu^{2,l}_\alpha(d\xi_1,d\xi_2).$$

We have to define the term that connects both boxes, namely $I^{l}_{0,0}(f,\nu^{2,l}_\alpha)$. Note that $\Omega^c_{0,y} = \Omega_0 \cup \Omega_y$, where $\Omega_0 = \{\eta : \sum_{x\in\wedge_l(0)} \eta(x) \ge 2^d+1\}$ and $\Omega_y = \{\eta : \sum_{x\in\wedge_l(y)} \eta(x) \ge 2^d+1\}$. Then we define

$$I^{l}_{0,0}(f,\nu^{2,l}_\alpha) = \int_{\Omega_0} \sum_{j=1}^{d} c(0,e_j,\xi_1)\Big[\sqrt{f(\xi^{0,-}_1,\xi^{0,+}_2)} - \sqrt{f(\xi)}\Big]^2 \nu^{2,l}_\alpha(d\xi_1,d\xi_2)$$

$$+ \int_{\Omega_y} \sum_{j=1}^{d} c(0,e_j,\xi_2)\Big[\sqrt{f(\xi^{0,+}_1,\xi^{0,-}_2)} - \sqrt{f(\xi)}\Big]^2 \nu^{2,l}_\alpha(d\xi_1,d\xi_2),$$

where $\xi^{0,\pm}_i = \xi_i \pm \partial_0$ and ∂_0 is the configuration with one particle at site 0 and in the rest empty.

This Dirichlet form corresponds to a particle system on $\wedge_l(0)\times\wedge_l(0)$, where the marginal processes evolve as SSEP and where particles can jump from the origin of one marginal process to the origin of the other and vice-versa, according to the jumps of the generator $\mathcal{L}_P$. Using Schwarz inequality and the definition of $\bar{f}_{y,l}$, we have that:

$$I^{1,l}_{x,z}(\bar{f}_{y,l},\nu^{2,l}_\alpha) \le I^{S}_{x,z}(\bar{f},\nu^{2,l}_\alpha) \quad \textit{and} \quad I^{2,l}_{x,z}(\bar{f}_{y,l},\nu^{2,l}_\alpha) \le I^{S}_{x,z}(\bar{f},\nu^{2,l}_\alpha)$$

Then, by equality (2.6.4) and the estimate on the Dirichlet form of f, it holds that $D_S(\bar{f},\nu_\alpha) \le CN^{d-\theta}$ which together with $l/N < \epsilon$, implies that:

$$\sum_{\substack{x,z\in\wedge_l(0)\\|x-z|=1}} I^{1,l}_{x,z}(f,\nu^{2,l}_\alpha) + \sum_{\substack{x,z\in\wedge_l(0)\\|x-z|=1}} I^{2,l}_{x,z}(f,\nu^{2,l}_\alpha) \le \frac{2(2l+1)^{d-1}2lD_S(\bar{f},\nu_\alpha)}{N^d}$$

$$\le C(d,l)\epsilon^\theta \tag{2.6.6}$$

In last expression $C(d,l)$ is a constant that depends on d and l. In what follows it may vary from line to line. It remains to obtain an upper bound for $I^l_{0,0}(\bar{f}_{y,l})$. By using Schwarz inequality and the definition of $\bar{f}_{y,l}$, we can bound $I^l_{0,0}(\bar{f}_{y,l},\nu^{2,l}_\alpha)$ by

$$\int_{\Omega_0}\sum_{j=1}^{d} c(0,e_j,\eta)\Big[\sqrt{\bar{f}(\eta^{0,y})}-\sqrt{\bar{f}(\eta)}\Big]^2\nu_\alpha(d\eta)$$

$$+\int_{\Omega_y}\sum_{j=1}^{d} c(y,y+e_j,\eta)\Big[\sqrt{\bar{f}(\eta^{0,y})}-\sqrt{\bar{f}(\eta)}\Big]^2\nu_\alpha(d\eta).$$

Let $E^{l,j}_{0,y}(\bar{f},\nu_\alpha)$ $j=1,2$ denote, respectively, the first and the second expectation above. In order to keep the proof clear we are going to estimate $E^{l,1}_{0,y}(f,\nu_\alpha)$ and state it as a lemma. We note that a similar argument provides the same bound for $E^{l,2}_{0,y}(f,\nu_\alpha)$.

