# Regularization by Noise in Ordinary and Partial Differential Equations

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# Regularization by Noise in Ordinary and Partial Differential Equations

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## 0.1 Preface.

I propose this mini-course with the purpose of encouraging young Brazilian researchers to study stochastic partial differential equations (SPDEs). Despite being a subject of great international relevance, in Brazil is not yet well explored by the analysis and probability communities.

In this mini-course I will present one of the aspects of great interest by the stochastic analysis community, the effect of noise in deterministic dynamical systems. I will examine some aspects regarding the effects of noise on ordinary differential equations (ODEs) and partial differential equations (PDEs). This research field is extremely interesting and incredibly wide, with works in many different directions. I will focus only on the fundamental issue of wellposedness.

From a more theoretical point of view, the noise may help stabilize some PDEs, in the sense that a stochastic partial differential equation (SPDE) can be well posed under more general hypotheses than its deterministic counterpart. This is phenomenon known as regularization by noise. Besides the theoretical importance the fundamental motivation is to study the equations of fluid dynamics under random perturbations. I recommend the works of R. Mikulevicius, B. Rozovskii (see [72] and [73]) and several works of F. Flandoli (see for example [22], [45] and [46]).

I would like to express my gratitude to Professor Pedro Catougno for introducing me in SPDEs some years ago. I would lite to thank the colleagues with whom I work on regularization by noise : P. Catuogno, E. Fedrizzi. F. Flandoli, M. Maurelli, D. Mollinedo, W. Neves and C. Tudor.

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I dedicate this book to my wife Isabella.

# Contents

# Chapter 1

# A crash in stochastic analysis .

In this chapter we give an elementary summary on the theory of stochastic calculus : Martingale, Brownian motion, Itô and Stratonovich Integral, Itô formula. For further study we recommend the excellent monographs: Kunita in [63], Mao in [71], Protter in [88], Karatzas and Shreve in [57].

### 1.0.1 Probability Space.

We note $\big(\Omega, \mathcal{F}, \{\mathcal{F}_t : t \in [0,T]\}, \mathbb{P}\big)$ a probability space where the filtration is a family $\mathcal{F}_t$ of increasing sub-$\sigma$-algebras of $\mathcal{F}$, that is, $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$ if $0 \leq s \leq t < \infty$. The filtration is said to be continuous if $\mathcal{F}_t = \cap_{s>t} \mathcal{F}_s$. When the probability space is complete, the filtration is said to verifies the usual conditions if it is right continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets.

A stochastic process with state in the metric space is a collection of random variables $X_t,\ t \in T$ defined on some probability space. The set $T$ is called its parametric set. In the later case the usual example is $T = [0,T]$.

For every fixed $\omega$, the mapping

$$t \to X_t(\omega)$$

defined on the parameter set $T$ is called a realization, trajectory, sample path or sample function of the process.

### 1.0.2 Continuous Time Martingales.

A real continuous time process $M = \{M_t\}$ is called martingale with respect to the filtration $\mathcal{F}_t$ if

- For each $t \geq 0$, $M_t$ is $\mathcal{F}_t$-measurable .
- For each $t \geq 0$, $\mathbb{E}|M_t| < \infty$.
- For each $s \leq t$, $\mathbb{E}[M_t \big| \mathcal{F}_s] = M_s$.

We have the following version of Doob maximal inequality.

**Proposition 1.0.1.** *Let $\{M_t, 0 \leq t \leq T\}$ be a martingale with continuous trajectories. Then, for all $p \geq 1$ and $\lambda > 0$ we have*

$$\mathbb{P}\big(\sup_{0\leq t\leq T} |M_t| > \lambda\big) \leq \frac{1}{\lambda^p}\mathbb{E}|M_T|^p.$$

### 1.0.3 Brownian motion.

Now we fast forward to 1827 where Robert Brown, a British botanist, is observing a suspended pollen grain in water. While looking at this pollen grain underneath a microscope, he notices that it undergoes a type of random walk. This random motion is now referred to a Brownian motion.

The mathematical definition of the Brownian motion is the following

**Definition 1.0.2.** *A stochastic $\{B_t, t \geq 0\}$ is called a Brownian motion if it verifies the next conditions*

- *$B_0 = 0$.*
- *For all $0 \leq t_1 < t_2 ... < t_n$ the increments, $B_{t_n} - B_{t_{n-1}}, ...$ ,$B_{t_2} - B_{t_1}$, are independent random variables.*
- *if $0 \leq s < t$, the increments $B_t - B_s$ has the normal distribution $N(0.t - s)$.*

- *The process $\{B_t\}$ has continuous trajectories.*

We observe that

$$\mathbb{E}[B_t] = 0,$$

$$\mathbb{E}[B_t B_s] = \mathbb{E}[(B_t - B_s)B_s + B_s^s]$$

$$= E[(B_t - B_s)]\mathbb{E}[B_s] + \mathbb{E}[B_s^s] = 0 + s = s$$

if $s \leq t$.
The $d$-dimensional process

$$B_t = (B_t^1, B_t^2, ..., B_t^d)$$

is called $d$-dimensional Brownian motion if $B_t^1, B_t^2, ..., B_t^d$ are independent standard Brownian motions.

### 1.0.4 Itô Integral.

In this section we define the stochastic integral respect to the standard Brownian motions. Since for almost all $\omega \in \Omega$, the Brownian trajectory $B_t(\omega)$ is nowhere differentiable, the integral can not be defined in the ordinary way. The integral was introduced by Itô in 1949 and it is know as Itô integral.

We denoted $L^2$ the space of stochastic processes $X_t$ such that

- $X_t$ is $\mathcal{F}_t$-adapted,
- $\mathbb{E} \int_0^T |x_t|^2 \, dt < \infty$.

The idea to define the stochastic integral for a class of simple process. Then we can extend the integral for process in $L^2$.

A real stochastic process $X_t$ is called simple (or step) if there exists a partition $0 = t_0 < t_1 < .. < t_k = T$ such that

$$X_t = \sum_{i=0}^{k-1} \xi_i 1_{(t_i, t_{i+1}]}.$$

Now, we give the definition of the stochastic integral for simple processes. We define

$$\int_0^T X_t dB_t = \sum_{i=0}^{k-1} \xi_i (B_{t_{i+1}} - B_{t_i})$$

and call it the stochastic integral of $X_t$ with respect to the Brownian motions. The integral has the following properties

- $\int_0^T c_1 X_t + c_2 Y_t \ dB_t = c_1 \int_0^T X_t dB_t + c_2 \int_0^T Y_t dB_t$ ,
- $\mathbb{E}| \int_0^T X_t \ dB_t|^2 ds = \mathbb{E} \int_0^T |X_t|^2 \ dt < \infty.$

In order to extend integral for processes in $L^2$ we present the following proposition.

**Proposition 1.0.3.** *For any $X_t \in L^2$ there exists a sequence $X_t^n$ of simple process such that*

$$\lim_{n\to\infty} \mathbb{E} \int_0^T |X_t^n - X_t|^2 \ dt = 0.$$

Now, we extend the Itô integral for $L^2$-process as

$$\int_0^T X_t dB_t = \lim_{n\to\infty} \int_0^T X_t^n dB_t.$$

We observe that

$$\begin{aligned}\mathbb{E}| \int_0^T X_t^n dB_t - \int_0^T X_t^m dB_t|^2 &= \mathbb{E}| \int_0^T X_t^n - X_t^m dB_t|^2 \\ &= \mathbb{E}| \int_0^T |X_t^n - X_t^m|^2 dt.\end{aligned}$$

Thus $\int_0^T X_t^n dB_t$ is a Cauchy sequence in $L^2(\Omega)$. Using the same arguments it easy to see that the integral does not dependent of the approximate sequence.

Now, we consider a stochastic process in the space $L^2$. Then, for any $t \in [0,T]$ we define the indefinite Itô integral as

$$\int_0^t X_t dB_t := \int_0^T 1_{[0,t]}(s) X_s dB_s.$$

The Itô integral has the nice properties.

- $M_t = \int_0^t X_s dB_s$ is a Martingale with respect to the filtration $\mathcal{F}_t$,
- $\mathbb{E}|\int_0^t X_s dB_s|^2 ds = \mathbb{E}\int_0^t |X_s|^2 \, ds < \infty$ ( Itô Isometry ),
- $\int_0^t c_1 X_s + c_2 Y_s dB_s = c_1 \int_0^T X_s dB_s + c_2 \int_0^T Y_s dB_s$ ,
- $\mathbb{E}(\int_0^t X_s dB_s) = 0.$

We observe that when $X_t$ has continuous path we have

$$\int_0^T X_t dB_t = \lim_{|P|\to 0} \sum_{i=0}^{k-1} X_{t_i}(B_{t_{i+1}} - B_{t_i}).$$

where $P$ is any partition with $|P| \to 0$. The the Itô integral is the Rieman sums evaluated in $t_i$.

### 1.0.5 Covariation and Stratonovich Integral.

We consider another type of stochastic integral, the Stratonovich stochastic integral. This stochastic integral was introduced by Fisk, and independently by Stratonovich in 1966. The mathematical theory can be found in [63] and [88]. Both, the Itô and Stratonovich integrals, are defined in a mathematically correct way. In applications one has to make a decision about which stochastic integral is appropriate.

We consider a continuous adapted $X_t$. The Stratonovich stochastic is defined as

$$\int_0^T X_t \circ dB_s = \lim_{|P|\to 0} \sum_{i=0}^{k-1} (\frac{X_{t_i} + X_{t_{t_i+1}}}{2})(B_{t_{i+1}} - B_{t_i}).$$

where $P$ is any partition with $|P| \to 0$.

Now, we consider $X_t, Y_t$ continuous adapted processes. The covariation is defined as

$$[X,Y] = \lim_{|P|\to 0} \sum_{i=0}^{k-1} (X_{t_{i+1}} - X_{t_i})(Y_{t_{i+1}} - Y_{t_i})$$

where $P$ is any partition with $|P| \to 0$.
The connection between Itô and Stranovich is

$$\int_0^T X_t \circ dB_s = \int_0^T X_t dB_s + \frac{1}{2}[X,B]. \tag{1.1}$$

We observe that if $Y_t$ has bounded variation then

$$[X,Y] = \lim_{|P|\to 0} \sum_{i=0}^{k-1} (X_{t_{i+1}} - X_{t_i})(Y_{t_{i+1}} - Y_{t_i})$$

$$= \lim_{|P|\to 0} \sup_{|P|} |(X_{t_{i+1}} - X_{t_i})| \sum_{i=0}^{k-1} |Y_{t_{i+1}} - Y_{t_i}|$$

$$\leq C \lim_{|P|\to 0} \sup_{|P|} |(X_{t_{i+1}} - X_{t_i})| = 0.$$

Finally, we recall if $B_t = X_t = Y_t$ then

$$\lim_{|P|\to 0} \mathbb{E}|\sum_{i=0}^{k-1} (B_{t_{i+1}} - B_{t_i})^2 - t|^2$$

$$= t^2 - 2t \lim_{|P|\to 0} \sum_{i=0}^{k-1} \mathbb{E}(B_{t_{i+1}} - B_{t_i})^2$$

$$+ \lim_{|P|\to 0} |\sum_{i=0}^{k-1} \mathbb{E}(B_{t_{i+1}} - B_{t_i})^2|^2 = 0,$$

from this we conclude that $[B_t, B_t] = t$.

### 1.0.6 Itô Formula

Itô formula is for stochastic calculus what the Newton-Leibnitz formula is for classical calculus. It also provides a practical method for computation of stochastic integrals.

Assume that $X_t$ can be expressed in the form

$$X_t = X_0 + \int_0^t u_s \; ds + \int_0^t v_s \; dB_s$$

where $v \in L^2$ and $\int_0^T |u_s| ds < \infty$.

**Theorem 1.0.4.** *(d=1) Let $F : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ and $F \in C^{1,2}$. Then*

$$F(X_t) = F(X_0) \; + \int_0^t \partial_s F(s, X_s) ds$$

$$+ \int_0^t \partial_x F(s, X_s) u_s ds + \int_0^t \partial_x F(s, X_s) \; v_s \circ dB_s,$$

*writing in Itô way is*

$$F(X_t) = F(X_0) \; + \int_0^t \partial_s F(s, X_s) ds + \int_0^t \partial_x F(s, X_s) u_s ds$$

$$+ \int_0^t \partial_x F(s, X_s) v_s dB_s + \frac{1}{2} \int_0^t \partial_x^2 F(s, X_s) v_s^2 ds.$$

Now, we present the multidimensional Itô formula. We assume

$$X_t = X_0 + \int_0^t u_s \; ds + \int_0^t v_s \; dB_s$$

that is,

$$X_t^i = X_0^i + \int_0^t u_s^i \; ds + \sum_{j=0}^{m} \int_0^t v_s^{i,m} \; dB_s^j$$

where $v^i \in L^2$ and $\int_0^T |u_s^{i,j}| ds < \infty$.

**Theorem 1.0.5.** *Let* $F : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}$ *and* $F \in C^{1,2}$.
*Then*

$$F(X_t) = F(X_0) \ + \int_0^t \partial_s F(s, X_s) ds$$

$$+ \ \int_0^t \nabla F(s, X_s) u_s ds + \ \int_0^t \nabla F(s, X_s) \ v_s \circ dB_s,$$

*writing in Itô way is*

$$F(X_t) = F(X_0) \ + \int_0^t \partial_s F(s, X_s) ds + \ \int_0^t \nabla F(s, X_s) u_s ds$$

$$+ \int_0^t \nabla F(s, X_s) v_s dB_s + \frac{1}{2} \sum_{i,k} \int_0^t a_{i,j} \partial^2_{i,j} F(s, X_s) ds.$$

*where* $a_{i,j} = \sum_{k=0}^m v^{i,k} v^{k,j}$.

We also present the Itô formula for the product of two semimartingales

**Theorem 1.0.6.** *We assume that*

$$X_t = X_0 + \int_0^t u_s \ ds + \int_0^t v_s \ \circ dB_s$$

*and*

$$Y_t = y_0 + \int_0^t w_s \ ds + \int_0^t z_s \ \circ dB_s.$$

*Then*

$$X_t Y_t = X_0 Y_0 + \int_0^t Y_s u_s \ ds + \int_0^t Y_s v_s \ \circ dB_s$$

$$+ \int_0^t X_s w_s \ ds + \int_0^t X_s z_s \ \circ dB_s.$$

Finally, we present Itô-Wentzell-Kunita formula .

**Theorem 1.0.7.** *We assume that*

$$X_t(x) = X_0(x) + \int_0^t u_s(x)\ ds + \int_0^t v_s(x)\ \circ dB_s$$

*and*

$$Y_t = Y_0 + \int_0^t w_s\ ds + \int_0^t z_s\ \circ dB_s$$

*Also we assume that* $X_t(.) \in C^3(\mathbb{R}^d)$ *and* $, w_s(.), z_s(.) \in C^2(\mathbb{R}^d)$. *Then*

$$X_t(Y_t) = X_0(Y_0) + \int_0^t u_s(Y_s)\ ds + \int_0^t v_s(Y_s)\ \circ dB_s$$

$$+ \int_0^t (\nabla X_s)(Y_s)\ w_s\ ds + \int_0^t (\nabla X_s)(Ys)\ z_s\ \circ dB_s.$$

### 1.0.7 Girsanov Transformation.

We suppose that $v_t$ is a $\mathcal{F}_s$-adapted process and that

$$\int_0^t |v_t^2| ds < \infty.$$

We denote

$$\mathcal{E}(\int_0^t v(s)\ dB_s) = e^{-\int_0^t v(s)\ dB_s - \frac{1}{2}\int_0^t |v_s^2| ds},$$

and

$$dQ = \mathcal{E}(\int_0^t v(s)\ dB_s) dP.$$

**Proposition 1.0.8.** *The process* $\int_0^t v(s)\ ds + B_t$ *is a Brownian motion under the measure* $Q$.

### 1.0.8 Continuity of Stochastic Processes.

The Kolmogorov continuity theorem is a theorem that guarantees that a stochastic process that verifies certain constraints on the moments of its increments be continuous.

We present the following version.

**Theorem 1.0.9.** *Suppose that* $X = \{X(t), t \in \mathbb{R}^d\}$ *is a stochastic process with values in the Banach Space* $(E, \|.\|)$ *such that the following estimation holds*

$$\mathbb{E}\big(\|X(t) - X(s)\|^\lambda\big) \leq C|t-s|^\alpha$$

*for* $\lambda > 0, \alpha > d$ *and* $t, s \in \mathbb{R}^d$. *Then there exists a constant* $C_0 = C_{d,\alpha,\lambda}(\omega)$ *such that*

$$\|X(t) - X(s)\| \leq \ C_0(\omega)|t-s|^\beta$$

*with* $\beta < \frac{\alpha - d}{\lambda}$.

# Chapter 2

# SDEs

We will concerned with the stochastic differential equation (SDE)

$$dX_t(x) = b(t, X_t(x))dt + \sigma(t, X_t(x))dB_t, \;\; X_s = x \in \mathbb{R}^d. \tag{2.1}$$

This is the short way of writing

$$X_t(x) = x + \int_s^t b(r, X_{s,r}(x))dr + \int_s^t \sigma(r, X_r(x))dB_r. \tag{2.2}$$

with $t \in [0, T]$, $x \in \mathbb{R}^d$ and where $(B_t)_{t \in [0,T]}$ is a $d$-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$ endowed with the filtration $(\mathcal{F}_t)_{t \in [0,T]}$.

## 2.1 Strong Solutions.

### 2.1.1 Existence and Uniqueness.

We will say that (2.1) or (2.2) has strong solution if there exists continuous adapted process $X_{s,t}$ such that verifies (2.2). We say that we have pathwise uniqueness if given two solutions $X_{s,t}$ and $Y_{s,t}$, then $\mathbb{P}(X_t = Y_t \; \forall \; s, t \; 0 \leq s < t)$.

We shall show existence and uniqueness when the coefficients $b$ and $\sigma$ are globally Lipschitz continuous.

**Theorem 2.1.1.** *Assume that $b$ and $\sigma$ are Lipschitz continuous. Then there exists pathwise solution of the SDE (2.1).*

*Proof.* We will utilize the iterative scheme. We define a sequence of adapted continuous processes by induction :

$$X_t^0 = x,$$

$$X_t^n = x + \int_s^t b(r, X_r^{n-1})dr + \int_s^t \sigma(r, X_r^{n-1})dB_r \tag{2.3}$$

Then we have

$$\mathbb{E} \sup_{s\leq u\leq t} |X_u^{n+1} - X_u^n|^p$$

$$\leq \int_s^t \mathbb{E}|b(r, X_r^{n-1}) - b(r, X_r^{n-1})dr$$

$$+ \mathbb{E} \sup_{s\leq u\leq t} [\int_s^u \sigma(r, X_r^n) - \sigma(r, X_r^{n-1})dB_r]$$

By Doob and Burkholder inequality we have

$$\mathbb{E} \sup_{s\leq u\leq t} |X_u^{n+1} - X_u^n|^p$$

$$\leq L^p(t-s)^{\frac{p}{q}} \int_s^t \mathbb{E}|X_r^n - X_r^{n-1}|^p dr$$

$$+ L^p(t-s)^{\frac{p}{2}-1} \int_s^u \mathbb{E}|X_r^n - X_r^{n-1}|^p dr$$

Thus

$$\mathbb{E} \sup_{s\leq u\leq t} |X_u^{n+1} - X_u^n|^p$$

$$\leq C \int_s^t \mathbb{E}|X_r^n - X_r^{n-1}|^p dr.$$

We denoted

$$m_t^n := \mathbb{E} \sup_{s \le u \le t} |X_u^{n+1} - X_u^n|^p.$$

Therefore we get

$$m_t^n \le \int_s^t m_r^{n-1} \, dr.$$

By iteration we obtain

$$m_t^n \le \int_s^t \le \frac{C^n}{n!} T^n m_t^0.$$

Then deduce that

$$\sum_{n=0}^{\infty} \mathbb{E} \sup_{s \le u \le t} |X_u^{n+1} - X_u^n|^p < \infty.$$

Therefore we conclude that $X_t^n$ converge uniformly in $[s, t]$ and in $L^p$ to $X_t$. Passing to the limit in equation (2.3) we obtain that $X_t$ is the strong solution of the SDE (2.1).

□

**Theorem 2.1.2.** *Assume that $b$ and $\sigma$ are Lipschitz continuous. Then pathwise uniqueness for the SDE (2.1) hold.*

*Proof.* Suppose that $X_{s,t}$ and $Y_{s,t}$ are two solution. Then for all $0 \le s < t \le T$ we have

$$X_{s,t} - Y_{s,t} = \int_s^t b(r, X_r) - b(r, Y_r) dr + \int_s^t \sigma(r, X_r) - \sigma(r, Y_r) dB_r.$$

Thus

$$\mathbb{E}|X_{s,t} - Y_{s,t}|^2 \le \; 2\mathbb{E}|\int_s^t b(r, X_r) - b(r, Y_r) dr|^2 dr$$
$$+ \mathbb{E}|\int_s^t \sigma(r, X_r) - \sigma(r, Y_r) dB_r|^2 dr.$$

By Cauchy-Schwarts and Itô isometry we obtain

$$\mathbb{E}|X_{s,t} - Y_{s,t}|^2 \leq C_T \int_s^t \mathbb{E}|X_{s,t} - Y_{s,r}|^2 dr.$$

From Gronwalls Lemma we conclude that $Y_t = X_t$ .Then for all rational numbers we have $Y_t = X_t$ for almost all $\omega$. By the continuity we conclude that $Y_t = X_t$ for all $t \in [s, T]$ and for allmost all $\omega$.

