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MINIMAL AND CONSTANT MEAN CURVATURE SURFACES IN HOMOGENEOUS 3-MANIFOLDS

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Aos meus pais, Antônio e Vivalda.

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Abstract

In this work we present some results on minimal and constant mean curvature surfaces in homogeneous 3-manifolds.

First, we classify the compact embedded surfaces with constant mean curvature in the quotient of $\mathbb{H}^2 \times \mathbb{R}$ by a subgroup of isometries generated by a horizontal translation along horocycles of $\mathbb{H}^2$ and a vertical translation. Moreover, in $\mathbb{H}^2 \times \mathbb{R}$, we construct new examples of periodic minimal surfaces and we prove a multi-valued Rado theorem for small perturbations of the helicoid.

In some metric semidirect products, we construct new examples of complete minimal surfaces similar to the doubly and singly periodic Scherk minimal surfaces in $\mathbb{R}^3$. In particular, we obtain these surfaces in the Heisenberg space with its canonical metric, and in Sol_3 with a one-parameter family of non-isometric metrics.

After that, we prove a half-space theorem for an ideal Scherk graph $\Sigma \subset M \times \mathbb{R}$ over a polygonal domain $D \subset M$, where M is a Hadamard surface with bounded curvature. More precisely, we show that a properly immersed minimal surface contained in $D \times \mathbb{R}$ and disjoint from Σ is a translate of Σ.

Finally, based in a joint paper with L. Hauswirth, we prove that if a properly immersed minimal surface in the quotient space $\mathbb{H}^2 \times \mathbb{R}\,/G$ has finite total curvature then its total curvature is a multiple of 2π, and moreover, we understand the geometry of the ends. Here G is a subgroup of isometries generated by a vertical translation and a horizontal isometry in $\mathbb{H}^2$ without fixed points.

Keywords: Minimal surfaces, constant mean curvature surfaces, periodic surfaces, uniqueness, finite total curvature.

Resumo

Neste trabalho apresentamos alguns resultados sobre superfícies mínimas e de curvatura média constante em variedades homogêneas tridimensionais.

Primeiro, classificamos as superfícies compactas mergulhadas com curvatura média constante no quociente de $\mathbb{H}^2 \times \mathbb{R}$ por um subgrupo de isometrias gerado por uma translação horizontal ao longo de horociclos de $\mathbb{H}^2$ e uma translação vertical. Além disso, em $\mathbb{H}^2 \times \mathbb{R}$, construímos novos exemplos de superfícies mínimas periódicas e provamos um teorema de Rado multi-valuado para pequenas perturbações do helicóide.

Em alguns produtos semidiretos métricos, contruímos novos exemplos de superfícies mínimas completas similares às superfícies mínimas de Scherk duplamente e simplesmente periódicas em $\mathbb{R}^3$. Em particular, obtemos estas superfícies no espaço de Heisenberg com sua métrica canônica, e em Sol_3 com uma família a um parâmetro de métricas não isométricas.

Depois disso, provamos um teorema de semi-espaço para um gráfico de Scherk ideal $\Sigma \subset M \times \mathbb{R}$ sobre um domínio poligonal $D \subset M$, onde M é uma superfície de Hadamard com curvatura limitada. Mais precisamente, mostramos que uma superfície mínima propriamente imersa contida em $D \times \mathbb{R}$ e disjunta de Σ é uma translação de Σ.

Finalmente, baseado num trabalho em colaboração com L. Hauswirth, provamos que se uma superfície mínima propriamente imersa em $\mathbb{H}^2 \times \mathbb{R}/G$ tem curvatura total finita, então sua curvatura total é um múltiplo de 2π e, além disso, entendemos a geometria dos fins. Aqui G é um subgrupo de isometrias gerado por uma translação vertical e uma isometria horizontal de $\mathbb{H}^2$ sem pontos fixos.

Palavras-chave: Superfícies mínimas, superfícies com curvatura média con-

stante, superfícies periódicas, unicidade, curvatura total finita.

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Introduction

One of the most natural and established topics in the differential geometry of surfaces is the global theory of minimal and constant mean curvature surfaces in the space forms $\mathbb{R}^3, \mathbb{S}^3$ and $\mathbb{H}^3$. This is a classic field that remains very active nowadays and uses a wide variety of techniques from different subjects, for example, variational calculus, complex analysis, topology, elliptic PDE theory and others.

The extension of this classic global theory for the case of immersed surfaces in homogeneous Riemannian three-dimensional manifolds has attracted the attention of many researchers in the last decade. These homogeneous manifolds are the most simple and symmetric Riemannian manifolds that we can consider besides the space forms, together forming the eight 3-dimensional Thurston geometries.

This theory is extremely rich, with lots of beautiful examples. Minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$, for instance, have been used by Collin and Rosenberg [4] to give counterexamples to a well-known conjecture of Schoen and Yau about harmonic diffeomorphisms between the complex plane and the disk.

In this work we will present our contributions to the theory of minimal and constant mean curvature surfaces. We will prove some results associated to uniqueness questions, classification problems, construction of new examples of minimal surfaces, halfspace theorems and related themes. Our new results stated here are proved in the papers [18, 35, 36, 37].

In the first chapter, we fix some notations, give some basic definitions, and state well known results that we use in the other chapters.

In Chapter 2, we start by proving an Alexandrov type theorem for a quotient space of $\mathbb{H}^2 \times \mathbb{R}$. More precisely, we classify the compact embedded surfaces with constant mean curvature in the quotient of $\mathbb{H}^2 \times \mathbb{R}$ by a subgroup of isometries generated by a parabolic translation along horocycles of $\mathbb{H}^2$ and a vertical translation. Section 2.4 is devoted to the construction of new examples of periodic minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$. In Section 2.5, we prove a multi-valued Rado theorem for small perturbations of the helicoid in $\mathbb{H}^2 \times \mathbb{R}$.

Chapter 3 focuses on construction of complete minimal surfaces in some metric semidirect products. In Section 3.3, we construct a doubly periodic minimal surface, and in Section 3.4, we construct a singly periodic minimal surface. These surfaces are similar to the doubly and singly periodic Scherk minimal surfaces in $\mathbb{R}^3$. In particular, we obtain these surfaces in the Heisenberg space with its canonical metric, and in Sol_3 with a one-parameter family of non-isometric metrics.

In Chapter 4, we prove a half-space theorem for an ideal Scherk graph $\Sigma \subset M \times \mathbb{R}$ over a polygonal domain $D \subset M$, where M is a Hadamard surface with bounded curvature. More precisely, we show that a properly immersed minimal surface contained in $D \times \mathbb{R}$ and disjoint from Σ is a translate of Σ.

Finally, in Chapter 5, based in a joint work with L. Hauswirth, we prove that if a properly immersed minimal surface in the quotient space $\mathbb{H}^2 \times \mathbb{R}/G$ has finite total curvature then its total curvature is a multiple of 2π and, moreover, we understand the geometry of the ends. Here G denotes a subgroup of isometries generated by a vertical translation and a horizontal isometry in $\mathbb{H}^2$ without fixed points.

CHAPTER 1

Preliminaries

In this chapter we fix notations, give definitions and state some well known results which will be used throughout this work. In Section 1.1, we list some basic definitions as minimal, stable and parabolic surface, and we recall the first and second variational formulae of area. In Section 1.2, we state the maximum principle, which we will use several times in this work. In section 1.3, we state an important result about curvature estimates for stable minimal surfaces. In section 1.4, we give the definition of the Flux formula and state the Flux theorem. Finally, in Section 1.5, we state the Douglas criterion for the existence of a minimal annulus with a certain contour.

1.1 Terminology and some basic facts

Let (M, g) be a Riemannian 3-manifold and consider Σ a surface in M. The *mean curvature vector* of Σ at a point p is defined by

$$\vec{H}_\Sigma(p) = \frac{1}{2}\sum_{i=1}^{2}(A_\Sigma)_p(e_i, e_i),$$

where A_Σ denotes the second fundamental form of Σ, and $\{e_1, e_2\}$ is an orthonormal basis of $T_p\Sigma$ with respect to the induced metric.

Let ν be a local unit normal vector field along Σ around $p \in \Sigma$. The *mean curvature* of Σ at p with respect to ν is defined by

$$H_\Sigma(p) = \langle \vec{H}_\Sigma(p), \nu(p)\rangle.$$

Remark 1. If there is no ambiguity we will denote the second fundamental form, the mean curvature vector and the mean curvature of Σ only by A, $\vec{H}$ and H, respectively.

Let $F : \Sigma \times (-\epsilon, \epsilon) \to M$ be a smooth normal variation of compact support of Σ, that is, $F(p, 0) = p$ for all p, $F(p, t) = p$ for all t and p outside some compact set, and the variational vector field $X = \frac{\partial F}{\partial t}(p, 0)$ is orthogonal to $T_p\Sigma$. Denote $\Sigma_t = F(p, t)$.

Proposition 1 (First variation formula of area)**.** *We have*

$$\frac{d}{dt}|\Sigma_t|\bigg|_{t=0} = -2\int_{\Sigma}\langle \vec{H}, X\rangle \, d\sigma,$$

where $|\Sigma_t|$ and $d\sigma$ denote the area of Σ_t and the area element of Σ with respect to the induced metric, respectively.

We say that Σ is a minimal surface if $\dfrac{d}{dt}|\Sigma_t|\big|_{t=0} = 0$ for every smooth normal variation Σ_t of Σ. Hence, Σ is minimal if, anf only if, $\vec{H} \equiv 0$.

Throughout this work we only consider oriented surfaces in oriented Riemannian manifolds. Hence, we can take ν a globally defined unit normal vector field along Σ, and then any variational vector field X of a smooth normal variation Σ_t of Σ can be written as $X = \phi\nu$, for some function $\phi \in C_0^\infty(\Sigma)$.

Proposition 2 (Second variation formula of area)**.** *We have*

$$\frac{d^2}{dt^2}|\Sigma_t|\bigg|_{t=0} = \int_{\Sigma} |\nabla_\Sigma \phi|^2 - (\mathrm{Ric}(\nu, \nu) + |A|^2)\phi^2 \, \mathrm{d}\sigma,$$

where Ric denotes the Ricci curvature of M, and $\nabla_\Sigma\phi$ denotes the gradient of ϕ on Σ with respect to the induced metric.

We say that a minimal surface Σ is stable if

$$\frac{d^2}{dt^2}|\Sigma_t|\bigg|_{t=0} \geqslant 0,$$

for every smooth normal variation of compact support Σ_t of Σ.

Notice that if Σ is area-minimizing then Σ is a stable minimal surface, and the condition of stability is equivalent to the first eigenvalue of the Jacobi operator $L = \Delta_\Sigma + \mathrm{Ric}(\nu, \nu) + |A|^2$ to be nonnegative. Here, Δ_Σ denotes the Laplacian on Σ with respect to the induced metric.

Let us remark that a simple and useful fact that implies stability is transversality to a Killing field, that is, if a minimal surface is transversal to a Killing field, then it is stable (see, for example, Lemma 2.1 [40]).

Definition 1. A surface $\Sigma \subset M$ is called *parabolic* if the only functions $u : \Sigma \to \mathbb{R}$ that satisfy $u \leq 0$ and $\Delta u \geq 0$ are the constant functions. Otherwise, we say that Σ is hyperbolic.

1.2 Maximum principle

A very useful result for studying surfaces with constant mean curvature is the maximum principle.

Theorem 1 (Maximum principle). *Let Σ_1 and Σ_2 be two constant mean curvature surfaces. Suppose there exists $p \in \Sigma_1 \cap \Sigma_2$ such that Σ_1 and Σ_2 are tangent at p, and Σ_2 lies in the mean convex side of Σ_1 in a neighborhood of p. Then $H_2 \geq H_1$, and the equality holds if, and only if, $\Sigma_1 = \Sigma_2$.*

In particular, the maximum principle implies that if two minimal surfaces are tangent at a point, and one surface lies on one side of the other in a neighborhood of that point, then these two minimal surfaces coincide.

For surfaces with boundary we have the following result.

Theorem 2 (Boundary maximum principle). *Let Σ_1 and Σ_2 be two constant mean curvature surfaces tangent at a point $p \in \partial\Sigma_1 \cap \partial\Sigma_2$. Suppose that in a neighborhood of p, Σ_1 and Σ_2 can be seen as graphs over the same domain in $T_p\Sigma_1 = T_p\Sigma_2$, and Σ_2 lies in the mean convex side of Σ_1 in this neighborhood of p. Then $H_2 \geq H_1$, and the equality holds if, and only if, $\Sigma_1 = \Sigma_2$.*

1.3 Curvature estimates

Rosenberg, Souam and Toubiana [50] obtained an estimate for the norm of the second fundamental form of stable H-surfaces in Riemannian 3-manifolds assuming only a bound on the sectional curvature. Their estimate depends on the distance to the boundary of the surface and only on the bound on the sectional curvature of the ambient manifold. More precisely, they proved the following result.

Theorem 3 (Rosenberg, Souam and Toubiana, [50]). *Let (M, g) be a complete smooth Riemannian 3-manifold of bounded sectional curvature $|K| \leq \Lambda < +\infty$. Then there exists a universal constant C which depends neither on M nor on Λ, satisfying the following:*

For any immersed stable H-surface Σ in M with trivial normal bundle, and for any $p \in \Sigma$ we have

$$|A(p)| \leq \frac{C}{min\{d(p, \partial\Sigma), \frac{\pi}{2\sqrt{\Lambda}}\}}.$$

On the assumption of uniform curvature estimates we have the following classical result.

Proposition 3. *Let M be a homogeneous 3-manifold. Let Σ_n be an oriented properly embedded minimal surface in N. Suppose there exist $c > 0$ such that for all n, $|A_{\Sigma_n}| \leq c$, and a sequence of points $\{p_n\}$ in Σ_n such that $p_n \to p \in M$. Then there exists a subsequence of Σ_n that converges to a complete minimal surface Σ with $p \in \Sigma$.*

1.4 Flux formula

An important tool for studying minimal and, more generally, constant mean curvature surfaces are the formulae for the flux of appropriately chosen ambient vector fields across the surface.

Let u be a function defined in D whose graph is a minimal surface, and consider $X = \frac{\nabla u}{W}$ defined on D, where $W^2 = 1 + |\nabla u|^2$. For an open domain $U \subset D$, and α a boundary arc of U, we define the flux formula across α as

$$F_u(\alpha) = \int_\alpha \langle X, \nu \rangle \, ds;$$

here α is oriented as the boundary of U and ν is the outer conormal to U along α.

Theorem 4 (Flux Theorem)**.** *Let $U \subset D$ be an open domain. Then*

1. *If ∂U is a compact cycle, $F_u(\partial U) = 0$.*
2. *If α is a compact arc of U, $F_u(\alpha) \leq |\alpha|$.*
3. *If α is a compact arc of U on which u diverges to $+\infty$,*
$$F_u(\alpha) = |\alpha|.$$
4. *If α is a compact arc of U on which u diverges to $-\infty$,*
$$F_u(\alpha) = -|\alpha|.$$

1.5 Douglas criterion

While a Jordan curve in Euclidean 3-space always bounds a minimal disk, it is generally quite difficult to decide whether a set of several contours is capable of bounding a minimal surface having a prescribed topological type. There is a very important criterion, due to Douglas [9] (see [27], Theorem 2.1, for the case of a general Riemannian manifold), which guarantees the existence of such minimal surface in certain instances. Although the Douglas criterion is quite general, we will only state the particular case that we will use here. For the general statement, see [27].

Theorem 5 (Douglas criterion)**.** *Let Γ_1 and Γ_2 be two disjoint Jordan curves. Consider S_1 and S_2 two least area minimal disks with boundary Γ_1 and Γ_2, respectively. If there is an annulus A with boundary $\Gamma_1 \cup \Gamma_2$ such that*

$$area(A) \leq area(S_1) + area(S_2),$$

then there exists a least area minimal annulus with boundary $\Gamma_1 \cup \Gamma_2$.

CHAPTER 2

The Alexandrov problem in a quotient space of $\mathbb{H}^2 \times \mathbb{R}$

In this chapter we prove an Alexandrov type theorem for a quotient space of $\mathbb{H}^2 \times \mathbb{R}$. More precisely we classify the compact embedded surfaces with constant mean curvature in the quotient of $\mathbb{H}^2 \times \mathbb{R}$ by a subgroup of isometries generated by a horizontal translation along horocycles of $\mathbb{H}^2$ and a vertical translation. Moreover, we construct some examples of periodic minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ and we prove a multi-valued Rado theorem for small perturbations of the helicoid in $\mathbb{H}^2 \times \mathbb{R}$.

2.1 Introduction

Alexandrov, in 1962, proved that the only compact embedded constant mean curvature hypersurfaces in $\mathbb{R}^n, \mathbb{H}^n$ and $\mathbb{S}^n_+$ are the round spheres. Since then, many people have proved an Alexandrov type theorem in other spaces.

For instance, W.T. Hsiang and W.Y. Hsiang [25] showed that a compact embedded constant mean curvature surface in $\mathbb{H}^2 \times \mathbb{R}$ or in $\mathbb{S}^2_+ \times \mathbb{R}$ is a rotational sphere. They used the Alexandrov reflection method with vertical planes in order to prove that for any horizontal direction, there is a vertical plane of symmetry of the surface orthogonal to that direction.

To apply the Alexandrov reflection method we need to start with a vertical plane orthogonal to a given direction that does not intersect the surface, and in $\mathbb{S}^2 \times \mathbb{R}$ this fact is guaranteed by the hypothesis that the surface is contained in the product of a hemisphere with the real line. We remark that in $\mathbb{S}^2 \times \mathbb{R}$, we know that there are embedded rotational constant mean curvature tori, but the Alexandrov problem is not completely solved in $\mathbb{S}^2 \times \mathbb{R}$. In other simply connected homogeneous spaces with a 4-dimensional isometry group ($\mathrm{Nil}_3, \widetilde{\mathrm{PSL}}_2(\mathbb{R})$, some Berger spheres), we do not know if the solutions to the Alexandrov problem are spheres.

In Sol_3, Rosenberg proved that an embedded compact constant mean curvature surface is a sphere [7].

Recently, Mazet, Rodríguez and Rosenberg [29] considered the quotient of $\mathbb{H}^2 \times \mathbb{R}$ by a discrete group of isometries of $\mathbb{H}^2 \times \mathbb{R}$ generated by a horizontal translation along a geodesic of $\mathbb{H}^2$ and a vertical translation. They classified the compact embedded constant mean curvature surfaces in the quotient space. Moreover, they constructed examples of periodic minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$, where by periodic we mean a surface which is invariant by a non-trivial discrete group of isometries of $\mathbb{H}^2 \times \mathbb{R}$.

We also consider periodic surfaces in $\mathbb{H}^2 \times \mathbb{R}$. The discrete groups of isometries of $\mathbb{H}^2 \times \mathbb{R}$ we consider are generated by a horizontal translation ψ along horocycles $c(s)$ of $\mathbb{H}^2$ and/or a vertical translation $T(h)$ for some $h > 0$. In the case the group is the $\mathbb{Z}^2$ subgroup generated by ψ and $T(h)$, the quotient space $\mathcal{M} = \mathbb{H}^2 \times \mathbb{R}\,/[\psi, T(h)]$ is diffeomorphic to $\mathbb{T}^2 \times \mathbb{R}$, where $\mathbb{T}^2$ is the 2-torus. Moreover, $\mathcal{M}$ is foliated by the family of tori $\mathbb{T}(s) = c(s) \times \mathbb{R}\,/[\psi, T(h)]$ which are intrinsically flat and have constant mean curvature $1/2$. In this quotient space $\mathcal{M}$, we prove an Alexandrov type theorem.

Moreover, we consider a multi-valued Rado theorem for small perturbations of the helicoid. Rado's theorem (see [47]) is one of the fundamental results of minimal surface theory. It is connected to the famous Plateau problem, and states that if $\Omega \subset \mathbb{R}^2$ is a convex subset and $\Gamma \subset \mathbb{R}^3$ is a simple closed curve which is graphical over $\partial\Omega$, then any compact minimal surface $\Sigma \subset \mathbb{R}^3$ with $\partial\Sigma = \Gamma$ must be a disk which is graphical over Ω, and then unique, by the maximum principle. In [8], Dean and Tinaglia proved a generalization of Rado's theorem. They showed that for a minimal surface of any genus whose boundary is almost graphical in some sense, the minimal surface must be graphical once we move sufficiently far from the boundary. In our work, we consider this problem for minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ whose boundary is a small perturbation of the boundary of a helicoid, and we prove that the solution to the Plateau problem is the only compact minimal disk with that boundary (see Theorem 7).

This chapter is organized as follows. In section 2.2, we introduce some notation used in this chapter. In Section 2.3, we classify the compact embedded constant mean curvature surfaces in the space $\mathcal{M}$, that is, we prove an Alexandrov type theorem for doubly periodic H-surfaces (see Theorem 6). In section 2.4, we construct some examples of periodic minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$. In section 2.5, we prove a multi-valued Rado theorem for small perturbations of the helicoid (see Theorem 7).

2.2 Terminology

Throughout this chapter, the Poincaré disk model is used for the hyperbolic plane, that is,

$$\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$$

with the hyperbolic metric $g_{-1} = \frac{4}{(1-x^2-y^2)^2} g_0$, where g_0 is the Euclidean metric in $\mathbb{R}^2$. In this model, the asymptotic boundary $\partial_\infty \mathbb{H}^2$ of $\mathbb{H}^2$ is identified with the unit circle. Consequently, any point in the closed unit disk is viewed as either a point in $\mathbb{H}^2$ or a point in $\partial_\infty \mathbb{H}^2$. We denote by $\mathbf{0}$ the origin of $\mathbb{H}^2$.

In $\mathbb{H}^2$ we consider γ_0, γ_1 the geodesic lines $\{x = 0\}, \{y = 0\}$, respectively. For $j = 0, 1$, we denote by Y_j the Killing vector field whose flow $(\phi_l)_{l \in (-1,1)}$ is given by hyperbolic translation along γ_j with $\phi_l(\mathbf{0}) = (l \sin \pi j, l \cos \pi j)$ and $(\sin \pi j, \cos \pi j)$ as attractive point at infinity. We call $(\phi_l)_{l \in (-1,1)}$ the flow of Y_j even though the family $(\phi_l)_{l \in (-1,1)}$ is not parameterized at the right speed.

We denote by $\pi : \mathbb{H}^2 \times \mathbb{R} \to \mathbb{H}^2$ the vertical projection and we write t for the height coordinate in $\mathbb{H}^2 \times \mathbb{R}$. In what follows, we will often identify the hyperbolic plane $\mathbb{H}^2$ with the horizontal slice $\{t = 0\}$ of $\mathbb{H}^2 \times \mathbb{R}$. The vector fields $Y_j, j = 0, 1$, and their flows naturally extend to horizontal vector fields and their flows in $\mathbb{H}^2 \times \mathbb{R}$.

Consider any geodesic γ that limits to the point $p_0 \in \partial_\infty \mathbb{H}^2$ at infinity parametrized by arc length. Let $c(s)$ denote the horocycle in $\mathbb{H}^2$ tangent to $\partial_\infty \mathbb{H}^2$ at p_0 that intersects γ at $\gamma(s)$. Given two points $p, q \in c(s)$, we denote by $\psi : \mathbb{H}^2 \times \mathbb{R} \to \mathbb{H}^2 \times \mathbb{R}$ the parabolic translation along $c(s)$ such that $\psi(p) = q$.

We write $\overline{pq}$ to denote the geodesic arc between the two points p, q of $\mathbb{H}^2 \times \mathbb{R}$.

2.3 The Alexandrov problem for doubly periodic constant mean curvature surfaces

Take two points p, q in a horocycle $c(s)$, and let ψ be the parabolic translation along $c(s)$ such that $\psi(p) = q$. We have $\psi(c(s)) = c(s)$ for all s. Consider G the $\mathbb{Z}^2$ subgroup of isometries of $\mathbb{H}^2 \times \mathbb{R}$ generated by ψ and a vertical translation $T(h)$, for some positive h. We denote by $\mathcal{M}$ the quotient of $\mathbb{H}^2 \times \mathbb{R}$ by G. The manifold $\mathcal{M}$ is diffeomorphic but not isometric to $\mathbb{T}^2 \times \mathbb{R}$ and is foliated by the family of tori $\mathbb{T}(s) = (c(s) \times \mathbb{R})/G$, $s \in \mathbb{R}$, which are intrinsically flat and have constant mean cuvature 1/2. Thus the tori $\mathbb{T}(s)$ are examples of compact embedded constant mean curvature surfaces in $\mathcal{M}$.

We have the following answer to the Alexandrov problem in $\mathcal{M}$.

Theorem 6. *Let $\Sigma \subset \mathcal{M}$ be a compact immersed surface with constant mean curvature H. Then $H \geq \frac{1}{2}$. Moreover,*

1. *If $H = \frac{1}{2}$, then Σ is a torus $\mathbb{T}(s)$, for some s;*
2. *If $H > \frac{1}{2}$ and Σ is embedded, then Σ is either the quotient of a rotational sphere, or the quotient of a vertical unduloid (in particular, a vertical cylinder over a circle).*

Proof. Let Σ be a compact immersed surface in $\mathcal{M}$ with constant mean curvature H. As Σ is compact, there exist $s_0 \leq s_1 \in \mathbb{R}$ such that Σ is between $\mathbb{T}(s_0)$ and $\mathbb{T}(s_1)$, and it is tangent to $\mathbb{T}(s_0), \mathbb{T}(s_1)$ at points q, p, respectively, as illustrated in Figure 2.1.

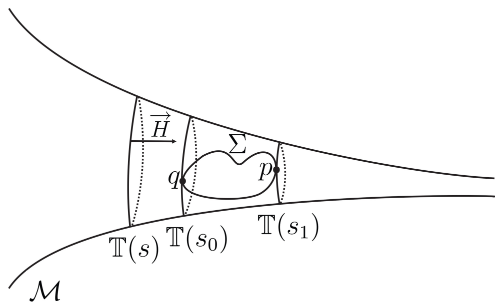

Figure 2.1: $\Sigma \subset \mathcal{M}$.

For $s < s_0$, the torus $\mathbb{T}(s)$ does not intersect Σ, and Σ stays in the mean convex region bounded by $\mathbb{T}(s)$. By comparison at q, we conclude that $H \geq \frac{1}{2}$. If $H = \frac{1}{2}$, then by the maximum principle, Σ is the torus $\mathbb{T}(s_0)$, and we have proved the first part of the theorem.

To prove the last part, suppose Σ is embedded and consider the quotient space $\widetilde{\mathcal{M}} = \mathbb{H}^2 \times \mathbb{R}/[T(h)]$, which is diffeomorphic to $\mathbb{H}^2 \times \mathbb{S}^1$. Take a connected component $\widetilde{\Sigma}$ of the lift of Σ to $\widetilde{\mathcal{M}}$, and denote by $\tilde{c}(s)$ the surface $c(s) \times \mathbb{S}^1$. Observe that $\tilde{c}(s)$ is the lift of $\mathbb{T}(s)$ to $\widetilde{\mathcal{M}}$. Moreover, let us consider two points $\tilde{p}, \tilde{q} \in \widetilde{\Sigma}$ whose projections in $\mathcal{M}$ are the points p, q, respectively.

It is easy to prove that $\widetilde{\Sigma}$ separates $\widetilde{\mathcal{M}}$. In fact, suppose by contradiction this is not true, then we can consider a geodesic arc $\alpha : (-\epsilon, \epsilon) \to \widetilde{\mathcal{M}}$ such that $\alpha(0) \in \widetilde{\Sigma}, \alpha'(0) \in T\widetilde{\Sigma}^{\perp}$ and we can join the points $\alpha(-\epsilon), \alpha(\epsilon)$ by a curve that does not intersect $\widetilde{\Sigma}$, hence we obtain a Jordan curve, which we still call α, whose intersection number with $\widetilde{\Sigma}$ is 1 modulo 2. Notice that the distance between $\widetilde{\Sigma}$ and $\tilde{c}(s_0)$ is bounded. Since we can homotop α so it is arbitrarily far from $\tilde{c}(s_0)$, we conclude that a translate of α does not intersect $\widetilde{\Sigma}$, contradicting the fact that the intersection number of α and $\widetilde{\Sigma}$ is 1 modulo 2. Thus $\widetilde{\Sigma}$ does separate $\widetilde{\mathcal{M}}$.