Lemma 2.6.3. *Let f be a density such that $D_\theta(f,\nu_\alpha) \le CN^{d-2}$ and let $y \in \mathbb{T}^d_N : 2l < |y| \le 2\epsilon N$. Then,*

$$E^{l,1}_{0,y}(f,\nu_\alpha) \le C(d,l)\epsilon^\theta.$$

Proof. The idea consists in expressing the exchange $\eta^{0,y}$ by means of allowed nearest neighbor exchanges $\eta^{x,x+1}$ of the generator $\mathcal{L}^N_\theta$. Since the integral is restricted to Ω^c_0 we have for certain 2^d+1 particles in $\wedge_l(0)$. We discuss for definiteness the path in the case $\eta(x)=1$, $\eta(y)=0$, the other possibility can be treated analogously. In order to bring the particle from 0 to y we first move 2^d of the particles in $\wedge_l(0)$ close to 0 by means of the jumps that corresponds to $\mathcal{L}_S$. Then we arrange them in order to form a d-dimensional hypercube of linear size 2, which is a mobile cluster. Now we can shift this cluster plus the particle originally in 0 in each of the d directions by using only the jumps in $\mathcal{L}_P$ and we bring them close to y. In figure 2.1 and 2.2 we show as an example the path which allows to shift in the e_1 and e_2 direction the mobile 2×2 square of particles (black circles) plus the particle originally in 0 (grey circle).

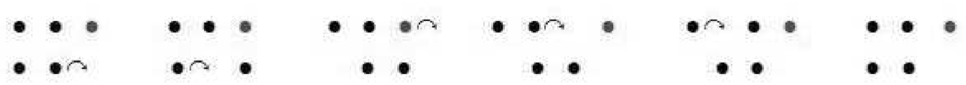

Figure 2.1: Moving in the direction e_1

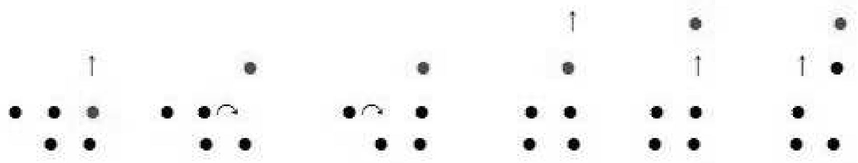

Figure 2.2: Moving in the direction e_2

Then we drop particle that was originally in 0 in site y and bring back the mobile cluster alone to $\wedge_l(0)$ again using moves in $\mathcal{L}_P$. Finally, using the jumps of $\mathcal{L}_S$, we put them in their initial positions.

Note that the above path is uniquely defined only after choosing the 2^d particles in the box $\wedge_l(0)$ which are used to form the mobile cluster and whose existence in guaranteed by the fact that the integral is restricted to Ω_0^c. This will bring an entropy factor which corresponds to the number of possible initial positions for the 2^d particles, which is bounded by l^{d2^d}.

Once for a given η the initial position of the 2^d particles which will be used to form the mobile cluster is fixed, namely $x_1, .., x_{2^d}$, we let $\tau_{\gamma^{0,y}}\tau_{\gamma^{0,y}-1}...\tau_1(\eta)$ be the sequence of nearest neighboring exchanges that represents the above described path. We can therefore rewrite, for any function g, the square difference $(g(\eta^{0,y}) - \bar{g}(\eta))^2$ as a telescopic sum:

$$\Big[g(\eta^{0,y}) - g(\eta)\Big]^2 = \Big[\sum_{k=1}^{\gamma^{0,y}} g\Big(\tau_k \prod_{i=1}^{k-1} \tau_i(\eta)\Big) - g\Big(\prod_{i=1}^{k-1} \tau_i(\eta)\Big)\Big]^2$$

where $\gamma^{0,y}$ denotes the number of steps of the path which is less than $C(d)(l + \epsilon N)$. Now we separate the part of the path in which the moves are performed via jumps in $\mathcal{L}_S$ from the one that can be performed by using $\mathcal{L}_P$ and, by using the elementary inequality $(x + y)^2 \le 2x^2 + 2y^2$, we bound last expression by

$$\Big[\sum_{k=1}^{C(d)l} g\Big(\tau_k \prod_{i=1}^{k-1} \tau_i(\eta)\Big) - g\Big(\prod_{i=1}^{k-1} \tau_i(\eta)\Big)\Big]^2 + \Big[\sum_{k=C(d)l+1}^{C(d)N\epsilon} g\Big(\tau_k \prod_{i=1}^{k-1} \tau_i(\eta)\Big) - g\Big(\prod_{i=1}^{k-1} \tau_i(\eta)\Big)\Big]^2$$

Note that in the previous expression we had to take into account the number of steps in the paths, which is order l for the terms involving moves of $\mathcal{L}_S$ and of order $N\epsilon$ for those using $\mathcal{L}_P$, thanks to our choice of the path.