□

### 2.1.2 Flow Properties.

We recall the relevant definition from H. Kunita [63]

**Definition 2.1.3.** *A stochastic flow of diffeomorphisms ( resp. the $C^{m,\alpha}$), associated to equation (2.1) is a map $(s, t, x, \omega) \to \phi_{s,t}(x)(\omega)$ defined for $0 \leq s \leq t$, $x \in \mathbb{R}^d$ $\omega \in \Omega$ with values in $\mathbb{R}^d$ such that*

- *given any $s \geq 0$, $x \in \mathbb{R}^d$ the process $X_t^{s,x} = \phi_{s,t}(x)$ is continuous $\mathcal{F}_{s,t}$ measurable solution of the equation (2.1),*
- *$\mathbb{P}$-a.s, for $0 \leq s \leq t$ the function, $\phi_{s,t}$ is a diffeomorphisms and the functions $\phi_{s,t}, \phi_{s,t}^{-1}, D^m\phi_{s,t}, D^m\phi_{s,t}^{-1}$, are continuous in $(s, t, x)$ (resp. the $C^{m,\alpha}$ class in $x$ uniformly in $0 \leq s \leq t \leq T$),*
- *$\mathbb{P}$-a.s, $\phi_{s,t} = \phi_{u,t}(\phi_{s,u})$ for all $0 \leq s \leq u \leq t$, $x \in \mathbb{R}^d$ and $\phi_{s,s} = x$*

We present the following relevant theorem on stochastic flows without proof. Unfortunately the rigorous proof contains a lot technical difficulties and is very long to see in a short course.

**Theorem 2.1.4.** *if $b, \sigma \in L^\infty([0, T], C_b^{m,\alpha}(\mathbb{R}^d))$. Then the map $x :\to X_t(x)$ is a stochastic flow of $C^{m,\alpha'}$-diffeomorphisms with $\alpha' < \alpha$.*

## 2.2 Weak Solutions.

**Definition 2.2.1.** *A weak solution of the SDE (2.1) is a triple $\big(X_t, B_t, \big(\Omega, \mathcal{F}, \{\mathcal{F}_t : t \in [0, T]\}, \mathbb{P}\big)\big)$ where*

1. *. $\big(\Omega, \mathcal{F}, \{\mathcal{F}_t : t \in [0, T]\}, \mathbb{P}\big)$ is a filtered probability space,*

2. *$X_t$ is a $\mathcal{F}_t$ continuous adapted process and $B_t$ is a standard Brownian motion,*

3. *$X_t$ verifies (2.2).*

Obviously, every strong solution is also a weak solution and a weak solution is a strong solution on the stochastic basis and relative to the Brownian motion which is part of the solution.

### 2.2.1 Weak Solutions via Girzanov Transformation.

**Proposition 2.2.2.** *We consider the SDE*

$$dX_t(x) = b(t, X_t(x))dt + dB_t, \ \ X(0) = x \in \mathbb{R}^d. \tag{2.4}$$

*We assume that $b$ is measurable function and satisfies*

$$\|b(t,x)\| \leq C(1 + |x|)$$

*for some positive constant $C$. Then the equation (2.4) has a weak solution.*

*Proof.* We consider $W_t$ a multidimensional Brownian motion. We set $X_t = x + W_t$. Then by Girsanov theorem

$$X_t - x - \int_0^t b(s, X_s) \ ds$$

is a Brownian motion under the measure

$$dQ = \mathcal{E}(\int_0^t v(s) \ dB_s)P.$$

Thus we have that

$$X_t = x + \int_0^t b(s, X_s) \ ds + B_t.$$

□

### 2.2.2 Existence.

**Proposition 2.2.3.** *We consider the SDE*

$$dX_t(x) = b(t, X_t(x))dt + \sigma(t, X_t)dB_t,\ X(0) = x \in \mathbb{R}^d. \tag{2.5}$$

*We assume $b, \sigma$ are continuous and satisfy*

$$\|\sigma(t,x)\|,\ \|b(t,x)\ | \leq C(1 + |x|)$$

*for some positive constant $C$. Then there exists a weak solution of the SDE (2.5).*

*Proof.* See [54] □

Finally we enunciate the Yamada-Watanabe theorem.

**Proposition 2.2.4.** *Weak existence and pathewise uniqueness imply pathewise existence.*

# Chapter 3

# SDE with Singular Drift.

SDEs with singular coefficients and driven by Brownian motion (more general noise) have been an important area of study in stochastic analysis and other related branches of mathematics. In this chapter we study in detail the case when the drift term is Hölder continuous and when the drift satisfies some globally integrability.

## 3.1 SDE with Hölder Drift.

In this section we follow the seminar paper by Flandoli, Gubinelli and Priola in [40]. We consider the SDE

$$dX_t(x) = b(t, X_t(x))dt + dB_t,\ \ X_s = x \in \mathbb{R}^d. \tag{3.1}$$

Let $T > 0$ be be fixed. For any $\alpha \in (0, 1)$, we denoted by $L^\infty([0, T], C_b^\alpha(\mathbb{R}^d))$ the space bounded Borel functions $f : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ such that

$$\|f\|_{\alpha,T} := \sup_{t\in[0,T]} \sup_{x\neq y, |x-y|\leq 1} \frac{|f(t, x) - f(t, y)|}{|x - y|^\theta} < \infty\,.$$

Moreover, for any $n \geq 1$ $f \in L^\infty([0,T], C_b^{n+\alpha}(\mathbb{R}^d))$ if all spatial derivatives belong to $L^\infty([0,T], C_b^\alpha(\mathbb{R}^d))$.

### 3.1.1 PDE Estimation.

We consider the following parabolic systems

$$\partial_t \psi_\lambda + \frac{1}{2}\Delta\psi_\lambda + bD\psi_\lambda - \lambda\psi_\lambda = f,\ (t,x) \in [0,T] \times \mathbb{R}^d, \tag{3.2}$$

where $f : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$.

We present the following Schauder estimates for the solution of (3.2).

**Theorem 3.1.1.** *We consider $f, b \in L^\infty([0,\infty), C_b^\alpha(\mathbb{R}^d))$. The there exists a unique solution $\psi_\lambda$ of the PDE (3.2) in the space $L^\infty([0,\infty), C_b^{2+\alpha}(\mathbb{R}^d))$. Moreover there exists a constant $C > 0$ such that*

$$\sup_{t\geq 0} \|\psi_\lambda\|_{C_b^{2+\alpha}} \leq C \sup_{t\geq 0} \|f\|_\alpha.$$

*Proof.* We refer to the reader to [43] and [60]. □

We also enunciate the next result.

**Lemma 3.1.2.** *We assume the assumptions of the theorem 3.1.1. Then the unique solution of the PDE (3.2) satisfies*

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} |D\psi_\lambda| \to 0 \text{ as } \lambda \to \infty.$$

### 3.1.2 Stochastic Flows.

**Theorem 3.1.3.** *We assume that $b \in L^\infty([0,\infty), C_b^\alpha(\mathbb{R}^d))$. Then*

*a) There exists a unique solution of the SDE (3.1).*

*b) There exists a stochastic flow $\phi_{s,t}$ of diffeomorphisms associated to equation (3.1). The flow is the class $C^{1+\alpha'}$ with $\alpha' < \alpha$.*

c) *Let $b^n \in L^\infty([0,\infty), C_b^\alpha(\mathbb{R}^d))$ be a sequence of the vector field and $\phi^n$ be the corresponding stochastic flow. If $b^n \to b$ in $L^\infty([0,\infty), C_b^\alpha(\mathbb{R}^d))$, the for any $p \geq 1$ we have*

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}^d} \sup_{s\in[0,T]} \mathbb{E}[\sup_{r\in[\ s,T]} |\phi_r^n - \phi_r|^p], \tag{3.3}$$

$$\sup_{x\in\mathbb{R}^d} \sup_{s\in[0,T]} \mathbb{E}[\sup_{r\in[\ s,T]} |\phi_r^n|^p] < \infty, \tag{3.4}$$

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}^d} \sup_{s\in[0,T]} \mathbb{E}[\sup_{r\in[\ s,T]} |D\phi_r^n - D\phi|^p]. \tag{3.5}$$

*Proof. Step 1: Zvonkin transformation .* We consider the parabolic systems

$$\partial_t \psi_\lambda - \frac{1}{2}\Delta\psi_\lambda + bD\psi_\lambda - \lambda\psi_\lambda = -b,\ (t,x) \in [0,T] \times \mathbb{R}^d. \tag{3.6}$$

By theorem 3.1.1 there exists a unique solution $\psi_\lambda$. We define

$$\varphi_\lambda = x + \psi_\lambda.$$

We claim that (for $\lambda$ large )

- $\varphi_\lambda$ has bounded first and second spatial derivative uniformly in time and $D^2\phi_\lambda \in C^\alpha(\mathbb{R}^d)$.
- For all $t \geq 0$, $\varphi_\lambda : \mathbb{R}^d \to \mathbb{R}^d$ is a diffeomorphisms of class $C^{2,\alpha'}$ with $\alpha' < \alpha$.
- The inverse $\varphi_\lambda^{-1}$ has bounded first and second spatial derivative uniformly in time.

The first item we follow from theorem 3.1.1. Since the $D\varphi_\lambda$ is not singular for $\lambda$ large and $\lim_{|x|\to\infty} \varphi(x) = \infty$ then the second point we follow from the classical Hadamard theorem.

Now, we observe that

$$D\varphi_\lambda^{-1} = [D\varphi_\lambda(\varphi_\lambda^{-1})]^{-1}$$

and

$$[D\varphi_\lambda(\varphi_\lambda^{-1})]^{-1} = \frac{1}{det(D\varphi_\lambda(\varphi_\lambda^{-1}))} Cof((D\varphi)(\varphi_\lambda^{-1}))^T$$

where $Cof$ denoted the matrix of cofactors of $D\varphi$. From this we have the last item.

*Step 2 : a) and b).* We define

$$\tilde{b}(t,x) = -\lambda\psi_\lambda(\varphi^{-1}(t,x)),$$

$$\tilde{\sigma}(t,x) = -D\varphi_\lambda(\varphi^{-1}(t,x))$$

and consider that following SDE

$$Y_t(x) = x + \int_s^t \tilde{b}(r, Y_r(x))dr + \int_s^t \tilde{\sigma}(r, Y_r(x))dB_r \qquad (3.7)$$

We note that $X_t$ solves equation (3.1) if only if $Y_t = \phi(X_t)$ solves equation (3.7). We observe that $\tilde{b}, \tilde{\sigma} \in L^\infty([0,T], C_b^{1+\alpha}(\mathbb{R}^d))$ then there a unique solution of the equation (3.7). Moreover, there exists a stochastic flow $\phi_{s,t}$ of diffeomorphism associated to equation (3.1) and the flow is the class $C^{1+\alpha'}$ with $\alpha' < \alpha$. This proves a) and b).

*Step 3: c).* We shall show only (3.3). Let $\psi^n$ the solution in $L^\infty([0,\infty), C_b^{2+\alpha}(\mathbb{R}^d))$ of the PDE (3.2) associated to $b^n$. Then $\psi - \psi^n$ satisfies

$$\partial_t(\psi-\psi^n) - \frac{1}{2}\Delta(\psi-\psi^n) + bD(\psi-\psi^n) - \lambda(\psi-\psi^n) = b - b^n + b^n D(\psi-\psi^n).$$

By theorem 3.1.1 we have

$$\|\psi - \psi^n\|_{L^\infty([0,\infty), C_b^{2+\alpha}(\mathbb{R}^d))} \leq C\|b^n - b\|_{L^\infty([0,T], C_b^\alpha(\mathbb{R}^d))}.$$

We set

$$\varphi^n = x + \psi^n.$$

Thus we have that $\tilde{\phi^n} = \varphi^n(\phi^n)$ verifies

$$\tilde{\phi^n}_t(x) = y + \int_s^t \tilde{b}^n(r, \tilde{\phi^n}_r(x))dr + \int_s^t \tilde{\sigma}^n(r, \tilde{\phi^n}_r(x))dB_r. \tag{3.8}$$

Then by classical arguments , see theorem II.3,1 in [63], we have

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}^d} \sup_{s\in[0,T]} \mathbb{E}\big[\sup_{r\in[s,T]} |\tilde{\phi^n_r} - \tilde{\phi}_r|^p\big] = 0.$$

We observe

$$\|\varphi^{n,-1}(\tilde{\phi^n}_t) - \varphi^{-1}(\tilde{\phi}_t)\|_\infty$$

$$\leq \|\varphi^{n,-1}(\tilde{\phi^n}_t) - \varphi^{-1}(\tilde{\phi^n_t})\|_\infty$$

$$+ \|\varphi^{-1}(\tilde{\phi^n}_t) - \varphi^{-1}(\tilde{\phi}_t)\|_\infty$$

$$\leq C\|b^n - b\|_{L^\infty([0,T],C_b^\alpha(\mathbb{R}^d))} + \|D\varphi^{-1}\|_\infty \|\tilde{\phi^n}_t - \tilde{\phi}_t\|,$$

thus we conclude (3.3).

□

### 3.1.3 One Exmaple.

We consider the ODE

$$X_t = \int_0^t min(1, |X_s|^\alpha)ds$$

with $0 \leq t \leq 1$. Then $X_t = 0$ is a solution and

$$X_t = \begin{cases} 0 & if\ 0 \leq t \leq \dfrac{1}{2}, \\ \left(\dfrac{t-\frac{1}{2}}{\beta}\right)^\beta & if\ \dfrac{1}{2} \leq t \leq 1 \end{cases}$$

with $\beta = \frac{1}{1-\alpha}$ is also a solution. However the equation with noise is well-posedness.

## 3.2 SDE with Integrable Drift.

In this section we follow Zhang in [97] and [98], see also Fedrizzi and Flandoli in [35]. We consider the SDE

$$dX_t(x) = b(t, X_t(x))dt + \sigma(t, X_t)dB_t,\ X_s = x \in \mathbb{R}^d, \tag{3.9}$$

with $0 \leq S \leq s \leq t \leq T$.
We assumed that

$$\begin{aligned} & b \in L^q\big([S,T]; L^p(\mathbb{R}^d)\big)\,, \\ \text{for} \qquad & p, q \in [2, \infty)\,, \qquad \frac{d}{p} + \frac{2}{q} < 1\,. \end{aligned} \tag{3.10}$$

This condition (with local integrability) was first considered by Krylov and Röckner in [59], where they proved the existence and uniqueness of strong solutions for the SDE (3.1) with $\sigma$ constant .

It is interesting to remark that condition (3.10) (more precisely with also equality) is known as the Ladyzhenskaya-Prodi-Serrin condition in the fluid dynamics literature.

The hypothesis on $\sigma$ are :

There exist constants $K \geq 1$ and $\alpha \in (0, 1)$ such that for all $(t, x) \in [S, T] \times \mathbb{R}^d$

$$K^{-1}|y| \leq |\sigma^t(t, x)y| \leq K|y|, \tag{3.11}$$

and for all $s \in [S, T]$ and $x, y \in \mathbb{R}^d$ we have

$$\|\sigma(t, x) - \sigma(t, y)\| \leq K|x - y|^\alpha. \tag{3.12}$$

We will have to use the spaces $L^q_p(S, T) = L^q((S, T), L^p(\mathbb{R}^d))$, $H^q_{\alpha,p} = L^q((S, T), W^{\alpha,p}(\mathbb{R}^d))$,$H^{\beta,q}_p = W^{\beta,q}((S, T), L^p(\mathbb{R}^d))$ and $\tilde{H}^q_{\alpha,p} = H^p_{\alpha,p} \cap H^{1,q}_p$. We denote $\mathbf{H}^\alpha_p := (Id - \Delta)^{-\frac{\alpha}{2}} L^p$.

We recall that by Sobolev embedding

$$\|f\|_{C^\delta_b(\mathbb{R}^d)} \leq \|f\|_{\mathbf{H}^\beta_p}\ if\ \beta - \delta > \frac{p}{d}.$$

### 3.2.1 SDE without Drift.

We consider the following SDE

$$dX_t(x) = \sigma(t, X_t)dB_t,\ X_s = x \in \mathbb{R}^d. \tag{3.13}$$

#### PDE Estimation.

We consider the PDE

$$\partial_t \psi + L^\sigma \psi + f = 0,\ \psi(T) = 0, \tag{3.14}$$

where

$$L^\sigma = \frac{1}{2}\sigma^{i,k}\sigma^{k,j}\partial_i\partial_j\psi$$

**Theorem 3.2.1.** *Let $1 < p < \infty$. We assume conditions (3.11) and (3.12). Then for any $f \in L_p^p(S,T)$ there exist a unique solution $\psi \in \tilde{H}_{2,p}^q$ to equation (3.14) with*

$$\|\psi\|_{\tilde{H}_{2,p}^q} \leq C\|f\|_{L_p^p(S,T)},$$

*where $C = C(d, \alpha, K, p)$. Moreover, if $f \in L_p^p(S,T) \cap L_p^q(S,T)$ then for any $\beta \in [0,2)$ and $\lambda > 1$ with $\frac{2}{q} + \frac{d}{p} < 2 - \beta + \frac{d}{\lambda}$ we have*

$$\|\psi(t)\|_{\boldsymbol{H}_\lambda^\beta} \leq C(T-t)^{\frac{2-\beta}{2} - \frac{d}{2p} - \frac{1}{q} + \frac{d}{2\lambda}}\|f\|_{L_p^q(S,T)},$$

*where $C = C(d, \alpha, K, p, q, \lambda, \beta)$.*

*Proof.* See theorem 3.1 in [97]. □

#### Krylov Type Estimate.

We need the following Krylov type estimate.

**Lemma 3.2.2.** *Assume that $\sigma$ verifies conditions (3.11) and (3.12), $f$ satisfies (3.10) and $X_{s,t}$ is solution of the SDE (3.13). Then for any $\delta \in (0, 1 - \frac{d}{2p} - \frac{1}{q})$ there exists a constant $C$ such that*

$$\mathbb{E}\Big(\int_s^t f(r, X_{S,r})dr \Big| \mathcal{F}_s\Big) \leq C(t-s)^\delta \|f\|_{L_p^q(S,T)}.$$

*Proof.* Let $p' = 2d$. Since $L_{p'}^{p'} \cap L_p^q$ is dense in $L_p^q$, it sufficient to show the inequality for

$$f \in L_{p'}^{p'} \cap L_p^q.$$

We consider the unique solution of the PDE

$$\partial_r \psi + L^\sigma \psi + f = 0, \quad u(t,x) = 0.$$

By theorem 3.2.1

$$\|\psi\|_{\tilde{H}_{2,p'}^{p'}} \leq C\|f\|_{L_{p'}^{p'}(S,t)},$$

and

$$\sup_{s,t} \|\psi\|_\infty \leq C(t-s)^\delta \|f\|_{L_p^q(S,t)}. \tag{3.15}$$

Let $\rho_n$ the standard moollifiers. We set

$$\psi_n(r,x) = (\psi * \rho_n)(r,x).$$

and

$$f_n(r,x) := -[\partial_t \psi_n + L^\sigma \psi_n]$$

Then we obtain

$$\|f_n - f\|_{L_{p'}^{p'}(s,t)}$$

$$\leq \|\partial_t(\psi_n - \psi)\|_{L_{p'}^{p'}(s,t)} + K\|\nabla^2(\psi_n - \psi)\|_{L_{p'}^{p'}(s,t)}$$

$$\leq \|(f * \rho_n - f)\|_{L_{p'}^{p'}(s,t)} + 2K\|\nabla^2(\psi_n - \psi)\|_{L_{p'}^{p'}(s,t)}$$

which converge to zero as $n \to \infty$.

By the Krylov classical estimation we have

$$\lim_{n\to\infty} \mathbb{E}\Big(\int_s^t |f_n(r, X_{S,r}) - f(r, X_{S,r})|\Big) \leq C \lim_{n\to\infty} \|f_n - f\|_{L_{p'}^{p'}(s,t)} = 0 \tag{3.16}$$

Applying the Itô to $\psi_n(r,x)$ we get

$$\psi_n(t, X_{S,t}) = \psi_n(s, X_{S,s}) - \int_s^t f_n(r, X_{S,r})\, dr + \int_s^t \partial_i f_n(r, X_{S,r})\, dB_r^i.$$

We observe that

$$\mathbb{E}\Big( \int_s^t \partial_i f_n(r, X_{S,r})\, dB_r^i \Big| \mathcal{F}_s \Big) = 0$$

Hence

$$\begin{aligned} &\mathbb{E}\Big( \int_s^t f(r, X_{S,r}) dr \Big| \mathcal{F}_s \Big) \\ &\leq \mathbb{E}\Big( \psi_n(s, X_{S,s}) - \psi_n(t, X_{S,t}) \Big| \mathcal{F}_s \Big) \\ &2 \sup_{s,t} \|\psi\|_\infty \leq C(t-s)^\delta \|f\|_{L_p^q(S,t)}. \end{aligned}$$

where used (3.15).

□

**Khasminskii Type Estimate.**

We need the following Khasminskii type estimate, see lemma 1.1 in [85].