Let us call A the mean convex component of $\widetilde{\mathcal{M}} \setminus \widetilde{\Sigma}$ with boundary $\widetilde{\Sigma}$ and B the other component. Hence $\widetilde{\mathcal{M}} \setminus \widetilde{\Sigma} = A \cup B$.

Let γ be a geodesic in $\mathbb{H}^2$ that limits to $p_0 \in \partial_\infty \mathbb{H}^2, \gamma(+\infty) = p_0$ (the point where the horocycles $c(s)$ are centered) and let us assume that γ intersects $\widetilde{\Sigma}$ in *at least* two points.

Consider $(l_t)_{t\in\mathbb{R}}$ the family of geodesics in $\mathbb{H}^2$ orthogonal to γ and denote by $P(t)$ the totally geodesic vertical annulus $l_t \times \mathbb{S}^1$ of $\widetilde{\mathcal{M}} = \mathbb{H}^2 \times \mathbb{S}^1$ (see Figure 2.2). Since $\widetilde{\Sigma}$ is a lift of the compact surface Σ, it stays in the region between $\tilde{c}(s_0)$ and $\tilde{c}(s_1)$, and the distance from any point of $\widetilde{\Sigma}$ to $\tilde{c}(s_0)$ and to $\tilde{c}(s_1)$ is uniformly bounded.

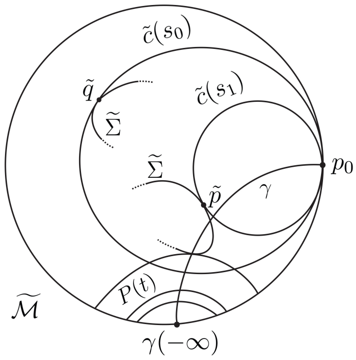

Figure 2.2: The family of totally geodesic annuli $P(t)$.

By our choice of γ, the ends of each $P(t)$ are outside the region bounded by $\tilde{c}(s)$, hence $P(t)\cap\widetilde{\Sigma}$ is compact for all t. Moreover, for t close to $-\infty$, $P(t)$ is contained in B and $P(t) \cap \widetilde{\Sigma}$ is empty. Then start with t close to $-\infty$ and let t increase until a first contact point between $\widetilde{\Sigma}$ and some vertical annulus, say $P(t_0)$. In particular, we know the mean curvature vector of $\widetilde{\Sigma}$ does not point into $\bigcup_{t\le t_0} P(t)$.

Continuing to increase t and starting the Alexandrov reflection procedure for $\widetilde{\Sigma}$ and the family of vertical totally geodesic annuli $P(t)$, we get a first contact point between the reflected part of $\widetilde{\Sigma}$ and $\widetilde{\Sigma}$, for some $t_1 \in \mathbb{R}$. Observe that this first contact point occurs because we are assuming that the geodesic γ intersects $\widetilde{\Sigma}$ in at least two points.

Then $\widetilde{\Sigma}$ is symmetric with respect to $P(t_1)$. As $\widetilde{\Sigma} \cap \left(\bigcup_{t_0\le t\le t_1} P(t)\right)$ is compact, then $\widetilde{\Sigma}$ is compact. Hence, given any horizontal geodesic α we can apply the Alexandrov procedure with the family of totally geodesic vertical annuli $Q(t) = \tilde{l}_t \times \mathbb{S}^1$, where $(\tilde{l}_t)_{t\in\mathbb{R}}$ is the family of horizontal geodesics orthogonal to α, and we obtain a symmetry plane for $\widetilde{\Sigma}$.

Hence we have shown that if some geodesic that limits to p_0 intersects $\widetilde{\Sigma}$ in two or more points, then $\widetilde{\Sigma}$ lifts to a rotational cylindrically bounded surface $\bar{\Sigma}$ in $\mathbb{H}^2 \times \mathbb{R}$. If $\bar{\Sigma}$ is not compact then $\bar{\Sigma}$ is a vertical unduloid, and if $\bar{\Sigma}$ is compact we know, by Hsiang-Hsiang's theorem [25], $\bar{\Sigma}$ is a rotational sphere. Therefore, we have proved that in this case $\Sigma \subset \mathcal{M}$ is either the quotient of a rotational sphere or the quotient of a vertical unduloid.

Now to finish the proof let us assume that every geodesic that limits to p_0 intersects $\widetilde{\Sigma}$ in *at most* one point. In particular, the geodesic β that limits to p_0 and passes through $\tilde{p} \in \tilde{c}(s_1)$ intersects $\widetilde{\Sigma}$ only at $\tilde{p}$. Write β^- to denote the arc of β between $\beta(-\infty)$ and $\tilde{p}$ (see Figure 2.3).

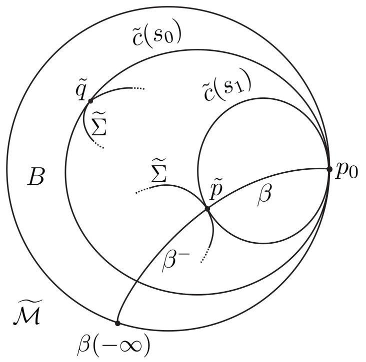

Figure 2.3: Geodesic β.

As $\beta \cap \widetilde{\Sigma} = \{\tilde{p}\}$, we have $\beta^- \cap \widetilde{\Sigma} = \emptyset$ and then $\beta^- \subset B$, since $\widetilde{\Sigma}$ separates $\widetilde{\mathcal{M}}$.

Hence at the point $\tilde{p} \in \widetilde{\Sigma} \cap \tilde{c}(s_1)$, the mean curvature vectors of $\widetilde{\Sigma}$ and $\tilde{c}(s_1)$ point to the mean convex side of $\tilde{c}(s_1)$ and $\widetilde{\Sigma}$ lies on the mean concave side of $\tilde{c}(s_1)$, then by comparison we get $H \leq \frac{1}{2}$. But we already know that $H \geq \frac{1}{2}$. Hence $H = \frac{1}{2}$ and $\widetilde{\Sigma} = \tilde{c}(s_1)$, by the maximum principle. Therefore, in this case we conclude $\Sigma = \mathbb{T}(s_1)$.

□

Remark 2. Note that a vertical unduloid, contained in a cylinder $D \times \mathbb{R}$ and invariant by a vertical translation $T(l)$ in $\mathbb{H}^2 \times \mathbb{R}$, passes to the quotient space $\mathcal{M} = \mathbb{H}^2 \times \mathbb{R}/[\psi, T(h)]$ as an embedded surface if the quotient of D is embedded and the number l is a multiple of h. Analogously, a rotational sphere of height l contained in a cylinder $D \times \mathbb{R}$ in $\mathbb{H}^2 \times \mathbb{R}$ passes to the quotient as an embedded surface if $l < h$ and the quotient of D is embedded in $\mathcal{M}$.

2.4 Construction of periodic minimal surfaces

In this section we are interested in constructing some new examples of periodic minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ invariant by a subgroup of isometries, which is either isomorphic to $\mathbb{Z}^2$, or generated by a vertical translation, or generated by a screw motion. In fact, we only consider subgroups generated by a parabolic translation ψ along a horocycle and/or a vertical translation $T(h)$, for some $h > 0$.

Periodic minimal surfaces in $\mathbb{R}^3$ have received great attention since Riemann, Schwarz, Scherk (and many others) studied them. They also appear in the natural sciences. In [33], Meeks and Rosenberg proved that a periodic properly embedded minimal surface of finite topology (in $\mathbb{R}^3/G, G$ a discrete group of isometries acting properly discontinuously on $\mathbb{R}^3, G \neq (1)$) has finite total curvature and the ends are asymptotic to standard ends (planar, catenoidal, or helicoidal). In a joint paper with Hauswirth [18], we consider the same study for periodic minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$. The first step is to understand what are the possible models for the ends in the quotient. This is one reason to construct examples.

2.4.1 Doubly periodic minimal surfaces

In $\mathbb{H}^2$ consider two geodesics α, β that limit to the same point at infinity, say $\alpha(-\infty) = p_0 = \beta(-\infty)$. Denote $B = \alpha(+\infty)$ and $D = \beta(+\infty)$. Take a geodesic γ contained in the region bounded by α and β that limits to the same point p_0 at infinity. Parametrize these geodesics so that $\alpha(t) \to B, \beta(t) \to D$ and $\gamma(t) \to p_0$ when $t \to +\infty$.

Fix $h > \pi$ and consider the following Jordan curve:

$$\begin{aligned}\Gamma_t &= \overline{(\alpha(t),0),(\gamma(t),0)} \cup \overline{(\alpha(t),0),(\alpha(t),h)} \cup \overline{(\beta(t),0),(\gamma(t),0)} \\ &\quad \cup \overline{(\beta(t),0),(\beta(t),h)} \cup \overline{(\alpha(t),h),(\gamma(t),h)} \cup \overline{(\beta(t),h),(\gamma(t),h)}\end{aligned}$$

as illustrated in Figure 4.1.

Consider a least area embedded minimal disk Σ_t with boundary Γ_t. Let Y be the Killing field whose flow $(\phi_l)_{l\in(-1,1)}$ is given by translation along the geodesic γ. Notice that Γ_t is transversal to the Killing field Y. Hence given any geodesic $\bar{\gamma}$ orthogonal to γ, we can use the Alexandrov reflection technique with the foliation of $\mathbb{H}^2 \times \mathbb{R}$ by the vertical planes $(\phi_l(\bar{\gamma}))_{l\in(-1,1)}$ to show that

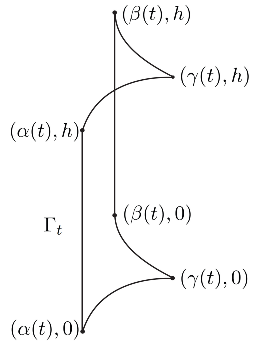

Figure 2.4: Curve Γ_t.

Σ_t is a Y-Killing graph. In particular, Σ_t is stable and unique (see Lemma 2.1 in [40]). This gives uniform curvature estimates for Σ_{t_0} for points far from the boundary (see Main Theorem in [50]). Rotating Σ_t by angle π around the geodesic arc $\overline{(\alpha(t),0),(\gamma(t),0)}$ gives a minimal surface that extends Σ_t, has int$\overline{(\alpha(t),0),(\gamma(t),0)}$ in its interior, and is still a Y-Killing graph. Thus we get uniform curvature estimates for Σ_t in a neighborhood of $\overline{(\alpha(t),0),(\gamma(t),0)}$. This is also true for the three other horizontal geodesic arcs in Γ_t.

Observe that for any t, Σ_t stays in the half-space determined by $\overline{BD}\times\mathbb{R}$ that contains Γ_t, by the maximum principle.

As $h>\pi$, we can use as a barrier the minimal surface $S_h\subset\mathbb{H}^2\times(0,h)$ which is a vertical bigraph with respect to the horizontal slice $\{t=\frac{h}{2}\}$. The surface S_h is invariant by translations along the horizontal geodesic $\gamma_0=\{x=0\}$ and its asymptotic boundary is $(\tau\times\{0\})\cup\overline{(0,1,0)(0,1,h)}$ $\cup(\tau\times\{h\})\cup\overline{(0,-1,0)(0,-1,h)}$, where $\tau=\partial_\infty\mathbb{H}^2\cap\{x>0\}$. For more details about the surface S_h, see [29, 30, 51].

For l close to 1, the translated surface $\phi_l(S_h)$ does not intersect Σ_t. Hence the surface Σ_t is contained between $\phi_l(S_h)$ and $\overline{BD}\times\mathbb{R}$.

Notice that when $t\to+\infty$, Γ_t converges to Γ, where

$$\begin{aligned}\Gamma &= (\alpha\times\{0\})\cup(\beta\times\{0\})\cup(\alpha\times\{h\})\cup\\ &\quad (\beta\times\{h\})\cup\overline{(D,0)(D,h)}\cup\overline{(B,0)(B,h)}.\end{aligned}$$

Therefore, as we have uniform curvature estimates and barriers at infinity, there exists a subsequence of Σ_t that converges to a minimal surface Σ, where Σ lies in the region of $\mathbb{H}^2\times[0,h]$ bounded by $\alpha\times\mathbb{R}$, $\beta\times\mathbb{R}$, $\overline{BD}\times\mathbb{R}$ and $\phi_l(S_h)$; with boundary $\partial\Sigma=\Gamma$.

Hence the surface obtained by reflection in all horizontal boundary geodesics of Σ is invariant by ψ^2 and $T(2h)$, where ψ is the horizontal translation along horocycles that sends α to β. Moreover, this surface in the quotient space

$\mathbb{H}^2 \times \mathbb{R} / [\psi^2, T(2h)]$ is topologically a sphere minus four points. Two ends are asymptotic to vertical planes and two are asymptotic to horizontal planes (cusps), all of them with finite total curvature.

Proposition 4. *There exists a doubly periodic minimal surface (invariant by horizontal translations along a horocycle and by a vertical translation) such that, in the quotient space, this surface is topologically a sphere minus four points, with two ends asymptotic to vertical planes and two asymptotic to horizontal planes, all of them with finite total curvature.*

2.4.2 Vertically periodic minimal surfaces

Take α any geodesic in $\mathbb{H}^2 \times \{0\}$. For $h > \pi$, consider the vertical segment $\alpha(-\infty) \times [0, 2h]$, and a point $p \in \partial_\infty \mathbb{H}^2$, $p \neq \alpha(-\infty), \alpha(+\infty)$. For some small $\epsilon > 0$, consider the asymptotic vertical segment joining (p, ϵ) and $(p, h + \epsilon)$. Now, connect (p, ϵ) to $(\alpha(-\infty), 0)$ and $(p, h + \epsilon)$ to $(\alpha(-\infty), 2h)$ by curves in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$, whose tangent vectors are never horizontal or vertical, and so that the resulting curve Γ is differentiable. Also, consider the horizontal geodesic β connecting p to $\alpha(+\infty)$.

Parametrize α by arc length, and consider γ a geodesic orthogonal to α passing through $\alpha(0)$. Let us denote by $d(t)$ the equidistant curve to γ in a distance $|t|$ that intersects α at $\alpha(t)$. For each t consider a curve Γ_t contained in the plane $d(t) \times \mathbb{R}$ with endpoints $(\alpha(t), 0)$ and $(\alpha(t), 2h)$ such that Γ_t is contained in the region R bounded by $\alpha \times \mathbb{R}, \beta \times \mathbb{R}, \mathbb{H}^2 \times \{0\}$ and $\mathbb{H}^2 \times \{2h\}$ with the properties that its tangent vectors do not point in the horizontal direction and Γ_t converges to Γ when $t \to -\infty$. In particular, Γ_t is transversal to the Killing field Y whose flow $(\phi_l)_{l \in (-1,1)}$ is given by translation along the geodesic γ.

Write α_t to denote the vertical segment $\alpha(t) \times [0, 2h]$ (see Figure 2.5).

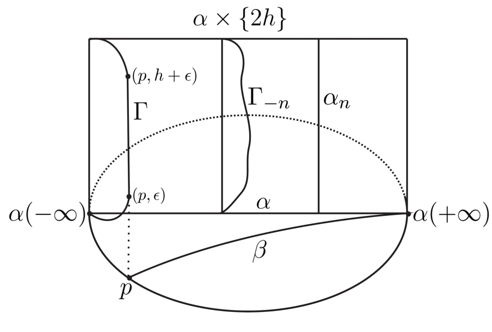

Figure 2.5: Curves Γ_{-n} and Γ.

For each n, let Σ_n be the solution to the Plateau problem with boundary $\Gamma_{-n} \cup (\alpha([-n,n]) \times \{0\}) \cup (\alpha([-n,n]) \times \{2h\}) \cup \alpha_n$. By our choice of the curves Γ_t, the boundary $\partial\Sigma_n$ is transverse to the Killing field Y. Using the foliation of $\mathbb{H}^2 \times \mathbb{R}$ by the vertical planes $\phi_l(\alpha), l \in (-1,1)$, the Alexandrov reflection technique shows that Σ_n is a Y-Killing graph. In particular, it is unique and stable [40], and we have uniform curvature estimates far from the boundary [50]. When we apply the rotation by angle π around $\alpha \times \{0\}$ to the minimal surface Σ_n, we get another minimal surface which extends Σ_n, is still a Y-Killing graph and has $\mathrm{int}(\alpha([-n,n]) \times \{0\})$ in its interior. Hence we obtain uniform curvature estimates for Σ_n in a neighborhood of $\alpha([-n,n]) \times \{0\}$. This is also true for $\alpha([-n,n]) \times \{2h\}$ and α_n.

Observe that Σ_n is contained in the region R, for all n.

By our choice of Γ, for each $q \in \Gamma$, we can consider two translations of the minimal surfaces S_h (considered in the last section) that pass through q so that one of them has asymptotic boundary under Γ, the other one has asymptotic boundary above Γ and their intersection with Γ is just the point q considered or is the whole vertical segment $\overline{(p,\epsilon)(p,h+\epsilon)}$. Hence, the envelope of the union of all these translated surfaces S_h forms a barrier to Σ_n, for all n.

Then, as we have uniform curvature estimates and barriers at infinity, we conclude that there exists a subsequence of Σ_n that converges to a minimal surface Σ with $(\alpha(+\infty) \times [0,2h]) \cup \Gamma = \partial_\infty \Sigma$, and then $\partial\Sigma = \Gamma \cup (\alpha \times \{0\}) \cup (\alpha \times \{2h\}) \cup (\alpha(+\infty) \times [0,2h])$.

Therefore, the surface obtained by reflection in all horizontal boundary geodesics of Σ is a vertically periodic minimal surface invariant by $T(4h)$. In the quotient space this minimal surface has two ends; one is asymptotic to a vertical plane and has finite total curvature, while the other one is topologically an annular end and has infinite total curvature.

Proposition 5. *There exists a singly periodic minimal surface (invariant by a vertical translation) such that, in the quotient space, this surface has two ends, one end is asymptotic to a vertical plane and has finite total curvature, while the other one is topologically an annular end and has infinite total curvature.*

2.4.3 Periodic minimal surfaces invariant by screw motion

Now we construct some examples of periodic minimal surfaces invariant by a screw motion, that is, invariant by a subgroup of isometries generated by the composition of a horizontal translation with a vertical translation.

Consider two geodesics α, β in $\mathbb{H}^2$ that limit to the same point at infinity, say $\alpha(+\infty) = p_0 = \beta(+\infty)$. For $h > \pi$, consider a smooth curve Γ contained in the asymptotic boundary of $\mathbb{H}^2 \times \mathbb{R}$, connecting $(\alpha(-\infty), 2h)$ to $(\beta(-\infty), 0)$ and such that its tangent vectors are never horizontal or vertical. Also, take a point $p \in \partial_\infty \mathbb{H}^2$ in the halfspace determined by $\beta \times \mathbb{R}$ that does not contain α.

For some small $\epsilon > 0$, consider the asymptotic vertical segment joining (p, ϵ) and $(p, h+\epsilon)$. Now, connect (p, ϵ) to $(p_0, 0)$ and $(p, h+\epsilon)$ to $(p_0, 2h)$ by curves in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$ whose tangent vectors are never horizontal or vertical, and such that the resulting curve $\widehat{\Gamma}$ is differentiable.

Parametrize α by arc length, and consider γ a geodesic orthogonal to α passing through $\alpha(0)$. Let us denote by $d(t)$ the equidistant curve to γ in a distance $|t|$ that intersects α at $\alpha(t)$. For each t, s consider two curves $\widehat{\Gamma}_t$ and Γ_s contained in the plane $d(t) \times \mathbb{R}$ and $d(s) \times \mathbb{R}$, respectively, with the properties that their tangent vectors are never horizontal, $\widehat{\Gamma}_t$ joins $(\alpha(t), 2h)$ to $(\beta(t), 0)$, Γ_s joins $(\alpha(s), 2h)$ to $(\beta(s), 0)$, $\widehat{\Gamma}_t$ converges to $\widehat{\Gamma}$ when $t \to +\infty$, Γ_s converges to Γ when $s \to -\infty$, and both curves are contained in the region R bounded by $\alpha \times \mathbb{R}, \theta \times \mathbb{R}, \mathbb{H}^2 \times \{0\}$ and $\mathbb{H}^2 \times \{2h\}$, where θ is the geodesic with endpoints p and $\beta(-\infty)$ (see Figure 2.6).

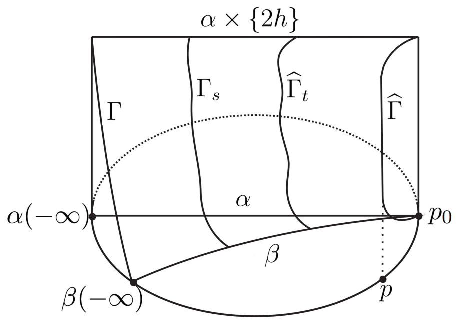

Figure 2.6: Curves $\widehat{\Gamma}_t, \Gamma_s, \widehat{\Gamma}$ and Γ.

For each n, let Σ_n be the solution to the Plateau problem with boundary $\Gamma_{-n} \cup (\alpha([-n, n]) \times \{2h\}) \cup \widehat{\Gamma}_n \cup (\beta([-n, n]) \times \{0\})$. The surface Σ_n is contained in the region R. As in the previous section, we can show that Σ_n is a Killing graph, then it is stable, unique and we have uniform curvature estimates far from the boundary. Rotating Σ_n by angle π around the geodesic $\alpha \times \{2h\}$ we get a minimal surface which extends Σ_n, is still a Killing graph, and has $\mathrm{int}(\alpha([-n, n]) \times \{2h\})$ in its interior. Hence we get uniform curvature estimates for Σ_n in a neighborhood of $\alpha([-n, n]) \times \{2h\}$. This is also true for $\beta([-n, n]) \times \{0\}$. Thus when $n \to +\infty$, there exists a subsequence of Σ_n that converges to a minimal surface Σ with $\Gamma \cup \widehat{\Gamma} \subset \partial_\infty \Sigma_n$. Using the same argument as before with suitable translations of the surface S_h as barriers, we conclude that in fact $\partial_\infty\Sigma = \Gamma\cup\widehat{\Gamma}$, and then $\partial\Sigma = \Gamma\cup(\alpha\times\{2h\})\cup(\beta\times\{0\})\cup\widehat{\Gamma}$.

The surface obtained by reflection in all horizontal boundary geodesics of Σ is a minimal surface invariant by $\psi^2 \circ T(4h)$, where ψ is the horizontal translation along horocycles that sends α to β. There are two annular embedded ends in the quotient, each of infinite total curvature.

Proposition 6. *There exists a minimal surface invariant by a screw motion such that, in the quotient space, this minimal surface has two annular embedded ends, each one of infinite total curvature.*

Now we will construct another interesting example of a periodic minimal surface invariant by a screw motion.

Denote by γ_0, γ_1 the geodesic lines $\{x = 0\}, \{y = 0\}$ in $\mathbb{H}^2$, respectively. Let c be a horocycle orthogonal to γ_1, and consider $p, q \in c$ equidistant points to γ_1. Take α, β geodesics which limit to $p_0 = (1, 0) = \gamma_1(+\infty)$ and pass through p, q, respectively. Fix $\epsilon > 0$ and $h > \pi$. Consider the points $A = \alpha(-t_0), C = \alpha(t_0), B = \beta(-t_0), D = \beta(t_0)$, and let us consider the following Jordan curve (see Figure 2.7):

$$
\begin{aligned}
\Gamma_{t_0} \;=\; & (\alpha([-t_0, t_0]) \times \{-\epsilon\}) \cup \overline{(C, -\epsilon)(D, 0)} \cup (\beta([-t_0, t_0]) \times \{0\}) \\
& \cup(\alpha([-t_0, t_0]) \times \{h\}) \cup \overline{(C, h)(D, h + \epsilon)} \cup (\beta([-t_0, t_0]) \times \{h + \epsilon\}) \\
& \cup\overline{(A, -\epsilon)(A, h)} \cup \overline{(B, 0)(B, h + \epsilon)}.
\end{aligned}
$$

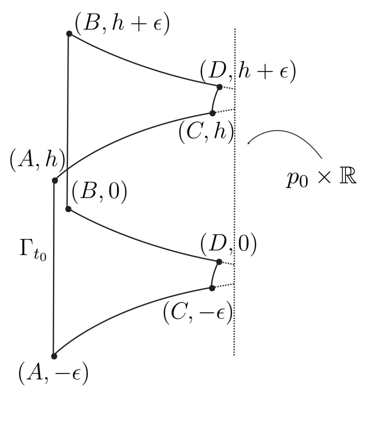

Figure 2.7: Curve Γ_{t_0}.

We consider a least area embedded minimal disk Σ_{t_0} with boundary Γ_{t_0}.

Denote by Y_1 the Killing vector field whose flow $(\phi_l)_{l\in(-1,1)}$ gives the hyperbolic translation along γ_1 with $\phi_l(0) = (l, 0)$ and p_0 as attractive point at infinity. As Γ_{t_0} is transversal to the Killing field Y_1, we can prove, using the maximum principle, that Σ_{t_0} is a Y_1-Killing graph with convex boundary, in particular, Σ_{t_0} is stable and unique [40]. This yields uniform curvature estimates far from the boundary [50]. Rotating Σ_{t_0} by angle π around the geodesic arc $\alpha([-t_0,t_0])\times\{-\epsilon\}$ gives a minimal surface that extends Σ_{t_0}, has $\mathrm{int}(\alpha([-t_0,t_0])\times\{-\epsilon\})$ in its interior, and is still a Y_1-Killing graph. Thus we get uniform curvature estimates for Σ_{t_0} in a neighborhood of $\alpha([-t_0,t_0])\times\{-\epsilon\}$. This is also true for the three other horizontal geodesic arcs in Γ_{t_0}.

Write $F=\alpha(-\infty), G=\beta(-\infty)$. Observe that, by the maximum principle, for any t_0, Σ_{t_0} stays in the halfspace determined by $\overline{FG}\times\mathbb{R}$ that contains Γ_{t_0}.

Since $h>\pi$, we can consider the minimal surface S_h (considered in Section 2.4.1) as a barrier. For l close to 1, the translated surface $\phi_l(S_h)$ does not meet Σ_{t_0}. The surface Σ_{t_0} is contained between $\phi_l(S_h)$ and $\overline{FG}\times\mathbb{R}$. When $t_0\to+\infty$, Γ_{t_0} converges to Γ, where

$$\begin{aligned}\Gamma \;=\;& (\alpha\times\{-\epsilon\})\cup\overline{(p_0,-\epsilon)(p_0,0)}\cup(\beta\times\{0\})\\ &\cup(\alpha\times\{h\})\cup\overline{(p_0,h)(p_0,h+\epsilon)}\cup(\beta\times\{h+\epsilon\})\\ &\cup\overline{(F,-\epsilon)(F,h)}\cup\overline{(G,0)(G,h+\epsilon)}.\end{aligned}$$

Using the maximum principle, we can prove that Σ_t is contained between $\phi_l(S_h)$ and $\overline{FG}\times\mathbb{R}$, for all $t>t_0$. Therefore, there exists a subsequence of the surfaces Σ_t that converges to a minimal surface Σ, where Σ lies in the region between $\mathbb{H}^2\times\{-\epsilon\}$ and $\mathbb{H}^2\times\{h+\epsilon\}$ bounded by $\alpha\times\mathbb{R}$, $\beta\times\mathbb{R}$, $\overline{FG}\times\mathbb{R}$ and $\phi_l(S_h)$; and has boundary $\partial\Sigma=\Gamma$.

Hence the surface obtained by reflection in all horizontal boundary geodesics of Σ is invariant by $\psi^2\circ T(2(h+\epsilon))$, where ψ is the horizontal translation along horocycles that sends α to β. Moreover, this surface in the quotient space has two vertical ends and two helicoidal ends, each one of finite total curvature.