We now apply the above inequality with the choice $g = \sqrt{\bar{f}}$ and use again the Cauchy-Schwarz inequality on each single term and the fact that the rates of $\mathcal{L}_P$ (2.2.3) are bounded from below by 1 thanks to our choice of the path for all the terms in the second telescopic sum. This leads to the bound

$$E^{l,1}_{0,y}(f,\nu_\alpha) \leq C(d)l \sum_{\substack{a_1\in\wedge_l(0)\\ \vdots \\ a_{2^d}\in\wedge_l(0)}} \int_{\Omega_0^c} 1_{\{a_1=x_1,..,a_{2^d}=x_{2^d}\}} \sum_{e_i} \Big[\sqrt{\bar{f}(\eta^{e_i})} - \sqrt{\bar{f}(\eta)}\Big]^2 \nu_\alpha(d\eta)$$

$$+C(d)N\epsilon \sum_{\substack{a_1\in\wedge_l(0)\\ \vdots \\ a_{2^d}\in\wedge_l(0)}} \int_{\Omega_0^c} 1_{\{a_1=x_1,..,a_{2^d}=x_{2^d}\}} \sum_{\tilde{e}_i} c(i,i+1,\eta)\Big[\sqrt{\bar{f}(\eta^{\tilde{e}_i})} - \sqrt{\bar{f}(\eta)}\Big]^2 \nu_\alpha(d\eta),$$

where $\{e_i\}_i$ denotes the bonds that we use inside $\wedge_l(0)$ when taking the 2^d particles whose initial positions are $x_1, ..x_{2^d}$, close to the particle at the site 0, by the jumps of the exclusion process, while $\tilde{e}_i = \{i, i+1\}$ corresponds to the bonds used by the generator $\mathcal{L}_P$ when performing the rest of the path. Since there are l^{d2^d} chances for the initial positions $x_1, \dots x_{2^d}$, we can bound last expression by:

$$l^{d2^d}C(d)l \sum_{x,x+1\in\wedge_l(0)} I^S_{x,x+1}(f,\nu_\alpha) + l^{d2^d}C(d)\epsilon N \sum_{x,x+1\in\wedge_l(0)} I^P_{x,x+1}(f,\nu_\alpha).$$

By the equality (2.6.4) and the bound on the Dirichlet form of f, it holds that:

$$\forall x \in \mathbb{T}^d_N: \qquad I^S_{x,x+1}(f,\nu_\alpha) \leq \frac{C}{N^\theta}, \qquad I^P_{x,x+1}(f,\nu_\alpha) \leq \frac{C}{N^2},$$

which together with $l \leq \epsilon N$, ends the proof. □

By the definition of the Dirichlet form $D^{2,l}_\theta(f,\nu^{2,l}_\alpha)$ in (2.6.5), together with (2.6.6) and the previous Lemma, we can be restricted to densities f that satisfy

$$D^{2,l}_\theta(f,\nu^{2,l}_\alpha) \leq C(d,l)\epsilon^\theta.$$

So, to conclude the proof of the Two-blocks estimate it is enough to show that

$$\limsup_{l\to+\infty} \limsup_{\epsilon\to 0} \sup_{D^{2,l}_\theta(f,\nu^{2,l}_\alpha)\leq C(d,l)\epsilon^\theta} \int_{\Omega^c_{0,y}} |\xi^l_1(0) - \xi^l_2(0)| f(\xi)\nu^{2,l}_\alpha(d\xi) = 0,$$

where the supremum is carried over densities with respect to $\nu^{2,l}_\alpha$. Note that the parameter ϵ is appearing only on the bound of the Dirichlet form.

Now, we proceed by bounding this last expression by another in which the supremum is carried over densities with Dirichlet form equal to 0. For each $\epsilon > 0$, there exists a density f_ϵ with Dirichlet form bounded by $C(d,l)\epsilon^\theta$, such that the supremum is attained. By compactness, there exits a subsequence f_{ϵ_k} such that

$$\lim_{k\to+\infty}\int_{\Omega^c_{0,y}}|\xi^l_1(0)-\xi^l_2(0)|f_{\epsilon_k}(\xi)\nu^{2,l}_\alpha(d\xi)=\limsup_{\epsilon\to 0}\int_{\Omega^c_{0,y}}|\xi^l_1(0)-\xi^l_2(0)|f_\epsilon(\xi)\nu^{2,l}_\alpha(d\xi).$$

We can extract a subsequence $f_{\epsilon_{k_m}}$ of f_{ϵ_k}, converging to f_∞ with $D^{2,l}_\theta(f_\infty,\nu^{2,l}_\alpha)=0$ and

$$\lim_{m\to+\infty}\int_{\Omega^c_{0,y}}|\xi^l_1(0)-\xi^l_2(0)|f_{\epsilon_{k_m}}(\xi)\nu^{2,l}_\alpha(d\xi)=\int_{\Omega^c_{0,y}}|\xi^l_1(0)-\xi^l_2(0)|f_\infty(\xi)\nu^{2,l}_\alpha(d\xi).$$

So, in order to conclude the proof we need to show that

$$\limsup_{l\to+\infty}\sup_{D^{2,l}_\theta(f,\nu^{2,l}_\alpha)=0}\int_{\Omega^c_{0,y}}|\xi^l_1(0)-\xi^l_2(0)|f(\xi)\nu^{2,l}_\alpha(d\xi)=0.$$