**Lemma 3.2.3.** *Let $X, Y, Z$ be three real-valued measurable $\mathcal{F}_t$-adapted process, and $f, g$ be two $\mathbb{R}^d$-valued measurable $\mathcal{F}_t$-adapted process,. Suppose that there exist $c_0 > 0$ and $\delta \in (0,1)$ such that for any $S \leq s \leq t \leq T$*

$$\mathbb{E}\Big( \int_s^t |Z_r| + |g_r|^2 dr \Big| \mathcal{F}_s \Big) \leq c_0 (t-s)^\delta$$

*and that*

$$\begin{aligned} X(t) = X(S) &+ \int_S^t Y(r) dr + \int_S^t f(r) dB_r \\ &+ \int_S^t X(r) Z(r) dr + \int_S^t X(r) g(r) dB_r. \end{aligned}$$

*Then, for any $p > 0$ and $\lambda_1, \lambda_2, \lambda_3 > 1$ we have*

$$\mathbb{E}\big(\sup_{t\in[T,S]} X^+(t)^p\big)$$

$$\leq C\Big(\|(X^+)(S)^p\|_{\lambda_1} + \|(\int_S^T Y(r)^+ dr)^p\|_{\lambda_2} + \|(\int_S^T |f(r)|^2 dr)^{\frac{p}{2}}\|_{\lambda_3}\Big),$$

*where $C = C(c_0, \delta, p, \lambda_i)$.*

**Maximal Function.**

Let $f$ be a locally integrable function on $\mathbb{R}^d$. The *Hardy-Littlewood maximal function* is defined by

$$\mathbb{M}f(x) = \sup_{0<r<\infty}\left\{\frac{1}{|\mathcal{B}_r|}\int_{\mathcal{B}_r} f(x+y)\, dy\right\},$$

where $\mathcal{B}_r = \{x \in \mathbb{R}^d \ : |x| < r\}$. The following results can be found in [93].

**Lemma 3.2.4.** *For all $f \in \mathbb{W}^{1,1}(\mathbb{R}^d)$ there exists a constant $C_d > 0$ and a Lebesgue zero set $E \subset \mathbb{R}^d$ such that*

$$|f(x)-f(y)| \leq C_d\, |x-y|\, \Big(\mathbb{M}|\nabla f|(x)+\mathbb{M}|\nabla f|(y)\Big) \qquad \textit{for any } x, y \in \mathbb{R}^d\backslash E.$$

*Moreover, for all $p > 1$ there exists a constant $C_{d,p} > 0$ such that for all $f \in L^p(\mathbb{R}^d)$*

$$\|\mathbb{M}f\|_{L^p} \leq C_{d,p}\, \|f\|_{L^p}.$$

**Sobolev Regularity of Random Fields.**

We follow Fedrizzi and Flandoli in [34].

Let $X : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ be random field. When we use below this name we always assume it is jointly measurable.

**Lemma 3.2.5.** *Assume that $X(\omega, .) \in L^p_{loc}$ and and there exists a sequence $\{X_n\}_n$ of the random fields such that*

1. . *$X_n \to X$ in distribution in probability, namely*

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} X_n(x) f(x)\ dx \to \int_{\mathbb{R}^d} X(x) f(x)\ dx.$$

2. . *For any $R > 0$ there exists a constant $C_R$*

$$\mathbb{E} \int_{B_R} |\nabla X_n|^p dx \le C.$$

*Then $X \in W^p_{loc}(\mathbb{R}^d)$.*

**Existence and Uniqueness.**

**Theorem 3.2.6.** *(Zhang [97] ) We assume that $\sigma$ verifies condition (3.11) and (3.12), and that $\nabla\sigma$ satisfies condition (3.10). Then*

a) *There exists a unique strong solution of the SDE (3.13) which has jointly continuous version with respect to $t, x$.*

b) *Assume that $\sigma'$ also verifies the hypothesis of the theorem. Let $X^{\sigma}_{s,t}(x)$ and $X^{\sigma'}_{s,t}(x)$ be the solutions to (3.13) associated with $\sigma$ and $\sigma'$ respectively. Then*

$$\sup_{x\in\mathbb{R}^d} \mathbb{E}\big( \sup_{t\in[s,T]} ||X^{\sigma}_{s,t}(x) - X^{\sigma'}_{s,t}(x)\|^2\big) \le C\ (t-s)^{\delta} \|\sigma - \sigma'\|^2_{L^q_p(S,T)} \tag{3.17}$$

*where $\delta \in (0,1)$ only depend on $p, q, d$.*

c) *For each $t \ge s$ and almost all $\omega$, $x \to X_{s,t}(x)$ is weakly differentiable. That is, $X_t \in W^p_{loc}(\mathbb{R}^d)$.*

*Proof.* a). Under the assumptions the uniqueness we follow from (3.17). Since $\sigma$ is bounded and uniformly continuous in $x$ with respect to $t$, the existence of the weak solution is classical. Then the existence of a strong solution follows by the Yamada-Watanabe theorem.

b) Without loss of generality, we assume that $s = S$ and write $X^{\sigma}_t := X^{\sigma}_{S,t}$. We set $d_t = X^{\sigma}_t - X^{\sigma'}_t$ then

$$d_t = \int_S^t \sigma(r, X_r^\sigma) - \sigma'(r, X_r^{\sigma'})dB_r.$$

By Itô formula, we have

$$|d_t|^2 = \int_S^t |\sigma(r, X_r^\sigma) - \sigma'(r, X_r^{\sigma'})|^2 dr + 2\int_S^t d_r \sigma(r, X_r^\sigma) - \sigma'(r, X_r^{\sigma'})dB_r$$

$$= \int_S^t Y(r)dr + \int_S^t f(r)dB_r + \int_S^t |d_r|^2 Z_r dr \int_S^t |d_r|^2 grdB_r$$

where

$$Y(r) = \|\sigma(r, X_r^\sigma) - \sigma'(r, X_r^{\sigma'})\|^2 - 2\|\sigma(r, X_r^\sigma) - \sigma(r, X_r^{\sigma'})\|^2,$$

$$f(r) = 2d_r(\sigma(r, X_r^{\sigma'}) - \sigma'(r, X_r^{\sigma'})),$$

$$Z(r) = \frac{2|\sigma(r, X_r^\sigma) - \sigma(r, X_r^{\sigma'})|^2}{|d_r|^2},$$

$$g(r) = \frac{2d(r)(\sigma(r, X_r^\sigma) - \sigma(r, X_r^{\sigma'}))}{|d_r|^2}.$$

Here we used the convection $\frac{0}{0} = 0$, if $|d(r)| = 0$ then $Z = g = 0$. Now, by lemma 3.2.4 and 3.2.2 we have

$$\mathbb{E}\Big(\int_s^t |Z(r)| + |g(r)|^2 \, dr \Big| \mathcal{F}_s\Big)$$

$$\leq C\mathbb{E}\Big(\int_s^t \mathbb{M}|\nabla\sigma|^2(X_r^\sigma) + \mathbb{M}|\nabla\sigma|^2(X_r^{\sigma'})dr \Big| \mathcal{F}_s\Big)$$

$$\leq C(t-s)^\delta \|\mathbb{M}|\nabla\sigma|^2\|_{L_{p/2}^{q/2}(S,T)}$$

$$\leq C(t-s)^\delta \||\nabla\sigma|^2\|_{L_{p/2}^{q/2}(S,T)}$$

$$= C(t-s)^{\delta}\|\nabla\sigma\|^2_{L^q_p(S,T)},$$

where $\delta \in (0, 1-\frac{d}{p}-\frac{2}{q})$, and that for any $\lambda \in (1, 1/(\frac{2}{q}+\frac{d}{q}))$,

$$\mathbb{E}\Big(\int_S^T \|\sigma(r,X_r^{\sigma'})-\sigma'(r,X_r^{\sigma'})\|^{2\lambda}dr\Big) \le C\ (T-S)^{\delta}\ \||\sigma-\sigma'|^{2\lambda}\|_{L^{\frac{q}{2\lambda}}_{\frac{p}{2\lambda}}(S,T)}$$

$$= C\ (T-S)^{\delta}\ \|\sigma-\sigma'\|^{2\lambda}_{L^p_p(S,T)}. \tag{3.18}$$

Then, by lemma (3.2.3) with $\lambda_1 = p$, $\lambda_2 = \lambda$ and $\lambda_3 = \frac{2\lambda}{\lambda+1}$ and by Hölder inequality , we obtain

$$\mathbb{E}\big(\sup_{t\in[S,T]}|d_t|^2\big) \le C\ \|\Big(\int_S^T |d_r|^2|\sigma(r,X^{\sigma'}_{S,r})-\sigma'(r,X^{\sigma'}_{S,r})|^2dr\Big)^{\frac{1}{2}}\|_{L^{\lambda_3}(\Omega)}$$

$$+\ C\ \ \|\int_S^T |\sigma(r,X^{\sigma'}_{S,r})-\sigma'(r,X^{\sigma'}_{S,r})|^2dr\|_{L^{\lambda_2}(\Omega)}$$

$$\le C\ \|\sup_{t\in[S,T]}|d_t|\|_{L^2(\Omega)}\ \|\int_S^T |\sigma(r,X^{\sigma'}_{S,r})-\sigma'(r,X^{\sigma'}_{S,r})|^2dr\|^{\frac{1}{2}}_{L^{\lambda}(\Omega)}$$

$$+\ C\ \ \|\int_S^T |\sigma(r,X^{\sigma'}_{S,r})-\sigma'(r,X^{\sigma'}_{S,r})|^2dr\|_{L^{\lambda}(\Omega)}$$

$$\le \frac{1}{2}\ \|\sup_{t\in[S,T]}|d_t|\|^2_{L^2(\Omega)}\ +C\ \|\int_S^T |\sigma(r,X^{\sigma'}_{S,r})-\sigma'(r,X^{\sigma'}_{S,r})|^2dr\|_{L^{\lambda}(\Omega)}$$

which, together with (3.18), yields (3.17).

c). We assume also that $\sigma(t,.) \in C_b^{\infty}(\mathbb{R}^d)$ and $s = S$, then

$$DX_t = 1 + \int_S^t \nabla\sigma(r,X_r)DX_r dB_r.$$

We set

$$Z(r) = \frac{\|\nabla\sigma(r,X_r)D_r\|^2}{|D_r|^2},$$

and

$$g(r) = \frac{< DX_r, \nabla\sigma(r, X_{S,r}) DX_r >}{|DX_r|^2}.$$

By Itô formula we obtain

$$|DX_t|^2 = |DX_{S,S}|^2 + \int_S^t |DX_r|^2 Z(r) dr + \int_S^t |DX_r|^2 g(r) dr.$$

Now , we observe for any $\delta \in (0, 1 - \frac{d}{p} - \frac{2}{q})$

$$\mathbb{E}\Big( \int_s^t |Z_r| + |g_r|^2 dr \Big| \mathcal{F}_s \Big) \leq \mathbb{E}\Big( \int_s^t |\nabla\sigma(r, X_{S,r})| dr \Big| \mathcal{F}_s \Big)$$

$$\leq C \|\nabla\sigma\|^2_{L^q_q(S,T)} (t-s)^\delta$$

where we used the Krylov estimation. Then by Khasminskii estimation we get

$$\sup_{x\in\mathbb{R}^d} \mathbb{E}\Big[ \sup_{t\in[S,T]} |\nabla X_{S,t}|^p \Big] \leq C,$$

where the constant $C = (K, \alpha, , p, q, d, \|\nabla\sigma\|_{L^q_q(S,T)})$. This implies that $X_{s,t}$ is weakly differentiable by lemma 3.2.5.

□

### 3.2.2 Main Result.

**PDE Estimation.**

We need the following theorem.

**Theorem 3.2.7.** *Assume that $\sigma_t$ satisfies conditions (3.11) and (3.12). Suppose also that one of the following conditions holds :*

1. *$b$ and $f$ satisfy (3.10), $\sigma_t$ in independent of $x$.*

2. *$\nabla\sigma$ and $b$ satisfy (3.10) for some $q = p > d + 2$.*

*Then there exist a unique solution* $\psi \in \tilde{H}^{2,q}_p(S,T)$ *to*

$$\partial_t \psi + L^\sigma \psi + b\nabla\psi + f = 0, \ \psi(T) = 0, \tag{3.19}$$

*satisfying*

$$\|\psi\|_{L^q_p(S,T)} + \|\nabla^2\psi\|_{L^q_p(S,T)} \le Ce^{C\|b\|_{L^q_p(S,T)}} \|f\|_{L^q_p(S,T)} \tag{3.20}$$

*and for all* $t \in [S,T]$,

$$\|\nabla\psi(t)\|_{C^{\frac{\delta}{2}}(\mathbb{R}^d)} \le C(T-S)^{\frac{\delta}{3}} e^{C\ (T-S)^{\frac{q\delta}{3}} \|b\|_{L^q_p(S,T)}} \|f\|_{L^q_p(S,T)}, \tag{3.21}$$

*where* $\delta := \frac{1}{2} - \frac{d}{2p} - \frac{1}{q}$ *and* $C = C(K,\alpha,p,q,d,\delta)$.

**Result.**

We consider the transformation

$$\varphi = x + \psi.$$

**Lemma 3.2.8.** *We assume hypothesis of the theorem (3.2.7). There exists an interval* $|s_0 - t_0| < \epsilon$ *such that*

*a)* $\varphi$ *a s* $C^1$*-diffeomorphism.*

*b)* $\|\nabla\varphi\|_\infty + \|\nabla\varphi^{-1}\|_\infty \le C$ *where* $C$ *is a universal constant.*

*c)* $\|\nabla^2\varphi\|_{L^q_p(S,T)} + \|\nabla\psi\|_{C^{\frac{\delta}{2}}} \le C$ *where* $C$ *depends on* $K,\alpha,p,q,d,\delta$.

*d) .* $X_{s_0,t}$ *solves SDE (3.9) on* $[s_0,t_0]$ *if only if* $Y_{s_0,t} = \varphi(X_{s_0,t})$ *solves the following SDE*

$$dY_t(x) = D\varphi(\varphi^{-1}(Y_t))dB_t, \ Y_{s_0} = x \in \mathbb{R}^d. \tag{3.22}$$

*Proof.* a). By (3.21) we can take $|t_0 - s_0| < \epsilon$ such that

$$\sup_{[s_0,t_0]} \|\nabla\psi\|_{C^{\frac{\delta}{2}}} \le \frac{1}{2}$$

In particular,

$$\frac{1}{2}|x-y| \leq |\psi(t,x)-\psi(t,x)| \leq \frac{1}{2}|x-y|.$$

Then

$$\frac{1}{2}|x-y| \leq |\varphi(t,x)-\varphi(t,x)| \leq \frac{3}{2}|x-y|.$$

From this we conclude that $\phi$ is a $C^1$-diffomorphism.

b) We follow of the definition of $\phi$ and (3.21).

c) We follow from (3.19) and (3.21).

d) We follow from generalized Ito formula, see [61] □

**Theorem 3.2.9.** *We assume hypothesis of the theorem (3.2.7). Then*

*a) For any $(t,x) \in [S,T] \times \mathbb{R}^d$, there is a unique strong solution denoted by $X_{s,t}$ to the SDE (3.9), which has jointly continuous version with respect to $t$ and $x$.*

*b) For each $t \geq s$ and almost all $\omega$, $x \to X_{s,t}(x,\omega)$ is weakly differentiable.*

*Proof.* We observe that the diffusion coefficient in equation (3.22) satisfies the hypothesis of the theorem 3.2.6 in the interval $[s_0, t_o]$. Then by lemma (3.2.8) there exists a unique solution of the SDE (3.9) in $[s_0, t_o]$ which is weakly differentiable. Finally doing a partition on the interval $[S,T]$ and by uniqueness we conclude the theorem.

□

## 3.3 Other results.

- The first result on SDE with irregular drift was obtained by Zvonkin in [99], who showed the existence of a unique strong solution of one-dimensional Brownian motion, when the drift coefficient $b$ is merely bounded and measurable. Later, the result was generalized by Veretennikov [95] to the multidimensional case. See also Gyongya and Krylov [50] and Gyongy and Martinez [51].

- In [74] Mohammed, Nilssen, and Proske showed existence and uniqueness of stochastic homeomorphism flows associated to equation (3.1 ) when the drift $b$ belong to $L^\infty([0,T]\times\mathbb{R}^d)$. The proofs are based on Malliavin calculus.

- In [43] Flandoli, Gubinelli and Priola extend the results showed in section 3.1 when the drift is unbounded and Hölder continuous.

- In [86] and [87] Priola, based on the same approach of [40], proved existence, uniqueness and flow properties for the equation

$$dX_t(x) = b(t, X_t(x))dt + L_t^\alpha,\ X_s = x \in \mathbb{R}^d, \tag{3.23}$$

  where $L_t^\alpha$ is $\alpha$-stable Lévy process and $b$ is bounded and Holder continuous. See also the recent work by Xie [96].

- In [5] Banos, Nilssen and Proske considered the equation

$$dX_t(x) = b(t, X_t(x))dt + B_t^H,\ X_s = x \in \mathbb{R}^d, \tag{3.24}$$

  where $B_t^H$ is the fractional Brownian motion with $H < \frac{1}{2}$ and $b$ is bounded and globally integrable. They proved existence and uniqueness of the solution.

- For some results on regularization by noise in infinity dimension see Da Prato and Flandoli in [23], Da Prato, Flandoli and Priola in [24], Da Prato, Flandoli, Priola and Röckner in [25].

# Chapter 4

# Stochastic continuity-transport equation.

The transport/continuity equation is one of the most fundamental and at the same time most elementary partial differential equation with applications in a wide range of problems from physics, engineering, biology or social science. In this chapter, we present some recent results on the effect of the noise in this equation.

## 4.1 Deterministic transport-continuity equation.

### 4.1.1 Regular Case.

We consider the systems of ODEs

$$\frac{d}{dt}X_t(x) = b(t, X_t(x)),\ \ X_0 = x \in \mathbb{R}^d. \tag{4.1}$$

where the vector field $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$.

We assume that $b$ is regular, Liptschitz with respect to the spatial variable uniformly respect with the time variable. In this classical situation there is strong connection with transport/ continuity equation. We consider the deterministic transport equation

$$\partial_t u(t,x) + b(t,x) \cdot \nabla u(t,x) = 0, u(0,x) = u_0(x).\,, \tag{4.2}$$

Then the quantity $u(t, X_t)$ is a constant respect the time : indeed

$$\frac{d}{dt} u(t, X_t) = -b(t, X_t)\nabla u(t, X_t) + \nabla u(t, X_t) b(t, X_t) = 0.$$

Then the unique solution for any initial condition is given by

$$u(t,x) = u_0(X_t^{-1}).$$

### 4.1.2 Weak solutions.

Recently research activity has been devoted to study continuity-transport equations with rough coefficients, showing a well-posedness result. A complete theory of distributional solutions, including existence, uniqueness and stability properties, is provided in the seminal works of DiPerna and Lions [31] and Ambrossio [2]. We introduce the notion of weak formulation.

**Definition 4.1.1.** *Assume that* $b, \operatorname{div} b \in L^1([0,T] \times \mathbb{R}^d)$ *A weak solution of the PDE (4.2) is a class functions* $u \in L^\infty([0,T], L^\infty \cap L^\infty(\mathbb{R}^d))$ *if for any* $\varphi \in C_0^\infty(\mathbb{R}^d)$ *and for all* $t \in [0,T]$*, we have*

$$\begin{aligned} \int_{\mathbb{R}^d} u(t,x)\varphi(x) dx = & \int_{\mathbb{R}^d} u_0(x)\varphi(x)\ dx \\ & + \int_0^t \int_{\mathbb{R}^d} u(s,x)\, b(s,x) \nabla\varphi(x) dx ds \qquad (4.3) \\ & + \int_0^t \int_{\mathbb{R}^d} u(s,x)\, \operatorname{div} b(s,x) \varphi(x) dx ds. \end{aligned}$$

The existence of the solution is trivial since the the transport equation is linear PDE. Therefore is sufficient to regularize the vector field and the initial data, then passing to the limit we obtain a solution.

We explain the procedure to get uniqueness.

Let us fix an even convolution mollifier $\rho_\epsilon$. We define $u^\epsilon = (u * \rho_\epsilon)(t,x)$, then we have that $u^\epsilon$ satisfies

$$\partial_t u^\epsilon + b\nabla u^\epsilon = R_\epsilon(b,u)$$

where $R_\epsilon(b,u)$ is the commutator

$$R_\epsilon(b,u) = b\nabla u^\epsilon - (b\nabla u) * \rho_\epsilon.$$

Thus, for all $\epsilon > 0$, we have

$$\partial_t \beta(u^\epsilon) + b\nabla\beta(u^\epsilon) = \beta'(u^\epsilon)R_\epsilon(b,u).$$

In order to pass to the limit we need strong convergence of the commutator, see [31].

**Lemma 4.1.2.** *We assume that $b \in L^1([0,T], W^{1,1}_{loc})$ and $u \in L^\infty([0,T]\times\mathbb{R}^d)$. Then*

$$R_\epsilon(b,u) \to 0 \ in \ L^1_{loc}([0,T]\times \ R^d).$$

Now, passing to the limit we have

$$\partial_t\beta(u(t,x)) + b(t,x)\cdot\nabla\beta(u(t,x)) = 0, \tag{4.4}$$

and this is definition of renormalized solution. Roughly speaking, renormalized solutions are distributional solutions to which the chain rule applies in the sense that, for every suitable $\beta$, $\beta(u)$ solves the equation (4.4).

Now, we are ready to give uniqueness results.