Proposition 7. *There exists a minimal surface invariant by a screw motion such that, in the quotient space, this minimal surface has four ends. Two vertical ends and two helicoidal ends, all of them with finite total curvature.*

2.5 A multi-valued Rado Theorem

The aim of this section is to prove a multi-valued Rado theorem for small perturbations of the helicoid. Recall that Rado's theorem says that minimal surfaces over a convex domain with graphical boundaries must be disks which are themselves graphical. We will prove that for certain small perturbations of the boundary of a (compact) helicoid there exists only one compact minimal disk with that boundary. By a compact helicoid we mean the intersection of a helicoid with certain compact regions in $\mathbb{H}^2 \times \mathbb{R}$. The idea here originated in the work of Hardt and Rosenberg [16]. We will apply this multi-valued Rado theorem to construct an embedded minimal surface in $\mathbb{H}^2 \times \mathbb{R}$ whose boundary is a small perturbation of the boundary of a complete helicoid.

Consider Y the Killing field whose flow $\phi_\theta, \theta \in [0, 2\pi)$, is given by rotations around the z-axis. For some $0 < c < 1$, let $D = \{(x, y) \in \mathbb{H}^2; x^2 + y^2 \leq c\}$. Take a helix h_0 of constant pitch contained in a solid cylinder $D \times [0, d]$, so that the vertical projection of h_0 over $\mathbb{H}^2 \times \{0\}$ is ∂D, and the endpoints of h_0 are in the same vertical line. Let us denote by Γ_0 the Jordan curve which is the union of h_0, the two horizontal geodesic arcs joining the endpoints of h_0 to the z-axis, and the part of the z-axis. Call $\mathcal{H}$ the compact part of the helicoid that has Γ_0 as its boundary. We know that $\mathcal{H}$ is a minimal surface transversal to the Killing field Y at the interior points. Take $\theta < \pi/4$, and consider $\mathcal{H}_1 = \phi_{-\theta}(\mathcal{H})$ and $\mathcal{H}_2 = \phi_\theta(\mathcal{H})$. Hence $\mathcal{H}_1, \mathcal{H}_2$ are two compact helicoids with boundary $\partial\mathcal{H}_1 = \phi_{-\theta}(\Gamma_0)$, $\partial\mathcal{H}_2 = \phi_\theta(\Gamma_0)$.

Consider h a small smooth perturbation of the helix h_0 with fixed endpoints such that h is transversal to Y and h is contained in the region between $\phi_{-\theta}(h_0)$ and $\phi_\theta(h_0)$ in $\partial D \times [0, d]$. Call Γ the Jordan curve which is the union of h, the two horizontal geodesic arcs and a part of the z-axis, hence $\Gamma = (\Gamma_0 \setminus h_0) \cup h$ (see Figure 2.8).

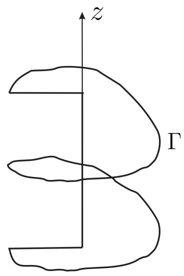

Figure 2.8: Curve Γ.

Denote by R the convex region bounded by $\mathcal{H}_1$ and $\mathcal{H}_2$ in the solid cylinder $D \times [0, d]$. The Jordan curve Γ is contained in the simply connected region R which has mean convex boundary. Then we can consider the solution to the Plateau problem in this region R, and we get a compact minimal disk H contained in R with boundary $\partial H = \Gamma$.

Proposition 8. *Under the assumptions above, H is transversal to the Killing* *field Y at the interior points. Moreover, the family $(\phi_\theta(H))_{\theta\in[0,2\pi)}$ foliates $D\times[0,d]\setminus\{z\text{-axis}\}$.*

Proof. As H is a disk, we already know that each integral curve of Y intersects H in at least one point.

Observe that $\phi_{\pi/2}(R)\cap R\setminus\{z\text{-axis}\}=\emptyset$ and, in particular, $\phi_{\pi/2}(H)\cap H\setminus\{z\text{-axis}\}=\emptyset$. Moreover, notice that the tangent plane of $\phi_{\pi/2}(H)$ never coincides with the tangent plane of H along the z-axis; at each point of the z-axis the surfaces are in disjoint sectors. So as one decreases t from $\pi/2$ to 0, the surfaces $\phi_t(H)$ and H have only the z-axis in common and they are never tangent along the z-axis. More precisely, as t decreases, $t>0$, there can not be a first interior point of contact between the two surfaces by the maximum principle. Also there can not be a point on the z-axis which is a first point of tangency of the two surfaces for $t>0$, by the boundary maximum principle. Thus the surfaces $\phi_t(H)$ and H have only the z-axis in common for $t>0$. The same argument works for $-\pi/2\le t<0$. Therefore the surfaces $\phi_t(H)$ foliate $D\times[0,d]\setminus\{z\text{-axis}\}$, for $t\in[0,2\pi)$.

In particular, we have concluded that each integral curve of Y intersects H in exactly one point. Denote by R_2 the region in R bounded by H and $\mathcal{H}_2$, and denote by N the unit normal vector field of H pointing toward R_2. As each integral curve of Y intersects H in exactly one point, we have $\langle N,Y\rangle\ge 0$ on H. As $\langle N,Y\rangle$ is a Jacobi function on the minimal surface H, we conclude that necessarily $\langle N,Y\rangle>0$ in int$H$. Therefore, H is transversal to the Killing field Y at the interior points. □

Theorem 7 (A multi-valued Rado Theorem)**.** *Under the assumptions above, H is the unique compact minimal disk with boundary Γ.*

Proof. Set $\Gamma_\theta=\phi_\theta(\Gamma)$ and $H_\theta=\phi_\theta(H)$, so H_θ is a minimal disk with $\partial H_\theta=\Gamma_\theta$. By Proposition 8, the family $(H_\theta)_{\theta\in[0,2\pi)}$ gives a foliation of the region $D\times[0,d]\setminus\{z\text{-axis}\}$.

Let $M\neq H$ be another compact minimal disk with boundary Γ. We will analyse the intersection between M and each H_θ.

First, observe that $M\cap H_\theta\neq\emptyset$ for all θ and by the maximum principle $M\subset D\times[0,d]$.

Fix θ_0. Given $q\in H_{\theta_0}\cap M$, then either $q\in$ int$M$ or $q\in\Gamma=\partial M$.

Suppose $q\in$ int$M$.

If the intersection is transversal at q, then in a neighborhood of q we have that $H_{\theta_0}\cap M$ is a simple curve passing through q. If we let θ_0 vary a little, we see in M a foliation as in Figure 2.9 (a).

On the other hand, if M is tangent to H_{θ_0} at q, as the intersection of any two minimal surfaces is locally given by an n-prong singularity, that is,

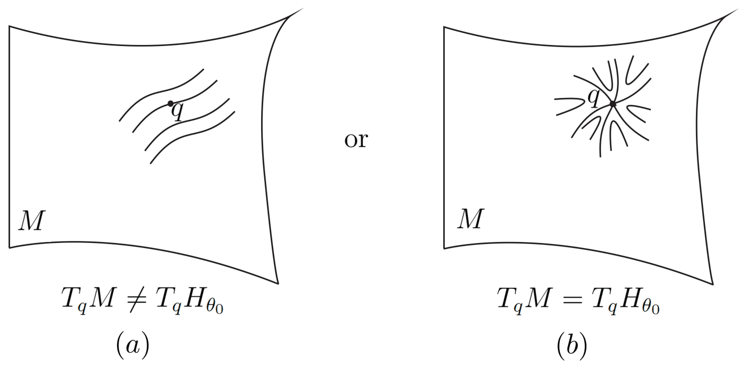

Figure 2.9: $q \in \mathrm{int}M$.

$2n$ embedded arcs which meet at equal angles (see Claim 1 of Lemma 4 in [23]), then in a neighborhood of q we have that $H_{\theta_0} \cap M$ consists of $2n$ curves passing through q and making equal angles at q. If we let θ_0 vary a little, we see in M a foliation as in Figure 2.9 (b).

Now suppose $q \in \Gamma$.

If $q \in \Gamma \cap \{z\text{-axis}\}$, to understand the trace of H_{θ_0} on M in a neighborhood of q we proceed as follows. Rotation by angle π of $\mathbb{H}^2 \times \mathbb{R}$ about the z-axis extends M smoothly to a minimal surface $\widetilde{M}$ that has q as an interior point. Each H_θ also extends by this rotation (giving a helicoid $\widetilde{H}_\theta$). So in a neighborhood of q, we understand the intersection of $\widetilde{M}$ and $\widetilde{H}_{\theta_0}$. The surfaces $\widetilde{\mathcal{M}}$ and $\widetilde{H}_{\theta_0}$ are either transverse or tangent at q as in Figure 2.9. Then when we restrict to $M \cap H_{\theta_0}$ and let θ_0 vary slightly, we see that the trace of H_{θ_0} on M near q is as in Figure 2.10, since the segment on the z-axis through q is in $M \cap H_{\theta_0}$.

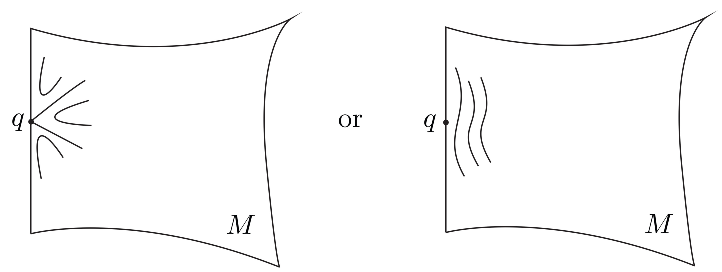

Figure 2.10: $q \in \Gamma \cap \{z\text{-axis}\}$.

On the other hand, if $q \in \Gamma \setminus \{z\text{-axis}\}$ then $\theta_0 = 0$, since $\Gamma_\theta \cap \Gamma \setminus \{z\text{-axis}\} = \emptyset$ for any $\theta \neq 0$. Note that we cannot have $M \cap H$ homeomorphic to a semicircle in a neighborhood of q, since this would imply that M is on one side of H at q and this contradicts the boundary maximum principle. Thus when we let $\theta_0 = 0$ vary a little, we have two possible foliations for M in a neighborhood of q as indicated in Figure 2.11.

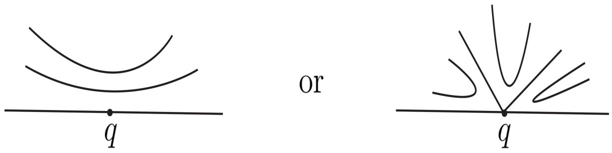

Figure 2.11: $q \in \Gamma \setminus \{z\text{-axis}\}$.

Now consider two copies of M and glue them together along the boundary.
Since M is a disk, when we glue these two copies of M we obtain a sphere with a foliation whose singularities have negative index by the analysis above. But this is impossible. Therefore, there is no other minimal disk with boundary Γ besides H.

□

Remark 3. This proof clearly works to prove Theorem 2 for slightly perturbed helicoids in $\mathbb{R}^3$.

Now let us construct an example of a complete embedded minimal surface in $\mathbb{H}^2 \times \mathbb{R}$ whose asymptotic boundary is a small perturbation of the asymptotic boundary of a complete helicoid.

Consider the (compact) helix $\beta(u) = (\cos u, \sin u, u)$ for $u \in [0, 4\pi]$. Notice that β is a multi-graph over $\partial_\infty \mathbb{H}^2$. Consider $\alpha(u)$ a perturbation of $\beta(u)$ such that α is transversal to $\partial_\infty \mathbb{H}^2 \times \{\tau\}$ for any $\tau \in [0, 4\pi]$, $\alpha(0) = \beta(0)$, $\alpha(4\pi) = \beta(4\pi)$ and so that the vertical distance between $\alpha(s)$ and $\alpha(s + 2\pi)$ is bigger than π for any $s \in [0, 2\pi]$.

Now for $t \in [0, 1]$, consider the curves $\alpha_t(u) = (1 - t)(0, 0, u) + t\alpha(u), u \in [0, 4\pi]$. Call Γ_t (respectively Γ_1) the Jordan curve which is the union of α_t (respectively α), the two horizontal geodesics joining the endpoints of α_t (respectively α) to the z-axis, and the part of the z-axis between $z = 0$ and $z = 4\pi$. Note that when t goes to 1, the curves Γ_t converge to the curve Γ_1. Denote by H_t the minimal disk with boundary Γ_t. By Theorem 7, H_t is stable and unique. In particular, we have uniform curvature estimates for points far from the boundary. As before, using rotation by angle π around horizontal geodesics, we can prove that there is uniform curvature estimates for H_t in a neighborhood of the two horizontal geodesic arcs of Γ_t.

As in the previous section, the envelope of the union of the translated surfaces S_π forms a barrier to the sequence H_t, hence we conclude that there exists a subsequence of H_t that converges to a minimal surface H_1 with boundary $\partial H_1 = \Gamma_1$. Rotation by angle π of $\mathbb{H}^2 \times \mathbb{R}$ around the z-axis extends H_1 smoothly to a minimal surface which has two horizontal (straight) geodesics in its boundary. Thus the surface obtained by reflection in all horizontal boundary geodesics of H_1 is a minimal surface whose asymptotic boundary is a small perturbation of the asymptotic boundary of the complete helicoid in $\mathbb{H}^2 \times \mathbb{R}$ which has β contained in its asymptotic boundary.

CHAPTER 3

Periodic minimal surfaces in semidirect products

In this chapter we prove existence of complete minimal surfaces in some metric semidirect products. These surfaces are similar to the doubly and singly periodic Scherk minimal surfaces in $\mathbb{R}^3$. In particular, we obtain these surfaces in the Heisenberg space with its canonical metric, and in Sol_3 with a one-parameter family of non-isometric metrics.

3.1 Introduction

In this chapter we construct examples of periodic minimal surfaces in some semidirect products $\mathbb{R}^2 \rtimes_\mathrm{A} \mathbb{R}$, depending on the matrix A. By periodic surface we mean a properly embedded surface invariant by a nontrivial discrete group of isometries.

One of the most simple examples of semidirect product is $\mathbb{H}^2 \times \mathbb{R} = \mathbb{R}^2 \rtimes_A \mathbb{R}$, when we take $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. In this space, Mazet, Rodríguez and Rosenberg [29] proved some results about periodic constant mean curvature surfaces and they constructed examples of such surfaces. One of their methods is to solve a Plateau problem for a certain contour. In [48], using a similar technique, Rosenberg constructed examples of complete minimal surfaces in $M^2 \times \mathbb{R}$, where M is either the two-sphere or a complete Riemannian surface with nonnegative curvature or the hyperbolic plane.

Meeks, Mira, Pérez and Ros [31] have proved results concerning the geometry of solutions to Plateau type problems in metric semidirect products

$\mathbb{R}^2 \rtimes_A \mathbb{R}$, when there is some geometric constraint on the boundary values of the solution (see Theorem 8).

The first example that we construct is a complete periodic minimal surface similar to the doubly periodic Scherk minimal surface in $\mathbb{R}^3$. It is invariant by two translations that commute and it is a four punctured sphere in the quotient of $\mathbb{R}^2 \rtimes_A \mathbb{R}$ by the group of isometries generated by the two translations. In the last section we obtain a complete periodic minimal surface analogous to the singly periodic Scherk minimal surface in $\mathbb{R}^3$.

These surfaces are obtained by solving the Plateau problem for a geodesic polygonal contour Γ (it uses a result by Meeks, Mira, Pérez and Ros [31] about the geometry of solutions to the Plateau problem in semidirect products), and letting some sides of Γ tend to infinity in length, so that the associated Plateau solutions all pass through a fixed compact region (this will be assured by the existence of minimal annuli playing the role of barriers). Then a subsequence of the Plateau solutions will converge to a minimal surface bounded by a geodesic polygon with edges of infinite length. We complete this surface by symmetry across the edges. The whole construction requires precise geometric control and uses curvature estimates for stable minimal surfaces.

These results are obtained for semidirect products $\mathbb{R}^2 \rtimes_A \mathbb{R}$ where $A = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$. For example, we obtain periodic minimal surfaces in the Heisenberg space, when $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, and in Sol_3, when $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, with their well known Riemannian metrics. When we consider the one-parameter family of matrices $A(c) = \begin{pmatrix} 0 & c \\ \frac{1}{c} & 0 \end{pmatrix}$, $c \geq 1$, we get a one-parameter family of metrics in Sol_3 which are not isometric.

3.2 Definitions and preliminary results

Generalizing direct products, a semidirect product is a particular way in which a group can be constructed from two subgroups, one of which is a normal subgroup. As a set, it is the cartesian product of the two subgroups but with a particular multiplication operation.

In our case, the normal subgroup is $\mathbb{R}^2$ and the other subgroup is $\mathbb{R}$. Given a matrix $A \in \mathcal{M}_2(\mathbb{R})$, we can consider the semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$, where the group operation is given by

$$(p_1, z_1) * (p_2, z_2) = (p_1 + \mathrm{e}^{z_1 A} p_2, z_1 + z_2),\ p_1, p_2 \in \mathbb{R}^2, z_1, z_2 \in \mathbb{R} \tag{3.1}$$

and

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}_2(\mathbb{R}).$$

We choose coordinates $(x, y) \in \mathbb{R}^2, z \in \mathbb{R}$. Then $\partial_x = \frac{\partial}{\partial x}, \partial_y, \partial_z$ is a parallelization of $G = \mathbb{R}^2 \rtimes_A \mathbb{R}$. Taking derivatives at $t = 0$ in (3.1) of the left multiplication by $(t, 0, 0) \in G$ (respectively by $(0, t, 0), (0, 0, t)$), we obtain the following basis $\{F_1, F_2, F_3\}$ of the right invariant vector fields on G:

$$F_1 = \partial_x,\ F_2 = \partial_y,\ F_3 = (ax + by)\partial_x + (cx + dy)\partial_y + \partial_z. \tag{3.2}$$

Analogously, if we take derivatives at $t = 0$ in (3.1) of the right multiplication by $(t, 0, 0) \in G$ (respectively by $(0, t, 0), (0, 0, t)$), we obtain the following basis $\{E_1, E_2, E_3\}$ of the Lie algebra of G:

$$E_1 = a_{11}(z)\partial_x + a_{21}(z)\partial_y,\ E_2 = a_{12}(z)\partial_x + a_{22}(z)\partial_y,\ E_3 = \partial_z, \tag{3.3}$$

where we have denoted

$$\mathrm{e}^{zA} = \begin{pmatrix} a_{11}(z) & a_{12}(z) \\ a_{21}(z) & a_{22}(z) \end{pmatrix}.$$

We define the *canonical left invariant metric* on $\mathbb{R}^2 \rtimes_A \mathbb{R}$, denoted by $\langle , \rangle$, to be that one for which the left invariant basis $\{E_1, E_2, E_3\}$ is orthonormal.

The expression of the Riemannian connection ∇ for the canonical left invariant metric of $\mathbb{R}^2 \rtimes_A \mathbb{R}$ in this frame is the following:

$$\begin{array}{lll} \nabla_{E_1} E_1 = aE_3 & \nabla_{E_1} E_2 = \frac{b+c}{2}E_3 & \nabla_{E_1} E_3 = -aE_1 - \frac{b+c}{2}E_2 \\ \nabla_{E_2} E_1 = \frac{b+c}{2}E_3 & \nabla_{E_2} E_2 = dE_3 & \nabla_{E_2} E_3 = -\frac{b+c}{2}E_1 - dE_2 \\ \nabla_{E_3} E_1 = \frac{c-b}{2}E_2 & \nabla_{E_3} E_2 = \frac{b-c}{2}E_1 & \nabla_{E_3} E_3 = 0. \end{array}$$

In particular, for every $(x_0, y_0) \in \mathbb{R}^2, \gamma(z) = (x_0, y_0, z)$ is a geodesic in G.

Remark 4. As $[E_1, E_2] = 0$, thus for all z, $\mathbb{R}^2 \rtimes_A \{z\}$ is flat and the horizontal straight lines are geodesics. Moreover, the mean curvature of $\mathbb{R}^2 \rtimes_A \{z\}$ with respect to the unit normal vector field E_3 is the constant $H = \mathrm{tr}(A)/2$.

The change from the orthonormal basis $\{E_1, E_2, E_3\}$ to the basis $\{\partial_x, \partial_y, \partial_z\}$ produces the following expression for the metric $\langle , \rangle$:

$$\begin{aligned} \langle , \rangle_{(x,y,z)} &= [a_{11}(-z)^2 + a_{21}(-z)^2]dx^2 + [a_{12}(-z)^2 + a_{22}(-z)^2]dy^2 + dz^2 \\ &\quad +[a_{11}(-z)a_{12}(-z) + a_{21}(-z)a_{22}(-z)](dx \otimes dy + dy \otimes dx) \\ &= \mathrm{e}^{-2\mathrm{tr}(A)z}\{[a_{21}(z)^2 + a_{22}(z)^2]dx^2 + [a_{11}(z)^2 + a_{12}(z)^2]dy^2\} + dz^2 \\ &\quad -\mathrm{e}^{-2\mathrm{tr}(A)z}[a_{11}(z)a_{21}(z) + a_{12}(z)a_{22}(z)](dx \otimes dy + dy \otimes dx). \end{aligned}$$

In particular, for every matrix $A \in \mathcal{M}_2(\mathbb{R})$, the rotation by angle π around the vertical geodesic $\gamma(z) = (x_0, y_0, z)$ given by the map $R(x, y, z) = (-x + 2x_0, -y + 2y_0, z)$ is an isometry of $(\mathbb{R}^2 \rtimes_{\mathrm{A}} \mathbb{R}, \langle , \rangle)$ into itself.

Remark 5. As we observed, the vertical lines of $\mathbb{R}^2 \rtimes_{\mathrm{A}} \mathbb{R}$ are geodesics of its canonical metric. For any line l in $\mathbb{R}^2 \rtimes_A \{0\}$ let P_l denote the vertical plane $\{(x, y, z) : (x, y, 0) \in l; z \in \mathbb{R}\}$ containing the set of vertical lines passing through l. It follows that P_l is ruled by vertical geodesics and, since rotation by angle π around any vertical line in P_l is an isometry that leaves P_l invariant, then P_l has zero mean curvature.

Although the rotation by angle π around horizontal geodesics is not always an isometry, we have the following result.

Proposition 9. *Let* $A = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \in \mathcal{M}_2(\mathbb{R})$ *and consider the horizontal geodesic* $\alpha = \{(x_0, t, 0) : t \in \mathbb{R}\}$ *in* $\mathbb{R}^2 \rtimes_A \{0\}$ *parallel to the* y*-axis. Then the rotation by angle* π *around* α *is an isometry of* $(\mathbb{R}^2 \rtimes_{\mathrm{A}} \mathbb{R}, \langle , \rangle)$ *into itself. The same result is true for a horizontal geodesic parallel to the* x*-axis.*

Proof. The rotation by angle π around α is given by the map $\phi(x, y, z) = (-x + 2x_0, y, -z)$, so $\phi_x = -\partial_x,\ \phi_y = \partial_y$ and $\phi_z = -\partial_z$.

If $A = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$, then

$$\mathrm{e}^{zA} = \begin{pmatrix} \displaystyle\sum_{k=0}^{\infty} \frac{(bc)^k z^{2k}}{(2k)!} & \displaystyle\sum_{k=1}^{\infty} \frac{b^k c^{k-1} z^{2k-1}}{(2k-1)!} \\ \displaystyle\sum_{k=1}^{\infty} \frac{c^k b^{k-1} z^{2k-1}}{(2k-1)!} & \displaystyle\sum_{k=0}^{\infty} \frac{(bc)^k z^{2k}}{(2k)!} \end{pmatrix}.$$

Hence, $a_{11}(z) = a_{22}(z)$ and $\mathrm{e}^{-zA} = \begin{pmatrix} a_{11}(z) & -a_{12}(z) \\ -a_{21}(z) & a_{11}(z) \end{pmatrix}$. Then

$$\begin{aligned} \langle , \rangle_{(x,y,z)} &= \{[a_{21}(z)^2 + a_{11}(z)^2]dx^2 + [a_{11}(z)^2 + a_{12}(z)^2]dy^2\} + dz^2 \\ &\quad -[a_{11}(z)a_{21}(z) + a_{12}(z)a_{11}(z)](dx \otimes dy + dy \otimes dx) \end{aligned}$$

and

$$\begin{aligned} \langle , \rangle_{\phi(x,y,z)} &= \{[a_{21}(z)^2 + a_{11}(z)^2]dx^2 + [a_{11}(z)^2 + a_{12}(z)^2]dy^2\} + dz^2 \\ &\quad +[a_{11}(z)a_{21}(z) + a_{12}(z)a_{11}(z)](dx \otimes dy + dy \otimes dx). \end{aligned}$$

Therefore, $\langle \phi_x, \phi_x \rangle_{\phi(x,y,z)} = \langle \partial_x, \partial_x \rangle_{(x,y,z)}$, $\langle \phi_y, \phi_y \rangle = \langle \partial_y, \partial_y \rangle$, $\langle \phi_z, \phi_z \rangle = \langle \partial_z, \partial_z \rangle$, that is, ϕ is an isometry. Analogously, we can show that the rotation by angle π around a horizontal geodesic parallel to the x-axis is also an isometry. $\square$

Remark 6. When the matrix A in $\mathbb{R}^2 \rtimes_{\mathrm{A}} \mathbb{R}$ is $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, we have the Heisenberg space and Sol_3, respectively, with their well known Riemannian metrics. When we consider the one-parameter family of matrices $A(c) = \begin{pmatrix} 0 & c \\ \frac{1}{c} & 0 \end{pmatrix}$, $c \geq 1$, we get a one-parameter family of metrics in Sol_3 which are not isometric. For more details, see [32].

Meeks, Mira, Pérez and Ros [31] have proved results concerning the geometry of solutions to Plateau type problems in metric semidirect products $\mathbb{R}^2 \rtimes_{\mathrm{A}} \mathbb{R}$, when there is some geometric constraint on the boundary values of the solution. More precisely, they proved the following theorem.

Theorem 8 (Meeks, Mira, Pérez and Ros, [31])**.** *Let $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$ be a metric semidirect product with its canonical metric and let $\Pi : \mathbb{R}^2 \rtimes_A \mathbb{R} \to \mathbb{R}^2 \rtimes_A \{0\}$ denote the projection $\Pi(x, y, z) = (x, y, 0)$. Suppose E is a compact convex disk in $\mathbb{R}^2 \rtimes_A \{0\}$, $C = \partial E$ and $\Gamma \subset \Pi^{-1}(C)$ is a continuous simple closed curve such that $\Pi : \Gamma \to C$ monotonically parametrizes C. Then,*

1. *Γ is the boundary of a compact embedded disk Σ of finite least area.*
2. *The interior of Σ is a smooth Π-graph over the interior of E.*

3.3 A doubly periodic Scherk minimal surface

Throughout this section, we consider the semidirect product $\mathbb{R}^2 \rtimes_{\mathrm{A}} \mathbb{R}$ with the canonical left invariant metric $\langle , \rangle$, where $A = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$. In this space, we prove the existence of a complete minimal surface analogous to Scherk's doubly periodic minimal surface in $\mathbb{R}^3$.

Fix $0 < c_0 < c_1$ and let a be a sufficiently small positive quantity such that

$$\begin{aligned} a \ &< \int_{c_0}^{c_1} \sqrt{a_{11}^2(z) + a_{21}^2(z)} + \sqrt{a_{11}^2(z) + a_{12}^2(z)} \mathrm{d}z \\ &\quad - \int_{c_0}^{c_1} \sqrt{(a_{11}(z) + a_{12}(z))^2 + (a_{11}(z) + a_{21}(z))^2} \mathrm{d}z. \end{aligned} \tag{3.4}$$

Observe that we can find such positive number a, since $|\partial_x| = \sqrt{a_{11}^2(z) + a_{21}^2(z)}$, $|\partial_y| = \sqrt{a_{11}^2(z) + a_{12}^2(z)}$ and $|\partial_x + \partial_y| = \sqrt{(a_{11}(z) + a_{12}(z))^2 + (a_{11}(z) + a_{21}(z))^2}$.

For each $c > 0$, consider the polygon P_c in $\mathbb{R}^2 \rtimes_A \mathbb{R}$ with the sides $\alpha_1, \alpha_2, \alpha_3^c, \alpha_4^c$ and α_5^c defined below.

$$
\begin{aligned}
\alpha_1 &= \{(t,0,0) : 0 \le t \le a\} \\
\alpha_2 &= \{(0,t,0) : 0 \le t \le a\} \\
\alpha_3^c &= \{(a,0,t) : 0 \le t \le c\} \\
\alpha_4^c &= \{(0,a,t) : 0 \le t \le c\} \\
\alpha_5^c &= \{(t,-t+a,c) : 0 \le t \le a\},
\end{aligned}
$$

as illustrated in Figure 3.1.