4. Decomposition of the Dirichlet form on hyperplanes

Now is the point in the proof in which we shall use the ergodic properties of the system. Let f be a density with $D^{2,l}_\theta(f,\nu^{2,l}_\alpha)=0$. Since we add a term to the Dirichlet form that relates both boxes, we have that f is constant along the hyperplanes having a fixed total number of particles in $\wedge_l(0,y)=\wedge_l(0)\cup\wedge_l(y)$. For each integer j, denote by $\nu^{2,l,j}_\alpha$ the measure $\nu^{2,l}_\alpha$, conditioned to the hyperplane $\Sigma_j=\{\xi:\sum_{x\in\wedge_l(0,y)}\xi(x)=j\}$. Then, we have that

$$\int_{\Omega^c_{0,y}}|\xi^l_1(0)-\xi^l_2(0)|f(\xi)\nu^{2,l}_\alpha(d\xi)=\sum_{j=0}^{2(2l+1)^d}c_j(f)\int_{\Omega^c_{0,y}}|\xi^l_1(0)-\xi^l_2(0)|\nu^{2,l,j}_\alpha(d\xi)$$

where for each j,

$$c_j(f)=\int 1_{\Sigma_j}f(\xi)\nu^{2,l}_\alpha(d\xi).$$

Since $\sum_{j=0}^{2(2l+1)^d}c_j(f)=1$, it remains to show that

$$\limsup_{l\to+\infty}\sup_{j\le 2(2l+1)^d}\int_{\Omega^c_{0,y}}|\xi^l_1(0)-\xi^l_2(0)|\nu^{2,l,j}_\alpha(d\xi)=0. \tag{2.6.7}$$

5. The Equivalence of Ensembles

The proof follows from a simple application of the equivalence of ensembles. Fix an integer K, that shall increase to ∞ after l. Now, decompose the sets $\wedge_l(0)$ and $\wedge_l(y)$ in cubes of length $2K+1$, which are denoted by $\{A_i, i=1,..,M\}$ and $\{B_i, i=1,..,M\}$, respectively, where $M=[\frac{2l+1}{2K+1}]$. By construction there exists two sets A_0 and B_0 such that $|A_0|\le CKl^{d-1}$ and $|B_0|\le CKl^{d-1}$. With this notation the integral (2.6.7) is bounded by

$$\sum_{i=1}^M(\frac{(2K+1)^d}{(2l+1)^d}\int\Big|\frac{1}{(2K+1)^d}\sum_{x\in A_i}\xi_1(x)-\frac{1}{(2K+1)^d}\sum_{x\in B_i}\xi_2(x)\Big|\nu^{2,l,j}_\alpha(d\xi),$$

plus a term of $O\Big(\frac{K}{l}\Big)$. Since the distribution of the variables $\{\xi(x), x \in A_i\}$ and $\{\xi(x), x \in B_i\}$ does not depend on i and by Corollary A2.1.7 of [14], we can approximate this integral with respect to the canonical measure, namely $\nu_\alpha^{2,l,j}$, by the integral with respect to the grand canonical measure, ν_α. Then as $l \to \infty$ and $j/2(2l+1)^d \to \alpha$, last integral converges to

$$\int \Big| \frac{1}{(2K+1)^d} \sum_{|x|\le K} \xi_1(x) - \frac{1}{(2K+1)^d} \sum_{|x|\le K} \xi_2(x) \Big| \nu_\alpha(d\xi),$$

and by the Law of Large Numbers vanishes as $K \to \infty$.

2.7 Spectral Gap

In this section we analyze the magnitude of the spectral gap for the one dimensional generators on finite boxes, $\mathcal{L}_{P,\Lambda_N}$ and $\mathcal{L}^N_{\theta,\Lambda_N}$, which have been defined in section 2. Note that the results below on the scaling of the spectral gap with the lattice size have not been used in the previous sections to derive the hydrodynamic limit. This was possible thanks to the fact that we were considering a gradient choice of the rates. The following analysis of the spectral gap can be regarded as a first step towards the analysis of the non gradient version of our models, e.g. for the choice

$$c'(x, x+e_j, \eta) = \begin{cases} 1 & \text{if } \eta(x-e_j) + \eta(x+2e_j) \ge 1 \\ 0 & \text{otherwise} \end{cases} .$$

which is a non gradient version with the same kinetic constraints as our original choice (2.2.3), namely $c'(x, x+e_i, \eta) = 0$ if and only if $c(x, x+e_i, \eta) = 0$.

For simplicity in the following we drop Λ_N from our notation. The main ingredient of our proofs will be a comparison with the spectral gap $\lambda(\mathcal{L}_{LR})$ for the *long range exclusion process* and the use of path arguments. Let us start by recalling the definition of $\mathcal{L}_{LR}$ and the result in [18] for its spectral gap. The action of $\mathcal{L}_{LR}$ on local functions $f : \Sigma_{N,k} \to \mathbb{R}$ is given by

$$(\mathcal{L}_{LR} f)(\eta) = \sum_{x,y\in\Lambda_N} \frac{1}{N} \eta(x)(1-\eta(y))(f(\eta^{x,y}) - f(\eta)), \tag{2.7.1}$$

where $\eta^{x,y}$ as defined in (2.2.2).