**Theorem 4.1.3.** *(Di perna-Lions [31] ) We assume that $b \in L^1([0,T], W^{1,1}_{loc})$, $\mathrm{div} b \in L^1([0,T], L^\infty(\mathbb{R}^d))$ and $\frac{b}{(1+|x|)} \in L^1([0,T], L^\infty(\mathbb{R}^d))$. Then there exists a unique $L^\infty$- weak solutions of the Cauchy problem (4.2).*

*Proof.* Since the transport equation is linear it is enough to show that a weak solution $u$ with initial condition $u_0 = 0$ vanishes identically. Consider a non-negative smooth cut-off function $\eta$ supported on the ball of radius 2 and such that $\eta = 1$ on the ball of radius 1. For each $R > 0$ introduce the rescaled functions $\eta_R(\cdot) = \eta(\frac{1}{R}\cdot)$. We take $\beta$ such that $min(|u|^2, |u|) \leq \beta(u) \leq |u|$ and as test function $\eta_R(\cdot)$ in (4.4), then

$$\int_{\mathbb{R}^d} \beta(u(t,x))\eta_R(x)dx = \int_0^t \int_{\mathbb{R}^d} \beta(u(s,x))\, b(s,x)\nabla\eta_R(x)dxds$$
$$+ \int_0^t \int_{\mathbb{R}^d} \beta(u(s,x))\, \mathrm{div} b(s,x)\eta_R(x)dxds$$

We observe that

$$\frac{|b(s,x)|}{R} 1_{R\leq|x|\leq 2R} \leq \frac{|b(s,x)|}{1+|x|} 1_{|x|\geq R}.$$

Hence

$$\int_{\mathbb{R}^d} \beta(u(t,x))\eta_R(x)dx \leq \int_0^t \int_{|x|\geq R} \beta(u(s,x))\, \frac{|b(s,x)|}{1+|x|} dxds$$
$$+ \int_0^t c(t) \int_{\mathbb{R}^d} \beta(u(s,x))\, \eta_R(x)dxds.$$

Passing to the limit as $R \to \infty$ we get

$$\int_{\mathbb{R}^d} \beta(u(t,x))dx \leq \int_0^t c(t) \int_{\mathbb{R}^d} \beta(u(t,x))\, dxds.$$

By Gronnwal lemma we conclude $u = 0$.

□

**Remark 4.1.4.** *We observe that the existence and uniqueness result can be extended easily for the equation*

$$\partial_t u(t,x) + b(t,x)\cdot\nabla u(t,x) + c(t,x)u(t,x) = 0, u(0,x) = u_0(x),$$

*when* $c = \mathrm{div} b$ *we get the following continuity equation*

$$\partial_t u(t,x) + \mathrm{div}(b(t,x)u(t,x)) = 0, u(0,x) = u_0(x).$$

### 4.1.3 More results.

- Afters some intermediate results ( [18] and [19] ) the theory has been generalized by L. Ambrosio [2] supported again on commutators, but with a different measure-theoretic framework, to the case of only $BV$ regularity for b instead of $W^{1,1}$.

- In the case of two-dimensional vector-field, we refer to the work of F. Bouchut and L. Desvillettes [10] that treated the case of divergence free vector-field with continuous coefficient, and to [52] in which this result is extended to vector-field with $L^2_{loc}$ coefficients with a condition of regularity on the direction of the vector-field.

- For $d > 2$ ( $d = 2$ in the nonautonomous case) there are examples of non uniqueness for nearly BV fields, see [1], [20] and [29].

- We would also like to mention the generalizations to transport-diffusion equations and the associated stochastic differential equations by C. Le Bris and P.L. Lions [65, 66] and A. Figalli [38].

- For some recent developments see [3], [11] and [92].

## 4.2 Stochastic Case.

### 4.2.1 Regular Case.

The method of stochastic characteristic for fisrt order stochastic partial differential equation was introduced by Bismuit [9], Funaki [48], Kunita [64] and Rozovskii [90]. Here, we present the linear case.

We consider the linear transport-continuity equation

$$\begin{cases} \partial_t u(t,x) + (b(t,x) + \dfrac{dB_t}{dt})\nabla u(t,x) + c(t,x)u(t,x) = 0\,, \\ u|_{t=0} = u_0\,. \end{cases} \tag{4.5}$$

Here, $(t,x) \in [0,T] \times \mathbb{R}^d$, $\omega \in \Omega$ is an element of the probability space $(\Omega, \mathbb{P}, \mathcal{F})$, $b : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ is a given vector field and $B_t = (B_t^1, ..., B_t^d)$ is a standard Brownian motion in $\mathbb{R}^d$. The stochastic integration is to be understood in the Stratonovich sense. When $c = 0$ is the transport equation and when $c = \text{div} b$ is the continuity equation.

The equation (4.5) is interpreted in the strong sense, as the following stochastic integral equation

$$u(t,x) = u_0(x)$$

$$-\int_0^t b(s,x)\nabla u(s,x)\, ds - \sum_{i=0}^{d} \int_0^t \partial_{x_i} u(s,x) \circ dB_s^i - \int_0^t c(s,x)\, u(s,x)\, ds$$

for $t \in [0,T]$ and $x \in \mathbb{R}^d$.

For $m \in \mathbb{N}$ and $0 < \alpha < 1$, let us assume the following hypothesis on $b$:

$$b \in L^1((0,T); C_b^{m,\alpha}(\mathbb{R}^d)) \tag{4.6}$$

where $C^{m,\alpha}(\mathbb{R}^d)$ denotes the class of functions of class $C^m$ on $\mathbb{R}^d$ such that the last derivative is Hölder continuous of order $\alpha$.

We consider the SDE

$$X_{s,t}(x) = x + \int_s^t b(r, X_{s,r}(x))\, dr + B_t - B_s, \tag{4.7}$$

It is well known that under conditions (4.6), $X_{s,t}(x)$ is a stochastic flow of $C^m$-diffeomorphism (see for example [63] and [62]). Moreover, the inverse $Y_{s,t}(x) := X_{s,t}^{-1}(x)$ satisfies the following backward stochastic differential equation

$$Y_{s,t}(y) = y - \int_s^t b(r, Y_{r,t}(y))\, dr - (B_t - B_s), \tag{4.8}$$

for $0 \leq s \leq t$.

**Lemma 4.2.1.** *Assume (4.6) for $m \geq 3$ and let $u_0 \in C^{m,\delta}(\mathbb{R}^d)$. Then the Cauchy problem (4.5) has a unique solution $u(t,.)$ for $0 \leq t \leq T$ such that it is a $C^m$-semimartingale which can be represented as*

$$u(t,x) = u_0(X_t^{-1}(x))\, e^{-\int_0^t c(s, X_{t-s}^{-1})ds}, \quad t \in [0,T], x \in \mathbb{R}^d.$$

*Proof.* We refer to [16] and [62].

□

### 4.2.2 Continuity equation with Hölder Drift.

The regularization effect of the noise on transport-continuity equation has been intensively studied in recent years. A first result in this direction was given by Flandoli, Gubinelli and Priola in [40] for the stochastic transport equation, they obtained wellposedness for an Hölder continuous drift term, with some integrability conditions on the divergence.

Here, we consider the stochastic continuity equation, $c = \mathrm{div} b$. We give the definition of solution.

**Definition 4.2.2.** *A stochastic process $u \in L^\infty([0,T] \times \Omega \times \mathbb{R}^d)$ is called a weak $L^\infty-$solution of the Cauchy problem* (4.5)*, when for any $\varphi \in C_c^\infty(\mathbb{R}^d)$, the real value process $\int u(t,x)\varphi(x)dx$ has a continuous modification which is a $\mathcal{F}_t$-semimartingale, and for all $t \in [0,T]$, we have $\mathbb{P}$-almost sure*

$$\int_{\mathbb{R}^d} u(t,x)\varphi(x)dx = \int_{\mathbb{R}^d} u_0(x)\varphi(x)\ dx$$
$$+ \int_0^t \int_{\mathbb{R}^d} u(s,x)\ b^i(s,x)\partial_i\varphi(x)\ dxds \tag{4.9}$$
$$+ \int_0^t \int_{\mathbb{R}^d} u(s,x)\ \partial_i\varphi(x)\ dx \circ dB_s^i.$$

**Remark 4.2.3.** *We shall write equation (4.9) in Itô formulation. From relation (1.1) we have*

$$\int_0^t \int_{\mathbb{R}^d} u(s,x)\ \partial_i\varphi(x)\ dx \circ dB_s^i.$$

$$= \int_0^t \int_{\mathbb{R}^d} u(s,x)\ \partial_i\varphi(x)\ dx\, dB_s^i + \frac{1}{2}[\int_{\mathbb{R}^d} u(s,x)\ \partial_i\varphi(x)\ dx, B_s^i].$$

*Taking as test function $\partial_j\varphi$ in (4.9) we have*

$$\int_{\mathbb{R}^d} u(t,x)\partial_j\varphi(x)dx = \int_{\mathbb{R}^d} u_0(x)\varphi(x)\ dx$$
$$+ \int_0^t \int_{\mathbb{R}^d} u(s,x)\ b^i(s,x)\partial_i\partial_j\varphi(x)\ dxds$$
$$+ \int_0^t \int_{\mathbb{R}^d} u(s,x)\ \partial_i\partial_j\varphi(x)\ dx \circ dB_s^i.$$

*We observe that only the term with stochastic integral contributes with the covariation. Thus*

$$[\int_{\mathbb{R}^d} u(t,x)\partial_j\varphi(x)dx, B_s^i] = \delta_{i,j} \int_0^t \int_{\mathbb{R}^d} u(s,x)\ \partial_{i,j}\varphi(x)\ dxds.$$

*Then the Itô formulation is*

$$\int_{\mathbb{R}^d} u(t,x)\varphi(x)dx = \int_{\mathbb{R}^d} u_0(x)\varphi(x)\ dx$$

$$+\int_0^t\int_{\mathbb{R}^d} u(s,x)\, b(s,x,\omega)\partial_i\varphi(x)\ dxds + \int_0^t\int_{\mathbb{R}^d} u(s,x)\ \partial_i\varphi(x)\ dx\, dB_s$$
$$+\frac{1}{2}\int_0^t\int_{\mathbb{R}^d} u(s,x)\ \Delta\varphi(x)\ dx\, ds.$$

This existence proof we follow from Catuogno and Olivera in [12].

**Lemma 4.2.4.** *We assume that $b \in L^1([0,T]\times\mathbb{R}^d)$ and $\mathrm{div} b \in L^1([0,T], L^\infty(\mathbb{R}^d))$. Then there exists a weak $L^\infty-$solution $u$ of the Cauchy problem* (4.5).

*Proof.* 1. First, let us consider the following auxiliary Cauchy problem for the continuity equation, that is to say

$$\begin{cases} \partial_t v(t,x) + \mathrm{div}\Big(v(t,x)\ b(t,x+B_t)\Big) = 0, \\ v(0,x) = u_0(x). \end{cases} \tag{4.10}$$

According to a minor modification of the arguments in DiPerna, Lions [31], see Proposition II.1 (taking only test functions defined on $\mathbb{R}^d$), it follows that, there exists a function $v \in L^\infty(U_T\times\Omega)$, which is a solution of the auxiliary problem (4.10) in the sense that, it satisfies for each test function $\varphi \in C_c^\infty(\mathbb{R}^d)$

$$\begin{aligned} \int_{\mathbb{R}^d} v(t,x)\varphi(x)dx &= \int_{\mathbb{R}^d} u_0(x)\varphi(x)\ dx \\ &\quad + \int_0^t\int_{\mathbb{R}^d} v(s,x)\, b(s,x+B_s)\cdot\nabla\varphi(x)\ dxds. \end{aligned} \tag{4.11}$$

One observes that, the process $\int v(t,x)\varphi(x)dx$ is adapted, since it is the weak limit in $L^2([0,T]\times\Omega)$ of adapted processes, see [84] Chapter III for details.

2. Now, let us define for each $y \in \mathbb{R}^d$,

$$F(y) := \int_{R^d} v(t,x)\,\varphi(x+y)\ dx.$$

Then, applying the Itô-Wentzell-Kunita Formula, see Theorem 8.3 of [63], to $F(B_t)$, it follows from (4.11)

$$\begin{aligned}\int_{\mathbb{R}^d} v(t,x)\,\varphi(x+B_t)\;dx &= \int_{\mathbb{R}^d} u_0(x)\,\varphi(x)\;dx \\ &+ \int_0^t \int_{\mathbb{R}^d} b(s,x+B_s)\cdot\nabla\varphi(x+B_s)v(s,x)\;dxds \\ &+ \int_0^t \int_{\mathbb{R}^d} v(s,x)\;\partial_i\varphi(x+B_s)dx\circ dB_s^i,\end{aligned} \tag{4.12}$$

where we have used that

$$\frac{\partial}{\partial y_i}\varphi(x+y) = \frac{\partial}{\partial x_i}\varphi(x+y).$$

3. Finally, defining $u(t,x) := v(t,x-B_t)$ we obtain from equation (4.12) that, $u(t,x)$ is a weak $L^\infty-$solution of the stochastic Cauchy problem (4.5). □

The uniqueness result is established using the properties of stochastic flow for the SDE with Hölder drift. We compose the solution $u$ with the stochastic flow to show uniqueness. Thus, avoiding the commutator and the problems there in.

**Theorem 4.2.5.** *Assume condition that* $b \in L^\infty([0,T], C_b^\alpha(\mathbb{R}^d))$. *Then If* $u, v \in L^\infty([0,T]\times\mathbb{R}^d\times\Omega)$ *are two weak* $L^\infty-$*solutions for the Cauchy problem* (4.5), *with the same initial data* $u_0 \in L^\infty(\mathbb{R}^d)$, *then u=v almost everywhere in* $[0,T]\times\mathbb{R}^d\times\Omega$.

*Proof.* It is sufficient by linearity to prove that, any weak $L^\infty-$solution $u$ with initial condition $u_0(x) = 0$ vanishes identically. Let $\phi_\varepsilon, \phi_\delta$ be standard symmetric mollifiers. Then, for each $t \in [0,T]$, $u_\varepsilon(t,\cdot) = u(t,\cdot) * \phi_\varepsilon$ verifies

$$\begin{aligned}\int_{\mathbb{R}^d} u(t,z)\phi_\varepsilon(y-z)dz &= \int_0^t \int_{\mathbb{R}^d} u(s,z)\; b^i(s,z)\partial_{z_i}\phi_\varepsilon(y-z)\;dzds \\ &+ \int_0^t \int_{\mathbb{R}^d} u(s,z)\;\partial_{z_i}\phi_\varepsilon(y-z)\;dz\circ dB_s^i.\end{aligned} \tag{4.13}$$

We recall that, for each $\epsilon > 0$ the equation for $u_\varepsilon$ is strong in the analytic sense.

Now, let us denote by $b^\delta$ the standard mollification of $b$ by $\phi_\delta$, and denote $X_t^\delta$ the associated flow given by the SDE (4.7), with $b^\delta$ instead of $b$. We also consider $Y_t^\delta$, which satisfies the backward SDE (4.8). Then, for each $\varphi \in C_c^\infty(\mathbb{R}^d)$, it follows for each $t \in [0,T]$ that

$$\int_{\mathbb{R}^d} (u * \phi_\varepsilon)(X_t^\delta) JX_t^\delta \; \varphi(x) \; dx = \int_{\mathbb{R}^d} (u * \phi_\varepsilon)(y) \; \varphi(Y_t^\delta) \; dy, \tag{4.14}$$

where $JX_t^\delta$ is the Jacobian map of $X_t^\delta$ .On the other hand, since $u_\varepsilon$ is strong in analytic sense, applying Itô's formula to the product of two semimartingales $(u * \phi_\varepsilon)(y) \; \varphi(Y_t^\delta)$, we get

$$\begin{aligned}\int_{\mathbb{R}^d} (u * \phi_\varepsilon)(y) \; \varphi(Y_t^\delta) \; dy = &- \int_0^t \int_{\mathbb{R}^d} u_\varepsilon(s,y) \; b^\delta(s,y) \cdot \nabla\varphi(Y_s^\delta) \; dyds \\ &- \int_0^t \int_{\mathbb{R}^d} u_\varepsilon(s,y) \; \partial_i \varphi(Y_s^\delta) dy \; \circ dB_s^i \\ &+ \int_0^t \int_{\mathbb{R}^d} \varphi(Y_s^\delta) \int_{\mathbb{R}^d} u(s,z) \; b(s,z) \cdot \nabla\phi_\varepsilon(y-z) \; dzdyds \\ &+ \int_0^t \int_{\mathbb{R}^d} \varphi(Y_s^\delta) \int_{\mathbb{R}^d} u(s,z) \; \partial_i \phi_\varepsilon(y-z) \; dzdy \; \circ dB_s^i.\end{aligned} \tag{4.15}$$

By integration by parts, we bring all the derivatives on $\varphi(Y^\delta)$, we have

$$\int_{\mathbb{R}^d} (u * \phi_\varepsilon)(X_t^\delta) J X_t^\delta \, \varphi(x) \, dx$$

$$= - \int_0^t \int_{\mathbb{R}^d} u_\varepsilon(s,y) \, b^\delta(s,y) \cdot \nabla\varphi(Y_s^\delta) \, dyds$$

$$- \int_0^t \int_{\mathbb{R}^d} u_\varepsilon(s,y) \, \partial_i \varphi(Y_s^\delta) dy \ \circ dB_s^i$$

$$+ \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} u(s,z) \, \phi_\varepsilon(y-z) \, b(s,z) \cdot \nabla\varphi(Y_s^\delta) \, dzdyds$$

$$+ \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} u(s,z) \, \phi_\varepsilon(y-z) \, \partial_i \varphi(Y_s^\delta) \, dzdy \ \circ dB_s^i,$$

where we have used that, $\phi_\varepsilon$ is symmetric.

Now for $\delta > 0$ fixed, we pass to the limit as $\varepsilon$ goes to $0^+$ to obtain from the above equation

$$\int_{\mathbb{R}^d} u(X_t^\delta) J X_t^\delta \, \varphi(x) \, dx = - \int_0^t \int_{\mathbb{R}^d} u(s,y) \, b^\delta(s,y) \cdot \nabla\varphi(Y_s^\delta) \, dyds$$
$$+ \int_0^t \int_{\mathbb{R}^d} u(s,y) \, b(s,z) \cdot \nabla\varphi(Y_s^\delta) \, dyds. \tag{4.16}$$

Then, we pass to the limit in (4.16) as $\delta$ goes to $0^+$, to obtain that, for each $\varphi \in C_c^\infty(\mathbb{R}^d)$, and $t \in [0,T]$

$$\lim_{\delta \to 0} \int_{\mathbb{R}^d} u(X_t^\delta) J X_t^\delta \varphi(x) dx = 0. \tag{4.17}$$

From theorem 3.1.3 we have

$$0 = \int_{\mathbb{R}^d} u(t,x) J X_t \varphi(X_t) \, dx = \int_{\mathbb{R}^d} u(X_t) \varphi(x) \, dx \tag{4.18}$$

for each $\varphi \in C_c^\infty(\mathbb{R}^d)$, and $t \in [0,T]$.

Finally, let $K$ be any compact set in $\mathbb{R}^d$. Then, we get

$$\begin{aligned}\int_K \mathbb{E}|u(t,x)|\ dx =& \int_K \mathbb{E}|u(t,X_t(Y_t))|\ dx\\ =&\ \mathbb{E}\int_{Y_t(K)} JX_t|u(t,X_t)|\ dx = 0\end{aligned}$$

where we have used (4.18) and the regularity of the stochastic flow. Then, we conclude our theorem. □

### 4.2.3 Continuity equation with unbounded drift

In this section we follow Mollinedo and Olivera in [75]. The main issue of to prove existence and uniqueness of $L^2$-weak solutions for one-dimensional stochastic continuity equation (4.5) with unbounded measurable drift without assumptions on the divergence.

We assume the following hypothesis:

**Hypothesis 4.2.6.** *The vector field $b$ satisfies*

$$|b(x)| \leq k(1+|x|)\,, \tag{4.19}$$

*and the initial condition holds*

$$u_0 \in L^2(\mathbb{R}, w\,dx) \tag{4.20}$$

*where $w$ is the weight defined by $w(x) = e^{2k_2x^2}$ with*
$k_2 = 2(k + 99Tk^2)$.

We denote $\mu = (1+|x|)^2$.

**Definition 4.2.7.** *A stochastic process $u \in L^2(\Omega \times [0,T] \times \mathbb{R}, \mu dx)$ is called a $L^2$- weak solution of the Cauchy problem (4.5) when: For any $\varphi \in C_0^\infty(\mathbb{R})$, the real valued process $\int u(t,x)\varphi(x)dx$ has a continuous modification which is an $\mathcal{F}_t$-semimartingale, and for all $t \in [0,T]$, we have $\mathbb{P}$-almost surely*

$$\begin{aligned}\int_{\mathbb{R}} u(t,x)\varphi(x)dx = \int_{\mathbb{R}} u_0(x)\varphi(x)\ dx + \int_0^t\int_{\mathbb{R}} u(s,x)\,b(x)\partial_x\varphi(x)\ dxds\\ + \int_0^t\int_{\mathbb{R}} u(s,x)\ \partial_x\varphi(x)\ dx \circ dB_s\,.\end{aligned} \tag{4.21}$$

Now, we denote by $b^\epsilon$ the standard mollification of $b$, and let $X_t^\epsilon$ be the associated flow given by the SDE (4.7) replacing $b$ by $b^\epsilon$. Similarly, we consider $Y_t^\epsilon$, which satisfies the backward SDE (4.8). We also recall the important results in [79] : let $X_t^\epsilon$ be the corresponding stochastic flows, then for all $p \geq 1$ there are constants $C_1 = C_1(k,p,T)$ and $C_2(k,p,T)$ such that

$$\mathbb{E}[|\partial_x X_t^\epsilon(x)|^p] \leq C_1 t^{-\frac{1}{2}} e^{C_2 x^2}, \tag{4.22}$$

the same results is valid for the backward flow $Y_t^\epsilon$ since it is solution of the same SDE driven by the drifts $-b^\epsilon$.

We shall here prove existence of solutions under hypothesis 4.2.6.