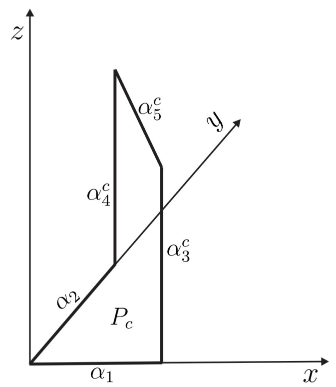

Figure 3.1: Polygon P_c.

We will denote $\alpha_1^0 = \{(t,0,0) : 0 \le t < a\}$, $\alpha_2^0 = \{(0,t,0) : 0 \le t < a\}$, $\alpha_3 = \{(a,0,t) : t > 0\}$ and $\alpha_4 = \{(0,a,t) : t > 0\}$, hence $P_\infty = \alpha_1^0 \cup \alpha_2^0 \cup \alpha_3 \cup \alpha_4 \cup \{(a,0,0),(0,a,0)\}$.

Let $\Pi : \mathbb{R}^2 \rtimes_A \mathbb{R} \to \mathbb{R}^2 \rtimes_A \{0\}$ denote the projection $\Pi(x,y,z) = (x,y,0)$. The next proposition is proved in Lemma 1.2 in [31], using the maximum principle and the fact that for every line $L \subset \mathbb{R}^2 \rtimes_A \{0\}$, the vertical plane $\Pi^{-1}(L)$ is a minimal surface.

Proposition 10. *Let E be a compact convex disk in $\mathbb{R}^2 \rtimes_A \{0\}$ with boundary $C = \partial E$ and let Σ be a compact minimal surface with boundary in $\Pi^{-1}(C)$. Then every point in* $\mathrm{int}\Sigma$ *is contained in* $\mathrm{int}\Pi^{-1}(E)$.

Observe that, for each $c > 0$, the polygon P_c is transverse to the Killing field $X = \partial_x + \partial_y$ and each integral curve of X intersects P_c in at most one point. From now on, denote by P the commom projection of every P_c over $\mathbb{R}^2 \rtimes_A \{0\}$, that is, $P = \Pi(P_c) = \Pi(P_d)$ for any $c, d \in \mathbb{R}$, and denote by E the disk in $\mathbb{R}^2 \rtimes_A \{0\}$ with boundary P. Let us denote by $\mathcal{R}$ the region $E \times \{z \geq 0\}$. Using Theorem 8, we conclude that P_c is the boundary of a compact embedded disk Σ_c of finite least area and the interior of Σ_c is a smooth Π-graph over the interior of E.

Let $\Omega_c = \{(t, -t + a, s) : 0 \leq t \leq a; 0 \leq s \leq c\}$.

Proposition 11. *If S is a compact minimal surface with boundary P_c, then $S = \Sigma_c$.*

Proof. By Proposition 10, $\mathrm{int}\Sigma_c$, $\mathrm{int}S \subset \mathrm{int}\Pi^{-1}(E)$, then, in particular, $\mathrm{int}\Sigma_c$, $\mathrm{int}S \subset \mathrm{int}\{\varphi_t(p) : t \in \mathbb{R}; p \in \Omega_c\}$, where φ_t is the flow of the Killing field X.

As S is compact, there exists t such that $\varphi_t(\Sigma_c) \cap S = \emptyset$. If $S \neq \Sigma_c$, then there exists $t_0 > 0$ such that $\varphi_{t_0}(\Sigma_c) \cap S \neq \emptyset$ and for $t > t_0, \varphi_t(\Sigma_c) \cap S = \emptyset$. Since for all $t \neq 0, \varphi_t(P_c) \cap S = \emptyset$, then the point of intersection is an interior point and, by the maximum principle, $\varphi_{t_0}(\Sigma_c) = S$. But that is a contradiction, since $t_0 \neq 0$. Therefore, $S = \Sigma_c$.

□

For each $n \in \mathbb{N}$, let Σ_n be the solution to the Plateau problem with boundary P_n. By Theorem 8 and Proposition 11, Σ_n is stable and unique. We are interested in proving the existence of a subsequence of Σ_n that converges to a complete minimal surface with boundary P_∞. In order to do that, we will use a minimal annulus as a barrier (whose existence is guaranteed by the Douglas criterion) to show that there exist points $p_n \in \Sigma_n, \Pi(p_n) = q \in \mathrm{int}E$ for all n, which converge to a point $p \in \mathbb{R}^2 \rtimes_A \mathbb{R}$, and then we will use Proposition 3.

Consider the parallelepiped with the faces A, B, C, D, E and F, defined below.

$$\begin{aligned}
A &= \{(u,-\epsilon,v) : \epsilon \le u \le a+\epsilon; c_0 \le v \le c_1\} \\
B &= \{(-\epsilon,u,v) : \epsilon \le u \le a+\epsilon; c_0 \le v \le c_1\} \\
C &= \{(u,-u,v) : -\epsilon \le u \le \epsilon; c_0 \le v \le c_1\} \\
D &= \{(u,-u+a,v) : -\epsilon \le u \le a+\epsilon; c_0 \le v \le c_1\} \\
E &= \{(u,-u+v,c_0) : -\epsilon \le u \le v+\epsilon; 0 \le v \le a\} \\
F &= \{(u,-u+v,c_1) : -\epsilon \le u \le v+\epsilon; 0 \le v \le a\},
\end{aligned}$$

where ϵ is a positive constant that we will choose later. Observe that each one of these faces is the least area minimal surface with its boundary. Let us analyse the area of each face.

1. In the plane $\{y = \text{constant}\}$ the induced metric is given by $g(x,z) = (a_{11}^2(z) + a_{21}^2(z))dx^2 + dz^2$. Hence,

$$\begin{aligned}
\text{area } A &= \int_{c_0}^{c_1}\int_{\epsilon}^{a+\epsilon} \sqrt{a_{11}^2(z) + a_{21}^2(z)}\mathrm{d}x\mathrm{d}z \\
&= a\int_{c_0}^{c_1} \sqrt{a_{11}^2(z) + a_{21}^2(z)}\mathrm{d}z.
\end{aligned}$$

2. In the plane $\{x = \text{constant}\}$ the induced metric is given by $g(y,z) = (a_{11}^2(z) + a_{12}^2(z))dy^2 + dz^2$. Hence,

$$\begin{aligned}
\text{area } B &= \int_{c_0}^{c_1}\int_{\epsilon}^{a+\epsilon} \sqrt{a_{11}^2(z) + a_{12}^2(z)}\mathrm{d}x\mathrm{d}z \\
&= a\int_{c_0}^{c_1} \sqrt{a_{11}^2(z) + a_{12}^2(z)}\mathrm{d}z.
\end{aligned}$$

3. The face C is contained in the plane parameterized by $\phi(u,v) = (u,-u,v)$ and the face D is contained in the plane parameterized by $\psi(u,v) = (u,-u+a,v)$. We have $\psi_u = \phi_u = \partial_x - \partial_y, \psi_v = \phi_v = \partial_z$. Then, $|\psi_u \wedge \psi_v| = |\phi_u \wedge \phi_v| = \sqrt{(a_{11}(z) + a_{12}(z))^2 + (a_{11}(z) + a_{21}(z))^2}$. Hence,

$$\begin{aligned}
\text{area } C &= \int_{c_0}^{c_1}\int_{-\epsilon}^{+\epsilon} \sqrt{(a_{11}(z) + a_{12}(z))^2 + (a_{11}(z) + a_{21}(z))^2}\mathrm{d}u\mathrm{d}v \\
&= 2\epsilon\int_{c_0}^{c_1} \sqrt{(a_{11}(z) + a_{12}(z))^2 + (a_{11}(z) + a_{21}(z))^2}\mathrm{d}z,
\end{aligned}$$

$$\begin{aligned} \text{area } D &= \int_{c_0}^{c_1}\int_{-\epsilon}^{a+\epsilon} \sqrt{(a_{11}(z)+a_{12}(z))^2+(a_{11}(z)+a_{21}(z))^2}\mathrm{d}u\mathrm{d}v \\ &= (a+2\epsilon)\int_{c_0}^{c_1} \sqrt{(a_{11}(z)+a_{12}(z))^2+(a_{11}(z)+a_{21}(z))^2}\mathrm{d}z. \end{aligned}$$

4. By Remark 4, the plane $\{z = \text{constant}\}$ is flat, and then the induced metric is the Euclidean metric. Hence,

$$\text{area } E = \text{area } F = \int_0^a \int_{-\epsilon}^{v+\epsilon} \mathrm{d}u\mathrm{d}v = \frac{a(a+4\epsilon)}{2}.$$

Therefore,

$$\text{area } C + \text{area } D + \text{area } E + \text{area } F < \text{area } A + \text{area } B$$

se, e somente se,

$$\begin{aligned} (a+4\epsilon)\left[a+\int_{c_0}^{c_1} \sqrt{(a_{11}+a_{12})^2+(a_{11}+a_{21})^2}\mathrm{d}z\right] &< a\int_{c_0}^{c_1}\sqrt{a_{11}^2+a_{21}^2}\mathrm{d}z \\ &+ a\int_{c_0}^{c_1}\sqrt{a_{11}^2+a_{12}^2}\mathrm{d}z \end{aligned}$$

se, e somente se,

$$\epsilon < \frac{a}{4}\frac{\displaystyle\int_{c_0}^{c_1}\sqrt{a_{11}^2+a_{21}^2}+\sqrt{a_{11}^2+a_{21}^2}\mathrm{d}z}{a+\displaystyle\int_{c_0}^{c_1}\sqrt{(a_{11}+a_{12})^2+(a_{11}+a_{21})^2}\mathrm{d}z} - \frac{a}{4}. \tag{3.5}$$

As we chose a satisfying (3.4), the factor in the right hand side of (3.5) is a positive number, then we can choose $\epsilon > 0$ such that the Douglas criterion is satisfied. Hence we obtain a minimal annulus $\mathcal{A}$ with boundary $\partial A \cup \partial B$ such that its projection $\Pi(\mathcal{A})$ contains points of int$E$, where E is the disk in $\mathbb{R} \rtimes_A \{0\}$ with boundary P. (See Figure 3.2).

As $\mathbb{R}^2 \rtimes_A \{z\}$ is a minimal surface, the maximum principle implies that, for each c, Σ_c is contained in the slab bounded by the planes $\{z = 0\}$ and $\{z = c\}$. Then for $c < c_0$, $\Sigma_c \cap \mathcal{A} = \emptyset$. As Σ_c is unique, Σ_c varies continuously with c, and when c increases the boundary $\partial\Sigma_c = P_c$ does not touch $\partial\mathcal{A}$. Therefore, using the maximum principle, $\Sigma_c \cap \mathcal{A} = \emptyset$ for all c, and Σ_c is under the annulus $\mathcal{A}$, which means that over any vertical line that intersects $\mathcal{A}$ and Σ_c, the points of Σ_c are under the points of $\mathcal{A}$.

Consider φ_t the flow of the Killing field $X = \partial_x + \partial_y$. Observe that $\{\varphi_t(\mathcal{A})\}_{t<0}$ forms a barrier for all points $p_n \in \Sigma_n$ such that $\Pi(p_n)$ is contained

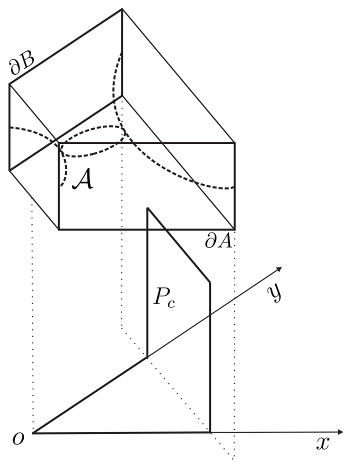

Figure 3.2: Annulus $\mathcal{A}$.

in a neighborhood $\mathcal{U} \subset E$ of the origin $o = (0, 0, 0)$. Moreover, for any $c_2 < c_3$ we can use the flow φ_t of the Killing field X and the maximum principle to conclude that Σ_{c_2} is under Σ_{c_3} in the same sense as before.

As, by Theorem 8, each Σ_c is a vertical graph in the interior, then $\Sigma_c \cap \Pi^{-1}(q)$ is only one point p_c, for every point $q \in \text{int}E$. Moreover, by the previous paragraph, the sequence $p_c = \Sigma_c \cap \Pi^{-1}(q)$ is monotone. Then, since we have a barrier, the sequence $\{p_n = \Sigma_n \cap \Pi^{-1}(q)\}$ converges to a point $p \in \Pi^{-1}(q)$, for all $q \in \mathcal{U}$.

In order to understand the convergence of the surfaces Σ_n we need to observe some properties of these surfaces.

First, notice that rotation by angle π around α_3, which we will denote by R_{α_3}, is an isometry. By the Schwarz reflection, we obtain a minimal surface $\widetilde{\Sigma}_n = \Sigma_n \cup R_{\alpha_3}(\Sigma_n)$ that has intα_3 in its interior. Note that the boundary of $\widetilde{\Sigma}_n$ is transverse to the Killing field $X = \partial_x + \partial_y$, and if φ_t denotes the flow of X, we have that $\varphi_t(\partial\widetilde{\Sigma}_n) \cap \widetilde{\Sigma}_n = \emptyset$ for all $t \neq 0$, hence, using the same arguments of the proof of Proposition 11, we can show that the minimal surface $\widetilde{\Sigma}_n$ is the unique minimal surface with its boundary. In particular, it is area-minimizing, and then it is stable. Hence, by Theorem 3, we have uniform curvature estimates for points far from the boundary of $\widetilde{\Sigma}_n$. In particular, we get uniform curvature estimates for Σ_n in a neighborhood of α_3. Analogously, we have uniform curvature estimates for Σ_n in a neighborhood of α_4.

Hence, for every compact contained in $\{z > 0\} \cap \mathcal{R}$, there exists a subsequence of Σ_n that converges to a minimal surface. Taking an exhaustion by compact sets and using a diagonal process, we conclude that there exists a subsequence of Σ_n that converges to a minimal surface Σ that has $\alpha_3 \cup \alpha_4$ in its boundary. From now on, we will use the notation Σ_n for this subsequence.

It remains to prove that in fact Σ is a minimal surface with boundary P_∞. In order to do that, we will use the fact that the interior of each Σ_n is a vertical graph over the interior of E. Let us denote by u_n the function defined in int$E$ such that $\Sigma_n = \text{Graph}(u_n)$. We already know that $u_{n-1} < u_n$ in int$E$ for all n.

Claim 1. *There are uniform gradient estimates for $\{u_n\}$ for points in $\alpha_1^0 \cup \alpha_2^0$.*

Proof. For $x_0 < 0$ and $\delta > 0$ consider the vertical strip bounded by $\beta_1 = \{(x_0, y, c_1) : -\delta \leq y \leq 0\}$, $\beta_2 = \{(x_0, t, -\frac{c_1}{a}t + c_1) : 0 \leq t \leq a\}$, $\beta_3 = \{(x_0, t-\delta, -\frac{c_1}{a}t + c_1) : 0 \leq t \leq a\}$ and $\beta_4 = \{(x_0, y, 0) : a - \delta \leq y \leq a\}$. This is a minimal surface transversal to the Killing field ∂_x, hence any small perturbation of its boundary gives a minimal surface with that perturbed boundary. Thus, if we consider a small perturbation of the boundary of this vertical strip by perturbing slightly β_1 by a curve contained in $\{x \geq x_0\}$ joining the points $(x_0, -\delta, c_1)$ and $(x_0, 0, c_1)$, we will get a minimal surface S with this perturbed boundary. This minimal surface S will have the property that the tangent planes at the interior of β_4 are not vertical, by the maximum principle with boundary.

Applying translations along the x-axis and y-axis, we can use the translates of S to show that Σ_n is under S in a neighborhood of α_2^0, and then we have uniform gradient estimates for points in α_2^0. Analogously, constructing similar barriers, we can prove that we have uniform gradient estimates in a neighborhood of α_1^0. □

Observe that besides the gradient estimates, the translates of the minimal surface S form a barrier for points in a neighborhood of $\alpha_1^0 \cup \alpha_2^0$.

We have that Σ_n is a monotone increasing sequence of minimal graphs with uniform gradient estimates in $\alpha_1^0 \cup \alpha_2^0$, and it is a bounded graph for points in a neighborhood $\mathcal{U}$ of the origin (because of the barrier given by the annulus $\mathcal{A}$). Therefore, there exists a subsequence of Σ_n that converges to a minimal surface $\widetilde{\Sigma}$ with $\alpha_1^0 \cup \alpha_2^0$ in its boundary. As we already know that Σ_n converges to the minimal surface Σ, we conclude that in fact $\Sigma = \widetilde{\Sigma}$, and then Σ is a minimal surface with $\alpha_1^0 \cup \alpha_2^0 \cup \alpha_3 \cup \alpha_4$ in its boundary. Notice that we can assume that Σ has P_∞ as its boundary, with Σ being of class C^1 up to $P_\infty \setminus \{(a,0,0),(0,a,0)\}$ and continuous up to P_∞.

Now considering the rotation by angle π around α_1 of Σ, we obtain the surface illustrated in Figure 3.3.

Continuing to rotate by angle π around the horizontal line in $\mathbb{R}^2 \rtimes_A \{0\}$, the resulting surface will be a minimal surface with four vertical lines as its

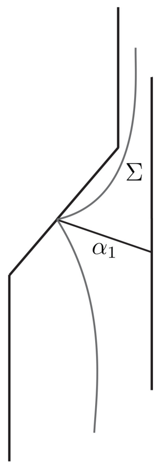

Figure 3.3: Rotation by angle π around α_1 of Σ.

boundary: $\{(a,0,t) : t \in \mathbb{R}\}, \{(0,a,t) : t \in \mathbb{R}\}, \{(-a,0,t) : t \in \mathbb{R}\}, \{(0,-a,t) : t \in \mathbb{R}\}$.

Now we can use the rotations by angle π around the vertical lines to get a complete minimal surface that is analogous to the doubly periodic minimal Scherk surface in $\mathbb{R}^3$. It is invariant by two translations that commute and it is a four punctured sphere in the quotient of $\mathbb{R}^2 \rtimes_{\mathrm{A}} \mathbb{R}$ by the group of isometries generated by the two translations.

Theorem 9. *In any semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$, where $A = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$, there exists a periodic minimal surface similar to the doubly periodic Scherk minimal surface in $\mathbb{R}^3$.*

3.4 A singly periodic Scherk minimal surface

Throughout this section, we consider the semidirect product $\mathbb{R}^2 \rtimes_{\mathrm{A}} \mathbb{R}$ with the canonical left invariant metric $\langle , \rangle$, where $A = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$. In this space, we construct a complete minimal surface similar to the singly periodic Scherk minimal surface in $\mathbb{R}^3$.

Fix $c_0 > 0$ and take $0 < \epsilon < a$ sufficiently smalls so that

$$a + 2\epsilon < \int_0^{c_0} \sqrt{a_{11}^2(z) + a_{21}^2(z)}dz.$$

For each $c > 0$, consider the polygon P_c in $\mathbb{R}^2 \rtimes_A \mathbb{R}$ with the six sides defined below.

$$\begin{aligned} \alpha_1^c &= \{(t,0,0) : 0 \leq t \leq c\} \\ \alpha_2^c &= \{(c,t,0) : 0 \leq t \leq a\} \\ \alpha_3^c &= \{(t,a,0) : 0 \leq t \leq c\} \\ \alpha_4^c &= \{(0,a,t) : 0 \leq t \leq c\} \\ \alpha_5^c &= \{(0,t,c) : 0 \leq t \leq a\} \\ \alpha_6^c &= \{(0,0,t) : 0 \leq t \leq c\}, \end{aligned}$$

and for each $\delta > 0$ with $\delta < a/2$, consider the polygon P_c^δ with the following six sides.

$$\begin{aligned} \alpha_1^{\delta,c} &= \left\{(t, \tfrac{\delta}{c}t, 0) : 0 \leq t \leq c\right\} \\ \alpha_2^{\delta,c} &= \{(c,t,0) : \delta \leq t \leq a-\delta\} \\ \alpha_3^{\delta,c} &= \left\{(t, \tfrac{ac-\delta t}{c}, 0) : 0 \leq t \leq c\right\}, \end{aligned}$$

$\alpha_4^c, \alpha_5^c, \alpha_6^c$, as illustrated in Figure 3.4.

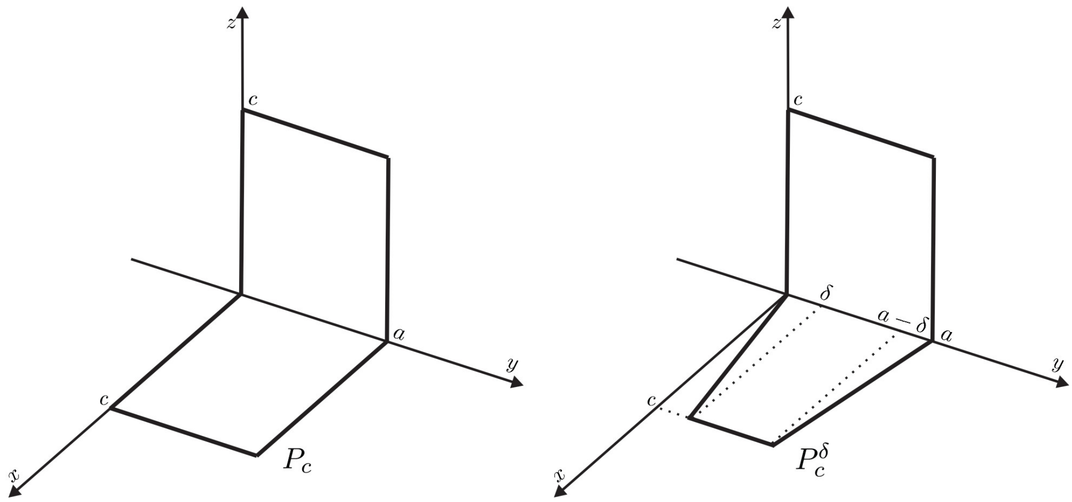

Figure 3.4: Polygons P_c and P_c^δ.

Denote by $\Omega(\delta, c)$ the region in $\mathbb{R}^2 \rtimes_A \{0\}$ bounded by $\alpha_1^{\delta,c}, \alpha_2^{\delta,c}, \alpha_3^{\delta,c}$ and the segment $\{(0,t,0) : 0 \leq t \leq a\}$. For each c and δ, we have compact minimal surfaces Σ_c and Σ_c^δ with boundary P_c and P_c^δ, respectively, which are solutions to the Plateau problem. By Theorem 8, we know that Σ_c

and Σ_c^δ are stable and smooth Π-graphs over the interior of $\Omega(0,c), \Omega(\delta,c)$, respectively. We will show that Σ_c is the unique compact minimal surface with boundary P_c.

Fix c. For each $0<\delta<a/2$, P_c^δ is a polygon transverse to the Killing field ∂_x and each integral curve of ∂_x intersects P_c^δ in at most one point. Thus we can prove, as we did in Proposition 11, that Σ_c^δ is the unique compact minimal surface with boundary P_c^δ.

Denote by u_c^δ, v_c the functions defined in the interior of $\Omega(\delta,c), \Omega(0,c)$, whose Π-graphs are $\Sigma_c^\delta, \Sigma_c$, respectively. Then, as ∂_x is a Killing field and each P_c^δ is transversal to ∂_x, we can use the flow of ∂_x and the maximum principle to prove that for $\delta'<\delta$ we have $0 \le u_c^\delta \le u_c^{\delta'} \le v_c$ in $\mathrm{int}\Omega(\delta,c)$, hence v_c is a barrier for our sequence u_c^δ. Because of the monotonicity and the barrier, the family u_c^δ converges to a function u_c defined in $\mathrm{int}\Omega(0,c)$ whose graph is a compact minimal surface with boundary P_c, and we still have $u_c \le v_c$ on $\Omega(0,c)$.

Now we will find another compact minimal surface with boundary P_c, whose interior is the graph of a function w_c defined in $\mathrm{int}\Omega(0,c)$ such that $v_c \le w_c$ and we will show that $u_c = w_c$. In order to do that, for each $0<\delta<a/2$, consider the polygon $\widetilde{P}_c^\delta$ with the six sides defined below.

$$\begin{aligned}
\widetilde{\alpha}_1^{\delta,c} &= \{(t, \tfrac{\delta t-\delta c}{c}, 0) : 0 \le t \le c\}\\
\alpha_2^c &= \{(c,t,0) : 0 \le t \le a\}\\
\widetilde{\alpha}_3^{\delta,c} &= \{(t, \tfrac{(a+\delta)c-\delta t}{c}, 0) : 0 \le t \le c\}\\
\widetilde{\alpha}_4^{\delta,c} &= \{(0, a+\delta, t) : 0 \le t \le c\}\\
\widetilde{\alpha}_5^{\delta,c} &= \{(0,t,c) : -\delta \le t \le a+\delta\}\\
\widetilde{\alpha}_6^{\delta,c} &= \{(0,-\delta,t) : 0 \le t \le c\}.
\end{aligned}$$

Denote by $\widetilde{\Omega}(\delta,c)$ the region in $\mathbb{R}^2 \rtimes_A \{0\}$ bounded by $\widetilde{\alpha}_1^{\delta,c}$, α_2^c, $\widetilde{\alpha}_3^{\delta,c}$ and the segment $\{(0,t,0) : -\delta \le t \le a+\delta\}$. For each δ, we have a compact minimal disk $\widetilde{\Sigma}_c^\delta$ with boundary $\widetilde{P}_c^\delta$ and the interior of $\widetilde{\Sigma}_c^\delta$ is a smooth Π-graph over the interior of $\widetilde{\Omega}(\delta,c)$. As $\widetilde{P}_c^\delta$ is transversal to the Killing field ∂_x, we can prove that $\widetilde{\Sigma}_c^\delta$ is the unique compact minimal surface with boundary $\widetilde{P}_c^\delta$.

Denote by w_c^δ the function defined in $\mathrm{int}\widetilde{\Omega}(\delta,c)$ whose graph is $\widetilde{\Sigma}_c^\delta$. Using the flow of ∂_x and the maximum principle, we can prove that for $\delta'<\delta$ we have $w_c^{\delta'} \le w_c^\delta$ in $\mathrm{int}\widetilde{\Omega}(\delta',c)$ and for all δ, $v_c \le w_c^\delta$ in $\mathrm{int}\Omega(0,c)$. Because of

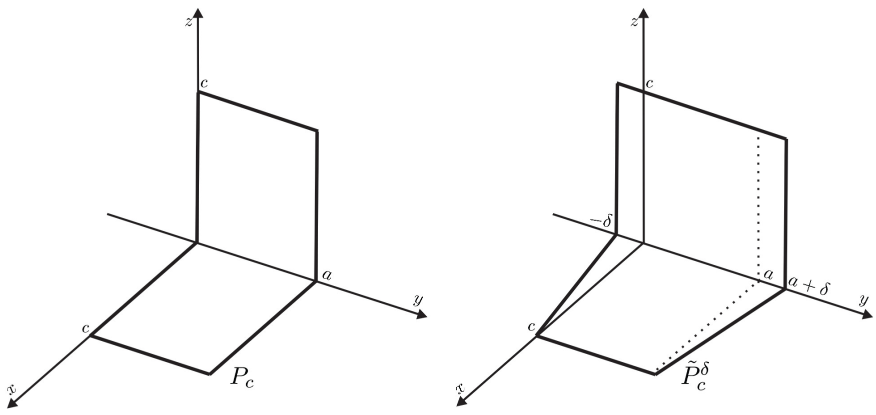

Figure 3.5: Polygons P_c and $\widetilde{P}_c^\delta$.

the monotonicity and the barrier, the family w_c^δ converges to a function w_c defined in $\mathrm{int}\widetilde{\Omega}(0,c) = \mathrm{int}\Omega(0,c)$ whose graph is a compact minimal surface with boundary P_c, and we still have $v_c \leq w_c$ in $\mathrm{int}\Omega(0,c)$.