Consider the Dirichlet form $\mathfrak{D}_{LR}$ associated to $\mathcal{L}_{LR}$, with respect to the uniform measure $\nu_{N,k}$, given explicitly by:

$$\mathfrak{D}_{LR}(f, \nu_{N,k}) = \frac{1}{N} \sum_{x,y\in\Lambda_N} \int_{\Sigma_{N,k}} (f(\eta^{x,y}) - f(\eta))^2 \nu_{N,k}(d\eta). \tag{2.7.2}$$

Quastel in [18], obtained that

$$\mathbf{Var}(f, \nu_{N,k}) \le \mathfrak{D}_{LR}(f, \nu_{N,k}), \tag{2.7.3}$$

by computing precisely the eigenvalues of the generator $\mathcal{L}_{LR}$. In order to prove our results (Proposition 2.2.6 and 2.2.7) for the spectral gap of the process with generator $\mathcal{L}_\theta^N$ $\forall k$ and with generator $\mathcal{L}_P$ for $k > 1/3$, we proceed in two steps.

Fix an integer k such that the density is restricted to $\rho = \frac{k}{N} > \frac{1}{3}$. With this restriction, for each $\eta \in \Sigma_{N,k}$, there exists a couple of particles whose distance is smaller or equal to two and $\Sigma_{N,k}$ is irreducible. Furthermore we will show that there exists a constant C that does not depend on N nor k such that

$$\mathfrak{D}_{LR}(f, \nu_{N,k}) \leq \frac{C\rho}{\rho - 1/3} N^2 \mathfrak{D}_P(f, \nu_{N,k}), \tag{2.7.4}$$

where $\mathfrak{D}_P(f, \nu_{N,k})$ is the Dirichlet that corresponds to $\mathcal{L}_P$.

Last result together with (2.7.3) allows to conclude the following Poincaré inequality:

Proposition 2.7.1. *Fix $k > N/3$. For the Markov process with generator $\mathcal{L}_{P,N}$ and for every $f \in L^2(\nu_{N,k})$, there exists a constant C not depending on N nor k such that:*

$$\boldsymbol{Var}(f, \nu_{N,k}) \leq \frac{C\rho}{\rho - 1/3} N^2 \mathfrak{D}_P(f, \nu_{N,k}).$$

The second inequality in the result of Proposition 2.2.6 is an immediate consequence of last result. The first inequality follows from the fact that for any function f it holds that $\mathfrak{D}_P(f, \nu_{N,k}) \leq \mathfrak{D}_\theta(f, \nu_{N,k})$.

The case in which $\rho = \frac{k}{N} \leq \frac{1}{3}$ is more demanding. Since the hyperplanes $\Sigma_{N,k}$ are no longer irreducible for $\mathcal{L}_P$, the corresponding spectral gap is zero. A natural issue is determining the magnitude of the spectral gap for $\mathcal{L}_P$ on the restricted irreducible set of configurations with k particles and at least one couple of particles at distance at most two. In this case the invariant measure is no longer the same one as for the long range jumps and the comparison with the latter process is no more useful. Instead, we will here consider for $k/N \leq 1/3$ the spectral gap for the modified generator $\mathcal{L}_\theta^N$ which, thanks to the addition of the exclusion part, is ergodic on the hyperplanes $\Sigma_{N,k}$ for any k and reversible with respect to $\nu_{N,k}$. By comparing the Dirichlet form of $\mathcal{L}_\theta^N$ with the one of $\mathcal{L}_{LR}$, we show that:

Proposition 2.7.2. *Fix $k \leq N/3$. For the Markov process with generator given by $\mathcal{L}_\theta^N$ and for every $f \in L^2(\nu_{N,k})$, there exists a constant C not depending on N nor k such that::*

$$\boldsymbol{Var}(f, \nu_{N,k}) \leq \frac{C\rho^\theta k^{1-\theta}}{\rho^3} N^\theta \mathfrak{D}_S(f, \nu_{N,k}) + \frac{C}{\rho} N^2 \mathfrak{D}_P(f, \nu_{N,k}).$$

Proposition 2.2.7 is an immediate consequence of Proposition 2.7.2.

2.7.1 Proof of Proposition 2.7.1

Fix an integer k and suppose that $\rho = \frac{k}{N} > \frac{1}{3}$. The Dirichlet form associated to $\mathcal{L}_P$ is given explicitly by

$$\mathfrak{D}_P(f, \nu_{N,k}) = \frac{1}{2} \sum_{\substack{x,y \in \Lambda_N \\ |x-y|=1}} \int_{\Sigma_{N,k}} c(x,y,\eta)(f(\eta^{x,y}) - f(\eta))^2 \nu_{N,k}(d\eta)$$

where $c(x,y,\eta)$ and $\eta^{x,y}$ were defined on (2.2.3) and (2.2.2), respectively. We have seen above that it is enough to show (2.7.4).