**Lemma 4.2.8.** *Assume that hypothesis 4.2.6 holds. Then there exists $L^2$-weak solution of the Cauchy problem* (4.5)*.*

*Proof. Step 1: Regularization.*

Let $\{\rho_\varepsilon\}_\varepsilon$ be a family of standard symmetric mollifiers and $\eta$ a nonnegative smooth cut-off function supported on the ball of radius 2 and such that $\eta = 1$ on the ball of radius 1. Now, for every $\varepsilon > 0$, we introduce the rescaled functions $\eta_\varepsilon(\cdot) = \eta(\varepsilon \cdot)$. Thus, we define the family of regularized coefficients given by

$$b^\epsilon(x) = \eta_\varepsilon(x)(b * \rho_\varepsilon(x))$$

and

$$u_0^\varepsilon(x) = \eta_\varepsilon(x)\big(u_0 * \rho_\varepsilon(x)\big)\,.$$

Clearly we observe that, for every $\varepsilon > 0$, any element $b^\varepsilon$, $u_0^\varepsilon$ are smooth (in space) and have compactly supported with bounded derivatives of all orders. We consider the regularized version of the stochastic continuity equation :

$$\begin{cases} du^\varepsilon(t,x) + \nabla u^\varepsilon(t,x) \cdot \big(b^\varepsilon(x)dt + \circ dB_t\big) + \mathrm{div} b^\varepsilon(x)\, u^\varepsilon(t,x)dt = 0\,, \\ u^\varepsilon\big|_{t=0} = u_0^\varepsilon \end{cases} \tag{4.23}$$

Following the classical theory of H. Kunita [62, Theorem 6.1.9] we obtain that

$$u^\varepsilon(t,x) = u_0^\varepsilon(Y_t^\varepsilon(x))JY_t^\varepsilon(x).$$

*Step 2: Boundedness.* Making the change of variables $y = y_t^\varepsilon(x)$ we have that

$$\int_{\mathbb{R}} \mathbb{E}[|u^\varepsilon(t,x)|^2]\,(1+|x|)^2 dx$$

$$= \int_\Omega \int_{\mathbb{R}} |u_0^\varepsilon(y)|^2 |X_t^\varepsilon(y)|^{-1}\ (1+|X_t^\varepsilon(y)|)^2\, dy\mathbb{P}(d\omega).$$

We claim that there are constants $k_1 = k_1(k,T)$ and $k_2 = 2(k + 99Tk^2)$ such that

$$\mathbb{E}\bigg[\bigg|\frac{d}{dx}X_t^\varepsilon(x)\bigg|^{-2}\bigg] = \mathbb{E}\bigg[\exp\bigg\{-2\int_0^t \mathrm{div} b^\varepsilon(X_s^\varepsilon(x))ds\bigg\}\bigg] \le k_1 t^{-3/8} e^{k_2 x^2}. \tag{4.24}$$

We postpone the proof for the lemma below.

Now, we also observe that

$$\mathbb{E}\bigg[|X_t^\varepsilon(x)|^4\bigg] \le C(|x|^4 + T^4). \tag{4.25}$$

Then we obtain

$$\mathbb{E}\bigg[\bigg|\frac{d}{dx}X_t^\varepsilon(x)\bigg|^{-1}(1+|X_t^\varepsilon(x)|)^2\bigg]$$

$$\le C\bigg(\mathbb{E}\bigg|\frac{d}{dx}X_t^\varepsilon(x)\bigg|^{-2} + \mathbb{E}\bigg|(1+|X_t^\varepsilon(x)|)^4\bigg|\bigg) \le C(k_1 t^{-3/8} e^{k_2 x^2} + T^4 + x^4).$$

Thus we deduce

$$
\begin{aligned}
&\int_{\mathbb{R}} \mathbb{E}[|u^{\varepsilon}(t,x)|^2](1+|x|)^2\,dx &(4.26)\\
&\quad\le \int_{\mathbb{R}} |u_0^{\varepsilon}(y)|^2 \mathbb{E}\left[\left|\frac{dX_s^{\varepsilon}(y)}{dy}\right|^{-1}(1+|X_t^{\varepsilon}(y)|)^2\right]dy\\
&\quad\le C\int_{\mathbb{R}} |u_0^{\varepsilon}(y)|^2\big(k_1 t^{-3/8} e^{k_2 x^2} + T^4 + y^4\big)\,dy\\
&\quad\le C k_1 t^{-3/8}\int_{\mathbb{R}} |u_0^{\varepsilon}(y)|^2 e^{k_2 y^2}\ dy + C\int_{\mathbb{R}} |u_0^{\varepsilon}(y)|^2 e^{k_2 y^2}\ dy. &(4.27)
\end{aligned}
$$

We observe that

$$
\begin{aligned}
\int_{\mathbb{R}} |u_0^{\varepsilon}(y)|^2 e^{k_2 y^2} dy &\le \int_{\mathbb{R}} \left[e^{k_2 y^2}\left(\int_{\mathbb{R}} \rho_{\varepsilon}(y-x)|u_0(x)|^2 dx\right)\right]dy\\
&= \int_{\mathbb{R}} \left[|u_0(x)|^2\left(\int_{B(x,\varepsilon)} \rho_{\varepsilon}(y-x) e^{k_2 y^2} dy\right)\right]dx\\
&= \int_{\mathbb{R}} \left[|u_0(x)|^2\left(\int_{B(0,\varepsilon)} \rho_{\varepsilon}(u) e^{k_2 (x+u)^2} du\right)\right]dx\\
&\le \int_{\mathbb{R}} \left[|u_0(x)|^2 e^{2k_2 x^2}\left(\int_{B(0,\varepsilon)} \rho_{\varepsilon}(u) e^{2k_2 u^2} du\right)\right]dx\\
&\le C\|u_0\|^2_{L^2(\mathbb{R},wdx)}. &(4.28)
\end{aligned}
$$

From (4.26) and (4.28) we conclude that

$$
\|u^{\varepsilon}\|^2_{L^2(\Omega\times[0,T]\times\mathbb{R},\mu dx)} \le C(k,T)\|u_0\|^2_{L^2(\mathbb{R},wdx)}\,.
$$

Therefore, the sequence $\{u^{\varepsilon}\}_{\varepsilon>0}$ is bounded in $L^2(\Omega\times[0,T]\times\mathbb{R},\mu dx)$. Then there exists a convergent subsequence, which we denote also by $u^{\varepsilon}$, such that converge weakly in $L^2(\Omega\times[0,T]\times\mathbb{R},\mu dx)$ to some process $u\in L^2(\Omega\times[0,T]\times\mathbb{R},\mu dx)$ .

*Step 3: Passing to the Limit.* Now, if $u^{\varepsilon}$ is a solution of (4.23), it is also a weak solution, that is, for any test function $\varphi\in C_0^{\infty}(\mathbb{R})$, $u^{\varepsilon}$

satisfies (written in the Itô form):

$$\int_{\mathbb{R}} u^{\varepsilon}(t,x)\varphi(x)dx = \int_{\mathbb{R}} u_0^{\varepsilon}(x)\varphi(x)\,dx + \int_0^t\int_{\mathbb{R}} u^{\varepsilon}(s,x)\,b^{\varepsilon}(x)\partial_x\varphi(x)\,dxds$$

$$+\int_0^t\int_{\mathbb{R}} u^{\varepsilon}(s,x)\,\partial_x\varphi(x)\,dx\,dB_s + \frac{1}{2}\int_0^t\int_{\mathbb{R}} u^{\varepsilon}(s,x)\,\partial_x^2\varphi(x)\,dx\,ds\,.$$

Thus, for prove existence of the SCE (4.5) is enough to pass to the limit in the above equation along the convergent subsequence found. This is made through of the same arguments of [40, theorem 15].

□

**Lemma 4.2.9.** *Assume $b \in C_c^{\infty}(\mathbb{R})$ and that satisfies the hypothesis 4.2.6. Then for $T>0$ there are constants $k_1 = k_1(k,T)$ and $k_2 = k_2(k,T)$ such that*

$$\mathbb{E}\left[\left|\frac{d}{dx}X_t(x)\right|^{-2}\right] \leq k_1 t^{-3/8} e^{k_2 x^2}\,, \tag{4.29}$$

*where $k_1 = \sqrt{c_1}\sqrt[4]{c_2}e^{35Tk^2}$ and $k_2 = 2(k+99Tk^2)$ ($c_1$ and $c_2$ are defined below in the proof).*

*Proof.* We consider the SDE associated to the vector field $b$ :

$$dX_t = b(X_t)\,dt + dB_t\,, \qquad X_0 = x\,.$$

We denote

$$\mathcal{E}\Big(\int_0^t b(X_u)dB_u\Big) = \exp\Big\{\int_0^t b(X_u)dB_u - \frac{1}{2}\int_0^t b^2(X_u)du\Big\},$$

and

$$dQ(\omega) = \mathcal{E}\Big(\int_0^t b(X_u)dB_u\Big)d\mathbb{P}(\omega).$$

Using the Girsanov's theorem we obtain that

$$\mathbb{E}\left[\left|\frac{dX_t}{dx}(x)\right|^{-2}\right] = \mathbb{E}_Q\left[\left|\frac{dX_t}{dx}(x)\right|^{-2}\right]$$

$$= \mathbb{E}\Big[\exp\Big\{-2\int_0^t b'(x+B_s)ds\Big\}\mathcal{E}\Big(\int_0^t b(x+B_s)dB_s\Big)\Big].$$

Now, we proceed as in the proof of the Lemma 3.6 of [79]. Let $b_1 = -b$, then we have

$$\mathbb{E}\Big[\Big|\frac{dX_t}{dx}(x)\Big|^{-2}\Big] = \mathbb{E}\Big[\exp\Big\{2\int_0^t b_1'(x+B_s)ds\Big\}\mathcal{E}\Big(\int_0^t b(x+B_s)dB_s\Big)\Big].$$

Applying the Itô formula to $\tilde{b}(z) = \int_\infty^z b_1(y)dy$ we deduce

$$\tilde{b}(x+B_t) = \tilde{b}(x) + \int_0^t b_1(x+B_s)dB_s + \frac{1}{2}\int_0^t b_1'(x+B_s)ds\,.$$

By Hölder inequality we have

$$\mathbb{E}\Big[\Big|\frac{dX_t}{dx}(x)\Big|^{-2}\Big] \tag{4.30}$$

$$\leq \|\exp\{4(\tilde{b}(x+B_t)-\tilde{b}(x))\}\|_{L^2(\Omega)} \tag{4.31}$$

$$\|\exp\Big\{-4\int_0^t b_1(x+B_s)dB_s\Big\}\times\mathcal{E}\Big(\int_0^t b(x+B_s)dB_s\Big)\|_{L^2(\Omega)}\,. \tag{4.32}$$

For the first term, we obtain

$$\begin{aligned}|\tilde{b}(x+B_t)-\tilde{b}(x)| &= |\int_0^1 b_1(x+\theta(B_t))d\theta|\ |B_t| \\ &\leq \int_0^1 (k+k|x+\theta B_t|)d\theta|B_t| \\ &\leq k|B_t| + k|x||B_t| + \frac{k}{2}(B_t)^2 \\ &\leq \frac{k}{2}x^2 + k|B_t| + k(B_t)^2.\end{aligned}$$

Then we get

$$\begin{aligned}\mathbb{E}[\exp\{8(\tilde{b}(x+B_t)-\tilde{b}(x))\}] &\leq \mathbb{E}[\exp\{8(\frac{k}{2}x^2+k|B_t|+k(B_t)^2)\}] \\ &= e^{4kx^2}\mathbb{E}[\exp\{8(k|B_t|+k(B_t)^2)\}] \\ &= e^{4kx^2}\frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}}\exp\{8k(|z|+z^2)-\frac{z^2}{2t}\}\,dz.\end{aligned}$$

Hence

$$\|\exp\{4(\tilde{b}(x+B_t)-\tilde{b}(x))\}\|_{L^2(\Omega)} \leq e^{2kx^2}t^{-1/4}\sqrt{c_1}, \tag{4.33}$$

where

$$c_1 = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\exp\{8k(|z|+z^2)-\frac{z^2}{2T}\}dz.$$

For the second term of (4.30) we have

$$\begin{aligned}&\mathbb{E}\Big[\exp\Big\{-8\int_0^t b_1(x+B_s)dB_s\Big\}\mathcal{E}\Big(\int_0^t b(x+B_s)dB_s\Big)^2\Big] \\ &\mathbb{E}\Big[\exp\Big\{-8\int_0^t b_1(x+B_s)dB_s\Big\}\exp\Big\{2\int_0^t b(x+B_s)dB_s-\int_0^t b^2(x+B_s)ds\Big\}\Big] \\ &= \mathbb{E}\Big[\exp\Big\{-10\int_0^t b_1(x+B_s)dB_s-\int_0^t b_1^2(x+B_s)ds\Big\}\Big] \\ &\leq \|\exp\Big\{-10\int_0^t b_1(x+B_s)dB_s-\alpha\int_0^t b_1^2(x+B_s)ds\Big\}\|_{L^2(\Omega)}\times \\ &\quad\times\|\exp\Big\{(\alpha-1)\int_0^t b_1^2(x+B_s)ds\Big\}\|_{L^2(\Omega)}.\end{aligned} \tag{4.34}$$

Now, we choose $\alpha = 100$ because $\frac{1}{2}(-20b_1(x+B_s))^2 = 2\alpha b_1^2(x+B_s)$. Then the process $\exp\{-20\int_0^t b_1(x+B_s)dB_s - 200\int_0^t b_1^2(x+B_s)ds\} = \mathcal{E}\Big(\int_0^t(-20b_1(x+B_s)dB_s)\Big)$ is a martingale with expectation equal to one. Thus

$$\| \exp\Big\{ -10\int_0^t b_1(x+B_s)dB_s - 100\int_0^t b_1^2(x+B_s)ds\Big\}\|_{L^2(\Omega)} = 1.$$

From (4.19) we deduce that the second term of (4.34) is bounded by

$$\begin{aligned}
\mathbb{E}\Big[\exp\Big\{2(\alpha-1)\int_0^t b_1^2(x+B_s)ds\Big\}\Big] &= \mathbb{E}\Big[\exp\Big\{198\int_0^t b_1^2(x+B_s)ds\Big\}\Big]\\
&\le \mathbb{E}\Big[\exp\Big\{198\int_0^t k^2(1+|x+B_s|)^2 ds\Big\}\Big]\\
&\le \mathbb{E}\Big[\exp\Big\{198tk^2(1+B_t^*)^2\Big\}\Big],
\end{aligned}$$

where $B_t^* = \sup_{s\le t}|x+B_s|$. We set

$$Y_s = \exp\{99tk^2(1+|x+B_s|)^2\}.$$

Then, by Doob's Maximal inequality we have

$$\mathbb{E}\Big[\exp\Big\{198tk^2(1+B_t^*)^2\Big\}\Big] = \mathbb{E}\Big[\sup_{s\le t} Y_s^2\Big]$$

$$\le 4\ \mathbb{E}[Y_t^2] = \mathbb{E}[\exp\{198tk^2(1+|x+B_t|)^2\}]$$

$$\le 4\ \mathbb{E}[\exp\{396tk^2(1+(x+B_t)^2)\}]$$

$$\le 4\ \mathbb{E}[\exp\{396tk^2(1+2(x^2+B_t^2))\}]$$

$$= 4e^{396tk^2}e^{792tk^2x^2}\mathbb{E}[\exp\{792k^2tB_t^2\}]$$

$$= 4e^{396tk^2}e^{792tk^2x^2}\frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}}\exp\{792tk^2z^2 - \frac{z^2}{2t}\}dz.$$

Substituting in (4.34) we have

$$\mathbb{E}\Big[\exp\Big\{-8\int_0^t b_1(x+B_s)dB_s\Big\}\mathcal{E}\Big(\int_0^t b(x+B_s)dB_s\Big)^2\Big]$$

$$\le e^{198Tk^2}e^{396Tk^2x^2}t^{-1/4}\sqrt{c_2}\,, \qquad (4.35)$$

where

$$c_2 = \frac{4}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp\{792Tk^2z^2 - \frac{z^2}{2T}\}dz.$$

Thus , replacing (4.33) and (4.35) in (4.30) we conclude

$$\begin{aligned}
\mathbb{E}\left[\left|\frac{dX_t}{dx}(x)\right|^{-2}\right] &\leq e^{2kx^2} t^{-1/4} \sqrt{c_1} e^{99Tk^2} e^{198Tk^2x^2} t^{-1/8} \sqrt[4]{c_2} \\
&= \sqrt{c_1} \sqrt[4]{c_2} e^{99Tk^2} t^{-3/8} e^{2(k+99Tk^2)x^2} \\
&= k_1 t^{-3/8} e^{k_2 x^2} ,
\end{aligned}$$

where $k_1 = \sqrt{c_1}\sqrt[4]{c_2}e^{99Tk^2}$ and $k_2 = 2(k + 99Tk^2)$. This proves (5.27). □

**Uniqueness.**

**Theorem 4.2.10.** *Under the conditions of hypothesis 4.2.6, uniqueness holds for $L^2$- weak solutions of the Cauchy problem* (4.5) *in the following sense: if $u, v$ are $L^2$- weak solutions with the same initial data $u_0 \in L^2(\mathbb{R}, w\,dx)$, then $u = v$ almost everywhere in $\Omega \times [0,T] \times \mathbb{R}$.*

*Proof. Step 0: Set of solutions.* Remark that the set of $L^2$- weak solutions is a linear subspace of $L^2(\Omega \times [0,T] \times \mathbb{R}, \mu dx)$, because the stochastic continuity equation is linear, and the regularity conditions is a linear constraint. Therefore, it is enough to show that a $L^2$- weak solution $u$ with initial condition $u_0 = 0$ vanishes identically.

*Step 1: Primitive of the solution.* We define $V(t,x) = \int_{-\infty}^{x} u(t,y)\,dy$. We consider a nonnegative smooth cut-off function $\eta$ supported on the ball of radius 2 and such that $\eta = 1$ on the ball of radius 1. For any $R > 0$, we introduce the rescaled functions $\eta_R(\cdot) = \eta(\frac{\cdot}{R})$. Let be $\varphi \in C_0^\infty(\mathbb{R})$, we observe that

$$\begin{aligned}
\int_{\mathbb{R}} V(t,x)\varphi(x)\eta_R(x)dx = &- \int_{\mathbb{R}} u(t,x)\theta(x)\eta_R(x)dx \\
&- \int_{\mathbb{R}} V(t,x)\theta(x)\partial_x\eta_R(x)dx\,,
\end{aligned}$$

where $\theta(x) = \int_{-\infty}^{x} \varphi(y)\, dy$. By definition of the solution $u$, taking as test function $\theta(x)\eta_R(x)$ we have that $V(t,x)$ satisfies

$$\int_{\mathbb{R}} V(t,x)\ \eta_R(x)\varphi(x)dx = -\int_0^t \int_{\mathbb{R}} \partial_x V(s,x)\, b(x)\eta_R(x)\varphi(x)\ dxds$$

$$-\int_0^t \int_{\mathbb{R}} \partial_x V(s,x)\ \eta_R(x)\varphi(x)\ dx \circ dB_s \tag{4.36}$$

$$-\int_0^t \int_{\mathbb{R}} \partial_x V(s,x)\, b(x)\partial_x \eta_R(x)\theta(x)\ dxds$$

$$-\int_0^t \int_{\mathbb{R}} \partial_x V(s,x)\ \partial_x \eta_R(x)\theta(x)\ dx \circ dB_s - \int_{\mathbb{R}} V(t,x)\theta(x)\partial_x\eta_R(x)dx. \tag{4.37}$$

Taking the limit as $R \to \infty$ we get

$$\int_{\mathbb{R}} V(t,x)\varphi(x)dx =$$

$$-\int_0^t \int_{\mathbb{R}} \partial_x V(s,x)\, b(x)\varphi(x)\ dxds - \int_0^t \int_{\mathbb{R}} \partial_x V(s,x)\ \varphi(x)\ dx \circ dB_s. \tag{4.38}$$

*Step 2: Smoothing.* Let $\{\rho_\varepsilon(x)\}_\varepsilon$ be a family of standard symmetric mollifiers. For any $\varepsilon > 0$ and $x \in \mathbb{R}^d$ we use $\rho_\varepsilon(x - \cdot)$ as test function, then have

$$\int_{\mathbb{R}} V(t,y)\rho_\varepsilon(x-y)\, dy = -\int_0^t \int_{\mathbb{R}} \big(b(y)\partial_y V(s,y)\big)\rho_\varepsilon(x-y)\ dyds$$

$$-\int_0^t \int_{\mathbb{R}} \partial_y V(s,y)\, \rho_\varepsilon(x-y)\, dy \circ dB_s$$

We set $V_\varepsilon(t,x) = (V * \rho_\varepsilon)(x)$, $b_\varepsilon(x) = (b * \rho_\varepsilon)(x)$ and $(bV)_\varepsilon(t,x) = (b.V * \rho_\varepsilon)(x)$. We deduce

$$V_\varepsilon(t,x) + \int_0^t b_\epsilon(x)\partial_x V_\varepsilon(s,x)\, ds + \int_0^t \partial_x V_\varepsilon(s,x)\ \circ dB_s$$

$$= \int_0^t \big(\mathcal{R}_\epsilon(V,b)\big)(x,s)\,ds,$$

where we denote $\mathcal{R}_\epsilon(V,b) = b_\varepsilon\ \partial_x V_\varepsilon - (b\partial_x V)_\varepsilon$.