Let us call Σ_1, Σ_2 the graphs of u_c, w_c, respectively. We will now prove that $\Sigma_1 = \Sigma_2$. Denote by ν_i the conormal to Σ_i along P_c, $i = 1, 2$. (See Figure 3.6).

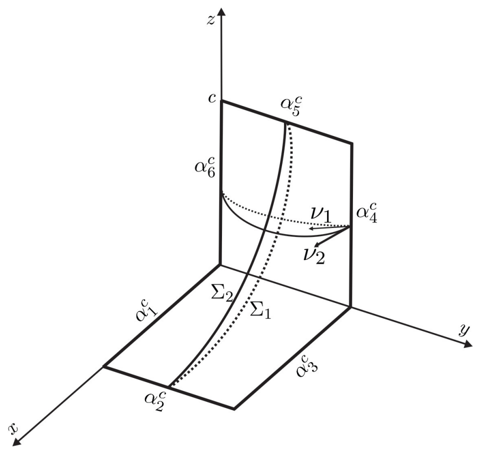

Figure 3.6: Σ_1 and Σ_2.

Suppose that $u_c \neq w_c$, then in fact we have $u_c < w_c$ in $\mathrm{int}\Omega(0,c)$. As ∂_x is tangent to α_1^c and α_3^c, then $\langle \nu_i, \partial_x \rangle = 0$, $i = 1, 2$, in α_1^c and α_3^c. In the other sides of P_c we have $\langle \nu_1, \partial_x \rangle < \langle \nu_2, \partial_x \rangle$. Therefore,

$$\int_{P_c} \langle \nu_1, \partial_x \rangle < \int_{P_c} \langle \nu_2, \partial_x \rangle .$$

But, using the Flux Formula for Σ_1 and Σ_2 with respect to the Killing field ∂_x, we have

$$\int_{P_c} \langle \nu_1, \partial_x \rangle = 0 = \int_{P_c} \langle \nu_2, \partial_x \rangle .$$

Then, $u_c = w_c$ and therefore, $\Sigma_c = \Sigma_1 = \Sigma_2$. In particular, Σ_c is the unique compact minimal surface with boundary P_c.

Denote by $\Omega(\infty)$ the infinite strip $\{(x, y, 0) : x \geq 0, 0 \leq y \leq a\}$, and by $\mathcal{R}$ the region $\{(x, y, z) : x \geq 0, 0 \leq y \leq a, z \geq 0\}$. Moreover, denote $\alpha_1 = \{(x, 0, 0) : x > 0\}$, $\alpha_3 = \{(x, a, 0) : x > 0\}$, $\alpha_4 = \{(0, a, z) : z > 0\}$ and $\alpha_6 = \{(0, 0, z) : z > 0\}$, hence $P_\infty = \alpha_1 \cup \alpha_3 \cup \alpha_4 \cup \alpha_6 \cup \{(0, 0, 0), (0, a, 0)\}$.

For each $n \in \mathbb{N}$, let Σ_n be the unique compact minimal surface with boundary P_n. We are interested in proving the existence of a subsequence of Σ_n that converges to a complete minimal surface with boundary P_∞. Using the existence of a minimal annulus, guaranteed by the Douglas criterion, we will show that there exist points $p_n \in \Sigma_n,\ \Pi(p_n) = q \in \operatorname{int} \Omega(\infty)$ for all n, which converge to a point $p \in \mathbb{R}^2 \rtimes_A \mathbb{R}$, and then we will use Proposition 3.

Consider the parallelepiped with faces A, B, C, D, E and F, defined below.

$$\begin{aligned}
A &= \{(u, -\epsilon, v) : \epsilon \leq u \leq d; 0 \leq v \leq c_0\} \\
B &= \{(u, a + \epsilon, v) : \epsilon \leq u \leq d; 0 \leq v \leq c_0\} \\
C &= \{(u, v, 0) : \epsilon \leq u \leq d; -\epsilon \leq v \leq a + \epsilon\} \\
D &= \{(u, v, c_0) : \epsilon \leq u \leq d; -\epsilon \leq v \leq a + \epsilon\} \\
E &= \{(\epsilon, u, v) : -\epsilon \leq u \leq a + \epsilon; 0 \leq v \leq c_0\} \\
F &= \{(d, u, v) : -\epsilon \leq u \leq a + \epsilon; 0 \leq v \leq c_0\},
\end{aligned}$$

where $d > \epsilon$ is a constant that we will choose later.

As we did in the last section, we can calculate the area of each one of these faces and we obtain:

$$\begin{aligned}
\text{area } A = \text{area } B &= (d - \epsilon) \int_0^{c_0} \sqrt{a_{11}^2(z) + a_{21}^2(z)} \mathrm{d}z, \\
\text{area } C = \text{area } D &= (d - \epsilon)(a + 2\epsilon), \\
\text{area } E = \text{area } F &= (a + 2\epsilon) \int_0^{c_0} \sqrt{a_{11}^2(z) + a_{12}^2(z)} \mathrm{d}z.
\end{aligned}$$

Hence,

$$\text{area } C + \text{area } D + \text{area } E + \text{area } F < \text{area } A + \text{area } B$$

se, e somente se,

$$(d-\epsilon)(a+2\epsilon) + (a+2\epsilon)\int_0^{c_0} \sqrt{a_{11}^2 + a_{12}^2}\mathrm{d}z < (d-\epsilon)\int_0^{c_0} \sqrt{a_{11}^2 + a_{21}^2}\mathrm{d}z$$

se, e somente se,

$$(d-\epsilon)\left[(a+2\epsilon) - \int_0^{c_0} \sqrt{a_{11}^2 + a_{21}^2}\mathrm{d}z\right] < -(a+2\epsilon)\int_0^{c_0} \sqrt{a_{11}^2 + a_{12}^2}\mathrm{d}z$$

se, e somente se,

$$d > \epsilon - \frac{(a+2\epsilon)\displaystyle\int_0^{c_0} \sqrt{a_{11}^2(z) + a_{12}^2(z)}\mathrm{d}z}{(a+2\epsilon) - \displaystyle\int_0^{c_0} \sqrt{a_{11}^2(z) + a_{21}^2(z)}\mathrm{d}z}.$$

As we chose $a + 2\epsilon < \int_0^{c_0} \sqrt{a_{11}^2(z) + a_{21}^2(z)}\mathrm{d}z$, we can choose $d > \epsilon$ so that the Douglas criterion is satisfied. Thus, there exists a minimal annulus $\mathcal{A}$ with boundary $\partial A \cup \partial B$ such that its projection $\Pi(\mathcal{A})$ contains points of $\mathrm{int}\Omega(\infty)$. (See Figure 3.7).

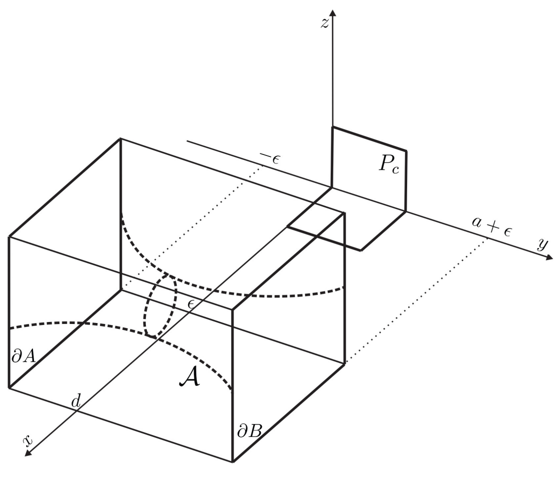

Figure 3.7: Annulus $\mathcal{A}$.

We know that, for each $c < \epsilon$, $\Sigma_c \cap \mathcal{A} = \emptyset$. When c increases P_c does not intersect $\partial\mathcal{A}$, then, using the maximum principle, $\Sigma_c \cap \mathcal{A} = \emptyset$ for all c, and

Σ_c is under the annulus $\mathcal{A}$. Thus, there exists a point $q \in \text{int}\Omega(\infty)$ such that $p_n = \Sigma_n \cap \Pi^{-1}(q)$ has a subsequence that converges to a point $p \in \Pi^{-1}(q)$. Observe that applying the flow of the Killing field ∂_x to the annulus $\mathcal{A}$ we can conclude that, in the region $\{x \geq d\}$, the surfaces Σ_n are bounded above by, for example, the plane $\{z = c_0\}$.

In order to understand the convergence of the surfaces Σ_n we need to prove some properties of these surfaces.

Claim 2. *The surfaces Σ_n are transversal to the Killing field ∂_x in the interior.*

Proof. Fix n. Suppose that at some point $p \in \text{int}\Sigma_n$ the tangent plane $T_p\Sigma_n$ contains the vector ∂_x. As the planes that contain the direction ∂_x are minimal surfaces, we have that Σ_n and $T_p\Sigma_n$ are minimal surfaces tangent at p, and then the intersection between them is formed by $2k$ curves, $k \geq 1$, passing through p making equal angles at p. By the shape of P_n (the boundary of Σ_n), we know that $T_p\Sigma_n$ intersects P_n either in only two points or in one point and a segment of straight line (α_1^n or α_3^n). Therefore, we will have necessarily a closed curve contained in the intersection. As Σ_n is simply connected this curve bounds a disk in Σ_n, but as the parallel planes to $T_p\Sigma_n$ are minimal surfaces, we can use the maximum principle to prove that this disk is contained in the plane $T_p\Sigma_n$ and then they coincide, which is impossible. Thus, the vector ∂_x is transversal to Σ at points $p \in \text{int}\Sigma_n$. □

Observe that, besides the interior points, the surfaces Σ_n are also transversal to ∂_x at the points in α_4 and α_6, by the maximum principle with boundary. Thus rotation by angle π around α_4 (respectively α_6) gives a minimal surface which is also transversal to the Killing field ∂_x in the interior, extends the surface Σ_n and has α_4^n (respectively α_6^n) in the interior. Therefore, we have uniform curvature estimates for Σ_n up to $\alpha_4 \cup \alpha_6$.

Hence, for every compact contained in $\{z > 0\} \cap \mathcal{R}$, there exists a subsequence of Σ_n that converges to a minimal surface. Taking an exhaustion by compact sets and using a diagonal process, we conclude that there exists a subsequence of Σ_n that converges to a minimal surface Σ that has $\alpha_4 \cup \alpha_6$ in its boundary. From now on we will use the notation Σ_n for this subsequence.

It remains to prove that in fact Σ is a minimal surface with boundary P_∞. In order to do that, we will use the fact that each Σ_n is a vertical graph in the interior. Let us denote by u_n the function defined in $\text{int}\Omega(n)$ such that $\Sigma_n = \text{Graph}(u_n)$, where $\Omega(n) = \{(x, y, 0) : 0 \leq x \leq n; 0 \leq y \leq a\}$.

Claim 3. *$u_{n-1} < u_n$ in $\text{int}\Omega(n-1)$.*

Proof. Recall that each Σ_n is the limit of a sequence of graphs $\widetilde{\Sigma}_n^\delta = \mathrm{Graph}(w_n^\delta)$, whose boundary is transversal to ∂_x. Using the flow of the Killing field ∂_x, we can prove that each $\widetilde{\Sigma}_n^\delta$ is above Σ_{n-1}, and then the limit surface Σ_n has to be above Σ_{n-1}. In fact, Σ_n is strictly above Σ_{n-1} in the interior, because as Σ_n and Σ_{n-1} are minimal surfaces, if they intersect at an interior point, there will be points of Σ_n under Σ_{n-1}, and we already know that, by the property of $\widetilde{\Sigma}_n^\delta$, this is not possible. □

Claim 4. *There are uniform gradient estimates for $\{u_n\}$ for points in $\alpha_1 \cup \alpha_3$.*

Proof. We will use the same idea as in Claim 1. For $y_0 > a$ and $\delta > 0$ consider the vertical strip bounded by $\beta_1 = \{(x, y_0, c_0) : d \leq x \leq d + \delta\}$, $\beta_2 = \{(t, y_0, \frac{c_0}{d} t) : 0 \leq t \leq d\}$, $\beta_3 = \{(t + \delta, y_0, \frac{c_0}{d} t) : 0 \leq t \leq d\}$ and $\beta_4 = \{(x, y_0, 0) : 0 \leq x \leq \delta\}$. This is a minimal surface transversal to the Killing field ∂_y, hence any small perturbation of its boundary gives a minimal surface with that perturbed boundary. Thus, if we consider a small perturbation of the boundary of this vertical strip by perturbing slightly β_1 by a curve contained in $\{y \leq y_0\}$ joining the points (d, y_0, c_0) and $(d + \delta, y_0, c_0)$, we will get a minimal surface S with this perturbed boundary. This minimal surface S will have the property that the tangent planes at the interior points of β_4 are not vertical, by the maximum principle with boundary.

Applying translations along the x-axis and y-axis, we can use the translates of S to show that Σ_n is under S in a neighborhood of α_3, and then we have uniform gradient estimates for points in α_3. Analogously, we can prove that we have uniform gradient estimates in a neighborhood of α_1. □

Observe that besides the gradient estimates, the translates of the minimal surface S form a barrier for points in a neighborhood of $\alpha_1 \cup \alpha_3$.

We have that Σ_n is a monotone increasing sequence of minimal graphs with uniform gradient estimates in $\alpha_1 \cup \alpha_3$, and it is a bounded graph for points in $\{x \geq d\}$ (because of the barrier given by the annulus $\mathcal{A}$). Therefore, there exists a subsequence of Σ_n that converges to a minimal surface $\widetilde{\Sigma}$ with $\alpha_1 \cup \alpha_3$ in its boundary. As we already know that Σ_n converges to the minimal surface Σ, we conclude that in fact $\Sigma = \widetilde{\Sigma}$, and then Σ is a minimal surface with $\alpha_1 \cup \alpha_3 \cup \alpha_4 \cup \alpha_6$ in its boundary. Notice that we can assume that Σ has P_∞ as its boundary, with Σ being of class C^1 up to $P_\infty \backslash \{(0,0,0), (0,a,0)\}$ and continuous up to P_∞. The expected "singly periodic Scherk minimal surface" is obtained by rotating recursively Σ by an angle π about the vertical and horizontal geodesics in its boundary.

Theorem 10. *In any semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$, where $A = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$,* *there exists a periodic minimal surface similar to the singly periodic Scherk minimal surface in* $\mathbb{R}^3$.

CHAPTER 4

A half-space theorem for ideal Scherk graphs in $M \times \mathbb{R}$

In this chapter we prove a half-space theorem for an ideal Scherk graph $\Sigma \subset M \times \mathbb{R}$ over a polygonal domain $D \subset M$, where M is a Hadamard surface with bounded curvature. More precisely, we show that a properly immersed minimal surface contained in $D \times \mathbb{R}$ and disjoint from Σ is a translate of Σ.

4.1 Introduction

A well known result in the global theory for proper minimal surfaces in the Euclidean 3-space is the *half-space theorem* by Hoffman and Meeks [24], which says that if a properly immersed minimal surface S in $\mathbb{R}^3$ lies on one side of some plane P, then S is a plane parallel to P. Moreover, they also proved the *strong half-space theorem*, which says that two properly immersed minimal surfaces in $\mathbb{R}^3$ that do not intersect must be parallel planes.

This problem of giving conditions which force two minimal surfaces of a Riemannian manifold to intersect has received considerable attention, and many people have worked on this subject.

Let us observe that there is no half-space theorem in Euclidean spaces of dimensions bigger than 4, since there exist rotational proper minimal hypersurfaces contained in a slab.

Similarly, there exists no half-space theorem for horizontal slices in $\mathbb{H}^2 \times \mathbb{R}$, since rotational minimal surfaces (catenoids) are contained in a slab [41, 42]. However there are half-space theorems for constant mean curvature (CMC) $\frac{1}{2}$ surfaces in $\mathbb{H}^2 \times \mathbb{R}$ [21, 43]. For instance, Hauswirth, Rosenberg and Spruck

[21] proved that if S is a properly immersed CMC $\frac{1}{2}$ surface in $\mathbb{H}^2 \times \mathbb{R}$, contained on the mean convex side of a horocylinder C, then S is a horocylinder parallel to C; and if S is embedded and contains a horocylinder C on its mean convex side, then S is a horocylinder parallel to C. And in [43] Nelli and Sa Earp showed that the mean convex side of a simply connected rotational CMC $\frac{1}{2}$ surface can not contain a complete CMC $\frac{1}{2}$ surface besides the rotational simply connected ones.

Other examples of homogeneous manifolds where there are half-space theorems for minimal surfaces are Nil_3 and Sol_3 [1, 5, 6]. For instance, we know that if a properly immersed minimal surface S in Nil_3 lies on one side of some entire minimal graph Σ, then S is the image of Σ by a vertical translation.

In [28], Mazet proved a general half-space theorem for constant mean curvature surfaces. Under certain hypothesis, he proved that in a Riemannian 3-manifold of bounded geometry, a constant mean curvature H surface on one side of a parabolic constant mean curvature H surface Σ is an equidistant surface to Σ.

Here we consider the half-space problem for an ideal Scherk graph Σ over a polygonal domain $D \subset M$, where M denotes a Hadamard surface with bounded curvature, that is, M is a complete simply connected Riemannian surface with curvature $-b^2 \leq K_M \leq -a^2 < 0$, for some constants $a, b \in \mathbb{R}$. More precisely, we prove the following result.

Theorem 11. *Let M denote a Hadamard surface with bounded curvature and let $\Sigma = Graph(u)$ be an ideal Scherk graph over an admissible polygonal domain $D \subset M$. If S is a properly immersed minimal surface contained in $D \times \mathbb{R}$ and disjoint from Σ, then S is a translate of Σ.*

We remark that Mazet's theorem does not apply in our case for Scherk surfaces. In fact, one of his hypothesis is that the equidistant surfaces have mean curvature pointing away from the original surface. However, an end of a Scherk surface is asymptotic to some vertical plane $\gamma \times \mathbb{R}$, where γ is a geodesic, so the equidistant surface is asymptotic to $\gamma_s \times \mathbb{R}$, where γ_s is an equidistant curve to γ. Hence, in the case of a Scherk surface, the mean curvature vector of an equidistant surface points toward the Scherk surface.

4.2 Definitions and preliminary results

In this section we present some basic properties of Hadamard manifolds and state some previous results. For more details, see [13] or [10, 11, 12].

Let M be a Hadamard manifold, that is, a complete simply connected Riemannian manifold with non positive sectional curvature. We say that two geodesics γ_1, γ_2 of M, parameterized by arc length, are *asymptotic* if there exists a constant $c > 0$ such that the distance between them satisfies

$$d(\gamma_1(t), \gamma_2(t)) \leq c, \text{for all } t \geq 0.$$

Note that to be asymptotic is an equivalence relation on the oriented unit speed geodesics of M. We call each one of these classes a point at infinity. We denote by $M(\infty)$ the set of points at infinity and by $\gamma(+\infty)$ the equivalence class of the geodesic γ. Throughout this section, we only consider oriented unit speed geodesics.

Let us assume that M has sectional curvature bounded from above by a negative constant. Then we have two important facts:

1. For any two asymptotic geodesics γ_1, γ_2, the distance between the two curves $\gamma_1|_{[t_0,+\infty)}, \gamma_2|_{[t_0,+\infty)}$ is zero for any $t_0 \in \mathbb{R}$.

2. Given $x, y \in M(\infty)$, $x \neq y$, there exists a unique geodesic γ such that $\gamma(+\infty) = x$ and $\gamma(-\infty) = y$, where $\gamma(-\infty)$ denotes the corresponding point at infinity when the orientation of γ is changed.

For any point $p \in M$, there is a bijective correspondence between the set of unit vectors in the tangent plane T_pM and $M(\infty)$, where a unit vector v is mapped to the point at infinity $\gamma_v(\infty)$, γ_v denoting the geodesic with $\gamma_v(0) = p$ and $\gamma_v'(0) = v$. Analogously, given a point $p \in M$ and a point at infinity $x \in M(\infty)$, there exists a unique geodesic γ such that $\gamma(0) = p$ and $\gamma(+\infty) = x$. In particular, $M(\infty)$ is bijective to a sphere.

There exists a topology on $M^* = M \cup M(\infty)$ satisfying that the restriction to M agrees with the topology induced by the Riemannian distance. This topology is called the cone topology of M^* (see [13], for instance).

In order to define horospheres we consider Busemann functions. Given a unit vector v, the Busemann function $B_v : M \to \mathbb{R}$ associated to v is defined as

$$B_v(p) = \lim_{t \to +\infty} \left(d(p, \gamma_v(t)) - t \right).$$

This is a C^2 convex function on M and it satisfies the following properties.

Property 1. The gradient $\nabla B_v(p)$ is the unique unit vector w in T_pM such that $\gamma_v(\infty) = \gamma_w(-\infty)$.

Property 2. If w is a unit vector such that $\gamma_v(\infty) = \gamma_w(\infty)$ then $B_v - B_w$ is a constant function on M.

Definition 2. Given a point at infinity $x \in M(\infty)$ and a unit vector v such that $\gamma_v(\infty) = x$, the *horospheres at* x are defined as the level sets of the Busemann function B_v.

We have the following important facts with respect to horospheres.

- By *Property 2*, the horospheres at x do not depend on the choice of the vector v.
- The horospheres at x give a foliation of M, and as B_v is a convex function, each one bounds a convex domain in M called a *horoball*.
- The intersection between a geodesic γ and a horosphere at $\gamma(\infty)$ is always orthogonal from *Property 1*.
- Take a point $p \in M$ and let H_x denote a horosphere at x. If γ is the geodesic passing through p with $\gamma(+\infty) = x$, then $H_x \cap \gamma$ is the closest point on H_x to p.
- Given $x, y \in M(\infty)$, if γ is a geodesic with these points at infinity, and H_x, H_y are disjoint horospheres, then the distance between H_x and H_y coincides with the distance between the points $H_x \cap \gamma$ and $H_y \cap \gamma$.

Now we will restrict M to be a Hadamard surface with curvature bounded from above by a negative constant, and we will write horocycle and horodisk to mean horosphere and horoball, respectively.

Let Γ be an ideal polygon of M, that is, Γ is a polygon all of whose sides are geodesics and the vertices are at infinity $M(\infty)$. We assume Γ has an even number of sides $\alpha_1, \beta_1, \alpha_2, \beta_2, ..., \alpha_k, \beta_k$. Let D be the interior of the convex hull of the vertices of Γ, so $\partial D = \Gamma$ and D is a topological disk. We call D an ideal polygonal domain.

Definition 3. An ideal Scherk graph over D is a minimal surface which is the graph of a function defined in D that takes the values $+\infty$ on each α_i, and $-\infty$ on each β_i.

For the sake of completeness, and in order to understand the hypothesis on our main result (Theorem 12), let us describe the necessary and sufficient conditions on the domain D, proved by Gálvez and Rosenberg [13], for the existence of an ideal Scherk graph over D.

At each vertex a_i of Γ, place a horocycle H_i so that $H_i \cap H_j = \emptyset$ if $i \neq j$.

Each α_i meets exactly two horodisks. Denote by $\tilde{\alpha_i}$ the compact arc of α_i outside the two horodisks, and denote by $|\alpha_i|$ the length of $\tilde{\alpha_i}$, that is, the distance between these horodisks. Analogously, we can define $\tilde{\beta_i}$ and $|\beta_i|$.

Now define

$$a(\Gamma) = \sum_{i=1}^{k} |\alpha_i|$$

and

$$b(\Gamma) = \sum_{i=1}^{k} |\beta_i|.$$

Observe that $a(\Gamma) - b(\Gamma)$ does not depend on the choice of the horocycles, because given two horocycles H_1, H_2 at a point $x \in M(\infty)$ and a geodesic γ with x as a point at infinity, then the distance between H_1 and H_2 coincides with the distance between the points $\gamma \cap H_1$ and $\gamma \cap H_2$.

Definition 4. An ideal polygon $\mathcal{P}$ is said to be inscribed in D if the vertices of $\mathcal{P}$ are among the vertices of Γ. Hence its edges are either interior in D or equal to some α_i or β_j.

The definition of $a(\Gamma)$ and $b(\Gamma)$ extends to inscribed polygons:

$$a(\mathcal{P}) = \sum_{\alpha_i \in \mathcal{P}} |\alpha_i| \quad \text{and} \quad b(\mathcal{P}) = \sum_{\beta_i \in \mathcal{P}} |\beta_i|.$$

We denote by $|\mathcal{P}|$ the length of the boundary arcs of $\mathcal{P}$ exterior to the horodisks bounded by H_i at the vertices of $\mathcal{P}$. We call this the truncated length of $\mathcal{P}$.

Definition 5. An ideal polygon Γ is said to be admissible if the two following conditions are satisfied.

1. $a(\Gamma) = b(\Gamma)$;
2. For each inscribed polygon $\mathcal{P}$ in D, $\mathcal{P} \neq \Gamma$, and for some choice of the horocycles at the vertices, we have

$$2a(\mathcal{P}) < |\mathcal{P}| \quad \text{and} \quad 2b(\mathcal{P}) < |\mathcal{P}|.$$

Moreover, an ideal polygonal domain D is said to be admissible if its boundary $\Gamma = \partial D$ is an admissible polygon.

The properties of an admissible polygon are the necessary and sufficient conditions for the existence of an ideal Scherk graph over $D \subset M$ [13].

4.3 Main Result

In this section we consider a Hadamard surface M with bounded curvature, that is, M is a complete simply connected Riemannian surface with curvature $-b^2 \leq K_M \leq -a^2 < 0$, for some constants $a, b \in \mathbb{R}$. We now can establish our main result.

Theorem 12. *Let M denote a Hadamard surface with bounded curvature and let $\Sigma = Graph(u)$ be an ideal Scherk graph over an admissible polygonal domain $D \subset M$. If S is a properly immersed minimal surface contained in $D \times \mathbb{R}$ and disjoint from Σ, then S is a translate of Σ.*

To prove this theorem we follow an idea of Rosenberg, Schulze and Spruck [49], by constructing a discrete family of minimal graphs in $D \times \mathbb{R}$.

Let $\Sigma = \text{Graph}(u)$ be an ideal Scherk graph over D with $\Gamma = \partial D$. Given any point $p \in D$, consider the geodesics starting at p and going to the vertices of Γ. Take the points over each one of these geodesics which are at a distance n from p. Now consider the geodesics joining two consecutive points as in Figure 4.1.

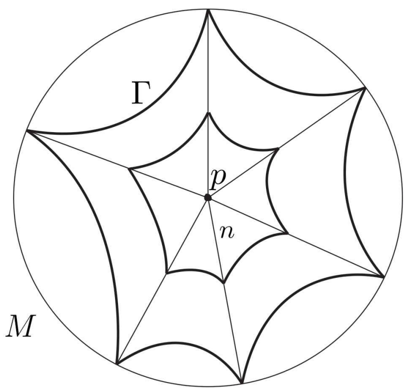

Figure 4.1: Construction of convex domain.

The angle at which two of these geodesics meet is less than π, hence we can smooth the corners to obtain a convex domain D_n with smooth boundary $\Gamma_n = \partial D_n$ and such that $D_1 \subset D_2 \subset ... \subset D_n \subset ...$ is an exhaustion of D.

Denote by A_n the annular-type domain $D_n \setminus \bar{D}_1$ and by Σ_n the graph of u restrict to A_n. Hence Σ_n is a stable minimal surface, and any sufficiently small perturbation of $\partial\Sigma_n$ gives rise to a smooth family of minimal surfaces $\Sigma_{n,t}$ with $\Sigma_{n,0} = \Sigma_n$. We use this fact to the deformation of $\partial\Sigma_n$ which is the graph over ∂A_n given by $\partial_1 \cup \partial_{n,t}$ for $t \geq 0$, where $\partial_1 = (\Gamma_1 \times \mathbb{R}) \cap \Sigma$, $\partial_{n,t} = (\Gamma_n \times \mathbb{R}) \cap T(t)(\Sigma)$ and $T(t)$ is the vertical translation by height t.