The idea consists in expressing the exchange $\eta^{x,y}$, for each $x, y \in \Lambda_N$ by means of a sequence of allowed jumps of $\mathcal{L}_P$. Consider for example the case $x < y$ and $\eta(x) = 1$, $\eta(y) = 0$. The restriction $\rho > \frac{1}{3}$ guarantees that there exists a couple of sites (a, b) both in Λ_N such that $\eta(a) = \eta(b) = 1$ and $|a - b| \leq 2$. Once the couple has been chosen we shift it close to sites $x-2, x-1$. We have thus reached a configuration with $\eta(x-2) = \eta(x-1) = \eta(x) = 1$, and we now shift the three particles to $(y-3, y-2, y-1)$. The allowed path which performs the shift of one step to the right is the composition of the following three basic steps: $\eta \to \eta_1 = \eta^{x,x+1}$, $\eta_1 \to \eta_2 = \eta_1^{x-1,x}$ and $\eta_2 \to \eta_3 = \eta_2^{x-2,x-1}$. When after a proper number of one step shifts we reach the configuration η_n with $\eta(y-3) = \eta(y-2) = \eta(y-1) = 1$, we can perform the exchange $\eta_n \to \eta_{n+1} = \eta_n^{y-1,y}$ which corresponds to dropping the particle at y. Then we make a similar backward path, this time shifting the two particles plus the vacancy untill bringing the vacancy in x and finally we bring back the two particles to their original position (a, b).

As we did for Lemma 2.6.3, we have to take into account the entropic term corresponding to the possible initial positions of the couple of particles (a, b). Some care is required at this point, since using the rough bound N for this original position would lead to an additional factor N in the inequality 2.7.4. Let us start by presenting a lower bound for the number of couples of particles at distance at most two

Lower bound for the number of couples for $k > N/3$

Fix a configuration $\eta \in \Sigma_{N,k}$. Let $\mathcal{B}$ be the set of sites which are occupied and such that for each of them there is a particle at distance at most two, $\mathcal{B} = \{x : \eta(x) = 1, \eta(x+1) + \eta(x-1) + \eta(x+2) + \eta(x-2) \geq 1\}$. Let also $\mathcal{A}$ denote the remaining occupied sites $\mathcal{A} = \{x : \eta(x) = 1, x \in \Lambda_N \setminus \mathcal{B}\}$. The following holds:

$$k = |\mathcal{A}| + |\mathcal{B}|$$

$$N \geq 3(|\mathcal{A}| - 1) + 1 + |\mathcal{B}|,$$

where for a set S, $|S|$ denotes its cardinality.

The first equality is obvious. In order to establish the second property we use the fact that, if x and y belong to $\mathcal{A}$, then $|x - y| \geq 3$. The inequality follows by organizing the k particles in the closest configuration which does not change the value of $|\mathcal{A}|$ and $|\mathcal{B}|$. The minimum number of couples of particles at distance at most two and such that there are not common sites among different couples is $|\mathcal{B}|/2$, see e.g. the figure below in which there

are 8 neighboring particles but just 4 couples. This, together with the above established relations among $k, N, |\mathcal{B}|$, gives the following lower bound for the number of couples

$$\sum_{z\in\Lambda_N} \eta(z)\eta(z+1) + \eta(z)\eta(z+2) \geq |\mathcal{B}|/2 \geq \frac{3}{4}(k - N/3 - 2/3).$$

Introducing this term in the Dirichlet form of the long range exclusion (2.7.2) we bound it by:

$$\frac{2}{N|\mathcal{B}|} \sum_{x,y\in\Lambda_N} \sum_{\eta\in\Sigma_{N,k}} \sum_{z\in\Lambda_N} \sum_{l=1}^{2} \eta(z)\eta(z+l)(f(\eta^{x,y}) - f(\eta))^2 \nu_{N,k}(\eta). \tag{2.7.5}$$

Let us now consider a term in the sum above

$$\eta(z)\eta(z+l)\Big(f(\eta^{x,y}) - f(\eta)\Big)^2$$

and let $x - (z+1) = n$ and $y - x = m$. By using the construction sketched above for a the possible path which connects η to $\eta^{x,y}$ via allowed elementary exchanges, it is possible to define a sequence η_i for $1 \leq i \leq \gamma_z^{x,y}$ with $\gamma_z^{x,y} = 6(m-1) + 4(n-1) + 2$ with the following properties: $\eta_1 = \eta$, $\eta_{\gamma_z^{x,y}} = \eta^{x,y}$ and $\forall i$ there exists $x_i \in \Lambda_N$ such that $\eta_i = \eta_{i-1}^{x(i),x(i)+1}$ and $c(x(i), x(i)+1, \eta_{i-1}) > 0$, namely the exchanges are permitted for the generator $\mathcal{L}_P$. Therefore we can rewrite each $\eta(z)\eta(z+l)(f(\eta^{x,y}) - f(\eta))$ as the telescopic sum:

$$\eta(z)\eta(z+l)\Big(f(\eta^{x,y}) - f(\eta)\Big)^2 = \Big(\sum_{i=1}^{\gamma_z^{x,y}-1} f(\eta_n) - f(\eta_{n-1})\Big)^2$$