*Step 3: Method of Characteristic.* Applying the Itô-Wentzell-Kunita formula to $V_\varepsilon(t, X_t^\epsilon)$ , see Theorem 8.3 of [63], we have

$$V_\varepsilon(t, X_t^\epsilon) = \int_0^t \big(\mathcal{R}_\epsilon(V,b)\big)(X_s^\epsilon, s) ds.$$

Then, considering that $X_t^\epsilon = X_{0,t}^\epsilon$ and $Y_t^\epsilon = Y_{0,t}^\epsilon = (X_{0,t}^\epsilon)^{-1}$ we have that

$$V_\varepsilon(t,x) = \int_0^t \big(\mathcal{R}_\epsilon(V,b)\big)(X_{0,s}^\epsilon(Y_{0,t}^\epsilon), s) ds = \int_0^t \big(\mathcal{R}_\epsilon(V,b)\big)(Y_{t-s}^\epsilon, s) ds.$$

Multiplying by the test functions $\varphi$ and integrating in $\mathbb{R}$ we get

$$\int V_\varepsilon(t,x)\ \varphi(x) dx = \int_0^t \int \big(\mathcal{R}_\epsilon(V,b)\big)(Y_{t-s}^\epsilon, s)\ \ \varphi(x)\ \ dx\ ds. \quad (4.39)$$

We observe that

$$\int_0^t \int \big(\mathcal{R}_\epsilon(V,b)\big)(Y_{t-s}^\epsilon, s)\ \varphi(x)\ \ dx\ ds$$
$$= \int_0^t \int \big(\mathcal{R}_\epsilon(V,b)\big)(x, s)\ JX_{t-s}^\epsilon \varphi(X_{t-s}^\epsilon)\ \ dx\ ds.$$

*Step 4: Convergence of the commutator.* Now, we observe that $\mathcal{R}_\epsilon(V,b)$ converge to zero in $L^2([0,T]\times\mathbb{R})$. In fact,

$$(b\ \partial_x V)_\varepsilon \to b\ \partial_x V\ \ in\ L^2([0,T]\times\mathbb{R}),$$

and by the dominated convergence theorem we obtain

$$b_\epsilon \partial_x V_\varepsilon \to b\ \partial_x V\ \ in\ L^2([0,T]\times\mathbb{R}).$$

*Step 5: Conclusion.* From step 3 we obtain

$$\int V_\varepsilon(t,x)\ \varphi(x)dx =$$

$$\int_0^t \int \big(\mathcal{R}_\epsilon(V,b)\big)(x,s)\ JX_{t-s}^\epsilon \varphi(X_{t-s}^\epsilon)\ dx\ ds. \tag{4.40}$$

Using Hölder's inequality we have

$$\mathbb{E}\left| \int_0^t \int \Big(\mathcal{R}_\epsilon(V,b)\Big)(x,s)\ JX_{t-s}^\epsilon \varphi(X_{t-s}^\epsilon)\ dx\ ds \right|$$

$$\leq \left( \mathbb{E}\int_0^t \int |\big(\mathcal{R}_\epsilon(V,b)\big)(x,s)|^2\ dx\ ds \right)^{\frac{1}{2}}$$

$$\times \left( \mathbb{E}\int_0^t \int |JX_{t-s}^\epsilon \varphi(X_{t-s}^\epsilon)|^2\ dx\ ds \right)^{\frac{1}{2}}.$$

From step 4 result

$$\left( \mathbb{E}\int_0^t \int |\big(\mathcal{R}_\epsilon(V,b)\big)(x,s)|^2\ dx\ ds \right)^{\frac{1}{2}} \to 0.$$

From formula (4.22) we obtain

$$\left( \mathbb{E}\int_0^t \int |JX_{s,t}^\epsilon \varphi(X_{t-s}^\epsilon)|^2\ dx\ ds \right)^{\frac{1}{2}} \leq C\Big( \int_{\mathbb{R}} |\varphi(x)|^2\ dx\ \Big)^{\frac{1}{2}}.$$

Passing to the limit in equation (4.40) we conclude that $V = 0$. Then we deduce that $u = 0$.

□

**Example 4.2.11.** *We have that*

$$b(x) = \sqrt{|x|}$$

*is not the DiPerna- Lions class because* $\mathrm{div} b = \frac{sig(x)}{2\sqrt{|x|}}$ *is not bounded. However, satisfies our condition because is the linear growth.*

### 4.2.4 More results.

- Fedrizzi and Flandoli in [34] showed a well-posedness result for stochastic transport equation under only some integrability conditions on the drift in the class of $W^{p,r}_{loc}$-solutions , with no assumption on the divergence. There, it is only assumed that

$$
\begin{aligned}
& b \in L^q\big([0,T]; L^p(\mathbb{R}^d)\big)\,, \\
\text{for} \qquad & p, q \in [2,\infty)\,, \qquad\qquad \frac{d}{p} + \frac{2}{q} < 1\,.
\end{aligned}
\tag{4.41}
$$

  Beck, Flandoli, Gubinelli and Maurelli [7] proved, using a technique based on the regularizing effect observed on expected values of moments of the solution, well-posedness result of the transport-continuity equation considering $L^p$-solutions. They also obtained well-posedness for the limit cases of $p, q = \infty$ and when the inequality in (4.41) becomes an equality. Also Neves and Olivera worked with this condition in [77] and [78].

- Mohammed, Nilssen, and Proske in [74] proved uniqueness for stochastic transport equation assuming only that the drift is bounded. However, they assumed that the solution is differentiable in the sense of Malliavin and the initial condition is smooth.

- Fedrizzi, Neves and Olivera in [36] introduced the notion of quasiregular solution for the stochastic transport equation and they showed existence and uniqueness when the drift $b \in L^2_{loc}$ and the divergence is bounded.

- Attanasio and Flandoli in [4] showed existence and uniqueness for stochastic transport equation when the vector field $b \in BV$ without assumption on the divergence.

- Mollindedo and Olivera in [76] proved well-posedness result for stochastic continuity equation when the drift is unbounded and it is Hölder continuous.

### 4.2.5 Persistence of Regularity.

As pointed out by Colombini, Luo and Rauch in [20], there exists an important example of $b \in L^\infty \cap W^{1,p}, (\forall p < \infty)$, such that the propagation of the continuity in the deterministic transport equation is missing. That is to say, even if the uniqueness is established in this case, the persistence condition is not, one may start with a continuous initial data, but the deterministic solution of the transport equation is not continuous. However, in the stochastic case we have the persistence property.

We recall from Fedrizzi and Flandoli [34] that a certain Sobolev regularity is maintained under the Ladyzhenskaya-Prodi-Serrin condition, that is,

$$u_0 \in \bigcap_{r\geq 1} W^{1,r}(\mathbb{R}^d) \Rightarrow u(t,.) \in \bigcap_{r\geq 1} W^{1,r}_{loc}(\mathbb{R}^d).$$

From Mollinedo and Olivera [76] we have that if $u_0 \in W^{1,2p}(\mathbb{R}^d)$ then $u(t,.) \in W^{1,2p}(\mathbb{R}^d, e^{-x^2})$ when the drift is unbounded and Hölder continuous.

# Chapter 5

# Other PDEs

In this chapter, we present other results on regularization by noise in partial differential equations : including advection equation, Burgers equation and hyperbolic systems of conservation law.

## 5.1 Advection equation.

This section we follow Flandoli and Olivera in [42]. Consider the linear stochastic vector advection equation in the unknown random field $B : \Omega \times [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$

$$dB + (v \cdot \nabla B - B \cdot \nabla v)\ dt + \sum_{k=1}^{\infty} (\sigma_k \cdot \nabla B - B \cdot \nabla \sigma_k) \circ dW_t^k = 0 \tag{5.1}$$

where $v : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma_k : \mathbb{R}^d \to \mathbb{R}^d$, $k \in \mathbb{N}$, are given divergence free vector fields and $\left(W_\cdot^k\right)_{k\in\mathbb{N}}$ is a family of independent real-valued Brownian motions on the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. The general structure of the noise assumed here is inspired by the theory of diffusion of passive scalars and vector fields in turbulent fluids, see for exaple [33] and is also motivated by the recent proposal for a variational principle approach to fluid mechanics, see [55]. This problema was considered in [44] under an Hölder condition on $v$ and a partial result is given in [47] for $v$ having suitable integrability. But in both cases the noise was the standard Brownian motion in $\mathbb{R}^d$, without a space structure.

We aim at studying existence and uniqueness, under low regularity assumption on $v$.

### 5.1.1 Preliminaries.

#### Itô Formulation

It is convenient to introduce the notation of the Lie bracket between vector fields

$$[A, B] = A \cdot \nabla B - B \cdot \nabla A$$

which is also equal to the Lie derivative $\mathcal{L}_A B$ and also, for divergence free fields, to $\operatorname{curl}(A \wedge B)$. In Stratonovich form equation (5.1) then reads

$$dB + [v, B]\ dt + \sum_{k=1}^{\infty} [\sigma_k, B] \circ dW_t^k = 0.$$

Its Itô formulation is

$$dB + [v, B]\ dt + \sum_{k=1}^{\infty} [\sigma_k, B]\ dW_t^k = \frac{1}{2} \sum_{k=1}^{\infty} [\sigma_k, [\sigma_k, B]]\ dt. \tag{5.2}$$

Let us show that (5.1) leads to (5.2). Recall that Stratonovich integral differs from Itô integral by 1/2 mutual variation: $X \circ dW = X dW + \frac{1}{2} d \langle X, W \rangle$; where, in the case of interest to us when $X$ is vector valued and $W$ is real valued, by $\langle X, W \rangle$ we mean the vector of components $\langle X^\alpha, W \rangle$. Then

$$[\sigma_k, B] \circ dW_t^k = [\sigma_k, B]\ dW_t^k + d \left\langle [\sigma_k, B], W^k \right\rangle_t .$$

Then

$$d \left\langle [\sigma_k, B], W^k \right\rangle_t = (\sigma_k \cdot \nabla) d \left\langle B, W^k \right\rangle_t - d \left\langle B, W^k \right\rangle_t \cdot \nabla \sigma_k.$$

From the equation for $dB$ and the property that the mutual variations between $W^k$ and BV functions or stochastic integrals with respect to $W^j$ for $j \neq k$ are zero (and $d \left\langle W^k, W^k \right\rangle_t = dt$) obtain

$$d\left\langle B,W^{k}\right\rangle_{t}=d\left\langle \int_{0}^{\cdot}\left[\sigma_{k},B_{s}\right]\ dW_{s}^{k},W^{k}\right\rangle_{t}=\left[\sigma_{k},B_{t}\right]\ dt.$$

Therefore we obtain (5.2).

We have introduced the second order differential operator, acting on smooth vector fields $B$, defined as

$$\mathcal{L}B\left(x\right):=\frac{1}{2}\sum_{k=1}^{\infty}\left[\sigma_{k},\left[\sigma_{k},B\right]\right]\left(x\right). \tag{5.3}$$

### Stochastic Exponentials

Let $(\Omega,\mathcal{F},\mathcal{F}_{t},P)$ be the filtered probability space introduced above, with the sequence $\left\{W_{t}^{k}\right\}_{k\in\mathbb{N}}$ of independent Brownian motions. Let $\mathcal{G}_{t}$ be the associated filtration:

$$\mathcal{G}_{t}=\sigma\left\{B_{s}^{k};s\in\left[0,t\right],k\in\mathbb{N}\right\}.$$

Let $\overline{\mathcal{G}}_{t}$ be the completed filtration. For some $T>0$, let

$$\mathcal{H}=L^{2}\left(\Omega,\overline{\mathcal{G}}_{T},P\right)$$

$$F=\cup_{n\in\mathbb{N}}L^{2}\left(0,T;\mathbb{R}^{n}\right)$$

$$\mathcal{D}=\left\{e_{f}\left(T\right);f\in F\right\}$$

where, for $n\in\mathbb{N}$, $f\in L^{2}\left(0,T;\mathbb{R}^{n}\right)$, with components $f_{1},...,f_{n}$, we set

$$e_{f}\left(t\right)=\exp\left(\sum_{k=1}^{n}\int_{0}^{t}f_{k}\left(s\right)dW_{s}^{k}-\frac{1}{2}\sum_{k=1}^{n}\int_{0}^{t}\left|f_{k}\left(s\right)\right|^{2}ds\right)$$

for $t\in[0,T]$. From Itô formula

$$de_{f}\left(t\right)=\sum_{k=1}^{n}f_{k}\left(t\right)e_{f}\left(t\right)dW_{t}^{k}.$$

The following result is known, see the argument in [81]:

**Lemma 5.1.1.** *$\mathcal{D}$ is dense in $\mathcal{H}$.*

### Structure and Assumptions on the Noise

Let $(\sigma_k)_{k\in\mathbb{N}}$ be a sequence of twice differentiable divergence free vector fields:

$$\sigma_k \in C^2\left(\mathbb{R}^d, \mathbb{R}^d\right), \qquad \operatorname{div}\sigma_k = 0 \tag{5.4}$$

such that

$$\sum_{k=1}^{\infty} |\sigma_k(x)|^2 < \infty \tag{5.5}$$

for every $x \in \mathbb{R}^d$. The matrix-valued function $Q(x,y) \in \mathbb{R}^{d\times d}$, $x, y \in \mathbb{R}^d$, given by

$$Q^{\alpha\beta}(x,y) := \sum_{k=1}^{\infty} \sigma_k^{\alpha}(x)\,\sigma_k^{\beta}(y)$$

is well defined, (we write $Q^{\alpha\beta}(x,y)$, $\alpha,\beta = 1,...,d$ for its components and similarly for $\sigma_k^{\alpha}(x)$). Our main assumptions on the noise are: $Q(x,y)$ is twice continuously differentiable in $(x,y)$, bounded with bounded first and second derivatives, that we summarize in the notation

$$Q \in C_b^2 \tag{5.6}$$

and

$$Q(x,x) \geq \nu Id_{\mathbb{R}^d} \tag{5.7}$$

for some $\nu > 0$, uniformly in $x \in \mathbb{R}^d$.

### Definition of Solution.

We present now the setting and a suitable definition of quasiregular weak solutions to equation (5.1) , adapted to treat the problem of well-posedness. We assume that the vector field $v$ satisfies

$$v \in L^{\infty}\left([0,T], L^p(\mathbb{R}^d;\mathbb{R}^d)\right) \qquad \text{for some } p \text{ such that } p > d,\ p \geq 2 \tag{5.8}$$

$$\operatorname{div} v(t,x) = 0\,. \tag{5.9}$$

Moreover, the initial condition is taken to be

$$B_0 \in L^4(\mathbb{R}^d) \cap L^{\infty}(\mathbb{R}^d)\,, \qquad \operatorname{div} B_0 = 0. \tag{5.10}$$

The next definition tells us in which sense a stochastic process is a quasiregular weak solution of (5.1). We formally use the identity

$$\begin{aligned}\int [A,B]\cdot C\ dx &= \int ((A\cdot\nabla)B-(B\cdot\nabla)A)\cdot C\ dx\\ &= -\int B\cdot(A\cdot\nabla)C - A\cdot(B\cdot\nabla)C\ dx\end{aligned}$$

which holds true for sufficiently smooth and integrable fields such that $\operatorname{div} A = \operatorname{div} B = 0$. Moreover, we use the adjoint operator $\mathcal{L}^*$, defined in Proposition 5.1.3 below, which maps test functions $\varphi \in C_c^\infty(\mathbb{R}^d,\mathbb{R}^d)$ into bounded continuous compact support vector fields $\mathcal{L}^*\varphi(x)$.

**Definition 5.1.2.** *A stochastic process $B:[0,T]\times\mathbb{R}^d\to\mathbb{R}^d$, $B\in L^2\left(\Omega\times[0,T],L^2_{loc}(\mathbb{R}^d)\right)$ is called a quasiregular weak solution of the Cauchy problem (5.1) when:*

*i)* $\operatorname{div} B(\omega,t)=0$*, in the sense of distributions, for a.e.* $(\omega,t)\in\Omega\times[0,T]$

*ii) for any $\varphi\in C_c^\infty(\mathbb{R}^d,\mathbb{R}^d)$, the real valued process $\int B(t,x)\cdot\varphi(x)dx$ has a continuous modification which is an $\mathcal{F}_t$-semimartingale,*

*iii) for any $\phi\in C_c^\infty(\mathbb{R}^d,\mathbb{R}^d)$ and for all $t\in[0,T]$, we have $\mathbb{P}$-almost surely*

$$\begin{aligned}&\int B(t,x)\cdot\phi(x)\ dx\\ &-\int_0^t\int (B(s,x)\cdot(v(s,x)\cdot\nabla)\phi(x)-v(s,x)\cdot(B(s,x)\cdot\nabla)\phi(x))\ dxdt\\ &-\sum_{k=1}^\infty\int_0^t\int B(s,x)\cdot(\sigma_k(x)\cdot\nabla)\phi(x)\ dx\ dW_s^k\\ &-\sum_{k=1}^\infty\int_0^t\int\ \sigma_k(x)\cdot(B(s,x)\cdot\nabla)\phi(x)\ dx\ dW_s^k\\ &\quad=\int B_0(x)\cdot\phi(x)\ dx+\frac{1}{2}\int_0^t\int\mathcal{L}^*\phi(x)\cdot B(s,x)\ dxdt \qquad (5.11)\end{aligned}$$

*iv) (Regularity in Mean) For all $n\in\mathbb{N}$ and each function $f\in L^2(0,T;\mathbb{R}^n)$, with components $f_1,...,f_n$, the deterministic function*

*$V(t,x) := \mathbb{E}[B(t,x)\, e_f(t)]$ is a measurable bounded function, which belongs to $L^2([0,T];H^1(\mathbb{R}^d)) \cap C([0,T];L^2(\mathbb{R}^d))$ and satisfies the parabolic equation*

$$\partial_t V + [v-h,V] = \mathcal{L}V \tag{5.12}$$

*in the weak sense, where $h(t,x) := \sum_{k=1}^n f_k(t)\, \sigma_k(x)$.*

We have called *quasiregular* this class of weak solutions because of the regularity of the expected values $V(t,x) := \mathbb{E}[B(t,x)\, e_f(t)]$. Equation (5.1) has an hyperbolic nature, it cannot regularize the initial condition; but in the average there is a regularization, on which we insist in the definition.

Let us see the formal motivation for equation (5.12). We apply formally Itô formula to the product of a solution with the stochastic exponential, in equation (5.2). We get

$$\begin{aligned} &d(Be_f) + [v,B]\, e_f\ dt + \sum_{k=1}^{\infty} e_f\, [\sigma_k, B]\ dW_t^k \\ &= \frac{1}{2}\sum_{k=1}^{\infty} e_f\, [\sigma_k,[\sigma_k,B]]\ dt + \sum_{k=1}^{n} f_k B e_f dW_t^k + \sum_{k=1}^{n} f_k e_f\, [\sigma_k, B]\ dt. \end{aligned}$$

Taking expectation we obtain

$$\begin{aligned} \partial_t V + [v,V] &= \frac{1}{2}\sum_k [\sigma_k,[\sigma_k,V]] + \sum_{k=1}^{n} f_k\, [\sigma_k, V] \\ &= \frac{1}{2}\sum_k [\sigma_k,[\sigma_k,V]] + \left[\sum_{k=1}^{n} f_k \sigma_k, V\right]. \end{aligned}$$

### The Differential Operator $\mathcal{L}$

A key role is played by the differential operator $\mathcal{L}$ defined by (5.3). We state here its property of uniform ellipticity, based on the assumptions on $Q$.