Then for t sufficiently small, there exists a minimal surface $\Sigma_{n,t}$ which is the graph of a smooth function $u_{n,t}$ defined on A_n with boundary $\partial_1 \cup \partial_{n,t}$. Note that $u_{n,t}$ satisfies the minimal surface equation on A_n and, by the maximum principle, $\Sigma_{n,t}$ stays between Σ and $\Sigma(t) = T(t)(\Sigma)$. We will show that there exists a uniform interval of existence for $u_{n,t}$, that is, we will prove that there exists $\delta_0 > 0$ such that for all n and $0 \leq t \leq \delta_0$, the minimal surfaces $\Sigma_{n,t} = \text{Graph}(u_{n,t})$ exist.

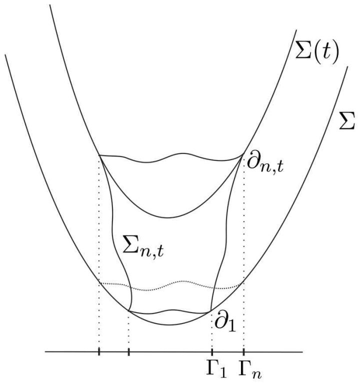

Figure 4.2: Minimal surface $\Sigma_{n,t}$.

Consider $\delta_0 > 0$ sufficiently small so that $u_{2,t}$ exists for any $t \in [0, \delta_0]$. We will show this δ_0 works for all $n \geq 2$. In order to do that we will prove that for $n > 2$ the set $B_n = \{\tau \in [0, \delta_0]; u_{n,t} \text{ exists for } 0 \leq t \leq \tau\}$ is in fact the interval $[0, \delta_0]$.

Claim 5. *The set B_n is open and closed. Hence $B_n = [0, \delta_0]$.*

Proof. By stability B_n is an open set. Now consider an increasing sequence $\tau_k \in B_n$ such that $\tau_k \to \tau$ when $k \to \infty$. The family of minimal graphs Σ_{n,τ_k} is contained in the region bounded by Σ and $\Sigma(\tau)$, and $\partial_1 \subset \partial\Sigma_{n,\tau_k}$ for all k, then there exists a minimal surface $\Sigma_{n,\tau}$ which is the limit of the surfaces Σ_{n,τ_k} with $\partial_1 \subset \partial\Sigma_{n,\tau}$. It remains to prove $\Sigma_{n,\tau}$ is a graph. As $D_2 \subset D_n$, we already know that for all k, $u_{n,\tau_k} \leq u_{2,\delta_0}$ in a neighborhood of Γ_1, then the gradient of u_{n,τ_k} is uniformly bounded in a neighborhood of Γ_1. Suppose there exists a sequence $p_k \in \Gamma_n$ with $u_{n,\tau_k}(p_k) \to p \in \partial_{n,\tau}$ such that $|\nabla u_{n,\tau_k}(p_k)| \to \infty$. This implies the minimal surface $\Sigma_{n,\tau}$ is vertical at p. Considering the horizontal geodesic γ that passes through p and is tangent to $\partial_{n,\tau}$ (recall $\partial_{n,\tau}$ is convex) we can apply the maximum principle with boundary to $\Sigma_{n,\tau}$ and $\gamma \times (-\infty\ , \tau]$ to conclude they coincide, which is impossible. Thus we have uniform gradient estimates for u_{n,τ_k} in $\Gamma_1 \cup \Gamma_n = \partial A_n$. By Lemma 3.1 in [49], we have uniform gradient estimates for $u_{n,k}$ on A_n, and then there exists a function $u_{n,\tau}$ such that $\Sigma_{n,\tau} = \text{Graph}(u_{n,\tau})$ is a minimal graph with boundary $\partial\Sigma_{n,\tau} = \partial_1 \cup \partial_{n,\tau}$, what implies $\tau \in B_n$ and the set B_n is closed. $\square$

Therefore, we have proved that for all $n \geq 2$ and $0 \leq t \leq \delta_0$, there exists a function $u_{n,t}$ defined on A_n such that $\Sigma_{n,t} = \text{Graph}(u_{n,t})$ is a minimal surface with boundary $\partial\Sigma_{n,t} = \partial_1 \cup \partial_{n,t}$.

Fix $t \in (0,\ \delta_0]$. For a fixed n_o, consider the sequence $\{u_{n,t}|_{A_{n_o}}\}$ for $n > n_o$. We already know $u_{n,t} \leq u_{n_o,t}$ in a neighborhood of Γ_1, hence we have uniform gradient estimates in such neighborhood. Moreover, as we have uniform curvature estimates for points far from the boundary, and $\Gamma_n \not\subset A_{n_o}$ for all $n > n_o$, we can get uniform curvature estimates for $\Sigma_{n,t}$ on A_{n_o} for all $n > n_o$. Thus there exists a subsequence $\{u_{n_j,t}|_{A_{n_o}}\}$ that converges to a function $\hat{u}_{n_o}$ defined over A_{n_o} whose graph $\hat{\Sigma}_{n_o}$ is a minimal surface with $\partial_1 \subset \partial\hat{\Sigma}_{n_o}$ and $u \leq \hat{u}_{n_o} \leq u + t$ over A_{n_o}.

Using the same argument above, the sequence $\{u_{n_j,t}|_{A_{2n_o}}\}$ for $n_j > 2n_o$ has a subsequence $\{u_{n_{jk},t}|_{A_{2n_o}}\}$ that converges to a function $\hat{u}_{2n_o}$ defined over A_{2n_o} whose graph $\hat{\Sigma}_{2n_o}$ is a minimal surface with $\partial_1 \subset \partial\hat{\Sigma}_{2n_o}$ and $u \leq \hat{u}_{2n_o} \leq u + t$ over A_{2n_o}.

As $\{u_{n_{jk},t}|_{A_{2n_o}}\} \subset \{u_{n_j,t}|_{A_{n_o}}\}$, we conclude that $\hat{u}_{2n_o} = \hat{u}_{n_o}$ in A_{n_o}.

Continuing this argument to A_{kn_o} for all $k > 2$ and applying the diagonal process, we prove that there exists a subsequence of $\{u_{n,t}\}$ that converges to a function $\hat{u}_\infty$ defined over $\Omega = D \setminus \bar{D}_1$ whose graph $\hat{\Sigma}_\infty$ is a minimal surface with $\partial\hat{\Sigma}_\infty = \partial_1$, $u \leq \hat{u}_\infty < u + t$ over Ω, and $\hat{u}_\infty = \hat{u}_{kn_o}$ in A_{kn_0} for all k.

For simplicity, let us write $\hat{u}$ and $\hat{\Sigma}$ to denote $\hat{u}_\infty$ and $\hat{\Sigma}_\infty$.

Note the minimal surface $\hat{\Sigma} = \text{Graph}(\hat{u})$ assumes the same infinite boundary values at Γ as the ideal Scherk graph $\Sigma = \text{Graph}(u)$. Consider the restriction of u to Ω and continue denoting by Σ the graph of u restricted to Ω. We will show that Σ and $\hat{\Sigma}$ coincide by analysing the flux of the functions $u, \hat{u}$ across the boundary of Ω, which is $\Gamma_1 \cup \Gamma$.

Let $\alpha_1, \beta_1, \alpha_2, \beta_2, ..., \alpha_k, \beta_k$ be the geodesic sides of the admissible ideal polygon Γ with $u(\alpha_i) = +\infty = \hat{u}(\alpha_i)$ and $u(\beta_i) = -\infty = \hat{u}(\beta_i)$. For each n, consider pairwise disjoint horocycles $H_i(n)$ at each vertex a_i of Γ such that the convex horodisk bounded by $H_i(n+1)$ is contained in the convex horodisk bounded by $H_i(n)$. For each side α_i, let us denote by α_i^n the compact arc of α_i which is the part of α_i outside the two horodisks, and by $|\alpha_i^n|$ the length of α_i^n, that is, the distance between the two horodisks. Analogously, we define β_i^n for each side β_i. Denote by c_i^n the compact arc of $H_i(n)$ contained in the domain D and let $\mathcal{P}^n$ be the polygon formed by α_i^n, β_i^n and c_i^n.

As the function u is defined in the interior region bounded by $\mathcal{P}^n$, and $\mathcal{P}^n$ is a compact cycle, then $F_u(\mathcal{P}^n) = 0$, by the Flux Theorem. In the other hand, as $u \leq \hat{u}$ we have $F_u(\mathcal{P}^n) \leq F_{\hat{u}}(\mathcal{P}^n)$, and then $F_{\hat{u}}(\mathcal{P}^n) \geq 0$. Moreover, the flux of $\hat{u}$ across $\mathcal{P}^n$ satisfies

$$\begin{aligned} F_{\hat{u}}(\mathcal{P}^n) &= \sum_i F_u(\alpha_i^n) + \sum_i F_u(\beta_i^n) + \sum_i F_u(c_i^n) \\ &\leq \sum_i (|\alpha_i^n| - |\beta_i^n|) + \sum_i |c_i^n|. \end{aligned}$$

Notice that $|c_i^n| \to 0$ when $n \to \infty$ and, since Γ is an admissible polygon, we have $\sum_i |\alpha_i^n| = \sum_i |\beta_i^n|$, for any n. Hence we conclude

$$F_{\hat{u}}(\mathcal{P}^n) \to 0 \quad \text{when} \quad n \to \infty.$$

Then $F_u(\Gamma) = \lim_{n\to\infty} F_u(\mathcal{P}^n) = 0 = \lim_{n\to\infty} F_{\hat{u}}(\mathcal{P}^n) = F_{\hat{u}}(\Gamma)$.

In the other hand, as $\mathcal{P}^n$ is homotopic to Γ_1, we have $F_{\hat{u}}(\Gamma_1) = F_{\hat{u}}(\mathcal{P}^n)$ for any n, and we conclude that $F_{\hat{u}}(\Gamma_1) = 0$. Analogously (or using the Flux Theorem as we did for $\mathcal{P}^n$), we also have $F_u(\Gamma_1) = 0$. Therefore, we have proved that the functions u and $\hat{u}$ have the same flux across the boundary $\partial\Omega = \Gamma_1 \cup \Gamma$.

As $\Sigma = \text{Graph}(u)$ and $\hat{\Sigma} = \text{Graph}(\hat{u})$ are two minimal graphs over $\Omega = D \setminus \bar{D}_1$ such that $u \leq \hat{u}$, $\partial\Sigma = \partial\hat{\Sigma}$ and $F_u(\partial\Omega) = F_{\hat{u}}(\partial\Omega)$, we conclude that necessarily $u \equiv \hat{u}$ over Ω, that is, $\hat{\Sigma}$ is the Scherk graph over Ω with $\partial\hat{\Sigma} = \partial_1$.

Remark. *We have proved that for any $t \in (0\ ,\delta_0]$ we can get a subsequence of the minimal surfaces $\Sigma_{n,t}$ that converges to a minimal surface $\hat{\Sigma}$ which is the Scherk graph over $D \setminus \bar{D}_1$ with $\partial\hat{\Sigma} = \partial_1$.*

Now we are able to prove the theorem.

Proof of Theorem 12. As $\Sigma \cap S = \emptyset$, we can suppose that S is entirely under Σ. Pushing down Σ by vertical translations, we will have two possibilities: either a translate of Σ touches S for the first time in the interior, and then, by the maximum principle, we conclude they coincide; or S is asymptotic at infinity to a translate of Σ. Let us analyse this last case.

Without loss of generality, we can suppose that S is asymptotic at infinity to Σ. If $S \neq \Sigma$, then as S is proper there is a point $p_0 \in \Sigma$ and a cylinder $C = B_\Sigma(p_0, r_0) \times (-r_0, r_0)$ for some $r_0 > 0$ such that $S \cap C = \emptyset$, where $B_\Sigma(p_0, r_0)$ is the intrinsic ball centered at p_0 with radius r_0. We can assume r_0 is less than the injectivity radius of Σ at p_0. In our construction of the surfaces $\Sigma_{n,t}$, we can choose D_1 so that $\partial_1 \subset B_\Sigma(p_0, \frac{r_0}{2})$, and take $t = \min\{\frac{r_0}{2}, \delta_0\}$.

Observe that when we translate $\Sigma_{n,t}$ vertically downwards by an amount t, the boundaries of the translates of $\Sigma_{n,t}$ stay strictly above S. Thus, by the maximum principle, all the translates remain disjoint from S. We call $\Sigma_{n,t}^{'}$ this final translate with boundary $\partial\Sigma_{n,t}^{'} = \partial_1^{'} \cup \partial_n^{'}$, where $T(t)(\partial_1^{'}) = \partial_1 \subset \Sigma$ and $\partial_n^{'} \subset \Sigma$. Hence, all the surfaces $\Sigma_{n,t}^{'}$ lie above S and, as we proved before, there exists a subsequence of $\Sigma'_{n,t}$ that converges to the ideal Scherk graph Σ' defined over $D \setminus \bar{D}_1$ with $T(t)(\Sigma') = \Sigma$. In particular, we conclude that S lies below Σ', which yields a contradiction, since we are assuming that S is asymptotic at infinity to Σ.

$\square$

CHAPTER 5

On doubly periodic minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ with finite total curvature in the quotient space

In this chapter we develop the theory of properly immersed minimal surfaces in the quotient space $\mathbb{H}^2 \times \mathbb{R}/G$, where G is a subgroup of isometries generated by a vertical translation and a horizontal isometry in $\mathbb{H}^2$ without fixed points. The horizontal isometry can be either a parabolic translation along horocycles in $\mathbb{H}^2$ or a hyperbolic translation along a geodesic in $\mathbb{H}^2$. In fact, we prove that if a properly immersed minimal surface in $\mathbb{H}^2 \times \mathbb{R}/G$ has finite total curvature then its total curvature is a multiple of 2π and, moreover, we understand the geometry of the ends. These results hold true more generally for properly immersed minimal surfaces in $M \times \mathbb{S}^1$, where M is a hyperbolic surface with finite topology whose ends are isometric to one of the ends of the above spaces $\mathbb{H}^2 \times \mathbb{R}/G$.

This whole chapter is based in a joint paper with L. Hauswirth [18].

5.1 Introduction

Among all the minimal surfaces in $\mathbb{R}^3$, the ones of finite total curvature are the best known. In fact, if a minimal surface in $\mathbb{R}^3$ has finite total curvature then this minimal surface is either a plane or its total curvature is a non-zero multiple of 2π. Moreover, if the total curvature is -4π, then the minimal surface is either the Catenoid or the Enneper's surface [44].

In 2010, Hauswirth and Rosenberg [20] developed the theory of complete embedded minimal surfaces of finite total curvature in $\mathbb{H}^2 \times \mathbb{R}$. In that work they proved that the total curvature of such surfaces must be a multiple of 2π, and they gave simply connected examples whose total curvature is $-2\pi m$, for each nonnegative integer m.

In the last few years, many people have worked on this subject and classified some minimal surfaces of finite total curvature in $\mathbb{H}^2 \times \mathbb{R}$ (see [19, 22, 39, 52]).

In [39], Morabito and Rodríguez constructed for $k \geq 2$ a $(2k-2)$-parameter family of properly embedded minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ invariant by a vertical translation which have total curvature $4\pi(1-k)$, genus zero and $2k$ vertical Scherk-type ends in the quotient by the vertical translation. Moreover, independently, Morabito and Rodríguez [39] and Pyo [45] constructed for $k \geq 2$ examples of properly embedded minimal surfaces with total curvature $4\pi(1-k)$, genus zero and k ends, each one asymptotic to a vertical plane. In particular, we have examples of minimal annuli with total curvature -4π.

It was expected that each end of a complete embedded minimal surface of finite total curvature in $\mathbb{H}^2 \times \mathbb{R}$ was asymptotic to either a vertical plane or a Scherk graph over an ideal polygonal domain. However in [46], Pyo and Rodríguez constructed new simply-connected examples of minimal surfaces of finite total curvature in $\mathbb{H}^2 \times \mathbb{R}$, showing this is not the case.

Here we consider $\mathbb{H}^2 \times \mathbb{R}$ quotiented by a subgroup of isometries $G \subset \mathrm{Isom}(\mathbb{H}^2 \times \mathbb{R})$ generated by a horizontal isometry in $\mathbb{H}^2$ without fixed points, ψ, and a vertical translation, $T(h)$, for some $h > 0$. The isometry ψ can be either a parabolic translation along horocycles in $\mathbb{H}^2$ or a hyperbolic translation along a geodesic in $\mathbb{H}^2$. We prove that if a properly immersed minimal surface in $\mathbb{H}^2 \times \mathbb{R}/G$ has finite total curvature then its total curvature is a multiple of 2π, and moreover, we understand the geometry of the ends. More precisely, we prove that each end of a properly immersed minimal surface of finite total curvature in $\mathbb{H}^2 \times \mathbb{R}/G$ is asymptotic to either a horizontal slice, or a vertical geodesic plane or the quotient of a *Helicoidal plane*. Where by *Helicoidal plane* we mean a minimal surface in $\mathbb{H}^2 \times \mathbb{R}$ which is parameterized by $X(x,y) = (x, y, ax+b)$ when we consider the halfplane model for $\mathbb{H}^2$.

Let us mention that these results hold true for properly immersed minimal surfaces in $M \times \mathbb{S}^1$, where M is a hyperbolic surface ($K_M = -1$) with finite topology whose ends are either isometric to $\mathcal{M}_+$ or $\mathcal{M}_-$, which we define in the next section.

5.2 Definitions and preliminary results

Unless otherwise stated, we use the Poincaré disk model for the hyperbolic plane, that is

$$\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$$

with the hyperbolic metric $g_{-1} = \sigma g_0 = \frac{4}{(1-x^2-y^2)^2} g_0$, where g_0 is the Euclidean metric in $\mathbb{R}^2$. In this model, the asymptotic boundary $\partial_\infty \mathbb{H}^2$ of $\mathbb{H}^2$ is identified with the unit circle.

We write $\overline{pq}$ to denote the geodesic arc between the two points p, q.

We consider the quotient spaces $\mathbb{H}^2 \times \mathbb{R}/G$, where G is a subgroup of $\mathrm{Isom}(\mathbb{H}^2 \times \mathbb{R})$ generated by a horizontal isometry on $\mathbb{H}^2$ without fixed points, ψ, and a vertical translation, $T(h)$, for some $h > 0$. The horizontal isometry ψ can be either a horizontal translation along horocycles in $\mathbb{H}^2$ or a horizontal translation along a geodesic in $\mathbb{H}^2$.

Let us analyse each one of these cases for ψ.

Consider any geodesic γ that limits to a point $p_0 \in \partial_\infty \mathbb{H}^2$ parametrized by arc length. Let $c(s)$ be the horocycles in $\mathbb{H}^2$ tangent to $\partial_\infty \mathbb{H}^2$ at p_0 that intersects γ at $\gamma(s)$, and write $d(s)$ to denote the horocylinder $c(s) \times \mathbb{R}$ in $\mathbb{H}^2 \times \mathbb{R}$. Taking two points $p, q \in c(s)$, let $\psi : \mathbb{H}^2 \times \mathbb{R} \to \mathbb{H}^2 \times \mathbb{R}$ be the parabolic translation along $d(s)$ such that $\psi(p) = q$. We have $\psi(d(s)) = d(s)$ for all s. If $G = [\psi, T(h)]$, then the manifold $\mathcal{M}$ which is the quotient of $\mathbb{H}^2 \times \mathbb{R}$ by G is diffeomorphic to $\mathbb{T}^2 \times \mathbb{R}$, where $\mathbb{T}^2$ is the 2-torus. Moreover, $\mathcal{M}$ is foliated by the family of tori $\mathbb{T}(s) = d(s)/G$, which are intrinsically flat and have constant mean cuvature $1/2$. (See Figure 5.1).

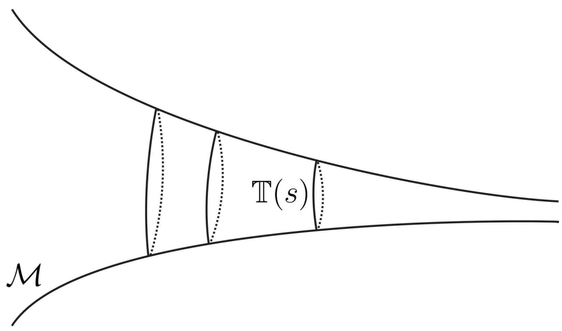

Figure 5.1: $\mathcal{M} = \mathbb{H}^2 \times \mathbb{R}/[\psi, T(h)]$, where ψ is a parabolic isometry.

Now take a geodesic γ in $\mathbb{H}^2$ and consider $c(s)$ the family of equidistant curves to γ, with $c(0) = \gamma$. Write $d(s)$ to denote the plane $c(s) \times \mathbb{R}$ in $\mathbb{H}^2 \times \mathbb{R}$. Given two points $p, q \in c(s)$, let $\psi : \mathbb{H}^2 \times \mathbb{R} \to \mathbb{H}^2 \times \mathbb{R}$ be the hyperbolic translation along γ such that $\psi(p) = q$. We have $\psi(d(s)) = d(s)$ for all s.

If $G = [\psi, T(h)]$, then the manifold $\mathcal{M}$ which is the quotient of $\mathbb{H}^2 \times \mathbb{R}$ by G is also diffeomorphic to $\mathbb{T}^2 \times \mathbb{R}$ and $\mathcal{M}$ is foliated by the family of tori $\mathbb{T}(s) = d(s)/G$, which are intrinsically flat and have constant mean cuvature $\frac{1}{2}\tanh(s)$. (See Figure 5.2).

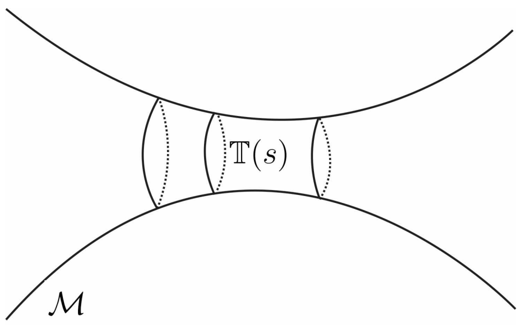

Figure 5.2: $\mathcal{M} = \mathbb{H}^2 \times \mathbb{R}\,/[\psi, T(h)]$, where ψ is a hyperbolic isometry.

In these quotient spaces we have two different types of ends. One where the injectivity radius goes to zero at infinity, which we denote by $\mathcal{M}_+$, and another one where the injectivity radius is strictly positive, which we denote by $\mathcal{M}_-$.

Hence $\mathcal{M}_+ = \bigcup_{s\geq 0} d(s)/[\psi, T(h)]$, where ψ is a parabolic translation along horocycles, and $\mathcal{M}_- = \bigcup_{s\geq 0} d(s)/[\psi, T(h)]$, for ψ hyperbolic translation along a geodesic in $\mathbb{H}^2$, or $\mathcal{M}_- = \bigcup_{s\leq 0} d(s)/[\psi, T(h)]$, where ψ can be either a parabolic translation along horocycles or a hyperbolic translation along a geodesic in $\mathbb{H}^2$. (See Figure 5.3).

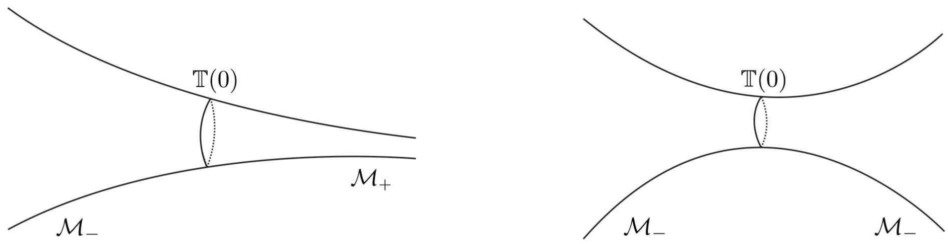

Figure 5.3: $\mathcal{M}_+$ and $\mathcal{M}_-$.

From now one we will not distinguish between the two quotient spaces above. We will denote both by $\mathcal{M}$.

Let Σ be a Riemannian surface and $X : \Sigma \to \mathcal{M}$ be a minimal immersion. As

$$\mathcal{M} = \mathbb{H}^2 \times \mathbb{R}\,/[\psi, T(h)] \cong \mathbb{H}^2/[\psi] \times \mathbb{S}^1,$$

we can write $X = (F, h) : \Sigma \to \mathbb{H}^2/[\psi] \times \mathbb{S}^1$, where $F : \Sigma \to \mathbb{H}^2/[\psi]$ and $h : \Sigma \to \mathbb{S}^1$ are harmonic maps. We consider local conformal parameters $z = x + iy$ on Σ. Hence

$$|F_x|_\sigma^2 + (h_x)^2 = |F_y|_\sigma^2 + (h_y)^2$$
$$\langle F_x, F_y \rangle_\sigma + h_x.h_y = 0 \tag{5.1}$$

and the metric induced by the immersion is given by

$$ds^2 = \lambda^2(z)|dz|^2 = (|F_z|_\sigma + |F_{\bar{z}}|_\sigma)^2|dz|^2. \tag{5.2}$$

Considering the universal covering $\pi : \mathbb{H}^2 \times \mathbb{R} \to \mathbb{H}^2/[\psi] \times \mathbb{S}^1$ we can take $\widetilde{\Sigma}$, a connected component of the lift of Σ to $\mathbb{H}^2 \times \mathbb{R}$, and we have $\widetilde{X} = (\widetilde{F}, \widetilde{h}) : \widetilde{\Sigma} \to \mathbb{H}^2 \times \mathbb{R}$ such that $\pi(\widetilde{\Sigma}) = \Sigma$ and $\widetilde{F} : \widetilde{\Sigma} \to \mathbb{H}^2, \widetilde{h} : \widetilde{\Sigma} \to \mathbb{R}$ are harmonic maps. We denote by $\widetilde{\partial}_t, \partial_t$ the vertical vector fields in $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{H}^2/[\psi] \times \mathbb{S}^1$, respectively. Observe that the functions $n_3 : \Sigma \to \mathbb{R}$, $\tilde{n}_3 : \widetilde{\Sigma} \to \mathbb{R}$, given by $n_3 = \langle \partial_t, N \rangle, \tilde{n}_3 = \left\langle \widetilde{\partial}_t, \widetilde{N} \right\rangle$, where $N, \widetilde{N}$ are the unit normal vectors of $\Sigma, \widetilde{\Sigma}$, respectively, satisfy $\tilde{n}_3 = n_3 \circ \pi$. Then if we define the functions $\omega : \Sigma \to \mathbb{R}, \tilde{\omega} : \widetilde{\Sigma} \to \mathbb{R}$ so that $\tanh(\omega) = n_3$ and $\tanh(\tilde{\omega}) = \tilde{n}_3$, we get $\tilde{\omega} = \omega \circ \pi$.

As we consider X a conformal minimal immersion, we have

$$n_3 = \frac{|F_z|^2 - |F_{\bar{z}}|^2}{|F_z|^2 + |F_{\bar{z}}|^2} \tag{5.3}$$

and

$$\omega = \frac{1}{2} \ln \frac{|F_z|}{|F_{\bar{z}}|}. \tag{5.4}$$

Note that the same formulae are true for $\tilde{n}_3$ and $\tilde{\omega}$.

We know that for local conformal parameters $\tilde{z}$ on $\widetilde{\Sigma}$, the holomorphic quadratic Hopf differential associated to $\widetilde{F}$, given by

$$\widetilde{Q}(\widetilde{F}) = (\sigma \circ \widetilde{F})^2 \widetilde{F}_{\tilde{z}} \bar{\widetilde{F}}_{\tilde{z}} (d\tilde{z})^2,$$

can be written as $(\widetilde{h}_{\tilde{z}})^2 (d\tilde{z})^2 = -\widetilde{Q}$. Then, since $\widetilde{h}$ and h differ by a constant in a neighborhood, $(h_z)^2(dz)^2 = -Q$ is also a holomorphic quadratic differential on Σ for local conformal parameters z on Σ. We note Q has two square roots globally defined on Σ. Writing $Q = \phi(dz)^2$, we denote by $\eta = \pm 2i\sqrt{\phi}dz$ a square root of Q, where we choose the sign so that

$$h = \mathrm{Re} \int \eta.$$

Using (5.2), (5.4) and the definition of Q, we have

$$ds^2 = 4(\cosh^2\omega)|Q|. \tag{5.5}$$

As the Jacobi operator of the minimal surface Σ is given by

$$J = \frac{1}{4\cosh^2\omega|\phi|}\left[\Delta_0 - 4|\phi| + \frac{2|\nabla\omega|^2}{\cosh^2\omega}\right]$$

and $Jn_3 = 0$, then

$$\Delta_0\omega = 2\sinh(2\omega)|\phi|, \tag{5.6}$$

where Δ_0 denotes the Laplacian in the Euclidean metric $|dz|^2$, that is, $\Delta_0 = 4\partial^2_{z\bar{z}}$.