By using this equality together with $c(x(i), x(i)+1, \eta_{i-1}) > 0$ and applying Cauchy-Schwarz inequality we can finally bound (2.7.5) from above by:

$$\frac{CN}{N|\mathcal{B}|} \sum_{x,y\in\Lambda_N} \sum_{\eta\in\Sigma_{N,k}} \sum_{z\in\Lambda_N} \sum_{l=1}^{2} \eta(z)\eta(z+l) \sum_{\tilde{e}_i} c(i, i+1, \eta)(f(\eta^{\tilde{e}_i}) - f(\eta))^2 \nu_{N,k}(\eta),$$

where $\tilde{e}_i = \{i, i+1\}$ denotes one bond that we have used when performing the path that takes the particle from x to y using the couple at the sites z and $z+1$ and C is a constant independent on N and k. Last expression is bounded above by

$$\frac{CN^3}{N|\mathcal{B}|} \sum_{\eta\in\Sigma_{N,k}} \sum_{z\in\Lambda_N} \eta(z) \sum_{x} c(x, x+1, \eta)(f(\eta^{x,x+1}) - f(\eta))^2 \nu_{N,k}(\eta),$$

and since the number of particles is k, it is equal to

$$\frac{C\rho}{\rho - 1/3} N^2 \mathfrak{D}_P(f, \nu_{N,k}).$$

2.7.2 Proof of Proposition 2.7.2

Now, fix an integer k such that $\rho = \frac{k}{N} \leq \frac{1}{3}$. In this case, we are going to show that

$$\mathfrak{D}_{LR}(f, \nu_{N,k}) \leq \frac{C\rho^\theta k^{1-\theta}}{\rho^3} N^\theta \mathfrak{D}_S(f, \nu_{N,k}) + \frac{C}{\rho} N^2 \mathfrak{D}_P(f, \nu_{N,k}), \tag{2.7.6}$$

which is enough to conclude. As before, the idea consists in expressing a path from x to y using the admissible jumps of $\mathcal{L}_\theta^N$.

For this choice of k we are no more guaranteed that there exist two particles at distance at most two, which was a key ingredient to construct the path in the high density regime. The idea will be to make use of the simple exclusion jumps to construct such mobile clusters and then proceed as before in a similar way as we did in in the proof of Lemma 2.6.3.

Fix a distance j and denote by $\mathcal{B}_j$ the set of particles at distance at most j, $\mathcal{B}_j = \{x : \eta(x) = 1, \exists l \in (-j, -1) \cup (1, j) \ s.t. \ \eta(x+l) = 1\}$ and by $\mathcal{A}_j$ the remaining particles. Then, the following holds:

$$k = |\mathcal{A}_j| + |\mathcal{B}_j|$$

$$N \geq (j+1)(|\mathcal{A}_j| - 1) + 1 + |\mathcal{B}_j|$$

This inequality is obtained in the same manner as before considering that now the minimum distance that the $\mathcal{A}_j$-particles have to be is $j+1$.

As before, the minimum number of couples that one can have is $|\mathcal{B}_j|/2$. By simple computations we obtain the lower bound

$$\sum_{z \in \Lambda_N} \sum_{l=1}^{j} \eta(z)\eta(z+l) \geq \frac{|\mathcal{B}_j|}{2} \geq \frac{j+1}{2j}\Big(k - \frac{N}{j+1} - \frac{j}{j+1}\Big). \tag{2.7.7}$$

Introducing this inequality in the Dirichlet form of the long range exclusion (2.7.2), we bound it from above by:

$$\frac{C}{N|\mathcal{B}_j|} \sum_{x,y \in \Lambda_N} \sum_{\eta \in \Sigma_{N,k}} \sum_{z \in \Lambda_N} \sum_{l=1}^{j} \eta(z)\eta(z+l)(f(\eta^{x,y}) - f(\eta))^2 \nu_{N,k}(\eta).$$

For each configuration η and each choice x, y, z and l, we can now construct a path which first bring together the two particles in z and $z+l$ by using the jumps of the simple exclusion and then uses this mobile cluster to perform the exchange of occupation variables in x and y, as in the proof of Lemma 2.6.3. Here $\mathcal{L}_S$, also denotes the generator of the Symmetric Simple Exclusion process restricted to the box Λ_N, given on local functions by

$$(\mathcal{L}_S f)(\eta) = \sum_{\substack{x,y \in \Lambda_N \\ |x-y|=1}} \frac{1}{2}\eta(x)(1-\eta(y))(f(\eta^{x,y}) - f(\eta)).$$