**Proposition 5.1.3.** *Assume $Q$ to be twice continuously differentiable, bounded with bounded first and second derivatives, and*

$$Q(x,x) \geq \nu Id_{\mathbb{R}^d}$$

*for some $\nu > 0$, uniformly in $x \in \mathbb{R}^d$. Then $\mathcal{L}$ is well defined, uniformly elliptic. In particular, there exists $C > 0$ such that*

$$-\int_{\mathbb{R}^d} \mathcal{L}B(x) \cdot B(x)\, dx \geq \frac{\nu}{2} \int_{\mathbb{R}^d} |DB(x)|^2\, dx - C \int_{\mathbb{R}^d} |B(x)|^2\, dx$$

*for all $B \in W^{1,2}\left(\mathbb{R}^d, \mathbb{R}^d\right)$. Moreover it has the form*

$$\begin{aligned}(\mathcal{L}B)^\alpha(x) = & \sum_{i,j=1}^d a_{ij}(x)\, \partial_i \partial_j B^\alpha(x) \\ & + \sum_{i,\beta=1}^d b_i^{\alpha\beta}(x)\, \partial_i B^\beta(x) + \sum_{\beta=1}^d c^{\alpha\beta}(x)\, B^\beta(x)\end{aligned}$$

*where $a_{ij}$ is twice continuously differentiable, bounded with bounded first and second derivatives, $b_i^{\alpha\beta}$ is continuously differentiable, bounded with bounded first derivatives and $c^{\alpha\beta}$ is bounded continuous. The formal adjoint operator $\mathcal{L}^*$ given by*

$$\begin{aligned}\mathcal{L}^*\phi(x) = & \sum_{i,j=1}^d \partial_i \partial_j \left(a_{ij}(x)\, \phi^\alpha(x)\right) \\ & - \sum_{i,\beta=1}^d \partial_i \left(b_i^{\alpha\beta}(x)\, \phi^\beta(x)\right) + \sum_{\beta=1}^d c^{\alpha\beta}(x)\, \phi^\beta(x)\end{aligned}$$

*maps vector fields $\phi$ that are twice continuously differentiable, bounded with bounded first and second derivatives, into vector fields $\mathcal{L}^*\phi$ hat are bounded continuous.*

We prepare the proof by the explicit computation of $[\sigma_k, [\sigma_k, B]]$. We have

$$\begin{gathered}[\sigma_k, [\sigma_k, B]] = \\ (\sigma_k \cdot \nabla)[\sigma_k, B] - ([\sigma_k, B] \cdot \nabla)\, \sigma_k\end{gathered}$$

$$\begin{aligned}&= (\sigma_k \cdot \nabla)(\sigma_k \cdot \nabla)\, B_t - (\sigma_k \cdot \nabla)(B_t \cdot \nabla)\, \sigma_k \\ &- ((\sigma_k \cdot \nabla)\, B_t \cdot \nabla)\, \sigma_k + ((B_t \cdot \nabla)\, \sigma_k \cdot \nabla)\, \sigma_k.\end{aligned}$$

All terms can be expressed by means of $Q$, after the following remarks. The function $Q(x,y)$ is defined on $\mathbb{R}^d \times \mathbb{R}^d$ with values in matrices $\mathbb{R}^{d\times d}$. When we differentiate $Q^{\alpha\beta}(x,y)$ with respect to the first set of components, we write $\left(\partial_i^{(1)} Q^{\alpha\beta}\right)(x,y)$:

$$\left(\partial_i^{(1)} Q^{\alpha\beta}\right)(x,y) = \lim_{\epsilon\to 0} \frac{Q^{\alpha\beta}(x+\epsilon e_i, y) - Q^{\alpha\beta}(x,y)}{\epsilon}$$

while when we differentiate $Q^{\alpha\beta}(x,y)$ with respect to the second set of components, we write $\left(\partial_i^{(2)} Q^{\alpha\beta}\right)(x,y)$. We have

$$\left(\partial_i^{(1)} Q^{\alpha\beta}\right)(x,y) = \partial_{x_i}\left(Q^{\alpha\beta}(x,y)\right) = \sum_{k=1}^{\infty} \left(\partial_i \sigma_k^{\alpha}\right)(x)\, \sigma_k^{\beta}(y)$$

$$\left(\partial_i^{(2)} Q^{\alpha\beta}\right)(x,y) = \partial_{y_i}\left(Q^{\alpha\beta}(x,y)\right) = \sum_{k=1}^{\infty} \sigma_k^{\alpha}(x) \left(\partial_i \sigma_k^{\beta}\right)(y).$$

Hence, when we evaluate at $y = x$,

$$\sum_{k=1}^{\infty} \left(\partial_i \sigma_k^{\alpha}\right)(x)\, \sigma_k^{\beta}(x) = \left(\partial_i^{(1)} Q^{\alpha\beta}\right)(x,x)$$

$$\sum_{k=1}^{\infty} \sigma_k^{\alpha}(x) \left(\partial_i \sigma_k^{\beta}\right)(x) = \left(\partial_i^{(2)} Q^{\alpha\beta}\right)(x,x).$$

Similarly,

$$\left(\partial_j^{(1)} \partial_i^{(1)} Q^{\alpha\beta}\right)(x,y) = \partial_{x_j}\partial_{x_i}\left(Q^{\alpha\beta}(x,y)\right) = \sum_{k=1}^{\infty} \left(\partial_j \partial_i \sigma_k^{\alpha}\right)(x)\, \sigma_k^{\beta}(y)$$

whence, at $y = x$,

$$\sum_{k=1}^{\infty} \left(\partial_j \partial_i \sigma_k^{\alpha}\right)(x)\, \sigma_k^{\beta}(x) = \left(\partial_j^{(1)} \partial_i^{(1)} Q^{\alpha\beta}\right)(x,x)$$

and so on for the other second derivatives. Let us denote by $[\sigma_k, [\sigma_k, B]]^{(\alpha)}$ the $\alpha$-component of the vector $[\sigma_k, [\sigma_k, B]]$.

**Lemma 5.1.4.**

$$\sum_k [\sigma_k, [\sigma_k, B]]^{(\alpha)}(x)$$

$$= \sum_{i,j=1}^{d} Q^{ij}(x,x)\,\partial_i\partial_j B^{\alpha}(x)$$

$$+ \sum_{i=1}^{d}\sum_{\gamma=1}^{d} \partial_\gamma^{(2)} Q^{\gamma i}(x,x)\,\partial_i B^{\alpha}(x) - \sum_{i,\beta=1}^{d} 2\left(\partial_\beta^{(2)} Q^{i\alpha}\right)(x,x)\,\partial_i B^{\beta}(x)$$

$$+ \sum_{\beta,\gamma=1}^{d} \partial_\beta^{(1)}\partial_\gamma^{(2)} Q^{\gamma\alpha}(x,x)\,B^{\beta}(x) - \sum_{\gamma,\beta=1}^{d} \left(\partial_\gamma^{(2)}\partial_\beta^{(2)} Q^{\gamma\alpha}\right)(x,x)\,B^{\beta}(x).$$

*Therefore the operator $\mathcal{L}$ has coefficients given by*

$$a_{ij}(x) = \frac{1}{2} Q^{ij}(x,x)$$

$$b_i^{\alpha\beta}(x) = \frac{1}{2}\sum_{\gamma=1}^{d} \partial_\gamma^{(2)} Q^{\gamma i}(x,x)\,\delta_{\alpha\beta} - \left(\partial_\beta^{(2)} Q^{i\alpha}\right)(x,x)$$

$$c^{\alpha\beta}(x) = \frac{1}{2}\sum_{\gamma=1}^{d} \partial_\beta^{(1)}\partial_\gamma^{(2)} Q^{\gamma\alpha}(x,x) - \frac{1}{2}\sum_{\gamma=1}^{d}\left(\partial_\gamma^{(2)}\partial_\beta^{(2)} Q^{\gamma\alpha}\right)(x,x).$$

*Proof.*

$$
\begin{aligned}
&\sum_k (\sigma_k \cdot \nabla) (\sigma_k \cdot \nabla) B_t^i \\
&\sum_k (\sigma_k \cdot \nabla) (B_t \cdot \nabla) \sigma_k^i - \sum_k ((\sigma_k \cdot \nabla) B_t \cdot \nabla) \sigma_k^i \\
&+ \sum_k ((B_t \cdot \nabla) \sigma_k \cdot \nabla) \sigma_k^i \\
&= \sum_k \sum_{\alpha\beta} \sigma_k^\alpha \partial_\alpha \left( \sigma_k^\beta \partial_\beta B_t^i \right) \\
&- \sum_k \sum_{\alpha\beta} \sigma_k^\alpha \partial_\alpha \left( B_t^\beta \partial_\beta \sigma_k^i \right) - \sigma_k^\alpha \partial_\alpha B_t^\beta \partial_\beta \sigma_k^i + B_t^\alpha \partial_\alpha \sigma_k^\beta \partial_\beta \sigma_k^i \\
&= \sum_k \sum_{\alpha\beta} \left( \sigma_k^\alpha \sigma_k^\beta \partial_\alpha \partial_\beta B_t^i + \sigma_k^\alpha \partial_\alpha \sigma_k^\beta \partial_\beta B_t^i - \sigma_k^\alpha B_t^\beta \partial_\alpha \partial_\beta \sigma_k^i \right) \\
&+ \sum_k \sum_{\alpha\beta} \left( -\sigma_k^\alpha \partial_\alpha B_t^\beta \partial_\beta \sigma_k^i - \sigma_k^\alpha \partial_\alpha B_t^\beta \partial_\beta \sigma_k^i + B_t^\alpha \partial_\alpha \sigma_k^\beta \partial_\beta \sigma_k^i \right) \\
&= \sum_{\alpha\beta} Q^{\alpha\beta} (x, x) \partial_\alpha \partial_\beta B_t^i \\
&+ \left( \partial_\alpha^{(2)} Q^{\alpha\beta} \right) (x, x) \partial_\beta B_t^i - \left( \partial_\alpha^{(2)} \partial_\beta^{(2)} Q^{\alpha i} \right) (x, x) B_t^\beta \\
&+ \sum_{\alpha\beta} \left( -2 \partial_\beta^{(2)} Q^{\alpha i} \partial_\alpha B_t^\beta + \partial_\alpha^{(1)} \partial_\beta^{(2)} Q^{\beta i} B_t^\alpha \right).
\end{aligned}
$$

The result of the lemma is just a rewriting of this expression. □

Now, we do the proof of the Proposition 5.1.3.

*Proof of Proposition 5.1.3.* Let us set

$$
R_0 := \sum_{\alpha\beta} \left( \left( \partial_\alpha^{(2)} Q^{\alpha\beta} \right) (x, x) \partial_\beta B_t^i - \left( \partial_\alpha^{(2)} \partial_\beta^{(2)} Q^{\alpha i} \right) (x, x) B_t^\beta \right) B
$$

$$
+ \sum_{\alpha\beta} \left( \left( -2 \partial_\beta^{(2)} Q^{\alpha i} \partial_\alpha B_t^\beta + \partial_\alpha^{(1)} \partial_\beta^{(2)} Q^{\beta i} B_t^\alpha \right) \right) B
$$

Then we have

$$
\begin{aligned}
-\int_{\mathbb{R}^d} \mathcal{L}B(x)\cdot B(x)\,dx &= -\sum_i \int_{\mathbb{R}^d} \sum_{\alpha\beta} Q^{\alpha\beta}(x,x)\,\partial_\alpha\partial_\beta B^i(x)\,B^i(x)\,dx + R_0 \\
&= \sum_i \int_{\mathbb{R}^d} \sum_{\alpha\beta} Q^{\alpha\beta}(x,x)\,\partial_\beta B^i(x)\,\partial_\alpha B^i(x)\,dx \\
&+ \sum_i \int_{\mathbb{R}^d} \sum_{\alpha\beta} \partial_\alpha Q^{\alpha\beta}(x,x)\,\partial_\beta B^i(x)\,B^i(x)\,dx + R_0 \\
&\geq \nu \sum_i \int_{\mathbb{R}^d} \left|\nabla B^i(x)\right|^2 dx \\
&- \sum_{i\alpha\beta} \int_{\mathbb{R}^d} \left|\partial_\alpha Q^{\alpha\beta}(x,x)\right| \left|\partial_\beta B^i(x)\right| \left|B^i(x)\right| dx - |R_0| \\
&= \nu \int_{\mathbb{R}^d} |DB(x)|^2\,dx - R_1 - |R_0|
\end{aligned}
$$

with $R_1$ defined by the identity. The estimates on $|R_0|$ are similar to the estimate on $R_1$, so we limit ourselves to this one. We have

$$R_1 \leq C_1 \sum_{i\alpha\beta} \int_{\mathbb{R}^d} \left|\partial_\beta B^i(x)\right| \left|B^i(x)\right| dx$$

because we have assumed that $Q$ has bounded derivatives,

$$\leq C_2 \int_{\mathbb{R}^d} |DB(x)|\,|B(x)|\,dx \leq \frac{\nu}{4} \int_{\mathbb{R}^d} |DB(x)|^2\,dx + C_3 \int_{\mathbb{R}^d} |B(x)|^2\,dx.$$

Here we have denoted by $C_i > 0$ some constants, possibly depending on $\nu$ and other factors, but not on $B$. In the analogous estimates for $|R_0|$,

$$|R_0| \leq \frac{\nu}{4} \int_{\mathbb{R}^d} |DB(x)|^2\,dx + C_4 \int_{\mathbb{R}^d} |B(x)|^2\,dx$$

we use the assumption that the second derivatives of $Q$ are bounded. We conclude that

$$-\int_{\mathbb{R}^d} \mathcal{L}B(x)\cdot B(x)\,dx \geq \frac{\nu}{2} \int_{\mathbb{R}^d} |DB(x)|^2\,dx - (C_3 + C_4) \int_{\mathbb{R}^d} |B(x)|^2\,dx.$$

$\square$

**Interpolation Inequality.**

**Lemma 5.1.5.** *If $f, h \in W^{1,2}\left(\mathbb{R}^d\right)$ and $g \in L^p\left(\mathbb{R}^d\right)$ for some $p > d$, then*

$$\int_{\mathbb{R}^d} f(x)\, g(x)\, \partial_i h(x)\, dx \leq C \|g\|_{L^p(\mathbb{R}^d)} \|f\|_{W^{1,2}(\mathbb{R}^d)} \|h\|_{W^{1,2}(\mathbb{R}^d)}$$

*where $C > 0$ is a constant independent of $f, g, h$ and for every $\epsilon > 0$ there is a constant $C_\epsilon > 0$ such that*

$$\int_{\mathbb{R}^d} f(x)\, g(x)\, \partial_i h(x)\, dx$$

$$\leq \epsilon \|h\|^2_{W^{1,2}(\mathbb{R}^d)} + \epsilon \|f\|^2_{W^{1,2}(\mathbb{R}^d)} + C_\epsilon \|g\|^{\frac{2p}{p-d}}_{L^p(\mathbb{R}^d)} \|f\|^2_{L^2(\mathbb{R}^d)}.$$

*Proof.* See [42] □

### 5.1.2 Existence and Uniqueness.

**Theorem 5.1.6.** *Under assumptions (5.4), (5.5), (5.8), (5.9), (5.10), (5.6), (5.7), a quasiregular weak solution of the Cauchy problem (5.1) exists.*

*Proof.* See [42] □

**Theorem 5.1.7.** *Under the assumptions of the previous theorem, let $B^i$, $i = 1, 2$, be two quasi-regular weak solutions of equation (5.1) with the same initial condition $B_0$. Assume that $\int B^i(t,x)\varphi(x)dx$ is $\overline{\mathcal{G}}_t$-adapted, for both $i = 1, 2$, for every $\varphi \in C_c^\infty\left(\mathbb{R}^3, \mathbb{R}^3\right)$. Then $B^1 = B^2$.*

*Proof. Step 0. Set of solutions.* Remark that the set of quasiregular weak solutions is a linear subspace of $L^2\left(\Omega \times [0,T] \times \mathbb{R}^3\right)$, because the stochastic advection equation is linear, and the regularity conditions is a linear constraint. Therefore, it is enough to show that a quasiregular weak solution $B$ with initial condition $B_0 = 0$ vanishes identically.

*Step 1. $V = 0$.* Let $V(t,x) = \mathbb{E}\left[B(t,x)\, e_f(t)\right]$, with $f \in L^2([0,T],\mathbb{R}^n) \cap L^\infty([0,T],\mathbb{R}^n)$. If we prove that $V = 0$, for arbitrary $f$, by Lemma 5.1.1 we deduce $B = 0$. The function $V$ satisfies

$$\partial_t V + [v + h, V] = \mathcal{L} V$$

with initial condition $V_0 = 0$, where $h(t,x) := \sum_{k=1}^{n} f_k(t)\,\sigma_k(x)$. It is thus sufficient to prove that a solution $V$ (in weak sense) of class $L^2([0,T];H^1(\mathbb{R}^3)) \cap C([0,T];L^2(\mathbb{R}^3))$ of this equation, such that $V_0 = 0$, is identically equal to zero. Let us see that this is a classical result of the variational theory of evolution equations.

Let $\mathcal{V} \subset \mathcal{H} \subset \mathcal{V}'$ be the Gelfand triple defined by

$$\mathcal{H} = L^2_\sigma\left(\mathbb{R}^3, \mathbb{R}^3\right)$$
$$\mathcal{V} = H^1_\sigma\left(\mathbb{R}^3, \mathbb{R}^3\right)$$

where the subscript $\sigma$ denotes the fact that we take these vector fields with divergence equal to zero. The norm $|.|_{\mathcal{H}}$ and scalar product $\langle .,. \rangle_{\mathcal{H}}$ are the usual ones, and the norm $\|.\|_{\mathcal{V}}$ in $\mathcal{V}$ is defined by

$$\|f\|^2_{\mathcal{V}} = \sum_{i=1}^{3} \int_{\mathbb{R}^3} \left|\nabla f^i(x)\right|^2 dx + \int_{\mathbb{R}^3} |f(x)|^2 \, dx.$$

Let $a : [0,T] \times \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ be the bilinear form defined on smooth fields $f, g$ as

$$a(t,f,g) = -\int_{\mathbb{R}^3} \mathcal{L}f(x) \cdot g(x)\, dx + \int_{\mathbb{R}^3} [v+h, f](x) \cdot g(x)\, dx$$

and extended to $\mathcal{V} \times \mathcal{V}$ by one integration by parts of the second order term in $\mathcal{L}$; moreover, since $v$ is not differentiable, we have to interpret also one term in $\int_{\mathbb{R}^3} [v+h, f](x) \cdot g(x)\, dx$ by integration by parts.

More precisely,

$$
\begin{aligned}
a(t,f,g) = & \sum_{i,j,\alpha=1}^{3} \int_{\mathbb{R}^3} a_{ij}(x)\, \partial_j f^{\alpha}(x)\, \partial_i g^{\alpha}(x)\, dx \\
& + \sum_{i,j,\alpha=1}^{3} \int_{\mathbb{R}^3} g^{\alpha}(x)\, \partial_j f^{\alpha}(x)\, \partial_i a_{ij}(x)\, dx \\
& - \sum_{i,\alpha,\beta=1}^{3} \int_{\mathbb{R}^3} b_i^{\alpha\beta}(x)\, \partial_i f^{\beta}(x)\, g^{\alpha}(x)\, dx - \\
& \sum_{\alpha,\beta=1}^{3} \int_{\mathbb{R}^3} c^{\alpha\beta}(x)\, f^{\beta}(x)\, g^{\alpha}(x)\, dx \\
& + \sum_{\alpha,\beta=1}^{3} \int_{\mathbb{R}^3} \left(v^{\alpha}(t,x) + h^{\alpha}(t,x)\right) \partial_{\alpha} f^{\beta}(x)\, g^{\beta}(x)\, dx \\
& + \sum_{\alpha,\beta=1}^{3} \int_{\mathbb{R}^3} \left(v^{\beta}(t,x) + h^{\beta}(t,x)\right) \partial_{\alpha} \left(f^{\alpha}(x)\, g^{\beta}(x)\right) dx
\end{aligned}
$$

where we recall that $\partial_i a_{ij}$ is bounded continuous. Then the weak form of equation $\partial_t V + [v+h, V] = \mathcal{L}V$, with $V_0 = 0$, is equivalent to

$$\langle V(t), \phi \rangle_{\mathcal{H}} + \int_0^t a(s, V(s), \phi)\, ds = 0. \tag{5.13}$$

for all $\phi \in \mathcal{V}$. Uniqueness for equations (5.12) and (5.13) are equivalent, in the class $V \in L^2(0,T;\mathcal{V}) \cap C([0,T];\mathcal{H})$. It is known, see [67], that uniqueness (and existence) in this class holds when $a$ is measurable in the three variables, continuous and coercive in the last two variables, namely

$$|a(t,f,g)| \leq C \|f\|_{\mathcal{V}} \|g\|_{\mathcal{V}} \tag{5.14}$$

$$a(t,f,f) \geq \nu \|f\|_{\mathcal{V}}^2 - \lambda |f|_{\mathcal{H}}^2 \tag{5.15}$$

for some constants $C, \lambda \geq 0$, $\nu > 0$, for a.e. $t$ and all $f, g \in \mathcal{V}$. Let us prove these two properties. It is sufficient to check them on the subset of smooth compact support divergence free fields $f, g$.

Let us prove (5.14). The first four terms in the explicit expression for $a(t,f,g)$ can be bounded above by $C\|f\|_{\mathcal{V}}\|g\|_{\mathcal{V}}$ because $a_{ij}, \partial_i a_{ij}, b_i^{\alpha\beta}, c^{\alpha\beta}$ are bounded. The difficult terms are the last two. Again, since $h$ is bounded, the terms

$$\sum_{\alpha,\beta=1}^{3}\int_{\mathbb{R}^3} h^\alpha(t,x)\,\partial_\alpha f^\beta(x)\,g^\beta(x)\,dx$$

$$+\sum_{\alpha,\beta=1}^{3}\int_{\mathbb{R}^3} h^\beta(t,x)\,\partial_\alpha\left(f^\alpha(x)\,g^\beta(x)\right)dx$$

can be bounded above by $C\|f\|_{\mathcal{V}}\|g\|_{\mathcal{V}}$. It remains to bound

$$\sum_{\alpha,\beta=1}^{3}\int_{\mathbb{R}^3} v^\alpha(t,x)\,\partial_\alpha f^\beta(x)\,g^\beta(x)\,dx$$

$$+\sum_{\alpha,\beta=1}^{3}\int_{\mathbb{R}^3} v^\beta(t,x)\,\partial_\alpha\left(f^\alpha(x)\,g^\beta(x)\right)dx.$$

But here we use repeatedly the first claim of Lemma 5.1.5 and bound also these terms with $C\|f\|_{\mathcal{V}}\|g\|_{\mathcal{V}}$. We have proved (5.14).

Finally, let us show property (5.15). From Proposition 5.1.3, the part of $a(t,f,f)$ related to $-\int_{\mathbb{R}^3}\mathcal{L}f(x)\cdot f(x)\,dx$ is bounded below by

$$\nu\int_{\mathbb{R}^3}|\nabla f(x)|^2\,dx - C\int_{\mathbb{R}^3}|f(x)|^2\,dx.$$

The remaining terms, namely

$$\sum_{\alpha,\beta=1}^{3}\int_{\mathbb{R}^3}\left(v^\alpha(t,x)+h^\alpha(t,x)\right)\partial_\alpha f^\beta(x)\,f^\beta(x)\,dx \tag{5.16}$$

$$+\sum_{\alpha,\beta=1}^{3}\int_{\mathbb{R}^3}\left(v^\beta(t,x)+h^\beta(t,x)\right)\partial_\alpha\left(f^\alpha(x)\,f^\beta(x)\right)dx \tag{5.17}$$

are bounded above in absolute value by

$$\frac{\nu}{2}\int_{\mathbb{R}^3}|\nabla f(x)|^2\,dx + C\int_{\mathbb{R}^3}|f(x)|^2\,dx$$

because of Lemma 5.1.5, with a suitable choice of $\epsilon > 0$. This implies $a(t, f, f) \geq \frac{\nu}{2} \|f\|_{\mathcal{V}}^2 - C |f|_{\mathcal{H}}^2$.