The sectional curvature of the tangent plane to Σ at a point z is $-n_3^2$ and the second fundamental form is

$$II = \frac{\omega_x}{\cosh\omega}dx\otimes dx - \frac{\omega_x}{\cosh\omega}dy\otimes dy + 2\frac{\omega_y}{\cosh\omega}dx\otimes dy.$$

Hence, using the Gauss equation, the Gauss curvature of (Σ, ds^2) is given by

$$K_\Sigma = -\tanh^2\omega - \frac{|\nabla\omega|^2}{4(\cosh^4\omega)|\phi|}. \tag{5.7}$$

5.3 Main Results

In this section, besides prove the main theorem of this chapter, we will firstly demonstrate some properties of an end when it is properly immersed in $\mathcal{M}_+$ or in $\mathcal{M}_-$, which are interesting by theirselves.

We will write $[d(0), d(s)]$ to denote the slab $\cup_{0\le t\le s} d(t)$ in $\mathbb{H}^2\times\mathbb{R}$ whose boundary is $d(0)\cup d(s)$.

Lemma 1. *There is no proper minimal end E in $\mathcal{M}_+$ with $\partial\mathcal{M}_+\cap E = \partial E$ whose lift is an annulus in $\mathbb{H}^2\times\mathbb{R}$.*

Proof. Let us prove it by contradiction. Suppose we have a proper minimal end E in $\mathcal{M}_+$ with $\partial\mathcal{M}_+\cap E = \partial E$ whose lift $\widetilde{E}$ is a proper minimal annulus in $\mathbb{H}^2\times\mathbb{R}$. Hence $\partial\widetilde{E}\subset d(0)$, $\widetilde{E}\subset\bigcup_{s\ge 0} d(s)$ and $\widetilde{E}\cap d(s)\neq\emptyset$ for any s, where $d(s) = c(s)\times\mathbb{R}$, $c(s)$ horocycle tangent at infinity to p_0.

Choose $p\neq p_0\in\partial_\infty\mathbb{H}^2$ such that $(\overline{pp_0}\times\mathbb{R})\cap\partial\widetilde{E} = \emptyset$.

Now consider $q\in\partial_\infty\mathbb{H}^2$ contained in the halfspace determined by $\overline{pp_0}\times\mathbb{R}$ that does not contain $\partial\widetilde{E}$ such that $(\overline{pq}\times\mathbb{R})\cap d(0) = \emptyset$. Let q go to p_0. If there exists some point q_1 such that $(\overline{pq_1}\times\mathbb{R})\cap\widetilde{E}\neq\emptyset$, then, as $p, q_1\notin d(s)$

for any s, and E is proper, that intersection is a compact set in $\widetilde{E}$. Therefore, when we start with q close to p and let q go to q_1, there will be a first contact point between $\overline{pq_0} \times \mathbb{R}$ and $\widetilde{E}$, for some point q_0. By the maximum principle this yields a contradiction. Therefore, we conclude that $\overline{pp_0} \times \mathbb{R}$ does not intersect $\widetilde{E}$. Choosing another point $\bar{p}$ in the same halfspace determined by $\overline{pp_0} \times \mathbb{R}$ as $\widetilde{E}$ such that $(\overline{\bar{p}p_0} \times \mathbb{R}) \cap \partial\widetilde{E} = \emptyset$, we can use the same argument above and conclude that $\widetilde{E}$ is contained in the region between $\overline{pp_0} \times \mathbb{R}$ and $\overline{\bar{p}p_0} \times \mathbb{R}$. Call $\alpha = \overline{pp_0}$ and $\bar{\alpha} = \overline{\bar{p}p_0}$.

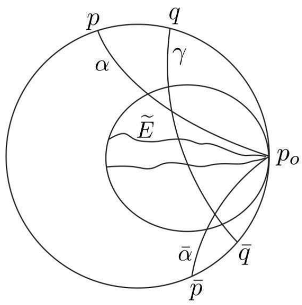

Figure 5.4: Curve γ.

Now consider a horizontal geodesic γ with endpoints $q, \bar{q}$ such that q is contained in the halfspace determined by $\alpha \times \mathbb{R}$ that does not contain $\widetilde{E}$, and $\bar{q}$ is contained in the halfspace determined by $\bar{\alpha} \times \mathbb{R}$ that does not contain $\widetilde{E}$ (see Figure 5.4). Up to a horizontal translation, we can suppose $\widetilde{E} \cap (\gamma \times \mathbb{R}) \neq \emptyset$. As E is proper, the part of $\widetilde{E}$ between $\partial\widetilde{E}$ and $\widetilde{E} \cap (\gamma \times \mathbb{R})$ is compact, then there exists $M \in \mathbb{R}$ such that the function $\widetilde{h}$ restrict to this part satisfies $-M \leq \widetilde{h} \leq M$. Consider the function v that takes the value $+\infty$ on γ and take the value M on the asymptotic arc at infinity of $\mathbb{H}^2$ between q and $\bar{q}$ that does not contain p_0. The graph of v is a minimal surface that does not intersect $\widetilde{E}$. When we let $q, \bar{q}$ go to p_0 we get, using the maximum principle, that $\widetilde{E}$ is under the graph of v and then $\widetilde{h}|_{\widetilde{E}}$ is bounded above by M, since v converges to the constant function M uniformly on compact sets as $q, \bar{q}$ converge to p_0 (see [29], section B). Using a similar argument, we can show that $\widetilde{h}|_{\widetilde{E}}$ is also bounded below by $-M$. Therefore $\widetilde{E}$ is an annulus contained in the region bounded by $\alpha \times \mathbb{R}, \bar{\alpha} \times \mathbb{R}, \mathbb{H}^2 \times \{-M\}$ and $\mathbb{H}^2 \times \{M\}$.

Take four points $p_1, p_2, p_3, p_4 \in \partial_\infty \mathbb{H}^2$ such that p_1, p_2 is contained in the halfspace determined by $\alpha \times \mathbb{R}$ that does not contain $\widetilde{E}$, and p_3, p_4 is contained in the halfspace determined by $\bar{\alpha} \times \mathbb{R}$ that does not contain $\widetilde{E}$. Moreover, choose these points so that there exists a complete minimal surface $\mathcal{A}$ taking value 0 on $\overline{p_1p_2}$ and $\overline{p_3p_4}$, and taking value $+\infty$ on $\overline{p_2p_4}$ and $\overline{p_1p_3}$ (see Figure 5.5). This minimal surface exists by [4].

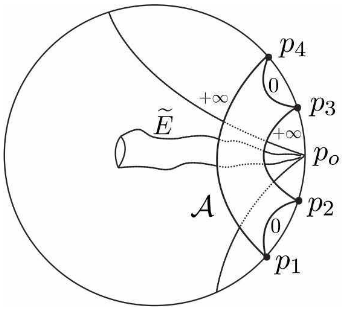

Figure 5.5: Minimal graph $\mathcal{A}$.

Up to a vertical translation, $\mathcal{A}$ does not intersect $\widetilde{E}$ and $\mathcal{A}$ is above $\widetilde{E}$. Pushing down $\mathcal{A}$ (under vertical translation) and using the maximum principle, we conclude that $\mathcal{A} = \widetilde{E}$, which is impossible. $\square$

Remark 7. We do not use any assumption on the total curvature of the end to prove the previous lemma.

Lemma 2. *If a proper minimal end E with finite total curvature is contained in $\mathcal{M}_-$, then E has bounded curvature and infinite area.*

Proof. Suppose E does not have bounded curvature. Then there exists a divergent sequence $\{p_n\}$ in E such that $|A(p_n)| \geq n$, where A denotes the second fundamental form of E. As the injectivity radius of $\mathcal{M}_-$ is strictly positive, there exists $\delta > 0$ such that for all n, the exponential map $\exp_{\mathcal{M}}$: $D(0, \delta) \subset T_{p_n}\mathcal{M} \to B_{\mathcal{M}}(p_n, \delta)$ is a diffeomorphism, where $B_{\mathcal{M}}(p_n, \delta)$ is the extrinsic ball of radius δ centered at p_n in $\mathcal{M}$. Without loss of generality, we can suppose $B_{\mathcal{M}}(p_n, \delta) \cap B_{\mathcal{M}}(p_k, \delta) = \emptyset$.

The properness of the end implies the existence of a curve $c \subset E$ homotopic to ∂E such that every point in the connected component of $E \setminus c$ that does not contain ∂E is at a distance greater than δ from ∂E. Call E_1 this component. Hence each point of E_1 is the center of an extrinsic ball of radius δ disjoint from ∂E.

Denote by C_n the connected component of p_n in $B_{\mathcal{M}}(p_n, \delta) \cap E_1$ and consider the function $f_n : C_n \to \mathbb{R}$ given by

$$f_n(q) = d(q, \partial C_n)|A(q)|,$$

where d is the extrinsic distance.

The function f_n restricted to the boundary ∂C_n is identically zero and $f_n(p_n) = \delta|A(p_n)| > 0$. Then f_n attains a maximum in the interior. Let q_n be such maximum. Hence $\delta|A(q_n)| \geq d(q_n, \partial C_n)|A(q_n)| = f_n(q_n) \geq f_n(p_n) = \delta|A(p_n)| \geq \delta n$, what yields $|A(q_n)| \geq n$.

Now consider $r_n = \frac{d(q_n, \partial C_n)}{2}$ and denote by B_n the connected component of q_n in $B_{\mathcal{M}}(q_n, r_n) \cap E_1$. We have $B_n \subset C_n$. If $q \in B_n$, then $f_n(q) \leq f_n(q_n)$ and

$$\begin{aligned} d(q_n, \partial C_n) &\leq d(q_n, q) + d(q, \partial C_n) \\ &\leq \tfrac{d(q_n, \partial C_n)}{2} + d(q, \partial C_n) \\ \Rightarrow d(q_n, \partial C_n) &\leq 2d(q, \partial C_n), \end{aligned}$$

hence we conclude that $|A(q)| \leq 2|A(q_n)|$.

Call g the metric on E and take $\lambda_n = |A(q_n)|$. Consider Σ_n the homothety of B_n by λ_n, that is, Σ_n is the ball B_n with the metric $g_n = \lambda_n g$. We can use the exponential map at the point q_n to lift the surface Σ_n to the tangent plane $T_{q_n}\mathcal{M} \approx \mathbb{R}^3$, hence we obtain a surface $\widetilde{\Sigma}_n$ in $\mathbb{R}^3$ which is a minimal surface with respect to the lifted metric $\tilde{g}_n$, where $\tilde{g}_n$ is the metric such that the exponential map $\exp_{q_n}$ is an isometry from $(\widetilde{\Sigma}_n, \tilde{g}_n)$ to (Σ_n, g_n).

We have $\widetilde{\Sigma}_n \subset B_{\mathbb{R}^3}(0, \lambda_n r_n)$, $|A(0)| = 1$ and $|A(q)| \leq 2$ for all $q \in \widetilde{\Sigma}_n$.

Note that $2\lambda_n r_n = f_n(q_n) \geq f_n(p_n) \geq \delta n$, hence $\lambda_n r_n \to +\infty$ as $n \to \infty$.

Fix n. The sequence $\left\{\widetilde{\Sigma}_k \cap B_{\mathbb{R}^3}(0, \lambda_n r_n)\right\}_{k \geq n}$ is a sequence of compact surfaces in $\mathbb{R}^3$, with bounded curvature, passing through the origin and the metric g_k converges to the canonical metric g_0 in $\mathbb{R}^3$. Then a subsequence converges to a minimal surface in $(\mathbb{R}^3, g_0)$ passing through the origin with the norm of the second fundamental form at the origin equal to 1. We can apply this argument for each n and using the diagonal sequence argument, we obtain a complete minimal surface $\widetilde{\Sigma}$ in $\mathbb{R}^3$, with $0 \in \widetilde{\Sigma}$ and $|A(0)| = 1$. In particular, $\widetilde{\Sigma}$ is not the plane. Then by Osserman's theorem [44], we have $\int_{\widetilde{\Sigma}} |A|^2 \geq 4\pi$.

We know that the integral $\int_{\Sigma} |A|^2$ is invariant by homothety of Σ, hence

$$\int_{B_n} |A|^2 = \int_{\Sigma_n} |A|^2 = \int_{\widetilde{\Sigma}_n} |A|^2.$$

Consider a compact $K \subset \widetilde{\Sigma}$ sufficiently large so that $\int_K |A|^2 \geq 2\pi$. Fix n such that $K \subset B(0, \lambda_n r_n)$. As a subsequence of $\widetilde{\Sigma}_k \cap B_{\mathbb{R}^3}(0, \lambda_n r_n)$ converges to $\widetilde{\Sigma} \cap B_{\mathbb{R}^3}(0, \lambda_n r_n)$, then for k sufficiently large, we have that

$$\int_{\widetilde{\Sigma}_k \cap B(0, \lambda_n r_n)} |A|^2 \geq 2\pi - \epsilon,$$

for some small $\epsilon > 0$. It implies $\int_{B_k} |A|^2 \geq 2\pi - \epsilon$, for k sufficiently large. As

$B_i \cap B_j = \emptyset$, we conclude that $\int_E |A|^2 = +\infty$. But this is not possible, since

$$\int_E |A|^2 = \int_E -2K_E + 2K\mathrm{sec}_{\mathcal{M}(E)} \leq -2 \int_E K_E < +\infty.$$

Therefore, E has necessarily bounded curvature.

Since E is complete, there exist $\epsilon > 0$ and a sequence of points $\{p_n\}$ in E such that p_n diverges in $\mathcal{M}_-$ and $B_E(p_k, \epsilon) \cap B_E(p_j, \epsilon) = \emptyset$, where $B_E(p_k, \epsilon) \subset E$ is the intrinsic ball centered at p_k with radius ϵ. As E has bounded curvature, then there exists $\tau < \epsilon$ such that $B_E(p_k, \tau)$ is a graph with bounded geometry over a small disk $D(0, \tau)$ of radius τ in $T_{p_k}E$, and the area of $B_E(p_k, \tau)$ is greater or equal to the area of $D(0, \tau)$. Therefore,

$$\mathrm{area}(E) \geq \sum_{n \geq 1} \mathrm{area}\left(B_E(p_n, \tau)\right) = \infty.$$

□

Definition 6. We write *Helicoidal plane* to denote a minimal surface in $\mathbb{H}^2 \times \mathbb{R}$ which is parametrized by $X(x, y) = (x, y, ax + b)$ when we consider the halfplane model for $\mathbb{H}^2$.

Now we can state the main result of this chapter.

Theorem 13. *Let $X : \Sigma \hookrightarrow \mathcal{M} = \mathbb{H}^2 \times \mathbb{R} / [\psi, T(h)]$ be a properly immersed minimal surface with finite total curvature. Then*

1. *Σ is conformally equivalent to a compact Riemann surface $\overline{M}$ with genus g minus a finite number of points, that is, $\Sigma = \overline{M} \setminus \{p_1, ..., p_k\}$.*
2. *The total curvature satisfies*

$$\int_\Sigma K d\sigma = 2\pi(2 - 2g - k).$$

3. *The ends contained in $\mathcal{M}_-$ are necessarily asymptotic to a vertical plane $\gamma \times \mathbb{S}^1$ and the ends contained in $\mathcal{M}_+$ are asymptotic to either*
    - *a horizontal slice $\mathbb{H}^2 / [\psi] \times \{c\}$, or*
    - *a vertical plane $\gamma \times \mathbb{S}^1$, or*
    - *the quotient of a Helicoidal plane.*

4. *If we parametrize each end by a punctured disk then either the holomorphic quadratic differential Q extends to zero at the origin (in the case where the end is asymptotic to a horizontal slice) or Q extends meromorphically to the puncture with a double pole and residue zero. In this last case, the third coordinate satisfies $h(z) = b\arg(z) + O(|z|)$ with $b \in \mathbb{R}$.*

Proof. The proof of this theorem uses arguments of harmonic diffeomorphisms theory as can be found in the work of Han, Tam, Treibergs and Wan [14, 15, 54] and Minsky [38].

From a result by Huber [26], we deduce that Σ is conformally a compact Riemann surface $\overline{M}$ minus a finite number of points $\{p_1, ..., p_k\}$, and the ends are parabolic.

We consider $\overline{M}^* = \overline{M} - \cup_i B(p_i, r_i)$, the surface minus a finite number of disks removed around the punctures p_i. As the ends are parabolic, each punctured disk $B^*(p_i, r_i)$ can be parametrized conformally by the exterior of a disk in $\mathbb{C}$, say $U = \{z \in \mathbb{C}; |z| \geq R_0\}$.

Using the Gauss-Bonnet theorem for $\overline{M}^*$, we get

$$\int_{\overline{M}^*} K d\sigma + \sum_{i=1}^{k} \int_{\partial B(p_i, r_i)} k_g \, ds = 2\pi(2 - 2g - k). \tag{5.8}$$

Therefore, in order to prove the second item of the theorem is enough to show that for each i, we have

$$\int_{\partial B(p_i, r_i)} k_g \, ds = \int_{B(p_i, r_i)} K d\sigma.$$

In other words, we have to understand the geometry of the ends. Let us analyse each end.

Fix i, denote $E = B^*(p_i, r_i)$ and let $X = (F, h) : U = \{|z| \geq R_0\} \to \mathbb{H}^2/[\psi] \times \mathbb{S}^1$ be a conformal parametrization of the end E. In this parameter we express the metric as $ds^2 = \lambda^2 |dz|^2$ with $\lambda^2 = 4(\cosh^2\omega)|\phi|$, where $\phi(dz)^2 = Q$ is the holomorphic quadratic differential on the end.

If $Q \equiv 0$ then $\phi \equiv 0$ and $h \equiv$ constant, what yields that the end E of Σ is contained in some slice $\mathbb{H}^2/[\psi] \times \{c_0\}$. Then, in fact, the minimal surface Σ is the slice $\mathbb{H}^2/[\psi] \times \{c_0\}$. Note that by our hypothesis on Σ this case is possible only when the horizontal slices of $\mathcal{M}$ have finite area. Therefore, we can assume $Q \not\equiv 0$.

Following the ideas of [15] and section 3 of [20], we can show that finite total curvature and non-zero Hopf differential Q implies that Q has a finite number of isolated zeroes on the surface Σ. Moreover, for $R_0 > 0$ large enough we can show that there is a constant α such that $(\cosh^2\omega)|\phi| \leq |z|^\alpha|\phi|$ and then, as the metric ds^2 is complete, we use a result by Osserman [44] to conclude that Q extends meromorphically to the puncture $z = \infty$. Hence we can suppose that ϕ has the following form:

$$\phi(z) = \left(\sum_{j\geq 1} \frac{a_{-j}}{z^j} + P(z)\right)^2,$$

for $|z| > R_0$, where P is a polynomial function.

Since ϕ has a finite number of zeroes on U, we can suppose without loss of generality that ϕ has no zeroes on U, and then the minimal surface E is transverse to the horizontal sections $\mathbb{H}^2/[\psi] \times \{c\}$.

As in a conformal parameter z, we express the metric as $ds^2 = \lambda^2|dz|^2$, where $\lambda^2 = 4(\cosh^2\omega)|\phi|$, then on U

$$-K_\Sigma\lambda^2 = 4(\sinh^2\omega)|\phi| + \frac{|\nabla\omega|^2}{\cosh^2\omega} \geq 0. \tag{5.9}$$

Hence,

$$\begin{aligned} -\int_U K dA &= \int_U 4(\sinh^2\omega)|\phi||dz|^2 + \int_U \frac{|\nabla\omega|^2}{\cosh^2\omega}|dz|^2 \\ &= \int_U 4(\cosh^2\omega)|\phi||dz|^2 - \int_U 4|\phi||dz|^2 + \int_U \frac{|\nabla\omega|^2}{4(\cosh^4\omega)|\phi|}dA \\ &= \text{area}(E) - 4\int_U |\phi||dz|^2 + \int_U \frac{|\nabla\omega|^2}{4(\cosh^4\omega)|\phi|}dA, \end{aligned}$$

where the last term in the right hand side is finite by (5.7), once we have finite total curvature.

By the above equality, we conclude that area(E) is finite if, and only if, $\phi = \left(\sum_{j\geq 2}\frac{a_{-j}}{z^j}\right)^2$. Equivalently, area$(E)$ is infinite if, and only if, $\phi = \left(\sum_{j\geq 1}\frac{a_{-j}}{z^j} + P(z)\right)^2$, with $P \not\equiv 0$ or $a_{-1} \neq 0$.

Claim 1: If the area of the end is infinite, then the function ω goes to zero uniformly at infinity.

Proof. To prove this we use estimates on positive solutions of sinh-Gordon equations by Han [14], Minsky [38] and Wan [54] to our context.

Given V any simply connected domain of $U = \{|z| \geq R_0\}$, we have the conformal coordinate $w = \int\sqrt{\phi}dz = u + iv$ with the flat metric $|dw|^2 =$

$|\phi||dz|^2$ on V. In the case where $P \not\equiv 0$, the disk $D(w(z), |z|/2)$ contains a ball of radius at least $c|z|$ in the metric $|dw|^2$ where c does not depend on z. In the case where $P \equiv 0$ and $a_{-1} \neq 0$, we consider the conformal universal covering $\tilde{U}$ of the annulus U given by the conformal change of coordinate $w = \ln(z)+f(z)$, where $f(z)$ extends holomorphically by zero at the puncture. Any point z in U lifts to the center $w(z)$ of a ball $D(w(z), \ln(|z|/2)) \subset \tilde{U}$ for $|z| > 2R_0$ large enough.

The function ω lifts to the function $\tilde{\omega} \circ w(z) := \omega(z)$ on the w-plane which satisfies the equation

$$\Delta_{|\phi|}\tilde{\omega} = 2\sinh 2\tilde{\omega}$$

where $\Delta_{|\phi|}$ is the Laplacian in the flat metric $|dw|^2$. On the disk $D_{|\phi|}(w(z), 1)$ we consider the hyperbolic metric given by

$$d\sigma^2 = \mu^2|dw|^2 = \frac{4}{(1-|w-w(z)|^2)^2}|dw|^2.$$

Then μ takes infinite values on $\partial D(w(z), 1)$ and since the curvature of the metric $d\sigma^2$ is $K = -1$, the function $\omega_2 = \ln\mu$ satisfies the equation

$$\Delta_{|\phi|}\omega_2 = e^{2\omega_2} \geq e^{2\omega_2} - e^{-2\omega_2} = 2\sinh\omega_2,$$

Then the function $\eta(w) = \tilde{\omega}(w) - \omega_2(w)$ satisfies

$$\Delta_{|\phi|}\eta = e^{2\tilde{\omega}} - e^{-2\tilde{\omega}} - e^{2\omega_2} = e^{2\omega_2}\left(e^{2\eta} - e^{-4\omega_2}e^{-2\eta} - 1\right),$$

which can be written in the metric $d\tilde{\sigma}^2 = e^{2\omega_2}|dw|^2$ as

$$\Delta_{\tilde{\sigma}}\eta = e^{2\eta} - e^{-4\omega_2}e^{-2\eta} - 1.$$

Since ω_2 goes to $+\infty$ on the boundary of the disk $D_{|\phi|}(w(z), 1)$, the function η is bounded above and attains its maximum at an interior point q_0. At this point $\eta_0 = \eta(q_0)$ we have

$$e^{2\eta_0} - e^{-4\omega_2}e^{-2\eta_0} - 1 \leq 0.$$

which implies

$$e^{2\eta_0} \leq \frac{1+\sqrt{1+4a^2}}{2},$$

where $a = e^{-2\omega_2(q_0)} \leq \sup\frac{1}{\mu^2} \leq \frac{1}{4}$. Thus at any point of the disk $D_{|\phi|}(w(z), 1)$, $\tilde{\omega}$ satisfies

$$\tilde{\omega} \leq \omega_2 + \frac{1}{2}\ln(\frac{2+\sqrt{5}}{4}).$$

We observe that the same estimate above holds for $-\tilde{\omega}$. Then at the point z, we have

$$|\omega(z)| = |\tilde{\omega}(w(z))| \leq \ln 4 + \frac{1}{2}\ln(\frac{2+\sqrt{5}}{4}) := K_0$$

uniformly on $R \geq R_0$. Using this estimate we can apply a maximum principle as considered by Minsky (see [38], Lemma 3.3). We know that for $|z|$ large, we can find a disk $D_{|\phi|}(w(z), r)$ with r large too and the metric $|dw|^2 = |\phi||dz|^2$ is flat. If (u, v) are Euclidean coordinates based at $w(z)$, we define a comparison function F on the disk $D_{|\phi|}(w(z), r)$ by

$$F(u, v) = \frac{K_0}{\cosh r}\cosh\sqrt{2}u\cosh\sqrt{2}v.$$

Then $F \geq K_0 \geq \omega$ on $\partial D_{|\phi|}(w(z), r)$, $\Delta_{|\phi|}F = 4F$ everywhere and $F(w(z)) = \frac{K_0}{\cosh r}$. Suppose the minimum of $F - \tilde{\omega}$ is a point p_0 where $\tilde{\omega}(p_0) \geq F(p_0)$. Then $0 \leq \tilde{\omega}(p_0) \leq 2\sinh 2\tilde{\omega}(p_0)$ and

$$\Delta_{|\phi|}(F - \tilde{\omega})(p_0) = 4F(p_0) - 2\sinh 2\tilde{\omega}(p_0) \leq 4(F(q_0) - \tilde{\omega}(q_0)) \leq 0.$$

Therefore we have necessarily $\tilde{\omega} \leq F$ on the disk. Considering the same argument to $F + \tilde{\omega}$ we can conclude $|\tilde{\omega}| \leq F$. Hence

$$|\tilde{\omega}(w(z))| \leq \frac{K_0}{\cosh r} \tag{5.10}$$

and then $|\tilde{\omega}| \to 0$ uniformly at the puncture, consequently $|\omega| \to 0$ uniformly at infinity. □

Claim 2: If $P \not\equiv 0$ then the end E is not proper in $\mathcal{M}$.

Proof. Suppose $P \not\equiv 0$. Up to a change of variable, we can assume that the coefficient of the leading term of P is one. Then, for suitable complex number $a_0, ..., a_{k-1}$, we have

$$P(z) = z^k + a_{k-1}z^{k-1} + ... + a_0 \text{ and } \sqrt{\phi} = z^k(1 + o(1)).$$

Let us define the function

$$w(z) = \int \sqrt{\phi(z)}dz = \int \left(\sum_{j\geq 1}\frac{a_{-j}}{z^j} + a_0 + ... + z^k\right)dz.$$

If $a_{-1} = a + ib$ and we denote by $\theta \in \mathbb{R}$ a determination of the argument of $z \in U$, then locally

$$\mathrm{Im}(w)(z) = b\log|z| + a\theta + \frac{|z|^{k+1}}{k+1}(\sin(k+1)\theta + o(1)) \tag{5.11}$$

and

$$\mathrm{Re}(w)(z) = a\log|z| - b\theta + \frac{|z|^{k+1}}{k+1}(\cos(k+1)\theta + o(1)). \tag{5.12}$$

If $C_0 > \max\{|\mathrm{Im}(w)(z)|; |z| = R_0\}$, then the set $U \cap \{\mathrm{Im}(w)(z) = C_0\}$ is composed of $k+1$ proper and complete curves without boundary $L_0, ..., L_k$ (see Figure 5.6).

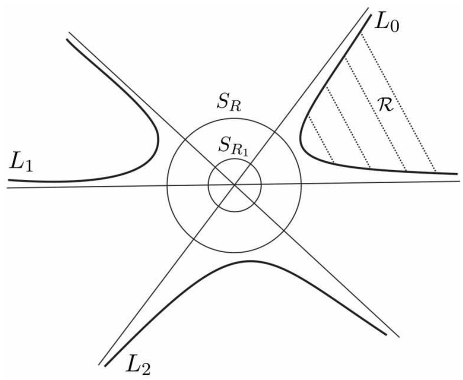

Figure 5.6: L_j for $k = 2$.