Since the jumps of the exclusion are used just to put the neighboring particles at a distance equal to two, the size of the path for this process is $O(j)$. With this purpose, write $f(\eta^{x,y})-f(\eta)$ as a telescopic sum, use the elementary inequality $(x+y)^2\leq 2x^2+2y^2$ and then the Cauchy-Schwarz inequality to bound last expression by:

$$\frac{Cj}{N|\mathcal{B}_j|}\sum_{x,y\in\Lambda_N}\sum_{\eta\in\Sigma_{N,k}}\sum_{z\in\Lambda_N}\sum_{l=1}^{j}\eta(z)\eta(z+l)\sum_{e_i}(f(\eta^{e_i})-f(\eta))^2\nu_{N,k}(\eta) \tag{2.7.8}$$

$$+\frac{CN}{N|\mathcal{B}_j|}\sum_{x,y\in\Lambda_N}\sum_{\eta\in\Sigma_{N,k}}\sum_{z\in\Lambda_N}\sum_{l=1}^{j}\eta(z)\eta(z+l)\sum_{\tilde{e}_i}c(i,i+1,\eta)(f(\eta^{\tilde{e}_i})-f(\eta))^2\nu_{N,k}(\eta), \tag{2.7.9}$$

where e_i denotes the bonds that we have used when bringing the neighboring particles at a distance equal to two, while $\tilde{e}_i=\{i,i+1\}$ denotes the bonds we have used when performing the remaining part of the path that takes the particle from x to y. Note, that there is a factor j multiplying the first expression which comes from the size of the path for the jumps of the exclusion, while for the other process the size of the path is of $O(N)$.

First we deal with the jumps that concerns $\mathcal{L}_P$. As before, we bound (2.7.9) from above by

$$\frac{CjN^3}{N|\mathcal{B}_j|}\sum_{\eta\in\Sigma_{N,k}}\sum_{z\in\Lambda_N}\eta(z)\sum_{x}c(x,x+1,\eta)(f(\eta^{x,x+1})-f(\eta))^2\nu_{N,k}(\eta),$$

and since $\eta\in\Sigma_{N,k}$, we obtain the bound:

$$\frac{Cj\rho}{\rho-1/j}N^2\mathfrak{D}_P(f,\nu_{N,k}).$$

Now, we bound (2.7.8), by

$$\frac{CjN^2}{N|\mathcal{B}_j|}\sum_{\eta\in\Sigma_{N,k}}\sum_{z\in\Lambda_N}\sum_{l=1}^{j}\eta(z)\sum_{x}(f(\eta^{x,x+1})-f(\eta))^2\nu_{N,k}(\eta).$$

Since the number of particles is k, we can bound last expression by:

$$\frac{Cj^2k}{(\rho-1/j)N^\theta}N^\theta\mathfrak{D}_S(f,\nu_{N,k}).$$

Reorganizing this facts together we obtain that:

$$\mathfrak{D}_{LR}(f,\nu_{N,k})\leq\frac{Cj^2k}{(\rho-1/j)N^\theta}N^\theta\mathfrak{D}_S(f,\nu_{N,k})+\frac{Cj\rho}{\rho-1/j}N^2\mathfrak{D}_P(f,\nu_{N,k}).$$

Optimizing over j, (2.7.6) follows.

Remark 2.7.3. *For sake of simplicity we have presented the spectral gap results only in the one dimensional setting. By a proper modification of the path arguments and an accurate estimate on the minimal number of mobile clusters it is possible to obtain for $d>1$ an analogous result as the one in Proposition 2.2.6 if the density $\rho=k/N$ is such that $k>C(d)(N/3)^d$.*

Remark 2.7.4. *Taking for instance the density fluctuations field as defined in (2.2.14) and the reference measure the Bernoulli product measure* ν_ρ*, by simple computations we obtain that*

$$\boldsymbol{Var}(\mathcal{Y}_t^N(H), \nu_\rho) = \rho(1-\rho)||H||_2^2,$$

while the Dirichlet form corresponding to $\mathcal{L}_P$ *equals to:*

$$\mathfrak{D}_P(\mathcal{Y}_t^N(H), \nu_\rho) = \frac{1}{N^2}\rho^2(1-\rho)||H'||_2^2.$$

So, if we consider H*, such that* $||H||_2^2 = ||H'||_2^2$*, then:*

$$\boldsymbol{Var}(\mathcal{Y}_t^N(H), \nu_\rho) = \frac{N^2}{\rho}\mathfrak{D}_P(\mathcal{Y}_t^N(H), \nu_\rho),$$

which implies that the spectral gap $\lambda_N(\mathcal{L}_{P,\Lambda_N}) \leq \frac{\rho}{N^2}$*. This is in agreement with the bound that we have obtained in (2.7.6), when considering the spectral gap with respect to the uniform measure.*

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