*Step 2. Conclusion. Until now we have proved that, for every* $f \in L^2([0,T],\mathbb{R}^n) \cap L^\infty([0,T],\mathbb{R}^n)$, the function $(t,x) \mapsto \mathbb{E}[B(t,x)e_f(t)]$ is the zero element of the space $L^2(0,T;\mathcal{V}) \cap C([0,T];\mathcal{H})$. We have to deduce that $B = 0$.

Being $(t,x) \mapsto \mathbb{E}[B(t,x)e_f(t)]$ the zero element of $C([0,T];\mathcal{H})$, we know that for every $t \in [0,T]$ we have

$$\int_{\mathbb{R}^3} \mathbb{E}[B(t,x)e_f(t)]g(x)\,dx = 0$$

for all $g \in C_c^\infty(\mathbb{R}^3,\mathbb{R}^3)$; and this holds true for all $e_f \in \mathcal{D}$. By linearity of the integral and the expected value we also have that

$$\int_{\mathbb{R}^3} \mathbb{E}\left[B(t,x)\,Y\right] g(x)\,dx = 0 \tag{5.18}$$

for every random variable $Y$ which can be written as a linear combination of a finite number of $e_f(t)$ and by density also the restriction $f \in L^\infty([0,T],\mathbb{R}^n)$ can be removed. Since by Lemma 5.1.1 the span generated by $e_f(t)$ is dense in $L^2(\Omega,\overline{\mathcal{G}}_t)$, (5.18) holds for any $Y \in L^2(\Omega,\overline{\mathcal{G}}_t)$. Namely, we have

$$\mathbb{E}\left[\int_{\mathbb{R}^3} B(t,x)g(x)dx\,Y\right] = 0$$

for every $Y \in L^2(\Omega,\overline{\mathcal{G}}_t)$. Since, by assumption, $\int_{\mathbb{R}^3} B(t,x)g(x)dx$ is $\overline{\mathcal{G}}_t$-measurable, we deduce

$$\int_{\mathbb{R}^3} B(t,x)g(x)dx = 0.$$

This holds true for every $g \in C_c^\infty(\mathbb{R}^3,\mathbb{R}^3)$, hence $B(t,\cdot) = 0$. □

## 5.2 Hyperbolic Systems of Conservation Law

A large number of problems in physics and engineering are modeled by systems of conservation laws

$$\partial_t u(t,x) + div(F(u(t,x))) = 0\,, \tag{5.19}$$

here $u$ is called the conserved quantity, while $F$ is the flux. Examples for hyperbolic systems of conservation laws include the shallow water equations of oceanography, the Euler equations of gas dynamics, the magnetohydrodynamics (MHD) equations of plasma physics, the equations of nonlinear elastodynamics and the Einstein equations of general relativity. When the initial data is smooth, it is well known that the solution can develop shocks within finite time. Therefore, global solutions can only be constructed within a space of discontinuous functions. Moreover, when discontinuities are present, weak solutions may not be unique. A central issue is to regain uniqueness by imposing appropriate selection criteria. The well-posedness theorems within the class of entropy solutions, for the scalar case, were established by Kruzkov(see [58]). For general $n \times n$ systems, the powerful techniques of functional analysis cannot be used. The well-posedness general system of conservation laws has been established only for initial data with sufficiently small total variation, see for instance [8], [28] and [91].

We also recall that in 1995 was introduced by Lions, Perthame and Tadmor [68] the notion of called kinetic solution for scalar conservation law. It relies on a new equation, the so-called kinetic formulation, that is derived from the conservation law at hand and that (unlike the original problem) possesses a very important feature - linearity. The two notions of solution, i.e. entropy and kinetic, are equivalent whenever both of them exist, nevertheless, kinetic solutions are more general as they are well defined even in situations when neither the original conservation law or the corresponding entropy inequalities can be understood in the sense of distributions.

### 5.2.1 Stochastic Conservation law.

Recently the effect of stochastic forcing on nonlinear conservation laws driven by space-time white noise has been largely studied, see for instance [13, 30, 37, 53]. For other hand, in [69] and [70] the authors introduced the theory of pathwise solutions to study the stochastic conservation law driven by is continuous noise. We present the recent result of Guess and Maurelli in [49] on regularization by noise in scalar conservation law. We consider the stochastic conservation law

$$\partial_t u(t,x) + (b(x,u(t,x)) + \frac{B}{dt})\nabla u = 0\,, \tag{5.20}$$

**Definition of Solution.**

We star defining of an entropy solution.

**Definition 5.2.1.** *A (stochastic) kinetic measure es a map $m : \Omega :\to \mathcal{M}\big([0,T]\times\mathbb{R}^d\times\mathbb{R}^d\big)$ weakly measurable, satisfying the following properties :*

- $m \in L^\infty\big(\Omega, \mathcal{M}\big([0,T]\times\mathbb{R}^d\times\mathbb{R}^d\big)\big)$,
- *$m$ is non-negative and support on $[0,T]\times\mathbb{R}^d\times[-R,-R]$ for some $R>0$.*
- *For any test functions $\varphi \in C_0^\infty([0,T]\times\mathbb{R}^d\times R)$, the process $(t,\omega)\to\int_{[0,T]\times\mathbb{R}^d\times R}\varphi\, dm$ is an adapted process.*

**Definition 5.2.2.** *Let $b \in L^1_{loc}(\mathbb{R}^{d+1})$ and $\mathrm{div} b \in L^1_{loc}(\mathbb{R}^{d+1})$ and let $u_0 \in L^1\cap L^\infty(\mathbb{R}^d)$, An entropy solution to equation (5.20) is a measurable function $u : \Omega\times[0,T]\times\mathbb{R}^d\times\to\mathbb{R}$ such that $\chi(t,\omega,x,\xi) = \chi(u(t,x,\omega),\xi) = 1_{\xi<u(t,x)} - 1_{\xi<0}$ satisfies the following properties*

1. *$\chi \in L^\infty([0,T]\times\Omega, L^1(\mathbb{R}^d\times\mathbb{R}))$ and is supported in $[0,T]\times\Omega\times[-R,R]$ for some $R$,*
2. *is weakly progressively measurable $L^\infty(\mathbb{R}^d\times R)$ valued process.*
3. *there exists a bounded kinetic measure $m$ such that for any test function $\varphi\in C_0^\infty([0,T]\times\mathbb{R}^d\times\mathbb{R})$ it, holds,*

$$< \chi_t, \varphi_t > = < \chi_0, \varphi_0 > + \int_0^t < \chi, \partial_t\varphi + \mathrm{div}(\varphi b) > dr$$

$$+ \int_0^t < \chi, \nabla\varphi > dB_r + \int_0^t < \chi, \Delta\varphi > dr - \int_{[0,t]\times\mathbb{R}^d\times\mathbb{R}} \partial_\xi\varphi dm$$

*with $\chi_0 = 1_{u_0<\xi} - 1_{\xi<0}$.*

The function is called kinetic solution.

We observe that Stratonovich formulation is

$$< \chi_t, \varphi_t >=< \chi_0, \varphi_0 > + \int_0^t < \chi, \partial_t \varphi + \mathrm{div}(\varphi b) > dr$$

$$+ \int_0^t < \chi, \nabla \varphi > \circ dB_r + \int_{[0,t]\times\mathbb{R}^d\times\mathbb{R}} \partial_\xi \varphi dm$$

### Main result

The main result in that paper is the following theorem.

**Theorem 5.2.3.** *We suppose that* $b \in L^\infty_{loc}(\mathbb{R}, L^\infty(\mathbb{R}^d)) \cap L^1_{loc}(\mathbb{R}, W^1_{loc}(\mathbb{R}^d))$ *and* $\mathrm{div} b \in L^p(\mathbb{R}^d, L^\infty_{loc}(\mathbb{R}))$ *for some* $p > d$*. Then for any initial condition* $u_0^1, u_0^2 \in L^1 \cap L^\infty(\mathbb{R}^d)$ *the two corresponding entropy solution* $u^1$ *and* $u^2$ *satisfy*

$$\mathbb{E} \int |u^1 - u^2| \; dx \leq C \; \int |u_0^1 - u_0^2| dx$$

*for any* $t \in [0, T]$.

*Proof.* See [49]. □

**Example 5.2.4.** *We consider the case*

$$b(x, u) = 2sig(x) min(R, \sqrt{|x|})u.$$

*for* $R > 0$ *and the initial condition* $u_0 = 1_{[0,t]}$*. There are several entropy solutions of (5.19), including*

$$u^1(t,x) = \begin{cases} 1 \quad if \quad 0 \leq x \leq (\frac{t}{2} + 1)^2 \\ \\ 0 \qquad otherwise\,. \end{cases}$$

$$u^2(t,x) = \begin{cases} 1 \quad if \quad -(\frac{t}{2})^2 \leq x \leq (\frac{t}{2} + 1)^2 \\ \\ 0 \qquad otherwise\,. \end{cases}$$

*We observe that* $b$ *satisfies the conditions of the theorem 5.2.3.*

### 5.2.2 (2x 2) hyperbolic systems of conservation law

In this subsection we follow Olivera in [82]. We consider the following systems of conservation law

$$\begin{cases} \partial_t v(t,x) + Div\big(f(v)\big) = 0 \\ \\ \partial_t v(t,x) + Div(vu) = 0\,. \end{cases} \tag{5.21}$$

We point that in the $L^1 \cap L^\infty$ setting this systems ill-posedness since the classical DiPerna-Lions-Ambrossio theory of uniqueness of distributional solutions for transport/ continuity equation does not apply when the drift has $L^1 \cap L^2$ regularity.

We study the influence of the noise in the hyperbolic systems (5.21). More precisely, we consider following stochastic systems of conservation law

$$\begin{cases} \partial_t v(t,x) + Div\big(F(v(t,x)\big) = 0\,, \\ \\ \partial_t u(t,x) + Div\big((u + \dfrac{dB_t}{dt}) \cdot u(t,x)\big) = 0\,, \\ \\ v|_{t=0} = v_0,\ u_{t=0} = u_0\ \ . \end{cases} \tag{5.22}$$

We show the existence and uniqueness of entropy-admissible solutions for the stochastic systems of conservation law (5.22)

#### Hypothesis

We assume the following conditions

**Hypothesis 5.2.5.** *The flux $F$ satisfies*

$$F \in C^1 \tag{5.23}$$

*and the initial condition holds*

$$v_0 \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R}), u_0 \in L^2(\mathbb{R}) \cap L^1(\mathbb{R}). \tag{5.24}$$

**Definition of the Solution.**

**Definition 5.2.6.** *Let $\eta \in C^1(\mathbb{R})$ be a convex function. If there exist $q \in C^{\mathbb{R}}$ suct that for all $v$*

$$\eta'(v)F'(v) = q'(v)$$

*then $\eta, q$ is called an entropy-entropy flux pair of the conservation law*

$$\partial_t v(t,x) + Div\big(f(v)\big) = 0,\ v(t,0) = v_0(x).$$

**Definition 5.2.7.** *The stochastic process $v \in L^\infty([0,T] \times \mathbb{R}) \cap L^\infty([0,T], L^1(\mathbb{R}))$ and $u \in L^\infty([0,T], L^2(\Omega \times \mathbb{R})) \cap L^1([0,T] \times \Omega \times \mathbb{R})$ are called a entropy weak solution of the stochastic hyperbolic systems* (4.5) *when:*

- *$v$ is entropy solution of the conservation law*

$$\partial_t v(t,x) + Div\big(F(v)\big) = 0,\ v(t,0) = v_0(x).$$

  *That is, if for every entropy flux pair $\eta, q$ we have*

$$\partial_t \eta(v) + Div(q(v)) \leq 0$$

  *in the sense of distribution.*

- *For any $\varphi \in C_0^\infty(\mathbb{R})$, the real valued process $\int u(t,x)\varphi(x)dx$ has a continuous modification which is an $\mathcal{F}_t$-semimartingale, and for all $t \in [0,T]$, we have $\mathbb{P}$-almost surely*

$$\begin{aligned}\int_{\mathbb{R}} u(t,x)\varphi(x)dx = &\int_{\mathbb{R}} u_0(x)\varphi(x)\ dx \\ &+ \int_0^t \int_{\mathbb{R}} u(s,x)\, v(t,x)\partial_x\varphi(x)\ dxds \qquad (5.25) \\ &+ \int_0^t \int_{\mathbb{R}} u(s,x)\ \partial_x\varphi(x)\ dx \circ dB_s\,.\end{aligned}$$

### Existence and Uniqueness

**Lemma 5.2.8.** *Assume that hypothesis 5.2.5 holds. Then there exists entropy-weak solution of the hyperbolic systems* (5.22)*.*

*Proof. Step 1: Conservation law .* According to the classical theory of conservation law, see for instance [28], we have that there exist a uniqueness entropy solution of the conservation law

$$\partial_t v(t,x) + Div\big(F(v)\big) = 0,\ v(t,0) = v_0(x).$$

If the the initial condition $v_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ then the solution $v \in L^\infty([0,T]\times\mathbb{R}) \cap L^\infty([0,T], L^1(\mathbb{R}))$.

*Step 2: Primitive of $v$.* It easy to see that for any test function $\varphi \in C_0^\infty(\mathbb{R})$ we have

$$\int_{\mathbb{R}} v(t,x)\varphi(x)dx = \int_{\mathbb{R}} v_0(x)\varphi(x)\ dx + \int_0^t\int_{\mathbb{R}} F(v(s,x))\partial_x\varphi(x)\ dxds.$$

Let $\{\rho_\varepsilon\}_\varepsilon$ be a family of standard symmetric mollifiers. Then we get

$$\int_{\mathbb{R}} v(t,y)\rho_\varepsilon(x-y)dy = \int_{\mathbb{R}} v_0(y)\rho_\varepsilon(x-y)dy$$

$$+\int_0^t\int_{\mathbb{R}} F(v(s,y))\partial_y\rho_\varepsilon(x-y)dyds.$$

Integrating we have

$$\int_0^z v_\varepsilon(t,x)dx = \int_0^z v_0^\varepsilon(x)dz + \int_0^t (F(v)*\rho_\varepsilon)(z)ds.$$

We denoted $\bar{v}_\varepsilon(t,x) := \int_0^z v_\varepsilon(t,x)dx$.

*Step 3: Regularization.* We define the family of regularized coefficients given by

$$v^\epsilon(t,.) = (v(t,x) *_x \rho_\varepsilon)(t,.).$$

Clearly we observe that, for every $\varepsilon > 0$, any element $v^\varepsilon$, $u_0^\varepsilon$ are smooth (in space) and with bounded derivatives of all orders. We consider the solution of

$$\begin{cases} du^\varepsilon(t,x) + \nabla u^\varepsilon(t,x)\cdot\big(v^\varepsilon(t,x)dt + \circ dB_t\big) + \mathrm{div} b^\varepsilon(x)\, u^\varepsilon(t,x)dt = 0\,, \\ u^\varepsilon\big|_{t=0} = u_0^\varepsilon. \end{cases} \tag{5.26}$$

Following the classical theory of H. Kunita [62, Theorem 6.1.9] we get

$$u^\varepsilon(t,x) = u_0^\varepsilon(X_t^{-1,\varepsilon}(t,x))JX_t^{-1,\varepsilon}(t,x),$$

is the unique solution to the regularized equation (5.26), where

$$dX_t = v^\varepsilon(t,X_t)\,dt + dB_t\,, \qquad X_0 = x\,.$$

*Step 4: Itô Formula .* Applying the Itô formula to $\bar{v}_\varepsilon(t,X_t^\epsilon)$ we obtain

$$\bar{v}_\varepsilon(t,X_t^\epsilon) = \int_0^{X_t^\epsilon} u_0^\epsilon(x)dx + \int_0^t (F(v)*\rho_\varepsilon)(s,X_s^\epsilon)ds + \int_0^t v_\varepsilon^2(s,X_s^\epsilon)ds$$

$$+ \int_0^t v_\varepsilon(s,X_s^\epsilon)dB_s + \frac{1}{2}\int_0^t (\partial_x v_\varepsilon)(s,X_s^\epsilon)ds.$$

*Step 5: Boundeness.* Now, we have

$$\|\bar{v}_\varepsilon(t,X_t^\epsilon)\|_{L^\infty(\Omega\times[0,T]\times\mathbb{R})} \leq \|v\|_{L^\infty([0,T],L^1(\mathbb{R}))},$$

$$\|\int_0^{X_t^\epsilon} v_0^\epsilon(x)dx\|_{L^\infty(\Omega\times[0,T]\times\mathbb{R})} \leq \|v_0\|_{L^1(\mathbb{R})},$$

$$\|\int_0^t (F(v)*\rho_\varepsilon)(s,X_s^\epsilon)ds\|_{L^\infty(\Omega\times[0,T]\times\mathbb{R})} \leq C\|F(v)\|_{L^\infty},$$

$$\| \int_0^t v_\varepsilon^2(s, X_s^\epsilon) ds \|_{L^\infty(\Omega\times[0,T]\times\mathbb{R})} C \leq \|v\|_{L^2([0,T],L^\infty(\mathbb{R}))},$$

*Step 6 : Estimation on the Jacobain.*
We denote

$$\mathcal{E}\bigg( \int_0^t v_\epsilon(s, X_s) dB_s \bigg) = \exp\bigg\{ \int_0^t v_\epsilon(s, X_s^\epsilon) dB_s - \frac{1}{2}\int_0^t v_\epsilon^2(s, X_s^\epsilon) ds \bigg\},$$

We note that $\partial_x X_t$ verifies

$$\partial_x X_t = \exp\bigg\{ \int_0^t (\partial_x v_\epsilon)(s, X_s)\ ds \bigg\}.$$

From steps 4-5 we deduce

$$\mathbb{E}|\partial_x X_t|^{-1} \leq C\mathbb{E}\bigg( \int_0^t v_\epsilon(s, X_s) dB_s \bigg).$$

We observe that the processes $\mathcal{E}\bigg( \int_0^t v_\epsilon(s, X_s) dB_s \bigg)$, is martingale with expectation equal to one. Hence,

$$\mathbb{E}|\partial_x X_t|^{-1} \leq C.$$

*Step 7: Passing to the limit .*
Making the change of variables $y = X_t^{-1,\varepsilon}(x)$ we deduce

$$\int_{\mathbb{R}} \mathbb{E}[|u^\varepsilon(t,x)|^2]\, dx = \int_{\mathbb{R}} |u_0^\varepsilon(y)|^2 \mathbb{E}|JX_t^\varepsilon|^{-1} dy.$$

From last step we have

$$\int_{\mathbb{R}} \mathbb{E}[|u^\varepsilon(t,x)|^2]\, dx \leq C. \tag{5.27}$$

Therefore, the sequence $\{u^\varepsilon\}_{\varepsilon>0}$ is bounded in $L^\infty([0,T], L^2(\Omega\times\mathbb{R})) \cap L^1([0,T]\times\Omega\times\mathbb{R})$. Then there exists a convergent subsequence, which we denote also by $u^\varepsilon$, such that converge weakly in

$L^\infty([0,T], L^2(\Omega \times \mathbb{R}))$ to some process $u \in L^\infty([0,T], L^2(\Omega \times \mathbb{R})) \cap L^1([0,T] \times \Omega \times \mathbb{R})$.

Now, if $u^\varepsilon$ is a solution of (5.26), it is also a weak solution, that is, for any test function $\varphi \in C_0^\infty(\mathbb{R})$, $u^\varepsilon$ satisfies :

$$\begin{aligned}\int_{\mathbb{R}} u^\varepsilon(t,x)\varphi(x)dx = \int_{\mathbb{R}} u_0^\varepsilon(x)\varphi(x)\ dx \\ + \int_0^t\int_{\mathbb{R}} u^\varepsilon(s,x)\, v^\varepsilon(s,x)\partial_x\varphi(x)\ dxds \\ + \int_0^t\int_{\mathbb{R}} u^\varepsilon(s,x)\ \partial_x\varphi(x)\ dx\, dB_s + \frac{1}{2}\int_0^t\int_{\mathbb{R}} u^\varepsilon(s,x)\ \partial_x^2\varphi(x)\ dx\, ds\,.\end{aligned}$$

Thus, for prove existence of (5.22) is enough to pass to the limit in the above equation along the convergent subsequence found. This is made through of the same arguments of [40, theorem 15].

□

**Theorem 5.2.9.** *Under the conditions of hypothesis 5.2.5, uniqueness holds for entropy -weak solutions of the hyperbolic problem* (5.22)*.*

*Proof.* See [82]. □

**One Example.**

Consider the transport (or transportation) equations in which the continuity equation is adjoined with the inviscid Burgers equation

$$\begin{cases}\partial_t v(t,x) + Div\big(\frac{1}{2}v^2(t,x)\big) = 0\,, \\ \partial_t u(t,x) + Div\big((v(t,x)u(t,x)\big) = 0\,, \\ v\big|_{t=0} = v_0,\ u_{t=0} = u_0\ \ ,\end{cases} \tag{5.28}$$

This transport equations model the dynamics of particles that adhere to one another upon collision and has been studied as a simple cosmological model for describing the nonlinear formation of large-scale structures in the universe. We point that in [56] and [94] the authors proved existence of weak solutions via $\delta-$ shock for Riemann initial condition.

## 5.3 More results

- On the the effect of noise in Schrodinger equation we refer to the works of A. de Bouard and Debussche in [17], Debussche and Tsutsumi in [27].
- On regularization by noise in Schrodinger equation and Korteweg-de Vries (KdV) see Chuk and Gubinelli in [14] and [15].
- On the effect on noise in Euler equation and related equations see Falndoli, Gubinelli and Priola in [41], Barbatoa, Bessaihb and Ferrario in [6], D. Crisan, F. Flandoli and D. Holm in [22].
- Results for the 1-dimensional Vlasov-Poisson equation see Delarue, Flandoli, Vincenzi in [32].
- Regularization in non-local conservation law see Olivera in [83].

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