Take $\mathcal{R}$ a simply connected component of $U \cap \{\mathrm{Im}(w)(z) \geq C_0\}$. The holomorphic map $w(z)$ gives conformal parameters $w = u + iv, v \geq C_0$, to $X(\mathcal{R}) \subset E$.

Then $\widetilde{X}(w) = (\widetilde{F}(w), v)$ is a conformal immersion of $\mathcal{R}$ in $\mathbb{H}^2 \times \mathbb{R}$ and we have

$$|\widetilde{F}_u|^2_\sigma = |\widetilde{F}_v|^2_\sigma + 1 \text{ and } \left\langle \widetilde{F}_u, \widetilde{F}_v \right\rangle_\sigma = 0.$$

Hence the holomorphic quadratic Hopf differential is

$$Q_{\widetilde{F}} = \phi(w)(dw)^2 = \frac{1}{4}\left(|\widetilde{F}_u|^2_\sigma - |\widetilde{F}_v|^2_\sigma + 2i\left\langle \widetilde{F}_u, \widetilde{F}_v \right\rangle_\sigma\right) = \frac{1}{4}(dw)^2$$

and the induced metric on these parameters is given by $ds^2 = \cosh^2\widetilde{\omega}|dw|^2$.

Consider the divergent curve $\gamma(v) = \widetilde{X}(u_0 + iv) = (\widetilde{F}(u_0, v), v)$. We have

$$d_{\mathbb{H}^2}(\widetilde{F}(u_0, C_0), \widetilde{F}(u_0, v)) \leq \int_{C_0}^{v} |\widetilde{F}_v| dv = \int_{C_0}^{v} |\sinh\tilde{\omega}| dv < \infty,$$

once we know $|\tilde{\omega}| \to 0$ at infinity by Claim 1.

Thus, when we pass the curve γ to the quotient by the third coordinate, we obtain a curve in E which is not properly immersed in the quotient space $\mathcal{M}$. Therefore, the claim is proved and we have $P \equiv 0$ necessarily. $\square$

Suppose $E \subset \mathcal{M}_+$. We have $E = X(U)$ homeomorphic to $\mathbb{S}^1 \times \mathbb{R}$. Up to translation (along a geodesic not contained in $\mathbb{T}(0)$), we can suppose that E is transverse to $\mathbb{T}(0)$. Then $E \cap \mathbb{T}(0)$ is k jordan curves $d_1, ..., d_j, \alpha_1, ..., \alpha_l, j + l = k$, where each d_i is homotopically zero in E and each α_i generates the fundamental group of $E, \pi_1(E)$.

We will prove that $l = 1$ necessarily and the subannulus bounded by α_1 is contained in $\cup_{s\geq 0} \mathbb{T}(s)$.

Assume $l \neq 1$. Then there exist $\alpha_1, \alpha_2 \subset \mathbb{T}(0)$ generators of $\pi_1(E)$. As $E \cong \mathbb{S}^1 \times \mathbb{R}$, there exists $F \subset E$ such that $F \cong \mathbb{S}^1 \times [0, 1]$ and $\partial F = \alpha_1 \cup \alpha_2$. So F is compact and its boundary is on $\mathbb{T}(0)$. By the maximum principle, $F \cap (\cup_{s<0} \mathbb{T}(s)) = \emptyset$. Hence $F \subset \cup_{s\geq 0} \mathbb{T}(s)$ and then, since $E \subset \mathcal{M}_+$, there exist a third jordan curve α_3 that generates $\pi_1(E)$ and another cylinder G such that $G \cap (\cup_{s<0} \mathbb{T}(0)) \neq \emptyset$ and ∂G is either $\alpha_1 \cup \alpha_3$ or $\alpha_2 \cup \alpha_3$, but we have just seen that such G can not exist. Therefore $l = 1$, that is, $E \cap \mathbb{T}(0) = \alpha \cup d_1 \cup ... \cup d_j$, where α generates $\pi_1(E)$. Moreover, the subannulus bounded by α is contained in $\cup_{s\geq 0} \mathbb{T}(s)$, and each $d_i \subset E$ bounds a disk on E contained in $\cup_{s\geq 0} \mathbb{T}(s)$.

Remark 8. The same holds true for $E \subset \mathcal{M}_-$, that is, if $E \subset \mathcal{M}_-$ and E is transversal to $\mathbb{T}(s)$ then $E \cap \mathbb{T}(s)$ is $l_s + 1$ curves $\alpha, d_1, ..., d_{l_s}$, where d_i is homotopically zero in E and α generates $\pi_1(E)$.

Take a point p in the horocycle $c(0) \subset \mathbb{H}^2$ and consider $e_1 = c(0)/[\psi]$, $e_2 = p \times \mathbb{R}/[T(h)]$. The curves e_1, e_2 are generators of $\pi_1(\mathbb{T}(0))$.

As $E \subset \mathcal{M}_+$ and $\pi_1(\mathcal{M}_+) = \pi_1(\mathbb{T}(0))$, we can consider the inclusion map $i_* : \pi_1(E) \to \pi_1(\mathbb{T}(0))$ and $i_*([\alpha]) = n[e_1] + m[e_2]$, where m, n are integers.

Case 1.1: $n = m = 0$. This case is impossible.

In fact, $n = m = 0$ implies that E lifts to an annulus in $\mathbb{H}^2 \times \mathbb{R}$ and we already know by Lemma 1 that is not possible.

Case 1.2: $n \neq 0, m = 0$.

We can assume, without loss of generality, that $\partial E \subset \mathbb{T}(0)$. Call $\widetilde{E}$ a connected component of $\pi^{-1}(E \cap \mathcal{M}_+)$ such that $\pi(\widetilde{E}) = E$. We have that $\widetilde{E}$ is a proper minimal surface and its boundary $\partial\widetilde{E} = \pi^{-1}(\partial E)$ is a curve in $d(0)$ invariant by ψ^n.

By the Trapping Theorem in [3], $\widetilde{E}$ is contained in a horizontal slab. Hence $\widetilde{h}|_{\widetilde{E}}$ is a bounded harmonic function, and then $h|_E$ is a bounded harmonic function defined on a punctured disk. Therefore h has a limit at infinity, and then we can say that Q extends to a constant at the origin, say zero. In particular, $\widetilde{h}$ has a limit at infinity.

The end of $\widetilde{E}$ is contained in a slab of width $2\epsilon > 0$ and by a result of Collin, Hauswirth and Rosenberg [2], $\widetilde{E}$ is a graph outside a compact domain of $\mathbb{H}^2 \times \mathbb{R}$. This implies that $\widetilde{E}$ has bounded curvature. Then there exists $\delta > 0$ such that for any $p \in E$, $B_E(p, \delta)$ is a minimal graph with bounded geometry over the disk $D(0, \delta) \subset T_pE$.

Now fix s and consider a divergent sequence $\{p_n\}$ in E. Applying hyperbolic translations to $\{p_n\}$(horizontal translations along a geodesic of $\mathbb{H}^2$ that sends p_n to a point in $\mathbb{T}(s)$), we get a sequence of points in $\mathbb{T}(s)$ which we still call $\{p_n\}$. As $\mathbb{T}(s)$ is compact, the sequence $\{p_n\}$ converges to a point $p \in \mathbb{T}(s)$ and the sequence of graphs $B_E(p_n, \delta)$ converges to a minimal graph $B_E(p, \delta)$ with bounded geometry over $D(0, \delta) \subset T_pE$.

As h has a limit at infinity, this limit disk $B_E(p, \delta)$ is contained in a horizontal slice. Then we conclude that $n_3 \to 1$ and $|\nabla h| \to 0$ uniformly at infinity, what yields a C^1-convergence of E to a horizontal slice. Now using elliptic regularity we get that E converges in the C^2-topology to a horizontal slice. In particular, the geodesic curvature of α_s goes to 1 and its length goes to zero, where α_s is the curve in $E \cap \mathbb{T}(s)$ that generates $\pi_1(E)$.

Denote by E_s the part of the end E bounded by ∂E and α_s. Applying the Gauss-Bonnet theorem for E_s, we obtain

$$\int_{E_s} K + \int_{\alpha_s} k_g - \int_{\partial E} k_g = 0.$$

By our analysis in the previous paragraph, we have $\int_{\alpha_s} k_g \to 0$, when $s \to \infty$. Then when we let s go to infinity, we get

$$\int_E K = \int_{\partial E} k_g,$$

as we wanted to prove.

Claim 3: If $m \neq 0$ then the area of the end is infinite.

Proof. In fact, consider $g : \Sigma \to \mathbb{R}$ the extrinsic distance function to $\mathbb{T}(0)$, that is, $g = d_{\mathcal{M}}(\ . \ , \mathbb{T}(0))$. Hence $|\nabla^{\mathcal{M}} g| = 1$ and $g^{-1}(s) = \Sigma \cap \mathbb{T}(s)$. We know for almost every s, $\Sigma \cap \mathbb{T}(s) = \alpha_s \cup d_1 \cup ... \cup d_l$, where α_s generates $\pi_1(E)$

and d_i is homotopic to zero in E. Then, by the coarea formula,

$$\begin{aligned}\int_{\{g\le s\}} 1dA &= \int_{-\infty}^{s}\left(\int_{\{g=\tau\}}\frac{ds_\tau}{|\nabla^\Sigma g|}\right)d\tau \;\ge\; \int_0^s |\alpha_\tau| d\tau \\ &\ge \int_0^s |e_2| d\tau \;=\; s|e_2|,\end{aligned}$$

where the last inequality follows from the fact we are supposing that $i_*[\alpha_s]$ has a component $[e_2]$, and in the last equality we use that the curve e_2 has constant length. Hence when we let s go to infinity, we conclude that the area of E is infinite. $\square$

So if $E \subset \mathcal{M}_+$ and $m \neq 0$, then the area of E is infinite. Also, we know by Lemma 2 that all the ends contained in $\mathcal{M}_-$ have infinite area. Thus we will analyse all these cases together using the commom fact of infinite area.

Suppose we have an end E with infinite area. We can assume without loss of generality that $\partial E \subset \mathbb{T}(0)$. We know that $\phi = \left(\sum_{j\ge 1}\frac{a_{-j}}{z^j}\right)^2$ with $a_{-1}\neq 0$ for $|z| \ge R_0$, and $|\omega| \to 0$ uniformly at infinity by Claim 1. In particular, we know that the tangent planes to the end become vertical at infinity.

Let $X : D^*(0,1) \subset \mathbb{C} \to \mathcal{M}$ be a conformal parametrization of the end from a punctured disk (we suppose, without loss of generality, that the punctured disk is the unit punctured disk). Now consider the covering of $D^*(0,1)$ by the halfplane $HP := \{w = u+iv, u < 0\}$ through the holomorphic exponential map $e^w : HP \to D^*(0,1)$. Hence, we can take $\hat{X} = X \circ e^w : HP \to \mathcal{M}$ a conformal parametrization of the end from a halfplane.

We denote by $h, \hat{h}$ the third coordinates of X and $\hat{X}$, respectively. We already know $h(z) = a \ln|z| + b\arg(z) + p(z)$ for $z \in D^*(0,1)$, where either a or b is not zero, and p is a polynomial function. Hence $|p(z)| \to 0$ when $|z| \to 0$ and $\hat{h}(w) = au + bv + \hat{p}(w)$, where $u = \text{Re }(w), v = \text{Im }(w)$ and $\hat{p}(w) = p(e^w)$.

As the halfplane is simply connected, consider $\widetilde{X} : HP \to \mathbb{H}^2\times\mathbb{R}$ the lift of $\hat{X}$ into $\mathbb{H}^2\times\mathbb{R}$. We have $\widetilde{X} = (\widetilde{F}, \widetilde{h})$, where $\widetilde{h}(w) = au + bv + \widetilde{p}(w)$, with $|\widetilde{p}(w)| \to 0$ when $|w| \to \infty$. Up to a conformal change of parameter, we can suppose that $\widetilde{h}(w) = au + bv$.

Observe $\partial\widetilde{E} = \widetilde{X}(\{u = 0\})$ and the curve $\{\widetilde{h} = c\}$ is the straight line $\{au + bv = c\}$. We have three cases to analyse.

Case 2.1: $a = 0, b \neq 0$, that is, the third coordinate satisfies $h(z) = b\arg(z) + O(|z|)$.

Without loss of generality we can suppose $b = 1$. Hence in this case, $\widetilde{h}(w) = v$ and $\partial\widetilde{E} = \widetilde{X}(\{u = 0\})$.

We have $\widetilde{X}(w) = (\widetilde{F}(w), v)$ a conformal immersion of $\widetilde{E}$, and

$$|\widetilde{F}_u|_\sigma^2 = |\widetilde{F}_v|_\sigma^2 + 1 \text{ and } \left\langle \widetilde{F}_u, \widetilde{F}_v \right\rangle_\sigma = 0.$$

Hence the holomorphic quadratic Hopf differential is

$$\widetilde{Q}_{\widetilde{F}} = \tilde{\phi}(w)(dw)^2 = \frac{1}{4}\left(|\widetilde{F}_u|_\sigma^2 - |\widetilde{F}_v|_\sigma^2 + 2i\left\langle \widetilde{F}_u, \widetilde{F}_v \right\rangle_\sigma\right) = \frac{1}{4}(dw)^2$$

and the induced metric on these parameters is given by $ds^2 = \cosh^2\widetilde{\omega}|dw|^2$.

Moreover, by (5.10) there exists a constant $K_0 > 0$ such that

$$|\widetilde{\omega}(w)| \leq \frac{K_0}{\cosh r}, \tag{5.13}$$

for $r = \sqrt{u^2 + v^2}$ sufficiently large.

Using Schauder's estimates and (5.13), we obtain

$$|\widetilde{\omega}|_{2,\alpha} \leq C\left(|\sinh \widetilde{\omega}|_{0,\alpha} + |\widetilde{\omega}|_0\right) \leq Ce^{-r}.$$

Then

$$|\nabla\widetilde{\omega}| \leq Ce^{-r}. \tag{5.14}$$

Now consider the curve $\gamma_c = \widetilde{E} \cap \mathbb{H}^2 \times \{v = c\}$, that is, $\gamma_c(u) = (\widetilde{F}(u,c), c)$. Let $(V, \sigma(\eta)|d\eta|^2)$ be a local parametrization of $\mathbb{H}^2$ and define the local function φ as the argument of $\widetilde{F}_u$, hence

$$\widetilde{F}_u = \frac{1}{\sqrt{\sigma}}\cosh \widetilde{\omega} e^{i\varphi} \text{ and } \widetilde{F}_v = \frac{i}{\sqrt{\sigma}}\sinh \widetilde{\omega} e^{i\varphi}.$$

If we denote by k_g the geosedic curvature of γ_c in $(V, \sigma(\eta)|d\eta|^2)$ and by k_e the Euclidean geodesic curvature of γ_c in $(V, |d\eta|^2)$, we have

$$k_g = \frac{k_e}{\sqrt{\sigma}} - \frac{\langle \nabla\sqrt{\sigma}, n\rangle}{\sigma},$$

where $n = (-\sin\varphi, \cos\varphi)$ is the Euclidean normal vector to γ_c. If t denotes the arclength of γ_c, we have

$$k_e = \varphi_t = \frac{\varphi_u\sqrt{\sigma}}{\cosh\widetilde{\omega}}$$

and

$$\frac{\langle \nabla\sqrt{\sigma}, n\rangle}{\sigma} = \frac{\langle \nabla\log\sqrt{\sigma}, n\rangle}{\sqrt{\sigma}} = \frac{1}{2\sqrt{\sigma}}\left(\cos\varphi(\log\sigma)_{\eta_2} - \sin\varphi(\log\sigma)_{\eta_1}\right).$$

Then,

$$k_g = \frac{\varphi_u}{\cosh\widetilde{\omega}} - \frac{1}{2\sqrt{\sigma}}\left(\cos\varphi(\log\sigma)_{\eta_2} - \sin\varphi(\log\sigma)_{\eta_1}\right). \tag{5.15}$$

In the complex coordinate w, we have

$$\widetilde{F}_w = \frac{e^{\widetilde{\omega}+i\varphi}}{2\sqrt{\sigma}} \text{ and } \widetilde{F}_{\bar{w}} = \frac{e^{-\widetilde{\omega}+i\varphi}}{2\sqrt{\sigma}}. \tag{5.16}$$

Moreover, the harmonic map equation in the complex coordinate $\eta = \eta_1 + i\eta_2$ of $\mathbb{H}^2$ (see [53], page 8) is

$$\widetilde{F}_{w\bar{w}} + (\log\sigma)_\eta \widetilde{F}_w \widetilde{F}_{\bar{w}} = 0. \tag{5.17}$$

Then using (5.16) and (5.17) we obtain

$$\begin{aligned} (-\widetilde{\omega} + i\varphi)_w &= -\sqrt{\sigma}\left(\tfrac{1}{\sqrt{\sigma}}\right)_w - (\log\sigma)_\eta \widetilde{F}_w \\ &= \tfrac{1}{2}(\log\sigma)_w - (\log\sigma)_\eta \widetilde{F}_w \\ &= \tfrac{1}{2}\left((\log\sigma)_\eta \widetilde{F}_w + (\log\sigma)_{\bar{\eta}} \bar{\widetilde{F}}_w\right) - (\log\sigma)_\eta \widetilde{F}_w \\ &= \tfrac{1}{2}(\log\sigma)_{\bar{\eta}} \bar{\widetilde{F}}_w - \tfrac{1}{2}(\log\sigma)_\eta \widetilde{F}_w, \end{aligned} \tag{5.18}$$

where $2(\log\sigma)_\eta = (\log\sigma)_{\eta_1} - i(\log\sigma)_{\eta_2}$ and $\bar{\widetilde{F}}_w = \frac{1}{2\sqrt{\sigma}} e^{-\widetilde{\omega}-i\varphi}$.

Taking the imaginary part of (5.18), we get

$$\varphi_u + \widetilde{\omega}_v = \frac{\cosh\widetilde{\omega}}{2\sqrt{\sigma}}\left(\cos\varphi(\log\sigma)_{\eta_2} - \sin\varphi(\log\sigma)_{\eta_1}\right). \tag{5.19}$$

By (5.15) and (5.19), we deduce

$$k_g = -\frac{\widetilde{\omega}_v}{\cosh\widetilde{\omega}}. \tag{5.20}$$

Therefore, by (5.13) and (5.14), when $c \to +\infty$, $k_g(\gamma_c)(u) \to 0$ and also when we fix c and let u go to infinity the geodesic curvature of the curve γ_c goes to zero. In particular, for c sufficiently large, the asymptotic boundary of γ_c consists in only one point (see [19], Proposition 4.1).

We will prove that the family of curves γ_c has the same boundary point at infinity independently on the value c. Fix u_0 and consider α_{u_0} the projection onto $\mathbb{H}^2$ of the curve $\widetilde{X}(u_0, v) = (\widetilde{F}(u_0, v), v)$, that is, $\alpha_{u_0}(v) = \widetilde{F}(u_0, v) \in \mathbb{H}^2$. We have $\alpha'_{u_0}(v) = \widetilde{F}_v$ and $|\alpha'_{u_0}(v)|_\sigma = |\sinh \widetilde{\omega}|$. Then

$$d(\alpha_{u_0}(v_1), \alpha_{u_0}(v_2)) \leq l(\alpha_{u_0}|_{[v_1,v_2]}) = \int_{v_1}^{v_2} |\sinh \widetilde{\omega}| dv \leq \int_{v_1}^{v_2} \sinh \mathrm{e}^{-r} dv,$$

where $r = \sqrt{u_0^2 + v^2}$. Thus, for any v_1, v_2, we have $d(\alpha_{u_0}(v_1), \alpha_{u_0}(v_2)) \to 0$ when $u_0 \to -\infty$.

Therefore, the asymptotic boundary of all horizontal curves γ_c in $\widetilde{E}$ coincide, and we can write $\partial_\infty \widetilde{E} = p_0 \times \mathbb{R}$.

Observe that as $\widetilde{h}|_{\partial \widetilde{E}}$ is unbounded, then we have two possibilities for $\partial \widetilde{E}$, either $\partial \widetilde{E}$ is invariant by a vertical translation or is invariant by a screw motion $\psi^n \circ T(h)^m, n, m \neq 0$.

Subcase 2.1.1: $\partial \widetilde{E}$ invariant by vertical translation and $E \subset \mathcal{M}_+$.

In this case, by the Trapping Theorem in [3], $\widetilde{E}$ is contained in a slab between two vertical planes that limit to the same vertical line at infinity, $p_0 \times \mathbb{R}$. Moreover, since $|\widetilde{\omega}| \to 0$, then we get bounded curvature by (5.7). The same holds true for E in $\mathcal{M}_+$.

Thus, using the same argument as in Case 1.2, we can show that in fact E converges in the C^2-topology to a vertical plane. Therefore, the geodesic curvature of α_s goes to zero and its length stays bounded, where α_s is the curve in $E \cap \mathbb{T}(s)$ that generates $\pi_1(E)$.

Applying the Gauss-Bonnet theorem for E_s, the part of the end E bounded by ∂E and α_s, we obtain

$$\int_{E_s} K + \int_{\alpha_s} k_g - \int_{\partial E} k_g = 0.$$

By our analysis in the previous paragraph, we have $\int_{\alpha_s} k_g \to 0$, when $s \to \infty$. Then, when we let s go to infinity, we get

$$\int_E K = \int_{\partial E} k_g,$$

as we wanted to prove.

Subcase 2.1.2: $\partial \widetilde{E}$ invariant by vertical translation and $E \subset \mathcal{M}_-$.

As $\partial \widetilde{E}$ invariant by vertical translation, then we can find a horizontal geodesic γ in $\mathbb{H}^2$ such that γ limits to p_0 at infinity and $\gamma \times \mathbb{R}$ does not intersect $\partial \widetilde{E}$. Call q_0 the other endpoint of γ. Take $q \in \partial_\infty \mathbb{H}^2$ contained in the halfspace determined by $\gamma \times \mathbb{R}$ that does not contain $\partial \widetilde{E}$. As the asymptotic boundary of $\widetilde{E}$ is just $p_0 \times \mathbb{R}$, then $\overline{qq_0} \times \mathbb{R}$ does not intersect $\widetilde{E}$ for q sufficiently close to q_0. Also note that for any q, $\overline{qq_0} \times \mathbb{R}$ can not be tangent at infinity to $\widetilde{E}$, because E is proper in $\mathcal{M}$. Thus, if we start with q close to q_0 and let q go to p_0, we conclude that in fact $\gamma \times \mathbb{R}$ does not intersect $\widetilde{E}$, by the maximum principle. Now if we consider another point $\bar{q_0} \in \partial_\infty \mathbb{H}^2$ contained in the same halfspace determined by $\gamma \times \mathbb{R}$ as $\partial\widetilde{E}$ and such that $\bar{\gamma} \times \mathbb{R} = \overline{\bar{q_0} p_0} \times \mathbb{R}$ does not intersect $\partial\widetilde{E}$, we can prove using the same argument above that $\bar{\gamma} \times \mathbb{R}$ does not intersect $\widetilde{E}$. Thus we conclude that $\widetilde{E}$ is contained in the region between two vertical planes that limit to $p_0 \times \mathbb{R}$.

As $|\widetilde{\omega}| \to 0$, we get bounded curvature by (5.7). So $E \subset \mathcal{M}_-$ is a minimal surface with bounded curvature contained in a slab bounded by two vertical planes that limit to the same point at infinity. Hence, using the same argument as in Case 1.2, we can show that E converges in the C^2-topology to a vertical plane. Therefore, as in Subcase 2.1.1 above, we get

$$\int_E K = \int_{\partial E} k_g.$$

Subcase 2.1.3: $\partial\widetilde{E}$ invariant by screw motion and $E \subset \mathcal{M}_+$.

In this case, by the Trapping Theorem in [3], $\widetilde{E}$ is contained in a slab between two parallel Helicoidal planes and, since $|\widetilde{\omega}| \to 0$, we get bounded curvature by (5.7). Then E is a minimal surface in $\mathcal{M}_+$ with bounded curvature contained in a slab between the quotient of two parallel Helicoidal planes.

Thus, using the same argument as in Case 1.2, we can show that in fact E converges in the C^2-topology to the quotient of a Helicoidal plane. In particular, the geodesic curvature of α_s goes to zero and its length stays bounded, where α_s is the curve in $E \cap \mathbb{T}(s)$ that generates $\pi_1(E)$.

Applying the Gauss-Bonnet theorem for E_s, the part of the end E bounded by ∂E and α_s, we obtain

$$\int_{E_s} K + \int_{\alpha_s} k_g - \int_{\partial E} k_g = 0.$$

By our previous analysis, we have $\int_{\alpha_s} k_g \to 0$, when $s \to \infty$. Then, when we let s go to infinity, we get

$$\int_E K = \int_{\partial E} k_g,$$

as we wanted to prove.

Subcase 2.1.4: $\partial\widetilde{E}$ invariant by screw motion and $E \subset \mathcal{M}_-$.

By Remark 8, we know that for almost every $s \leq 0$, $\widetilde{E} \cap d(s)$ contains a curve invariant by screw motion, so it is not possible to have $p_0 \times \mathbb{R}$ as the only asymptotic boundary. Thus this subcase is not possible.

Case 2.2: $a \neq 0$. We will show this is not possible.

Consider the change of coordinates by the rotation $e^{i\theta}w : HP \to \widetilde{HP}$, where $\tan\theta = \frac{a}{b}$ (notice that if $b = 0$, then $\theta = \pi/2$) and $\widetilde{HP} = e^{i\theta}(HP) \subset \{\tilde{w} = \tilde{u} + i\tilde{v}\}$. From now on, when we write one curve in the plane $\tilde{w} = \tilde{u} + i\tilde{v}$, we mean the part of this curve contained in $\widetilde{HP}$.

In this new parameter $\tilde{w}$, we have $\partial\widetilde{E} = \widetilde{X}(\{b\tilde{u} + a\tilde{v} = 0\})$, the curve $\{\widetilde{h} = c\}$ is the straight line $\{\tilde{v} = \frac{c}{\sqrt{a^2+b^2}}\}$. (See Figure 5.7).

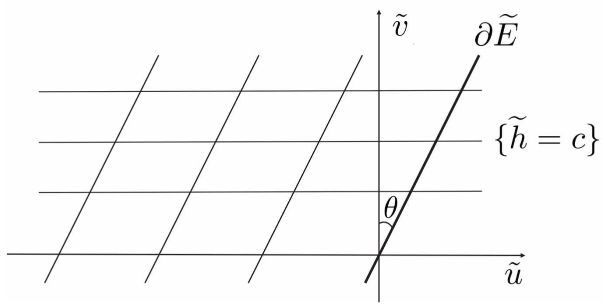

Figure 5.7: Parameter $\tilde{w} = \tilde{u} + i\tilde{v}$.

Now consider the curve $\beta(t) = (0, t), t \geq 0$. The angle between $\widetilde{X}(\beta)$ and $\partial\widetilde{E}$ is $\theta \neq 0$ and $\widetilde{X}(\beta)$ is a divergent curve in $\widetilde{E}$. However, the curve $\widetilde{F}(\beta) = \widetilde{F}(0, t)$ satisfies

$$l(\widetilde{F}(\beta)) = \frac{1}{|a|}\int_0^t |\widetilde{F}_{\tilde{v}}| d\tilde{v} = \frac{1}{|a|}\int_0^t |\sinh\widetilde{\omega}| d\tilde{v} \leq C,$$

for some constant C not depending on t, since we know by (5.10) that $|\tilde{\omega}| \to 0$ at infinity. This implies that when we pass the curve $\widetilde{X}(\beta)$ to the quotient space $\mathcal{M}$, we obtain a curve in E which is not proper in $\mathcal{M}$, what is impossible, since the end E is proper.

Therefore, analysing the geometry of all possible cases for the ends of a proper immersed minimal surface with finite total curvature Σ in $\mathcal{M}$, we have proved the theorem. $\square$

Remark 9. The case of a Helicoidal end contained in $\mathcal{M}_+$ is in fact possible, as shows the second example constructed in Section 2.4.3. (See Proposition ??).

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